Richard Schnorrenberger

Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve

Florianópolis, SC

2017

Richard Schnorrenberger

Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve

Dissertação de mestrado em conformidade com as normas ABNT apresentado à comissão avaliadora como requisito parcial para a obtenção de título de mestre em economia.

Universidade Federal de Santa Catarina – UFSC

Departamento de Economia e Relações Internacionais

Programa de Pós-Graduação em Economia

Orientador: Guilherme Valle Moura

Florianópolis, SC

2017

Ficha de identificação da obra elaborada pelo autor, através do Programa de Geração Automática da Biblioteca Universitária da UFSC.

Schnorrenberger, Richard

Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve/ Richard Schnorrenberger – Florianópolis, SC, 2017.

113 p.

Orientador: Guilherme Valle Moura

Dissertação de Mestrado – Universidade Federal de Santa Catarina – UFSC
Departamento de Economia e Relações Internacionais
Programa de Pós-Graduação em Economia, 2017.

1. Fixed-income portfolio optimization. 2. Brazilian yield curve. 3. Dynamic Nelson-Siegel model. 4. Macroeconomic factors. 5. Yield curve forecasting. 6. Mean-variance approach. I. Moura, Guilherme Valle. II. Universidade Federal de Santa Catarina. III. Programa de Pós-Graduação em Economia. IV. Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve.

Richard Schnorrenberger

Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve

Dissertação de mestrado em conformidade com as normas ABNT apresentado à comissão avaliadora como requisito parcial para a obtenção de título de mestre em economia.

Prof. Dr. Jaylson Jair da Silveira, Dr.
Coordenador do Curso

Banca Examinadora:

Prof. Dr. Guilherme Valle Moura
Universidade Federal de Santa Catarina

Prof. Dr. João Frois Caldeira
Universidade Federal do Rio Grande do Sul

Prof. Dr. André Alves Portela
Universidade Federal de Santa Catarina

Prof. Dr. Newton Carneiro Affonso da Costa Junior
Universidade Federal de Santa Catarina

Florianópolis, SC
2017

Acknowledgements

Agradeço,

Ao meu orientador Guilherme Valle Moura, pela fundamental contribuição para o desenvolvimento deste trabalho, pelas ideias, pela compreensão, pela amizade e confiança. Me sinto grato por toda a sua dedicação, paciência e disponibilidade para atender todas as minhas demandas acadêmicas e estimular meu interesse nas ciências econômicas.

À todo o corpo docente do PPGECO, pelo incentivo ao desenvolvimento intelectual e científico, e pelo compromisso com a excelência acadêmica. Aos funcionários do PPGECO, por todo o apoio.

Ao CNPq pelo apoio financeiro, que é fundamental para permitir a dedicação exclusiva dos alunos aos seus estudos.

Aos meus colegas do PPGECO, pelas conversas, pelo compartilhamento de conhecimentos, e pela amizade durante esse tempo de convivência.

Aos membros da Banca Examinadora, pelo tempo dispensado na leitura deste trabalho e pelos comentários, sugestões e críticas.

À minha namorada Karina Buss, pelo companheirismo, paciência, amor, e principalmente, por ser a pessoa que mais acredita no meu potencial. Sou muito grato pela sua dedicação ao nosso relacionamento. Sou uma pessoa muito mais feliz por ter você em minha vida!

Aos meus pais Alípio e Érica, e irmão Rodrigo, por constituírem a base da minha educação. Sou muito grato por todo o carinho, pelo apoio dado para a minha trajetória acadêmica, e pelos momentos que passamos juntos.

"Much to learn you still have." ...
"This is just the beginning!"
(Yoda, Jedi Master)

Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve

Abstract

The study investigates the statistical and economic value of forecasted yields generated by dynamic yield curve models which incorporate a large macroeconomic dataset. The analysis starts off by modeling and forecasting the term structure of the Brazilian nominal interest rates using several specifications for the Dynamic Nelson-Siegel (DNS) framework, suggested by Diebold & Li (2006). The first exercise concerns the incorporation of macro factors extracted from a large macroeconomic dataset, including forward-looking variables, to compare the forecast performance between some macroeconomic representations of the DNS model and itself. The results for forecast horizons above three months support the evidence for the incorporation of one macro factor that summarizes broad macroeconomic information regarding mainly inflation expectations. The conclusion that macroeconomic information tends to improvement in yield curve forecasting extend results found in previous literature. In order to assess the economic value of those forecasted yields, a fixed-income portfolio optimization using the mean-variance approach of Markowitz (1952) is performed. The analysis indicate that good yield curve predictions are important to achieve economic gains from forecasted yields in terms of portfolio performance. Preferred forecasted yields for short forecast horizons perform quite well for optimal mean-variance portfolios with one-step-ahead estimates for fixed-income returns, while forecasted yields generated by a macroeconomic DNS specification outperforms in terms of portfolio performance with twelve-step-ahead estimates. Therefore, there is an economic and statistical gain from considering a large macroeconomic dataset to forecast the Brazilian yield curve dynamics, specially for longer forecast horizons and for medium- and long-term maturities.

Keywords: Fixed-income portfolio optimization. Brazilian yield curve. Dynamic Nelson-Siegel model. Macroeconomic factors. Yield curve forecasting. Mean-variance approach.

Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve

Resumo

O estudo investiga o valor estatístico e econômico dos rendimentos previstos por modelos dinâmicos da curva de juros que incorporam um grande conjunto de dados macroeconômicos. A análise parte da modelagem e previsão da estrutura a termo das taxas de juros nominais brasileiras, usando diversas especificações para o modelo Dinâmico de Nelson-Siegel (DNS), sugerido por Diebold & Li (2006). O primeiro exercício diz respeito à incorporação de macrofatores extraídos de um grande conjunto de dados macroeconômicos, incluindo variáveis de expectativas, para comparar o desempenho de previsão entre algumas representações macroeconômicas do modelo DNS e ele mesmo. Os resultados para horizontes de previsão acima de três meses apoiam a evidência para a incorporação de um fator macro que resume principalmente informações gerais sobre expectativas de inflação. A conclusão de que informação macroeconômica tende a aprimorar a previsão da curva de juros estende os resultados encontrados na literatura recente. Para avaliar o valor econômico dos rendimentos previstos, é realizada uma otimização de carteira de renda fixa usando a abordagem de média-variância de Markowitz (1952). A análise indica que boas previsões para as curvas de juros são importantes para obter ganhos econômicos com os rendimentos previstos em termos de desempenho do portfólio. Rendimentos previstos com maior precisão para horizontes de previsão curtos atingem bons resultados para portfólios ótimos que utilizam estimativas de um passo a frente para os retornos de renda fixa, enquanto que rendimentos previstos gerados por uma especificação macroeconômica do modelo DNS atingem bom desempenho para a otimização que utiliza estimativas de doze passos a frente. Portanto, há um ganho econômico e estatístico ao considerar um grande conjunto de dados macroeconômicos para prever a dinâmica da curva de juros brasileira, especialmente para horizontes de previsão mais longos e para maturidades de médio e longo prazo.

Palavras-chaves: Otimização de portfólio de renda fixa. Curva de juros brasileira. Modelo dinâmico de Nelson-Siegel. Fatores macroeconômicos. Previsão da curva de juros. Abordagem de média-variância.

List of Figures

List of Tables

List of abbreviations and acronyms

τ	vector of relevant maturities $\tau_1, \tau_2, ..., \tau_N$
N	number of relevant maturities τ_i
$P_t(\tau)$	price of the zero coupon bond of maturity τ at time t
$y_t(\tau)$	continuously compounded yield to maturity at time t of the zero coupon bond of maturity τ
NS	Nelson-Siegel model
DNS	Dynamic Nelson-Siegel model
AFNS	Arbitrage-free Nelson-Siegel model
L_t	level factor
S_t	slope factor
C_t	curvature factor
f_t	state vector at time t
μ	factor mean
y_t	vector of observed yields for N relevant maturities τ_i at time t
A	state transition matrix
Λ	factor loadings matrix of the measurement equation
η_t	state equation factor disturbances
ε_t	measurement equation disturbances
Q	covariance matrix of transition disturbances
H	covariance matrix of measurement disturbances
Y_t	vector of observed variables
P_t	covariance matrix of the state vector f_t
w_t	vector of optimal portfolio weights
$\mu_{r_{t\|t-h}}$	h-step-ahead expected returns

$\Sigma_{r_{t\|t-h}}$	covariance matrix of the h-step-ahead expected returns
δ	investor's risk aversion coefficient
c_i	periodic coupons in periods t_i
X	multivariate panel of macroeconomic variables
X^1	first macroeconomic factor
X^2	second macroeconomic factor
K	number of macroeconomic factors
DI-futuro	Brazilian Inter Bank Deposit Future Contract
T	in-sample observations
S	out-of-sample observations
PCA	principal component analysis
RMSFE	root mean squared forecast error
TRMSFE	trace root mean squared forecast error
DM	Diebold-Mariano
RW	random walk

Contents

1 Introduction

There is a wide heterogeneity between term structure models that try to fit and forecast the dynamic behavior of yield curves. Traditional term structure models decompose interest rates into a set of yield latent factors, such as level, slope and curvature (LITTERMAN; SCHEINKMAN, 1991). Even providing good in-sample fit (Nelson & Siegel (1987); Dai & Singleton (2000)) and satisfactory results for out-of-sample forecasts (Duffee (2002); Diebold & Li (2006)), the economic meaning of such models is limited since they neglect a macroeconomic environment that could affect interest rates of different maturities. Many yield curve models simply ignore macroeconomic linkages. Nonetheless, there are macroeconomic forces that shape the term structure, so that changes in macro variables can affect future yield curves and expectations of future interest rates (GÜRKAYNAK; WRIGHT, 2012). Thereby, researchers have begun to use a joint macro-finance modeling strategy, which provides the most comprehensive understanding of the term structure of interest rates.

The development of term structure models that integrate macroeconomic and financial factors is recent in economic research. Ang & Piazzesi (2003), Diebold et al. (2006) and Hördahl et al. (2006) provide the pioneering studies that incorporate macroeconomic information to explain the dynamics of the yield curve through time. Diebold et al. (2006) provide a macroeconomic interpretation of the Dynamic Nelson-Siegel (DNS) model, suggested by Diebold & Li (2006). They combine observable macroeconomic variables, basically related to real activity, inflation, and monetary policy, and yield factors into the Vector Autoregression (VAR) that governs the dynamics of factors. While these studies consistently find significant relationships between macroeconomic variables and government bond yields, they ignore potential macroeconomic information that could be useful for yield curve modeling and forecasting.

More recently, a literature that uses large macroeconomic datasets has emerged, based on the idea that monetary authorities use rich information sets to take decisions about short-term rates (BERNANKE; BOIVIN, 2003). Moench (2008) proposed to use the "Factor-Augmented VAR" (BERNANKE et al., 2005) procedure to jointly model the yield curve dynamics and macro factors extracted from a large macroeconomic dataset. Pooter et al. (2010) and Favero et al. (2012) also use "data-rich environments" for the term structure by extracting common macro factors through dimensionality reduction techniques, such as principal component analysis. In general, these studies consistently reveal that the inclusion of few macroeconomic principal components leads to better out-of-sample yield forecasts compared to benchmark models that use individual macro variables or do not incorporate macroeconomic information. The key point is that if dynamics of the yield curve and macroeconomy are correlated, the incorporation of macro factors into term structure models can improve forecast accuracy and possibly generate better information for fixed-income portfolio analytics.

Term structure models play an important role in fixed-income asset pricing, strategic asset allocation, and, of course, portfolio analytics. The evolution of the yield curve is essential to compute the risk and return characteristics of one's fixed-income portfolio (BOLDER, 2015). In order to take an active position in a fixed-income portfolio, based on the mean-variance approach of Markowitz (1952), dynamic yield curve models are used to generate yield forecasts for selected maturities, which are then used to compute expected fixed-income returns. The fixed-income portfolio problem essentially consists in predicting the distribution of returns for a set of securities and select the optimal vector of portfolio weights conditional on one's expected returns and risk preferences.

Although the mean-variance approach of Markowitz has been widely explored in the context of equity portfolios, little is known about portfolio optimization in fixed-income markets. A recent literature, kick-started by Kokn & Koziol (2006), that exploits the risk-return trade-off in bond returns has emerged. Kokn & Koziol (2006) employ the Vasicek (1977) model to perform a mean-variance bond portfolio selection. Caldeira et al. (2016) extend this approach by employing dynamic factor models for the term structure and derive simple closed-form expressions for expected bond returns and their covariance matrix based on forecasted yields. Thornton & Valente (2012) assess the economic value of the predictive power of forward rates for bond excess returns. These studies contribute to validate the use of the term structure models to perform mean-variance optimization in the fixed-income context. The present study solves an alternative version of the mean-variance optimization problem, following Caldeira et al. (2016), and uses datasets of Brazilian nominal interest rates.

This study contributes to the present literature by assessing the economic value of forecasted yields generated by yield curve models incorporating a large macroeconomic dataset. That is, it combines the benefits from incorporating macroeconomic information into term structure models and the use of those forecasted yields to assess their economic value through a portfolio optimization analysis. The incorporation of macroeconomic factors into term structure models has the theoretical premise of increasing the model's predictive power. In this sense, the main question is the following: Is there some economic gain, in terms of portfolio performance, from incorporating macroeconomic information into term structure models? Hence, the major purpose is to investigate the magnitude of the statistical and economic gain with the incorporation of a large macroeconomic dataset into the Dynamic Nelson-Siegel model.

The empirical evidence indicates that the incorporation of one macro factor, which summarizes broad macroeconomic information regarding mainly inflation expectations, contributes to improve yield curve predictions for 6- and 9-month-ahead forecast horizons, specially for medium and long-term maturities. Furthermore, estimates for alternative specifications of the DNS framework suggest that imposing further restrictions on factor dynamics can lead to forecast improvements in favor of some parsimonious specifications with less number of estimated parameters. In the context of portfolio selection, good yield curve predictions proved to be important to achieve better results in terms of portfolio performance. Parsimonious yield curve models without macroeconomic information and with better forecast accuracy for short forecast horizons perform quite well for optimal mean-variance portfolios with one-step-ahead estimates for fixed-income returns. On the other hand, forecasted yields generated by a macroeconomic specification for the term structure provide better information to perform a mean-variance portfolio optimization which uses twelve-step-ahead estimates for fixed-income returns.

The outline of the study is as follows. Part I is composed by a literature review that focuses on: (i) term structure of interest rates, Chapter 2; (ii) term structure models, mainly the class of Nelson-Siegel models, Chapter 3; (iii) the relationship between the term structure and macroeconomy, Chapter 4; and (iv) fixed-income portfolio optimization, Chapter 5. Part II discusses the theoretical models for the yield curve, the empirical data and the estimation methodology, which comprises a principal component analysis, the state-space model and Kalman filter, and the closed-form expressions for the distribution of fixed-income returns. Part III discusses the empirical results regarding in-sample and out-of-sample yield curve estimates and the application to fixed-income portfolio optimization. Finally, Chapter 14 involves the concluding remarks.

Part I

Literature Review

2 The term structure of interest rates

The term structure of interest rates expresses the relationship between spot rates from different maturities at any point in time, being obtained by prices or yields of fixed-income instruments negotiated in financial markets. Public bonds, for example, are instruments used in financing public debt and pay for their holder a monetary amount in some future date, known as maturity. Furthermore, the yield that the bond pays until its maturity is closely related to the time value of money, i.e., the idea that money available at the present time is worth more than the same amount in the future due to its potential earning capacity. Investors must be compensated for elements that deteriorate the value of money over time. The major example is inflation: an increase in price level before bond's maturity deteriorates its nominal value, causing loss of purchasing power for the bond's holder[1]. Hence, the core principle of fixed-income theory is based on the assumption that money can earn interest over time.

Diebold & Rudebusch (2013) discuss three key theoretical bond market constructs and the relationships among them: the discount curve, the forward rate curve, and the yield curve. The yield curve expresses the graphic construction of the term structure of interest rates by tracing yields against a set of maturities $\tau = (\tau_1, \tau_2, ..., \tau_N)$ for a given issuer at a point in time. Let $P_t(\tau)$ denote the price of a zero coupon bond, without default risk, of maturity τ in period t, and with maturity value equal to unity. Zero coupon bonds do not pay periodic coupons, so that investors receive bond's face value and earnings only at maturity. Thus, $P_t(\tau)$ is the present value of \$1 receivable τ periods ahead. Furthermore, $y_t(\tau)$ is its continuously compounded yield to maturity at time t[2]. The basic assumption is that $P_t(\tau)$ yields the present value of future cash flow promised by the issuer, discounted by a discount factor,

$$y_t(\tau) = -\frac{\log(P_t(\tau))}{\tau}, \tag{2.1}$$

where τ can assume any value in the set of possible maturities.

There is an immediate relationship between the yield curve and the discount curve, and the knowledge of one allows someone to build another. From yield curve (2.1) we obtain the discount curve,

$$P_t(\tau) = e^{-\tau y_t(\tau)}. \tag{2.2}$$

For a bond with maturity τ that pays periodic coupons c_i in periods t_i, the discount curve

[1] Another element is the credit risk, which deteriorates bond's value by increasing default probability of the bond issuer.

[2] The yield $y_t(\tau)$ is commonly used as an annualized rate, and it is thus considered in this study.

follows:

$$P_t(\tau) = \sum_{i=1}^{n} c_i e^{-(t_i - t) y_t(\tau)}.$$

The forward rate at time t, applied to interval between τ_1 and τ_2, relates to the spot rate[3] $y_t(\tau)$, and is defined as

$$f_t(\tau_1, \tau_2) = \frac{\tau_2 y_t(\tau_2) - \tau_1 y_t(\tau_1)}{\tau_2 - \tau_1} = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} y_t(u) du. \tag{2.3}$$

The forward rate expresses the marginal rate of return for holding a bond for an additional period $\tau_2 - \tau_1 > 0$. In other words, forward rates represent the rate on a commitment to buy a one-period bond at a future date, as if investors were performing transactions at a current interest rate applied to a future date. The limit case of (2.3), when τ_2 is quite close to τ_1, expresses the nominal curve of instantaneous forward rates,

$$f_t(\tau) = -P_t'(\tau)/P_t(\tau), \tag{2.4}$$

where $P_t'(\tau)$ represents the first derivate of the function $P_t(\tau)$. The function $f_t(\tau)$ describes the instantaneous rate of return of an investment for a short time.

Eq. (2.4) reveals that discount curve and the forward rate curve are fundamentally related, so that knowledge of the discount curve lets one calculate the forward rate curve. Finally, Eqs. (2.2) and (2.4) imply the relationship between the yield curve and forward rate curve:

$$y_t(\tau) = \frac{1}{\tau} \int_0^{\tau} f_t(u) du. \tag{2.5}$$

In particular, the spot rate $y_t(\tau)$ is an arithmetic average of the instantaneous forward rates. Therefore, the yield curve $y_t(\tau)$ is the average rate of decline in forward rates for the interval between 0 and τ.

Thus, knowledge of any one of $P_t(\tau)$, $y_t(\tau)$, or $f_t(\tau)$ implies knowledge of the other two curves, the three are effectively interchangeable (DIEBOLD; RUDEBUSCH, 2013). Hence, with no loss of generality it is possible to work with any of those curves. According to (BOLDER, 2015), a wide range of important information is embedded in those curves ranging from fundamental issues such as the time value of money, expected monetary policy actions, and inflationary expectations to more complicated, but equally important ideas such as risk premia, assessments of creditworthiness, and relative liquidity.

In practice, yield curves are not observed, because in real financial markets one can observe just a few bond maturities (e.g., three-month, twelve-month, five-year, ...), which only allows to plot discrete points of yields against their maturities. The theoretical yield curve is a smooth and continuous curve constructed through these discrete points observed in financial markets. For this reason, yield curves must be estimated from observed bond

[3] The spot rate is commonly used to represent the lower term deposit rate of return possible in the economy.

prices. At any point of time t, there will be a set of information about bonds with different maturities τ and different cash flow payments, which can be used to construct those curves. This exercise of adjusting the term structure of interest rates across the whole maturity spectrum is the role of yield curve modeling.

2.1 Interpolation

As a matter of fact, the issuance of debt bonds is not continuous in time, besides being different by maturity and cash flow payment. As a result, not all possible maturities will be observed at any time t and one will only have a few discrete points of yields against maturities. However, obtaining a complete and continuous yield curve that reflects the same class of securities is of great importance for policy makers and financial analysts who need to associate interest rates for any maturity. That is, the empirical yield curve needs to be converted into a smooth and continuous curve that connects the discrete points observed in financial markets. The first approach to yield curve construction is due to McCulloch (1971), who employs a cubic spline discount function interpolation to model the yield curve between missing maturities. An improved and most popular alternative to yield curve construction is due to Fama & Bliss (1987), who construct yields from estimated forward rates at the observed maturities. According to Diebold & Rudebusch (2013), this approach sequentially constructs the forward rates necessary to price successively longer-maturity bonds. Those forward rates are often called "unsmoothed Fama-Bliss" forward rates, and they are transformed to unsmoothed Fama-Bliss yields through Eq. (2.5).

Choudhry (2011) discusses some popular interpolation methods to fit a smooth yield curve using observed bond prices. The most common approaches refer to linear interpolation, logarithmic interpolation, polynomials, cubic splines, and statistical models, which use the parametric form of Nelson & Siegel (1987). As stated in Caldeira (2011), commonly, polynomials with known forms (such as Laguerre polynomials) are used as functions that link maturities to interest rates. Although, there is no theoretical model behind this approach. It is assumed that the term structure can be explained by a polynomial function f. Since one estimates the coefficients of f that best fit the actual rates observed in financial markets, it is possible to obtain the interest rate associated with any maturity. The term structure obtained with this method is called interpolated yield curve.

Cubic splines interpolation, proposed by McCulloch (1971), employes piecewise combinations of cubic functions to fit the yield curve. In other words, a piecewise polynomial smoothly connects the yield curve between each pair of vertices (or knot points) of the observed yield data. Hence, the goal of cubic spline interpolation is to get an interpolation formula that is continuous in both the first and second derivatives, both within the intervals and at the interpolating vertices. The computation of interpolated yields between two vertices requires the settlement of some smoothness criteria for the polynomial function, which ensures a continuously differentiable curve: (i) the level of the cubic spline and its two first derivatives are identical at the knot points and (ii) the second partial derivative of each curve point must be continuous between two vertices. The technique produces interpolated curves that preserve certain smoothness, precision and rigor in its process.

Assuming that maturities $\tau_1, ..., \tau_N$ and yields $y_t(\tau_i), ..., y_t(\tau_N)$ are observed, the yield for any maturity τ, where τ is a point between two vertices ($\tau_i \leq \tau \leq \tau_{i+1}$), follows the function,

$$y_t(\tau) = a_i + b_i(\tau - \tau_i) + c_i(\tau - \tau_i)^2 + d_i(\tau - \tau_i)^3, \tag{2.6}$$

where

$$\begin{aligned} y_t^{'}(\tau) &= b_i + 2c_i(\tau - \tau_i) + 3d_i(\tau - \tau_i)^2, \\ y_t^{''}(\tau) &= 2c_i + 6d_i(\tau - \tau_i), \\ y_t^{'''}(\tau) &= 6d_i. \end{aligned}$$

The idea of cubic splines interpolation is to complete each cubic spline by assessing the coefficients $(a_i, b_i, c_i, d_i,)$ for $1 \leq i \leq N-1$. That is, in each interval $(\tau_i,\ \tau_{i+1})$, the method can fit a flexible line through the points $(\tau_i,\ y_t(\tau_i))$ and $(\tau_{i+1},\ y_t(\tau_{i+1}))$ using the formula given by (2.6). The imposed constraints that contribute to form a system of simultaneous equations are:

(i). The interpolation function passes through given vertices, so that $a_i = y_t(\tau_i)$ for $i = 1, 2, ..., N-1$ and $a_{N-1} + b_{N-1}h_{N-1} + c_{N-1}h_{N-1}^2 + d_{N-1}h_{N-1}^3 = y_t(\tau_N) = a_N$, where $h_i = \tau_{i+1} - \tau_i$;

(ii). The interpolation function passes through given vertices, so that $a_i + b_ih_i + c_ih_i^2 + d_ih_i^3 = a_{i+1}$ for $i = 1, 2, ..., N-2$;

(iii). The interpolation function is continually differentiable, so that $b_i + 2c_ih_i + 3d_ih_i = b_{i+1}$ for $i = 1, 2, ..., N-2$.

This constitutes a system with $3N - 4$ equations with $4N - 4$ unknown parameters. Therefore, there is still N linear constraints to be specified.

Defining the derivative of the interpolation function to the right of the upper endpoint as,

$$b_N = b_{N-1} + 2c_{N-1}h_{N-1} + 3d_ih_{N-1}^2.$$

In the general case, the specification of the remaining N constraints is equivalent to the specification of $(b_1, b_2, ..., b_N)$. In particular, if one defines $(b_1, b_2, ..., b_N)$, then, $(c_1, c_2, ..., c_N)$

and $(d_1, d_2, ..., d_N)$ follows straightforward, for each i, and two equations with two unknown parameters come up:

$$\begin{aligned} m_i &= \frac{a_{i+1} - a_i}{h_i}, \\ c_i &= \frac{3m_i - b_{i+1} - 2b_i}{h_i}, \\ d_i &= \frac{b_{i+1} + b_i - 2m_i}{h_i}, \end{aligned}$$

for $i = 1, 2, ..., N - 1$.

3 Term structure models

In general, many forces are at work at moving interest rates. Identifying these forces and understanding their impact on yields, is therefore of crucial importance (POOTER et al., 2010). Term structure models aim to specify the behavior of interest rates, seeking to identify the driving forces, also called factors, that help to explain prices of fixed-income securities of all kinds. These factors are stochastic in nature, and thus carry an uncertainty character. Hence, yield curve modeling is based on probability theory and specifies a statistical process that describes the stochastic character of those factors who impact on interest rates. The fact that the term structure is influenced by a number of factors[1] reveals a non trivial process for the yield curve modeling and forecasting.

A term structure model can describe the form of the yield curve at a given point in time or/and the dynamics of the yield curve through time. The first perspective encompasses a mathematical exercise of fitting a static yield curve, while the second question seeks to understand how the yield curve moves across time. A dynamic yield curve model, therefore, seeks to use statistical techniques to describe the future evolution of the yield curve in a manner that is consistent with its observed behaviour (BOLDER, 2015).

Significant progress has been made in term structure modeling, whereas the set of term structure models is divided into three major popular classes: affine, arbitrage-free (AF) and statistical models, which include the Nelson & Siegel (1987) approach. The literature on affine term structure models was kick-started by Vasicek (1977) and Cox et al. (1985), later characterized by Duffie et al. (1996) and classified by Dai & Singleton (2000)[2]. Affine models describe the dynamics of the term structure as a function of a small number of factors, such as the short rate. However, affine models are inconsistent with the no-arbitrage hypothesis once they have difficulty in adjusting the yield curve across the entire maturity spectrum. This fact leds to the formulation of arbitrage-free models, which use adjustment factors to allow the model to match the empirical yield curve more effectively.

Nonetheless, some problems emerge from the restrictions that arbitrage-free models impose to factor loadings, as the deterioration of the empirical fit to yield data. Nelson & Siegel (1987) suggest a statistical representation of the yield curve, seeking to improve empirical fit of term structure models. A step further, the Dynamic Nelson-Siegel approach

[1] For example, the constant adjustment of investors' expectations about inflation is an impact factor on the components of the yield curve.

[2] A survey of issues regarding the specification of affine term structure models in continuous time are explored in Piazzesi (2010).

seeks not only to improve the yield curve fit over time, but also to perform efficient forecasts about the future yield curve dynamics. The class of Nelson-Siegel models has been a popular choice among central bankers supported by its appealing features concerning smoothness and parsimony.

3.1 Affine term structure models

Affine models are based on the assumption of an economic equilibrium in a particular interest rate market, and they are developed using risk-neutral probabilities. These models assume that securities with similar maturities must have similar returns, otherwise no investor would buy those securities with lower expected return. Such a difference can not persist in an economic equilibrium environment. Affine models specify that bond prices depend on state variables, typically associated with the short rate r, which follow a normal Gaussian distribution. Gürkaynak & Wright (2012) define that affine models are so called because they define yields of different maturities as affine functions (constant plus a linear term) of factors[3], which form a state vector and capture the yield curve movements over time.

Two of the most popular bond pricing models are those constructed by Vasicek (1977) and Cox et al. (1985). They are known as one-factor models and describe the process of the term structure as a function of one single state variable, the short rate r. Since we can not predict the future path of r with certainty, it is natural to set r as a random variable. Its future value can take various possible outcomes, namely an associated probability distribution. Thus, one-factor models aim to specify the stochastic process that describes the dynamics of the variation process of r, which ultimately is the treatment of randomness of the bond prices and forward rates. The fact is that one can not know the future level of forward rates, but they can be estimated by modeling the current spot rates. Wilmott (2007) discusses the mathematical foundation of affine term structure models by modeling the dynamics of the short rate, which is assumed to be a continuous random variable. Thus, the role of one-factor models is to specify the stochastic process that describes the variation of the short rate. The standard Wiener process is a popular choice of stochastic process.

The short rate suffers dynamic shocks that cause variations in its value. If the variations of r are normally distributed and shocks follow a Wiener process, denoted by dW, r is a stochastic process that changes its value instantly according to its mean $\bar{\mu}$ and standard deviation σ, and whose pattern of variation follows a stochastic differential equation:

$$dr = \bar{\mu}(r,t)\,dt + \sigma(r,t)dW. \tag{3.1}$$

[3] Piazzesi (2010) characterizes that $y_t(\tau)$ is a linear function of a state vector f_t with parameters $A(\tau)$ and $B(\tau)$ that depends on maturity τ: $y_t(\tau) = A(\tau) + B(\tau)' f_t$.

The term $\bar{\mu}$ can be seen as an observed trend that influences the direction of the instantaneous variation dr, avoiding that the stochastic component carry the spot rate to infinite levels. Hence, one-factor models describe the dynamic process of the variation in r as a function of time and dW, where the first term of (3.1) is the deterministic component and $\sigma(r,t)dW$ is the stochastic element.

Vasicek (1977) and Cox et al. (1985) (CIR) assume a similar structure for the deterministic component, which incorporates a mean reversion term:

$$\bar{\mu} = \kappa(\theta - r), \tag{3.2}$$

where parameter κ controls mean reversion, i.e., the adjustment speed of the short yield according to its distance from the average long-term rate θ, which ultimately controls the yield curve shape. The difference between them concerns the addition of a multiplicative standard deviation component ($\sqrt{r}$) into the stochastic component for the CIR model.

Once the behavior of the short rate is identified, one can build the complete term structure from expected yields for any future period using (2.2). Hence, one-factor models capture the dynamics of the short rate following the functional form (3.1), which in turn is used to model the complete forward curve. Note that this approach uses only one source of randomness and assumes that all forward rates move in the same direction, resulting in high correlation between bond returns of different maturities. For this reason, affine models are not capable to reproduce the different shapes of the observed yield curve in a dynamic and accurate way. Therefore, for having such a dependency between yields of different maturities, one-factor models are inconsistent with the hypothesis of no-arbitrage. Besides that, the single factor structure severely limits the scope for interesting term structure dynamics, which rings allow in terms of both introspection and observation (DIEBOLD; RUDEBUSCH, 2013).

Examining the yield curve movements over time, one can clearly notice that there are more than just a common factor operating in term structures, which involve multiple factors in the real world. Multifactor models are directly developed from one-factor models, but they incorporate more than just one factor[4]. Their implementation and calibration is a demanding and time consuming process, rightfully because they incorporate a larger amount of information and estimated parameters. In response to the difficulty that one-factor models have in adjusting the complete term structure, multifactor models have been developed to improve yield curve modeling along its entire maturity spectrum. The Heath et al. (1992) model is a general structure regarding multifactor models, which models not only short rates but long-term rates too, using the entire term structure as an input to the process.

[4] Some examples of state factors used in multifactor models: long-term yield, real interest rate adjusted by inflation-linked bonds, current inflation rate, spread between short and long-term yields, among others.

3.2 Arbitrage-free models

The whole class of AF models for the term structure assumes that financial markets eliminate opportunities of riskless arbitrage across maturities and over time. Arbitrage-free models hold theoretical cross-sectional restrictions on factor loadings for absence of arbitrage opportunities in well-organized markets, specifying the risk-neutral evolution of the yield factors and its risk premia[5]. There is a concern in adjusting the observed yield curve, so that the observed yields are close to those estimated by the AF model. This accurate adjustment ensures the consistency of the model with the family of observed curves in the market.

In theory, the hypothesis of absence of arbitrage opportunities is characterized by consistency between parameters describing the dynamic evolution of the curve under a risk-neutral measure, and a family of parameterized curves under a physical measure. In the context of Björk & Christensen (1999), consistency between a term structure model M and a certain family of parameterized forward rate curve G refers to test if family G contains all curves estimated by M. When pair (M, G) is consistent, which ensures the absence of arbitrage opportunities between bonds of different maturities over time, the term structure model produces forward rate curves that belongs to the relevant family, eliminating the need of changing model parameters each period t.

Ho & Lee (1986) introduced the first AF model, applying the structures of Vasicek (1977) and Cox et al. (1985). The model describes the following stochastic process for the short rate:

$$dr = \theta(t)\, dt + \sigma\, dW(t), \tag{3.3}$$

where the second term of (3.3) is constant and independent of r. The function $\theta(t)$ is dependent on time describing the average movement of the spot rate and its direction. Hull & White (1990) describe an extension of Vasicek (1977) where the spot rate follows:

$$dr = \kappa(\theta - r)\, dt + \sigma\, dW(t). \tag{3.4}$$

In short, the assumption of no-arbitrage guarantees that, after accounting for risk, the dynamic evolution of yields over time is consistent with the cross-sectional shape of the yield curve at any point in time. Some problems emerge from the no-arbitrage imposition that may degrade the empirical performance of a misspecified model, explaining why models like the canonical affine AF models often exhibit poor forecasting performance. Duffee (2002) argue that affine AF models do not exhibit good empirical fit. Besides that, the estimation procedure is problematic in the economic perspective since the maximum likelihood function has several maximum points and there are nonlinear relationships between parameters and yields. Kim & Orphanides (2005) discuss that the maximum

[5] Specific restrictions of an arbitrage-free model may be found in Ang & Piazzesi (2003).

likelihood function appears to have multiple points of local maximum with similar values but not equivalent, which generate different implications for economic behavior. Joslin et al. (2014) argue that a standard practice to bypass the numerous likelihood maxima is to set to zero most parameters that are statistically insignificant, and then analyse the constrained model.

3.3 The class of Nelson-Siegel models

3.3.1 Nelson-Siegel

Nelson & Siegel (1987) suggest a flexible and parsimonious structure with less parameterization to fit a smooth yield curve to unsmoothed yields. The static Nelson-Siegel (NS) representation specifies the evolution of the yield curve factors, such as the dynamics of risk premiums, and proves that a linear combination of three smooth exponential factors[6] can properly adjust the different formats of the entire yield curve at any time:

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_3 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right), \tag{3.5}$$

where parameter λ controls the exponential decay rate of the curve, or the rate at which factor loadings decay to zero[7]. Thus, (3.5) suggests a functional form for fitting the cross section of unsmoothed bond prices or yields.

Parsimony and flexibility provide the most appealing features of NS framework. A parsimonious approximation of the entire yield curve promotes smoothness between yield vertices, ensuring empirical tractability and trustworthy estimates. The flexibility of NS representation to represent the various formats of the yield curve[8] can be seen by the interpretation of model coefficients as measures of short, medium and long-term components of the curve. According to the way that each factor shock affects the curve, Litterman & Scheinkman (1991) named β_1, β_2 and β_3 as level, slope and curvature factors of the term structure. They are unobserved, or latent, whereas the associated loadings are restricted by a functional form that imposes smoothness of loadings across maturities (DIEBOLD; RUDEBUSCH, 2013).

The static Nelson-Siegel form has long been very popular among financial market practitioners and central banks for curve fitting at a point in time because of its considerable statistical appealing. However, Björk & Christensen (1999) and Filipović (1999) prove that under the absence of no-arbitrage conditions there is no nontrivial model governed by stochastic processes and consistent with the family of Nelson-Siegel curves, i.e., NS

[6] More specifically, a constant plus a Laguerre function, which consists in polynomials multiplied by exponential decay terms on the domain $[0, \infty)$ that can approximate any forward rate curve.

[7] Parameter λ can also be interpreted as β_3 maximum point.

[8] As stated in Diebold & Li (2006, p. 348), Nelson-Siegel model is capable of replicating a variety of yield curve shapes: upward sloping, downward sloping, humped, and inverted humped.

models are not theoretically arbitrage-free. In another way, the NS framework does not contain the necessary restrictions to eliminate opportunities for riskless arbitrage.

3.3.2 Dynamic Nelson-Siegel

Diebold & Li (2006) introduced the dynamic component to the static Nelson-Siegel framework through time-varying parameters. Furthermore, up to Diebold & Li (2006) few term structure models gave importance to out-of-sample forecasting. They perform a new interpretation of the NS framework by introducing dynamic [9] and efficient forecasting perspective for out-of-sample period[10]. The mechanics of Dynamic Nelson-Siegel (DNS) follow the functional form of Nelson & Siegel (1987), which has a good fit to the observed interest rates for different maturities and moments of time, but with the incorporation of a time-series environment through time-varying factors:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right). \tag{3.6}$$

DNS carries cross-sectional and time-series perspectives, representing a spatial and temporal linear projection of $y_t(\tau)$ on the time-varying variables β_{1t}, β_{2t} and β_{3t}, which can be interpreted respectively as long, short and medium-term latent factors[11].

The interpretation of the yield latent factors refers to the inspection of the factor loadings $(1, ((1-e^{-\lambda\tau})/\lambda\tau), ((1-e^{-\lambda\tau})/\lambda\tau - e^{-\lambda\tau}))$. The long-term variable β_{1t} drives the term structure level since $\lim_{\tau\to\infty} y_t(\tau) = \beta_{1t}$, which loading is constant at 1 at all maturities. An increase in β_{1t} shifts the entire yield curve equally, as its factor loading is identical at all maturities. The loading on β_{2t} is a function that starts at 1 but decays monotonically with maturity. Fluctuations on β_{2t} generate greater deviations in short-term yields. Diebold & Li (2006) define β_{2t} as the difference between a ten-year yield and a three-month yield, $y_t(120)-y_t(3)$[12]. In this case, an increase of β_{2t} indicates greater positive feedback from long-term yields compared to short yields. In addition, it is important to note that the instantaneous yield depends on both the level and slope factors, because $y_t(0) = \beta_{1t} + \beta_{2t}$. At least, Diebold & Li (2006) define medium-term factor related to curvature as twice the two-year yield minus the sum of the ten-year and three-month yields, $2y_t(24) - y_t(3) - y_t(120)$. The loading on β_{3t} increases at middle maturities and

[9] The dynamic is fundamental to model the evolution of the securities market (CHRISTENSEN et al., 2011).

[10] Diebold & Li (2006) argue that equilibrium and arbitrage-free models focus only on fitting the term structure at a given point of time to ensure the absence of arbitrage opportunities. As they seek to incorporate dynamic and the out-of-sample forecast perspective to yield curve, the authors use a model capable to describe the future dynamics of the yields for different maturities over time.

[11] The DNS form is included in the set of so-called three-factor (level, slope and curvature) models.

[12] Other authors, as Frankel & Lown (1994), define the slope factor as $y_t(\infty) - y_t(0)$, which is exactly equal to $-\beta_{2t}$. However, the following discussion interpret the slope of the yield curve as the spread in interest rates between long and short-term maturities.

then decays to zero. So, an increase in β_{3t} has little effect on short and long-term yields, but increases the medium-term yields.

DNS is a leading example of a "dynamic factor model". According to Diebold & Rudebusch (2013), dynamic factor models provide appealing features because yield data actually display factor structure. Some key reasons to prove the statistical appealing: (i) factor structure generally provides a highly accurate empirical description of yield curve data, because just a few constructed variables or factors can summarize bond price information; (ii) statistical tractability, by providing a valuable compression of information, effectively collapsing an intractable high-dimensional modeling situation into a tractable low-dimensional situation. Beyond good fit and forecast performance of DNS, its simplicity confirms the increasing popularity of the DNS structure.

State-space representation of DNS

Diebold & Li (2006) show that it is possible to interpret the DNS model in state-space system format, assuming that the dynamic latent factors are state variables and follow a stochastic first-order vector-autoregressive. The state-space model can be summarized by the matrix notation:

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t, \tag{3.7}$$

$$y_t = \Lambda f_t + \varepsilon_t, \tag{3.8}$$

for $t = 1, ..., T$. The parameter f_t is the state vector (level, slope and curvature factors), μ is the factor mean, A is the state transition matrix, η_t is the state equation factor disturbances, Λ is the sensitivity matrix of the measurement equation, ε_t is the measurement equation disturbances, and y_t is the $N \times 1$ vector of observed yields for N different maturities τ_i at time t, so that $y_t = [y_t(\tau_1), y_t(\tau_2), ..., y_t(\tau_N)]'$, where τ_1 is the shortest maturity considered and τ_N is the longest.

The measurement equation (3.8) adds a stochastic error term to the deterministic DNS curve, which relates the set of N yields to the unobserved yield factors[13], which are emphasized as level (L_t), slope (S_t) and curvature factors (C_t). So, the factor loadings matrix Λ relates the yield curve dynamic to the constructed factors. The transition equation (3.7) determines the common factor dynamics as a first-order process, which incorporates higher-order dynamics if it is necessary. The covariance structure of the measurement and transition disturbances specify that the vectors η_t and ε_t are mutually orthogonal, orthogonal to the initial state vector and white noise processes:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right], \tag{3.9}$$

[13] According to (DURBIN; KOOPMAN, 2012), the state-space representation allows one to study the development of the state factors over time using the set of observed yields in financial markets.

$$E(f_0\eta_t^{'}) = 0, \tag{3.10}$$

$$E(f_0\varepsilon_t^{'}) = 0. \tag{3.11}$$

The system requires that the covariance matrix of measurement disturbances H is diagonal, so that the disturbances ε_t of different maturities are uncorrelated [14]. Further, the covariance matrix of transition disturbances Q is not diagonal, so that the disturbances η_t can be correlated in time, allowing for correlated shocks between state factors. Later we will see that Diebold et al. (2006) expand the system (3.7)-(3.11) by simply introducing macroeconomic factors in the state factors vector. Therefore, this state-space representation is not unique, so that measurement and transition equations accommodate transformations; as the inclusion of a constant in both equations, which is generally inconsequential.

3.3.3 Arbitrage-free Nelson-Siegel

As reported by Diebold & Rudebusch (2013), the lack of freedom from arbitrage motivated Diebold et al. (2005) and Christensen et al. (2011) to introduce the class of arbitrage-free Nelson-Siegel (AFNS) yield curve models, which maintain the dynamic nature and good empirical performance of DNS framework, but adds the theoretical requirement of no-arbitrage condition. Basically, the authors impose absence of arbitrage opportunities to DNS, making it theoretically more satisfactory. Otherwise, they keep the theoretical rigor of AF models but incorporate DNS elements to make it empirically interesting. The DNS model can easily be transformed into an arbitrage-free structure when desirable (DIEBOLD; RUDEBUSCH, 2013). The no-arbitrage version of DNS contains the associated restrictions on factor loadings that ensure absence of arbitrage. The calculations of Christensen et al. (2011) show that affine AF models yields the DNS structure plus a correction term that ensures the necessary restrictions of no-arbitrage. It is also called "yield-adjustment term", which is time-invariant and depends only on bond maturity[15].

[14] According to Diebold et al. (2006), this assumption is common in the literature for simplifying model estimation by reducing the number of parameters. In the estimation of affine models for the term structure, the assumption of independent and identically measurement errors are also added to observed yields.

[15] The analytical AFNS model decomposition and its yield-adjustment term are carried out in Appendices A and B of Christensen et al. (2011).

4 The term structure and the macroeconomy

Surprisingly, the literature on term structure models delayed in incorporating macroeconomic foundations. There have always been divergent modeling strategies between finance and macro literature concerning to interest rates. Finance models described the short-term interest rate as a linear function of a few unobserved latent factors, whereas long-term rates reflected changes in risk premiums. In contrast, the macro literature has an appeal to model short-term rates as a function of the central bank objetives regarding inflation and output. Moreover, macro models explain long-term yields supported by the "expectations hypothesis" of the term structure, which suggests that long-term rates are risk-adjusted averages of expected future short rates. As stated in Diebold & Rudebusch (2013), both the DNS and the affine no-arbitrage dynamic latent factor models provide useful statistical descriptions of the yield curve, but in their original, most basic, forms they offer little insight into the nature of the underlying economic forces that drive its movements. The economic meaning of such models is limited since they neglect a macroeconomic environment that could affect interest rates of different maturities.

A literature that explores different approaches to jointly model the term structure and the macroeconomy has emerged lately. Besides that, some active progress to solve this missing link has been made. The first natural approach to incorporate macroeconomic foundations was performed by no-arbitrage affine models that combined latent yield and macro factors into the affine function for the short-term interest rate. This affine function is assumed to depend on macro factors that measures economic variables such as inflation, real activity, credit, among others. Ang & Piazzesi (2003), Hördahl et al. (2006) and Rudebusch & Wu (2008) provide examples that incorporate this idea through an affine no-arbitrage structure with macroeconomic variables.

Ang & Piazzesi (2003) introduced the effects of the macroeconomy to the term structure combining macro-financial factors that determine spot rate dynamics through macro factors from the Taylor rule. Their results show that output shocks have a significant impact on medium-term yields and curvature, while inflation surprises largely affect the level of the entire yield curve. Despite the effects are limited, incorporating macro factors extracted from inflation and real activity data series into the model improve interest rate forecasts. Diebold et al. (2006) provide a macroeconomic interpretation of the DNS model by combining macro and yield factors into transition equation of the state-space model, where macroeconomy affects latent factors through state transition matrix that governs the factor vector-autoregressive dynamics. Their estimates for U.S. Treasury bonds try to find a correlation between latent factors and macroeconomic variables related to real activity, inflation, and a monetary policy instrument, showing that the level factor is highly correlated with inflation and the slope factor is highly correlated with real activity. The curvature factor appears to be unrelated to any of the main macroeconomic variables.

The other possibility to model the effects of macroeconomics on the yield curve is composed by the class of structural models with dynamic factors. These models emphasize the macro structure of the economy, incorporating macro shocks through the law of motion of factors, which is founded on some economic model based on agents utility maximization. Usually, this is a new-Keynesian macroeconomic model with micro-foundations which defines macro factors dynamics through IS curve equation, Phillips curve and monetary policy rule. Rudebusch & Wu (2008) and Bekaert et al. (2010) provide examples of macro-finance specification for structural models using the no-arbitrage condition. Rudebusch & Wu (2008) obtain a good fit to the macro-finance data, revealing that the level factor respond to market expectations for central bank reaction to its inflation objective function, while the slope factor captures central bank reaction to business cycle. The great feature of structural models for the term structure is that they allow one to calculate yield curve responses to diverse macroeconomic shocks.

More recently, a literature that uses larger macroeconomic information sets has emerged. Bernanke & Boivin (2003) argue that central banks monitor and explore rich information sets into their monetary policy decisions. However, the inclusion of a large number of individual variables largely increase the number of parameters to be estimated. Bernanke & Boivin (2003) employ a factor-model approach, explored by Stock & Watson (2002a) and Stock & Watson (2002b), that allows for extracting few factors which summarize the systematic information in large datasets. In short, their evidences suggest that central banks base its policy decision upon a broad set of conditioning information, so that the hypothesis that monetary policy authorities exploit only a limited amount of information is rejected. Once central banks decision about short interest rates affects the entire term structure, incorporating information about the overall state of the economy could improve yield forecasts. The extraction of common factors that explains most of high-dimensional data variation can be performed through dimensionality reduction techniques, such as principal component analysis. Examples for models that use "data-rich environment" for the term structure are Moench (2008) and Favero et al. (2012)[1]. For the Brazilian economy, Almeida & Faria (2014) and Vieira et al. (2017) already reproduced the basis of the original studies. Pooter et al. (2010) argue that macroeconomic variables interact best with the yield curve when introduced as factors from data-rich environments, revealing that the inclusion of few principal components leads to better forecast performance compared to the use of individual variables.

Moench (2008) use a "factor-augmented" (BERNANKE et al., 2005) procedure as

[1] Furthermore, Ludvigson & Ng (2009) also apply principal component analysis to obtain macro factors, which are used to predict excess bond returns.

the state equation that describes the dynamics of the short-term interest rate conditional on a large macroeconomic information set. Their forecast exercise provides better out-of-sample yield forecasts at intermediate and long horizons, particularly for short and medium-term maturities, than benchmark models such as DNS and the affine yield factor model of Duffee (2002). Almeida & Faria (2014) replicate the model proposed by Moench (2008) for the Brazilian economy and estimate the DNS model incorporating macroeconomic factors. As the original study, the authors show that their approaches have better predictive performance than benchmark models, despite having a deterioration of the results with increased maturity for the factor-augmented model. At least, Vieira et al. (2017) combine the factor-augmented VAR methodology with DNS model for the Brazilian yield curve, extracting macro factors from a large data containing forward-looking variables of market expectations about future macro-financial scenario. This model improves the predicting accuracy of extant models in literature, particularly at short-term horizons.

Differentiation of macro-finance models is given by the set of macroeconomic information incorporated, specific macro structural model, and restrictions imposed on those dynamic interactions between macro and financial factors. As described before, macroeconomic information can be incorporated by simply adding common macro variables, such as inflation and real activity, or by macro factors extracted from a large number of macroeconomic time series variables, which is proved to improve predictive accuracy of the models. The relationship between macro-financial factors concerns to the way that shocks propagate between both classes of factors; if the vector-autoregressive that governs factor dynamics captures a unidirectional or bidirectional "feedback". Ang & Piazzesi (2003) and Hördahl et al. (2006) imposes restrictions so that only macro variables affect yield curve components and the macroeconomy is determined independently of the yield curve factors. On the other hand, Diebold et al. (2006) allow bidirectional dynamics for macro-yield interactions, i.e., an unrestricted VAR in the transition equation of DNS model. They find a significant bidirectional interaction, where yield curve also contains important information about future macro scenarios.

The basis of correlation between the macroeconomy and the yield curve is largely explained by the expectations hypothesis, which relates both the long and short end of the yield curve. If the central bank response to the state of the economy governs the short-term rate, ultimately expectations about future longer-term interest rates also depend upon macro variables. So, it is plausible that the response of monetary policy to macroeconomic shocks contributes to explain and forecast the dynamics of the yield curve through time. Christensen & Rudebusch (2012) argue that not only central bank bond purchase programmes affect yields, but announcements of central bank plans also affect financial markets through the signalling channel. They analyse the response of government bond yields to bond purchase programmes of the Federal Reserve and the Bank of England, finding significant responses of the US and UK yield curve components to central bank announcements. In parallel, Oliveira & Ramos (2011) find that unanticipated shocks of monetary policy are capable to affect the Brazilian nominal yield curve maturing up to 2 years.

Changing the viewpoint, if the term structure contains relevant information on investor expectations about the future economic conditions, it can be a useful tool to capture information that helps monetary policy decision. Nimark (2008) discusses that central banks can increase the welfare of their objective function by using securities market information about macro fundamentals. Thus, the study of macro effects on yield curve dynamics is important for policy makers who need to extract macroeconomic expectations from financial markets, especially to make decisions that may affect interest rates. The key point is that if macroeconomy and yield curve components are correlated, the incorporation of macro factors can generate term structure models that forecast better than those without macroeconomy effects (ANG; PIAZZESI, 2003), which is of great interest here.

5 Fixed-income portfolio optimization

The portfolio approach suggested by Markowitz (1952) is one of the cornerstones of modern finance theory. Although the mean-variance approach of Markowitz has been widely explored in the construction of equity portfolios, little is known about portfolio optimization in fixed-income markets. Kokn & Koziol (2006) point out some reasons why bond portfolio optimization is only recently explored. First, at the time when Markowitz's approach became more widely recognized as a useful tool for portfolio management, interest rates were not particularly volatile and a portfolio approach seemed somehow unnecessary. Second, severe difficulties to implement Markowitz's approach might have discouraged further work: the large number of parameters needed when using a large number of assets and the variation of bond moments over time.

To the extent that the interest rate markets become more volatile and unstable, bonds with different cash flow payments are created and advanced term structure models has been developed, it is natural to think about the potential for risk diversification and optimization of fixed-income portfolios. In addition, securities of different cash flow payments, reflecting their defined coupons and maturity values, are imperfect substitutes, which suggests that there may be "preferred-habitat" investors who have maturity-specific demand or duration-specific demand[1] (CHRISTENSEN; RUDEBUSCH, 2012). Accordingly, a recent literature that exploits the bond portfolio selection in a mean-variance context has been emerged.

Kokn & Koziol (2006) are the precursors in this literature by performing a mean-variance bond portfolio selection employing the Vasicek (1977) model. The authors estimate the expected returns, return variances, and covariances of different German bonds, showing that a small number of risky bonds is sufficient to achieve portfolios with quite promising predicted risk-return profiles. Caldeira et al. (2016) extend their approach to the general class of dynamic factor models of the term structure, and derive simple closed-form expressions for expected bond returns and their covariance matrix based on forecasted yields. Their empirical evidence for the US market shows that proposed optimal bond portfolios has better performance than traditional yield curve strategies, used in bond desks, in terms of Sharpe ratio. Another reference in this context is Thornton & Valente (2012), which assess the economic value of the predictive power of forward rates for bond excess returns in an out-of-sample forecasting exercise. In particular, they investigate the economic gains accruing to an investor who exploits the predictability of bond excess returns relative to the no-predictability alternative consistent with the expectations hypothesis. Their findings confirm that it is very difficult to improve performance upon a simple naïve

[1] For example, investors who seek for bonds that pay higher coupon rates.

benchmark.

The strategic asset allocation exercise is essentially a prediction problem and an optimization problem, where the investor seeks an optimal combination of securities in an uncertain environment, which point out the non-trivial aspect of the problem (BOLDER, 2015). The fixed-income portfolio problem essentially consists in predicting the distribution of return outcomes for a set of securities and select the optimal vector of portfolio weights conditional on one's predicted distribution and risk preferences. An important stage of the portfolio choice problem is the tactical planning, which refers to the decision for taking a passive or active position and implementing the chosen strategy. A passive positioning essentially tries to replicate all the risk factors of a benchmark strategy; deviating from one or more of the risk factors associated with the benchmark is called an active strategy[2]. The active positioning is based on a forecast of future market changes, because the portfolio and benchmark will respond differently to them, so that the portfolio manager must decide in which direction and by how much the risk factor value of the portfolio will deviate from those of the strategic benchmark (FABOZZI et al., 2006).

In line with Choudhry (2003), active portfolio management can be broken down into four basic categories: (i) the expectations approach, which aims to predict the direction of interest rates changes; (ii) the yield curve approach, which seeks to gain from predicting the changes in the shape and levels of the yield curve; (iii) the yield spread strategy, which attempts to make gains from changes in yield spread between individual bonds or bond sectors; and (iv) the fair value approach, which aims to assess the valuation of individual securities and identify mispriced bonds. Here, the analysis is interested in the second approach, whereas the yield curve forecasting exercise will indicate future changes in the shape of the yield curve[3], which are then used to generate forecasts of bond returns. In other words, the yield curve strategy is aimed at achieving gains from identifying changes at specific maturities of the term structure of interest rates.

The fixed-income portfolio optimization contributes to the present study in order to assess the economic value of the forecasted yields generated by the yield curve models. In particular, the yield curve strategy is employed to compute optimal portfolios using closed-form expressions for expected fixed-income returns and their covariance matrix. Moreover, the empirical implementation focuses on evaluating whether the incorporation of macroeconomic information into the DNS model generates economic gains in terms of fixed-income portfolio performance by choosing allocations in risky assets based on a trade-off between expected return and risk.

[2] In other words, active positioning involves deviating from the market exposures embedded in the strategic benchmark.

[3] The main types of shifts in the yield curve are the following: upward and downward *parallel* shifts, flattening and steepening yield curve twists, and changes in the humped shape, so-called *butterfly* twists. For instance, the most common changes are a combination of a downward shift and steepening, or an upward shift and flattening (CHOUDHRY, 2003).

5.1 The mean-variance approach of Markowitz

The approach suggested by Markowitz (1952) is the most common formulation of portfolio choice problems, which point out that investors allocate their wealth in risky assets based on the trade-off between expected return and risk. The mean-variance portfolio problem calculates the optimal portfolio weights based solely on one-step-ahead return forecasts, assuming that investors are risk averse[4]. Hence, at the time of the portfolio choice, it is assumed that investors are only concerned with the expected returns for the h-step-ahead forecast horizon and its covariance matrix, defined by $\mu_{r_{t|t-h}}$ and $\Sigma_{r_{t|t-h}}$, respectively.

The mean-variance portfolio problem can be formulated by minimizing the portfolio variance for a particular h-step-ahead expected return, subject to additional restrictions on the vector of optimal weights w_t:

$$\underset{w_t}{\text{Min}} \quad w_t^{'} \Sigma_{r_{t|t-h}} w_t - \frac{1}{\delta} w_t^{'} \mu_{r_{t|t-h}}$$

$$\text{subject to}: \; w_t^{'}\imath = 1; \; w_t \geq 0. \tag{5.1}$$

where $\imath$ is an appropriately sized vector of ones and δ is the investor's risk aversion coefficient. Vector $\mu_{r_{t|t-h}}$ collects the h-step-ahead expected returns for maturities $\tau_1, ..., \tau_N$, so that its dimension is $N \times 1$, while the covariance matrix $\Sigma_{r_{t|t-h}}$ is $N \times N$. The optimization problem is subject to both constraints, the non-negative individual weights, which restricts short sales, and the budget constraint, which ensures that all wealth is invested in risky assets.

As the mean-variance problem solves a quadratic utility function, the necessary and sufficient condition for optimization is to solve the optimal weights w_t for the first order condition. According to Brandt (2009), the Markowitz paradigm yields two important economic insights: (i) the diversification of a portfolio with imperfectly correlated assets reduces the portfolio investment adjusted-risk without dropping its returns; and (ii) once a portfolio is fully diversified, higher expected returns can only be achieved at the cost of greater risk. In other words, it is possible to decrease the nonsystematic risk by adding more assets to portfolio, proving the benefit of diversification. The mean-variance model is efficient in capturing these two fundamental aspects of portfolio choice theory.

A common criticism to the single period problem is its inherent myopic aspect, once myopic portfolio weights are calculated based solely on single-step-ahead expected returns, with no concerns to longer, multi-period investment horizons and intermediate portfolio rebalancing. However, the dynamic optimization problem carries multiple long-term forecasts, which are less accurate, and also allows to accumulate errors from different

[4] For two assets with the same expected return, investors prefer the less risky. This implies that a greater expected return can only be achieved when the investor takes a greater risk.

forecast horizons, impacting negatively on the dynamic optimization results. Brandt (2009) describes that a common justification from practitioners is that the expected utility loss from errors that could creep into the solution of a complicated dynamic optimization problem outweighs the expected utility gain from investing optimally as opposed to myopically[5]. For instance, Lan (2015) compares the performance of myopic versus dynamic portfolio policies and concludes that myopic behavior can even lead to utility gains for the real-time investor.

5.2 Risk factors

In measuring risk, the portfolio manager is essentially interested in the statistical distribution of the portfolio returns, which depend on a relevant set of risk factors. Any variable that can impact on the value of a security is considered a risk factor. The risk factors associated with a hypothetical asset class exhibit different characteristics, differing in level of volatility or dependence with other risk factors, and can be separated into financial, monetary policy and macroeconomic risk factors, among others. Assessing the uncertainty of future returns is essentially equivalent to measuring the risk of one's portfolio (BOLDER, 2015). The portfolio return is the compensation that an investor has received for being exposed to such risk factors. This section aims to describe key measures that emerge from the discount curve (2.2), which links the portfolio return to some relevant risk factors influencing the value of fixed-income securities.

Starting from scratch, equation (2.1) shows an inverse relationship between the present value of a cash flow and its continuously compounded yield, so that an increase (decrease) in $y_t(\tau)$ drops (raises) the present value of each cash flow, leading to a reduction (increase) in the current security value, $P_t(\tau)$. Thus, the security's value decreases with positive changes in interest rates, but at decreasing rates, because of the convexity aspect of the function. One can formalize this relationship by calculating the first derivate of the security's value with respect to a change in its yield,

$$\begin{aligned} \frac{\partial P_t(\tau)}{\partial y_t(\tau)} &= \frac{\partial}{\partial y_t(\tau)} \left(\sum_{i=1}^{n} c_i e^{-(t_i - t) y_t(\tau)} \right) \\ &= -\sum_{i=1}^{n} c_i (t_i - t) e^{-(t_i - t) y_t(\tau)}. \end{aligned} \tag{5.2}$$

In other words, the equation explicits the sensitivity of $P_t(\tau)$ to an infinitesimal change in its yield. Now, it is clear to note from (5.2) that the sensitivity depends on two key risk factors, or exposures: the yield to maturity and time.

[5] Brandt (2009) characterize some situations where it is optimal to invest myopically with a single-period horizon.

One can obtain the percentage change in the security's value (gain or loss) for a small change in $y_t(\tau)$ by dividing both sides of (5.2) by $P_t(\tau)$:

$$\begin{aligned} D_M &= \frac{1}{P_t(\tau)} \frac{\partial P_t(\tau)}{\partial y_t(\tau)} \\ &= -\frac{1}{P_t(\tau)} \sum_{i=1}^{n} c_i (t_i - t) e^{-(t_i - t) y_t(\tau)}, \end{aligned} \tag{5.3}$$

which represents the modified duration, denoted by D_M; a well-known measure of the risk of a fixed-income security. According to Bolder (2015), equation (5.3) provides, in short, the analytic representation of a security's exposure to its yield. The equation can also be used as a local measure of exposure[6], when one is interested in computing the exposure of the fixed-income security to changes in yields solely at particular areas of the yield curve. For example, an investor fearing a specific movement at the 1-year rate, would like to know the sensitivity of its bond's value to a 50 basis-point movement in the particular yield with maturity $\tau_i = 12$ months.

The concept of duration can have different meanings, which sometimes is quoted as a sensitivity and sometimes it is described as a cash-flow weighted time to maturity of a fixed-income security. From the second perspective, duration establishes an average maturity of the future cash flow promised by the issuer. For a bond that pays periodic coupons, there are intermediate portions of the expected cash flow being paid at specific moments before maturity. In this case, the bond maturity is not equivalent to duration. Some important insights emerge from the association between duration and bond volatility: (i) the duration increases with maturity, but at decreasing rates, so that the higher the duration, the more exposed the security is against changes in interest rates; (ii) the higher the yield to maturity, the shorter the duration, because higher intermediate payments have higher relative weight on the cash flow.

Regarding equation (2.1) again, it is noteworthy that the security's value is not a linear function of its yield. That is, some degree of non-linearity emerges from the relationship between security's price and yield. For this reason, even though being quite reasonable for relatively small yield changes, the linear approximation performed by the modified duration is not fully accurate for sizeable changes in $y_t(\tau)$. The second derivative of the discount curve with respect to $y_t(\tau)$ seeks to efficiently capture the full exposure of a fixed-income security to the yield factor:

$$\begin{aligned} \frac{\partial^2 P_t(\tau)}{\partial y_t(\tau)^2} &= \frac{\partial^2}{\partial y_t(\tau)^2} \left(\sum_{i=1}^{n} c_i e^{-(t_i - t) y_t(\tau)} \right) \\ &= \sum_{i=1}^{n} c_i (t_i - t)^2 e^{-(t_i - t) y_t(\tau)}. \end{aligned} \tag{5.4}$$

[6] Generally called as the *key-rate* duration.

As done before, one can normalize the total variation by dividing (5.4) by $P_t(\tau)$:

$$\frac{1}{P_t(\tau)}\frac{\partial^2 P_t(\tau)}{\partial y_t(\tau)^2} = \frac{1}{P_t(\tau)}\sum_{i=1}^{n} c_i(t_i - t)^2 e^{-(t_i-t)y_t(\tau)}, \tag{5.5}$$

obtaining the bond convexity factor[7]. Therefore, duration and convexity provide a basis for understanding the exposure of a fixed-income security to changes in market interest rates, whereas the convexity measure seeks to correct the approximation performed by the modified duration.

In respect to time exposure, to understand the sensitivity of a bond's value to changes in time, one can simply replicate the mathematical derivations made before:

$$\begin{aligned}\frac{\partial P_t(\tau)}{\partial t} &= \frac{\partial}{\partial t}\left(\sum_{i=1}^{n} c_i e^{-(t_i-t)y_t(\tau)}\right) \\ &= y_t(\tau)\sum_{i=1}^{n} c_i e^{-(t_i-t)y_t(\tau)} = y_t(\tau)P_t(\tau).\end{aligned} \tag{5.6}$$

And to compute a kind of time duration, one need to divide both sides of (5.6) by $P_t(\tau)$:

$$D_t = \frac{1}{P_t(\tau)}\frac{\partial P_t(\tau)}{\partial t} = \frac{1}{P_t\,(\tau)}\, y_t(\tau)\, P_t(\tau) = y_t(\tau), \tag{5.7}$$

where the percentage change of the security's value to a small variation in time is well approximated by its yield.

At least, an important risk factor comes up from the credit risk linked to a fixed-income security, which is the probability of default associated with the bond issuer. The credit risk can be considered one element of the spread between the security's yield and the corresponding risk-free rate, which is generally the shortest maturity bond issued by the national government. This spread between some particular bond and the risk-free borrower represents an additional premium required by the market, and decreases with better credit quality. One can decompose the mathematical expressions for the impact of spread movements on the bond price, assuming an additive decomposition of the bond's yield into the risk-free component and an idiosyncratic credit spread component. From the decomposition, one can see that there is nothing different about the sensitivity of the bond price whether the yield change comes from a movement in the risk-free rate, the credit spread, or the overall yield (BOLDER, 2015).

[7] It is also worth to note that bonds with higher convexity benefit the investor when interest rates fluctuate: the decrease (increase) in $P_t(\tau)$ is relatively smaller (higher) in response to a positive (negative) variation in $y_t(\tau)$.

Part II

Theoretical Models, Data and Estimation Methodology

6 Theoretical models for the yield curve

6.1 Specification of the yield factors model

According to Diebold & Li (2006), the DNS model describes the yields of different maturities as a linear function of unobserved yields latent factors, where a first-order vector-autoregressive process governs the dynamic movements of L_t, S_t and C_t over time. The system (3.7)-(3.11) constituted by the transition and measurement equations can be better visualized as follows:

$$\begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix},$$

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix}.$$

The DNS model specified by the system (3.7)-(3.11) has several parameters to be estimated and will be denominated as *DNS-VAR(3)* model henceforward. The transition matrix A is 3×3 and has 9 free parameters, the mean factors μ is 3×1 and has 3 free parameters, and the measurement matrix Λ is $N \times 3$ and includes only one free parameter, λ. In addition, the matrix Q is 3×3 and contais 6 free parameters (one variance term for each of the latent factors and three covariance terms among themselves, making matrix Q symmetric), and the matrix H is $N \times 1$ and has N free parameters (one variance term for each one of the N yields of different maturities). The *DNS-VAR(3)* model is widely used as a benchmark, and here it will be the object of comparison to models that incorporate macroeconomic factors.

The choice for the DNS structure is motivated by its statistical appealing features, being quite simple, flexible and stable to estimate. Besides that, DNS reveals good out-of-sample forecasting performance, which is of great interest here. The class of AFNS models covers the theoretical lack of no-arbitrage conditions of DNS, but it is a bite more complicated to estimate because it involves higher parameterization; i.e., the application of AFNS has a relative cost of estimation procedure. Following Diebold & Rudebusch (2013), DNS is almost arbitrage-free, as is. Ultimately, the features of the DNS yield curve model fit very well for the current purpose.

6.2 Specification of the macroeconomic models

The belief that yield curve dynamics are closely linked to macroeconomic developments motivates the incorporation of macroeconomic information in term structure models. Following Diebold et al. (2006), the introduction of relationship between the components of the yield curve and macroeconomic factors consists simply in incorporating macro factors as exogenous explanatory variables into the state vector and corresponding expansion of the matrices that form the state-space model (3.7)-(3.11). The approach here simply replaces the individual macroeconomic variables used in Diebold et al. (2006) by a small number of macroeconomic factors obtained from a large set of possible regressors. Therefore, the model structure only contemplates effects of macro factors to yield factors in future time periods via dynamic interaction in the transition equation; macro factors affect yield latent factors one-period-ahead, which in turn determine the yields.

The assumption that in a DNS environment, the yield curve can be simply decomposed by L_t, S_t, and C_t, remains valid. The three yield factors are all that one needs to explain most yield variation (DIEBOLD; RUDEBUSCH, 2013), so that the inclusion of macro factors will be useful for yield curve modeling in order to explain the dynamics of the yield factors. Thus, macroeconomic factors extracted from a large set of macroeconomic variables are linked to yield factors, so that a kind of two-step DNS procedure is employed. First, yield factors (level, slope, curvature) and macro factors (e.g., broad real activity and broad inflation expectations) are extracted, and then all factors are analyzed in a joint vector autoregression.

The expansion of the *DNS-VAR(3)* model to macroeconomic representations of the DNS form is given by the incorporation of one and two macro factors, denoted by X^1 and X^2, to the state vector. The state vector is now $f_t^{'} = (L_t, S_t, C_t, X_t^1)$ for the model denominated *DNS-VAR(4)*^{1}, $f_t^{'} = (L_t, S_t, C_t, X_t^2)$ for the model denominated *DNS-VAR(4)*^{2} and $f_t^{''} = (L_t, S_t, C_t, X_t^1, X_t^2)$ for the model denominated *DNS-VAR(5)*. Sometimes, these macroeconomic specifications will be regarded as *yields-macro* models. Table 1 summarizes the general DNS specifications used in the estimation procedure.

The inclusion of the $K = 1, 2$ macroeconomic factors is motivated by the principal components analysis, which extract a small number of common factors from a panel series composed by 182 macroeconomic variables. The approach is supported by the set of conditioning information that monetary authorities take into account when deciding interest rates levels. The ordering of the state factors in $f_t^{'}$ and $f_t^{''}$ is performed this way because the information of the yield curve is observed at the beginning of each month. The expansion of the DNS model also requires an appropriate increase in the dimensions of the matrices that form the system (3.7)-(3.11)[1], leading to a substantial increase in the

[1] By the way, for the *DNS-VAR(3+K)* model the matrix A is now $(3+K) \times (3+K)$, whereas μ and η_t are $(3+K) \times 1$. The non-diagonal matrix Q is now $(3+K) \times (3+K)$. The measurement equation

number of parameters to be estimated.

Table 1 summarizes the denominations of the general DNS specifications described above and that will be used in estimation procedures.

Table 1 – General DNS specifications set.

Model Specification	Factors
DNS-VAR(3)	Level (L_t), Slope (S_t), Curvature (C_t)
DNS-VAR(4)^1	Level (L_t), Slope (S_t), Curvature (C_t) and 1st Macro Factor (X_t^1)
DNS-VAR(4)^2	Level (L_t), Slope (S_t), Curvature (C_t) and 2nd Macro Factor (X_t^2)
DNS-VAR(4)	Comprehends the *DNS-VAR(4)*^{1} and *DNS-VAR(4)*^{2} models
DNS-VAR(5)	Level (L_t), Slope (S_t), Curvature (C_t), 1st Macro Factor (X_t^1) and 2nd Macro Factor (X_t^2)
yields-macro	Comprehends the *DNS-VAR(4)* and *DNS-VAR(5)* models

matrix (Λ) is now $N \times (3 + K)$, while the other matrices that form the measurement equation still remains the same; ε_t is $N \times 1$ and H is $N \times N$. In particular, the K rightmost columns of Λ contain only zeros so that the yields still load only on the yield curve factors. The row(s) of Λ regarding the K macro factors are null for the effects of the three yield latent factors on macro factors.

7 Data description

The estimation procedure uses the following data. The macroeconomic factors are extracted from a macro panel containing 182 monthly time series for the Brazilian economy. Table 5 in Appendix A shows the macroeconomic panel data, whereas the individual variables are classified in various economic categories as follows: money growth (about 6% of the total set of variables), consumption and sales (10.5%), credit (5.5%), employmet, wage and income (9.4%), price (22.5%), production and real activity (18.7%), financial and risk (5%), fiscal (5.5%), and external sector (17%). Regarding the timing of the macro series, it is worth to note that the observation of macroeconomic data by agents only happens after a certain time of the reference month, because several variables take some time to be released. In general, the macro series are released with a lag of one up to three months. For example, the observed inflation variable IPCA is released until the tenth day of the following month from the reference. For this reason, the econometrician needs to be careful about the use of contemporaneous information, that may exaggerate the benefits of using macroeconomic information when forecasting yields (POOTER et al., 2010). Thereby, the analysis assumes that agents have an expectation or trustworthy proxy about the current macroeconomic scenario.

Most part of the macroeconomic dataset originates from Rossi & Carvalho (2009) and Almeida & Faria (2014), while the forward-looking variables are based on some variables used by Vieira et al. (2017). The forward-looking variables refers to market expectations about several key economic variables, available in the weekly market reports published by the Central Bank of Brazil, so-called Focus report[1]. Given the high-inflationary past of the Brazilian economy, the monetary authority monitors the market expectations about daily indicators of real activity, external sector, fiscal accounts, and mainly inflation. The market forecasts contained in the Focus report consists in 1-month-ahead until 5-year-ahead expectations. All forward-looking variables considered come from the weekly Focus report, released by the Central Bank of Brazil every Monday, focusing on the mean of market expectations for 1-1.5 year ahead, 2-2.5 year ahead and 3-5 year ahead. This gives a solid information about the future state of the Brazilian macroeconomy.

Time series regarding Brazilian interest rates are removed from the macro dataset to avoid complications that could emerge for the estimation process from not using an arbitrage-free model. Moench (2008) and Koopman & Wel (2013) also remove all variables relating to interest rates, arguing that central banks do not take into account the information contained in yields when making monetary policy decisions.

[1] In particular, it contains surveys with approximately 100 financial market participants who provide their predictions about the future value of some key economic variables.

The yields data consists in monthly observed yields of Brazilian Inter Bank Deposit Future Contract (DI-futuro) negotiated at the Brazilian Mercantile and Futures Exchange (BM&F), which is the entity that offers and determines the number of maturities with authorized DI-futuro contracts[2]. According to Caldeira et al. (2013), the DI-futuro contract with maturity τ is a zero-coupon future contract in which the underlying asset is the DI-futuro interest rate accrued on a daily basis, capitalized between trading period t and τ. The contract value is set by its value at maturity, R$ 100,000.00, discounted according to the accrued interest rate negotiated between the seller and the buyer. Technically, the DI-futuro rate has an underlying asset, the average daily rate of Brazilian interbank deposits[3] (borrowing/lending) calculated and released by the Clearinghouse for Custody and Settlements (CETIP). Despite being traded daily, the DI-futuro rate is expressed in annually compounded terms, based on 252 business days. Caldeira et al. (2010) explicit that when buying a DI-futuro contract for the price at time t and keeping it until maturity τ, the gain or loss is given by:

$$100,000 \left(\frac{\prod_{i=1}^{\zeta(t,\tau)} (1 + y_i)^{1/252}}{(1 + DI^*)^{\zeta(t,\tau)/252}} - 1 \right), \tag{7.1}$$

where y_i denotes the DI-futuro rate, $(i - 1)$ days after the trading day, and DI^* is the interest rate agreed between the seller and the buyer. The function $\zeta(t, \tau)$ represents the number of working days between t and τ. Therefore, the DI-futuro contract negotiates the accrued interest rates, based on market expectations about the future behavior of DI rates, in the period ranging from the trade date to maturity. The DI-futuro contract is very similar to the zero-coupon bond, except for the daily payment of marginal adjustments. Every day the cash flow is the difference between the adjustment price of the current day and the adjustment price of the previous day, indexed by the DI-futuro rate of the previous day. With respect to liquidity, as reported by BM&F, the DI-futuro market traded a total of 309,308,981 contracts in 2015 (about US$ 6.2 billion, in exchange rate from December, 2015), compared to 286,125,664 contracts in 2014.

Information about DI-futuro contracts are taken from the first business day of the month in which the contract is due. The interpolated yield curves are obtained by cubic splines interpolation, which allows one to convert observed yields into relevant maturities. The present study converts data into the following $N = 14$ different maturities; $\tau = 3$, 6, 9, 12, 15, 18, 21, 24, 27, 30, 36, 42, 48 and 60 months. The estimation procedure is carried out for the in-sample period from 2003:04 to 2012:11, while the predictive analysis is performed for the out-of-sample period from 2012:12 to 2016:03; a total of $T = 116$ in-sample and $S = 40$ monthly out-of-sample observations.

[2] The DI-futuro contract is a broad fixed-income market, trading currently about 29 maturities with authorized contracts every day.

[3] The DI rate reflects the average cost of interbank transactions and is an extremely important reference for various banking operations of the national financial system.

8 Principal component analysis

In a large macroeconomic dataset some groups of economic variables often move together, because they measure the same driving forces governing the behavior of the system. Thus, a system with abundant and redundant information contains only few driving forces that generates the entire original data. The principal component analysis (PCA) takes advantage of this redundancy of information, which identify the patterns in data and replace a group of variables that measure the same phenomenon by a few new variables. The method generates a set of new variables, called principal components, which form a linear combination of the original variables. The usefulness of PCA regards to data dimensionality reduction without much loss of information, solving the problem of analysing data in large multivariate systems.

The principal components can be extracted through the calculation of eigenvectors from the covariance matrix of the original data. The eigenvectors of the covariance matrix reflect vectors that characterise the original data and successfully account for variance in the observed variables. By definition, all the eigenvectors of a matrix are orthogonal to each other, so that there is no redundant information, where eigenvectors with the highest eigenvalues are the principle components of the dataset. Thus, ordering the eigenvectors by eigenvalue, highest to lowest, gives the principal components in order of significance. There will be some principal components with small eigenvalues, what enables one to leave out some components and still account for most of the variance in the observed variables, ending with less dimensions than the original data.

The procedure of principal components extraction in large panels of time series requires stationarity. For this reason different preadjustments are applied to the time series in the dataset when necessary, particularly first difference and first difference of the logarithm. Table 5 in Appendix A displays the necessary transformations for each macroeconomic variable to obtain stationary series. Finally, I standardize all variables to have mean zero and unit variance.

Bernanke et al. (2005) discuss the two-step approach for extracting common factors from the panel of macro data prior to estimating the term structure model. In this study, the common macro factors are also extracted prior to estimating the yield curve models, serving as input for the state vector of the DNS models. As in Bernanke et al. (2005) and replicated by Moench (2008), the dimensionality reduction exercise is achieved using standard static principal components following the approach suggested by Stock & Watson (2002a) and Stock & Watson (2002b). It is assumed that X_t, the R-dimensional multivariate time series containing the macroeconomic variables, admit a factor model representation with common factors F,

$$X_t = \alpha F_t + \epsilon_t^x, \tag{8.1}$$

where ϵ_t^x is a $R \times 1$ vector idiosyncratic components. According to Moench (2008), if V denote the eigenvectors corresponding to the k_x largest eigenvalues of the $R \times R$ cross-sectional variance-covariance matrix XX' of the data, subject to the normalization $F'F/R = I_r$, then estimates $\hat{F}$ of the factors and $\hat{\alpha}$ the factor loadings are given by $\hat{F} = \sqrt{R}V$ and $\hat{\alpha} = \sqrt{R}X'V$. As mentioned before, the common factors are estimated as the eigenvectors corresponding to the k_x largest eigenvalues of the variance-covariance matrix XX'. In another way, factors represent a linear combination of optimally weighted variables from the large macro dataset[1]. In addition, the PCA procedure is computationally simple and achieves plausible results.

In practice, the true number of common factors which capture the common variation in the dataset X is not known, but a small number of factors are capable to explain most of variation of all variables in the dataset. Because of this and computational constraints, the number of macro factors to be included in the estimations is limited. Later I will show that the analysis use two common macro factors, where both explain about 24.8% of the overall variation in observed variables[2].

Exterkate et al. (2013) investigate some additional issues regarding PCA and yield curve forecasting, exploring various ways of incorporating macroeconomic information in the Nelson-Siegel framework. First, they investigate whether it is useful to take the forecast objective explicitly into account when constructing the macro factors. The idea is to investigate whether it is desirable to include all available data in PCA or just those variables that are most correlated with the variable that one aim to predict. Second, whether it pays off to construct factors from groups of related macro variables, instead of one large pool of all available variables. The findings can be summarized: (i) for longer maturities, it is better to form groups of related variables and then extract factors from these groups, explicitly considering the forecast objective when constructing factors; (ii) for shorter maturities, and for medium maturities at shorter horizons, it is even better to extract principal components from all available information; (iii) in times where yields are highly volatile, macroeconomic variables are of substantive help in forecasting the yield curve. Hence, the extraction of macro factors from the large panel data seems to be a correct way to proceed the analysis.

---

1 Exterkate et al. (2013) discuss different methods to achieve a dimension reduction as least angle regression, principal component regression, principal covariate regression, partial least squares, hard thresholding and soft thresholding.

2 Note that I made a somewhat ad hoc choice for the maximum number of factors used in the following analysis, based solely on the marginal contribution of each factor to the forecasting exercise. Models with too many macro factors, above than two, provide forecasts that do not improve prediction accuracy. These empirical results are not reported to save space and can be provided upon request.

9 State-space model and Kalman filter

The DNS state-space structure represented by (3.7)-(3.11) implies that Kalman filter is immediately applicable for optimal filtering and smoothing of the latent yield factors (DIEBOLD; RUDEBUSCH, 2013). The unobserved state vector f_t and unknown parameters of the system can be estimated by several procedures. Diebold & Li (2006) introduced the two-step DNS approach. They treat λ as a calibrated parameter, so that in the first stage the measurement equation can be estimated by OLS to obtain a three-dimensional time series of estimated factors for each period t. In the second stage, the temporal dynamics of the estimated factors can be specified as AR(1) or VAR(1) processes, for example[1]. However, the simplicity of the two-step procedure comes with a cost. The approach ignores and transfers in an unknown way the estimated residuals from the first stage to the estimates of the subsequent stage, distorting the second-step inference and revealing itself an inefficient approach of estimation.

Diebold et al. (2006) introduced the one-step DNS, proving that it is possible to estimate simultaneously both the transition and measurement equations by maximum likelihood using Kalman filter. This in turn, seeks to estimate λ and obtain the conditional distribution of vector f_t given the set of information contained in the vector of observed variables $Y_t = \{y_1, ..., y_t\}$, building the likelihood function to be maximized. For the macroeconomic DNS structure, the one-step DNS is not absolutely one-step once macroeconomic factors are obtained separately from the state-space estimation. Thus, macro factors primarily extracted from principal component analysis are simply combined with yield latent factors in the state transition equation. The present study apply the one-step DNS, which is considered efficient by allowing one to do all estimation simultaneously.

Following the procedure described in Caldeira et al. (2010, p. 35), the one-step DNS uses Kalman filter to construct the likelihood function to be maximized and obtain the parameters estimates. Define $f_{t|t-1}$ as the expectation for the state vector f_t given the set of information $Y_{t-1} = \{y_1, ..., y_{t-1}\}$ and its estimated covariance matrix to period t equal to $P_{t|t-1}$. For values of $f_{t|t-1}$ and $P_{t|t-1}$, when observation y_{t-1} is available, the prediction error can be calculated as $v_t = y_t - \Lambda f_{t|t-1}$. Thus, after the next observation y_t, a more accurate inference of $f_{t|t}$ and $P_{t|t}$ could be performed:

$$f_{t|t} = f_{t|t-1} + P_{t|t-1}\Lambda' F_t^{-1} v_t,$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}\Lambda' F_t^{-1} \Lambda P'_{t|t-1},$$

[1] For the macroeconomic DNS structure, the two-step approach just links the yield factors and macro factors extracted from a large set of macroeconomic variables, analyzing all factors in a joint vector autoregression.

where $f_{t|t}$ and $P_{t|t}$ are the filtered states and its filtered covariance matrix, and $F_t = Var(y_t|y_{t-1},...,y_1) = \Lambda P_{t|t-1}\Lambda' + H$ is the covariance matrix of prediction errors of the observed variable. It is possible to note that filtered states in time t are the predicted states for t plus an adjustment term based on the reliability of the observations, equal to $K_t v_t$. The parameter $K_t = P_{t|t-1}\Lambda' F_t^{-1}$ is also called Kalman gain. In short, Kalman filter algorithm estimates for $t = 1,...,T$ the one-step-ahead state forecasts for period t ($f_{t|t-1}$), its variance-covariance matrix ($P_{t|t-1}$), and the one-step-ahead observation forecasts for period t ($y_{t|t-1}$) and its estimated variance-covariance matrix (F_t); feeding the forecasted and filtered estimates into the data likelihood function.

For a certain time series $Y_T = \{y_1,...,y_T\}$, Kalman filter algorithm works recursively for $t = 1,...,T$ with initial values for the set of unknown parameters collected in θ. The vector θ is composed by parameters of matrices A, Q and H, together with the vector of average factor states μ and parameter λ, which are treated as time-invariant. The estimation of θ uses a numerical optimization method that maximizes the log-likelihood function[2], which is constructed via decomposition of the one-step-ahead prediction error:

$$l(\theta) = -\frac{NT}{2}log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} log|F_t| - \frac{1}{2}\sum_{t=1}^{T} v_t' \, log(F_t^{-1})v_t. \tag{9.1}$$

Theoretically, the maximum likelihood estimator obtained is preferable to two-step DNS approach[3], as the joint estimation of the transition and measurement equation parameters produces efficient estimates of yields.

The Kalman filter procedure starts with initial values for states (f_0), initial values for coefficients of state transition matrix (A_0), initial state disturbance loading matrix (B_0), initial values for observation innovation matrix (D_0) and initial value for parameter λ_0, forming the initial set of parameters (θ_0)[4]. For *DNS-VAR(3)* model, θ_0 comes from the two-step DNS approach; initial state values are simply the average of level, slope and curvature factors filtered in two-step DNS approach and λ_0 is calibrated at 0.0726, as suggested by Diebold & Li (2006). For *yields-macro* models, A_0 and B_0 are set to zero, D_0 with respect to yield components comes from the two-step DNS approach while the part related to macro factors is set to zero, and f_0 and λ_0 follow the assumptions made for *DNS-VAR(3)* model.

[2] The MATLAB estimation code uses "fminunc" function to optimize the procedure of finding the unknown parameters.

[3] Diebold & Rudebusch (2013) discuss other estimation alternatives for state-space DNS, like expectation maximization (EM) and Bayesian one-step methods.

[4] To compute the model estimation code on MATLAB platform one need to transform the system of Eqs. (3.7)-(3.9) in to the formulation supported by the SSM functionality of the Econometrics Toolbox. The transformation requires that the vectors of disturbances η_t and ε_t must equal $\eta_t = Bu_t$ and $\varepsilon_t = D\epsilon_t$ respectively, where B is the state disturbance loading matrix and D is the observation innovation matrix. The vectors u_t and ϵ_t of disturbances are defined as uncorrelated, unit-variance white noise processes, and their covariance matrices are identity matrices. It is possible to notice that the covariance of η_t must equal the covariance of the scaled white noise process Bu_t; similarly happens with covariance of e_t, so that $Q = BB'$ and $H = DD'$.

Immediately, if information is available by time t, for the forecast horizon h-step-ahead the construction of out-of-sample forecasts consists simply in forecasting the factors for a given forecast horizon and apply them to the yield curve equation explored in (3.6):

$$f_{t+h|t} = E_t(f_{t+h}|y_t, ..., y_1) = \mu + \prod_{j=t+1}^{t+h} A_j\,(f_{t|t} - \mu) = \mu + A^h(f_{t|t} - \mu), \tag{9.2}$$

$$y_{t+h|t} = E_t(y_{t+h}|y_t, ..., y_1) = \Lambda E_t(f_{t+h}|y_t, ..., y_1) = \Lambda\mu + \Lambda(A^h(f_{t|t} - \mu)). \tag{9.3}$$

Again, the application of Kalman filter is convenient to extract the optimal general h-step-ahead prediction of both the yield factors and the observed yields ($f_{t+h|1:t}$, $y_{t+h|1:t}$). From (3.9), it is worth to remember that the disturbances η_t and ε_t have zero mean, which implies that estimation errors in-sample do not pass along forecasts. As filtered state vector $f_{t|t}$ equals smoothed states at period t in this case[5], one can use any of these estimates to compute predicted states for horizon $t + h$.

[5] Filtered state vector at period t, when using all information up to t corresponds to smoothed states. By definition, the smoothed states at period t using all available information T is $f_{t|T} = E[f_t|y_T, ..., y_1]$.

10 The distribution of log-returns

Following the discussion in Caldeira et al. (2016), factor models for the term structure of interest rates are designed to model only the bond yields. Thus, the forecasting stage of yield curve models aim for modeling merely moments of the expected yields. However, the fixed-income portfolio problem requires estimates of the expected return of each security, as well as estimates of their covariance matrix. The following mathematical decompositions show that it is possible to obtain expressions for the expected return of fixed-income securities and their covariance matrix based on the distribution of the expected yields.

The mean-variance portfolio optimization is performed for two different forecast horizons: (i) first, one-step-ahead forecasts for log-returns of DI-futuro contracts are used to optimize fixed-income portfolios with monthly rebalancing; and subsequently (ii) twelve-step-ahead forecasts for log-returns are used to find optimal portfolios with annual rebalancing. For this reason, the portfolio choice problem requires moments of the expected yields for one-month- and one-year-ahead forecast horizons. The system of Eqs. (3.7)-(3.9) implies that the distribution of one-step-ahead forecasts for y_t, of any maturity τ_i, is normally distributed, i.e. $y_{t|t-1} \sim N(\mu_{y_{t|t-1}}, \Sigma_{y_{t|t-1}})$, with moments given by[1]:

$$\mu_{y_{t|t-1}} = E_{t-1}[y_t] = \Lambda f_{t|t-1}, \tag{10.1}$$

and

$$\Sigma_{y_{t|t-1}} = \Lambda(AP_{t-1|t-1}A' + Q)\Lambda' + H, \tag{10.2}$$

where $f_{t|t-1}$ denotes the predicted value of the unknown factors f_t conditional on period $t-1$ information, and the covariance matrices Q and H, defined in (3.9), are time-invariant. Eq. (10.1) follows straightforward from Eq. (9.3), which define the expectation of yields for the h-step-ahead forecast horizon. Note that the covariance matrix of the true, but non-observable states (f_t), would be given simply by Q. However, as stated in Eq. (10.1), predicted states based on filtered estimates of f_{t-1} are used when computing expected yields. Thus, Eq. (10.2) takes into account the uncertainty in the Kalman filter estimates of the unobserved factors through $AP_{t-1|t-1}A'$, containing the covariance matrix of the filtered states ($P_{t-1|t-1}$) and not only the covariance matrix of the unobserved factors, Q[2]. Therefore, the first term in Eq. (10.2), $AP_{t-1|t-1}A' + Q$, adjusts for the fact that filtered estimates are used in (10.1), and not the true value of states.

[1] Durbin & Koopman (2012, p. 112) define the general formulations for the conditional mean square error matrix and conditional mean of the covariance matrix of predicted states.

[2] For comparison, Caldeira et al. (2013) use the true value of the state vector and show that the second moment of $y_{t|t-1}$ just takes into account the covariance matrix of the unobserved factors, Q.

Similarly, the distribution of twelve-step-ahead forecasts is normally distributed, i.e. $y_{t|t-12} \sim N(\mu_{y_{t|t-12}}, \Sigma_{y_{t|t-12}})$, with moments given by:

$$\mu_{y_{t|t-12}} = E_{t-12}[y_t] = \Lambda f_{t|t-12}, \tag{10.3}$$

and

$$\begin{aligned}
\Sigma_{y_{t|t-12}} &= \Lambda P_{t|t-12}\Lambda' + H \\
&= \Lambda(AP_{t-1|t-12}A' + Q)\Lambda' + H \\
&= \Lambda(A(AP_{t-2|t-12}A' + Q)A' + Q)\Lambda' + H \\
&= \Lambda(A^2 P_{t-2|t-12}A'^{\,2} + AQA' + Q)\Lambda' + H \\
&= \Lambda(A^2(AP_{t-3|t-12}A' + Q)A'^{\,2} + AQA' + Q)\Lambda' + H \\
&= \Lambda(A^3 P_{t-3|t-12}A'^{\,3} + A^2QA'^{\,2} + AQA' + Q)\Lambda' + H_{t|t-12} \\
&= \ldots \\
&= \Lambda(A^{12}P_{t-12|t-12}A'^{\,12} + \sum_{i=1}^{12} A^{i-1}QA'^{\,i-1})\Lambda' + H.
\end{aligned} \tag{10.4}$$

Note that, in this case, the term $(A^{12}P_{t-12|t-12}A'^{\,12} + \sum_{i=1}^{12} A^{i-1}QA'^{\,i-1})$ adjusts for the fact that the model uses filtered estimates for the twelve-step-ahead forecasts of yields and accumulates the uncertainty in the Kalman filter estimates for each step forecast.

Using the fact that the price of a security with maturity τ_i at time t, $P_t(\tau_i)$, is the present value at time t of \$1 receivable τ_i periods ahead, the bond price for a particular maturity τ_i can be computed following the discount curve (2.2), $P_t(\tau_i) = \exp(-\tau_i \cdot y_t(\tau_i))$. To compute the realized return, $r_t(\tau_i)$, of holding that security from $t-h$ to t while its maturity decreases from τ_i to τ_{i-h}, one can use the bond price and the log-return expressions,

$$r_t(\tau_i) = \log\left(\frac{P_t(\tau_{i-h})}{P_{t-h}(\tau_i)}\right) = \log P_t(\tau_{i-h}) - \log P_{t-h}(\tau_i) = -\tau_{i-h} \cdot y_t(\tau_{i-h}) + \tau_i \cdot y_{t-h}(\tau_i). \tag{10.5}$$

It is clear to note from (10.1)-(10.5) that the vector of h-step-ahead forecasts of log-returns, $r_{t|t-h}$, also follows a Normal distribution $N(\mu_{r_{t|t-h}}, \Sigma_{r_{t|t-h}})$ with mean given by:

$$\mu_{r_{t|t-h}} = -\tau_{i-h} \otimes \mu_{y_{t|t-h}}(\tau_{i-h}) + \tau_i \otimes y_{t-h|t-h}(\tau_i), \tag{10.6}$$

where $\mu_{y_{t|t-h}}(\tau_{i-h})$ is the mean vector of the expected yields with maturity τ_{i-h} at time t conditional on period $t-h$ information, $y_{t-h|t-h}(\tau_i)$ is the vector of observed yields with maturity τ_i at time $t-h$, and $\otimes$ represents the Hadamard (elementwise) multiplication. The conditional covariance matrix of the expected log-returns, which is positive-definite, is given by:

$$\Sigma_{r_{t|t-h}} = \tau'_{i-h}\tau_{i-h} \otimes \Sigma_{y_{t|t-h}}. \tag{10.7}$$

The discussion above solves the problem for obtaining estimates of the expected log-returns for fixed-income securities and their covariance matrix based on yield curve models such as the DNS model, which are essential inputs to the portfolio choice problem based on the mean-variance approach suggested by Markowitz. All ingredients necessary to calculating the closed-form expressions (10.6)-(10.7) are easily retrieved from the Kalman filter estimation discussed in Chapter 9. In particular, predicted states $f_{t|t-h}$, and the covariance matrix of filtered states, $P_{t-h|t-h}$, which are used to determine the moments (10.1)-(10.2), are direct products of the Kalman filter recursions and are readily available[3].

[3] The MATLAB function called "filter" can also be used to compute the moments $\mu_{y_{t|t-h}}$ and $\Sigma_{y_{t|t-h}}$. The output of the function reports the forecasted observations and the covariance matrix of filtered states.

Part III

Empirical Results and Discussion

11 Preliminary evidence

Fig. 1 shows the resulting three-dimensional surface for the nominal yields of Brazilian DI-futuro contracts as a function of maturity over time. The graph reveals some stylized facts common to yield curve data, as its dynamics through time and its various possible shapes and levels. Over the first years of the sample period the Brazilian nominal yields decreased quickly, changing from rates above 20% to yields near 10% per year in 2006. The decreasing path of Brazilian yields goes until December, 2012, as the inflation was reduced over the years. Since then, yield curves started an increasing shift in level, following the advance of inflation expectations. For most part of the period, the yield curves reflected an upward sloping and concave shape, being slightly downward sloping in 2005, 2008 and 2015.

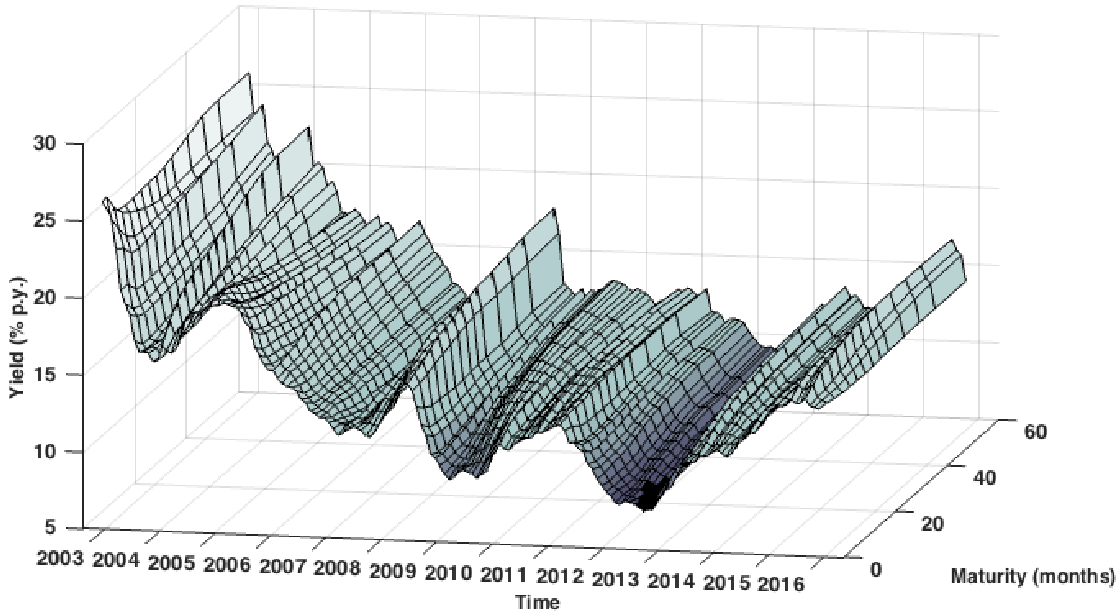


Figure 1 – Brazilian nominal yields in three dimensions, observed from DI-futuro contracts at maturities ranging from 3 months to 5 years during the sample period 2003:04-2016:03.

In respect to the extraction of common factors from the large macroeconomic dataset, the analysis use two macro factors, where both explain about 24.8% of the overall variation in observed variables. In table 5 of Appendix A I have grouped 182 macroeconomic variables into 9 economic categories. In order to provide some insights about the macroeconomic content in each factor extracted, Fig. 2 displays the correlation coefficient between every macro variable and each of the first principal components. The first macro factor (X^1) exceeds 16.6% of the total variance of the original data and correlates mostly with the following economic categories: production and real activity, price, employment, wage and income. The second factor (X^2) accounts for 8.24% of the variation in original data and correlates mostly with real activity variables, external sector and most prominently with inflation expectations. Besides that, the individual forward-looking variables are highly correlated with the common macro factors, specially X^1.

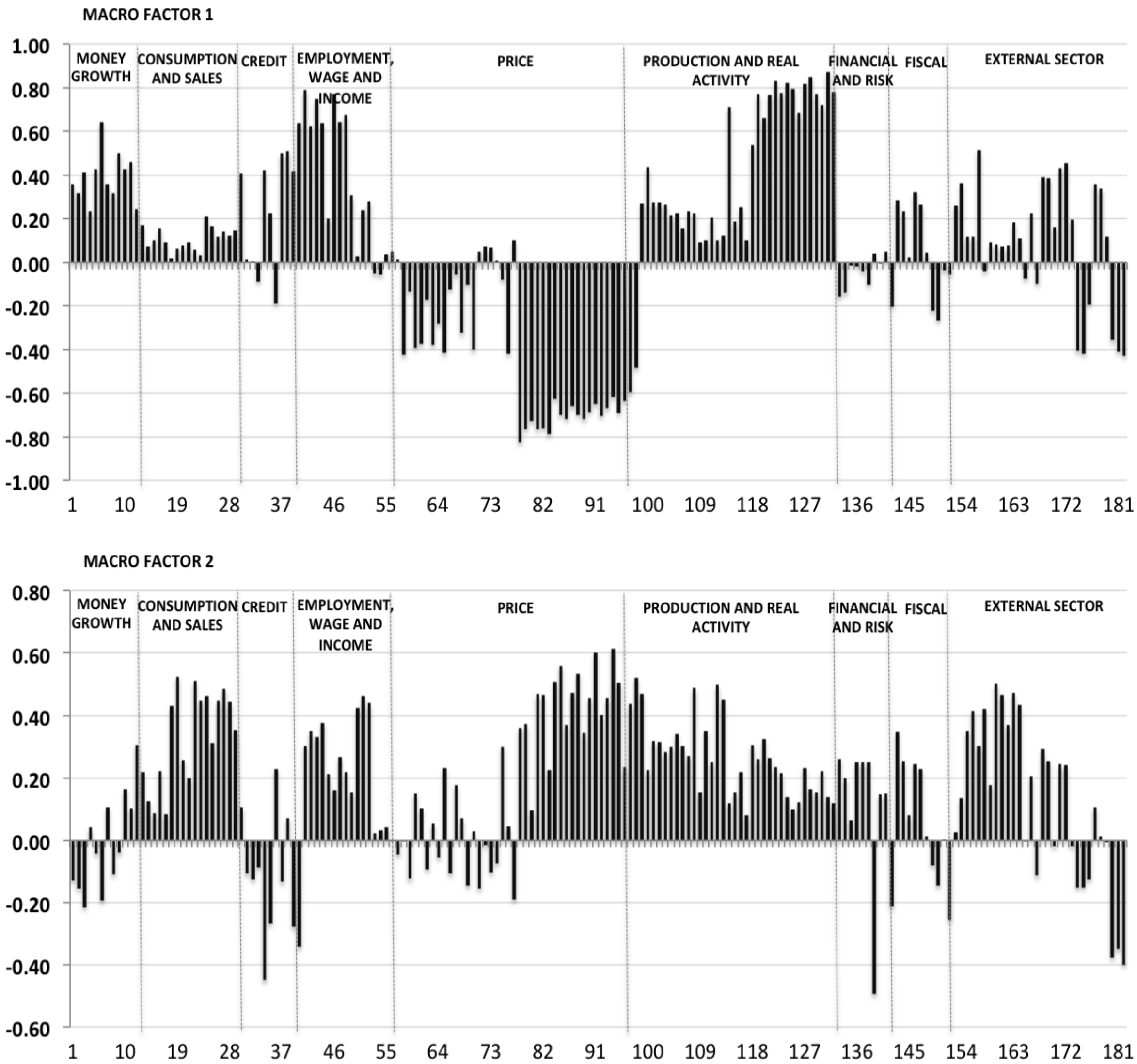


Figure 2 – Correlation between macro common factors and individual macroeconomic series.

The correlations described above give an indication that X^1 is possibly related to business cycle, while X^2 is likely to represent the price level and central bank's efforts to control inflation. In order to confirm these primary assumptions, Fig. 3 plots the X^1 and X^2 time series against some highly correlated individual macroeconomic variables. The first macro factor exhibited in panels (a) and (b) is a relatively smooth time process that clearly shows characteristics related to business cycle once it strongly correlates with market expectations for GDP and the General Registration of Employed and Unemployed (CAGED). The graph from panel (a) undoubtedly evidence that X^1 follows the path that market expectations are delineating for the one-year-ahead Brazilian economic scenario. For example, the negative values for X^1 in the beginning of 2003, 2008 (recent financial crisis) and after 2013, are clearly associated with recession periods. For these reasons, X^1 can be labelled as *business cycle* factor.

On the other hand, panels (c) and (d) reveal a noisier process for X^2. Factor X^2 presents a relative strong and positive correlation with market expectations for next 12 months IGP-DI and market expectations for 2-2.5 years ahead IPA-M, both inflation indexes. Fig. 2 also displays other strong correlations between X^2 and inflation expectations, specially for prices administrated by contracts and monitored. Thus, it is plausible to assume that the second macro factor reflect mainly inflation future scenarios, and can be labelled as *inflation* factor. These common factors extracted here are in line with findings in Koopman & Wel (2013) and Pooter et al. (2010), where the first macro factor resembles the real activity and the second factor is mostly related to price indexes.

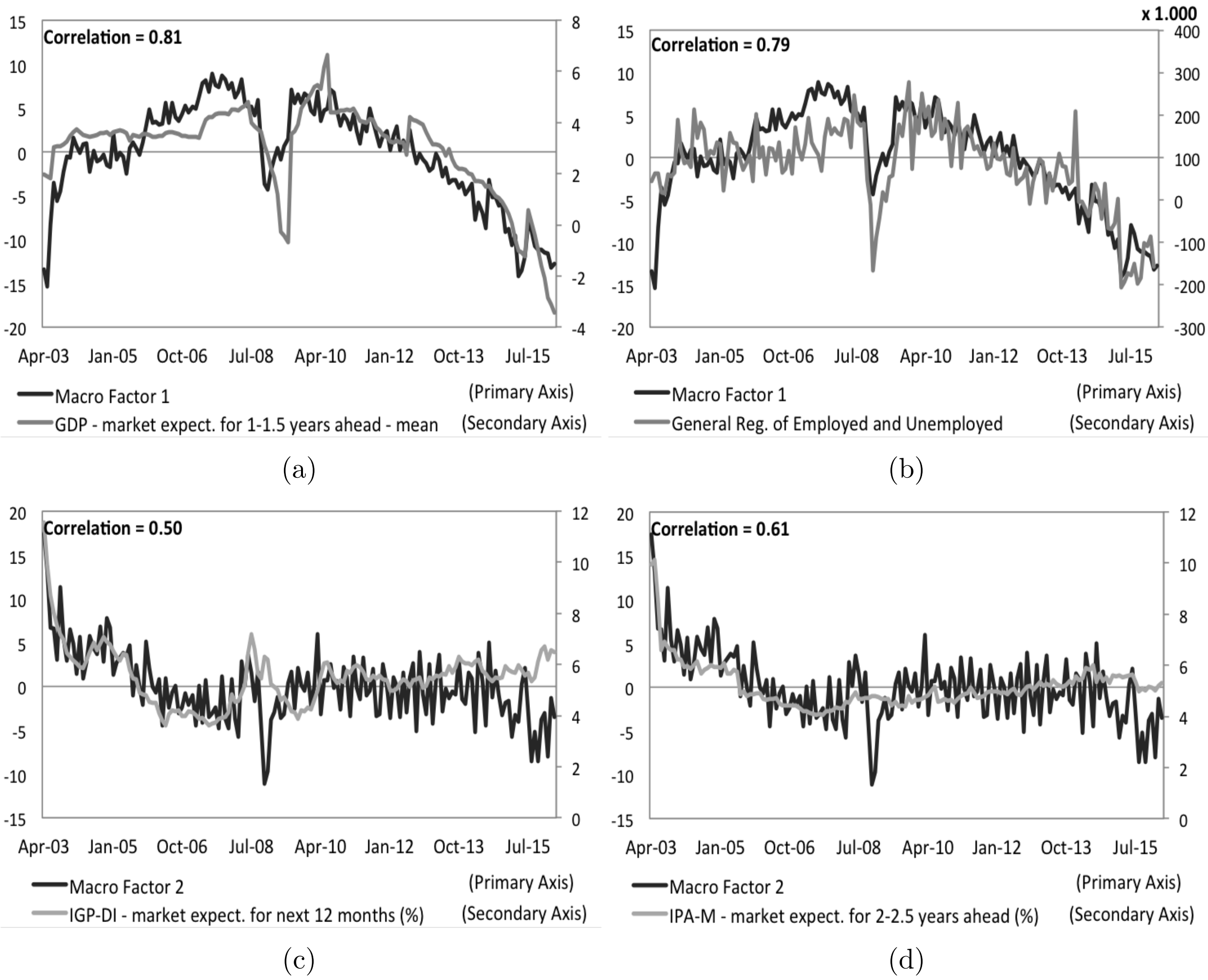


Figure 3 – Plots of common macro factors and individual most correlated macroeconomic variables.

Next, table 2 presents summary statistics for the yield dataset at various maturities, including the yield latent factors and the first two standardized macroeconomic factors.

The table reports the mean, standard deviation, minimum and maximum, as well as the 25%, 50% (median) and 75% quantiles, and sample autocorrelations at displacements of 1 and 12 months. As analysed in Diebold & Rudebusch (2013), several important yield curve facts emerge: (i) time-averaged yields increase with maturity revealing an increasing and slightly concave shape; which reports some kind of term premium, perhaps due to risk aversion, liquidity preferences, or preferred habitats; (ii) yield volatilities decrease with maturity until τ = 30-month and then slightly increase; the first behavior is normal as long rates involve averages of expected future short rates according to expectations hypothesis, but the second one could suggest some kind of unusual uncertainty scenario expected for long-term; (iii) yields are highly persistent, as evidenced by the very large 1-month and significant 12-month spread autocorrelations, specially for shorter maturities.

Table 2 – Summary statistics of yield curve and macro series.

	Mean	Sd	Quantiles					$\hat{\rho}(1)$	$\hat{\rho}(12)$
			Min	Q(25%)	Median	Q(75%)	Max		
DI-futuro yields (by maturity)									
3	0.1300	0.0385	0.0703	0.1058	0.1214	0.1477	0.2617	0.9548	0.5116
6	0.1304	0.0371	0.0707	0.1057	0.1224	0.1508	0.2501	0.9576	0.5278
9	0.1309	0.0358	0.0719	0.1056	0.1242	0.1536	0.2441	0.9540	0.5335
12	0.1317	0.0348	0.0731	0.1083	0.1247	0.1546	0.2427	0.9506	0.5314
15	0.1325	0.0340	0.0734	0.1104	0.1255	0.1552	0.2429	0.9463	0.5268
18	0.1333	0.0333	0.0747	0.1123	0.1260	0.1560	0.2439	0.9414	0.5236
21	0.1339	0.0327	0.0766	0.1128	0.1262	0.1562	0.2442	0.9377	0.5241
24	0.1345	0.0322	0.0786	0.1139	0.1269	0.1571	0.2440	0.9345	0.5244
27	0.1351	0.0319	0.0800	0.1145	0.1267	0.1578	0.2454	0.9309	0.5244
30	0.1355	0.0317	0.0810	0.1151	0.1269	0.1577	0.2471	0.9278	0.5271
36	0.1361	0.0317	0.0819	0.1166	0.1270	0.1571	0.2516	0.9220	0.5317
42	0.1367	0.0318	0.0834	0.1180	0.1273	0.1575	0.2564	0.9172	0.5343
48	0.1371	0.0319	0.0846	0.1185	0.1272	0.1583	0.2575	0.9167	0.5342
60	0.1375	0.0322	0.0864	0.1182	0.1273	0.1569	0.2579	0.9185	0.5355
Yield curve latent factors (level, slope and curvature)									
L_t	0.1408	0.0355	0.0942	0.1200	0.1287	0.1522	0.2884	0.8954	0.4792
S_t	-0.0138	0.0277	-0.0794	-0.0315	-0.0176	0.0016	0.0558	0.8829	-0.0146
C_t	-0.0017	0.0384	-0.1124	-0.0258	0.0020	0.0227	0.1103	0.8720	-0.0276
First two standardized principal components (PC) from the macro series									
1st PC	0	1	-2.6934	-0.5574	0.1206	0.7820	1.5682	0.9113	0.4939
2st PC	0	1	-2.7784	-0.7503	-0.0165	0.6053	4.3350	0.4402	0.2468

Notes: The table presents the descriptive statistics for DI-futuro contracts over the period 2003:04-2016:03. The monthly yield curves were constructed using cubic splines interpolation. For each maturity, the table displays the mean, standard deviation (Sd), minimum (Min), 25% quantile, median, 75% quantile, maximum (Max), and sample autocorrelations at displacements of 1 ($\hat{\rho}(1)$) and 12 ($\hat{\rho}(12)$) months. In addition, it shows the statistics for the latent factors of yield curve, defined as level, slope and curvature, and for the first two standardized principal components extracted from the macro dataset, presented in table 5, Appendix A.

The observed yields data show an asymmetric distribution, where most of the observations concentrate in lower rates. In line with this fact, the level factor is skewed to the right due to high observed yields of the first years of the sample period, intensifying data in the first two quantiles. The slope factor is negative for most part of the observations and concentrate in the second quantile of its sample distribution. The observations regarding X^1 pursue areas with positive values. So, as X^1 is highly correlated with business cycle, its sample statistics reflect that there are more procyclical periods in the Brazilian economy overall the sample period. The observations of X^2 point to a sample distribution slightly skewed to the left, where data variation is larger for negative values and central observations concentrate in the third and positive quantile. When it comes to sample autocorrelations, the latent yield factors and X^1 exhibit high persistences at displacement of 1 month, while X^2 shows a moderate persistence.

Before estimating the different DNS specifications, I analyse the correlations between observed yields, latent yield factors and various lags of the macro factors to check whether macro factors capture predictive information about interest rates. In particular, the purpose is to investigate whether the extracted macro factors are potentially useful explanatory variables in a term structure model. Table 6 in Appendix B summarizes those correlations, reporting that interest rates for some relevant maturities jointly with level and slope factors, are most strongly correlated with the short end of the yield curve. This fact highlights the potencial of yield factors to explain dynamic yield curve movements through time. The contemporaneous and 1 month lagged correlations from panels A and B exhibit a negative correlation that ranges from 0.21 to 0.27 between X^1 and observed yields. The aspect of these relationships demonstrate convergence to economic theory, where ascending economic periods are compatible with low interest rates. On the other hand, X^2 reveals a relative much stronger and positive correlation with observed yields and yield factors, whereas the correlation coefficient decrease with the lag length. These findings help motivating the usefulness of the incorporation of macro factors to the yield curve model specifications.

Moreover, two interesting findings with respect to the inflation factor emerge. First, the medium and long-term interest rates depend more heavily on X^2 than short maturities; a similar finding is stated in Koopman & Wel (2013), where macro factors have a much stronger impact on interest rates associated with medium maturities. Second, it is interesting to observe that the inflation factor is highly correlated with both the level and curvature of the yield curve for all panels. Economic theory broadly suggests that the nominal yield curve level should be linked to the level of expected inflation (DIEBOLD; RUDEBUSCH, 2013). Dijk et al. (2014), for example, argue that long-run inflation expectations drive the level factor. This evidence indicates that factors extracted from the macroeconomic dataset might be useful for forecasting interest rates, where X^2 exhibit a greater potencial to explain yield curve movements.

12 Estimating term structure models

12.1 In-sample estimates

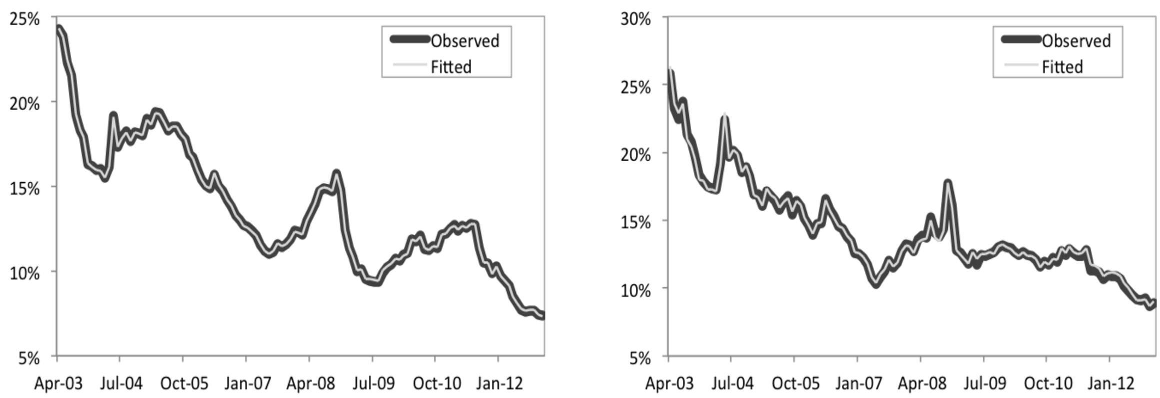
This section explores the results obtained from estimating the DNS specification models represented by *DNS-VAR(3)* and the set of *yields-macro* models defined in Sections 6.1 and 6.2. Table 7 in Appendix C reports the estimates of the coefficients in the state transition matrix for *DNS-VAR(3)*, *DNS-VAR(4)*^{2} and *DNS-VAR(5)* models. In addition, table 8 in Appendix C displays the descriptive statistics of the measurement disturbances. Thus, estimates for *DNS-VAR(4)* model just consider the inflation factor, X^2, which correlates with yield factors and improve forecast accuracy as will be reported later. The results indicate that on average the estimated models provide a good fit to the yield curve across the entire maturity spectrum, except for very short maturities. For maturities above 9 months the models fit the observed data efficiently well. This bad fit behavior for short maturities also is reported by Diebold et al. (2006), where estimated errors for yields of 3-months maturity are relatively higher. The fact of more pronounced adjustment at the medium and long end of the yield curve is a general feature of the DNS framework.

Table 8 also reports that the mean and standard deviation of the measurement disturbances from *DNS-VAR(4)*^{2} and *DNS-VAR(5)* estimates outperform *DNS-VAR(3)* model for maturities over 12 months. That is, yield curve estimates of the macroeconomic specifications for medium and long-term maturities are more accurate than *DNS-VAR(3)* estimates. Furthermore, the estimated errors of *DNS-VAR(5)* model are higher compared to *DNS-VAR(4)*^{2} model. These results sign for a path where macroeconomic information can improve yield curve predictions, at least for longer maturities.

Most of estimates for the leading diagonal of transition matrix A present high coefficients, ranging from 0.48 to 0.95 for f_t, confirming the high persistence level of the entire set of latent factors. Even macro factors are consistently significant at a level of 1% regarding to their persistence coefficients, whereas those for X^2 have values between 0.44 and 1.01. On the other hand, the estimated off-diagonal elements of A are all smaller than 0.12 in absolute values for *DNS-VAR(3)* and *DNS-VAR(4)*^{2} models. Besides the small bilateral effects between factors of f_t, the estimates reveal many significant off-diagonal relationships between the components of the yield curve. For example, the 1-month lagged slope factor consistently affect the level factor at significant and positive values, as well as the lagged curvature factor continually impact on slope factor. For the *DNS-VAR(5)* model, the whole set of estimated off-diagonal coefficients regarding f_t are significant at 1% level and range from 0.04 to 0.34 in absolute values.

The estimates for *yields-macro* models do not report significant effects of macro factors on yield factors, except for curvature factor, which responds to X^1 one period lagged in *DNS-VAR(5)* model. A similar finding was reported by Koopman & Wel (2013), highlighting the importance of the dynamic interactions between the lagged curvature factor and business cycle features. In addition, the yield factors do not impact on macro factors at significant levels, except for *DNS-VAR(5)* model where X^2 respond to shocks in curvature 1-month lagged. The lack of significant relationships between macro factors and yield factors is not necessarily a sign of poor prediction performance for the macroeconomic specifications.

The fitted yield curve also can be seen in Fig. 4, which plots the time series for 12- and 60-month maturity of observed and fitted yields estimated by *DNS-VAR(4)*^{2} model. It is noteworthy that the DNS framework is capable to capture the cross-sectional variation of Brazilian nominal yields, fitting quite well the different shapes of the yield curve during the in-sample period.



(a) Realized and fitted annualized 12 months yields (b) Realized and fitted annualized 60 months yields

Figure 4 – Observed and fitted time series for two selected interest rates, the 12-month yield and the 60-month yield, estimated by *DNS-VAR(4)*^{2} model for in-sample period.

12.1.1 Alternative model specifications

The unrestricted DNS model (3.7)-(3.11), and its macroeconomic structure, are a general model specification for the yield curve dynamics. Nonetheless, those general specifications could carry the problems of overparameterization and loss of degrees of freedom. An unrestricted state transition matrix provides a very general linear model of yields typically with good in-sample fit, but the large number of estimated coefficients may reduce its value for out-of-sample forecasting (DIEBOLD; RUDEBUSCH, 2013). Further, previous literature have shown that parsimonious models often outperform more sophisticated models (POOTER et al., 2010). Thus, this suggests that there could be some parsimonious factor structures that can reduce the risk of overfitting and improve the out-of-sample forecast accuracies relative to the unrestricted model.

This section could in principle consider an almost unlimited number of alternative models. I primarily consider combinations of a restricted vector-autoregressive for the transition matrix A and a restricted structure for the covariance matrix Q. The previous section reports small off-diagonal effects between factors, although some statistically significant relationships, which support the following discussion that consider a set of restrictions on matrix A, matrix Q, and on the number of factors. The restrictions are applied to the following general specification models: *DNS-VAR(3)*, *DNS-VAR(4)*^{2} and *DNS-VAR(5)*. Even for diagonal structures, the estimates maintain effects of macro factors on the components of the yield curve aiming to explore the potential contributions of the macroeconomy to yield curve forecasting. The alternative models representing the different restrictions applied are as follows:

1. The first restriction is to set diagonal structures for both the matrix A and Q. These restriction models are denoted by *DNS-AR(3)*, *DNS-AR(4)*^{2} and *DNS-AR(5)*. As the model structure only contemplates effects of macroeconomic factors to the yield curves in future time periods via dynamic interaction in the transition equation, the diagonal structures for A and Q are just applied to the yield factors. The restriction implies that I do not model any yields-to-macro feedbacks, allowing only for a unidirectional link from macro factors to yields. Hence, the *DNS-AR(3)* specification represents a complete first-order autoregressive structure, whereas yield factors also depend on macroeconomic factors in *DNS-AR(4)*^{2} and *DNS-AR(5)* specifications[1].

2. The second restriction is to allow the transition equation to be unrestricted while the covariance matrix Q is only diagonal when it comes to yield factors interactions. The difference to the first specifications is that now the yield factors can affect each other, so that matrix A is fully estimated. The general models with these restrictions are now denoted by *DNS-VAR(3)*$^{Q-diag}$, *DNS-VAR(4)*$^{2,Q-diag}$ and *DNS-VAR(5)*$^{Q-diag}$, respectively.

3. The third restriction inverts the second one, allowing the transition covariance matrix Q to be unrestricted while matrix A has a diagonal structure applied to yield factors interactions. Now, the general specifications are denoted by *DNS-VAR(3)*$^{A-diag}$, *DNS-VAR(4)*$^{2,A-diag}$ and *DNS-VAR(5)*$^{A-diag}$, respectively.

4. The fourth restriction is to estimate the general models, limit the matrix Q to be diagonal and set to zero the least significant estimated parameters of matrix A, following the analysis process stated in Christensen & Rudebusch (2012). The selection of significant coefficients of matrix A follows the general-to-specific modeling strategy, discussed in Hendry (2001), which restricts the least significant parameter in the estimation to zero and then re-estimates the model. The best fitting specification

[1] The difference in the estimation refers to matrices A and Q, where just the off-diagonal elements regarding the yields latent factors interactions are set to zero in *DNS-AR(4)*^{2} and *DNS-AR(5)*.

is based on values for Akaike and Bayes information criteria (AIC and BIC). The general models following the proposed selection process are now denoted by *DNS-VAR(3)*^{S}, *DNS-VAR(4)*^{2,S} and *DNS-VAR(5)*^{S}, respectively, and their transition matrices follows the subsequent structures:

$$A^{3S} = \begin{pmatrix} k_{11}^S & 0 & 0 \\ k_{21}^S & k_{22}^S & k_{23}^S \\ 0 & 0 & k_{33}^S \end{pmatrix}, \; A^{4S} = \begin{pmatrix} k_{11}^S & 0 & 0 & k_{14}^S \\ k_{21}^S & k_{22}^S & k_{23}^S & k_{24}^S \\ k_{31}^S & 0 & k_{33}^S & 0 \\ 0 & 0 & 0 & k_{44}^S \end{pmatrix}, \; A^{5S} = \begin{pmatrix} k_{11}^S & 0 & 0 & 0 & k_{15}^S \\ k_{21}^S & k_{22}^S & k_{23}^S & 0 & k_{25}^S \\ 0 & 0 & k_{33}^S & k_{34}^S & 0 \\ 0 & 0 & 0 & k_{44}^S & 0 \\ 0 & 0 & 0 & k_{54}^S & k_{55}^S \end{pmatrix}.$$

5. The fifth and last restriction imposed is to allow for a smaller number of factors, whereas the curvature factor is not included in the factors vector. The literature commonly chooses three yield factors as the appropriate structure (DIEBOLD; LI, 2006), but the purpose is to investigate whether the curvature factor has an impact on fitting and forecasting yields data. So, the general specifications are reduced to two yield factors (level and slope), keeping the unrestricted structures for A and Q. These alternative specifications are denoted by *DNS-VAR(2)*^{C}, *DNS-VAR(3)*^{2,C} and *DNS-VAR(4)*^{C}, respectively.

Table 10 in Appendix E reports the goodness-of-fit statistics covering the in-sample estimates. The statistics clearly show that the *DNS-VAR(3)* model maximizes the log likelihood value (log L) and minimizes the values for AIC and BIC. Then, the *DNS-VAR(3)*$^{A-diag}$, *DNS-VAR(3)*$^{Q-diag}$, *DNS-VAR(3)*^{S} and *DNS-AR(3)* models also present good in-sample fit. Therefore, specifications with small number of estimated parameters present highest values for log L and smaller values for AIC and BIC. Looking at the *yields-macro* alternative specifications, those that consider only one macro factor report better results than those specifications that incorpore two macro factors, which clearly exhibit relative poor adjustments compared to their pair. Moreover, the class of models that apply the fifth restriction report the poorer in-sample adjustments. The key point to note here is that the general *DNS-VAR(3)* model provide the most parsimonious specification choice of all estimated models, followed by other specifications with small number of estimated parameters.

The evaluation of in-sample adjustment, reported by table 11 in Appendix E, examine the (trace) root mean squared error (RMSE) relative to benchmark, which in this case is the *DNS-AR(3)* model, supported by its relatively small number of estimated parameters. The RMSE statistic for maturity τ_i and for model m is calculated as follows:

$$\text{RMSE}_m(\tau_i) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\hat{y}_t(\tau_i) - y_t(\tau_i))^2}, \tag{12.1}$$

where $\hat{y}_t(\tau_i)$ is the predicted yield for the maturity τ_i at time t, and $y_t(\tau_i)$ is the observed yield. To evaluate the measurement disturbances of each model the trace root mean squared error (TRMSE) is also reported:

$$\text{TRMSE}_m(\tau_i) = \sqrt{\frac{1}{N}\frac{1}{T}\sum_{i=1}^{N}\sum_{t=1}^{T}(\hat{y}_t(\tau_i) - y_t(\tau_i))^2}. \tag{12.2}$$

Lower values for RMSE and TRMSE indicate better in-sample fit.

The estimates for relative TRMSEs reveal that most part of the alternative models outperform the in-sample estimation of the benchmark, whereas *DNS-VAR(3)* and *DNS-VAR(5)*$^{A-diag}$ achieve the best in-sample fit to observed data. A large set of alternative models outperform the benchmark for short and medium-term maturities, represented by 3, 6 and 36 months. The alternative specifications provide the best adjustment compared to benchmark at very short maturities. In general, the estimates present a similar empirical fit between the whole set of alternative models, except for the class of specifications considering the fifth restriction, which reveal a poor in-sample performance, i.e., the curvature factor is important to fit in-sample data.

12.2 Forecast performance evaluation

In the previous section, it has been shown that the whole set of DNS models provide a fairly good in-sample fit to Brazilian nominal yields data. In this section, I perform the out-of-sample forecast exercise of the *DNS-VAR(3)* and *yields-macro* models using a rolling window analysis. This implies that the multiple step ahead forecasts explored here are closely related to an investor's pseudo real-time decision. However, the analysis is not based on fully real-time data once some macroeconomic variables of panel 5 in Appendix A are constructed from the revised dataset and some macro information have not been released yet at the time when a forecast is made.

The forecast exercise for the multiple forecast horizons of 1-, 3-, 6-, 9- and 12-month-ahead are performed with rolling window samples of size $T = 116$. The first estimation sample is from April, 2003, to November, 2012; the second rolling window contains observations for period $t = 2$ (May, 2003) through $T + 1$ (December, 2012), and so on. Predictions are made for $T + h$ at the end of each rolling window, where h is the forecast horizon. Hence, the out-of-sample forecasts are carried out over the time interval from December, 2012, to March, 2016. The number of rolling window samples is $S = 40$, whereas the last 11 rolling window samples have some restrictions related to forecast horizons. That is, there are 40 out-of-sample forecasts for 1-month horizon, 39 for 2-month horizon, and so on until 29 out-of-sample forecasts for 12-month horizon.

The evaluation of out-of-sample forecasts requires some measures to compute the errors for each maturity τ_i. Given a time series of S out-of-sample forecasts for h-period-ahead forecast horizon, the root mean squared forecast error (RMSFE) calculates a forecast error measure for maturity τ_i at forecast horizon h and for model m:

$$\text{RMSFE}_m(\tau_i) = \sqrt{\frac{1}{S}\sum_{t=1}^{S}(\hat{y}_{t+h|t}(\tau_i) - y_{t+h}(\tau_i))^2}, \tag{12.3}$$

where $y_{t+h}(\tau_i)$ is the yield for the maturity τ_i observed at time $t+h$, and $\hat{y}_{t+h|t}(\tau_i)$ is the corresponding forecast made at time t. The performance analysis also reports the trace root mean squared forecast error (TRMSFE), which calculates the trace of the covariance matrix of the forecast errors across all N maturities:

$$\text{TRMSFE}_m(\tau_i) = \sqrt{\frac{1}{N}\frac{1}{S}\sum_{i=1}^{N}\sum_{t=1}^{S}(\hat{y}_{t+h|t}(\tau_i) - y_{t+h}(\tau_i))^2} \tag{12.4}$$

The Diebold & Mariano (1995) test is applied to compare forecast accuracy between two competing models. The Diebold-Mariano (DM) statistic tests whether the out-of-sample forecast error from model m_1 for maturity τ_i ($e^{m_1}_{t+h|t}(\tau_i)$) is statistically different from the forecasts of the competing model m_2 ($e^{m_2}_{t+h|t}(\tau_i)$), where $e^{m_l}_{t+h|t}(\tau_i) = y_{t+h}(\tau_i) - \hat{y}^{m_l}_{t+h|t}(\tau_i)$ for $l = 1, 2$. As the forecast exercise is computed for out-of-sample observations, in practice one has a series of forecast errors for maturity τ_i. The accuracy of each forecast series for τ_i is measured by a particular loss function ($L(e^{m_l}_{t+h|t}(\tau_i))$), which in this case is assumed to be a squared error loss function equal to $(e^{m_l}_{t+h|t}(\tau_i))^2$. The null hypothesis of DM test determine the equality between the expectation of $L(e^{m_1}_{t+h|t}(\tau_i))^2$ and $L(e^{m_2}_{t+h|t}(\tau_i))^2$, against the alternative hypothesis of difference between those expectations. Hence, the test is based on the loss differential $d_{t+h} = L(e^{m_1}_{t+h|t}(\tau_i))^2 - L(e^{m_2}_{t+h|t}(\tau_i))^2$, testing for the null where the loss differential has zero expectation for all t:

$$H_0 : E[d_t] = 0. \tag{12.5}$$

Assuming that $\bar{d} = \frac{1}{S}\sum_{t=1}^{S} d_t$ is the sample mean of the loss differential, the DM test statistic for different forecasting methods (DM_m) can be computed as:

$$DM_m = \frac{\bar{d}}{\sqrt{\frac{\hat{\delta}}{S}}} \xrightarrow{d} N(0, 1), \tag{12.6}$$

where $\hat{\delta}$ is a consistent estimate of the asymptotic (long run) covariance matrix of the loss differential ($\text{cov}(d_t, d_{t-j})$). In another way, $\sqrt{\hat{\delta}/S}$ is a consistent estimate of the standard deviation of $\bar{d}$. The long-run variance is used in the statistic because the sample of d_t is serially correlated for h>1, and for a variety of reasons (DIEBOLD; MARIANO, 1995). The Newey & West (1987) estimate is employed to calculate robustly $\hat{\delta}$, which allows controlling for the serial correlation in the forecasting errors. The DM statistic requires only one assumption: the loss differential must be covariance stationary (DIEBOLD, 2015)[2]. A

[2] It is worth remembering that the DM test is designed to emphasize that the errors are driven by forecasts, not models (DIEBOLD, 2015). After Diebold & Mariano (1995), some researchers (Clark & McCracken (2001), Giacomini & White (2006)) elaborated alternative predictive accuracy tests, which emphasize the fully-articulated econometric models used to the out-of-sample forecasting.

negative value for the DM_m statistic indicates predictive superiority of the first model of the pair, which in this case is model m_1. The probability to reject the null of equal predictive accuracy is higher when absolute values for DM_m are higher. Absolute values higher than 1.96 indicate rejection of the null hypothesis at the 5% level.

Table 3 reports the summary statistics of forecast performance for the general specifications: *DNS-VAR(3)*, *DNS-VAR(4)* and *DNS-VAR(5)*. The table presents the RMSFE and TRMSFE statistics for the τ maturities and for 1-, 3-, 6-, 9- and 12-month-ahead forecast horizons, showing where a competitor *yields-macro* model outperforms the *DNS-VAR(3)* model and when Diebold-Mariano test rejects the null of equal forecasting accuracy between them. Some basic considerations can be made: (i) the *DNS-VAR(5)* and *DNS-VAR(4)*^{1} clearly underperform the general DNS framework for the entire maturity and forecast horizon spectrum; (ii) the *DNS-VAR(4)*^{2} consistently outperform the *DNS-VAR(3)* model for most maturities and for forecast horizons longer than one month. The DM test rejects the null hypothesis at a 5% level of the *DNS-VAR(4)*^{2} model for particular cases: (i) 6-month-ahead predictions for some medium- and long-term maturities, and (ii) 9-month-ahead predictions for the long end of the yield curve. Therefore, it is possible to affirm that the *DNS-VAR(4)*^{2} model forecasts quite well for medium- and long-term maturities and for forecast horizons longer than one month.

The DM test also rejects the null for most forecasted yields of *DNS-VAR(4)*^{1} and *DNS-VAR(5)* models, confirming the inferior performance of these specifications in relation to *DNS-VAR(3)*. Both models that include the business cycle factor forecast poorly, supporting the evidence of relatively small impact of X^1 on the Brazilian yield curve. In other words, the incorporation of macro factors containing information strongly correlated with business cycle do not contribute to predict the Brazilian yield curve. In such cases, the superiority of forecast perfomance of the *DNS-VAR(3)* model is clearly shown.

The forecasts produced by the *DNS-VAR(4)*^{2} model provide the lowest RMSFEs and TRMSFEs for most predictions above 1-month horizon, while *DNS-VAR(3)* model provide the lowest values for 1-month-ahead forecasts. Thus, the inclusion of the inflation factor into the general DNS framework appears to lead to lower RMSFEs for most interest rates and most forecast horizons above 1 month. In addition, the results for the DM tests confirm the significant improvements of *DNS-VAR(4)*^{2} model for 6- and 9-month-ahead predictions, specially for medium- and long-term maturities. Overall, the results imply the support for the incorporation of a macro factor that summarizes broad macroeconomic information regarding mainly inflation expectations. This evidence confirms the previous viewpoint that suggested a strong correlation between the nominal yield curve level and expected inflation.

Appendix D also reports the forecast results for *yields-macro* models using only forward-looking variables as macroeconomic information. Those evidence converge to previous results, where the *DNS-VAR(4)* model that incorporates an inflation expectation factor provides lowest RMSFEs and TRMSFEs for the entire maturity spectrum and for forecast horizons above 3 months.

The forecast exercise confirms the estimates reported by Moench (2008), Pooter et al. (2010), Koopman & Wel (2013), Almeida & Faria (2014), among other studies. Moench (2008) finds that a no-arbitrage FAVAR model with macroeconomic appeal provides better out-of-sample yield forecasts at intermediate and long horizons than various benchmarks including the affine three factor model of Duffee (2002) and the general DNS framework. Pooter et al. (2010) show that adding macro factors does indeed improve the forecast accuracy of several term structure models such as those suggested by Diebold et al. (2006) and Ang & Piazzesi (2003), specially for subperiods with substantial uncertainty about the future path of interest rates. Koopman & Wel (2013) validate the incorporation of macroeconomic variables in a smooth dynamic factor model, which improves the performance for forecasting the US term structure compared to a set of dynamic models without macroeconomic information. For the Brazilian term structure, Almeida & Faria (2014) confirm the better predictive performance of the model proposed by Moench (2008) when compared to the usual benchmarks.

12.2.1 Alternative model specifications

The summary statistics of forecast performance of the various alternative specifications described in Section 12.1.1 are reported by table 12 of appendix E. The table presents the forecast analysis for 1-, 6-, 9- and 12-month-ahead and for maturities 3, 6, 12, 24, 36, 48 and 60 months, performed with rolling window samples. Each forecasted yield is compared to *benchmark*, which is the random walk (RW) model. Using the estimated yields for each maturity τ_i at each rolling sample for the forecast horizons, the RMSFE and TRMSFE are calculated, where Diebold-Mariano statistic tests for superior predictive performance between the alternative specifications and benchmark.

To compare the out-of-sample forecasting performance of the alternative specifications, I choose the random walk as the benchmark because of the high persistence processes observed in yield data. The $t+h$-step-ahead forecasts for an yield of maturity τ_i of the RW model are given by: $y_{t+h|t}(\tau_i) = y_t(\tau_i)$. That is, a h-step-ahead forecast is simply equal to the most recently observed value $y_t(\tau_i)$. The RW model is a good benchmark for judging the relative prediction power of other models, since yields of all maturities are close to being non-stationary. Thus, in practice, it is difficult to beat the RW in terms of out-of-sample forecasting accuracy. Many other studies that consider interest rate forecasting have shown that consistently outperforming the RW is a difficult task (see, for example, Duffee (2002); Ang & Piazzesi (2003); Hördahl et al. (2006); Moench

(2008)). Nevertheless, since Diebold & Li (2006) study, favorable evidence for interest rate predictability against RW model has been reported.

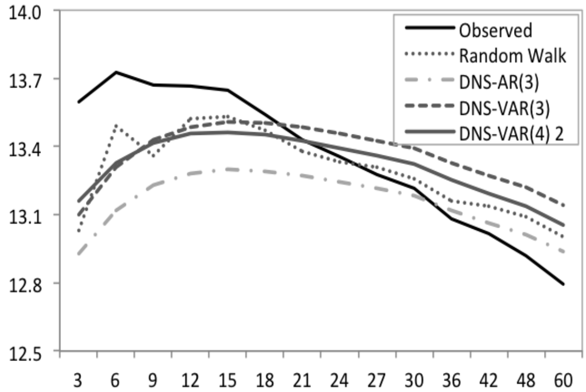
The RW model shows good prediction accuracy for 1-month-ahead forecasts, where no alternative specification outperform the benchmark in terms of TRMSFE. However, in terms of RMSFE some specifications with fewer estimated parameters (*DNS-AR(3)*, *DNS-VAR(3)*, *DNS-VAR(3)*$^{Q-diag}$, *DNS-VAR(3)*$^{A-diag}$ and *DNS-VAR(2)*^{C}) outperform the benchmark for medium- and long-term maturities. The forecast accuracy of the alternative specifications increases with the forecast horizon. For forecast horizons equal to and above 6-month-ahead, there are various alternative models that outperform the benchmark for almost all maturity spectrum, as *DNS-AR(3)*, *DNS-VAR(3)*, *DNS-VAR(4)*^{2}, *DNS-VAR(3)*$^{Q-diag}$, *DNS-VAR(4)*$^{2,Q-diag}$, *DNS-VAR(3)*$^{A-diag}$, *DNS-VAR(3)*^{S} and *DNS-VAR(2)*^{C}. For longer forecast horizons, 9- and 12-month-ahead, the *DNS-VAR(5)*$^{Q-diag}$ model also outperforms the benchmark for the entire maturity spectrum, suggesting a specification in which the business cycle factor contributes to yield curve forecasting. The imposition of restrictions in the covariance matrix Q also proved to generate good yield forecasts. In general, the alternative specifications outperform the RW model in terms of TRMSFE when the forecast horizon is equal to and longer than 6 months.

The *DNS-AR(3)*, *DNS-VAR(4)*^{2} and *DNS-VAR(2)*^{C} models provide the forecasted yields with lowest RMSFEs and TRMSFEs for 9- and 12-month-ahead forecast horizons, whereas *DNS-AR(3)* and *DNS-VAR(2)*^{C} represent specifications with relative small number of estimated parameters. The *DNS-VAR(2)*^{C} model consistently beats the benchmark for forecast horizons equal to and above 6 months and for the entire maturity spectrum, which suggests that the curvature factor is not so important to yield curve forecasting.

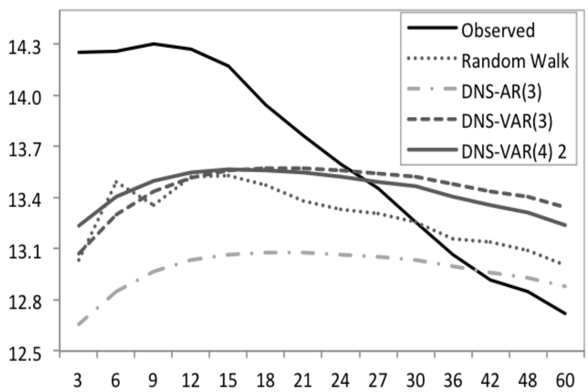
The DM test rejects the null of equal forecasting accuracy in favor to the following outperforming models for longer forecast horizons and specially for medium- and long-term maturities: *DNS-AR(3)*, *DNS-VAR(3)*, *DNS-VAR(4)*^{2}, *DNS-VAR(3)*$^{Q-diag}$, *DNS-VAR(4)*$^{2,Q-diag}$, *DNS-VAR(3)*$^{A-diag}$, *DNS-VAR(3)*^{S} and *DNS-VAR(2)*^{C}. Nonetheless, the DM test also rejects the null in favor to the RW model relative to some DNS specifications that present a poor forecast ability, as the *DNS-VAR(4)*$^{2,A-diag}$ and *DNS-VAR(5)*$^{A-diag}$, particularly for shorter forecast horizons.

In short, the evidence indicate that imposing further restrictions can lead to improvements in forecast accuracy, pointing to specifications with less number of estimated parameters. Moreover, the contribution of macroeconomic information seems to be relevant through the *DNS-VAR(4)*^{2}, *DNS-VAR(4)*$^{2,Q-diag}$ and *DNS-VAR(5)*$^{Q-diag}$ models for longer forecast horizons, and specially for medium and long-term maturities. Therefore, the results provide enough information to affirm that it is difficult to beat the benchmark for very short forecast horizons, while different DNS specifications can easily beat the benchmark forecasts for longer forecast horizons, specially for medium and long-term maturities.

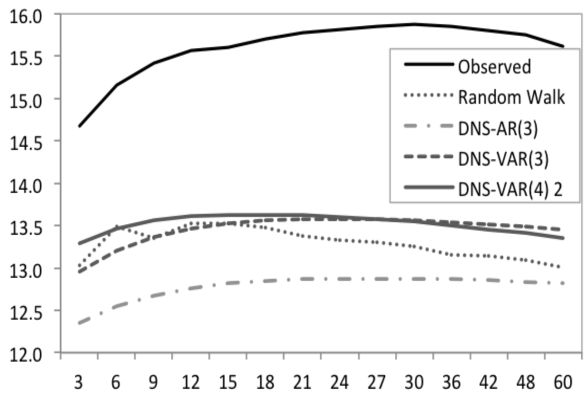
To visualize some general findings, Fig. 5 illustrates the actual yields and those predicted by the random walk, *DNS-AR(3)*, *DNS-VAR(3)* and *DNS-VAR(4)*^{2} models for some selected forecast horizons in March, 2015. The forecasted yield curves reveal the deterioration of the benchmark forecasts in relation to *DNS-VAR(3)* and *DNS-VAR(4)*^{2} with the increase of the forecast horizon. Furthermore, the *DNS-VAR(4)*^{2} model seems to capture relatively well the changes in the 12-month ahead yield curve.



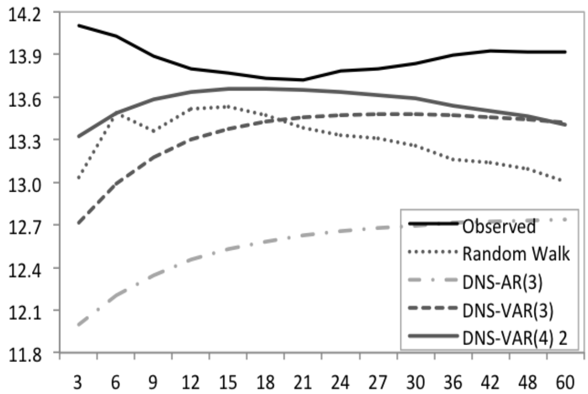
(a) 1-month ahead yield curves



(b) 3-month ahead yield curves



(c) 6-month ahead yield curves



(d) 12-month ahead yield curves

Figure 5 – Observed and forecasted annualized yield curves (in %) for four selected forecast horizons, 1-, 3-, 6- and 12-months-ahead, estimated by the random walk, *DNS-AR(3)*, *DNS-VAR(3)* and *DNS-VAR(4)*^{2} models in March, 2015.

Table 3 – (Trace)-Root Mean Squared Forecast Errors of *DNS-VAR(3)* and *yields-macro* models. DI-futuro data from 2003:04 to 2016:03, whereas out-of-sample period is 2012:12-2016:03.

	Panel A: *DNS-VAR(3)* model					**Panel B: *DNS-VAR(5)* model**				
Maturities	Forecast horizon					Forecast horizon				
	1-M	3-M	6-M	9-M	12-M	1-M	3-M	6-M	9-M	12-M
3	0.486	0.969	1.817	2.472	2.978	1.424*	2.422*	3.357*	6.536*	14.204*
6	0.478	0.987	1.761	2.313	2.739	1.491*	2.435*	3.275*	6.217*	13.395*
9	0.502	1.021	1.691	2.185	2.581	1.565*	2.487*	3.211*	5.936*	12.710*
12	0.536	1.043	1.641	2.112	2.468	1.622*	2.508*	3.166*	5.705*	12.141*
15	0.546	1.049	1.610	2.058	2.381	1.654*	2.530*	3.112*	5.482*	11.585*
18	0.574	1.084	1.620	2.033	2.329	1.687*	2.542*	3.048*	5.239*	11.063*
21	0.600	1.114	1.628	2.020	2.287	1.708*	2.537*	2.985*	5.009*	10.565*
24	0.615	1.125	1.643	2.020	2.264	1.717*	2.527*	2.942*	4.805*	10.090*
27	0.617	1.132	1.654	2.005	2.250	1.718*	2.519*	2.912*	4.608*	9.640*
30	0.622	1.139	1.652	1.994	2.241	1.722*	2.511*	2.891*	4.422*	9.212*
36	0.624	1.134	1.658	2.001	2.242	1.728*	2.511*	2.838*	4.112*	8.452*
42	0.630	1.138	1.675	2.012	2.257	1.726*	2.507*	2.818*	3.880*	7.811*
48	0.633	1.132	1.670	2.012	2.262	1.717*	2.510*	2.816*	3.699*	7.292*
60	0.640	1.125	1.670	2.023	2.267	1.727*	2.550*	2.869*	3.519*	6.624
TRMSFE	0.581	1.087	1.672	2.094	2.406	1.660*	2.507*	3.022*	5.026*	10.581*

	Panel C: *DNS-VAR(4)*[1] model					**Panel D: *DNS-VAR(4)*[2] model**				
Maturities	Forecast horizon					Forecast horizon				
	1-M	3-M	6-M	9-M	12-M	1-M	3-M	6-M	9-M	12-M
3	1.864	2.925	4.304*	5.949*	8.216*	0.523	1.069	**1.724**	**2.202**	**2.483**
6	1.834	2.936	4.251*	5.823*	7.924*	0.520	1.044	**1.638**	**2.015**	**2.285**
9	1.793	2.920*	4.185*	5.706*	7.688*	0.556	1.043	**1.560**	**1.894**	**2.121**
12	1.773	2.908*	4.103*	5.595*	7.430*	0.592	1.051	**1.520**	**1.798**	**2.010**
15	1.738	2.873*	4.020*	5.454*	7.215*	0.616	**1.049**	**1.484**	**1.731**	**1.912**
18	1.714	2.848*	3.929*	5.323*	7.001*	0.645	**1.067**	**1.484***	**1.695**	**1.841**
21	1.687	2.803*	3.853*	5.206*	6.793*	0.665	**1.090**	**1.489***	**1.669**	**1.787**
24	1.661	2.763*	3.784*	5.087*	6.614*	0.677	**1.095**	**1.495***	**1.647**	**1.760**
27	1.638	2.725*	3.697*	4.979*	6.459*	0.680	**1.102**	**1.497***	**1.641**	**1.757**
30	1.612	2.694*	3.607*	4.884*	6.322*	0.687	**1.106**	**1.502***	**1.636***	**1.756**
36	1.594	2.617*	3.511*	4.712*	6.077*	0.703	**1.121**	**1.510**	**1.639***	**1.765**
42	1.571	2.564*	3.429*	4.572*	5.889*	0.709	**1.125**	**1.520***	**1.657***	**1.786**
48	1.548	2.510*	3.347*	4.455*	5.724*	0.711	**1.122**	**1.515***	**1.661***	**1.797**
60	1.518	2.431*	3.228*	4.275*	5.468*	0.715	**1.121**	**1.516**	**1.677**	**1.813**
TRMSFE	1.685	2.756*	3.818*	5.169*	6.822*	0.646	**1.086**	**1.534**	**1.762**	**1.932**

Notes: The table presents the forecasting performances of *DNS-VAR(3)* model and *yields-macro* models. In particular, it reports the root mean squared forecast errors (RMSFE) and trace RMSFE (TRMSFE) obtained by using individual *DNS-VAR(3)*, *DNS-VAR(4)*[1], *DNS-VAR(4)*[2] and *DNS-VAR(5)* models. The values reported are divided by 1×10^{-2}. The RMSFE is reported for each model for the τ maturities and for 1-, 3-, 6-, 9- and 12-month-ahead forecast horizons. The latest line of each panel reports the TRMSFE for the different forecast horizons. The evaluation sample refers to 2012:12-2016:03 (40 out-of-sample forecasts), being 40 out-of-sample forecasts for 1-month horizon, 39 for 2-month horizon, and so on until 29 out-of-sample forecasts for 12-month horizon. Numbers in **bold** indicate that the alternative *yields-macro* models from panels B, C and D outperform the *DNS-VAR(3)* model, otherwise indicate underperformance. The star on the right of the cell entries indicate where Diebold-Mariano test rejects the null of equal forecasting accuracy between the competitor *yields-macro* model and *DNS-VAR(3)* model, with 5% probability of the null hypothesis.

13 Application to fixed-income portfolio optimization

13.1 Methodology for evaluating portfolio performance and implementation details

This section aims to assess the economic value of the forecasting ability of the major yield curve models estimated previously. The empirical implementation of the mean-variance optimization problem defined by (5.1) is performed by using one- and twelve-step-ahead estimates of the vector of expected returns and its covariance matrix considering five alternative values for the risk aversion coefficient δ: 0.0001, 0.01, 0.1, 0.5 and 1. Following the recursive estimation strategy of the yield curve models, the optimal mean-variance portfolios are also computed recursively as new h-step-ahead estimates for DI-futuro returns are known. Moreover, optimal mean-variance portfolios using one-step-ahead forecasts for DI-futuro returns are rebalanced on a monthly basis, while portfolios using twelve-step-ahead forecasts are rebalanced on an annual basis. Thus, the empirical analysis with monthly rebalancing computes the optimal portfolio for each period over the S out-of-sample observations ranging from December, 2012 to March, 2016, giving a sample of 40 optimal portfolio weights, w_t.

Otherwise, the optimization with annual rebalancing computes the optimal portfolio for twelve consecutive months at 11:2012, 11:2013, and 11:2014. Nevertheless, the portfolio performance statistics are computed for every month of the out-of-sample period. The last rebalancing procedure is performed at November, 2014, because from March, 2015, on there are no forecasts for 12-month-ahead yields being considered by the yield curve models. Moreover, rebalancing frequency is important when dealing with fixed-income assets, because the securities in the portfolio can age and be closer to maturity. For this reason, the shortest maturity considered here is $\tau_i = 15$ months, because DI-futuro contracts with maturity lower or equal to 12 months will already be matured before the subsequent rebalancing process. This only allows the computation of performance statistics until January, 2016, because at February, 2016 the 15-month security will already be matured, so that the number of out-of-sample observations is equal to 38.

It is clear that the scenario with annual rebalancing requires more diligence regarding implementation procedure and computation of performance statistics. Note that, after one period, an optimal portfolio containing securities that yield an average duration of τ_i at the time of the mean-variance optimization, becomes a portfolio with average duration τ_{i-1} and so on until the next rebalancing process, changing the characteristics of the original portfolio over time. Thus, the computation of the time series of portfolio returns need to take care about the constant decrease of the time-to-maturity of its securities.

The performance analysis use some alternative criteria to evaluate the performance of the optimal mean-variance fixed-income portfolios. First of all, I describe the evaluation criteria related to portfolio excess return relative to the risk-free rate (R_{f_t}), which is consider to be the short Brazilian Interbank Deposit (CDI) rate. The average gross (i.e., before transaction costs) excess return relative to the risk-free rate ($\bar{rx}$) is calculated as follows:

$$\bar{rx} = \frac{1}{S}\sum_{t=1}^{S} rx_t,$$

where $rx_t = w_{t-1}^{'}R_t - R_{f_t}$ denotes the gross excess portfolio return at time t and $R_t = [r_t(\tau_i), ..., r_t(\tau_N)]'$ is a vector collecting DI-futuro returns of all maturities considered.

According to Han (2006), it would be appropriate to consider transaction costs when rebalancing the portfolio weights frequently. The empirical scenario with annual rebalancing can alleviate the impact of transaction costs on portfolio performance. However, the less frequent rebalancing means that the portfolio weights will be outdated, which could negatively affect the portfolio performance because investors would be investing away from the optimal one (CALDEIRA et al., 2016). In line with Thornton & Valente (2012) and Corte et al. (2008), the performance analysis also considers the excess return net of transaction costs (rx_t^{net}), which takes into account the negative impact of transaction costs on portfolio average performance, and is calculated as:

$$rx_t^{net} = (1 - c \cdot turnover_t)(1 + rx_t) - 1, \tag{13.1}$$

where c is the fee that must be paid for each transaction and $turnover_t$ is the portfolio turnover at time t, defined as the fraction of wealth traded between periods $t-1$ and t, i.e,

$$turnover_t = \sum_{i=1}^{N}(|w_{i,t} - w_{i,t-1}|).$$

The parameter $w_{i,t}$ is the optimal weight of maturity τ_i at time t. The level of transaction costs being considered is 5 bps, reflecting a fixed percentage for each rebalance trade. Similarly to $\bar{rx}$, the average excess portfolio return net of transaction costs is defined as $\bar{rx}^{net} = \frac{1}{S}\sum_{t=1}^{S} rx_t^{net}$. Moreover, statistics regarding the volatility (standard deviation) of the net excess return ($\hat{\sigma}$) and the risk-adjusted net excess return (SR) measured by the Sharpe ratio are calculated as follows,

$$\hat{\sigma} = \sqrt{\frac{1}{S}\sum_{t=1}^{S}(rx_t^{net} - \hat{\mu_p})^2},$$

$$SR = \frac{\bar{rx}^{net}}{\hat{\sigma}},$$

where $\hat{\mu_p}$ denotes the sample mean of the portfolio net excess return. Ultimately, the performance analysis takes into account the average duration in years of the portfolios, which allows one to better understand the composition of the optimal portfolios. The average duration of a fixed-income portfolio is calculated as $\frac{1}{S}\sum_{t=1}^{S} w_t'\tau$, where here τ regards to the vector of individual security durations. A higher (lower) average portfolio duration suggests that the optimal portfolio is invested in long (short) maturities. As described in Section 5.2, a portfolio with higher duration carries a higher exposure to changes in market interest rates.

13.2 Results for mean-variance portfolios

Table 4 reports the out-of-sample performance of mean-variance portfolios of DI-futuro contracts that use estimates of yields from the random walk (RW)[1], *DNS-AR(3)*, *DNS-VAR(3)* and *DNS-VAR(4)*[2] model specifications. For the scenario which considers one-step-ahead estimates and more frequent portfolio rebalancing (Panel A in Fig. 4), the overview indicates that positive excess return statistics are essentially obtained when the risk aversion coefficient is higher than 0.1, where the annualized net excess returns range from 0.40% to 1.57%. The RW and *DNS-AR(3)* models also report positive net excess returns for some δ's smaller than 0.5. The best overall performance in terms of Sharpe ratio is achieved by the mean-variance portfolio obtained with the RW model with $\delta = 0.5$ ($SR = 0.472$). When lower risk aversion is considered, most of the results indicate negative Sharpe ratios and higher volatility levels. This scenario with lower risk aversion (δ's between 1×10^{-4} and 0.1) shows annualized net excess returns ranging from -2.96% to 1.09% and an annualized standard deviation ranging from 20.1% to 4.46% across all model specifications. As expected, an increase in the risk aversion coefficient leads to decreases in portfolio volatility as well as decreases in the average duration, that is, optimal portfolios are invested in short-term maturities. This result is intuitive since lower maturity securities are less risky, allowing investors with higher risk aversion to lower portfolio risk by investing in shorter maturities. This evidence is even more pronounced for the RW model, which quickly decreases volatility and duration with the increase of δ, investing basically in 3- and 6-month maturities for δ's higher than 0.1. For instance, the average portfolio duration across specifications for an investor with risk aversion coefficient $\delta = 1$ is 0.85 year, whereas the same figure for an investor with $\delta = 1 \times 10^{-4}$ is 2.25 years.

On the other hand, the scenario which considers twelve-step-ahead estimates for DI-futuro returns and an annual portfolio rebalancing (Panel B in Table. 4[2]) reports negative net excess returns across all model specifications and across all levels of the risk

[1] It is noteworthy that the covariance matrix of the expected log-returns obtained from forecasted yields of the RW model is simply the sample covariance from the in-sample observations.

[2] Panel B does not report the results for $\delta = 0.01$ because they are all equal to $\delta = 1 \times 10^{-4}$.

Table 4 – Performance of optimal DI-futuro contracts mean-variance portfolios.

Panel A: one-step-ahead estimates with monthly rebalancing					
Yield Curve Model	Mean gross exc. R (%)	Mean net exc. R (%)	Std. Dev. (%)	Sharpe Ratio	Duration (years)
δ=0.0001					
Random Walk	-1.143	-1.218	21.981	-0.055	2.600
DNS-AR(3)	-2.860	-2.910	23.375	-0.125	1.919
DNS-VAR(3)	0.880	0.820	23.134	0.035	2.356
DNS-VAR(4)^2	-0.905	-0.983	24.664	-0.040	2.163
δ=0.01					
Random Walk	1.081	1.008	13.040	0.077	1.615
DNS-AR(3)	-2.908	-2.960	23.132	-0.128	1.879
DNS-VAR(3)	0.108	0.042	22.330	0.002	2.261
DNS-VAR(4)^2	-0.897	-0.975	24.642	-0.040	2.162
δ=0.1					
Random Walk	1.135	1.092	4.463	0.245	0.626
DNS-AR(3)	-1.121	-1.156	20.109	-0.058	1.462
DNS-VAR(3)	-0.527	-0.584	21.167	-0.028	1.978
DNS-VAR(4)^2	-0.126	-0.195	21.282	-0.009	1.919
δ=0.5					
Random Walk	0.521	0.511	1.083	0.472	0.287
DNS-AR(3)	1.182	1.139	14.777	0.077	1.064
DNS-VAR(3)	0.574	0.524	16.069	0.033	1.416
DNS-VAR(4)^2	0.531	0.459	18.076	0.025	1.434
δ=1					
Random Walk	0.405	0.401	0.889	0.451	0.261
DNS-AR(3)	1.234	1.196	8.872	0.135	0.760
DNS-VAR(3)	1.622	1.572	13.544	0.116	1.187
DNS-VAR(4)^2	1.052	0.987	15.382	0.064	1.227
Panel B: twelve-step-ahead estimates with annual rebalancing					
Yield Curve Model	Mean gross exc. R (%)	Mean net exc. R (%)	Std. Dev. (%)	Sharpe Ratio	Duration (years)
δ=0.0001					
Random Walk	-1.662	-1.667	20.750	-0.080	3.303
DNS-AR(3)	-0.394	-0.400	8.379	-0.048	1.487
DNS-VAR(3)	-2.763	-2.765	23.940	-0.116	3.618
DNS-VAR(4)^2	1.019	1.014	15.938	0.064	2.434
δ=0.1					
Random Walk	-1.463	-1.468	8.958	-0.164	1.882
DNS-AR(3)	-0.394	-0.400	8.379	-0.048	1.487
DNS-VAR(3)	-3.093	-3.097	22.447	-0.138	3.397
DNS-VAR(4)^2	-0.372	-0.377	8.430	-0.045	1.502
δ=0.5					
Random Walk	-1.039	-1.044	7.527	-0.139	1.749
DNS-AR(3)	-0.560	-0.566	7.829	-0.072	1.408
DNS-VAR(3)	-2.098	-2.103	11.011	-0.191	2.066
DNS-VAR(4)^2	-0.394	-0.400	8.379	-0.048	1.487
δ=1					
Random Walk	-0.657	-0.662	7.041	-0.094	1.555
DNS-AR(3)	-0.688	-0.690	10.311	-0.067	1.302
DNS-VAR(3)	-1.170	-1.175	8.227	-0.143	1.801
DNS-VAR(4)^2	-0.531	-0.536	7.894	-0.068	1.422

Notes: Performance statistics for mean-variance portfolios using the random walk, *DNS-AR(3)*, *DNS-VAR(3)* and *DNS-VAR(4)*^{2} model specifications to compute the forecasted yields for the out-of-sample period from 2012:12 to 2016:03. Panel A reports the statistics for the portfolio optimization using one-month-ahead estimates for DI-futuro returns, while Panel B reports the statistics using one-year-ahead estimates. The optimal portfolios are rebalanced on a monthly basis for Panel A estimates, and on an annual basis for Panel B. The statistics of gross and net excess returns, standard deviation, and Sharpe ratio are annualized and the average portfolio duration is measured in years. The excess return is calculated using the short Brazilian Interbank Deposit (CDI) rate as the risk-free asset. The level of transaction costs for all rebalance trade is 5 bps. Parameter δ denotes the value of the risk aversion coefficient.

tolerance considered, except for the *DNS-VAR(4)*^{2} model with $\delta = 1 \times 10^{-4}$ and 0.01. This portfolio selection exercise reports similar annualized results for $\delta = 1 \times 10^{-4}$ and

$\delta = 0.01$, which for the *DNS-VAR(4)*^{2} model are: $\bar{rx}$ equal to 1.019%, $\bar{rx}^{net}$ =1.014%, volatility (measured by the standard deviation) equal to 15.93%, $SR = 0.064$ and average duration equal to 2.43 years. The *DNS-VAR(4)*^{2} model also minimizes losses for higher δ's. The general results show that annualized net excess returns range from -3.097% to 1.014%, and the annualized standard deviation ranges from 7.04% to 23.94%, whereas the Sharpe ratio ranges from -0.191 to 0.064. As before, an increase in the risk aversion coefficient leads to decreases in in the average duration, indicating optimal portfolios invested mostly in long-term maturities for lower levels of δ. Moreover, the impact of transaction costs is relatively small for estimates with annual portfolio rebalancing, whereas net excess returns are very close to gross excess returns.

The key difference compared to Panel A concerns the average portfolio duration: it is higher across all model specifications and δ's for the estimates with annual rebalancing; e.g., the average portfolio duration across specifications for $\delta = 1$ is now 1.52 years and for $\delta = 1 \times 10^{-4}$ is 2.71. The comparison also suggests that optimal mean-variance portfolios with monthly rebalancing deliver higher net excess returns than those with annual rebalancing, pointing out a gain in rebalancing the portfolio weights frequently to keep optimal allocation updated.

Table 4 shows that negative net excess returns prevail in most optimal mean-variance portfolios. In rising interest rate environments, as the out-of-sample period, fixed-income prices suffer from the increase in interest rates in the short term, i.e, rising rate environments can result in negative fixed-income returns. A bond's total return comprises not just price changes, but also income, so that the income on a bond can help offset falling prices, cushioning the overall total return. It turns out that the optimal mean-variance portfolios can not benefit from increased yields over the long term because of rebalancing: investors do not hold fixed-income securities until their maturity, which makes them vulnerable to mark to market process. That is, the rebalancing process applied here turns portfolio's total returns highly dependent on price changes, once securities in portfolio do not mature. For this reason, the income returns are not enough to offset the price decline in DI-futuro contracts. Moreover, the optimal portfolios with annual rebalancing present higher negative excess returns than those with monthly rebalancing: the yield curve models underestimate the climb in 12-month-ahead interest rates over the out-of-sample observations, generating optimal mean-variance portfolios with higher average duration and exposure to price changes.

The composition of optimal portfolio allocations can also be seen in Figs. 8 and 9 in Appendix F, which plot the average portfolio weight in each maturity, and for each level of the risk aversion coefficient across all model specifications. The visual inspection of these figures indicates that optimal portfolio allocations are invested mostly in shorter maturities as one move to higher levels of risk aversion. In the two extreme cases where

$\delta = 1 \times 10^{-4}$ and $\delta = 1$, the optimal allocations are mostly concentrated in the longest and in the shortest maturities, respectively. For intermediate levels of the risk aversion coefficient, the optimal allocations are more diversified across maturities.

At least, Fig. 6 illustrates the performance of the optimal DI-futuro mean-variance portfolios by plotting the cumulative net returns over the out-of-sample period obtained with the alternative specifications when δ is equal to 1×10^{-4} and 1. The figure suggests that *DNS-AR(3)* and *DNS-VAR(3)* specifications deliver better performance for mean-variance portfolios using one-step-ahead estimates for returns and for $\delta = 1 \times 10^{-4}$ during most part of the out-of-sample period. For $\delta = 1 \times 10^{-4}$, the RW model reports more stable and higher cumulative returns. Further, the *DNS-VAR(4)*^{2} presents better portfolio performance using twelve-step-ahead estimates. The best overall performance in terms of cumulative net returns until January, 2016, are achieved by mean-variance portfolios obtained with the RW model and with $\delta = 1$ for one-step-ahead estimates (33.17%) and with the *DNS-VAR(4)*^{2} with $\delta = 1 \times 10^{-4}$ for one-year-ahead estimates (31.631%). Therefore, there is a benefit from monthly rebalancing and paying more transaction costs, avoiding deviations from the optimal mean-variance portfolio. In the general context, the alternative yield curve models achieve similar cumulative net returns at January, 2016, except for the RW and *DNS-VAR(3)* at the twelve-step-ahead scenario with $\delta = 1 \times 10^{-4}$, which considerably underperforms their competitors.

It is also noteworthy the big drop in cumulative net returns for the one-step-ahead estimates in September, 2015. At the end of August, 2015, there is a deterioration of the Brazilian macroeconomic fundamentals due to the perception of a downturn in medium- and long-term fiscal scenario. In September 9, 2015, Brazil loses investment-grade rating from Standard & Poor's (S&P). Financial markets reacted with capital flight to safer investments and a consequent increase in premium required for holding Brazilian securities. Fig. 7 helps to visualize the scenario where short DI-futuro yields slightly rise while long-term yields suffer a large increase from July, 2015, to September, 2015, reflecting a large deterioration in long-term expectations about Brazilian macroeconomic foundations. In fact, term structure models fail to capture this change in yield curves, specially in the slope factor. Once investors with more risk-averse preferences tend to hold short-term maturities, they are not affected so much as the less risk-averse investors with $\delta = 1 \times 10^{-4}$ evidenced by panel (a) of Fig. 6. The only exception is the RW model, which is invested in very short maturities during this period and does not report that big fall in net returns.

In general, the results from the optimal mean-variance portfolios based on estimates for DI-futuro returns reveal that in most cases it is difficult to obtain positive Sharpe ratios for the period analysed. Just for some cases the performance of the optimal portfolios beats the risk-free asset: (i) in the high risk aversion context across all model specifications for the scenario with monthly rebalancing; and (ii) in the low risk aversion context for the

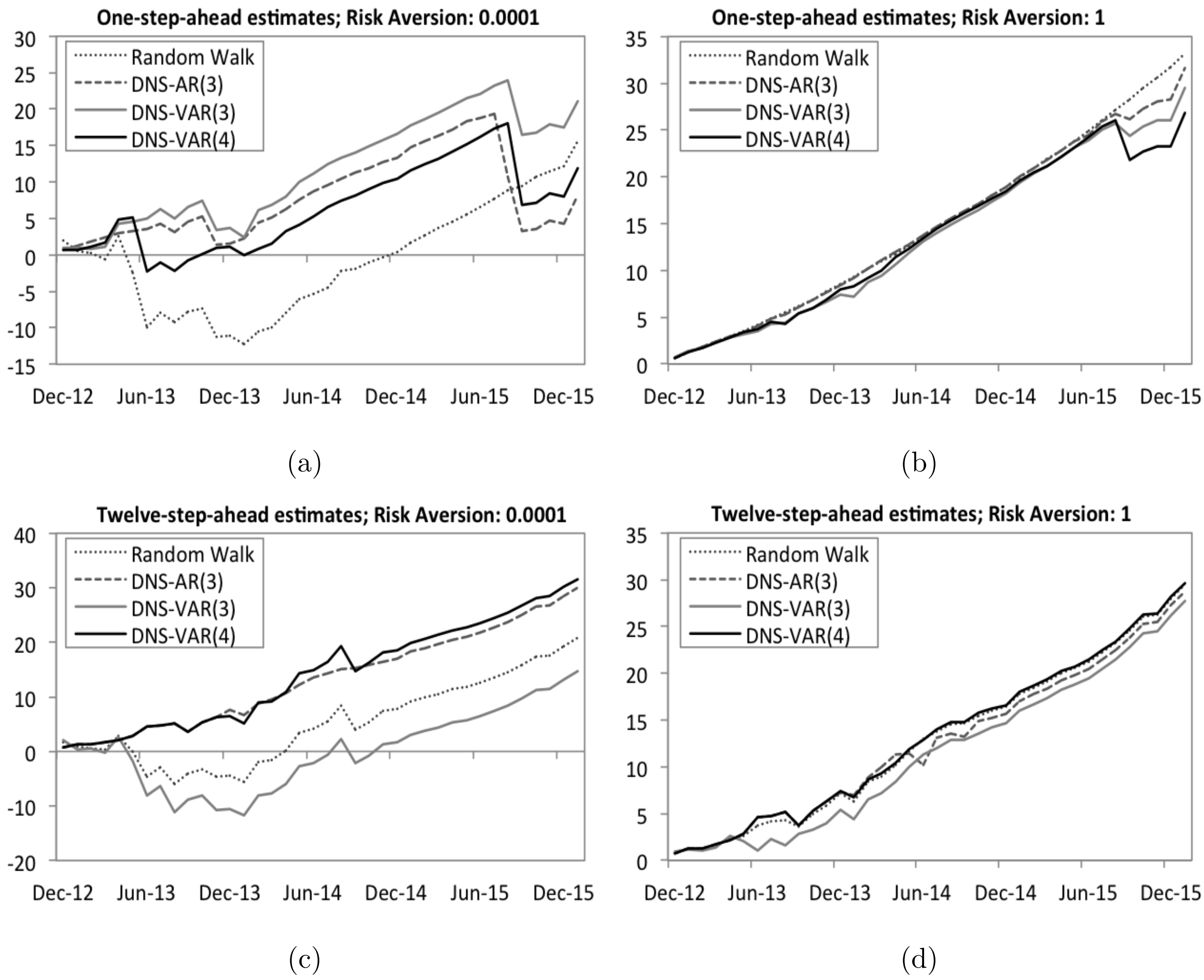


Figure 6 – Cumulative net returns (in %): mean-variance portfolios with $\delta = 1 \times 10^{-4}$ and $\delta = 1$ for 1- and 12-step-ahead forecasts over the out-of-sample period.

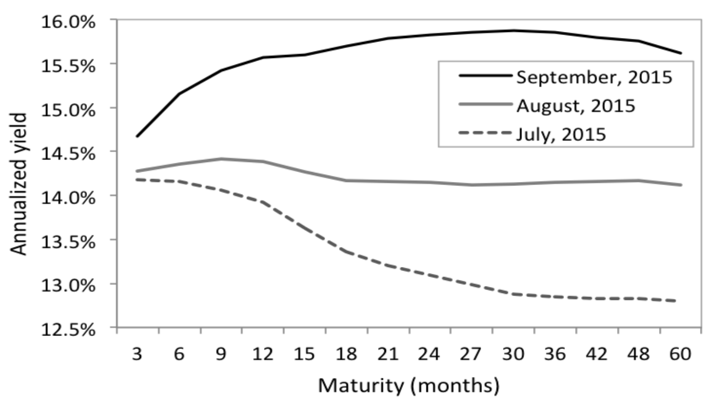


Figure 7 – Observed yield curves from July, 2015, to September, 2015.

DNS model with an inflation factor and for the scenario with annual rebalancing.

The link between the performance of the alternative yield curve models in forecasting yields and the performance of optimal portfolios indicates that better accuracy in yield curve forecasting leads to an improvement in terms of portfolio performance based on the mean-variance approach. Looking back at Section 12.2, specifications with a small number of parameters show better forecast accuracy for short forecast horizons, in contrast to better performance of the *DNS-VAR(4)*^{2} specification for longer forecast horizons. It is noteworthy that these findings are consistent with the performance evaluation described in this section, highlighting the relevance of good yield curve predictions to achieve better results in terms of portfolio performance.

In order to verify the portfolio performance for an alternative out-of-sample period, which also considers a falling interest rates environment, I reproduce the portfolio optimization exercise for the out-of-sample period from May, 2011, to March, 2016, with $T = 97$ and $S = 59$. The period with falling interest rates comprises 09:2011 to 04:2013. The estimates for the scenario which considers one-step-ahead estimates for DI-futuro returns are slightly similar to those reported before, whereas higher risk-averse investors obtain positive excess returns across all model specifications. Nonetheless, more encouraging results are found for investors with smaller δ. For instance, the RW model can achieve mean net excess returns of 3.68% for $\delta = 0.01$ and 2.14% for $\delta = 1 \times 10^{-4}$. This means that optimal mean-variance portfolios can achieve quite satisfying results in falling interest rates environments. The big difference concerns the scenario with twelve-step-ahead estimates, which now obtains positive excess returns across all model specifications and across all levels of risk aversion considered, except for the RW model, which reports poor performances for $\delta = 1 \times 10^{-4}$ and 0.01. Less risk-averse investors achieve better portfolio performance, and invest basically in DI-futuro contracts with maturity below 24 months. As well as before, the *DNS-VAR(4)*^{2} model reports higher Sharpe ratio and returns ($SR = 0.148$ and $\bar{rx}^{net}$ =1.461% for δ's smaller than 0.1). At first glance, the out-of-sample period is quite relevant to determine the portfolio performance in the context of fixed-income. These empirical results are not reported to save space and can be provided upon request.

Furthermore, the yield curve is the most common risk factor that fixed-income securities are exposed to. It turns out that the yield curve is itself driven by a set of macroeconomic risk factors. From this perspective, the incorporation of a broad macroeconomic information into term structure models can play an important role to improve performance of fixed-income portfolios. The evidence show that the incorporation of one macro factor related to inflation expectations into the DNS model leads to an improvement in terms of portfolio performance when considering twelve-step-ahead estimates of DI-futuro returns. That is, macroeconomic information contributes to improve efficiency in terms of portfolio performance for optimal fixed-income portfolios with annual rebalancing. Hence, there is an economic gain from considering macroeconomic information to forecast the yield curve dynamics, specially for medium- and long-term maturities.

14 Concluding remarks

The recent literature on yield curve forecasting suggests that the incorporation of a large macroeconomic dataset into term structure models improve forecast performance (POOTER et al., 2010). Most part of the current studies test for statistical benefits from incorporating macroeconomic information into term structure models, but little is known about the economic value of those forecasted yields. Besides testing for statistical improvement, this study uses a fixed-income portfolio analysis in order to assess the economic value of forecasted yields generated by yield curve models with macro factors extracted from a large macroeconomic dataset.

The out-of-sample forecast exercise support the evidence that a DNS yield curve model incorporating one macro factor, which summarizes broad macroeconomic information regarding mainly inflation expectations, outperforms the general DNS model for (i) 6-month-ahead predictions and for some medium- and long-term maturities, and (ii) 9-month-ahead predictions for the long end of the yield curve. In general, this macroeconomic specification forecasts quite well for medium- and long-term maturities and for forecast horizons longer than one month. The forecast exercise indicates that a specification of the general DNS framework with an inflation factor is particularly useful to predict the Brazilian nominal yield curve dynamics. Similar findings are also reported in Diebold et al. (2006), Moench (2008), Koopman & Wel (2013), Almeida & Faria (2014), among others.

Furthermore, estimates for alternative specifications of the DNS framework suggest that imposing further restrictions on factor dynamics can lead to improvements in forecast accuracy in favor of some parsimonious specifications with less number of estimated parameters. Most of the alternative DNS specifications outperform the random walk model when the forecast horizon is equal to and longer than 6 months, and specially for medium and long-term maturities. In line with numerous studies since Ang & Piazzesi (2003) and Diebold & Li (2006), these results report favorable evidence for yield curve forecasting against the random walk model when considering longer forecast horizons.

The results for mean-variance portfolios with one-step-ahead estimates for DI-futuro returns and monthly rebalancing indicate that positive excess returns are obtained for higher risk aversion coefficients. Otherwise, the portfolio optimization with twelve-step-ahead estimates and annual rebalancing reports negative excess returns across all models and across all levels of risk tolerance, except for forecasted yields from the DNS model which incorporates an inflation factor and for lower risk aversion coefficients. This macroeconomic specification also minimizes the loses for higher risk aversion coefficients. In general, negative net excess returns prevail in most optimal mean-variance portfolios;

in few cases it is possible to observe positive Sharpe ratios for the period analysed. This evidence is a consequence of the rising interest rates environment, where income returns on DI-futuro contracts are not enough to offset their price decline over the out-of-sample period.

Moreover, the estimates suggest that optimal mean-variance portfolios with monthly rebalancing deliver higher net excess returns than those with annual rebalancing, pointing out a gain in rebalancing the portfolio weights frequently to keep optimal allocation updated. This basically happens because yield curve models underestimate the climb in 12-month-ahead interest rates over the out-of-sample observations, generating optimal mean-variance portfolios with higher average duration and exposure to price changes.

The overview indicates that good yield curve predictions are important to achieve economic gains from forecasted yields in terms of portfolio performance. It is clear that yield curve models with better forecast accuracy for short forecast horizons perform quite well for optimal mean-variance portfolios with one-step-ahead estimates for DI-futuro returns. In parallel, the DNS model with an inflation factor, which has better forecast accuracy for longer forecast horizons, outperforms in terms of portfolio performance with twelve-step-ahead estimates. Therefore, there is an economic and statistical gain from considering a large macroeconomic dataset to forecast the yield curve dynamics, specially for longer forecast horizons and for medium- and long-term maturities.

Bibliography

ALMEIDA, C.; FARIA, A. Forecasting the Brazilian term structure using macroeconomic factors. *Brazilian Review of Econometrics*, v. 34, n. 1, p. 45–77, 2014. 41, 42, 54, 77, 89

ANG, A.; PIAZZESI, M. A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary economics*, Elsevier, v. 50, n. 4, p. 745–787, 2003. 23, 35, 40, 42, 43, 77, 89

BEKAERT, G.; CHO, S.; MORENO, A. New keynesian macroeconomics and the term structure. *Journal of Money, Credit and Banking*, Wiley Online Library, v. 42, n. 1, p. 33–62, 2010. 41

BERNANKE, B. S.; BOIVIN, J. Monetary policy in a data-rich environment. *Journal of Monetary Economics*, Elsevier, v. 50, n. 3, p. 525–546, 2003. 23, 41

BERNANKE, B. S.; BOIVIN, J.; ELIASZ, P. Measuring the effects of monetary policy: A factor-augmented vector autoregressive (FAVAR) approach. *The Quarterly Journal of Economics*, Oxford University Press, v. 120, n. 1, p. 387–422, 2005. 23, 41, 56

BJÖRK, T.; CHRISTENSEN, B. J. Interest rate dynamics and consistent forward rate curves. *Mathematical Finance*, Wiley Online Library, v. 9, n. 4, p. 323–348, 1999. 35, 36

BOLDER, D. J. *Fixed-Income Portfolio Analytics: A Practical Guide to Implementing, Monitoring and Understanding Fixed-Income Portfolios*. [S.l.]: Springer, 2015. 24, 28, 32, 45, 47, 48, 49

BRANDT, M. Portfolio choice problems. *Handbook of Financial Econometrics*, v. 1, p. 269–336, 2009. 46, 47

CALDEIRA, J. F. Estimação da estrutura a termo da curva de juros no Brasil através de modelos paramétricos e não paramétricos. *Análise Econômica*, v. 29, n. 55, 2011. 29

CALDEIRA, J. F.; MOURA, G. V.; PORTUGAL, M. S. Efficient yield curve estimation and forecasting in brazil. *Revista Economia, January/April*, 2010. 55, 58

CALDEIRA, J. F.; MOURA, G. V.; SANTOS, A. A. P. Measuring risk in fixed income portfolios using yield curve models. *Computational Economics*, Springer, v. 46, n. 1, p. 65–82, 2013. 55, 61

CALDEIRA, J. F.; MOURA, G. V.; SANTOS, A. A. P. Bond portfolio optimization using dynamic factor models. *Journal of Empirical Finance*, Elsevier, v. 37, p. 128–158, 2016. 24, 44, 61, 82

CHOUDHRY, M. *Bond and money markets: strategy, trading, analysis*. [S.l.]: Butterworth-Heinemann, 2003. 45

CHOUDHRY, M. *Analyzing and interpreting the yield curve*. [S.l.]: Wiley Online Library, 2011. 29

CHRISTENSEN, J. H.; DIEBOLD, F. X.; RUDEBUSCH, G. D. The affine arbitrage-free class of nelson–siegel term structure models. *Journal of Econometrics*, Elsevier, v. 164, n. 1, p. 4–20, 2011. 37, 39

CHRISTENSEN, J. H. E.; RUDEBUSCH, G. D. The response of interest rates to us and uk quantitative easing. *The Economic Journal*, Wiley Online Library, v. 122, n. 564, p. F385–F414, 2012. 42, 44, 72

CLARK, T. E.; MCCRACKEN, M. W. Tests of equal forecast accuracy and encompassing for nested models. *Journal of econometrics*, Elsevier, v. 105, n. 1, p. 85–110, 2001. 75

CORTE, P. D.; SARNO, L.; THORNTON, D. L. The expectation hypothesis of the term structure of very short-term rates: Statistical tests and economic value. *Journal of Financial Economics*, Elsevier, v. 89, n. 1, p. 158–174, 2008. 82

COX, J. C.; INGERSOLL, J. E. J.; ROSS, S. A. A theory of the term structure of interest rates. *Econometrica: Journal of the Econometric Society*, JSTOR, p. 385–407, 1985. 32, 33, 34, 35

DAI, Q.; SINGLETON, K. J. Specification analysis of affine term structure models. *The Journal of Finance*, Wiley Online Library, v. 55, n. 5, p. 1943–1978, 2000. 23, 32

DIEBOLD, F. X. Comparing predictive accuracy, twenty years later: A personal perspective on the use and abuse of diebold–mariano tests. *Journal of Business & Economic Statistics*, Taylor & Francis, v. 33, n. 1, p. 1–1, 2015. 75

DIEBOLD, F. X.; LI, C. Forecasting the term structure of government bond yields. *Journal of Econometrics*, Elsevier, v. 130, n. 2, p. 337–364, 2006. 12, 14, 23, 36, 37, 38, 51, 58, 59, 73, 78, 89

DIEBOLD, F. X.; MARIANO, R. S. Comparing predictive accuracy. *Journal of Business & Economic Statistics*, Taylor & Francis Group, v. 13, n. 3, p. 253–263, 1995. 75

DIEBOLD, F. X.; PIAZZESI, M.; RUDEBUSCH, G. D. Modeling bond yields in finance and macroeconomics francis x. diebold, monika piazzesi, and glenn d. rudebusch. *American Economic Review*, v. 95, p. 415–420, 2005. 39

DIEBOLD, F. X.; RUDEBUSCH, G. D. *Yield Curve Modeling and Forecasting: The Dynamic Nelson-Siegel Approach*. [S.l.]: Princeton University Press, 2013. 27, 28, 29, 34, 36, 38, 39, 40, 51, 52, 58, 59, 68, 69, 71

DIEBOLD, F. X.; RUDEBUSCH, G. D.; ARUOBA, S. B. The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of Econometrics*, Elsevier, v. 131, n. 1, p. 309–338, 2006. 23, 39, 40, 42, 52, 58, 70, 77, 89

DIJK, D. et al. Forecasting interest rates with shifting endpoints. *Journal of Applied Econometrics*, Wiley Online Library, v. 29, n. 5, p. 693–712, 2014. 69

DUFFEE, G. R. Term premia and interest rate forecasts in affine models. *The Journal of Finance*, Wiley Online Library, v. 57, n. 1, p. 405–443, 2002. 23, 35, 42, 77

DUFFIE, D.; KAN, R. et al. A yield-factor model of interest rates. *Mathematical Finance*, Blackwell Publishers, v. 6, p. 379–406, 1996. 32

DURBIN, J.; KOOPMAN, S. J. *Time series analysis by state space methods*. [S.l.]: Oxford University Press, 2012. 38, 61

EXTERKATE, P. et al. Forecasting the yield curve in a data-rich environment using the factor-augmented nelson–siegel model. *Journal of Forecasting*, Wiley Online Library, v. 32, n. 3, p. 193–214, 2013. 57

FABOZZI, F. J.; MARTELLINI, L.; PRIAULET, P. *Advanced bond portfolio management: best practices in modeling and strategies*. [S.l.]: John Wiley & Sons, 2006. v. 143. 45

FAMA, E. F.; BLISS, R. R. The information in long-maturity forward rates. *The American Economic Review*, JSTOR, p. 680–692, 1987. 29

FAVERO, C. A.; NIU, L.; SALA, L. Term structure forecasting: No-arbitrage restrictions versus large information set. *Journal of Forecasting*, Wiley Online Library, v. 31, n. 2, p. 124–156, 2012. 23, 41

FILIPOVIĆ, D. A note on the nelson–siegel family. *Mathematical Finance*, Wiley Online Library, v. 9, n. 4, p. 349–359, 1999. 36

FRANKEL, J. A.; LOWN, C. S. An indicator of future inflation extracted from the steepness of the interest rate yield curve along its entire length. *The Quarterly Journal of Economics*, JSTOR, p. 517–530, 1994. 37

GIACOMINI, R.; WHITE, H. Tests of conditional predictive ability. *Econometrica*, Wiley Online Library, v. 74, n. 6, p. 1545–1578, 2006. 75

GÜRKAYNAK, R. S.; WRIGHT, J. H. Macroeconomics and the term structure. *Journal of Economic Literature*, American Economic Association, v. 50, n. 2, p. 331–367, 2012. 23, 33

HAN, Y. Asset allocation with a high dimensional latent factor stochastic volatility model. *Review of Financial Studies*, Society for Financial Studies, v. 19, n. 1, p. 237–271, 2006. 82

HEATH, D.; JARROW, R.; MORTON, A. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, JSTOR, p. 77–105, 1992. 34

HENDRY, D. F. Achievements and challenges in econometric methodology. *Journal of Econometrics*, Elsevier, v. 100, n. 1, p. 7–10, 2001. 72

HO, T. S.; LEE, S.-B. Term structure movements and pricing interest rate contingent claims. *The Journal of Finance*, Wiley Online Library, v. 41, n. 5, p. 1011–1029, 1986. 35

HÖRDAHL, P.; TRISTANI, O.; VESTIN, D. A joint econometric model of macroeconomic and term-structure dynamics. *Journal of Econometrics*, Elsevier, v. 131, n. 1, p. 405–444, 2006. 23, 40, 42, 77

HULL, J.; WHITE, A. Pricing interest-rate-derivative securities. *Review of financial studies*, Society for Financial Studies, v. 3, n. 4, p. 573–592, 1990. 35

JOSLIN, S.; PRIEBSCH, M.; SINGLETON, K. J. Risk premiums in dynamic term structure models with unspanned macro risks. *The Journal of Finance*, Wiley Online Library, v. 69, n. 3, p. 1197–1233, 2014. 36

KIM, D. H.; ORPHANIDES, A. *Term structure estimation with survey data on interest rate forecasts*. [S.l.]: Cambridge University Press, 2005. v. 5341. 35

KOKN, O.; KOZIOL, C. Bond portfolio optimization: A risk-return approach. *Journal of fixed income*, Euromoney Institutional Investor PLC, v. 15, n. 4, p. 48–60, 2006. 24, 44

KOOPMAN, S. J.; WEL, M. Van der. Forecasting the US term structure of interest rates using a macroeconomic smooth dynamic factor model. *International Journal of Forecasting*, Elsevier, v. 29, n. 4, p. 676–694, 2013. 54, 67, 69, 71, 77, 89

LAN, C. An out-of-sample evaluation of dynamic portfolio strategies. *Review of Finance*, Oxford University Press, v. 19, n. 6, p. 2359–2399, 2015. 47

LITTERMAN, R. B.; SCHEINKMAN, J. Common factors affecting bond returns. *The Journal of Fixed Income*, Institutional Investor Journals, v. 1, n. 1, p. 54–61, 1991. 23, 36

LUDVIGSON, S. C.; NG, S. Macro factors in bond risk premia. *Review of Financial Studies*, Society for Financial Studies, v. 22, n. 12, p. 5027–5067, 2009. 41

MARKOWITZ, H. Portfolio selection. *The Journal of Finance*, Wiley Online Library, v. 7, n. 1, p. 77–91, 1952. 12, 14, 24, 44, 46

MCCULLOCH, J. H. Measuring the term structure of interest rates. *The Journal of Business*, JSTOR, v. 44, n. 1, p. 19–31, 1971. 29

MOENCH, E. Forecasting the yield curve in a data-rich environment: A no-arbitrage factor-augmented var approach. *Journal of Econometrics*, Elsevier, v. 146, n. 1, p. 26–43, 2008. 23, 41, 42, 54, 56, 57, 77, 78, 89

NELSON, C. R.; SIEGEL, A. F. Parsimonious mosdeling of yield curves. *Journal of Business*, JSTOR, p. 473–489, 1987. 23, 29, 32, 36, 37

NEWEY, W. K.; WEST, K. D. Hypothesis testing with efficient method of moments estimation. *International Economic Review*, JSTOR, p. 777–787, 1987. 75

NIMARK, K. Monetary policy with signal extraction from the bond market. *Journal of Monetary Economics*, Elsevier, v. 55, n. 8, p. 1389–1400, 2008. 43

OLIVEIRA, F.; RAMOS, L. *Choques não Antecipados de Política Monetária e a Estrutura a Termo das Taxas de Juros no Brasil*. [S.l.], 2011. 43

PIAZZESI, M. Affine term structure models. *Handbook of Financial Econometrics*, Elsevier, v. 1, p. 691–766, 2010. 32, 33

POOTER, M. D.; RAVAZZOLO, F.; DIJK, D. J. V. Term structure forecasting using macro factors and forecast combination. *FRB International Finance Discussion Paper*, n. 993, 2010. 23, 32, 41, 54, 67, 71, 77, 89

ROSSI, J. L.; CARVALHO, M. D. Identification of monetary policy shocks and its effects: Favar methodology for the Brazilian economy. *Brazilian Review of Econometrics*, v. 29, n. 2, p. 285–313, 2009. 54

RUDEBUSCH, G. D.; WU, T. A macro-finance model of the term structure, monetary policy and the economy. *The Economic Journal*, Wiley Online Library, v. 118, n. 530, p. 906–926, 2008. 40, 41

STOCK, J. H.; WATSON, M. W. Forecasting using principal components from a large number of predictors. *Journal of the American Statistical Association*, Taylor & Francis, v. 97, n. 460, p. 1167–1179, 2002. 41, 56

STOCK, J. H.; WATSON, M. W. Macroeconomic forecasting using diffusion indexes. *Journal of Business & Economic Statistics*, Taylor & Francis, v. 20, n. 2, p. 147–162, 2002. 41, 56

THORNTON, D. L.; VALENTE, G. Out-of-sample predictions of bond excess returns and forward rates: An asset allocation perspective. *Review of Financial Studies*, Society for Financial Studies, v. 25, n. 10, p. 3141–3168, 2012. 24, 44, 82

VASICEK, O. An equilibrium characterization of the term structure. *Journal of Financial Economics*, Elsevier, v. 5, n. 2, p. 177–188, 1977. 24, 32, 33, 34, 35, 44

VIEIRA, F.; FERNANDES, M.; CHAGUE, F. Forecasting the Brazilian yield curve using forward-looking variables. *International Journal of Forecasting*, Elsevier, v. 33, n. 1, p. 121–131, 2017. 41, 42, 54, 104

WILMOTT, P. *Paul Wilmott introduces quantitative finance*. [S.l.]: John Wiley & Sons, 2007. 33

Appendix

APPENDIX A – Macroeconomic variables panel

Table 5 – Macroeconomic variables panel

Series	Description	Unit	TF	Source
Money growth				
1	M0 - monetary base - mean	R$ - million	3	Bacen
2	M0 - expanded monetary base - end of period	R$ - million	3	Bacen
3	M0 - monetary base - currency issued - mean	R$ - million	3	Bacen
4	M0 - monetary base - bank reserves - mean	R$ - million	3	Bacen
5	Sight deposits - mean of working days	R$ - million	3	Bacen
6	Savings deposits - end of period	R$ - million	3	Bacen
7	Term deposits - total - with incorpored earnings	R$ - million	3	Bacen
8	M1 - end of period	R$ - million	3	Bacen
9	M2 - end of period - new concept	R$ - million	3	Bacen
10	M3 - end of period - new concept	R$ - million	3	Bacen
11	M4 - end of period - new concept	R$ - million	3	Bacen
Consumption and sales				
12	Real revenues - industry *	Index (2006=100)	3	CNI
13	Electric Energy Consumption	Gwh	3	Eletrobras
14	Electric Energy Consumption - other sectors	Gwh	3	Eletrobras
15	Electric Energy Consumption - commerce	Gwh	3	Eletrobras
16	Electric Energy Consumption - industry	Gwh	3	Eletrobras
17	Electric Energy Consumption - households	Gwh	3	Eletrobras
18	Apparent Consumption - gasoline - mean - qt/day	Barrel - thousand	3	ANP
19	Apparent Consumption - petroleum derivatives - mean - qt/day	Barrel - thousand	3	ANP
20	Apparent Consumption - ethanol fuel - mean - qt/day	Barrel - thousand	3	ANP
21	Apparent Consumption - fuel oil - mean - qt/day	Barrel - thousand	3	ANP
22	Apparent Consumption - diesel oil - mean - qt/day	Barrel - thousand	3	ANP
23	Apparent Consumption - LPG gas - mean - qt/day	Barrel - thousand	3	ANP
24	Domestic Sales - trucks	Units	3	Fenabrave
25	Domestic Sales - busses	Units	3	Fenabrave
26	Domestic Auto-sales	Units	3	Fenabrave
27	Domestic Sales - light commercial vehicles	Units	3	Fenabrave
28	Domestic Sales - automotive vehicles	Units	3	Fenabrave
29	Real sales - industry - São Paulo (SP)	Index (2006=100)	3	Fiesp
30	Real sales - retail *	Index (2011=100)	3	IBGE/PMC
Credit				
31	Credit operations to the public sector	R$ - million	3	Bacen
32	Credit operations to the public sector - federal government	R$ - million	3	Bacen
33	Credit operations to the public sector - state and municipal governments	R$ - million	3	Bacen
34	Credit operations to the public sector - industry	R$ - million	3	Bacen
35	Credit operations to the public sector - housing	R$ - million	3	Bacen
36	Credit operations to the public sector - rural	R$ - million	3	Bacen
37	Credit operations to the public sector - commerce	R$ - million	3	Bacen
38	Credit operations to the public sector - individuals	R$ - million	3	Bacen
39	Credit operations to the public sector - other services	R$ - million	3	Bacen
40	Credit operations to the private sector	R$ - million	3	Bacen
Employment, wage and income				
41	General Registration of Employed and Unemployed (CAGED)	Net employment	1	CAGED
42	Personnel employed - industry *	Index (2006=100)	3	CNI
43	Formal employment - general index	Index	3	MTE
44	Formal employment - processing industry	Index	3	MTE
45	Formal employment - food and beverages	Index	3	MTE
46	Formal employment - construction	Index	3	MTE
47	Formal employment - commerce	Index	3	MTE
48	Formal employment - services	Index	3	MTE
49	Formal employment - direct public administration	Index	3	MTE
50	Unemployment rate - Metropolitan region of SP (MRSP)	Percentage (%)	3	Seade/PED
51	Unemployment rate - hidden - MRSP	Percentage (%)	3	Seade/PED
52	Hours worked - industry *	Index (2006=100)	2	CNI
53	Real minimum wage	R$	3	IPEA
54	Minimum wage - power parity of purchase (PPP)	US$	3	IPEA
55	Real wage - mean - industry - SP	Index (2006=100)	3	Fiesp
56	Real average income - salaried - main job in MRSP *	Index (2000=100)	3	Seade/PED
57	Payroll - general industry	Index (2001.01=100)	3	IBGE/Pimes

Continued on the next page

Price				
58	IPCA - general *	Index (1993=100)	3	IBGE/SNIPC
59	IPCA - food and beverages	Var. % (p.m.)	1	IBGE/SNIPC
60	IPCA - housing	Var. % (p.m.)	1	IBGE/SNIPC
61	IPCA - health personal care	Var. % (p.m.)	1	IBGE/SNIPC
62	IPCA - transport	Var. % (p.m.)	1	IBGE/SNIPC
63	IPCA - regulated prices	Var. % (p.m.)	1	Bacen
64	IPCA - market prices	Var. % (p.m.)	1	Bacen
65	IPCA - household items	Var. % (p.m.)	1	IBGE/SNIPC
66	IPCA - personal expenses	Var. % (p.m.)	1	IBGE/SNIPC
67	IPCA - clothing	Var. % (p.m.)	1	IBGE/SNIPC
68	IPCA - market prices - marketables	Var. % (p.m.)	1	Bacen
69	IPCA - market prices - unmarketables	Var. % (p.m.)	1	Bacen
70	INPC - general *	Index (1993=100)	3	IBGE/SNIPC
71	IPA Source - agricultural products	Index (1994.08=100)	3	FGV/IGP
72	IPA Source - industrial products	Index (1994.08=100)	3	FGV/IGP
73	IPA-EP - general	Index (1994.08=100)	3	FGV/IGP
74	IGP-DI - general	Index (1994.08=100)	3	FGV/IGP
75	INCC - general	Index (1994.08=100)	3	FGV/IGP
76	IPC - general	Index (1994.08=100)	3	FGV/IGP
77	Brazil Commodities Index *	Index (2005.12=100)	3	Bacen
78	IPCA - market expect. for next 12 months - mean *	Var. % (p.y.)	1	Focus
79	IPCA - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
80	IPCA - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
81	INPC - market expect. for next 12 months - mean *	Var. % (p.y.)	1	Focus
82	INPC - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
83	INPC - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
84	IGP-DI - market expect. for next 12 months - mean *	Var. % (p.y.)	1	Focus
85	IGP-DI - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
86	IGP-DI - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
87	IGP-M - market expect. for next 12 months - mean *	Var. % (p.y.)	1	Focus
88	IGP-M - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
89	IGP-M - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
90	IPA-DI - market expect. for next 12 months - mean *	Var. % (p.y.)	1	Focus
91	IPA-DI - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
92	IPA-DI - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
93	IPA-M - market expect. for next 12 months - mean *	Var. % (p.y.)	1	Focus
94	IPA-M - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
95	IPA-M - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
96	Prices adm. by contracts and monitored - for 1-1.5 years ahead - mean	Var. % (p.y.)	1	Focus
97	Prices adm. by contracts and monitored - for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
98	Prices adm. by contracts and monitored - for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
Production and Real Activity				
99	Gross Domestic Product (GDP)	R$ - million	3	Bacen
100	Economic Activity Index of Central Bank (IBC-Br) *	Index (2002=100)	3	Bacen
101	Industrial production (IP) - general industry - quantum *	Index (2012=100)	3	IBGE/PIM-PF
102	IP - processing industry - quantum *	Index (2012=100)	3	IBGE/PIM-PF
103	IP - intermediate goods - quantum *	Index (2012=100)	3	IBGE/PIM-PF
104	IP - consumer goods - quantum *	Index (2012=100)	3	IBGE/PIM-PF
105	IP - consumer durables - quantum *	Index (2012=100)	3	IBGE/PIM-PF
106	IP - consumer goods semi and non-durables - quantum *	Index (2012=100)	3	IBGE/PIM-PF
107	IP - capital goods - quantum *	Index (2012=100)	3	IBGE/PIM-PF
108	IP - machinery and equipment - quantum	Index (2012=100)	3	IBGE/PIM-PF
109	IP - beverages - quantum	Index (2012=100)	3	IBGE/PIM-PF
110	IP - pulp, paper and paper products - quantum	Index (2012=100)	3	IBGE/PIM-PF
111	IP - metallurgy - quantum	Index (2012=100)	3	IBGE/PIM-PF
112	IP - furniture - quantum	Index (2012=100)	3	IBGE/PIM-PF
113	IP - textile - quantum	Index (2012=100)	3	IBGE/PIM-PF
114	Installed Capacity Utilization - industry	Percentage (%)	1	CNI
115	Consumer Confidence Index	Index	3	Fecomercio
116	Economic Conditions Index	Index	3	Fecomercio
117	Future Expectations Index	Index	3	Fecomercio
Production and Real Activity				
118	GDP - Agriculture - market expect. for 1-1.5 years ahead - mean	Var. % (p.y.)	1	Focus
119	GDP - Agriculture - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
120	GDP - Agriculture - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
121	GDP - Industry - market expect. for 1-1.5 years ahead - mean	Var. % (p.y.)	1	Focus
122	GDP - Industry - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
123	GDP - Industry - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus

Continued on the next page

124	GDP - Services - market expect. for 1-1.5 years ahead - mean	Var. % (p.y.)	1	Focus
125	GDP - Services - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
126	GDP - Services - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
127	GDP - market expect. for 1-1.5 years ahead - mean	Var. % (p.y.)	1	Focus
128	GDP - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
129	GDP - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
130	Industrial Production - market expect. for 1-1.5 years ahead - mean	Var. % (p.y.)	1	Focus
131	Industrial Production - market expect. for 2-2.5 years ahead - mean	Var. % (p.y.)	1	Focus
132	Industrial Production - market expect. for 3-5 years ahead - mean	Var. % (p.y.)	1	Focus
Financial and Risk				
133	Treasury Bill - 3 months	% (p.y.)	2	FRED
134	Treasury Bill - 2 years	% (p.y.)	2	FRED
135	Treasury Bill - 10 years	% (p.y.)	2	FRED
136	LIBOR - based on U.S. dollar - 1-month	% (p.y.)	2	FRED
137	LIBOR - based on U.S. dollar - 3-month	% (p.y.)	2	FRED
138	LIBOR - based on U.S. dollar - 12-month	% (p.y.)	2	FRED
139	EMBI+ - Brazilian Risk	% (p.y.)	2	JP Morgan
140	Stock Index - Ibovespa - closing	Index (1999.01=100)	3	Anbima
141	Companies value - Ibovespa	R$ - million	3	BM&Fbovespa
Fiscal				
142	Gross debt - general government	% GDP	2	Bacen
143	Public primary budget result - market expect. for 1-1.5 years ahead - mean	% GDP	2	Focus
144	Public primary budget result - market expect. for 2-2.5 years ahead - mean	% GDP	2	Focus
145	Public primary budget result - market expect. for 3-5 years ahead - mean	% GDP	2	Focus
146	Public nominal budget result - market expect. for 1-1.5 years ahead - mean	% GDP	2	Focus
147	Public nominal budget result - market expect. for 2-2.5 years ahead - mean	% GDP	2	Focus
148	Public nominal budget result - market expect. for 3-5 years ahead - mean	% GDP	2	Focus
149	Government net debt - market expect. for 1-1.5 years ahead - mean	% GDP	2	Focus
150	Government net debt - market expect. for 2-2.5 years ahead - mean	% GDP	2	Focus
151	Government net debt - market expect. for 3-.5 years ahead - mean	% GDP	2	Focus
External sector				
152	Exchange rate - end of period	R$/US$	2	Bacen
153	Internacional Reserves - liquidity concept	US$ - million	3	Bacen
154	Imports - prices	Index (2006=100)	3	Funcex
155	Imports - capital goods - quantum	Index (2006=100)	3	Funcex
156	Imports - quantum	Index (2006=100)	3	Funcex
157	Exports - prices	Index (2006=100)	3	Funcex
158	Exports - quantum	Index (2006=100)	3	Funcex
159	Exports - aggregate factor - basic products - (FOB)	US$ - million	3	MDIC
160	Exports - aggregate factor - insdustrialized products - (FOB)	US$ - million	3	MDIC
161	Exports - aggregate factor - manufactured products - (FOB)	US$ - million	3	MDIC
162	Exports - aggregate factor - semi-manufactured products - (FOB)	US$ - million	3	MDIC
163	Imports - (FOB)	US$ - million	3	MDIC
164	Exports - (FOB)	US$ - million	3	MDIC
165	Trade Balance - amount - (new methodology - BPM6)	US$ - million	2	Bacen
166	Terms of trade	Index (2006=100)	3	Funcex
167	Current Account - amount	US$ - million	2	Bacen
168	Exports - market expect. for 1-1.5 years ahead - mean	US$ - billion	3	Focus
169	Exports - market expect. for 2-2.5 years ahead - mean	US$ - billion	3	Focus
170	Exports - market expect. for 4-5 years ahead - mean	US$ - billion	3	Focus
171	Imports - market expect. for 1-1.5 years ahead - mean	US$ - billion	3	Focus
172	Imports - market expect. for 2-2.5 years ahead - mean	US$ - billion	3	Focus
173	Imports - market expect. for 4-5 years ahead - mean	US$ - billion	3	Focus
174	Current Account - market expect. for 1-1.5 years ahead - mean	US$ - billion	2	Focus
175	Current Account - market expect. for 2-2.5 years ahead - mean	US$ - billion	2	Focus
176	Current Account - market expect. for 3-5 years ahead - mean	US$ - billion	2	Focus
177	Foreign Direct Investment - market expect. for 1-1.5 years ahead - mean	US$ - billion	3	Focus
178	Foreign Direct Investment - market expect. for 2-2.5 years ahead - mean	US$ - billion	3	Focus
179	Foreign Direct Investment - market expect. for 3-5 years ahead - mean	US$ - billion	3	Focus
180	Exchange rate - market expect. for 1-1.5 years ahead - mean	R$/US$	2	Focus
181	Exchange rate - market expect. for 2-2.5 years ahead - mean	R$/US$	2	Focus
182	Exchange rate - market expect. for 3-5 years ahead - mean	R$/US$	2	Focus

Notes: Each macroeconomic series has been deflated according to its monetary unit of measure. (*) Series with seasonal adjustment at source; the other series have been deseasonalized using a stable seasonal filter with additive decomposition. Forward-looking variables do not need seasonal adjustment, because they all refer to annual values. Transformation (TF): (1) Original serie (in level); (2) First difference; (3) First difference of the natural logarithm. The sample is composed by data from 2003:04 to 2016:03, where the credit series (31-40) has been discontinued after 2014:12, serie 29 is not available for 2015:12-2016:03, and series 96-98 about inflation of administrated prices are not available for 2003:04-2003:05.

APPENDIX B – Correlation between yield curve and macro factors

Table 6 – Correlation between yield curve and macro factors.

	$y_t(3)$	$y_t(6)$	$y_t(12)$	$y_t(24)$	$y_t(48)$	$y_t(60)$	L_t	S_t	C_t
Panel A: Contemporaneous correlation of macro factors and yields									
X_t^1	-0.266	-0.268	-0.258	-0.243	-0.252	-0.253	-0.270	-0.128	0.284
X_t^2	0.451	0.447	0.446	0.460	0.485	0.491	0.500	0.061	-0.320
$y_t(3)$	1.000	0.994	0.978	0.950	0.905	0.887	0.790	0.534	-0.322
Panel B: Correlation of 1 month lagged macro factors and yields									
X_{t-1}^1	-0.217	-0.226	-0.227	-0.217	-0.227	-0.227	-0.263	-0.150	0.301
X_{t-1}^2	0.383	0.385	0.387	0.411	0.447	0.456	0.476	0.132	-0.302
$y_{t-1}(3)$	0.991	0.993	0.983	0.953	0.903	0.883	0.772	0.563	-0.277
Panel C: Correlation of 6 months lagged macro factors and yields									
X_{t-6}^1	0.045	0.021	0.013	0.032	0.042	0.041	-0.305	-0.067	0.328
X_{t-6}^2	0.253	0.248	0.253	0.293	0.364	0.384	0.433	0.210	-0.266
$y_{t-6}(3)$	0.825	0.857	0.877	0.858	0.818	0.802	0.695	0.630	-0.119
Panel D: Correlation of 12 months lagged macro factors and yields									
X_{t-12}^1	0.281	0.269	0.260	0.256	0.230	0.218	-0.453	-0.037	0.222
X_{t-12}^2	0.203	0.201	0.194	0.224	0.291	0.312	0.501	0.193	-0.328
$y_{t-12}(3)$	0.657	0.678	0.696	0.706	0.734	0.746	0.672	0.672	-0.013

Notes: The table summarizes the correlation patterns between the yields and macro factors used for estimating *yields-macro* models. X_t^1 and X_t^2 denote the macro factors extracted form the large macro panel for the Brazilian economy, $y_t(3)$ to $y_t(60)$ denote the yields of maturities 1- to 60-months, respectively, and L_t, S_t and C_t are the yield factors. All correlation coefficients regarding observed yields $y_t(3)$ to $y_t(60)$ are statistically different from zero, except for X_{t-6}^1 of panel C.

APPENDIX C – In-sample estimates

Table 7 – Estimated transition matrices A.

Panel A: *DNS-VAR(3)* model					
	L_{t-1}	S_{t-1}	C_{t-1}		
L_t	0.9540**	0.0961**	-0.0112		
	0.0431	0.0470	0.0521		
S_t	0.0907*	0.8576**	0.1122**		
	0.0508	0.0580	0.0561		
C_t	-0.0428	0.0195	0.9184**		
	0.0933	0.0684	0.0581		
Panel B: *DNS-VAR(4)*^{2} model					
	L_{t-1}	S_{t-1}	C_{t-1}	X^2_{t-1}	
L_t	0.7290**	0.0976**	-0.0881*	0.0011	
	0.0850	0.0505	0.0503	0.0009	
S_t	0.0825	0.7141**	0.0726*	0.0004	
	0.0784	0.0438	0.0416	0.0007	
C_t	-0.0358	0.0437**	0.6018**	-0.0012	
	0.0817	0.0198	0.1819	0.0028	
X^2_t	0.0167	-0.0007	-0.0134	1.0119**	
	8.1912	13.7008	6.6454	0.1641	
Panel C: *DNS-VAR(5)* model					
	L_{t-1}	S_{t-1}	C_{t-1}	X^1_{t-1}	X^2_{t-1}
L_t	0.8641**	0.0718**	-0.0429**	-0.0001	0.0005
	0.0054	0.0157	0.0183	0.0004	0.0005
S_t	0.1660**	0.8287**	0.1702**	0.0000	0.0001
	0.0073	0.0180	0.0079	0.0004	0.0004
C_t	-0.3421**	0.2768**	0.4815**	0.0017**	0.0009
	0.0386	0.0603	0.0354	0.0007	0.0007
X^1_t	0.1562	-0.1376	-0.0098	0.8696**	-0.1541**
	1.6531	3.8703	1.9845	0.0111	0.0135
X^2_t	0.2156	-0.1918	0.1815**	-0.4440**	0.4476**
	3.0958	8.6696	0.0835	0.0224	0.0264

Notes: The table presents the estimated transition matrices for the *DNS-VAR(3)*, *DNS-VAR(4)*^{2} and *DNS-VAR(5)* models for the in-sample period from 2003:04 to 2012:11. The standard errors are shown under each corresponding coefficient.
* Estimated coefficients statistically significant at a level of 10%.
** Estimated coefficients statistically significant at a level of 1%.

Table 8 – Descriptive statistics of the measurement disturbances.

Maturities	*DNS-VAR(3)*		*DNS-VAR(4)*^{2}		*DNS-VAR(5)*	
	Mean	Sd	Mean	Sd	Mean	Sd
3	0.2327	0.5476	0.2571	0.5949	0.2521	0.5852
6	0.0915	0.2263	0.1074	0.2662	0.1041	0.2582
9	0.0078	0.0558	0.0176	0.0810	0.0156	0.0764
12	-0.0158	0.0554	**-0.0102**	**0.0429**	**-0.0113**	**0.0444**
15	-0.0139	0.0611	**-0.0111**	**0.0523**	**-0.0117**	**0.0536**
18	-0.0056	0.0516	**-0.0046**	**0.0499**	**-0.0047**	**0.0506**
21	-0.0033	0.0310	-0.0033	0.0316	**-0.0033**	0.0323
24	0.0006	0.0192	**0.0001**	**0.0177**	**0.0002**	**0.0190**
27	0.0057	0.0245	**0.0050**	**0.0229**	**0.0051**	**0.0228**
30	0.0041	0.0262	**0.0036**	**0.0256**	**0.0036**	**0.0251**
36	-0.0030	0.0393	**-0.0028**	0.0395	**-0.0030**	0.0405
42	-0.0030	0.0489	**-0.0017**	**0.0468**	**-0.0023**	**0.0481**
48	-0.0081	0.0711	**-0.0057**	**0.0662**	**-0.0066**	**0.0646**
60	-0.0400	0.1757	**-0.0355**	**0.1743**	**-0.0371**	**0.1704**

Notes: The table reports the mean and standard deviation (Sd) of the errors from the measurement equation of *DNS-VAR(3)*, *DNS-VAR(4)*^{2} and *DNS-VAR(5)* models for the 14 different maturities, which estimated coefficients are reported in table 7. Numbers in **bold** indicate where absolute values of *yields-macro* models outperfm *DNS-VAR(3)* model results.

APPENDIX D – Analysis with only forward-looking macro variables

As stated in Vieira et al. (2017), I also perform the alternative analysis considering the extraction of principal components only from forward-looking variables. The idea is to check whether the variation of expectations about the future state of the economy contribute to explain current yield curve movements. The first two common factors extracted from the panel of macroeconomic forward-looking variables, here denominated as $X^{1,e}$ and $X^{2,e}$, account for 54.14% of the variation in original data and strongly correlates with X^1 and X^2; the correlation between the first macro factors is around 0.96 while for the second ones is near 0.72. Here, $X^{1,e}$ explain about 37.48% of the overall variation in observed variables and correlates mostly with market expectations for inflation and economic activity. In parallel, $X^{2,e}$ explain about 16.6% and correlates highly with the entire set of groups, composed by price, production, fiscal and external sector, but more strongly with inflation expectations.

Table 9 reports the summary statistics of forecast performance for macro specifications using only forward-looking variables as macroeconomic information. The evidence here converge to previous results, where the *DNS-VAR(4)*^{2,e} model, which incorporates $X^{2,e}$, provides lowest RMSFEs and TRMSFEs for the entire maturity spectrum and for forecast horizons above 6 months. The DM test rejects the null hypothesis at a 5% level for some 6-, 9- and 12-month-ahead predictions of the *DNS-VAR(4)*^{2,e} model, while some forecasted yields of *DNS-VAR(4)*^{1,e} and *DNS-VAR(5)* models rejects the null of DM statistic.

Table 9 – (Trace)-Root Mean Squared Forecast Errors of yields-only and yields-macro models regarding only forward-looking macro variables (Focus data). DI-futuro data from 2003:04 to 2016:03, whereas out-of-sample period is 2012:10-2016:03.

	Panel A: *DNS-VAR(3)* model					**Panel B: *DNS-VAR(5)* model**				
	Forecast horizon					Forecast horizon				
Maturities	1-M	3-M	6-M	9-M	12-Month	1-M	3-M	6-M	9-M	12-M
3	0.508	0.967	1.808	2.501	3.066	**0.490**	1.016	2.036	3.071	4.455
6	0.465	0.961	1.740	2.347	2.793	0.502	1.135	2.182	3.149	4.517
9	0.483	0.986	1.665	2.198	2.610	0.543	1.242	2.260*	3.176	4.554
12	0.518	1.006	1.604	2.111	2.482	0.585*	1.309*	2.298*	3.190	4.582
15	0.530	1.011	1.565	2.048	2.386	0.601*	1.334*	2.306*	3.188	4.556
18	0.558	1.043	1.567	2.020	2.325	0.632*	1.370*	2.338*	3.164	4.518
21	0.583	1.070	1.570	1.999	2.286	0.658*	1.383*	2.323*	3.129	4.481
24	0.597	1.078	1.578	1.997	2.263	0.666*	1.373*	2.302*	3.098	4.449
27	0.598	1.083	1.585	1.989	2.242	0.665*	1.353*	2.278*	3.057	4.407
30	0.603	1.088	1.580	1.975	2.223	0.664	1.341*	2.248*	3.012	4.359
36	0.607	1.081	1.579	1.969	2.221	0.655	1.309*	2.201*	2.944	4.291
42	0.614	1.082	1.591	1.976	2.231	0.651	1.279	2.165*	2.889	4.225
48	0.616	1.073	1.582	1.970	2.234	0.649	1.251	2.118	2.834	4.173
60	0.622	1.061	1.574	1.972	2.245	0.649	1.220	2.049	2.754	4.084
TRMSFE	0.567	1.043	1.615	2.083	2.413	0.618	1.284	2.224*	3.050	4.406

	Panel C: *DNS-VAR(4)*^{1} model					**Panel D: *DNS-VAR(4)*^{2} model**				
	Forecast horizon					Forecast horizon				
Maturities	1-M	3-M	6-M	9-M	12-M	1-M	3-M	6-M	9-M	12-M
3	0.625	0.978	**1.720**	**2.477**	3.090	**0.489**	**0.954**	**1.542***	**2.073**	**2.747**
6	0.596	1.103	1.911	2.631	3.339	0.472	**0.900**	**1.458***	**1.971**	**2.567**
9	0.652	1.202	2.034	2.770	3.560	0.498	**0.963**	**1.377***	**1.846**	**2.417**
12	0.686	1.279*	2.134	2.865	3.695	0.550	1.018	**1.323***	**1.758**	**2.279**
15	0.717	1.323*	2.192*	2.926	3.786	0.571	1.054	**1.286**	**1.657**	**2.132**
18	0.749	1.375*	2.240*	2.976	3.844	0.604	1.086	**1.273**	**1.582**	**2.022**
21	0.776	1.414*	2.266*	3.007	3.869	0.619	1.102	**1.263***	**1.513***	**1.946**
24	0.787	1.431*	2.281*	3.019	3.879	0.628	1.110	**1.259***	**1.471***	**1.878**
27	0.793	1.452*	2.284*	3.026	3.889	0.629	1.105	**1.271***	**1.430***	**1.823**
30	0.801	1.462*	2.278*	3.019	3.883	0.635	1.114	**1.287***	**1.402***	**1.781**
36	0.811	1.484*	2.266*	3.002	3.855	0.644	1.122	**1.302**	**1.372***	**1.738**
42	0.819	1.492*	2.246*	2.985	3.822	0.648	1.112	**1.322***	**1.357***	**1.718**
48	0.821	1.480*	2.221*	2.958	3.785	0.650	1.105	**1.329**	**1.344***	**1.701***
60	0.827	1.480*	2.184*	2.915	3.722	0.657	1.097	**1.331***	**1.330***	**1.681***
TRMSFE	0.751	1.363*	2.167*	2.903	3.723	0.596	1.062	**1.332***	**1.597**	**2.058**

Notes: Here, the table presents the forecasting performances of the different models. It reports the root mean squared forecast errors (RMSFE) and trace RMSFE (TRMSFE) obtained by using individual *DNS-VAR(3)* model and *yields-macro* model with one and two macro factors. The values reported are divided by 1×10^{-2}. The RMSFE is reported for each model for the τ maturities and for 1-, 3-, 6-, 9- and 12-month-ahead forecast horizons. The latest line of each panel reports the TRMSFE for the different forecast horizons. The evaluation sample refers to 2012:10-2016:03 (42 out-of-sample forecasts), being 42 out-of-sample forecasts for 1-month horizon, 41 for 2-month horizon, and so on until 31 out-of-sample forecasts for 12-month horizon. Numbers in bold indicate that the alternative *DNS-VAR(4)*^{2} model from panel B outperform the *DNS-VAR(3)* model, otherwise indicate underperformance. The star on the right of the cell entries indicate where Diebold-Mariano test rejects the null of equal forecasting accuracy between the competitor *DNS-VAR(4)*^{2} model and *DNS-VAR(3)* model, with 5% probability of the null hypothesis.

APPENDIX E – Estimates of the alternative DNS specifications

Table 10 – In-sample statistics of the alternative DNS specifications.

Specifications	Goodness-of-fit statistics			
	k	log L	AIC	BIC
DNS-AR(3)	24	8526.6	-17025.2	-16959.1
DNS-AR(4)^{2}	34	8221.0	-16373.9	-16280.3
DNS-AR(5)	45	8033.9	-15977.9	-15853.9
DNS-VAR(3)	33	8627.4	-17188.7	-17097.9
DNS-VAR(4)^{2}	46	8172.6	-16253.1	-16126.5
DNS-VAR(5) *	62	8106.2	-16054.1	-15883.4
DNS-VAR(3)$^{Q-diag}$	30	8549.9	-17039.8	-16957.2
DNS-VAR(4)$^{2,Q-diag}$	43	8182.4	-16278.8	-16160.4
DNS-VAR(5)$^{Q-diag}$ *	59	7975.7	-15833.4	-15671.0
DNS-VAR(3)$^{A-diag}$	27	8607.3	-17160.6	-17086.3
DNS-VAR(4)$^{2,A-diag}$	37	8305.3	-16536.5	-16434.6
DNS-VAR(5)$^{A-diag}$ *	48	8097.6	-16099.1	-15967.0
DNS-VAR(3)^{S}	26	8541.65	-17031.3	-16959.7
DNS-VAR(4)^{2,S}	36	8090.78	-16109.6	-16010.4
DNS-VAR(5)^{S} *	45	8026.92	-15963.8	-15839.9
DNS-VAR(2)^{C}	24	7871.8	-15695.7	-15629.6
DNS-VAR(3)^{2,C}	34	7588.3	-15108.5	-15014.9
DNS-VAR(4)^{C}	47	7357.9	-14621.7	-14492.3

Notes: The table shows the summary statistics of 18 alternative specifications of the DNS model for Brazilian DI-futuro yields, which are estimated for the in-sample data from 2003:04 to 2012:11. Each specification is listed with its maximum log likelihood value (log L), number of estimated parameters (k), Akaike information criterion (AIC) and Bayes information criterion (BIC), which minimum values are given in **bold**. The star on the right of models DNS-VAR(5), DNS-VAR(5)$^{Q-diag}$, DNS-VAR(5)$^{A-diag}$ and DNS-VAR(5)^{S} indicate that they are estimated with the MATLAB function called "refine", which refine initial parameters to aid state-space model estimation. If state-space estimation fails to converge, or converges to an unsatisfactory solution, then "refine" might find a better set of initial parameter values to pass to estimate.

Table 11 – Relative RMSE and TRMSE of the alternative DNS specifications.

Specifications	Maturities								TRMSFE
	3	6	12	18	24	36	48	60	
DNS-AR(3)	0.0065	0.0029	0.0004	0.0005	0.0002	0.0004	0.0007	0.0017	0.0020
DNS-AR(4)^2	**0.988**	**0.979**	1.058	**0.999**	**0.992**	**0.986**	1.029	1.012	**0.988**
DNS-AR(5)	**0.984**	**0.975**	1.057	1.002	1.005	**0.998**	1.014	1.011	**0.984**
DNS-VAR(3)	**0.909**	**0.840**	1.390	1.043	0.975	**0.965**	1.080	1.042	**0.911**
DNS-VAR(4)^2	**0.990**	**0.987**	1.064	1.007	**0.898**	**0.967**	1.004	1.029	**0.992**
DNS-VAR(5)	**0.954**	**0.922**	1.181	1.026	**0.974**	**0.980**	1.020	1.023	**0.955**
DNS-VAR(3)$^{Q-diag}$	1.000	1.002	1.004	1.001	1.007	1.004	**0.998**	**0.999**	1.000
DNS-VAR(4)$^{2,Q-diag}$	1.024	1.045	**0.961**	**0.997**	1.002	1.010	**0.948**	**0.978**	1.025
DNS-VAR(5)$^{Q-diag}$	**0.999**	1.001	1.006	1.005	1.003	1.002	**0.990**	**0.998**	**0.999**
DNS-VAR(3)$^{A-diag}$	**0.939**	**0.894**	1.259	1.026	**0.976**	**0.964**	1.064	1.039	**0.941**
DNS-VAR(4)$^{2,A-diag}$	**0.940**	**0.893**	1.263	1.031	**0.975**	**0.963**	1.063	1.038	**0.941**
DNS-VAR(5)$^{A-diag}$	**0.936**	**0.894**	1.276	1.033	**0.977**	**0.969**	1.057	1.037	**0.938**
DNS-VAR(3)^S	**0.994**	**0.991**	1.023	**0.998**	1.003	**0.998**	1.010	1.002	**0.995**
DNS-VAR(4)^{2,S}	1.023	1.038	**0.892**	**0.997**	1.019	**0.969**	1.087	1.020	1.025
DNS-VAR(5)^S	**0.994**	**0.991**	1.024	1.003	**0.994**	**0.996**	1.008	1.009	**0.995**
DNS-VAR(2)^C	1.590	2.106	4.825	**0.417**	2.221	2.184	3.786	2.288	1.826
DNS-VAR(3)^{2,C}	1.589	2.106	4.837	**0.412**	2.223	2.185	3.786	2.289	1.826
DNS-VAR(4)^C	1.590	2.105	4.816	**0.419**	2.227	2.184	3.786	2.288	1.826

Notes: The table presents the relative RMSE and TRMSE of the alternative estimated specifications of the DNS model for the in-sample data from 2003:04 to 2012:11. Each value is compared to *benchmark*, which is the *DNS-AR(3)* model, while the values in the row corresponding to the *DNS-AR(3)* model are real values for RMSE and TRMSE. Numbers smaller than one (shown in **bold**) indicate that the alternative model outperform the benchmark, whereas numbers larger than one indicate underperformance.

Table 12 – Relative RMSFE and TRMSFE of the alternative DNS specifications.

Specifications	Maturities							TRMSFE
	3	6	12	24	36	48	60	
Panel A: Forecast horizon: 1-month-ahead								
Random Walk	0.0032	0.0037	0.0049	0.0063	0.0066	0.0066	0.0067	0.0058
DNS-AR(3)	1.1927	1.0512	1.0534	**0.9816**	**0.9873**	1.0074	1.0207	1.0064
DNS-AR(4)^2	1.7219	1.4325	1.2610	1.0942	1.0773	1.0821	1.0851	1.1453
DNS-AR(5)	7.9686	6.9047	5.1090	3.8593	3.7712	3.8355	3.9769	4.3478
DNS-VAR(3)	1.5009*	1.2891*	1.0941	**0.9772**	**0.9496**	**0.9516**	**0.9573**	1.0087
DNS-VAR(4)^2	1.6141*	1.4023*	1.2078	1.0755	1.0698	1.0701	1.0696	1.1212
DNS-VAR(5)	2.4525	2.1321	1.7318	1.4589	1.4607	1.4832	1.5097	1.5667
DNS-VAR(3)$^{Q-diag}$	1.5292*	1.2739*	1.0849	**0.9755**	**0.9551**	**0.9575**	**0.9664**	1.0098
DNS-VAR(4)$^{2,Q-diag}$	1.6652	1.2954	1.1325	1.0253	1.0196	1.0186	1.0212	1.0661
DNS-VAR(5)$^{Q-diag}$	1.6530	1.5033	1.3785	1.2582	1.2482	1.2536	1.2569	1.2940
DNS-VAR(3)$^{A-diag}$	1.3288*	1.0912	1.0404	**0.9850**	**0.9765**	1.0044	1.0266	1.0078
DNS-VAR(4)$^{2,A-diag}$	1.6819*	1.5337*	1.3746*	1.2005*	1.1777*	1.1913*	1.2118*	1.2511
DNS-VAR(5)$^{A-diag}$	2.2338*	2.0167*	1.6937*	1.4473*	1.3918*	1.3793*	1.3670*	1.5067
DNS-VAR(3)^S	1.4081*	1.2794*	1.1389	1.0164	**0.9983**	1.0138	1.0235	1.0506
DNS-VAR(4)^{2,S}	1.3632*	1.3358	1.3133	1.1300	1.1159	1.1060	1.1021	1.1653
DNS-VAR(5)^S	9.9898	7.4159	4.2906	2.3616	1.8662	1.6338	1.4883	3.2172
DNS-VAR(2)^C	1.9537*	1.4584*	1.0430	**0.9296**	**0.9188**	**0.9346**	**0.9554**	1.0018
DNS-VAR(3)^{2,C}	2.2865*	1.8051*	1.3808*	1.1948*	1.1925	1.1872	1.2040	1.2807
DNS-VAR(4)^C	3.5477*	2.8711*	2.1302*	1.6825	1.6552	1.6195	1.5944	1.8505
Panel B: Forecast horizon: 6-month-ahead								
Random Walk	0.0143	0.0146	0.0156	0.0173	0.0183	0.0185	0.0185	0.0169
DNS-AR(3)	**0.7863***	**0.8315**	**0.8500**	**0.8689**	**0.8732***	**0.8875***	**0.8923***	**0.8632**
DNS-AR(4)^2	1.5616	1.4156	1.2112	1.0169	**0.9536**	**0.9378**	**0.9228**	1.0908
DNS-AR(5)	4.2099	3.8557	3.2413	2.6188*	2.3744*	2.3155*	2.3044*	2.8420
DNS-VAR(3)	1.2686*	1.2082*	1.0526	**0.9493**	**0.9048***	**0.9031**	**0.9009***	**0.9889**
DNS-VAR(4)^2	1.2035	1.1237	**0.9751**	**0.8636***	**0.8239***	**0.8193***	**0.8179***	**0.9075**
DNS-VAR(5)	2.2131	2.1564	2.0031	1.7281	1.5797	1.5420	1.5180	1.7776
DNS-VAR(3)$^{Q-diag}$	1.1160	1.0701	**0.9397**	**0.8659***	**0.8370***	**0.8417***	**0.8434***	**0.8984**
DNS-VAR(4)$^{2,Q-diag}$	1.2902	1.2529	1.1374	1.0124	**0.9787**	**0.9800**	**0.9781**	1.0564
DNS-VAR(5)$^{Q-diag}$	1.0373*	1.0538*	1.0806*	1.0726*	1.0481*	1.0490*	1.0454*	1.0602
DNS-VAR(3)$^{A-diag}$	**0.9168***	**0.9360***	**0.9284***	**0.9528**	**0.9530**	**0.9737**	**0.9815**	**0.9497**
DNS-VAR(4)$^{2,A-diag}$	1.1587	1.1741	1.2679	1.4971	1.6432	1.7806	1.8886	1.5231
DNS-VAR(5)$^{A-diag}$	1.8320*	1.8143*	1.7180*	1.5606*	1.4605*	1.4390*	1.4194*	1.5830
DNS-VAR(3)^S	1.2033*	1.1934*	1.0867	**0.9928**	**0.9382**	**0.9279***	**0.9180***	1.0126
DNS-VAR(4)^{2,S}	1.5215	1.5299	1.4306	1.1532	1.0189	**0.9520**	**0.9189**	1.1987
DNS-VAR(5)^S	1.8908	2.0598*	2.1008*	1.7902*	1.5048	1.3493	1.2459	1.7452
DNS-VAR(2)^C	**0.9362**	**0.9311***	**0.8485***	**0.7741***	**0.7237***	**0.7120***	**0.7038***	**0.7866**
DNS-VAR(3)^{2,C}	1.7603	1.6707	1.5166	1.2951	1.1976	1.1715	1.1554	1.3520
DNS-VAR(4)^C	1.8625	1.7231	1.4877	1.1796	1.1000	1.0935	1.0830	1.2972

Specifications	Maturities							TRMSFE
	3	6	12	24	36	48	60	
Panel C: Forecast horizon: 9-month-ahead								
Random Walk	0.0208	0.0206	0.0212	0.0218	0.0225	0.0226	0.0228	0.0217
DNS-AR(3)	**0.6808***	**0.7126***	**0.7353***	**0.7797***	**0.7988***	**0.8145***	**0.8153***	**0.7702**
DNS-AR(4)^2	1.7515	1.5929	1.3061	1.0381	**0.9432**	**0.9128**	**0.8832**	1.1719
DNS-AR(5)	4.2925	3.9950	3.4710	2.9590*	2.7025*	2.6168*	2.5616*	3.1668
DNS-VAR(3)	1.1887*	1.1222	**0.9973**	**0.9252**	**0.8893***	**0.8901***	**0.8875***	**0.9639**
DNS-VAR(4)^2	1.0586	**0.9776**	**0.8493***	**0.7545***	**0.7285***	**0.7347***	**0.7360***	**0.8110**
DNS-VAR(5)	1.8626	1.8735	1.7887	1.6576	1.5686	1.5312	1.4990	1.6816
DNS-VAR(3)$^{Q-diag}$	1.0364	**0.9821**	**0.8824***	**0.8300***	**0.8049***	**0.8093***	**0.8094***	**0.8606**
DNS-VAR(4)$^{2,Q-diag}$	1.2071	1.1853	1.0737	**0.9635**	**0.9030**	**0.8828**	**0.8670**	1.0003
DNS-VAR(5)$^{Q-diag}$	**0.8858**	**0.9187**	**0.9714**	**0.9908**	**0.9884**	**0.9952**	**0.9858**	**0.9767**
DNS-VAR(3)$^{A-diag}$	**0.8855***	**0.8990***	**0.8979***	**0.9228***	**0.9283***	**0.9478***	**0.9501**	**0.9199**
DNS-VAR(4)$^{2,A-diag}$	1.0157	1.0377	1.2565	1.7787	2.1383	2.4111	2.5866	1.8261
DNS-VAR(5)$^{A-diag}$	1.8688*	1.8777*	1.8271*	1.7536*	1.6960*	1.6812*	1.6564*	1.7649
DNS-VAR(3)^S	1.0986	1.0807	1.0116	**0.9485**	**0.8976***	**0.8777***	**0.8568***	**0.9616**
DNS-VAR(4)^{2,S}	1.6605	1.6639	1.5291	1.2207	**0.9907**	**0.8651**	**0.8010***	1.2746
DNS-VAR(5)^S	2.3400	2.5549	2.6132	2.2543	1.8407	1.5720	1.3774	2.1918
DNS-VAR(2)^C	**0.9158***	**0.9054***	**0.8509***	**0.7856***	**0.7266***	**0.7032***	**0.6839***	**0.7938**
DNS-VAR(3)^{2,C}	2.1988	2.1690	2.0129	1.8438	1.7492	1.7155	1.6838	1.8958
DNS-VAR(4)^C	2.4401	2.3247	2.0385	1.6991	1.5274	1.4653	1.4210	1.8210
Panel D: Forecast horizon: 12-month-ahead								
Random Walk	0.0261	0.0261	0.0261	0.0254	0.0257	0.0256	0.0256	0.0257
DNS-AR(3)	**0.6016***	**0.6162***	**0.6373***	**0.6938***	**0.7227***	**0.7426***	**0.7476***	**0.6819**
DNS-AR(4)^2	2.2077	1.9787	1.6416	1.3173	1.1424	1.0622	1.0097	1.4813
DNS-AR(5)	5.3483	4.9616	4.4375	4.0282*	3.7239*	3.5948*	3.5153*	4.2045
DNS-VAR(3)	1.1404	1.0513	**0.9449**	**0.8927***	**0.8729***	**0.8838***	**0.8869**	**0.9347**
DNS-VAR(4)^2	**0.9509**	**0.8771**	**0.7695***	**0.6939***	**0.6870***	**0.7023***	**0.7094***	**0.7506**
DNS-VAR(5)	1.5601	1.5506	1.5221	1.5064	1.4572	1.4522	1.4475	1.5006
DNS-VAR(3)$^{Q-diag}$	**0.9885**	**0.9156**	**0.8330***	**0.7986***	**0.7858***	**0.7976***	**0.8010***	**0.8303**
DNS-VAR(4)$^{2,Q-diag}$	1.2524	1.1986	1.0984	1.0321	**0.9725**	**0.9582**	**0.9493**	1.0581
DNS-VAR(5)$^{Q-diag}$	**0.8068**	**0.8467**	**0.8955**	**0.9063**	**0.8793**	**0.8648**	**0.8466**	**0.8802**
DNS-VAR(3)$^{A-diag}$	**0.8729***	**0.8720***	**0.8700***	**0.8950***	**0.9028***	**0.9202***	**0.9218***	**0.8909**
DNS-VAR(4)$^{2,A-diag}$	1.0650	1.0217	1.3280	2.2302	2.8402	3.2949	3.6073	2.2859
DNS-VAR(5)$^{A-diag}$	2.2500*	2.1550*	2.0701*	2.0703*	2.0241*	2.0225*	2.0155*	2.0747
DNS-VAR(3)^S	1.0099	**0.9678**	**0.9115***	**0.8701***	**0.8276***	**0.8102***	**0.7907***	**0.8824**
DNS-VAR(4)^{2,S}	2.1721	2.1661	2.0122	1.6367	1.2475	**0.9848**	**0.8297**	1.6847
DNS-VAR(5)^S	3.5884	3.8276	3.8746	3.3798	2.6787	2.2027	1.8584	3.2844
DNS-VAR(2)^C	**0.9002***	**0.8763***	**0.8358***	**0.7878***	**0.7417***	**0.7227***	**0.7032***	**0.7971**
DNS-VAR(3)^{2,C}	3.9441	3.9056	3.8067	3.8178	3.7315	3.7260	3.7204	3.8022
DNS-VAR(4)^C	4.1541	3.8890	3.4816	3.1189	2.8789	2.7876	2.7309	3.2731

Notes: The table presents the relative RMSFE and TRMSFE of the alternative estimated specifications of the DNS model for the out-of-sample period of 2012:12 to 2016:03 and for the 1-, 6-, 9- and 12-month-ahead forecast horizons. Each value is compared to *benchmark*, which is the random walk, while the values in the rows corresponding to the random walk model are real values for RMSFE and TRMSFE. Numbers smaller than one (shown in **bold**) indicate that the alternative model outperform the benchmark, whereas numbers larger than one indicate underperformance. The star on the right of the cell entries indicate where Diebold-Mariano test rejects the null of equal forecasting accuracy between values of the alternative models and benchmark, with 5% probability of the null hypothesis.

APPENDIX F – Optimal portfolio allocations

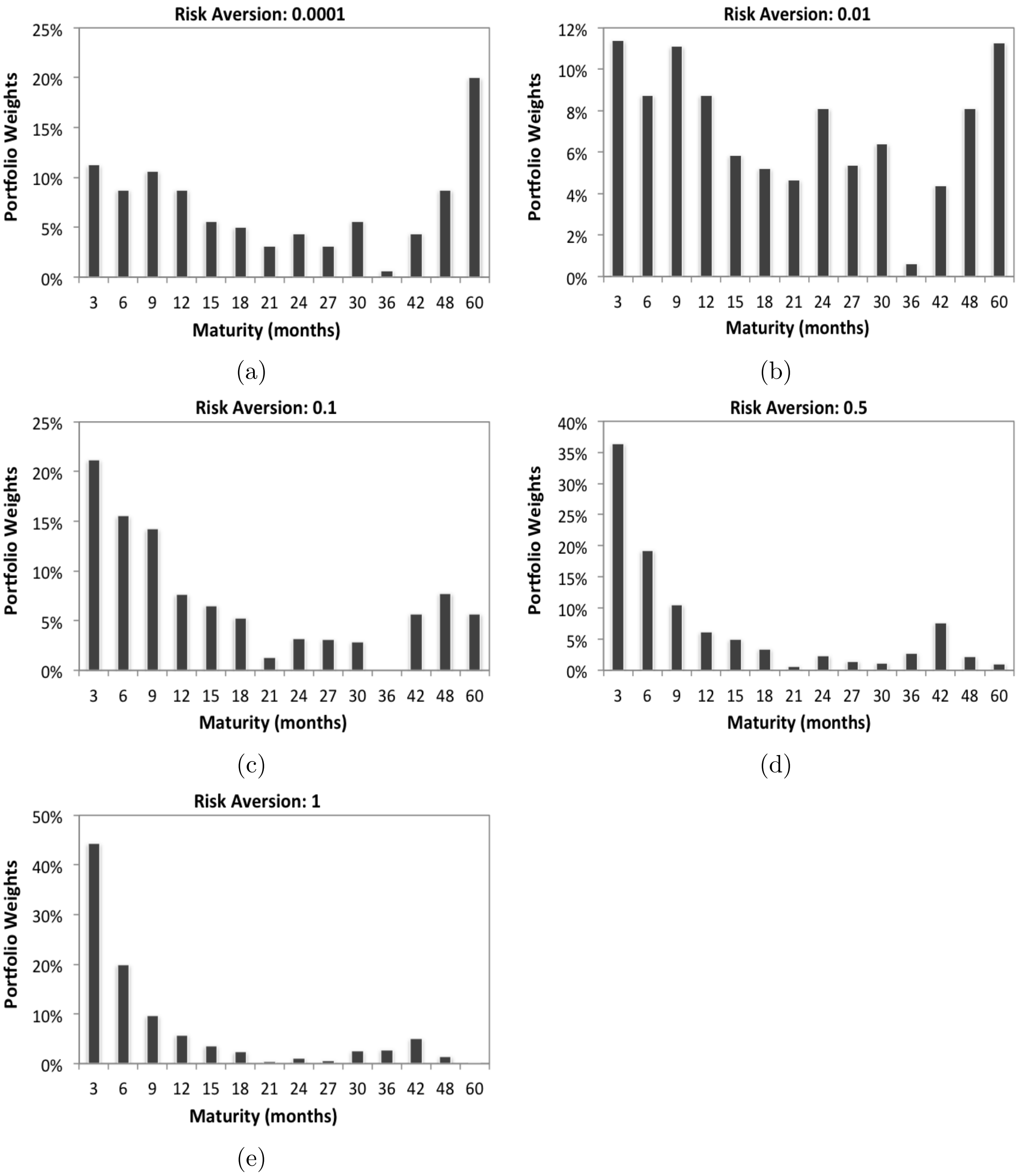
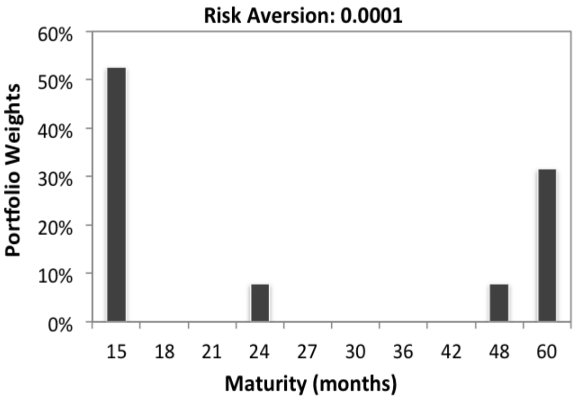
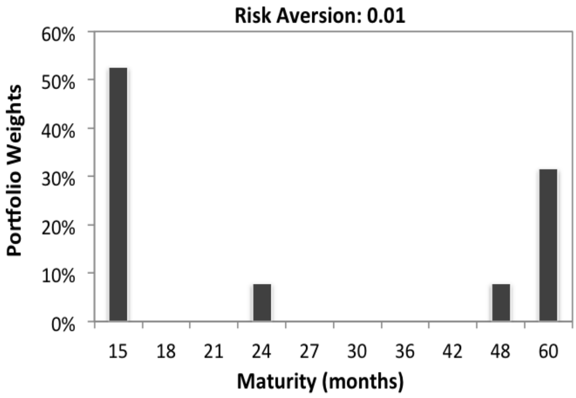


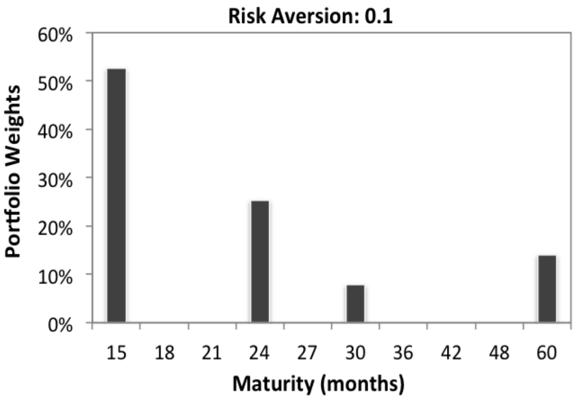
Figure 8 – Optimal portfolio allocations: average mean-variance portfolio weight in each maturity, for 1-step-ahead forecasts over the out-of-sample period.



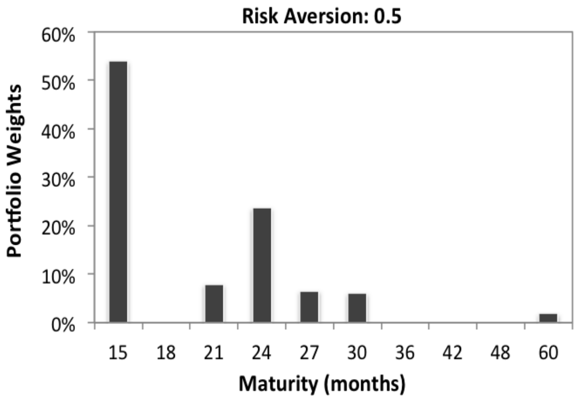
(a)



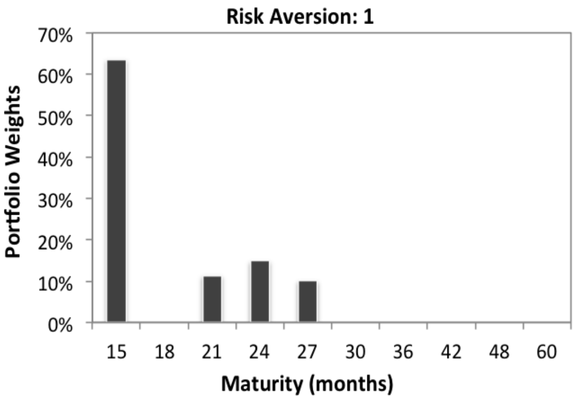
(b)



(c)



(d)



(e)

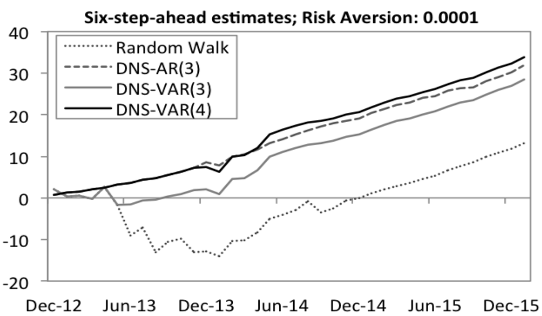
Figure 9 – Optimal portfolio allocations: average mean-variance portfolio weight in each maturity, for 12-step-ahead forecasts over the out-of-sample period.

APPENDIX G – Optimal mean-variance portfolio for 6-step-ahead forecasts

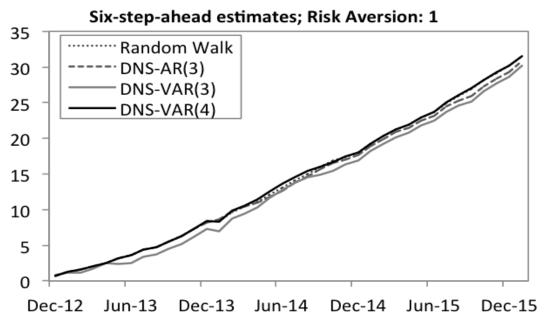
Table 13 – Performance of optimal DI-futuro contracts mean-variance portfolios.

Six-step-ahead estimates with semi-annual rebalancing					
Yield Curve Model	Mean gross exc. R (%)	Mean net exc. R (%)	Std. Dev. (%)	Sharpe Ratio	Duration (years)
δ=0.0001					
Random Walk	-3.145	-3.153	23.729	-0.133	3.237
DNS-AR(3)	0.162	0.154	6.275	0.025	1.053
DNS-VAR(3)	-0.093	-0.106	15.551	-0.007	2.263
DNS-VAR(4)^2	1.075	1.065	10.752	0.099	1.553
δ=0.01					
Random Walk	-2.831	-2.842	16.885	-0.168	2.367
DNS-AR(3)	0.162	0.154	6.275	0.025	1.053
DNS-VAR(3)	-0.093	-0.106	15.551	-0.007	2.263
DNS-VAR(4)^2	1.075	1.065	10.752	0.099	1.553
δ=0.1					
Random Walk	-1.379	-1.388	8.921	-0.156	1.551
DNS-AR(3)	0.162	0.154	6.275	0.025	1.053
DNS-VAR(3)	-0.093	-0.106	15.551	-0.007	2.263
DNS-VAR(4)^2	0.499	0.489	7.546	0.065	1.277
δ=0.5					
Random Walk	-0.127	-0.134	4.432	-0.030	1.008
DNS-AR(3)	-0.270	-0.273	3.243	-0.084	0.830
DNS-VAR(3)	-0.443	-0.457	6.647	-0.069	1.359
DNS-VAR(4)^2	0.159	0.150	5.544	0.027	1.026
δ=1					
Random Walk	-0.141	-0.143	2.984	-0.048	0.806
DNS-AR(3)	-0.355	-0.357	3.079	-0.116	0.803
DNS-VAR(3)	-0.506	-0.518	5.065	-0.102	1.194
DNS-VAR(4)^2	-0.106	-0.111	3.616	-0.031	0.896

Notes: Performance statistics for mean-variance portfolios using the *DNS-AR(3)*, *DNS-VAR(3)* and *DNS-VAR(4)*^{2} model specifications to compute the forecasted yields for the out-of-sample period from 2012:12 to 2016:03. The table reports the statistics for the portfolio optimization using six-month-ahead estimates for DI-futuro returns. The optimal portfolios are rebalanced on a semi-annual basis. The statistics of gross and net excess returns, standard deviation, and Sharpe ratio are annualized and the average portfolio duration is measured in years. The excess return is calculated using the short Brazilian Interbank Deposit (CDI) rate as the risk-free asset. The level of transaction costs is 5 bps. Parameter δ denotes the value of the risk aversion coefficient.

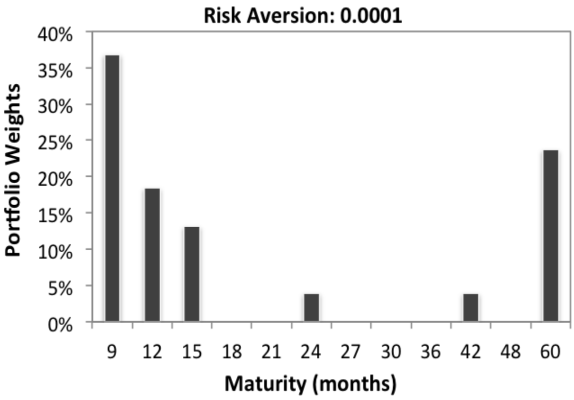


(a)

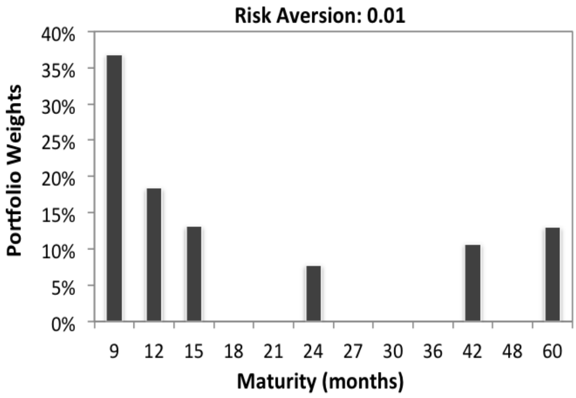


(b)

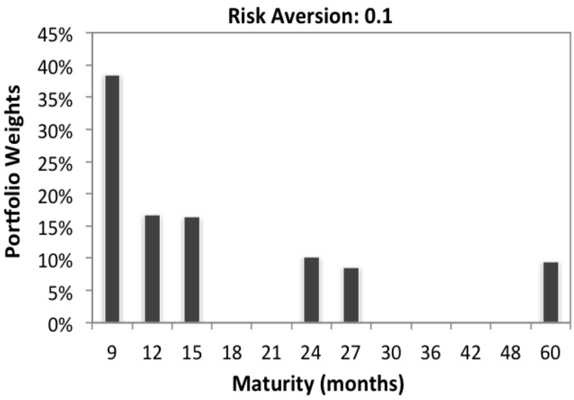
Figure 10 – Cumulative net returns (in %): mean-variance portfolios with $\delta = 1 \times 10^{-4}$ and $\delta = 1$ for 6-step-ahead forecasts over the out-of-sample period.



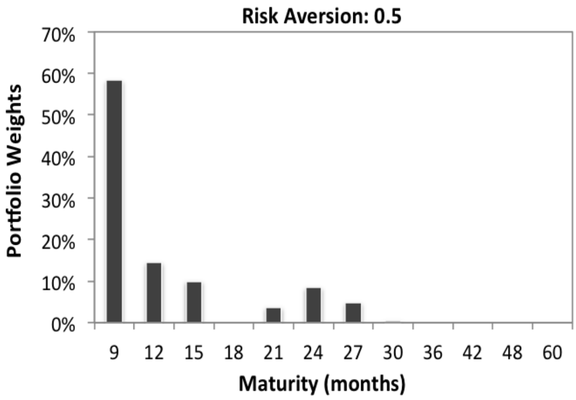
(a)



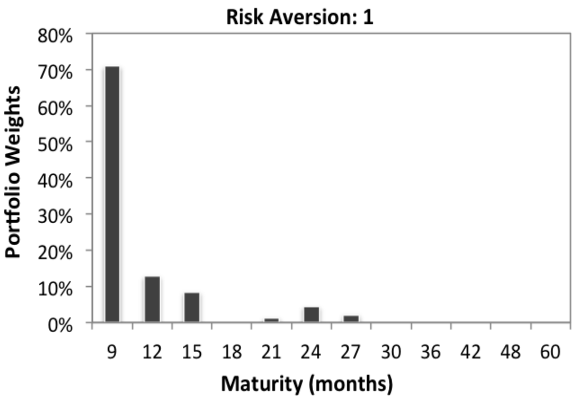
(b)



(c)



(d)



(e)

Figure 11 – Optimal portfolio allocations: average mean-variance portfolio weight in each maturity, for 6-step-ahead forecasts over the out-of-sample period and for $\delta = 1 \times 10^{-4}$ and $\delta = 1$.