2012 – "Dig where you stand" 2. Proceedings of the second "International conference on the history of mathematics education", New University of Lisbon, Portugal, October 2–5, 2011

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Proceedings of the Second "International Conference on the History of Mathematics Education"
October 2-5, 2011, New University of Lisbon, Portugal

Editors:
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Introduction

In their Introduction to the Proceedings of the first *Conference on On-going Research in the History of Mathematics Education*, held 20 to 24 June 2009 in Garðabær (Iceland), the editors expressed their conviction, given the variety of themes, the number of countries involved and the presence of experienced scholars alongside with young researchers, that this area of research would continue to develop, encouraging 'digging' into one's own history and into its broader contexts.

These expectations were, in fact, confirmed when the – already – second thematic International Conference was held from 2 to 5 October 2011 in Lisbon. While there had been 35 participants in Iceland – presenting 18 contributions, there were now 40 researchers from 18 countries representing all five continents presenting some thirty research papers.

The venue of the Conference was the *Universidade Nova de Lisboa*. It proved to be a highly productive event, organized in an excellent manner by the Local Committee: Mária Almeida, Rui Candeias, Cristolinda Costa, António Domingos, Henrique Guimarães, José Matos, Cecilia Monteiro, and Teresa Monteiro. Rodrigo Figueiredo (secretariat) was in charge of the Proceedings. These Proceedings comprise thirty of the contributions, which have been categorized according to the following thematic dimensions:

Policy and mathematics education

K. Bjarnadóttir, A. L. Mattedi Dias, R. d'Enfert, J. Kilpatrick, E. Luciano, J. Prytz, G. Schubring

Reforms of mathematics teaching

M. C. Almeida, J. B. Pitombeira de Carvalho, J. M. Matos, H. J. Smid, G. Vanpaemel & D. de Bock & L. Verschaffel

Teaching arithmetic and algebra

E. Ausejo, M. Menghini, M. Picado & L. Rico

Teaching geometry

E. Barbin, H. C. Hansen

Mathematics curricula

G. De Young, J. Paradinas, F. Pineau, L. Rogers

Methods of teaching

J. Krüger, L. Giacardi, G. Moussard

Mathematics teachers
F. Furinghetti, A. Karp, W. R. Valente

Textbooks
A. Amaral & A. Gomes & E. Ralha, A. Christiansen, Th. Preveraud

The contributions in this volume have been peer-reviewed.

Given that the Third International Conference on The History of Mathematics Education is already being prepared, for September 2013, we confidently anticipate the future development of this prospering research area.

December 2012

Kristín Bjarnadóttir, Fulvia Furinghetti, José M. Matos, Gert Schubring

The making of mathematics curriculum: the case of Telescola in Portugal in the mid-1960s

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Abstract

The Modern Mathematics movement spread through many countries, mainly from the 1960s, attempting to adapt contents and teaching methods of this school discipline to the changed character of science and society. In Portugal, Telescola – a national network of schools complemented by classes on television – had its first school year in 1965/66. This paper studies the gradual shaping (bricolage) of mathematics curriculum as the new ideas of Modern Mathematics were incorporated into the televised lessons and classroom practice. Research will be based on the "lessons" texts and interviews with their author and presenter, António Augusto Lopes.

Introduction

The aim of this paper is to contribute to the history of school mathematics in Portugal, by studying the mathematics curriculum of Telescola – a national network of schools complemented by classes on television – that incorporated the ideas introduced by the Modern Mathematics. When discussing the history of school disciplines, Chervel (1990) argues that they are spontaneous and original creations of the school system and stresses the importance of their study. This author states that the history of the contents is a central component of the history of school disciplines, in this perspective the way how the curricula are introduced is essential in the study of school disciplines (Chervel, 1990).

Gimeno's model to study the curriculum (1998) identifies levels of intervention on the curriculum, each resulting from the action of different actors. We are interested in three particular moments, the first is the *prescribed curriculum,* which is set by the central administration and prescribes or gives directions about curriculum contents. The following phase, the *curriculum presented to teachers* reaches teachers through organized materials that make available an interpretation of the curriculum, which is usually materialized in textbooks. A third moment, *modelled curriculum*, is the result of teachers' understanding of the previous two. The aim of this paper is to realize the ways in which modern mathematical ideas were approached during the 1965/66 school year of Telescola. It will be focused in particular on the new language of mathematics, as well as

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differences in mathematical presentation observed in curriculum construction.

The study is based on data collected through documentary research, mainly official documents, and on a content analysis of the mathematics "lessons" texts published in 1965/66, for the teachers, in the Bulletin IMAVE, supplemented with interviews with their author and presenter, António Augusto Lopes.

Context

During the late 1950's and the early 1960's, Portugal was seeking both economic and technological development. In this aim, the access to basic education had to be widened (Teodoro, 1999). But, the government was unable to build schools fast enough to absorb the increasing numbers of students, in part, due to the nature; small and scattered Portuguese villages in rural areas. By that time, in our country, middle schools and high schools could only be found in larger towns; students from rural areas often had to migrate to larger towns in order to continue their education. Another problem was a shortage of trained teachers.

We need to say that Portugal was under a dictatorial regime from 1933 and the national school system, in 1965, consisted of a primary cycle (age 6-9), followed by parallel branches for secondary schools: the Liceu, which prepared students for the university, and technical schools. And, at a higher level: the University and the Technical Institutes.

From 1965, *Telescola*, a network of schools for 10 and 11 years-old pupils, supported by televised lessons was gradually put in place by the Portuguese Ministry of Education, in an effort to enlarge schooling after primary school, as demanded by economic development. Students that attended the *Postos* (the name of those schools) and finished the two-year course could enrol in the secondary schools. As an experiment, Telescola allowed the unification of the two initial cycles of secondary education and, by 1968, this system covered the entire country, especially in remote areas. Telescola was one of the departments within the *IMAVE* – Institute for Audio-Visual Means of Education (Telles, 1965)

The Modern Mathematics Movement proposed a change in the mathematics syllabus and in the teaching methods. The reform of Modern Mathematics in Portugal can be divided into three intertwined periods: the beginnings, from 1957 until 1963, in which the flow of new ideas can be detected; experimentation, from 1963 to 1968, during which the new ideas were implemented in classrooms; and dissemination, from 1968, that saw the gradual generalization of the reform to all students (Matos, 2009).

Telescola was established within the third period and the new ideas of Modern Mathematics were gradually incorporated into Telescola's TV classes in the first school year, 1965/66, providing an experimental field for their later generalization to the entire population of 5th and 6th graders. Mathematics classes in *Telescola* were actually the first experience in the dissemination of the new ideas through an entire school sub-system in Portugal[1] (Matos & Almeida, 2010).

An approach to teaching using television transmission required major changes in the pedagogical model spread in the rest of the school system. In Portugal, the lessons were all live broadcasted from Telescola's studio in Oporto, and received at the reception stations (schools or postos), which were located mainly in primary schools. The maximum number of students in a class was 20.

The 'teacher' delivered TV lectures, (the 'teacher' was chosen among those who were considered experts in their subjects). The 'monitors' – classroom teachers – led the classes at the reception stations (the 'monitor' was usually a primary school teacher). Each subject lesson was 50 minute long and encompassed two distinct moments, the first was a television 'lesson', 20 minutes long (which was considered to be the pupil attention time limit), and the second was an immediate 'follow-up' in class, 30 minutes long, guided by a monitor.

Until 1975, the content of the broadcasted lesson of each subject was created by a group of 'teachers'. We emphasize that creating and presenting the TV lectures were not the only activities of the teachers, they were also responsible for the production of pedagogical material intended for the 'monitors' and the production of assessment tests. The printed summary of broadcasted 'lessons', various explanations and suggestions for other activities that were published in a monthly newsletter (Bulletin IMAVE) were a part of the materials produced by the 'teachers' (OCDE, 1977).

António Augusto Lopes was the only mathematics 'teacher' at Telescola during its first school year (Foi aprovado o horário da TV Escolar e Educativa, 1965, p. 20). He also trained mathematics teachers for the liceus and was a member of the commission for the reform of mathematics teaching for the upper cycle of liceus, which was appointed to organize an experiment in order to incorporate the ideas introduced by the Modern Mathematics in the curriculum. About the programme for Telescola he wrote in 1965 "we clearly chose – not without some concern – the path of Modern Mathematics" (IMAVE, Out/Nov 1965, preface, p. 2). At present, António Lopes states: "mainly my work was planned on

[1]In this paper we will occasionally mention "classical mathematics" to express the approach to teaching former to the Modern Mathematics one.

my personal experience and in the works of some authors, for instance, Piaget, Servais, Dienes, Papy and Puig Adam" (2009, interview).

The new language of mathematics

We analyzed the texts of the 87 mathematics lessons performed during the school year 1965/6 and published in advance by the bulletin IMAVE, all written by António Augusto Lopes. The lessons were held on Mondays, Wednesdays and Fridays of each week, began immediately on October 25, 1965 and finished June 29, 1966.

Almost all of these texts have a similar structure: 1) a summary, 2) a 'Descriptive Schema' with the script broadcasted by the 'teacher,' 3) an identification of the material needed during or after the TV lecture, and 4) indications for the immediate 'follow-up' conducted by the 'monitor' in class, which included methodological suggestions, most of it training exercises.

The theory of sets, one of Modern Mathematics ideas, is a new topic in the mathematical content. Sets and their operations (union, intersection, difference, and complement) were introduced during the first five lessons (two weeks). As the set theory was adopted as the appropriate language to express mathematics, students had to be acquainted with it. So, in Lesson No. 1, the 'teacher' discussed the "concept of set (similar – but informal – word: collection) and element of a set (similar – but informal – words: individual, entity, object)" (IMAVE, Out/Nov 1965, Lesson. N.º 1, p. 86). Immediately following, it was underlined that sets can consist of elements of various natures: people, physical objects, numbers, signs; and there was also a brief mention to sets and non-sets.

2 — Em seguida, o professor apresenta o primeiro exercício — **que todos devem fazer!** Trata-se de preencher uma tabela: — numa coluna figuram **designações de conjuntos**; noutra coluna figuram as **designações dos elementos** desses conjuntos.	2 - The teacher presents a first exercise, **that all must do!** It was requested to fill the blanks on a table: one column has **designation of sets**, and the other column has **the designation of the elements** of those sets.

Fig. 1. Sets, elements of a set and their designation (IMAVE, Out/Nov 1965, p. 86)

At the end of the 'lesson', the 'teacher' stressed that "an object (whatever its nature) should not be mistaken with the symbol that represent it" (IMAVE, Out/Nov 1965, 1965, p. 86). In the 'follow-up' period, the 'monitor' would present some verbal exercises (IMAVE, Out/Nov 1965, 1965, p. 88) and revision ones, both focused on the distinction between a set and its elements and on the belong notion (∈). In the following lessons, some main notions and definitions were introduced and their representation shown (symbols, Venn diagrams).

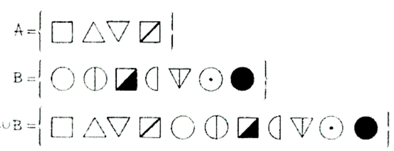
The analysis shows two major changes: other mathematical notions were built up based on the notion of sets and their operations, and some topics were introduced for the first time (some related with sets, some not). In the scope of the first kind of changes let's observe how addition, which had been already studied at primary school by an approach of counting collections of concrete objects combined into a single collection, was introduced. Using the cardinality of a set, the sum was defined as the number of elements of the reunion of disjoint sets (Fig. 2), usually consisting of abstract entities. The same can be observed in the way the other arithmetic operations were introduced.

1 — Retoma-se a situação anterior, através de novos exemplos sobre reunião de conjuntos disjuntos; **o número de elementos do conjunto reunião** *de dois conjuntos disjuntos é, por definição,* a **soma dos números de elementos** desses conjuntos:

$$\#(A \cup B) = \# A + \# B$$

Desta maneira, os alunos são colocados a rever experiências anteriores — colhidas na escola primária e na vida quotidiana.

O esquema de base a explorar em todas as situações é evidentemente o que a seguir se indica:



1 - We resume the previous with new examples about the reunion of disjoint sets; the **number of elements in the reunion** *of two disjoint sets*, is the **sum of the number of elements** of those sets:

$$\#(A \cup B) = \# A + \# B$$

(…)

The basic schema to explore in all situation is as follows:

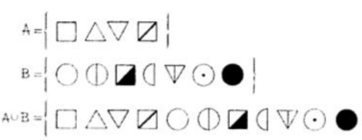


Fig. 2. The addition and the reunion of disjoint sets (IMAVE, Out/Nov 1965, Lesson N.º 13, p. 117)

The possibility of addition that was not in question at the primary school, but now the connection with common everyday experience used in the classical approach is replaced by the abstract character of sets. The use of sets to communicate mathematical ideas requests a more complex mathematical context, thus the new ideas have to be carefully introduced. Now the teacher must draw the students' attention to the fact that addition may not be well defined. It depends upon the *universe* (assuming a universe in which the operation is defined is essential), so following the 'definition' of sum in Lesson N.º 13 (IMAVE, Out/Nov 1965, p. 117), the ‘teacher’ discusses this requirement bringing out that addition may neither be an internal composition law nor universal. An example is shown by the teacher: in the universe, E = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, 3 and 4 belong to E, and 3 + 4 belongs to this set; however, 5 and 8 are in E, but 5 + 8 does not belong to E (IMAVE, Out/Nov 1965, p. 117).

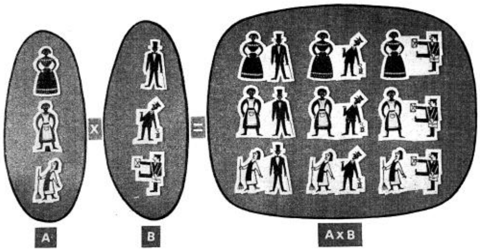
There were changes of school mathematics content, new topics associated to sets were introduced. To illustrate we chose two: Cartesian product of two sets and writing numbers in distinct bases.

2 — *Os pares possíveis no baile:*

Sendo **A** = { Maria, Isabel, Luísa }
B = { Carlos, Pedro, Rui }

o produto **A** × **B** é o de todos os pares possíveis: o primeiro elemento de cada par pertence a **A**; o segundo elemento pertence a **B**.

O esquema junto mostra a possível concretização do problema:



2 - The possible pairs at the dance:
A = {Maria, Isabel, Luísa}
B = {Carlos, Pedro, Rui}
the product **AxB** consists of all possible pairs: the first element of the pair belongs to **A**; the second element belongs to **B**.
The schema shows a possible implementation to the problem:

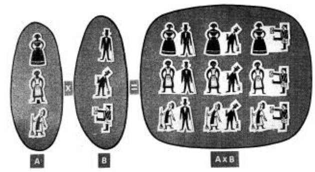


Fig. 3. Cartesian product: an example (IMAVE, Dez 1965, Lesson N.º 21, p. 47)

Concerning the Cartesian product, although the predicate notation is presented, what matters, essentially, is the dynamics of the operation. The Cartesian product notion is presented by means of simple examples, drawn from things that people do in everyday life (Fig. 3).

The mention of non-decimal numeration systems was justified by its applications. The ‘teacher’ used simple examples combined with practical graphic representation and verbal exercises to better acquire the concept the students are asked to create their own graphic representation to illustrate how to count, in different systems, the objects of a collection (less than 25).

Differences in mathematics communication

Before addressing the differences between the role of the ‘teachers’ and ‘monitors’, a few comments will be made to help understand the purposes of learning mathematics and pedagogical guiding principles that underlies the modernized curriculum. The 'monitors' were acquainted with the contents and the spirit of the new curriculum mainly via bulletin IMAVE that advanced the following reasons to learn mathematics: usefulness in everyday life, usefulness in scientific and technological applications, training the mind, to make children aware of a important component of modern culture patterns. It is stressed in the bulletin that it is better to teach mathematics by making the pupils participate in its elaboration, for "the act of learning is a creator act, which requires the following activities: observe, choose, deduce and act" (IMAVE, Out/Nov 1965, p. 83). It was also stressed that the problems posed to the pupils should be related with their everyday life, as well as, the use concrete situations as a starting point to abstraction and to resort to a concrete experience to suggest a definition and or even a demonstration. Also important is studying the students' mistakes, as a means to diagnose difficulties that arise in their mathematical education. Referring to the use of concrete models in teaching, it is marked that models allow progressive leap from the perception to the conceptualization, as so the model is fundamental to develop progressive and active abstraction based on intuition: intuition of the concrete, mental intuition of representations and mathematical intuition. Practical work and group work were recommended (Almeida, 2010).

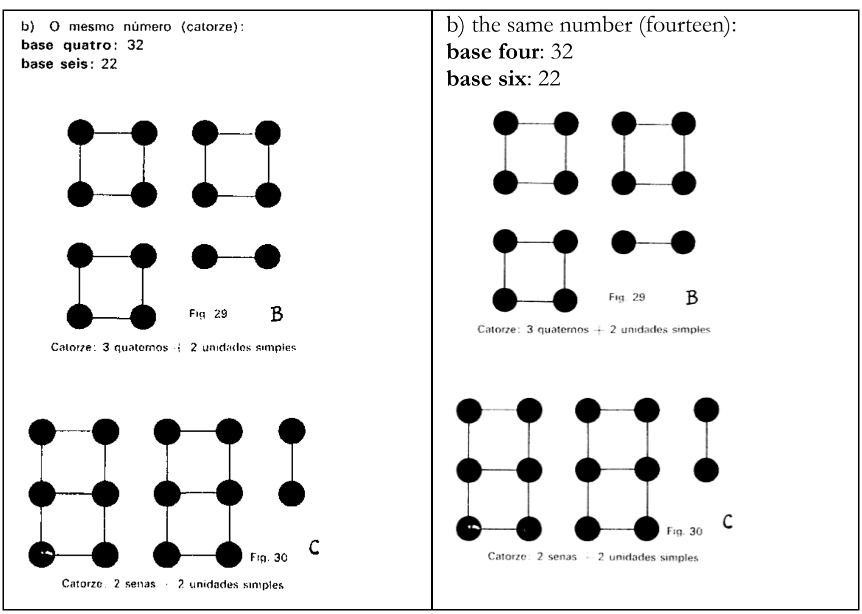
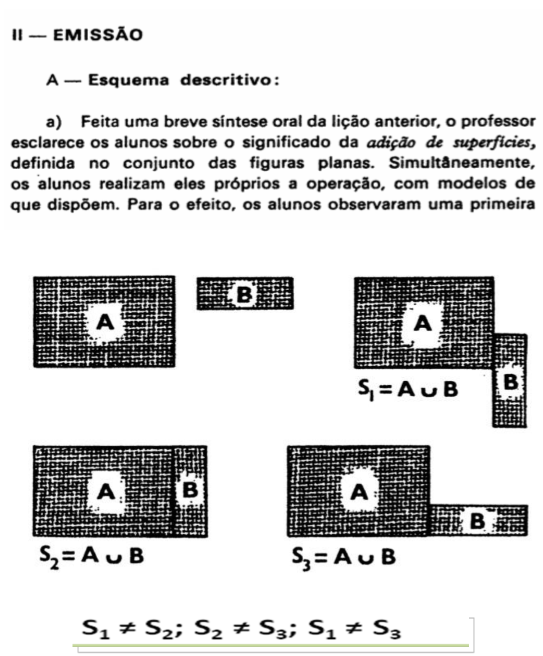


Fig. 4. Writing numbers in distinct bases (IMAVE, Out/Nov 1965, Lesson N.º 7, p. 103)

From the analysis of the texts of the lessons it is possible to infer some routine of the lessons tasks. The tasks of the teacher were to introduce new mathematical contents and reassess old mathematical notions when it was needed. The tasks of the monitor were to summarize specifically indicated points of the mathematical contents introduced by the teacher, but mainly to supervise the pupils' work in class as well as resolutions of practice and revision exercises, assuring that the exercises assigned by the teacher were executed.

To illustrate this conception we will take Lesson N.º 41, performed in February 23, 1966. During the TV lecture, Antonio Lopes planned to discuss the addition of geometric figures using the language of sets and their operation.


II — EMISSÃO

A — Esquema descritivo:

a) Feita uma breve síntese oral da lição anterior, o professor esclarece os alunos sobre o significado da *adição de superfícies*, definida no conjunto das figuras planas. Simultâneamente, os alunos realizam eles próprios a operação, com modelos de que dispõem. Para o efeito, os alunos observaram uma primeira

TV lecture

A – Descriptive schema:

a) After a brief oral synthesis of the previous lecture, the teacher clarifies for the students the meaning of *surface addition*, defined on the set of plane shape figures. Simultaneously, the students perform by themselves the operation, with models available to them.

Fig. 5. Addition in the set of plane figures (IMAVE, Fev 1966, Lesson N.º 41, p. 57)

The addition operation of two rectangles formulated in the language of sets was a complex question, such as it was explained in the text (there isn't a single result, Fig. 5 shows three different results; and problems about a possible 'neutral element' were also addressed). The conclusion was that the operation was possible but not uniform, because the final surface wasn't unique[2].

The difference between the tasks of the 'teacher' and those of the 'monitor' can be observed in Fig. 6, which refers to the 'follow-up' of Lesson N.º 41. The matter addressed by the 'teacher' in this 'lesson' is not resumed in the activities designed for its 'follow-up' and to be lead by the monitor. The activities to be proposed by the monitor are practice exercises – one exercise to perform mass conversion, one other to compute the multiple of a whole number (formulated in terms of sets), and the last involves resolution of numerical expressions – are far from the subjects matter covered by the 'teacher' in the same lesson.

b) <u>*No quadro*</u> 1. *Calcular:* 1.) 7,4 g + 575 mg + 3,47 dg (em centigramas); 2.º) 2 34 hg + 15,6 dag + 0,8 kg (em decagramas:.) 2. *Determinar:* 1.º) O conjunto A dos múltiplos de 8 compreendidos entre 50 e 90. 2.º) O conjunto B dos múltiplos de 16 compreendidos entre 45 e 100. 3.º) O conjunto A ∩ B. 3. *Calcular:* x = (2 + 4 × 3) : 7 + (2 × 3 — 1) × 4 : 5 y = 6 × 4 : 8 + 2 × (3 + 1 × 2) — (2 + 4 × 2) : 2	On the blackboard: (practice exercises)
Respostas: 1. 1.º) 832,2 cg; 2.º) 119 dag. 2. 1.º) A = {56, 64, 72, 80, 88}; 2.º) B = {48, 64, 80, 96} 3.º) A ∩ B = {64, 80} 3. x = 6; y = 8.	Answers:

Fig. 8. Activities to be proposed for the 'follow-up' of Lesson N.º 41 (IMAVE, Fev 1966, p. 58)

The differences between the educational tasks of 'teachers' and the 'monitors' can be analyzed through the heuristic moments of curriculum construction proposed by Gimeno (1998),which permit to distinguish phases in which the curriculum is realized in different ways by different actors. The prescribed curriculum of centralized educational systems contains a set of requirements or guidelines "which act as a reference to

[2] This topic – operations on plane figures - is not resumed in the mathematics program that enters in 1968/69, for 10/11 years old students.

regulate the curricular system, perform as starting point for the development of materials, system control, etc..." (Gimeno, 1998, p.104). However, teachers seldom use these guiding principles directly. "There are [other] means which are structured to perform this action" (p. 150) the so called curriculum presented to teachers, is usually displayed in textbooks.

In the Portuguese context, Telescola poses some difficulty to Gimenos' distinction. The first challenge is that 'teaching' is shared between a 'teacher' and a 'monitor'. The 'teacher' explains contents in a distinct and higher level (figuratively, but literally too, as the television should stand on a high placed self) and the monitor explores and consolidates contents in the plane of the classroom and in interaction with students. "Didactic guidelines" regarding mathematics clearly mentions: "It is up to the monitor to ensure the complete development of activities by the students, **as determined by the teacher and without limiting each pupil's own rhythm**" (IMAVE, Out/Nov 1965, p. 83, bold at the original). The second is that the mathematics program is being constructed by Antonio Augusto Lopes during 1965/66 with the agreement of the administration, as well as the decision about the teaching plans to fit the content.

The usual sequence Ministry > Mediators > Professor (or Program > Textbooks > Lesson plan) was subverted and, at least in 1965/66, we have both, an actor: the 'teacher', who produces the program, applies it in 'lessons' and partially teaches it; and another actor: the 'monitor', which takes notice not only of the text of 'lessons' content, but also of the TV lecture and perform the classroom (practical) teaching. There isn't still any research at the classroom level; however, there is evidence to suggest that at the practice level, on TV lectures, the 'teacher' represents the doctrinal dimension of teaching, allied to exposition of the technical language, usually associated to the programmatic content of mathematics. In the classes led by the 'monitor', mathematics is associated to manipulations of materials and essentially to solving exercises. But at the same time, it would be excessive to simplify this analysis by reducing the 'teacher's' lessons to mere exposition and the 'monitor´s' teaching to practical and concrete. On the one hand, there are no studies on school practices in Telescola. On the other, a part of the teacher's interventions direct the students to mathematical activity, valuing the autonomous learning of students.

Conclusion

The Modern Mathematics Movement proposed a change in the mathematics syllabus and in the teaching methods. In the making of Telescola's mathematics curriculum we can trace the influence of educators who favor modernization, for it is impregnated with the modern spirit of pedagogy and mathematics. Concerning changes in mathematical content, the distinctive characteristic is the adoption of set theory as the appropriate language to express mathematics, leading to the use of sets to communicate mathematical ideas and new topics associated to sets and their operations. The research allows sustaining that there are distinctions between the role of the 'teacher' and the 'monitor', even though the 'teaching' is shared between a 'teacher' and a 'monitor'. In a simple way we say that the 'monitor', in contact with students, explores and consolidates contents in the classroom; and, the 'teacher', on TV lectures, explains mathematical contents.

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Reforming mathematics education in the early 20th century: implications on "measuring", as presented in a Portuguese textbook

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Abstract

"Measuring" is part of mathematics from the very beginning of human activities; it is therefore in the core of the history of mathematics itself and, at the same time, it became a fundamental topic in teaching/learning mathematics. The present article analyses the approach to measure and to metric systems as exposed in an old Portuguese textbook dated from the beginning of the 20th century. This textbook was adopted at the beginning of one of the most important Portuguese educational reforms and it was designed to be used in Primary Schools, either by pupils and by teachers, half a century after Portugal had adopted, by decree, the so called "French metric system".

Our study focuses on identifying the conceptual structure in measure and in the metric system, the procedures and methodology, the illustration/representation, the context of examples, exercises and problems, as well as the suggested tasks used.

Measuring: why?

Measure is present in almost every aspect of our life. From early stages of human history we have easily found the need to measure with greater accuracy and such need contributed to the development of the society and to its wide organization in agriculture, astronomy, commercial exchanges as well in medicine, science, technology, engineering or economics. Under these circumstances it seemed natural to find measuring systems to be used by certain communities and, when appropriated, to spread to other regions and/or other countries. This expansion demanded, as it still does, reliable and standard measuring systems in order to facilitate trade as well as scientific, technological and even political relationships.

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One of the most important examples in achieving standard systems for weights and measures took place in France (Høyrup, 1994). It had been developed by a group of scientists and was approved by decree on the 10th of December 1799: it consisted of a decimalized system of measurement based on the definitions of "meter" and "kilogram". It is known as SI ("Système International d'unités") and became, over the years, the official system of units in almost every country in the world.

Measuring in Portugal: How?

Portugal, over the centuries, was also involved in establishing a standard system of weights and measures. By 1253, our king D. Afonso III, had promulgated the Law of *Almotaçaria*[2] seen as the first milestone in the history of structuring and standardizing the national weights and metric system in Portugal (Cruz, 2007). In 1361 a license was published demanding that "bread" measures (for grain) should be gauged by the "*alqueire"* of Santarém[3] and the *Cortes of Elvas*, stated that wine measures should be gauged by the "*almude"*[4] of Lisbon and also determined that the weights used for certain commodities, such as meat, wool and linen should be gauged by the "*arroba"*[5] of Lisbon.

The Portuguese Maritime expansion (1415-1543) was also the expansion of both domestic and foreign trade, as well as social changes, which forced standardization and simplification of our system of measuring.

The so called *Cologne mark* – standard weight used all over Europe – had been re-adopted in Portugal (14th October 1488) during the realm of D. João II.

> This adoption was a symptom of the relevance of the rising mercantilism and the internationalization of weights and measure as a consequence of growing trade.

An important reform by D. Manuel I (1469-1521) – *Reforma Manuelina* – had, finally, some significant success in unifying "measuring" in Portugal: it was implemented providing copies of the royal unit standards (instruments) to every town. But under the realm of the D. Sebastião

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[2] "Almotaçaria" comes from the arab "al-mohtacib" and refers to an elected magistrate whose job was to supervise application of the law.

[3] Santarém is an old town situated in one of the most fertile regions in Portugal. "Alqueire" measures liquids (mainly olive oil, nowadays) and varies between 13, 215 and 22, 605 liters. It has been used as a measure for an area of a field (where an "alqueire" of grain may be planted).

[4] "Almude" corresponds, nowadays, to 25 liters.

[5] "Arroba" is equivalent to, at present, approximately 15 kilograms.

(1554-1578) yet another reform on adapting/unifying units took place; it involved, particularly, the units for volumes and it was to last, in theory at least, many centuries:

> … it stayed in use until the application of the International Metric System. (Cruz, 2007, p. 136)

Table 1. Portuguese's weight and volume systems (adapted from Cruz, 2007)

Reform		Designation	Conversion	Metric equivalence
Weight system (13th-15th century)	Unit	*arrátel*		+/- 0,3596kg
	Multiples	*arroba* *quintal* *carga*	32 4 arrobas 3 quintais	
	Submultiples	*marco* *onça*	16/25 1/8 marco	
Weight: D. Manuel I's system	Unit	*marco*		229,383g
	Multiples	*arrátel* *arroba* *quintal*	2 32 arráteis 4 arrobas	
	Submultiples	*onça* *oitavo* *escrópulo* *grão*	1/8 1/8 onça 1/3 oitavo 1/24 escrópulo	
Liquid: D. Manuel I's system	Unit	*almude*		
	Submultiples	*canada* *meia-canada* *quartilho* *meio-quartilho*	1/12 1/24 1/48 1/96	
Dry: D. Manuel I's system	Unit	*alqueire*		
	Multiples	*moio*	60	
	Submultiples	*meio-alqueire* *quarta* *oitava* *maquia* *meia-maquia*	1/2 1/4 1/8 1/16 1/36	
Liquid: D. Sebastião's system	Unit	*almude*		16,676 dm^3
	Submultiples	*meio-almude* *canada* *meia-canada* *quartilho* *meio-quartilho*	1/2 1/12 1/24 1/48 1/96	
Dry: D. Sebastião's system	Unit	*alqueire*		13,077 dm^3
	Multiples	*fanga* *moio*	4 60	
	Submultiples	*meio-alqueire* *quarta* *oitava*	1/2 1/4 1/8	

We may summarize the main reforms of the Portuguese measuring systems as follows (see Table 1) and it seems incredible to note that – in spite of so many reforms, so many attempts and so many years – many of the old names/designations are, still nowadays, used in Portugal. Moreover many of these names and these units are still in use and, as in past times, its meaning still varies from region to region.

Later on, in 1812[6], the Royal Academy of Sciences of Lisbon integrated a commission for the *Exame dos Forais e Melhoramento da Agricultura*[7] to study and to gather information on weights and measures which had been used in Portugal up to that date. Then, in 1814, the *Mappa do Systema decimal em Nomenclatura Portugueza*[8] - based on the French Metric System - was presented in order "to promote external and internal trade". However the French Decimal Metric System would only be officially adopted, by decree, on the 12th of December 1852, and only in 1859 the old nomenclature was, officially, abolished.

Measure and metric system in Portuguese education, from the early 20th century

We will now concentrate our attention in the teaching of mathematics in Portugal, particularly referring to the metric system, from the second half of the 19th century onwards. Education, in Portugal, was far from being accessible to every citizen but it aimed, for those who attended schools on those days, at more than mere instruction of contents: it also intended to spread moral values. Teachers were therefore highly regarded professionals who were trained, subjected to a close scrutiny, to accomplish results. Textbooks were examined and went through systematic revisions in order to be adopted as official books for teachers to follow and pupils to learn from. These books[9], after having granted a government approval, would reach both teachers and pupils in both private and public primary schools and we can easily find traces on the importance of teaching/learning the Metric System inasmuch as some of these textbooks were even entirely dedicated to the topic.

In the dawn of the twentieth century, Portugal went through an agitated political moment and the social events had particular

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[6] In 1812, Napoleon had officially replaced the names of the old units of weight, measure and subdivisions only in 1840, the metric system became mandatory in France.

[7] *Exam of "Forais" and Improvement of Agriculture*. By "forais" we mean royal documents which ruled the administration of the towns

[8] *Map of the decimal system, in Portuguese terminology*. The main linear unit was named *Mão travessa*, which was the hundred-thousandth part of the quarter of meridian.

[9] At that time, in Portugal, it was possible to choose among textbooks. Later, between 1933 and 1974, a unique mathematics textbook was adopted, in all Portuguese schools.

repercussions in our educational system. On October, the 5th, 1910, our Monarchy came to an end (after almost 8 centuries), a Republican government was implemented and, in order to generate and consolidate "a new form of being Portuguese" (Carvalho, 2001, p. 651), a new Educational Reform[10] took place. This major reform, based on experimental instruction and new pedagogical methods, aimed particularly at implementing the first notions of freedom, citizenship and solidarity to everybody.

By decree, on March, the 29th, 1911, the Portuguese Primary Education was structured in three levels of education (Ministério da Educação, 1986, pp. 4-5): Elementary (a mandatory level for 7 to 9/10 years old children, with a final exam), Complementary (10/11 to 12/13 years old pupils, with a final exam that would grant access to the next level) and Superior (13/14 to 15/16 years old students). Literature, Sciences, Arts and "Technics"[11] were the main topics for the Elementary level program. In Sciences, pupils should learn about Arithmetic (the basic arithmetic operations), the Metric System and Elementary Geometry as well as the, so called, Natural Sciences.

Within this framework, we found the textbook *Arimética, Sistema Métrico e Geometria – para as escolas primárias*[12]. It was officially published in 1912 and, judging by its many editions, it became quite popular: it was consequently adopted by many teachers and pupils at public and private primary schools.

Arimética, Sistema Métrico e Geometria – para as escolas primárias

Characterizing the textbook

The textbook *Arimética, Sistema Métrico e Geometria – para as escolas primárias*, (see Fig. 1) that we have analyzed was officially approved by decree in 1910 and our copy refers to the 17th edition of it, with illustrations, published in 1912.

The author is said to be a certified primary teacher, Ricardo Dinis Carvalho, who had worked as Inspector in Schools and is presented, right

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[10] This reform was, probably, one of the most complete reforms on Portuguese education – it included all levels of education, from Kindergarten to University and a Ministry of Public Instruction (1913) was established, for the first time, in Portugal.

[11] "Thecnics" included hygiene instruction, gymnastics, educative games and agricultural work.

[12] "Arithmetic, Metric System and Geometry – for primary schools". Such a title refers to a division, of the three topics of the divided mathematics curriculum. The term "Arimética" is no longer used in Portuguese language, the actual word is "Aritmética"

at the cover of the book, as an honorary member of "El Fomento de Las Artes de Madrid". The author also reported on having written the book according to Primary Education curricula and this means that it might be used at the three years composing the Elementary Level of Education/Schooling.

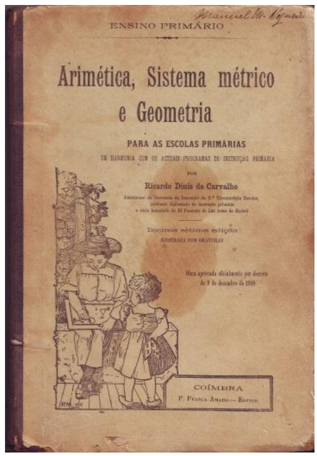

Fig. 1. The textbook cover

The textbook is divided in the three parts of the mathematics contents (for the three years) ordered by: Part I – Arithmetic, Part II – Metric System (where we will focus our attention) and Part III – Geometry.

Physically, this textbook is rather small and rather dense: about approximately 10 cm wide by 17 cm long with 135 pages. The topics, keeping in mind that the textbook was used for teaching/learning mathematics during 3 years, were necessarily presented in a quite condensed form.

Analyzing the textbook

The methodology to study the approach of the concept of measure and metric system presented in the textbook aimed at content analysis, taking into account: the concepts, the illustrations/representations and also phenomenological aspects.

In our study we will consider the mathematical structure associated to the concept of measure, examining relations between this concept and other mathematical concepts presented, implicitly or explicitly, in the text. At the same time, we are interested in phenomenological analysis of the concept. It is our intention to identify phenomena embedded in measure: the origins and situations related to both measure and the metric system. We will report on such situations that can be modelled by mathematical structures and associations between structures and phenomena (Freudenthal, 1983). We are also concerned about procedures/methodology and tasks related to the concepts presented in this textbook. Finally, acknowledging the relevance of illustrations and the representations of concepts and methods in mathematics, we will explore them (Botsmanova, 1972 and Santos-Bernard, 1996).

- *On content/concepts*

The author reflected upon several mathematical arguments/notions based on the concept of measure. However we did not find any axiomatic approach to his approach within the Metric System topic, but also in Arithmetic and/or Geometry. In this textbook one definition is, normally, given to each concept and the concept of measure is strongly embedded in the Arithmetic (Part I). In fact, in the section named as "Continuous and discontinuous magnitudes", the author explained that *to measure a magnitude is to compare it to another of the same type* (Carvalho, 1912, p.52). We may, in this case, foresee some of the concerns that, for many centuries, had highlighted one of Euclid[13] most difficult topics: "ratio" and "proportion" inasmuch as the concept of measure is specially connected to fractions, either in its definition or in its origins but goes rather beyond it, as Pythagorean themselves, have acknowledged when attempting to measure/compare the diagonal of a square from knowing its side or the circumference from the diameter. Our author wrote (Carvalho, 1912, p. 540)[14]:

> … these numbers have their origins in measuring magnitudes when the chosen unit doesn't fit into an entire number of times in the magnitude to be measured.
>
> Of course that this only happens with continuous magnitudes, because, discontinuous magnitudes are not really measured, but counted ...

In Part II – Metric System - different magnitudes, as well as their units, are introduced, explained and defined. For instance, the definition of a *meter* is based on the legal definition of 1889[15] (Carvalho, 1912, p.80):

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[13] We are thinking on the famous mathematical Euclid treaty - *The Elements* - and, in particular, on some of the oldest discussions concerning the, so called, arithmetical books

[14] *… estes números proveem da medição das grandezas, quando a unidade escolhida se não contêm um número exacto de vezes na grandeza que se quer medir.*

É claro que isto tem logar quando as grandezas são contínuas, pois que, quando elas são descontínuas, não se medem realmente, contam-se….

[15] *5. O metro é a unidade principal das medidas de comprimento. As suas dimensões são as duma barra de platina, depositada nos arquivos do Observatório de Paris.*

> 5. Meter is the principle unit of length. Its size is equal to the size of a platinum plate, deposited in the Observatory of Paris.

However, the author added a footnote referring to its previous definition, 1791, as being "one ten-millionth of a quarter meridian of Earth" and complemented the official (according to SI) metric system with applications to other units used, for example, in agriculture, itineraries, firewood, monetary and surveying systems.

Part III – Geometry - is related, in our analysis, to volume and arc dimensions. The concept of measure appears, in this instance, as a context to introduce new geometrical concepts or methods.

We may summarize these notions in Table 2:

Table 2. Concepts related with measure

Concept	Book Part	Definition	Distinctions
Magnitude	Arithmetic	*everything that can be increased and decreased* (Carvalho, 1912, p.3)	Continuous (*linked constituents, forming a whole*) and discontinuous (*composed by separate parts*)
Unit	Arithmetic	*a certain magnitude with which we can compare* - counting or measuring (Carvalho, 1912, p.4)	Natural (*determined by the nature of magnitude*); conventional (*defined by agreement*) and mixed (both, *determined by the nature of magnitude and by agreement*) (Carvalho, 1912,p. 52)
Number	Arithmetic	*the result of that comparison* (Carvalho, 1912, p.4)	
To measure	Arithmetic	*To measure a magnitude is to compare it to another of the same species.* (Carvalho, 1912,p. 52)	
Fraction	Arithmetic	*the number that consists of one or more equal parts of the unit* (Carvalho, 1912, p.53)	
Time[16] measure	Arithmetic		
Arc measure	Arithmetic		
Metric System	Metric System	… *all different weights and measure units based on meter* (Carvalho, 1912, p. 53)	

[16] Before time measure approach, the difference between *complexo* and *incomplexo* numbers is explained as follows: a *complexo* number is a number with different kinds of units, one of them is the main unit, and the other its submultiples (*for example: 8days 14 hours and 25 minutes*, Carvalho, 1910, p. 71); an *incomplexo* number is a number with one kind of unit (*for example: 8days, 14 hours, 25 minutes, etc,* Carvalho, 1910, p. 71).

Metric Magnitudes	Metric System		
Multiples and Submultiples	Metric System	Presented with Greek and Latin words	
Units of different magnitudes	Metric System		
Meter	Metric System	*Principal unit to measure length. The meter is defined as the length of a dense platinum-iridium alloy bar*. (Carvalho, 1912, p. 80)	
Measures for itineraries	Metric System		
Surfaces measures	Metric System	*… used on magnitudes with two dimensions – length and width* (Carvalho, 1912, p. 85)	
Surveying measurements	Metric System		
Volume measures	Metric System	*… used on magnitudes with three dimensions – length, width and height* (Carvalho, 1912, p. 89)	
Firewood measures	Metric System		*Ester – the principal unit of firewood measures* (Carvalho, 1912, p. 93)
Volume of a body	Geometry	*… is the portion of space that the body occupies* (Carvalho, 1912, p.112)	
Measuring an arc	Geometry	*Measure an arc is to find the number of degrees, minutes and seconds that it has* (Carvalho, 1912, p. 134)	

- *On procedures/methodology:*

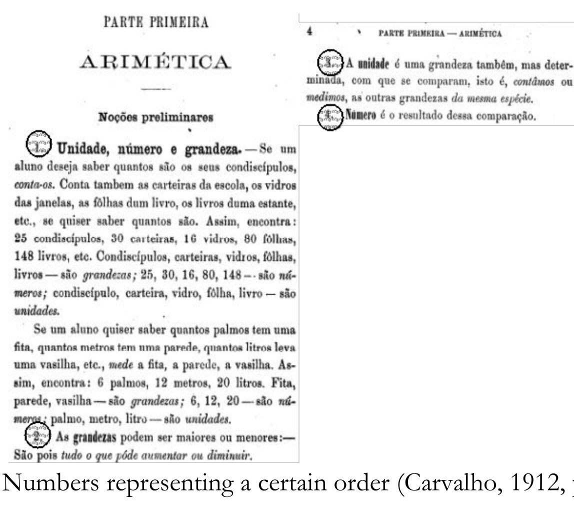
The author started each part of his textbook using a methodology which cannot be identified with the most traditional axiomatic approaches. He began with some preliminary notions, explained and gave examples with which pupils might be familiarized with, and only then the definitions of the concepts were introduced. In our opinion this may be seen as a remarkable approach where Carvalho, without neglecting scientific rigor, provided accurate definitions that seemed appropriated to the specific level of education. This methodology also presented mathematics as a solid structure: without having started by definitions or by axioms, the mathematical concepts had, nevertheless, been interiorized and always defined and introduced so that they related to each other, particularly to the previously defined ones. The aim seemed clear: to furnish pupils with practical knowledge, knowledge they could rely upon, knowledge they could use in real life, knowledge they could fully understand and without omitting mathematical rigor.

This textbook gave teachers very important hints about what was essential, as opposed to what was superfluous. But it gave them more than this piece of information; it also reflected deeply upon the methods, to be applied to pupils, to reach knowledge.

An interesting detail that, in our opinion, should be highlighted is the organization used by the author. He employed a certain order to present topics, using numbers that can be easily identified, at any moment, during the study (see Fig. 2). The reader would, therefore, be fully aware that, for example, the item with number 3 meant that 2 other items were treated (and should be studied/remembered) previously.

These numbers were not used to numbering paragraphs not even to numbering a definition or a proposition. In fact they seemed to indicate the order in which the contents/concepts should be either taught or learned.

We have referred to the use that teachers could have obtained from this textbook and it is now time to say something else, as opposed to what might be the approach nowadays: such a textbook seemed to be mainly designed for teachers but not to any teacher; it was assumed – because we are faced with such a condensed text – that the teacher should have a sound scientific and pedagogical knowledge (Shulman, 2004 and Ball, 2001) in order to make a proper use of it. Teachers were then, under these circumstances, free to adapt their teaching to each group of pupils and pupils should, on the other hand, rely upon their teachers for helping them to disclose every detail, every aspect, every concept, every result/property within the textbook.


PARTE PRIMEIRA

ARIMÉTICA

Noções preliminares

1. **Unidade, número e grandeza.** — Se um aluno deseja saber quantos são os seus condiscípulos, *conta-os.* Conta tambem as carteiras da escola, os vidros das janelas, as fôlhas dum livro, os livros duma estante, etc., se quiser saber quantos são. Assim, encontra: 25 condiscípulos, 30 carteiras, 16 vidros, 80 fôlhas, 148 livros, etc. Condiscípulos, carteiras, vidros, fôlhas, livros — são *grandezas;* 25, 30, 16, 80, 148 — são *números;* condiscípulo, carteira, vidro, fôlha, livro — são *unidades.*

Se um aluno quiser saber quantos palmos tem uma fita, quantos metros tem uma parede, quantos litros leva uma vasilha, etc., *mede* a fita, a parede, a vasilha. Assim, encontra: 6 palmos, 12 metros, 20 litros. Fita, parede, vasilha — são *grandezas;* 6, 12, 20 — são *números;* palmo, metro, litro — são *unidades.*

2. **As grandezas** podem ser maiores ou menores: — São pois *tudo o que póde aumentar ou diminuir.*

4 PARTE PRIMEIRA — ARIMÉTICA

3. **A unidade** é uma grandeza tambêm, mas determinada, com que se comparam, isto é, *contâmos* ou *medimos*, as outras grandezas *da mesma espécie.*

4. **Número** é o resultado dessa comparação.

Fig. 2. Numbers representing a certain order (Carvalho, 1912, pp. 3-4)

Carvalho described how units, numbers and magnitudes should be approached and are related to each other. The author explained, in a clear and rigorous (let us pay attention, for instance, to the use of a "if… then…" argumentation) vernacular language, the difference between counting and measuring. He used specific examples, certainly familiar to every pupil (Carvalho, 1912, p. 3)[17]:

> 1. Unit, number and magnitude - If a pupil wants to know how many colleagues there are in his classroom, he counts them. He also counts desks, windows, pages of a book, books on a shelf, etc., if he wishes to know how many there are. Then, he will find: 25 colleagues, 30 desks, 16 windows, 80 pages, 148 books. Colleagues, desks, windows, pages and books – are magnitudes; 25, 30, 16, 80 and 148 – are numbers; Colleague, desk, window, page and book – are units.
>
> If a pupil wants to know how many palms does a tape have, how many meters does a wall have, how many litters does a bottle take, … he measures the tape, the wall, the bottle. Then, he will find: 6 palms, 12 meters, 20 litters. Tape, hall, bottle – are magnitudes; 6, 12, 20 – are numbers; palm, meter, litter – are units.

Magnitude, *unit* and *number* were then defined, as it was showed in our Table 2. Using the natural relation between number, counting and measure, the author explained and distinguished (rational positive) numbers as divided in: *integer numbers*, *fractions* and *mixed numbers*. Each kind of numbers were introduced, as shown below, by means of representation of segments, see Fig. 3.

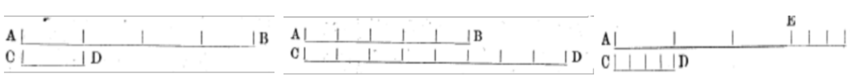


Fig. 3. From left to right: Integer numbers, fraction and mixed numbers. (Carvalho, 1912, pp. 53-54)

At Part I – Arithmetic, the author explained unit conversions using examples (see Fig. 4). Once again we can find the connection between Arithmetic knowledge and the concept of measure, engaging pupils with mathematical structure.

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[17] *Unidade, número e grandeza. – Se um aluno deseja saber quantos são os seus condiscípulos, conta-os. Conta também as carteiras da escola, os vidros das janelas, as fôlhas dum livro, os livros de uma estante, etc., se quiser saber quantos são. Assim, encontra: 25 condiscíplos, 30 carteiras, 16 vidros, 80 fôlhas, 148 livros, etc. Condiscíplos, carteiras, vidros, fôlhas, livros – são grandezas; 25, 30, 16, 80, 148 – são números; Condiscíplo, carteira, vidro, fôlha, livro – são unidades. Se um aluno quiser saber quantos palmos tem uma fita, quantos metros tem uma parede, quantos litros leva uma vasilha, etc., mede a fita, a parede, a vasilha. Assim, encontra: 6 palmos, 12 metros, 20 litros. Fita, parede, vasilha – são grandezas; 6, 12, 20 – são números; palmo, metro, litro – são unidades.*

The author started Part II - Metric System, by: (1) explaining what a measure system should be and how it should be used; reporting on the need for different units, multiples and submultiples for each magnitude and on principal units; justifying that the metric decimal system satisfies all these requirements[18]; (2) presenting metric magnitudes and each principal units as well as their use; (3) informing about the multiples and submultiples in Latin and Greek, see Fig. 5.

It seems interesting to refer that the author presented the principal units in Latin and Greek, explaining why do we use the same prefix and suffix with different magnitudes, which seems to reveal Carvalho's importance attributed to word etymology.

With examples, that should be tested by pupils, Carvalho explained the need to choose the correct unit as well as when and if we should use multiples and submultiples. After this introduction, the author presented the different magnitudes (length, area, volume, capacity, weight) and measures (firewood, itinerary, agriculture, surveyor, money[19]) defining the principal unit, multiples and submultiples, explaining and exemplifying how to convert one measure to another, introducing measuring instruments and explaining their proper use with practical examples and proposing *problemas*[20].

The author presented two proofs by construction[21]. One, at Surfaces Measures topic, aims at proving that $1m^2=100dm^2$ and the other, at Volume Measures topic, proves that $1m^3=1000dm^3$ (see Fig. 6).

The exploration and interpretation of these two pictures might be aimed at helping teachers to help their pupils on understanding the relationship between area and volume serving as a step to later introduction of formulas/algebraic argumentation.

Different procedures on several aspects are presented in the analyzed textbook; we may, for each magnitude, find appropriate instruments being introduced but not only that: their use is even shown with some practical examples. It is very interesting to realize that, unlike our actual textbooks, pupils of those days were supposed to be familiarized with several measure instruments: a ruler, a folding meter, a survey's chain[22] or a tape measure or several balances such as Roman, Roberval and Decimal.

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[18] The author explained why the name for the, so-called, metric decimal system.

[19] Nowadays, only money units are part of the Mathematics Program at Portuguese schools whereas agricultural units are named as area units.

[20] The author uses the word *problemas* (problems) as a synonym to exercises.

[21] P*roof by construction* proves the existence of a mathematical object with certain properties by developing a method for creating such an object.

[22] Survey's chain remembers the rope with knots used by the Egyptians.

81. **Redução de um número incomplexo a complexo.** — Divide-se o número dado pela sua relação com a unidade imediatamente superior, e depois o quociente que resultar divide-se pela relação seguinte, e assim por diante; o último quociente e os restos sucessivos formam o número complexo.

Exemplo 1.º *Reduzir 8775 minutos a dias:*

$$\begin{array}{l|l} 8775^m & 60 \\ 277 & 146^h \mid 24 \\ \;375 & \quad 2^h \mid 6^d \\ \;\;15^m & \end{array}$$

Logo: $8775^m = 6^d\, 2^h\, 15^m$.

Exemplo 2.º *Seja o número incomplexo* $\frac{5}{7}$ *da circunferência para reduzir a complexo:*

$$\begin{array}{r|l} 5 & 7 \\ \times 360^\circ & 257^\circ\, 8'\, 34'' \\ \hline 1800^\circ: & \\ 40 & \\ 50 & \\ 1 & \\ \times 60' & \\ \hline 60': & \\ 4 & \\ \times 60'' & \\ \hline 240'': & \\ 30 & \\ 2'' & \end{array}$$

Logo, abstraindo do resto, será $\frac{5}{7}$ da circunferência $=$

$= 257^\circ\, 8'\, 34''$.

82. **Redução de um complexo a fracção e desta a número decimal.** — Reduz-se o número complexo á ínfima espécie e divide-se o número obtido pelo número de vezes que a unidade a que êle se quer reduzir contêm a da última espécie e depois converte-se em número decimal pelo processo conhecido (n.º 76).

Exemplo 1.º *Reduzir* $12^d\, 9^h\, 5^m\, 7^s$ *a fracção decimal da hora:*

$$\begin{array}{r} 12 \\ \times 24 \\ \hline 48 \\ 24\; \\ \hline 288^h \\ +9 \\ \hline 297^h \\ \times 60 \\ \hline 17820^m \\ +5 \\ \hline 17825^m \\ \times 60 \\ \hline 1069500^s \\ +7 \\ \hline 1069507^s \end{array}$$

E, como a hora tem 3600^s, será $12^d\, 9^h\, 5^m\, 7^s =$

$$= \frac{1069507^s}{3600} =$$

$= 297^h{,}085$.

Fig. 4. Conversion of an "incomplex" number to a "complex" number and conversion of a "complex" number to a fraction and, then, to a decimal number. There is a reference to a previous known procedure (nº 76) (Cavalho, 1912, pp. 74-75).

Gregos	*Deca*	que significa	dez..........	10
	Hecto	» »	cem..........	100
	Quilo	» »	mil..........	1000
	Míria	» «	dez mil........	10000

Latinos	*Deci*	que significa	a décima parte...	0,1
	Centi	» »	a centésima parte.	0,01
	Mili	» »	a milésima parte..	0,001

Fig. 5. Latin and Greek words for multiples and submultiples (Carvalho, 1912, pp. 78-79)

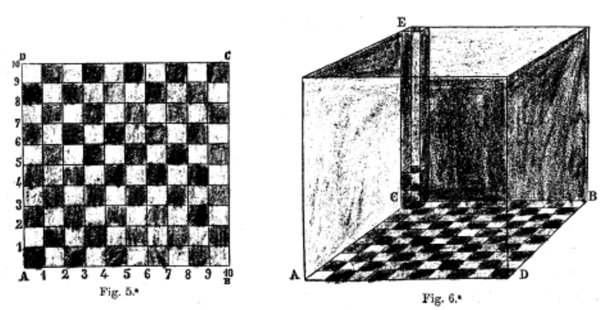


Fig. 6. Relation between area and volume

Portuguese monetary system was also taught with every paper money and every coin being identified. Some problematic situations were presented to introduce a certain topic together with pictorial representations and, usually, the end of each chapter is composed by some *problemas*.

- *On illustrations/representations*

By illustration/representation we mean drawings, photographs, images, diagrams, graphs or historical documents. Assuming that a visual image "is an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein, 1987, p.101), in our study we have considered not only the nature of illustrations, but also the link between the mathematical concept under scrutiny and the type of illustration used in the textbook. In the work of Botsmanova (1972) we have seen the nature of illustrations being categorized as:

- Object-illustrative pictures which illustrate mathematical objects, but not mathematical structure;
- Object-analytical images which reveal the mathematical structure;
- Abstract spatial diagrams reflecting an abstract numerical relationship.

In addition, regarding illustrations related to an exercise, problem or task we have considered Santos-Bernard work (1996), distinguishing illustrations as relevant or merely cosmetic:

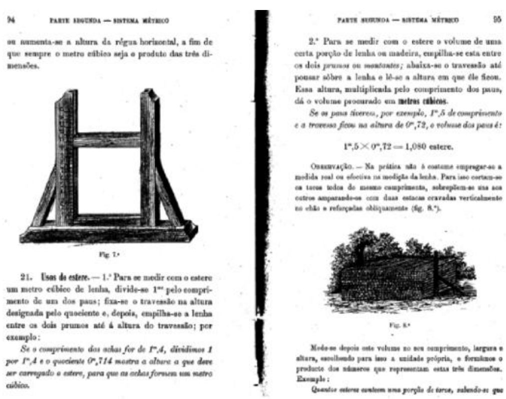
> The relevant illustrations give information in order to accomplish the task. This information may be only given on the illustration or may be the same information given in the text and repeated in the illustration.
>
> The cosmetic illustrations are the ones that give a general context of the task or their purpose is to decorate the page. These types of illustrations are characteristic, in that they do not give any information for answering or accomplishing the task. (Santos-Bernard, 1996, p. 252)

This one hundred years old mathematics textbook used mainly text and numeric representations to explaining content and relied, naturally, on much less sophisticated techniques in what book production standards have at present. However, as it is stressed at the very front-page of the textbook, we may find several illustrations throughout this didactical resource.

Most of the illustrations related to measure, either in the Metric System Part or the Arithmetic Part, had a clear purpose, namely to complement the text and to help understanding the mathematical contents.

The author used words to explain procedures and methods (mathematical or of measurement), to describe instruments and to describe how they are used. Nevertheless, in most cases, illustrations after these explanations surely helped to visualize instruments and/or to understand procedures (see Figures 7 to 10).

One can, for example, find in Part III– Geometry, a picture of a protractor complementing the explanation on how to measure the amplitude of angles. These illustrations, object oriented, could have suppressed the lack of didactical resources in schools (the lack of didactic resources was evident, according to the results of School Inspections between 1863 and 1864). Although one image does not replace the object, it provided useful information to the pupils about the various instruments which could be found in Portugal.



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ou aumenta-se a altura da régua horizontal, a fim de que sempre o metro cúbico seja o produto das três dimensões.

Fig. 7.ª

21. **Usos do estere.** — 1.º Para se medir com o estere um metro cúbico de lenha, divide-se 1^m^ pelo comprimento de um dos paus; fixa-se o travessão na altura designada pelo quociente e, depois, empilha-se a lenha entre os dois prumos até à altura do travessão; por exemplo:

Se o comprimento das achas for de 1^m^,4, dividimos 1 por 1^m^,4 e o quociente 0^m^,714 mostra a altura a que deve ser carregado o estere, para que as achas formem um metro cúbico.

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2.º Para se medir com o estere o volume de uma certa porção de lenha ou madeira, empilha-se esta entre os dois *prumos* ou *montantes*; abaixa-se o travessão até pousar sôbre a lenha e lê-se a altura em que êle ficou. Essa altura, multiplicada pelo comprimento dos paus, dá o volume procurado em **metros cúbicos**.

Se os paus tiverem, por exemplo, 1^m^,5 de comprimento e a travessa ficou na altura de 0^m^,72, o volume dos paus é:

1^m^,5 × 0^m^,72 = 1,080 estere.

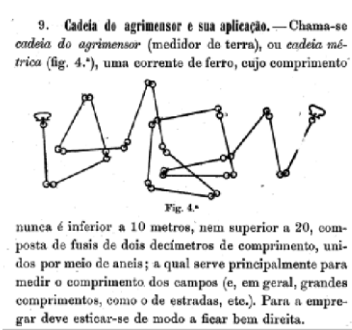
OBSERVAÇÃO. — Na prática não é costume empregar-se a medida real ou efectiva na medição da lenha. Para isso cortam-se os tocos todos do mesmo comprimento, sobrepõem-se uns aos outros amparando-os com duas estacas cravadas verticalmente no chão e reforçadas obliquamente (fig. 8.ª).

Fig. 8.ª

Mede-se depois este volume no seu comprimento, largura e altura, escolhendo para isso a unidade própria, e formámos o producto dos números que representam estas três dimensões. Exemplo:

Quantos esteres contem uma porção de [illegible], sabendo-se que

Fig. 7. From left to right: Illustration helping pupils to visualize estere (the instrument to measure firewood, and, at the same time, the unit used to measure); illustration describing how to use estere. (Carvalho, 1912, pp. 94-95)



9. **Cadeia de agrimensor e sua aplicação.** — Chama-se *cadeia do agrimensor* (medidor de terra), ou *cadeia métrica* (fig. 4.ª), uma corrente de ferro, cujo comprimento

Fig. 4.ª

nunca é inferior a 10 metros, nem superior a 20, composta de fusis de dois decímetros de comprimento, unidos por meio de aneis; a qual serve principalmente para medir o comprimento dos campos (e, em geral, grandes comprimentos, como o de estradas, etc.). Para a empregar deve esticar-se de modo a ficar bem direita.

Fig. 8. Surveyor's chain description and visualization, it is also explained its proper use. (Carvalho, 1912, p. 83)

In Fig. 6, an object-analytical image could also be used for helping readers to understand proof, emphasizing mathematical structure and involving pupils with both arithmetic and geometric concepts: divisions, fractions, multiples, powers and other sort of numbers became associated with segments, perimeters, lengths, widths, areas or volumes in a way that could develop the so called "spatial visualization" as well as could emphasize the 2D representation of 3D objects.

One also finds many other illustrations that referred to official documents: on the Portuguese monetary system (Figures 11 and 12) readers would not only learn about the absolute and relative values of the money/coins circulating in Portugal but would gather important complementary knowledge on other "measures"; for example: on the kind of metal used (metal leagues was, for very early stages of Portuguese arithmetic treaties, in 16th century, a crucial theme to be reported/taught), on linear measurements made, in particular, with very specific terminology for the money context (diameters, thickness), or even weights, in a truly interdisciplinary mathematical activity that went beyond informative material shown in a simple table.

Fig. 9. Pictures of iron weights to measure weight between 50kg and 50gr (left) and weights of less than 1gr – made by yellow metal – (right). (Carvalho, 1912, pp. 103-104)

Fig. 10. Balances' illustrations after its description. Decimal balance with other name: railway scale (it was used at train stations to measure luggage and packages weight) (Carvalho, 1912, p. 106)

Moedas portuguesas que andam em circulação

Espécie de metal	Denominação da moeda	Valor em réis	Diâmetro em milímetros	Espessura em milímetros	Pêso em gramas
Ouro	Coroa	10$000	30	2,0	17,735
	Meia coroa	5$000	23	1,3	8,868
	Quinto de coroa	2$000	18,5	0,8	3,547
	Décimo de coroa	1$000	14	0,7	1,774
Prata	Dez tostões	1$000	37	2,5	25,000
	Cinco tostões	500	30	2,0	12.500
	Dois tostões	200	23	1,3	5,000
Níquel	Um tostão	100	22,0	—	4,000
	Meio tostão	50	18,0	—	2,500
Bronze	Vintêm	20	30,0	—	12,000
	Dez réis	10	25,0	—	6,000
	Cinco réis	5	20,0	—	3,000

Fig. 11. Coins in circulation in the beginning of 20th century (Carvalho, 1912, p. 110)

Sistema monetário português decretado pelo govêrno da Républica

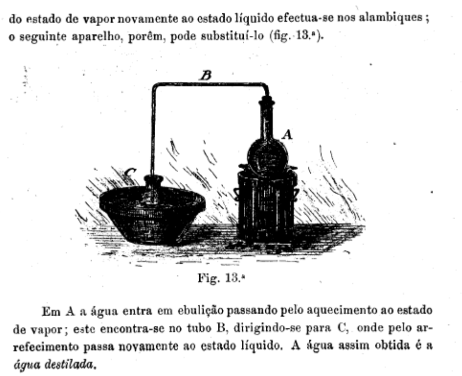
Designação das moedas de ouro	Equivalencias no actual sistema — Réis	Diametros — Milimetros	Toque: Toque legal — Milésimos	Toque: Tolerancia — Milésimos	Pesos: Pêso legal — Gramas	Pesos: Tolerancia de fabrico — Milésimos	Tolerancia para o desgaste abaixo da tolerancia mínima do fabrico — Milésimos
10 escudos	10$000	30	900	±2	18,0650	±2	5
5 escudos	5$000	24			9,0325		
2 escudos	2$000	19			3,6130		
1 escudo	1$000	15			1,8065		

Fig. 12. Equivalence, in real unit, of the gold coins and their characteristics. (Carvalho, 1912, p. 111)

Several algorithms were also presented to solve an exercise or a problem or yet to explain certain methods. In Fig. 4 one may see two of these cases. We could consider that this was an object-analytical illustration, revealing the mathematical structure related with reductions but, simultaneously, engaging pupils in mathematical procedures and reasoning.

The author's didactic and scientific concerns were such that the reader could even find illustrations introduced in footnotes. For example, Carvalho (1912) explained the phenomenon to obtain distilled water (see Fig. 13), when gram is defined[23].

[23] The gram is the weight of a cubic centimeter of distilled water. (p. 101)



do estado de vapor novamente ao estado líquido efectua-se nos alambiques ; o seguinte aparelho, porêm, pode substituí-lo (fig. 13.ª).

Fig. 13.ª

Em A a água entra em ebulição passando pelo aquecimento ao estado de vapor; este encontra-se no tubo B, dirigindo-se para C, onde pelo arrefecimento passa novamente ao estado líquido. A água assim obtida é a *água destilada*.

Fig. 13. Footnote explaining the phenomenon to obtain distilled water. (Carvalho, 1910, p. 101)

Finally on the relationship established between illustrations and tasks (examples, exercises and problems used to introduce certain concepts in Part I), one would definitely characterize most of them as being relevant and it seemed difficult (none related to the concept of measure) to name as "cosmetically guided" any of the illustrations and representation in the textbook. Those were times where, perhaps, it was not admissible to spend resources on coloring, on substituting texts by figures, confusing definitions with representations and general with particular. These were times where everything in the textbook seems to be a fundamental piece of the whole jigsaw which teachers had to compose in order to make sense within each pupil's head.

- *On phenomenology*

Our author provided a large variety of real life – sciences, commercial, agricultural and economic contexts – spread in examples, exercises and/or problems. His main concern seemed to have been providing real practical situations.

Carvalho offered, within the theme of measure and metric system, fruitful examples of the use of the decimal systems but it is worth reporting on the notation used to symbolize units, submultiples and multiples of the meter much different from what is used in textbooks of our days but showing the development of our notation: $1^{m}\times 0^{m},72=1,080$*estere* (or $1^{est},080$).

This notation is related to its use in everyday life, in different areas (agriculture, fire measure, surveying, monetary system, trade, among others) and to the metric system internationalization. However, the use of practical situations does not neglect the mathematical rigor. It seems evident that the author's concern might have been to prepare Portuguese pupils with the necessary mathematical skills to deal with practical situations. He established relationships among concepts, not only in mathematics but, above all, in contexts of the reality lived and experienced by pupils. Even so, there were no references to estimation in measurement.

This appears to us as being closely connected with Freudenthal´s (1983, p. ix) phenomenological perspective (…) *of a mathematical concept,* inasmuch as

> structure, or idea means describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind, and as far as this description is the concerned with the learning process of the young generation, it is didactical phenomenology, a way to show the teacher the places where the learner might step into the learning process of mankind.

Final considerations

If there was still any doubt about measurement being at the very soul of mathematics itself this textbook offered us an important opportunity to clarify such doubts or, as it was our case, to systematize such importance from very early stages of learning mathematics. After the analysis made to this "old" textbook, we gathered several hints to understand our mathematical concept of measurement as being both elementary and fundamental.

The evident phenomenology in examples and exercises that were chosen, with practical applications both to the school context and to society, in general, the author got to the point of showing that measure and their metric systems were, in fact, inherent to human activity.

The organization of the content under analysis (measure and metric system) in this book is also, in our opinion, quite remarkable. It revealed some innovating and actual approaches, considering the chosen methodology: definitions and representations seemed clearly intertwined, theory and practice side by side, intuition and rigor both parts of the same knowledge construction to be achieved. The author opted by beginning each topic with precise preliminary notions, providing examples or problems to explain the use and the applications of the mathematical concept of measure; he also established connections between concepts and among different mathematical topics, (for example arithmetic and geometry). Mathematical knowledge was presented, in the analyzed textbook, as a network of concepts, definitions, proofs, algorithms, rules, theories interrelated, but also related with historical data and reality (in its many aspects). One might, nevertheless, feel that the author did not go as far as he might have gone, for example, in avoiding the "incommensurability". However let us keep in mind that, this textbook contains the content for the first three years of schooling and that the so called irrational numbers (surely related to measure, surely worthwhile being tackled and, surely of non-obvious treatment) were not, by then as it is not nowadays, an "elementary" concept. Let us also reaffirm that we, in our research, have been focusing our attention on the so called elementary mathematical concepts, the ones that are first taught/learnt by pupils in schools.

Assuming that a textbook of Elementary Mathematics, is both a resource to support the pupil's process of learning, and a vehicle for mathematical culture (including its history, its relationships with other sciences and with society), it seems clear that fundamental concepts and adequate methods of mathematical education – having a consistent content sequence, promoting logical thinking and integrating mathematical knowledge in many and various human activities – as they are presented in this textbook are a model to be referred to. The author, by writing such textbook, has, in our opinion, definitely accomplished his mission of promoting information and formation to pupils and to teachers.

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Juan de Icíar's *Practical Arithmetic* (1549): Writing and reckoning in Spanish Renaissance

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Abstract

Juan de Icíar (b. 1522 or 1523), the most important calligrapher during the Spanish Renaissance, is also the author of a purely mathematical book, Book titled **practical arithmetic** very useful for anyone willing to be trained in reckoning *(Icíar, 1549). It is a mercantile arithmetic conceived for educational purposes, an essential book to learn the mathematical skills and the teaching thereof in Spain, in the mid-sixteenth century.*

Icíar's reputation as a calligrapher stems from his Orthographia pratica*, published in Zaragoza in 1548, which was republished ten times between 1550 and 1596. This work was titled* Recopilación subtilissima, intitulada Ortographia pratica… *in 1548, and* Arte subtilissima por la qual se enseña a escreuir perfectamente *between 1550 and 1555, but changed to* Libro Subtilissimo, por el qual se enseña a **escreuir y contar** perfectamente el qual lleua el mesmo orden que lleua vn maestro con su discipulo *(Subtlest Book, that teaches how to* ***write and reckon*** *perfectly in the same order as a teacher does with his student) from 1559 onwards, when it was sold bound with* Arte breue y prouechoso de cuenta Castellana y Arithmetica, donde se muestran las cinco reglas de guarismo por la cuenta castellana, y reglas de memoria. Y agora nueuamẽte en esta postrera impression se han añadido vnas cuentas muy graciosas y prouechosas, sacadas del libro de Fray Iuan de Ortega: y mas al cabo va añadida vna cuenta abreuiada de marauedis *(Brief and Useful Art of Castilian reckoning and arithmetic, where the five rules of algorism in Castilian reckoning are shown, and memory rules. And now in this last edition very amusing and useful calculations, from Fray Iuan de Ortega book, have been added, and at the end an abridged accounting in maravedis), a work similar to the one originally published by father Juan Gutiérrez in 1539.*

This paper studies the mathematical work of Juan de Icíar within the context of the education of the new bourgeoisie (merchants, liberal professions) during the Spanish Renaissance, and establishes the relationship between Practical Arithmetic, Libro Subtilissimo, *and* Arte Breue y Prouechoso.

Introduction: Commercial arithmetic in Spain

Commercial arithmetics in Spain date back to the 14th century (Caunedo & Córdoba 2000; Caunedo, 2007; Docampo Rey, 2004; Docampo Rey 2006a; Docampo Rey 2006b) and were first printed in the 15th century. Actually, the first printed book on mathematics in Spain was probably the second on the subject to be printed in world history: Francesc Santcliment's *Summa de l'art d'Aritmètica*, first published in Catalan (Barcelona, 1482), and later in Spanish (Zaragoza, ca. 1487)

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education.* Caparica: UIED.

(Malet, 1998). Nevertheless, this kind of literature flourished in the sixteenth century, when the volume of internal and external trade increased. Although Spain was formally a unified state since 1474, Castile and Aragon kept different laws and hence coins, weights and measures, also within the different reigns that constituted both Crowns (such as Navarre in Castile or Valencia in Aragon). Under these circumstances, not only merchants but also liberal professions were progressively able to appreciate the advantages of the new arithmetic based on Arabic numerals, and to become aware of the importance of assimilating it as part of their education.

Seven mercantile arithmetics were published in Spain before the first Arithmetic including Algebra was printed in Valencia in 1552, namely the *Libro primero de Arithmetica Algebratica, en el qual se contiene el arte Mercantiuol con otras muchas reglas del Arte menor, y la Regla del Algebra vulgarmente llamada de Arte Mayor*, by the German Marco Aurel. They coexisted with speculative arithmetics in Latin published by Spanish academic authors such as Pedro Ciruelo (*Tractatus arithmeticae practice,* Paris, 1495; *Cursus quatttuor mathematicarum artium liberalium*, Alcalá, 1516), Gaspar Lax de Sariñena (*Arithmetica speculativa*, Paris, 1515), and Cardinal Juan Martínez Silíceo (*Ars Arithmetica in Theoricem et Praxim scissa: omni hominum conditioni superque utilis et necessaria,* Paris, 1514, 1518, 1519, 1526; Valencia, 1544; Sánchez Salor & Cobos Bueno, 1996), and also with commodity trading manuals such as the Books of *Almutaçafes* (inspectors of weights and measures) by Adrián de Aynsa (1510, 1577, 1595) and Pascual de Abensalero (1609) (Ausejo *et al.,* 2001).

Three of these seven arithmetics were authored by clerics, which points to the fact that the Catholic church started to take care of this new type of non-university education: the Franciscan friar Juan Andrés, from Zaragoza, published *Sumario breue d'la pratica dela arithmetica de todo el curso de larte mercãtiuol bien declarado, el qual se llama Maestro de cuento* in Valencia (1515), where the first printed reference to Luca Pacioli in Spain is to be found (Caunedo, 2009); the Dominican friar Juan de Ortega, from Palencia (Castile), published *Conpusición de la arte de la Arismética y de Geometría* in Lyon (1512). In 1539 Father Juan Gutiérrez de Gualda, a priest from the small town of Villarejo de Fuentes (Castile), published *Arte breue y muy prouechoso de cuẽta castellana y arismetica, dõde se muestran las cinco reglas de guarismo por la cuẽta castellana, y reglas de memoria* in Toledo (Gutiérrez, 1539).

Santcliment was a teacher (Malet, 1998, pp. 28-29). Joan Ventallol, from Majorca, was probably a scrivener (Mesquida Cantallops, 1996, p. 20). In 1521 he published at his expense *Practica mercantiuol,* in Catalan, in Lyon. In 1546 Gaspar de Texeda, a scrivener from Zaragoza

(Protocolo de Bartolomé Malo, 1547, ff. 325 / 326 v.), published *Suma de arithmetica pratica y de todas mercaderías* in Valladolid (Castile), the first arithmetic containing book keeping printed in Spain (González Ferrando, 1956). Finally Icíar, certainly a scrivener and teacher in Zaragoza, closes the series of non-algebraic arithmetics of the first half of the 16th century with his *Practical Arithmetic* (Icíar, 1549).

Ortega's work was the most circulated of these books, as it was published in French (Lyon, 1515, the first mercantile arithmetic printed in France), in Italian (Roma, 1515; Messina, 1522), and in Spanish as *Tratado subtilissimo de arismetica y de geometria* (Sevilla, 1534, 1537, 1542, 1552; Granada, 1563). Furthermore, Ortega is considered as the most creative author of the period from the point of view of mathematics, especially due to his method for finding roots (Malet & Paradís, 1984, Vol. 1, pp. 124-127).

In this context, Gutiérrez and Icíar's works have been overlooked, the first for being "elementary", the second because it was a really rare book (Echegaray, 1908, p. 140) until the beginning of this century –only one single copy at the British Library (Shelf mark C.62.h.5.) was known until 2005–. The confusion between both works appears in several authors (Cotarelo y Mori, 2004, p. 366; Alonso García, 1953, p. 41; Usón Villalba, 2004, pp. 458-465 and 470; Martínez Pereira & Torné, 2007, p. 60).

In this paper, we will present Icíar's Arithmetic contents[1] and show his relationship with Gutiérrez's arithmetic within the framework of the non-university education of the incipient bourgeoisie in Spain, a country where *botteghe dell'abaco* in the Italian style have yet to be documented.

Juan de Icíar: Life and work

All we know about Juan de Icíar is that he was born in Durango, Biscay (Basque Country, Castile at that time) in 1522 or 1523 –depending on the dating of the engraving of his portrait first published in his *Orthographia pratica*, 1548, which says he was then 25 years old[2]. He moved to Zaragoza by the end of 1546, apparently due to some kind of personal disgrace –according to the dedication of his Arithmetic (Icíar, 1549, f. ii), where he settled as *scriptor de libros* and teacher. We know of several of his apprenticeship contracts and one contract as private tutor for training in reading and writing. In 1554 he was married to Catalina Carrión (Pedraza, 2007, pp. 17-20 and 23; San Vicente, 1969). There is documentary

[1] The detailed study and edition is work in progress (Spain's Ministry of Economy and Competitivity, formerly MICINN, Research Project HAR2010-15457).

[2] The woodblocks for this edition started to be prepared in 1547 (Pedraza, 2007, pp. 20-25).

evidence of his activity in Zaragoza until 1559 (San Vicente, 1969; Icíar, 1569). In 1773 he retired to Logroño after being ordained (Cotarelo y Mori, 2004, p. 351).

Figure 1. Portrait of Juan de Icíar in Arithmetica Practica (1549)

He also started, together with engraver Juan de Vingles, to prepare the woodblocks for the edition of his *Orthographia pratica* (1548), according to his deep knowledge –probably acquired in Italy– of the works by Vicentino, Tagliente and Palatino (Cotarelo y Mori, 2004, p. 351). On 5 June 1547, they both formed a business partnership with the tallow-chandler Alonso Fraylla to publish the first edition of the book (1548), whose success led to publish a second enlarged and improved edition titled *Arte subtilissima por la qual se enseña a escreuir perfectamente* (1550), actually the best edition (Pedraza, 2007, pp. 20-25).

This venture was possible due to the combination of Fraylla's innovative entrepreneurship with Icíar's and Vingles's expertise –as calligrapher and engraver respectively– within the context of a thriving printing industry: this second edition was printed by Jorge Cocci's heir, Pedro Bernuz, reputed as the best printer of the time in Aragon, which also played a part in the final success of this typical Renaissance project. As a result, Icíar won reputation and recognition as master of calligraphers in Spain. He was even called by King Philip II to work at El Escorial as scriptor and private tutor to the crown prince (Alonso García, 1953, pp. 21-22; Cotarelo y Mori, 2004, p. 351).

Juan de Icíar's *Arithmetica Practica*

The relationship between Icíar and Bernuz started before the publication of *Arte subtilissima* in 1550, as he printed Icíar's *Practical Arithmetic* in 1549. This work was published by Icíar and the bookseller

Miguel de Suelves, pseudonym Çapila, who was, from 1553 onwards, Icíar's publisher for the rest of his life.

Icíar sold his part of the edition to the merchants Juan and Antón Alberite for 100 gold ducats, to be paid in case the book was reprinted under the same title anywhere in Spain and abroad (Pedraza, 2007, pp. 31-32), which accounts for the work never being published again. Furthermore, the book was printed in folio, whereas *Arte subtilissima* was printed in quarto, so that when Suelves thought of exploiting both Icíar's works together, there was no possibility to use Icíar's *Practical Arithmetic* as such.

In educational media, a book in folio suggests a work to be read, assimilated, consulted, and preserved, more a teacher's book or a work to be kept by students under private tuition than a handbook for students attending a school. This *Practical Arithmetic* (Icíar, 1549) was presented as *agora nuevamente hecho por Juan de Yciar Vizcayno (now newly made by Juan de Yciar from Vizcaya),* which could refer either to the originality of the work or to a prior edition.

Icíar's *Practical Arithmetic* starts with a dedication that refers to Pythagoras and Plato in order to justify mathematics –especially Arithmetic, but also Geometry– as the base and foundation of knowledge, since it enables to know the account, number, weight and measure of everything, but warns of the impossibility to learn and understand mathematics without a good teacher (Icíar, 1549, f. ii).

Figure 2. Title page of Juan de Icíar's Arithmetica Practica (1549)

The book is divided in two parts. The first one, in nine chapters (Icíar, 1549, ff. I-XI) –not fifteen, as stated in (Cotarelo y Mori, 2004, p. 367), is devoted to Arabic numerals, place values, fundamental operations (addition, subtraction, multiplication, division) and checks by their inverse operation.

The second part in the table of contents is actually divided in 5 parts in text, with 15 chapters in total devoted to series (3 chapters, Icíar, 1549, ff. XI-XV), rule of three (2 chapters, Icíar, 1549, ff. XV-XXI), and rule of fellowship (2 chapters, Icíar, 1549, ff. XXI-XXVIII).

At this point, he dismisses the rules of barter for being elementary, the rules of testaments for being similar to the rule of fellowship, and the rules of alloying (for gold and silver) for being too specific in a book addressed to the merchants in general; instead, he offers a series of fifteen recipes (in spite of announcing 17 in the table of contents) to calculate the profit of loans in various periods of time (Icíar, 1549, ff. XXVIII-XXXII).

He goes on with seven chapters on fractions (Icíar, 1549, ff. XXXII-XXXVIII), the rule of single false position (Icíar, 1549, ff. XXXVIII-XLI), square roots (Icíar, 1549, ff. XLI-XLIII), *square rules* (exercises with weights, measures, widths and lengths, Icíar, 1549, ff. XLIII-XLVI), lengths, weights, and coins of several Italian, French, British, and Spanish territories (Icíar, 1549, ff. XLVI-XLVII), fundamental operations with square roots (Icíar, 1549, ff. XLVIII-L), and a final chapter where he discusses the reckoning teaching methods practiced in Aragon, and gives a two folio table of submultiples of all units of currency, weights and measures in Aragon as fractions of the upper unit –for instance from 1 *dinero* to 12 *dineros* as fractions of 1 *sueldo*– to be learnt by heart in order to learn to directly multiply any combination of units without reducing before and after operating (Icíar, 1549, ff. L-LVI).

The book is high-quality printed, really beautifully illustrated in the chapter devoted to *square rules*, and has plenty of examples –fifteen to twenty in several parts–, but most remarkable is Icíar's educational vocation, both in structure and contents. The detailed explanations, the order of subjects, the combination of "theory" and practice, and the choice of contents depending on the audience he intends to reach, together with his precise references to Pellos and Ortega (Icíar, 1549, ff. LXIII) show his solid education in mercantile arithmetic. His *Practical Arithmetic* is essentially original –as far as practical arithmetics can be original in contents– and, contrary to what has been said (Salavert Fabiani, 1990, p. 76; Martínez Pereira, 2006, p. 364), it is clearly and substantially different from Gutiérrez's arithmetic, as it will be shown in the next section of this paper.

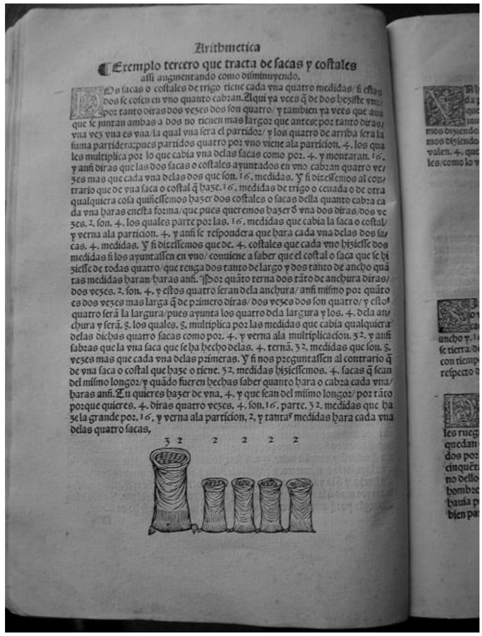
Arithmetica
Exemplo tercero que tracta de sacas y costales assi augmentando como diminuyendo.

Figure 3. Juan de Icíar's *Arithmetica Practica* (1549), f. XLIII v.

Icíar, Suelves and Juan Gutiérrez: *Arte subtilissima por la qual se enseña a escreuir perfectamente* and *Arte breue y prouechoso de cuẽta castellana y arithmetica*

Unlike what happened with *Arithmetica Practica*, Suelves published a third edition of *Arte subtilissima por la qual se enseña a escreuir perfectamente* in 1553 and a fourth in 1555 (Icíar, 1555), the latter including an anonymous *Arte breue y prouechoso de cuẽta castellana y arithmetica, donde se muestrã las cinco reglas de guarismo por la cuẽta castellana, y reglas de memoria*, a work he also published separately that very year (1555).

This work has not only been traditionally attributed to Juan Gutiérrez, but also considered a mere reprint of his first edition (Gutiérrez, 1539), although the original title is slightly different (*muy prouechoso* instead of simply *prouechoso,* and *Arismetica* instead of *arithmetica).* The fact is that the detailed comparison between the 1539 and the 1555 editions shows a new typography in the latter, together with a new foreword and eight new chapters, so that both editions are equal in contents only from the ninth chapter onwards in the 1555 version –which is the third in 1539–, the final prayer being excluded in 1555.

The original 1539 edition starts, after a foreword which is actually a dedication, with a statement of principles of *quenta castellana* and *arismetica* (Roman and Arabic numerals) that briefly explains his purpose, namely, to show the five fundamental operations ordinarily used in Spain (addition, subtraction, multiplication, division by a single unit, and division in general), "in order to avoid cheating", by *cuenta castellana* and *guarismo* (Arabic numerals), the latter being shorter.

It begins by explaining Arabic numerals and place values, with their names –from units all the way up to *quento de quentos* (10^{12}), and insists on announcing that every rule will be illustrated with examples in Roman and Arabic numerals (Gutiérrez, 1539, p. 4). Chapter 2 is devoted to addition, with the *real* check (inverse operation) and the checks of sevens and of nines –which involve subtraction and multiplication, not explained yet.

This part was replaced with a new foreword and 8 chapters:

Foreword. - The author refers having compiled foreign books; he highlights the brevity and usefulness of the work, and the contents on coins, weights and measures used in Spain; he announces he will use roman numerals in addition and subtraction, but not in multiplication and division, since they are no longer used; he also states that with this book anybody can become an accountant without the need of a teacher.

1. - Unity and number (with references to Euclid), odd and even numbers.
2. - Arabic numerals and place values.
3. - Letters in *cuẽnta llana* (Roman numerals) and an almost one-page long table to illustrate place values in *guarismo* (Arabic numerals).
4. - Coins, weights and measures in Spain: Coins in Aragon, Valencia, Barcelona, Perpignan, Navarra, Castile, and Portugal; weights and measures only in Aragon, Castile, and Valencia.
5. - Addition: definition; insistence on announcing that each example will be given in Roman and Arabic numerals, *para que cualquiere mas facilmente las pueda entender y declarar (so that anybody can more easily understand and declare them).*
6. - Addition: Two examples in Arabic and Roman numerals.
7. - Addition: Check of nines (not as detailed, as the check of sevens was in the 1539 edition), with a statement in favor of the inverse operation as the true check.
8. - Addition of *ducados, sueldos* and *dineros* in Aragon: unlike Castile, where merchants worked in *maravedis*, merchants in Aragon worked with *ducados, sueldos* and *dineros* (1 *ducado* = 22 *sueldos*, 1 *sueldo* = 12 *dineros*), which required to sum *dineros* first, carry on the entire part of the result in *sueldos* to be added up to the given *sueldos*, and repeat the same procedure with *ducados*.

The two additional examples with weights and measures in Arabic numerals that are given in order to illustrate how the same process is valid in such cases –as well as for any monetary system– come from the 1539 edition, although the first one was originally published in Roman numerals.

Finally, Chapter X includes a final reference to the Portuguese monetary system, which is not present in the 1539 edition.

The remaining 6 chapters are devoted, in both editions, to subtraction, multiplication, division by a single unit, division in general, application to currency exchange, and addition of (geometric) series *for those willing to be liberal* (independent) *accountants* (Gutiérrez, 1539, p. 40; 1555, p. 44) –a part of arithmetic indirectly related with loans and interests, officially condemned by the Catholic Church. And that was the end of both treatises on *principles* of *quenta castellana* and arithmetic.

In absence of data to discern the responsibility of the three main characters in the authorship of the 1555 edition –Icíar, Gutiérrez and Suelves, it is worth recalling that the 1539 edition might have been royalty free, since it was printed with imperial privilege for ten years[3], although the partial appropriation is clear.

As regards changes in the 1555 edition, apart from the initial cultured reference in the first chapter, the new contents tend to reinforce the practical character of the work: the new reckoning techniques are applied to monetary systems more complex than the Castilian one, and the difficulties involved in the checks of the sum are avoided, which follows the structure of Icíar's *Practical Arithmetic*.

It is also worth noting that, despite what is announced in the new foreword of 1555, the constant reference to Roman numerals is kept and underlined, which seems to show there was a public to address with these elementary books in the transition from classic to modern arithmetic –perhaps Icíar's *Practical Arithmetic* had proved to be excessive for the self-training referred to in the new 1555 foreword which, by the way, contradicted Icíar's *Practical Arithmetic* foreword as regards the need for a teacher.

Furthermore, it should be noticed that *Arte breue y prouechoso de cuẽta castellana y arithmetica* seems to have been sold independently and also as part of Icíar's *Arte subtilissima* –both works having the same unique colophon of Zaragoza, 15 May 1555, with Esteban de Nágera as printer and Suelves as publisher, but there is also an edition of *Arte subtilissima* without arithmetic (Martínez Pereira, 2006, pp. 343-355).

[3] On printing privileges in Castile and Aragon see (García Cuadrado, 1996, p. 128-129).

From *Arte subtilissima por la qual se enseña a escreuir perfectamente* to *Libro Subtilissimo, por el qual se enseña a escreuir y contar perfectamente*

As matter of fact, the 1955 edition of Icíar's *Arte subtilissima* must have been judged positive, since in 1559 Suelves launched a new edition titled *Subtlest Book that teaches how to write and reckon in the same order as a teacher does with his student* (Icíar, 1559a), containing *Brief and Useful Art of Castilian reckoning and arithmetic, where the five rules of figure on Castilian reckoning are shown, and memory rules. And now in this last edition very amusing and useful calculations, from Fray Iuan de Ortega book, have been added, and at the end an abridged account in maravedis* (Icíar, 1559b), a further reworking of Gutiérrez' *Arte breue y muy prouechoso de cuẽta castellana y arismetica* (Gutiérrez, 1539). It is noteworthy that *Libro Subtilisimo* is the first printed work in Spanish devoted to the teaching of writing and reckoning altogether.

In this edition, *Libro Subtilissimo* ends with a colophon insisting on Icíar's authorship *(Ivan de Yziar wrote it in Zaragoza in 1559)* but again no author is mentioned on the title page of *Arte breue y prouechoso de cuenta Castellana y Arithmetica*, although Icíar's portrait is printed on the back of the title page. According to the colophon, the book was printed by the widow of Esteban de Nágera, and was finished in Zaragoza, on 27 June 1559. The foreword was suppressed, and the book was a reprint of the 1555 edition –including the misprint numbering chapter 14 as VIII, like in the 1539 edition– until the end, except for the Thank the Lord, which was placed after the sixteen new additional pages devoted to introduce new practical rules through brief recipes, examples and problems –including the use of fractions:

1. - *Six very brief and useful rules for house rental, servants' wages, and other daily expenses* were added, consisting of simplified recipes for the rule of three to calculate the daily wages in *dineros* from the monthly salary in *sueldos* or from the annual salary in pounds, the monthly salary in *sueldos* from the annual salary in pounds, and the corresponding reverse operations: for instance, the daily wage in *dineros* is equivalent to two fifths of the monthly salary in *sueldos*.
2. - Five kinds of Boethian proportions: $ma:a$, $(m+1):m$, $(m+n):n$, $(mn+1):m$, $(mn+k):m$, illustrated with 12 examples where the rule of three is implicitly used and linear equations and systems in 2 variables (dependent or not) are solved. As advertised in the title of the book, problems numbers 3, 4, 7, and 11 are inspired on (Ortega, 1512, ff. 180, 177, 78).
3. - Four additional solved problems, the first one in (Ortega, 1512, f. 178), and the second a problem of rule of fellowship.
4. - Four proposed problems (rule of fellowship).

5. - Table of conversion of *reales* and crowns into *maravedis.*

It should be noticed that the exercises inspired on Ortega are not solved the same way, and that no mention is made of the rule of three and the rule of fellowship, which might mean a desire to show that this arithmetic was, in practical terms, as good as Ortega's highly reputed 203 folio book, and required a smaller theoretical effort. The reference to Boethian proportions, the sacred core of scholarly mathematics (Høyrup, 2009, p. 107), appears as a touch of class.

In 1559, no independent edition of *Arte breue y prouechoso de cuenta Castellana y Arithmetica* seems to have been sold.

In 1564 Suelves published both Gutiérrez's *Arte breve y muy prouechoso de quẽta castellana y Arithmetica, dõde se muestrã las cinco reglas de guarismo por la quenta Castellana, y reglas de memoria,* printed by Bernuz (Gutiérrez, 1564a), and Icíar's *Libro Subtilissimo,* printed by the widow of Bartolomé de Nágera, the latter including Gutiérrez's work or not (Icíar, 1564).

This edition corresponds to the 1539 original, except for the initial table of contents and the final inclusion of nine exercises with beautiful illustrations from Icíar's *Practical Arithmetic* (Icíar, 1549, ff. XLIII-XLV.) It also shows an improved presentation as regards print quality and organization, with new chapter headings (resulting in eleven chapters instead of eight) and subheadings.

Again in 1566 *Libro Subtilissimo* was printed including Gutiérrez's work or not (Icíar, 1566).

Suelves published this 1564 edition of Gutiérrez as *Arte brebe y mvy probechosa de cuenta castellana y Arismetica, donde se demvestran las cinco reglas de guarismo por la cuenta castellana, y reglas de memoria* a third and final time in 1569, licensed by the Archbishop of Zaragoza and the Inquisitors, and printed by Miguel de Güesa (Gutiérrez, 1569)[4]. In the meanwhile, Andrés de Angulo had also published Gutiérrez's work, including the table of conversion of *reales* and crowns into *maravedis* of the 1559 edition, in Alcalá (Castile) in 1564, with His Majesty's privilege (Gutiérrez, 1564b), the only edition where the author is identified as Reverend Juan Gutiérrez de Gualda, priest of Villarejo de Fuentes.

Conclusion

According to historians of writing (Cotarelo y Mori, 2004, p. 366; Martínez Pereira & Torné, 2007, pp. 60-63), the 1555 edition of *Arte subtilissima,* by suppressing the theoretical texts on writing, began the

[4] The mention to the license proves that the edition was according to the law (García Cuadrado, 1996, pp. 146-148).

process of degradation of Icíar's original work, that from that moment was gradually used to improve handwriting, and as sample-book of ornamental letters. This is consistent with the downward evolution of the arithmetic contents when considering an audience of young –but not small– students aspiring to earn a living as clerks. The merchant audience could still use *Practical Arithmetic* (Icíar, 1549) since, as far as we now, Suelves's did not sell his half to the brothers Alberite.

In any case, Icíar's *Ortographia pratica* is the first handbook on writing printed in Spain. His second work, *Practical Arithmetic*, shows his pedagogical side also in arithmetic, and the conjunction of both works in *Libro Subtilissimo* is the first handbook on writing and reckoning printed in Spain, which gives us some knowledge of how modern arithmetic entered general education in Spain.

Nevertheless, in the second half of the 16th century the most circulated author was Bachelor Pérez de Moya, especially his *Arithmetica practica y speculatiua* (Salamanca, 1562), with 30 editions in two centuries (Smith, 1908, pp. 308-311).

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Teaching of conics in 19th and 20th centuries in France: On the conditions of changing (1854 – 1997)

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Abstract

In the 19th century two different kinds of textbooks were published, proposing different methods of teaching conics. In the handbooks of Geometry for secondary schools, conics were studied as « usual curves » and defined by foci or by focus and directrix. While, in the textbooks of Analytical Geometry written to prepare entrance to the Polytechnic School, conics were defined by the general equation of the second degree. In the beginning of 20th century there was an important Reform of mathematics teaching, characterized by three points: exploration of the experimental nature of geometry, common study of plane and space geometries, and the introduction of transformations. In the syllabus of 1902 and 1905, conics were also defined as sections of cones. In 1931, conics were one of the two great aspects of geometrical teaching, with many possibilities of conceptual changes in teaching: between plane and space geometries, and between geometrical and algebraic approaches. The 1945 syllabus was the great period for conics, with full freedom given to teachers to organize their lessons on conics. In the textbooks for secondary schools, three general definitions of conics were given, in particular as locus. The Reform of Modern Mathematics was a period of decline for the teaching of geometry, where linear algebra took the place of classical geometry. Conics were a small part of the syllabus and they were seen as curves of second degree. Ten years later, in the Counter-Reform, geometrical definitions were given, but conics remained a minor part of the Curricula. The Curriculum of 1991 proposed "teaching by activities". This approach was not sufficient to keep a teaching of conics, and they disappeared in 1997 syllabus. Our purpose is to situate all these approaches in the mathematical and institutional contexts of teaching and to focus on the conditions of change in mathematical teaching.

Introduction: what is a conic? Some historical references

In the 3rd century B.C., Apollonius based the theory of all three conics on sections of one circular cone, right or oblique (Fig. 1).

Given a circle *BC* and any point *A* outside the plane of the circle, a double cone is generated by a line through *A* and moving around the circumference of the circle In the case of the parabola, a section of the cone by a plane is chosen such that *FG* is parallel to *AC* and *DE* is perpendicular to *BC*. Apollonius introduced a (finite) line *FM* such that the ratio of *FM* to *FA* is equal to the ratio of the square of *BC* (area) to the rectangle (area) with sides *BA* and *AC*. He proved that if *K* is a point of the parabola then the square on *KL* is equal to the rectangle with sides *FM* and *FL*. This geometrical relation can be called "the symptom of the parabola" (Fig. 2).

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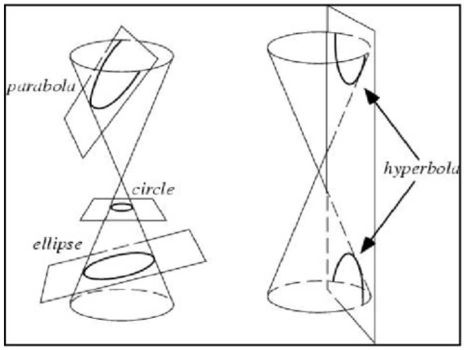


Fig. 1. Conics of Apollonius

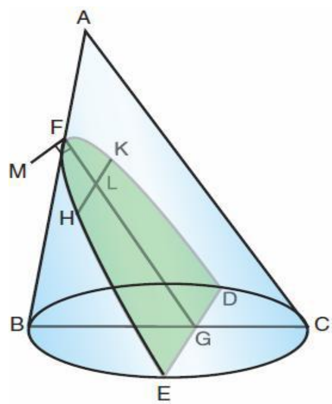


Fig. 2. The parabola of Apollonius

Apollonius' three definitions of conics as sections of the cone were criticized from the time of the 16th century, for instance by Kepler, because they seemed too complicated. More and more, the geometers preferred to take the properties given by Pappus or by Claude Mydorge (1639) to define the conics, that means by foci for the ellipse and the hyperbola and by focus and directrix for the parabola. In his *Dioptrics*[1] (1637), Descartes used the "gardeners constructions" for the ellipse and the hyperbola. But in his *Geometry* (1637), he explained that the "geometrical curves" can be associated with algebraic equations and that the conics correspond to equations of the second degree. Euler proved this result in his *Introduction to Infinitesimal Analysis* of 1748 (Bongiovanni, 2007). The equation associated with a conic enabled Descartes to find the

[1] We translate the titles of the books in English and we give the original references in the bibliography.

normal of the curve by an algebraic calculation. After the introduction of infinitesimal analysis by Leibniz and Newton at the end of the 17th century, the problems on curves were treated by the Calculus, which achieved great success throughout the following century.

A third great period for the history of conics was the "pure geometry" of the 19th century. The major idea of Monge and Poncelet was to conceive conics as a projection of a circle. In his *Treatise on projective properties of figures* (1822), Poncelet systematically researched "the projective properties" (Poncelet, 1865), that means the properties which are preserved after a projection. Projective geometry can be a powerful method to obtain results on conics as a consequence of similar results proven for a circle. By introducing infinite and imaginary notions, he provided also a means of considering all the conics as the same object. The theory of transversals of a curve gives fruitful results for conics, with the theory of the polars obtained by Servois, Gergonne and Poncelet.

This period can be exemplified by the theorem of Dandelin, which would be used fifty years later in teaching (Dandelin, 1822). The Belgian mathematician proved that if two spheres are inscribed in a circular cone so that they are tangent to a given plane cutting the cone in a conic section, the points of contact F and F' with the plane are the foci of the conic section (Fig. 3).

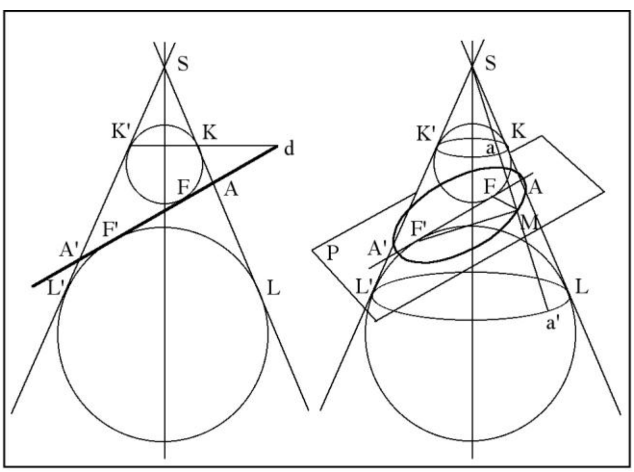


Fig. 3. Dandelin's theorem for the ellipse

Teaching of conics in secondary schools during the second part of 19th century

In the second part of the 19th century, secondary teaching was marked by several Reforms, in 1852, 1872, 1880-1881, 1884, 1886, and 1890, while the number of students stagnated and decreased. Two kinds of separate teaching methods and schools were created, which correspond to the different social classes. "Classical teaching" meant teaching ancient languages and literature. There were also classes to prepare students for the examination of the Grandes Écoles (polytechnic and engineering schools). "Special teaching", which will become « scientific teaching », contained more mathematics and was intended for the middle class of future technicians.

The case of Jules Dufailly's textbooks is of interest for analyzing the disparities between the teachings of mathematics in the different classrooms. He edited three books in the same years: *Elements of geometry for the class of literature* (1874), *Elements of geometry for scientific teaching* (1875) and *Geometry for Baccalaureat and Great Schools* (5d ed. 1885).

The reforms bring in several tensions, which set "modern teaching" in opposition to "classical teaching" and "utilitarian teaching" to "universal teaching", and "pure geometry" to "analytical geometry". The French mathematician Joseph Bertrand wrote against "utilitarian teaching": "Fifteen years ago, in France, scientific studies seemingly suffered a serious crisis, about which the friends of the science were concerned. [...] Science had to be studied for its practical utility and it was a dangerous error to see it as intellectual gymnastics and as a way to strengthen the spirit and to increase its subtlety; the elegant questions of the General Examination of the Lycées of Paris have been replaced by numerical calculations [...]. We despaired of the most humble study, but none the less useful or less wide-ranging, for a long time, that is the study of the named elementary theories, which celebrate our classical teaching" (quoted in Chasles, 1870, pp. 380-381).

An example of the teaching in the classrooms for preparing for the examination of the Grandes Écoles is given by the *Lessons of Analytical geometry with two or three dimensions for the candidates of the Polytechnic School and Normal School, preceded by an Introduction containing the First notions on Usual Curves for the candidates of the Baccalaureate* of Roguet (1854). In the introduction, each conic is defined and studied separately: the ellipse is defined by the loci and drawn by the gardeners' construction. But in the lessons of Analytic Geometry themselves, the different conics are defined together by the equation of second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Three kind of curves and three separate studies are given, according as we have $B^2 - 4AC < 0$ for ellipse, $B^2 - 4AC = 0$ for parabola and $B^2 - 4AC > 0$ for hyperbola.

Geometry for candidates for Baccalaureat of Sciences and for Great Schools by Dufailly (6ed. 1888) is quite different of the Roquet's handbook. It is composed of two parts: the first one is devoted to plane figures and the second one to space figures. Three definitions of conics are given: by foci for the ellipse and the hyperbola, by focus and directrix for the parabola. Three separate studies correspond to each conic. They begin with the construction of the conic and their properties of symmetry. In the case of an ellipse, centre and tangent are given, and it is stated that the normal projection of a circle in one plane is an ellipse. A global study of the conics follows the separate studies, by introducing Dandelin's theorem for conics.

Defailly introduced the notion of the eccentricity e of a conic, which is defined as the constant ratio of MF to MH, where MF is the distance of a point M of the conic to a given point F and MH is the distance of M to a given line. This notion provided a new way to have a global conception of the different conics. According to the value of e, we obtain each of the three kinds of conics. If $e < 1$ then the conic is an ellipse, if $e = 1$ it is a parabola and if $e > 1$ it is a hyperbola (Fig. 4).

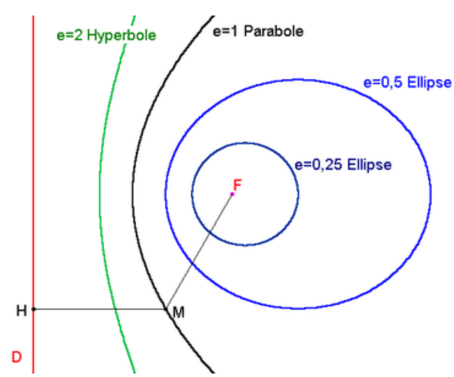


Fig. 4. Eccentricities of the conics

In conclusion, modern teaching did not only contain more mathematics, but it also proposed a method of geometrical teaching nearer to the conceptions of the "pure geometry" of the 19th century than to "classical teaching".

The conics in a treatise of "modern geometry"

In the same years, Rouché and Comberousse edited a *Treatise of geometry* with new conceptions coming from what they called "modern Geometry". They wrote in the preface: "The splendid discoveries of the modern Geometry did not get into the teaching; neglected by the curricula, they did not occupy the place that they deserved in mathematical studies; we speak of them barely and sometimes in analytical Geometry, where wrongly they seem to be a new conquest of the tool created by Descartes. [...] But is it possible to learn one programme of examination and at the same time to try to learn the impact of science, by quickly gained knowledge, by a general view on methods? It is the thought, which guided us, in the composition of this Book" (Rouché, Comberousse, 1900, p.XV). The first edition appeared in 1866, and the seventh one in 1903d0.

The authors did not indicate who were the expected readers. But, as it concerned elementary contents, it could be taken as an initiation to "modern geometry" for students and for teachers. The treatise was composed of two parts, the first on plane geometry and the second on space geometry, preceded by a chapter entitled "Historical notions". Part I contained four books and each of them had an appendix, which presented the recent notions and methods of the 19th century. For instance, the Appendix of the third book introduced the theory of transversals, the method of polars and inversion. In the edition of 1900, the part I concluded with four Notes, like "On the impossibility of the quadrature of the circle » and the " Geometry of the triangle" written by Neuberg. While the part II contained one "Note on non-Euclidean geometry" written by Poincaré.

Book VIII of Part II of the *Treatise* was entitled "The usual curves and surfaces" but it was entirely devoted to the conics. It was composed of seven chapters. Chapters I, II and III concerned the fundamental properties of the ellipse, the hyperbola and the parabola. For instance, for the ellipse, the curve was defined by foci, then by the tangents and also the construction of the tangents was given. The chapter also contained the so-called Poncelet's theorems on tangents of a conic (with foci). The first theorem stated that if *P* is outside of the conic, *PA* and *PB* are tangents to the conic and *F'* is one of the foci then *F'P* is the bisector of angle *AF'B*. The second theorem stated that, the angles *APB* and *FPF'* have the same bisector. So, in these three chapters, the presentation of conics was classical, but some new properties of the curves were given (Fig. 5).

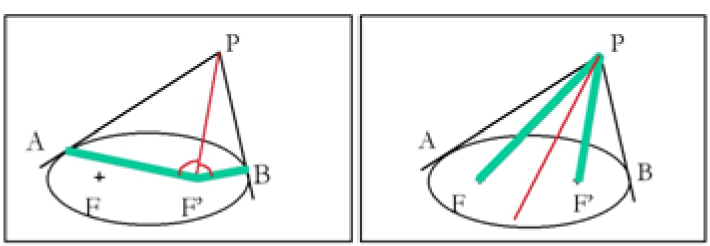


Fig. 5. Poncelet's theorems

The contents of the three following chapters explain why the conics were studied in the part on the geometry of space. Indeed, chapter IV is entitled “Ellipse considered as an orthogonal projection of the circle”. This result was proven and many corollaries were given, on the constructions of tangents, on the construction of the intersections between an ellipse and a straight line, and on the area of an ellipse. This chapter is clearly a heritage of the conceptions of Monge and his students. The next chapter V can be seen has an application of the projective geometry of Poncelet. Indeed, now the authors considered “the parabola as a limit of an ellipse”, because “the limit of an ellipse, such that the summit and the near focus are fixed, while the other focus recedes in the direction of the major axis, is a parabola” (Rouché & Comberousse, 1900, p. 345).

Chapter VI was titled “The common origin of the three curves. Plane section of a cone”. There was an explicit intention to understand all the conics together. The authors used the theory of polars to prove the theorem on eccentricity. Then they proved that each conic is a section of a cone as an application of the Dandelin’s theorems. The last chapter concerned the helix.

The teaching of conics in secondary schools from the Reform of 1902 to the Reform of 1945

Concerning geometrical teaching, the main ideas of the Reform of 1902 were to emphasize the experimental nature of geometry, to blend plane geometry and space geometry and to introduce transformations (Bkouche, 1991). The spirit of the Reform was the same as in the textbook of Rouché and Comberousse, which was to take account of many conceptions of the “pure” or “synthetic” or “modern” geometry of the 19th century. So, it is not strange that the teaching of the conics in the Curriculum of 1902 was more or less the same as in the Book of Rouché and Comberousse. In some sense, the Reform followed another reform contained in this book, edited thirty years before, where several editions show that they met with great success. Moreover, the Reform followed the creation of “scientific teaching”, which influenced the “classical teaching” of secondary schools.

The parabola was excluded in the curriculum of 1902 and was included in the Curriculum of 1905. But in 1905, the conics were only given by their geometrical definitions and the reduced equation. It seems that we have an effect of the intention to give only one kind of definition for all the conics, with the consequence that some of them can be excluded. The teaching of conics took a special place in the Curricula of 1925 and 1931, and the intention is the same. The teaching of geometry was separated into two parts: “transformations of figures” and “conics”. They also introduced the reduced equation of the conics.

The Curriculum of 1945 was the great period for the conics. The main feature was that the accompanying instructions proposed that teachers should organize their teaching as they desired. Indeed, as it was written, "full liberty is allowed to teachers to organize their lessons on conics. To study these curves and to solve classical problems, they will begin on the characteristic properties they judge the most convenient". After several years with different kinds of definitions, and consequently with different kinds of studies of conics, the choice was left to the teachers themselves.

This Curriculum had to be linked to the edition of an important book in 1942, entitled *The conics*, which was a collection of papers written by the French mathematician René Lebesgue from his 1921, mathematical and pedagogical papers. It appeared after his death and three years before the new Curriculum of 1945. Lebesgue explained that "the arbitrary and disparate definitions of the three conics shocked him during his years of studies in the Lycée" (Lebesgue, 1942, p. 1). As a professor of the École Normale de Sèvres, that prepared young women to become mathematics teachers, he decided to give his opinion. But he did not want to propose a new teaching method opposed to the former one, because he was convinced that any method, however 'perfect' it is, will become bad after it has become official. For him, "the only teaching that a teacher can give is to think in front of his or her students"(idem).

Three definitions of the conics were given in the Curriculum of 1945. The first kind was by focus, for ellipse and hyperbola. The two other definitions gave a global conception of the different conics: the definition by eccentricity, which was put forward by Lebesgue in "Les coniques dans l'enseignement secondaire", published in *L'enseignement scientifique* of 1933 (Lebesgue, 1942, pp. 2-24) and the definition given by Leconte, emphasized by Lebesgue in "Encore quelques observations concernant les coniques" published also in *L'enseignement scientifique* of 1935 (Lebesgue, 1942, pp. 24-34). The conics are defined as the locus of the centres *M* of circles which go through a given point F and which are tangent to a given circle with center *F'* (the director circle) or to a straight line. If *N* is the point where the two circles are tangent, then *MF* equals *MN* (Fig. 6).

In the case where the director circle is replaced by a straight line, the conic is defined as the locus of the centre *M* of the circles which go through a given point F and which are tangent to a straight line. So, this conic is a parabola defined by focus and directrix. In the Curriculum of 1945, the conics are also defined by an equation and as sections of a cone.

A typical textbook was the *Geometry – Class of mathematics* of Deltheil and Caire (1950), devoted to the last class of the Lycée (16-17 years old). It began with a historical chapter naming the works of Menaechmus, Archimedes, Apollonius, Galileo, Kepler, Newton, Desargues, Descartes, Poncelet, Chasles, Steiner, Plucker, Cayley, Quetelet, Dandelin. Three definitions of the conics were given and their equivalences were proven: by eccentricity, by director circle, by focus and directrix. They were followed by theorems on the tangents and on the envelopes of conics. The book gave the equations of the conics, which were obtained from the eccentricities, and the definitions of the conics as sections of a cone.

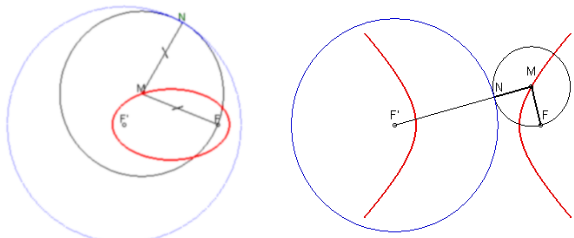


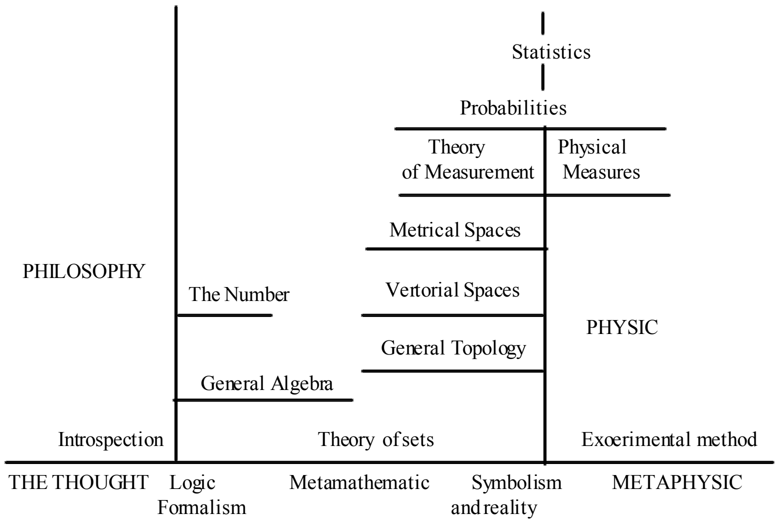
Fig. 6. Definitions of an ellipse and of a hyperbola by Leconte

Conics in a textbook of "Modern Mathematics"

By the end of the 1950s, Lucienne Félix wrote a handbook entitled *Modern account of elementary mathematics* (1959), which was conceived as she wrote, as a "book of initiation". She wrote in the introduction that "a construction of the Mathematic is satisfactory in our epoch only by the conditions that have revealed the unity of this science; there is a continuity of method, in spite of the diversity of the structures considered, from the introduction of notions like integers and fractions, points and right lines, until the most complex notions" (Félix, 1959, p. v). She added that the results which are known in childhood have their place in this building later, so "it is logically and psychologically necessary to reconstruct all the set of results accumulated in elementary schools in a mathematically satisfactory form. This knowledge is already imbued by sensorial elements, justified by the appeal to physical experience and to a non analyzed intuition of the space" (idem). Her conception of mathematics is described in a table in the beginning of the book (Table 1).

Book I of the "Modern Account" concerned the fundamental structures, the first chapter introduced the vocabulary and the symbolism of the theory of sets, the relations of equivalence, the relations of order and the operations on sets. It continued with five other chapters on numbers, on vector spaces, on the applications of a set in a set, transformations of points and numerical functions, on metric spaces, on Boole's Algebra, measures and probabilities. Book II was devoted to Arithmetic and Algebra with a first part on the theory of numbers and a second part on the algebraic expressions and the resolution of equations. Book III concerned Calculus with the local and the global studies of a function, the graphs of a function, the primitives of a function and finished with the complex numbers conceived as a field[2].

Table 1. Felix's conception of mathematics



The last Book IV was entitled "The geometries", which occupied about one third of the "Modern Account". It was composed of three parts: on the affine and the projective geometries, on the metrical geometries and on the conics. So the conics appeared as a part of "the geometries", in a sense of the different kinds of geometries, and they occupied the last part of the Book IV and of the complete book of Félix. It contained only fifteen pages. We can understand the presence of this part as in accordance with the Curriculum of the period. We can also note that the author was a student of Lebesgue, who was faithful to the memory of his master. She published a book on her master Lebesgue in 1971 (Félix, 1971).

Indeed this part on conics is very complete, but it is considered as an application of modern mathematic. She wrote: "After the straight lines and the circumferences, the most remarkable curves in plane geometry are the conics. Their study offers different points of view and uses all the previous theories. For us it is a field of application" (Félix, 1959, p. 402).

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[2] A complete analysis of the content of the "Modern Account" is given in (Barbin, 2012b).

The part on conics begins with the definitions in a cone of revolution with the aim to define the conics by metric relations. Five relations are given: by eccentricities, two by the ratio of tangential relations, by focus and directrix, and by focus for the ellipse. The presentations use the vocabulary and the symbolisms of the theory of sets. Then this part on conics goes on with the analytical, the affine, the projective and the tangential points of view.

The Great Reform of Modern Mathematics was "prepared" by Félix's "textbook of initiation" and also by many papers published in the *Bulletin* of the French Association of mathematics teachers (APMEP). But, as for the Reform of 1902, we have also to take in account the social context, the several reforms of-educational politics, and the mathematical context, the willingness to teach the geometry of the 20th century.

The teaching of conics in and after the reform of "Modern Mathematics" (1968-1997)

By the end of the 1950s there was a lack of engineers, of researchers and of technicians in France. It partially explains the scholar transformations.

In 1959, Berthoin's reform extended compulsory schooling until 16 years old. The year after, there was the suppression of the examination for entering the Colleges for the pupils of primary schools. In 1963, Fouchet's reform created the Colleges that welcomed all the students in the same kinds of schools until 15 years old.

In the same years, there was an international movement, expressed in 1959 during the Colloque of Royaumont organized in France by OECE, to promote a reform of the mathematical teaching. For the French mathematicians Choquet and Dieudonné, it was necessary to teach "modern mathematics" and Dieudonné exclaimed "Down with Euclid!" (Barbin, 1989). Indeed, by the second part of the 19th century, new foundations of the geometry had been invented. In his *Erlangen Programme* (1872), Felix Klein associated an algebraic group with each kind of geometry (euclidian, non euclidian, and projective). While, in his *Foundations of the Geometry* (1899), David Hilbert proposed a "geometry of incidence" based on the relations between geometrical objects. The theories of linear algebra and vector algebra began at the beginning of the 20th century.

In 1964, Choquet wrote in his *The teaching of Geometry* that "our preference has to go to methods leading to the fundamental notions that twenty centuries of mathematics finished to bring out: the notion of set, the relations of order and equivalence, the algebraic laws, vector space, symmetry, and transformations" (Choquet, 1964, p. 10). Also in 1964, Dieudonné explained in his *Linear Algebra and Elementary Geometry* that « in our time of great proliferation in all the sciences, everything which can condense and lead to unification has a virtue that we could not over-estimate. For the « pseudo-sciences » given above, we hope to forget their existence soon and also their names; the sooner the better » (Dieudonné, 1964, p. 12). For him, the "pseudo-sciences" were "Pure geometry", "Analytical geometry ", "Trigonometry", "Projective geometry", and "Non-euclidean geometry", etc.

In 1964, Rivaud edited his *Exercises of Geometry*, devoted to the students preparing the examinations for the Grandes Écoles in the upper classrooms of the Lycées. He wrote: "The new curricula for [these] classrooms simplified and reorganized geometrical teaching. The increased importance accorded to Algebra and to Calculus justifies a reduction of the time devoted to Geometry. [...] A more rational introduction of fundamental notions was researched in the new program. The introduction of axiomatic methods permits us to stress the development of distinct geometries, each having its proper construction and methods: affine, metric and projective geometries" (Rivaud, 1964, p. 5). He concluded: "the usual introduction of elementary geometry gives an obvious priority to a metric conception of space" (op. cit., p. 6). In chapter I, all the geometrical curves were defined by a parametrical representation. In chapter IV, the conics were defined by the "symptoms" of Apollonius and then by their equations.

The French Association of mathematics teachers APMEP expended great activity to promote "modern mathematics" with its Bulletin, meetings and also regular films for the television. For this Association, "modern mathematics" will be favorable for democracy (Barbin, 2012a). The teachers of APMEP knew that the teaching of geometry was specially concerned with "modern methods". They considered that as we have to teach geometry, the true problem is to improve its introduction: so it will be almost obvious that the study of vector spaces will restore the place of the geometry in a teaching context adapted to the modern conditions of the science (APMEP, 1967).

Frenkel edited a *Geometry for the student teacher*, where he developed the ideas expressed ten years before by Dieudonné. He wrote: "Mathematically speaking, elementary geometry is not distinguished from linear algebra, but only by artificial boundaries on the dimension and sometimes by the field concerned" (Frenkel, 1973, p. 14). What is the place for the conics in such textbook? They only appeared in the last question of the last problem given by Frenkel. Moreover, they were only mentioned as a specific case and in brackets: "Given a projective space X with dimension $n \geq 2$ on the field of real numbers, and q a quadratic form on ^X of kernel ^S. We call a projective quadric (conic if n = 2) of equation $q(x) = 0$, the image Q (empty eventually) in X of the isotropic vectors of q" (op. cit., pp. 347-348).

The conics were only a small part of the Curriculum of "Modern Mathematics": the curves were given by the equation:

$$ax^2 + by^2 + 2cx + 2dy + e = 0$$

Their study concerned the axes, centers of symmetry, asymptotes, reduced equations, existence of tangents, but no the properties of tangents. Ellipse, hyperbola, parabola were also defined by focus and directrix.

Choquet and Dieudonné were among those who protested against the Reform. Choquet wrote in 1973 that it is "an attack against geometry and the using of intuition", and Dieudonné wrote in 1974 that it is "a new scholastic [...] a still more aggressive and stupid form put under the flag of the modernism" (Bareil, 1992). In the Curricula of the Counter-Reform (1983 and 1986), the conics were defined by focus and directrix, reduced equations, and eccentricity. It was, more or less, the contents of Dufailly's book of 1888. In the curriculum of 1991, only "activities" on conics appeared: they are defined by focus, directrix and eccentricity, but also by the Cartesian and parametric equations. Finally, the conics disappeared from teaching with the Curriculum of 1997 (Barbin, 2008).

On the conditions of change

The interest to focus on a particular theme in the history of mathematics teaching comes from the possibility of analyzing the different contexts in which a change occurs, but also which make possible a change. That means that we are interested in the process, by the uphill struggle, by the reforms of the conceptions of the actors that precede a Reform.

Using a global approach (Belhoste, 1990), the historians studied the social and the mathematical context of the Reforms. In the case of the Reforms of 1902 and 1968, new teaching corresponds more or less to the new mathematics. Moreover, as we note, it seems clear that the actors shared the idea that the new teaching [curriculum] was particularly adapted to the middle-class and the lower-class.

But using a local approach, we note that the Reforms depended also on the production of books, presenting a synthesis of new conceptions that can make a concrete realization possible. This is the case with the treatise of Rouché and Comberousse for the Reform of 1902, with the book of Lebesgue for the Reform of 1945 and for the textbook of Félix for the Reform of 1968. Our purpose is not to claim that they are the only examples, that a particular Book makes a Reform, but to analyze their effects as a necessary condition of change, that is, to have teachers able to teach the new contents. In the period concerned by this paper there are many mediums, like journals for teachers and associations of teachers. But we need to examine the institutional process itself.

Examine for instance the Reform of 1945. A rich teaching of the conics, where a « full liberty is let to the teachers », required that the teachers had a very good knowledge of the conics. So it is interesting to note that since 1924, the conics were a regular subject for the Concours d'Agrégation, necessary for those wanting to become teachers in the Lycées, specially in the upper class of these secondary schools. Moreover, this subject became more and more frequent from 1938.

Table 2 is obtained from the list of problems given by Ballicioni and Chazel in their *Choice of problems of Mathematics with Solutions* (1962). In this book, devoted to the future teachers of secondary school, the authors wrote: "This Book essentially contains texts and solutions of problems given as the first examination to candidates of Agrégation de Mathématiques, for men and women. Therefore, they are geometrical problems" (Balliccioni & Chazel, 1962, p. vii). They explained the explicit choice of "pure geometry" against "analytical geometry": "Generally, the first examination was entirely devoted to pure geometry [...]. It was agreed that the solution of geometrical questions has to be furnished with the help of methods only of pure geometry, in contrast with those of analytical geometry, which where completely excluded" (idem). The authors note that things changed after 1962, because the solutions were given with a double treatment, synthetic and analytic.

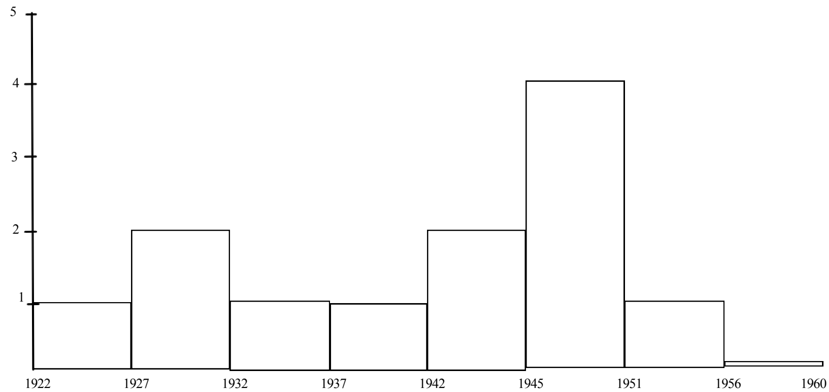
Table 2. Conics in the Concours d'Agrégation from 1924 to 1960

Year	*Men*	*Women*	*Year*	*Men*	*Women*
1924	Conics		1939	Conics	
1926	Conics		1942		Conics
1929			1943		Conics
1935	Inversion	Conics	1948	Conics	
1936			1959	Conics	
1938	Conics		1960	Conics	

For the same purpose, it is interesting to examine the problems of the Concours Géneral, proposed to the best students of the Lycées in the same period, from 1922 to 1960 (Table 3).

Indeed, several problems led to conics and many problems were about conics. In the book of Lespinard and Pernet (1962), we only noticed the problems on conics and we obtained the table above. This table shows the importance of conics and that this subject increased from 1942, three years before the Reform of 1945 which corresponds to the great period for these curves.

Table 3. Problems on conics in the Concours Général from 1922 to 1960



If we compare this table with the table on conics in the Concours d'Agrégation, we can conclude that the triumph of the conics in the Curriculum of 1945 was the result of a general movement at the stage of the selection of the students and of the future teachers. But we can also examine that from the years 1955-1960, ten years before the Reform of the Modern Mathematics, the theme of conics declined.

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Values and virtues of a rural society: reflected in 18th and 19th century arithmetic textbooks in Iceland

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Abstract

This presentation recounts a survey of six textbooks, published in the 18th, 19th and early 20th century, and their relations. While all the authors' interests in the progress of Icelandic society are beyond doubt, their visions were to maintain the values and virtues of the old self-sufficient rural society, and to teach the public to make the most of its current resources. Three further characteristics of the authors and their values were detected; firstly a youthful enthusiasm; secondly, access to social network such as official positions and nepotism to ensure distribution of their products; and thirdly, the target groups were self-educating youngsters in the absence of schools.

Introduction

The Lutheran Protestantism and Humanism brought printing to Iceland in the mid-16th century and channelled German cultural currents to Iceland through Denmark. The main emphasis was laid on religious publications. Enlightenment extended cultural and educational interests to secular books in the 18th century. Among them were arithmetic textbooks. In the following we shall explore six arithmetic textbooks that were published in the period 1780–1930, the goals and target groups of their authors, and their access to publication of their works. The first four are the first substantial arithmetic textbooks in Icelandic and the fifth and sixth ones were chosen for their relations to the first two books and their impact in the 20th century.

Niss (1996, p. 13) conjectures that there are just a few types of fundamental reasons for mathematics education: contributing to the technological and socio-economic development of society and to its political, ideological and cultural maintenance and development, in addition to providing individuals with prerequisites which may help them to cope with education or occupation, private and social life and life as a citizen. We shall argue that while all the textbooks inspected were intended to provide individuals with prerequisites to help them to cope with their occupation, mainly as farmers or skilled workers, they were also to contribute to the socio-economic development of society. They may, however, merely have contributed to its ideological maintenance, that is maintaining the status quo of society, since the authors feared its urbanization.

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The Enlightenment

The Enlightenment was an eighteenth century philosophical movement, characterized by rationalism, an impetus toward learning, and a spirit of scepticism and empiricism in social and political thought. Iceland was part of the Danish state and the Enlightenment movement in Iceland originated in Denmark, where in turn it was largely derived from Protestant Germany (I. Sigurðsson, 2010). The Enlightenment left a marked impact from the 1770s onwards on many Icelanders who studied in Copenhagen, some of whom became leading officials in Iceland. The following period was characterized by the activities of The Society of the Learned Arts (*Það íslendska Lærdómslistafélag*), founded by Icelanders in Copenhagen. The society published a journal in which emphasis was laid on various utilitarian matters. Many top officials, both secular figures and the bishops of the Lutheran state church, played an important role in the movement. The Icelandic Enlightenment was by no means radical. While its champions wanted ordinary people to be more 'enlightened' they did not wish to see any major change in the structure of society.

Education was an important aspect within the Icelandic Enlightenment. The publishing activities of its champions can be seen as a major exercise in adult education. For that purpose the movement acquired a printing press in 1794. The long term influence in Iceland on book publishing is of major and lasting importance. The publications included discussion on the possibility of progress in the main branches of economy, farming and fishing. Such discussions were emphasized in the *Journal of the Society of the Learned Arts* (I. Sigurðsson, 2010).

The end of the Age of Enlightenment in Iceland is often marked by the death of its main champion, Magnús Stephensen (1762–1833), but it may be argued that its influence, e.g. on education, lasted longer than in the neighbouring countries. Certain conditions in Iceland were detrimental to dissemination of its ideology during its first decades, such as tragic effects of a gigantic volcanic eruption in 1783–84 and disruption in the trade between Iceland and Denmark during the Napoleonic wars. Changes that had taken place in Iceland by the turn of the 20th century were far less extensive than for example in Norway and Denmark. The development of education for the common people was much slower to emerge in Iceland; the technological changes that had taken place in the neighbouring countries only reached Iceland to a limited extent; and there were relatively fewer towns than in most other European countries. This may be seen as a relatively favourable set of conditions for the reception of 'enlightened' ideas in the last two thirds of the 19th century and even up to about 1920 (I. Sigurðsson, 2010).

Society

The Icelandic society was predominantly rural throughout the 19th century, and besides livestock farming, fishing played a large role in the economy (I. Sigurðsson, 2010). A large proportion of farmers travelled to seasonal fishing stations or sent their servants (Karlsson, 2000, p. 163). Hardly anywhere until the late 19th century did fishing develop into a year-round occupation with fishing towns. In 1835, the total population numbered 56,035, while the population of Reykjavík, the largest town being formed as the centre of governmental affairs, was 639 or 1.1% (Jónsson and Magnússon, 1997).

The traditional fishing from open boats changed in the second half of the 19th century with an increase of decked boats, schooners (Karlsson, 2000, pp. 239–241). The schooners contributed to modernizing Icelandic society. They were the first large-scale enterprises in the country, run by a single employer with a hundred or more employees, provided with a year-round occupation. The schooners created a professional class of fishermen for the first time in history and more so when the motor boats arrived at the turn of the 20th century. Society remained, however, predominantly rural into the 1920s when population in urban nuclei for the first time exceeded the rural population.

Enlightenment and mathematics education

By definition, the Enlightenment suited mathematics education well in its rationalism, impetus toward learning and belief that reason would advance human progress. Olavius (1780) and Stephensen (Stefánsson, 1785), the authors of the first two substantial Icelandic arithmetic textbooks, were both prominent leaders of the early Enlightenment movement. Both attended the University of Copenhagen and were avid writers of articles and books with the purpose of enhancing progress in farming.

Two significant arithmetic textbooks were published in the 1800s; one by Jón Guðmundsson, an ally of the independence leader Jón Sigurðsson, and another by the Rev. Eiríkur Briem. They, too, stayed in Copenhagen for a year or two, but only after they had published their books, so their impetus to write the books was in both cases domestic. Finally we shall briefly mention authors of the early 20th century textbooks; in particular Elías Bjarnason a common farm servant, who, after first enjoying school education intermittently from seasonal fishing, established a primary school on his own initiative and later wrote a perennial arithmetic textbook.

Arithmetic textbooks in use in Iceland largely adhered to the European tradition of mathematics textbooks, originating in the late Middle Ages.

They began by an explanation of the concepts of a number and of numerical place value, followed by the arithmetic operations in whole numbers and fractions: addition, subtraction, multiplication, division and extracting square roots. The remaining content concerned mathematical techniques for business use: use of the Rule of Three, monetary exchange, problems of partnership and barter (Swetz, 1992).

Ó. Olavius

Ólafur Olavius (1741–1788), who studied philosophy, was one of the most prolific writers of the Society of the Learned Arts. He lead a project of buying a printing press to Hrappsey Island in western Iceland in 1773, the first one to print secular books, a major step forward for the activities of the Enlightenment movement. Olavius was, however, forced to leave the printing press due to disagreement concerning financing of the undertaking. His substantial arithmetic textbook (Olavius, 1780) of over 400 pages, *Greinilig Vegleidsla til Talnalistarinnar / A Clear Guide to the Art of Number*, was modelled after a German textbook, *Der Demonstrativen Rechenkunst* (Clausberg, 1732), 1544 pages. Although the *Clear Guide* is by far shorter than Clausberg's *Rechenkunst*, they contain many of the same examples (Bjarnadóttir, 2007). The target group of the *Clear Guide* is the self-educating general public in a country devoid of schools:

> ... I know not of anyone out there [in Iceland], who teaches the general public something in the computing art ... I count off a little part in the [two Latin] schools and therefore it is self-said that every common person, who wants to learn something, must be his own teacher. Writing for them such a book, where little or nothing is explained and only show with few examples how to do... I could not do (Olavius, 1780: xiii–xiv).

The *Clear Guide* gives a number of examples on common algorithms and their variations as aids to mental arithmetic. The book and its model witness didactic concerns. Unfortunately it may not have had large distribution due to defamation resulting from the disagreement on the printing press in addition to the calamities of the 1780s and the competition from Stephensen's textbook published in 1785.

Ó. Stephensen

Ólafur Stefánsson Stephensen (1731–1812) returned home from his law studies in Copenhagen in 1754. He entered the royal administration to eventually become the governor of Iceland in 1790, a post normally reserved for Danes. During 1783–1787 he was on a sick leave, which he used for extensive writing, mainly for the *Journal of the Society for the Learned Arts*. In his writings, he made efforts to clarify for his fellow countrymen how to run their assets economically. Stephensen considered fishing and farming indispensable parts of the same industry and the livestock farming as the basis of society. Migration from the countryside to the seaside, where people risked to live in poverty, idleness and hunger, too much freedom and lack of morality and security among the landless population by the seaside, are recurring themes in his writings, which otherwise enhanced progress and 'enlightening' of the common people, but under control of the authorities (J. Sigurðsson, 2011). He suggested, e.g. in 1769, that the buying of tobacco and spirit should be rationed proportionally to people's credit by the merchant (Ólason and Jóhannesson, 1943, pp. 366–7).

In 1785 Stephensen published at his own cost a textbook (Stefánsson, 1785), *Stutt Undirvísun í Reikníngslistinni og Algebra / A Short Teaching on the Computing Art and Algebra* (248 pages), which became a required reading at the two Latin schools (*Lovsamling for Island 5*, 1855: p. 244; *Alþingisbækur Íslands 1781–1790*, 1986: p. 327). In his preface Stephensen recounts that in 1758 he had written down what he learned in Copenhagen in order not to forget it. As many manuscript copies of that work were still in distribution he decided to have it printed. In his address to the reader the author feels necessary to safeguard himself against those already knowledgeable:

> ... this booklet is not composed in the understanding that there are not many in this country that well can compute, and can, without it, teach it to others, especially officials of the religious and secular classes; rather it is intended for use to youngsters and adolescents ... (Stefánsson, 1785: To the reader).

To be sure, the two Latin Schools deserved to have an arithmetic textbook to assist the teaching, but the authorization of the book by Stephensen, which may have been linked to his high administrative position, excluded Olavius's book from the tiny market. Earthquakes, which destroyed one of the schools in 1784, may, however, have reduced the educational impact of Stephensen's textbook. In 1802, the two Latin Schools were merged into one where mathematics education was marginal. According to a memoir: "Everyone who reached the upper grade was given Governor Ólafur's Arithmetic, but it was up to the pupils whether they ever opened the book or not." (Helgason, 1907–1915: pp. 85–86). From 1822 onwards, mathematics teaching at the Latin School adhered to Danish regulations and Danish textbooks by Bjørn, Ursin and Ramus were used (Bjarnadóttir, 2006).

Stephensen's book, *Clear Guide*, was more compact than that of Olavius, only 4–7 examples on each of the four operations in whole numbers. Decimal fractions are new vis-à-vis the *Clear Guide*, progressions/sequences are treated more fully and *Short teaching* exceeds Olavius's book by an introduction to algebra. The decimal fractions are notable. While their introduction in Europe is usually credited to Stevin (1548/49–1620), they do not seem to have been commonly introduced in 18th century arithmetic textbooks. The influential *Einleitung zur Rechenkunst* by L. Euler (1738) does not introduce decimal fractions, nor does Clausberg's (1732) *Rechenkunst*.

M. Stephensen

Two arithmetic manuscripts, Lbs. 408, 8vo, and Lbs. 409, 8vo, have been passed down to us through Þórunn, the daughter of Magnús Stephensen (1762-1833), Ólafur Stephensen's son. Lbs. 409, 8vo, carries the name *Stutt Undervisan umm Arithmeticam Vulgarem eða Almenneliga Reikningslist*, nearly identical to the title of Ólafur Stephensen's book, while their texts are different. The Lbs. 408, 8vo, contains Prof. J. M. Geuss's lecture notes on arithmetic during 1781–82. Magnús Stephensen recounts in his autobiography that he in 1781–82, aged 19, studied lecture notes by Prof. J. M. Geuss at the University of Copenhagen (Stephensen, M., 1888). He did not attend the lectures as he did not understand the professor's 'obscure' language after having missed the entrance examination and the first three months due to his late arrival to Copenhagen. He obtained excellent grade in June 1782.

During 1784–85, when Magnús Stephensen stayed in Copenhagen, aged 22, he recounts that he altered his father's manuscript of the *Short teaching* significantly while preparing it for publication. Furthermore, he added chapters on decimal fractions, ratios, proportions and sequences, and chapters on algebra and linear and quadratic equations, a total of six out of the twenty-nine chapters, covering 78 of its 248 pages. The novelties in *Short teaching* can therefore be credited to the son and his studies of Prof. Geuss's lecture notes. The manuscript Lbs. 409, 8vo, is probably a copy of Ólafur Stephensen's original. The two works follow similar scheme but have hardly an example in common except a rhyme on the place value notation. In addition to the alterations that Magnús Stephensen lists in his autobiography, an examination reveals that he added an introduction, modelled on Geuss's lecture notes; he separated examples involving measuring, monetary and barter units from the general treatment of the four species; expanded the chapter on sequences; and added to variations of the Rule of Three.

Geuss's lecture notes and the *Short teaching* begin by the same terms: 'A quantity (quantitas) is all that may be increased and decreased' (Lbs. 408, 8vo, p. 1), '... decreased and increased' (Stefánsson, 1785, p. 1). The discussion continues to 'quantitates intensivae' and 'quantitates extensivae', and later to 'mathesis pura, theoretica, abstracta' and 'mathesis practica, applicata, and mixta', in both cases also translated into the vernaculars. The textbook is not a direct translation of the lecture notes but the impact is clear. The sections on decimal fractions have for example the same length and cover the same topics, while the examples taken are comparable, not identical. Neither gives word problems on decimal fractions. Similar influences appear in arithmetic and geometric ratios. The extant notes of Geuss's lectures neither treat arithmetic and geometric sequences nor algebra and linear and quadratic equations, while Magnús Stephensen in his autobiography recounts that he possessed copies of lectures in arithmetic, algebra, geometry, trigonometry etc.

It is remarkable that a university freshman, who did not understand his professor, had the courage to rewrite a textbook according to lecture notes that he had copied from others. This endeavour was, however, by no means Magnús Stephensen's first or last. When he stayed shipwrecked in Norway in 1783–4 he copied cook-book recipes, translated them into Icelandic, wrote forewords on behalf of his brother and published an excellent cook-book 'for upper-class house wives' in year 1800 by the name of his sister-in-law.

In 1794, the old Enlightenment society had declined and the Society for the Education of the Nation (*Landsuppfræðingarfélag*) was founded with the purpose to buy the Hrappsey printing press and move to Leirárgarðar in the neighbourhood of Magnús Stephensen, who became the undisputed leader of the Enlightenment movement in 1796. The printing press formally became official property in 1798 as a gift from the contributors, and Stephensen became its director. In 1799, the society bought the Hólar Printing Press, governing religious printing. Magnús Stephensen then became in charge of all printing in the country (Jónsson, 1867). He had many works of his printed at the printing press, e.g. the cook-book and a psalm book where he excluded the Devil to the regrets of many. Stephensen's main occupation was, however, to be the country's highest court magistrate. The impact of the Stephenson family remained strong throughout the 19th century.

J. Guðmundsson

The second half of the 19th century was characterized by a growing struggle for autonomy from Denmark, lead from Copenhagen by Jón Sigurðsson and guided by the spirit of Enlightenment and rationalism. Sigurðsson's prime supporter in Iceland, Jón Guðmundsson (1807–1875), wrote an arithmetic textbook in 1841, *Reikníngslist, einkum handa leikmönnum / Arithmetic Art, mainly for Laymen*. The book was printed at the printing press in Viðey where Magnús Stephensen resided during his final years, and published at the cost of Ólafur M. Stephensen, the son of Magnús Stephensen and the son-in-law of his brother. Ólafur M. Stephensen hired the printing press and ran the publishing house after his father's death until 1844 (Jónsson, 1867).

The *Arithmetic Art* was, like most textbooks published into the first quarter of the twentieth century, aimed at extensive home education. The author defined his target group mainly for self-instruction of those who already could read and felt necessary to safeguard himself against others:

> ... Those [who know arithmetic better than the author] I do not have to tell that the book is not written for them but for laymen ... (Guðmundsson, 1841: To the Reader).

The book includes the traditional content. A section on percentages is new vis-à-vis the 18th century books. It contains topics such as the Euclidian algorithm as does Olavius's book, and testing of operations, but no algebraic explanations. No evidence has been found as yet concerning its use except that it was used in 1842–43 at the Latin School, where Danish textbooks were otherwise prevalent. No other schools existed where the book might be used. It is not mentioned in the author's biography.

The author mentioned Stephensen's *Short teaching* and current Danish textbooks in his text, where he explained why he preferred the Danish approach to the Rule of Three to the Governor's 'long one'. The rhyme on values of digits in the placement notation is repeated from Stephensen's book, as is the case of a very foreign example: In a war, a general leads 132,000 men against 91,000 men, numbers greatly exceeding the Icelandic population (Stefánsson, 1785, p. 27; Guðmundsson, 1841, p. 27–28). The 1841 *Arithmetic Art* contains more examples on trade with merchants than the 1785 *Short Teaching*. The former monopoly trade became free for all subjects of the Danish Kingdom in 1787 which contributed to increased trade. Both textbooks contain examples on fish catch and fish as a barter unit. While Stephensen exhibits a number of examples regarding time, Guðmundsson is more concerned with international geography, but no less with moralizing on expenditure on the imported goods: coffee, tobacco and alcohol.

E. Briem

The Reverend Eiríkur Briem (1846–1929) published an arithmetic textbook (Briem, 1869) in cooperation with printer Einar Þórðarson, when Briem was only 23 years old. The printing press had been transferred to Reykjavík in 1844 and from 1852 it was run by Einar Þórðarson who finally bought it in 1877. Freedom to print was enacted in 1855.

Briem enjoyed home education and stayed in Reykjavík for only three years during his studies at the Latin School and the Theological Academy where he graduated in 1867 (Bjarnadóttir, 2010). While staying at home, Briem studied Stephensen's treatises on economy thoroughly, especially the paper on the profitability of sheep farming exceeding cattle farming and fishing (Bárðarson, 1931). Briem became influenced by Spencer's educational theories when he stayed in Copenhagen in 1879–1880. Spencer e.g. voiced his views on religion and economics, both subjects occupying Briem. Briem's *Arithmetic* was published in an extended version in 1880 after Briem's stay in Copenhagen. It was widely used into the 1910s. His aim was to create a practical handbook for the self-instructing youth, devoid of theoretical explanations, but with the important additions, logarithms:

> ... for the chapter about algebra, equations and logarithmic calculations I have, however, expected that people had some instruction; in that chapter I have, as elsewhere, avoided supporting the rules prescribed by reasoning; when I have made exceptions in several places, it is because the reasoning could as well be an exercise or it was so clear that it could be used to support the memorizing of the rule (Briem, 1880: iii).

The first two lower secondary schools were established in the early 1880s. The author's brother used the arithmetic textbook for teaching at a new lower secondary school in northern Iceland, and the author himself taught the book to freshmen at the Latin School for eight years.

The content of Briem's *Arithmetic* follows the European tradition described by Swetz (1992), adding decimal fractions, percentages and interests as did Guðmundsson. Briem was not concerned with philosophical discussion on the number concept, but went directly to the placement notation, guided by the same rhyme as his predecessors, Stephensen and Guðmundsson. A study of the textbook reveals the author's roots in a self-sustaining society of sheep and cattle farming, supported by seasonal fishing, as well as his wish to provide people with financial education and advice concerning cautious allocation of their income.

The period from 1880 to the Great War was characterized by considerable societal change, both in educational matters and in economic terms. Economic progress was based on fishing at a larger scale than before, using decked vessels. Briem was an active participant in the development as a Member of Parliament and guardian of the first National Bank of Iceland, besides teaching at the Theological Academy and being philosophy professor at the University of Iceland from its establishment in 1911. However, his *Arithmetic* did not change with the times. It contains only a few examples of running fishing vessels, involving both profits and losses. There is also an example of a farmer's expenditure on coffee for ten years. The reader is asked to compute the interests of the amount which otherwise could be used to improve the farmer's land and increase his livestock (Briem, 1905, p. 37, 41).

Ó. Daníelsson and S. Á. Gíslason

Ólafur Daníelsson (1879–1957) studied mathematics in Copenhagen and returned in 1904. While waiting for a suitable post in the home country with only one high school and no college nor university, he wrote an arithmetic textbook for adolescents (Daníelsson, 1906), published in extended versions in 1914 and 1920. The last version was to become the most influential textbook into the 1970s. Daníelsson was doctor of mathematics while his close friend, Sigurbjörn Á. Gíslason (1876–1969), who also wrote a noteworthy arithmetic textbook series, was a theologian. Both of them introduced the novelty of the metric system, authorized in Iceland in 1907. However, their didactical visions were different and so were their backgrounds.

The population in Reykjavík had become 11,600 in 1910 out of a total population of 85,183, or 13.6% (Jónsson and Magnússon, 1997). Gíslason lived in Reykjavík from the age of 15, first as a school boy and later as a teacher. He was the first of the arithmetic textbook writers mentioned above, to aim his series at school teaching. He emphasized mental arithmetic, introduced flexible methods and warned of rote learning and introducing tables too early, but recommended use of the abacus and examples referring to daily life (Gíslason, 1911, pp. 3–4). Gíslason's series counted six small volumes, four of them intended for primary level from the age of seven. Daníelsson's text was not fit for primary level teaching and will not be treated individually in this respect but was duly reflected in the text of his student Bjarnason.

E. Bjarnason

E. Bjarnason (1879–1970), author of a perennial textbook series, was a son of a farmer, educated at home by his mother, a minister's daughter. He arrived at the age of 22 from the southeast Iceland to the fishing village and commercial centre of Eyrarbakki in the southwest to fish in the wintertime in 1901–2 (Skaftfells, 1945). Eyrarbakki hosted the country's oldest extant primary school, established in 1852. The teacher allowed Bjarnason to stay as much at school as he could in his spare time when weather precluded fishing. During the next season, in 1902, Bjarnason did not go to sea but established his own school with the support of his parish minister. When the Teacher Training College was established in 1908, he attended it for one year, to graduate in 1909. By that time he was a married man, a father to three children and running a farm.

Bjarnason returned to his school and farm, but in 1919 he became a teacher, and later head-teacher, at the Reykjavík Primary School. His textbook series was intended for age 10–14, the age level of compulsory schooling at that time. It was first published in 1927–29 – probably existing earlier in a manuscript, even since his rural school teaching – and it was reprinted into the 1970s. It is written under a strong influence from Ólafur Daníelsson, his teacher at the Teacher College. This case is reminiscent of Prof. Geuss's impact on young Stephensen; in both cases young men who without hesitation transfer their mentors' ideas to share with their countrymen.

Bjarnason lived at the junction of two eras, without and with schools:

> I wanted this book to be of aid to children and adolescents who want to learn the basic concepts of arithmetic on their own ... I have tried to have all explanations and hints as simple as I could and not more than I found necessary. I do not consider it practical to introduce children to many different methods for the same task. Later, when the basic concepts have become firmly established, more methods may be pointed out if it is considered desirable.
>
> Many parents, especially in Reykjavík, have told me that they do not know any arithmetic textbook that they feel they can use, even though they wanted to guide their children in their arithmetic studies. I wanted this book also to assist them.
>
> Finally, I hope that the textbook becomes usable as a textbook in schools – while not for small children. The intention is that teachers can, if they wish, leave out the explanations as they are presented in the book without rearranging the order of the tasks (Bjarnason, 1927, p. 3).

The order of expected target groups of the 1927 textbook is as follows: first self-instructing youth; secondly their parents; thirdly the schools. At this time half the population lived in rural areas. Nearly half the age cohort 10–14 years attended itinerant schools. They were based on self-instruction for long periods, which was not always favourable for mathematics learning. Lower secondary schools were confined to one state-run school, a handful of private schools and a rising number of technical schools in the small urban nuclei. Presumably the majority of students never progressed beyond the four operations in whole numbers and common and decimal fractions, and probably had problems with that. Bjarnason's books may therefore well have met the anticipated target groups at its time.

Gíslason's *Arithmetic* and Bjarnason's *Arithmetic* were authorized as textbooks for primary schools in 1929. Bjarnason's book was republished by the State Textbook Publishing House, established in 1937, following the Great Depression. Its role was to distribute free textbooks to children, but the publishing house was kept underfunded for decades and the book remained the sole choice of an arithmetic textbook into the 1970s.

The target groups of textbooks in three centuries

All the authors feel a need to define their target groups and in most cases school use is secondary. The authors of 1780s and 1840s seem to expect the ordinary reader of books to be an educated man. Their books, however, are not intended for the well educated but for the youngsters who already know how to read. Later authors, Briem and Bjarnason, tacitly assume reading knowledge, while they still aim at self-education. All the authors assume dire need for education in the underdeveloped Icelandic society, while they beware of offending their countrymen.

Briem, in 1880, was expecting progress in the nearest future. However, progress in education had still a long way to go in Bjarnason's time when schooling in general was still far behind the other Nordic countries. In the late 1920s the state paid 40 *krónur* (crowns) per child for its education, while in Denmark the amount was 211 kr., in Norway 245 kr. and in Sweden it was 214 kr. (*Menntamál*, 1929). In a country with conditions of communication and transport similar to Norway, Iceland spent only 16% of what Norway spent on schooling.

Bjarnason's time was, however, a turning point in demographic development and education. After 1925, the majority of the population lived in towns and villages with 200 inhabitants or above (Jónsson and Magnússon, 1997). A considerable number of lower secondary schools were established in districts and towns in the late 1920s and a legislative act on primary and secondary education in 1946 opened up direct track from primary to secondary education.

Bjarnason hardly foresaw the longevity of his work, which is conjectured to be partly due to wide-spread nepotism, tracing back to the Stephensens. Bjarnason's son was the Director of Educational Affairs in 1944–1974, the heydays of Bjarnason's textbook, working closely with the monopolized State Textbook Publishing House, where he was a member of the governing board in 1956–1964. The son himself was brought up in traditional rural setting and the medieval arithmetic text tradition, and may not, in spite of his high position in the educational system, have been familiar with any different approach to arithmetic teaching (Bjarnadóttir, 2006, p. 148, 209).

Relations of the textbooks to the society

The 18th century authors wrote their textbooks in or right after their stay in Copenhagen; this applies also to Magnús Stephenson's additions to his father's book. Those books are written under foreign, Danish and German, influences. At least Olavius seldom referred directly to Icelandic society, except to Icelandic trading units which were drawn from barter trade and the basic unit was 'fish', where 240 fish units equalled one cow (Bjarnadóttir, 2006, p. 78). There was a severe lack of monetary currency which did not become common until in the last phase of the period in concern. Olavius did not stay or work in Iceland more than a couple of years when he was in charge of the printing press. He was not concerned with ordinary farming life, but with didactic ways of mental arithmetic, translated from Clausberg's *Rechenkunst*.

Stephensen mentioned Iceland directly in one example: "Anno 1750, about 10,000 barrels of meal arrived to Iceland, intended to be share between the 40,000 people who were known for certain to live there at that time; how much meal was the share of each that year?" (Stefánsson, 1785, p. 60). The example provides interesting information about import of meal, showing that Iceland was not exclusively a self-sufficient society. The meal was hardly imported as charity, the year 1750 was better than 1751 and the following years when harsh climate caused famine. Poor farmers may have exchanged the more expensive meat for meal to mix with blood, liver and kidneys for puddings.

The Stephensens belonged to the elite of landowners and royal officials. No one doubts their sincere wish to educate the general public but they took the order of society for granted. An example concerns a wedding reception of 70 guests. They are to sit at three tables and the available amount of money is 200 rixdollars; the elite is to have food for 4 rd., people at the middle table for 3 rd. and at the lowest table for 2 rd. each. The question was how many could sit at each table.

Impacts of the Stephensens' societal interests may not be detected in their own textbook but their followers' textbooks. Guðmundsson's and Briem's books are written to teach people to make the most of what they have and what they earn. More than one example in Guðmundson's *Arithmetic Art* warns the reader against consuming alcohol: Two brothers earned some money as farm servants and fishermen during a year, but the elder one earned more. However, the younger one had more left after the year as the elder brother squandered his money more on diverse useless items such as spirits, coffee and bread by the baker. Another example of this kind tells a story of a shady dealer buying spirit in Reykjavík (the Sodoma of Icelandic society) to sell in rural areas and making (too) much profit. Still another example concerns a worker in Reykjavík who began to consume half a pint of spirit each day from the age of thirty to his death at sixty. The expenditure was gigantic, which must have been the intention to demonstrate. However, health issues and social problems were not mentioned in this context.

In many of his story problems, Guðmundsson recounts informative matters, such as the time span since Columbus discovered America, the strength of the silk thread compared to the thread in a spider web, the furs of the Russian Emperor, the distance of a lightning computed by the time delay of the thunderclap, etc. Guðmundsson is thus faithful to the Enlightenment vision, to enlighten the public.

There is no noticeable difference between the society that Guðmundsson and Briem describe in their 19th century textbooks. Both textbooks contain examples on the cost of supporting paupers, besides the usual examples on harvesting hay, feeding cows and sheep and buying or weaving material for clothes. In Briem's 1869 text, the salary of servants harvesting hay is paid in butter and their food counted in fishes. The student is to calculate the cost of the hay in rixdollars, while real money was scarce. Shopping at the merchant was still barter trade; the farmer's credit was fish, wool, meat and fat, for which he bought rye, oatmeal, beans, coffee and linen. Briem does not moralize to the same degree as Guðmundsson, while he warns seriously against borrowing:

> A farmer borrowed autumn-wool by the end of March against paying it back by the end of June by the same amount of spring-wool; now the price of the autumn-wool was 50 p., but of the spring-wool 80 p.; how much did he have to pay in % p.a.? Answer: 240% p.a. (Briem, 1880, p. 24).

The farmers had to ensure enough wool for their family and servants to spin for knitting and weaving in the late-winter and early spring before the bustle of lamb-births, shearing and hay-making; the old farming cycle in a nutshell.

Bjarnason's problems describe a society which is less self-sufficient than the one described in the 19th century textbooks. People buy daily groceries at merchants who buy and sell their goods in irregular quantities. A typical problem is:

> A farmer buys three boxes of butter. One of the boxes contains 20,5 kg butter, the second 16,755 kg and the third 16 kg. The merchant now sells gradually 4,5 kg, 3,675 kg, 14,075 kg and 9 kg. How much is then left? (Bjarnason, 1933, p. 54).

This merchant buys his goods from farmers or possibly from a dairy, and the customers buy large quantities to keep at their homes as was customary in the countryside. During the latter part of the period of

Bjarnason's textbook, after 1960, this sole arithmetic textbook for age 10–13 no longer had the aim of enhancing progress in society, but to maintain the ideology of a disappearing societal structure.

Discussion

The authors Magnús Stephensen, Eiríkur Briem and Elías Bjarnason were very young when they decided to share their knowledge with their fellow countrymen. Magnús Stephensen was only 22 years old when he was entrusted to prepare his father's textbook for publishing and took the initiative to rewrite it, and indeed only 21 years old when he prepared a well-known cook-book. One wonders about the generosity to write and rewrite whole books under other people names. The 'other people' were in this case, however, relatives, members of a family with strong family ties.

Eiríkur Briem was 23 years old when he wrote his textbook, which became the most influential arithmetic textbook for 40 years, and the common man, Elías Bjarnason, was only 23 years old, when he decided to establish a school after becoming acquainted with that kind of activity in bad-weather breaks from fishing. Jón Guðmundsson completed the Latin School late due to illness and poverty, and at the age of 35 when he published his textbook, he was not yet the influential person he later became. Sigurbjörn Á. Gíslason began teaching just after graduation from the Latin School at the age of 21. He may then have begun creating exercises for his six volumes textbook series, which he published at the age of 34.

All our textbook authors became well-known in Iceland. Magnús Stephensen as the highest magistrate at the country's court and extremely active editor of educational publications of a variety of kinds; Ólafur Olavius as a prolific writer whose products were published in Copenhagen while he never had a position in Iceland; Jón Guðmundsson as the editor of an influential journal supporting the independence movement; Eiríkur Briem as a professor of theology, member of parliament and the warden of the first National Bank; Sigurbjörn Á. Gíslason as teacher at secondary schools, but later reputed for his humanitarian work; and Elías Bjarnason as the head teacher of Reykjavík Primary School, the sole primary school in Reykjavík until 1930.

One may wonder if the longevity of the books may be attributed to the high positions of their authors. This certainly contains a grain of truth; Magnús Stephensen directed the country's only printing press until his death in 1833. No arithmetic textbook competing with the book of the father and son was published there while he was alive, while the son, Ólafur M. Stephensen, printed Guðmundsson's *Arithmetic Art* at his own cost, when his father's 56 year old book was probably out of stock. Elías Bjarnason taught at the far largest primary school in the country, and his influential son could take care of his interests at the monopoly State Textbook Publishing House into the 1970s.

Eiríkur Briem taught arithmetic at the sole Latin School, where he used the book, and his brother used it in the country's sole state-run lower secondary school, which ensured a certain, although limited, distribution. Gíslason taught at many schools in Reykjavík and it was not until 1942, at the age of 66, that he turned to priesthood at the nursing home he had established. As for Jón Guðmundsson, he had neither an official position nor worked at a school. He was, indeed, *persona non grata* by the Danish authorities and nothing is known about the distribution of his textbook. There was no secondary school at his time except the Latin School where Danish textbooks were exclusively used, except Guðmundsson's book for one year, until Briem introduced his own book. Olavius had no official position in Iceland and his book was used in neither of the two Latin Schools of that time.

Conclusions

The first champions of the Enlightenment, who wrote the textbooks, were influenced by cultural currents during their studies at the University of Copenhagen, while their followers had never been abroad before the publication of their textbooks. Their knowledge of international cultural trends was in their cases confined to their reading, not the least writings by the Stephensens. They were concerned with society and wanted to contribute to its development by education on improved farming in Ólafur Stephensen's understanding; that fishing and agriculture were indispensable parts of the same industry and the livestock farming the basic foundation of society. They had reservations about the urbanisation of society and wanted to contribute to the ideological maintenance of the old self-sufficient society by demonstrating how to act economically within its limits.

Several more factors were involved in the production and distribution of textbooks, in our case arithmetic textbooks, in the 18th, 19th and early 20th century. The first factor was a youthful <u>enthusiasm</u>, inspired by the Enlightenment, to promote an underdeveloped society. The second factor was a <u>social network</u>, either high social position or even nepotism as we saw in the case of Stephensen and Bjarnason, or a position within the school system, however tiny, possibly intertwined with the first factor. Stephensen, Briem, Gíslason and Bjarnason provide examples of this. In this respect, access to a printing press and/or a publishing house was of uttermost importance as seen by Stephensen, Guðmundsson, Briem and

Bjarnason. The third factor was the target group, in all cases primarily the self-instructing youth, a witness of a society lacking schools for the general public.

The textbooks were all related to present society at the times of their publication, even the 20th century textbooks took some notice of the emerging urban society. That Bjarnason's book became out of phase with society, being the sole choice for more than three decades, is a different story about how an official decision may hinder development. It is regrettable that Bjarnason's textbook, reflecting the predominantly rural society of the early 20th century, remained the only choice for city children of the 1960s and 70s. It reminds somewhat of the monopolistic position of the *Short Teaching* by the Stephensens. There is a second similarity between the Stephensens and Bjarnason, i.e. the son supporting and promoting the father's work. Bjarnason's son was in a key position to continue the publication of his father's work indefinitely.

Didactical interests seem to have been secondary. Far into the 20th century calculating correctly by hand was of crucial importance, so that it may have been considered more useful to know one method securely than a variety of methods. From the modern point of view, Olavius/Clausberg's as well as Gíslason's interests in versatility in algorithms and emphasis on mental arithmetic are, however, more likely to underpin real understanding and motivation so that mathematics becomes more of a tool to cope with life than mere exercises in procedures and algorithms.

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Ernst Breslich and his influence in Brazil: The debate on fusion of the branches of school mathematics

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Abstract

We present briefly the career and ideas of the American educator Ernst Breslich, who played an important role in American mathematics education in the 1910-1930 period, and how he influenced mathematics education in Brazil. Euclides Roxo, a very important mathematics educator in Brazil, was responsible for the mandatory national secondary school mathematics curriculum of the 1931 educational reform. This curriculum, strongly influenced by Felix Klein´s ideas, made mathematics present in all school years of secondary education and abolished the separate teaching of arithmetic, algebra and geometry in different school years or as independent school disciplines. In his project to "correlate" (unify) the various branches of school mathematics, Roxo followed Breslich´s ideas and adopted his textbooks as model for the ones he, Roxo, wrote for the new mathematics programs.

Introduction

The role played by Felix Klein in the mathematics teaching international reform movements at the end of the 19th and beginning of the 20th centuries is well known (Miorim, 1998; Schubring, 1989). In particular, his ideas had a strong influence in the Brazilian secondary school mathematics teaching reforms of the late 20ties and 30ties of the 20th century. The fusion or correlation of the various branches of school mathematics was one of the major concerns of Klein and IMUK.[1]

On the other hand, the influence of Ernst Breslich on Brazilian mathematics teaching reform is very little known outside Brazil. Actually, Breslich has been little studied, notwithstanding his educational activities in a very important period, at a new, dynamic and innovative institution – the University of Chicago.[2]

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[1] IMUK (Internationale Mathematik Unterricht Kommission, Commission Internationale pour l'Enseignement des Mathématiques) was created at the 4th International Congress of Mathematicians, which convened in Rome in April 1908. In the 1950s its name was changed to ICMI (International Commission on Mathematical Instruction).

[2] References to Breslich can be found in (Miranda, 2003; González & Herbst, 2006; Carvalho, 2006), the last specifically on Breslich's influence in Brazil.

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Ernst Breslich

Ernst Rudolf Breslich was born in 1874 in Germany, and became an American citizen in 1896. It is not known when he arrived in the United States. In 1900 he received his Master's degree from the University of Chicago. He was the director of the mathematics department of the Laboratory Schools of the University of Chicago during the period 1913-1920, and taught at the same school till 1925. In 1926 he received his doctorate, also from the University of Chicago (Breslich 1926). In 1929, he becomes again director of the mathematics department of the Laboratory Schools. He retired in 1939 and came back to the University for six months, in 1943. He died in 1966, at the age of 92 years.

Breslich was the president of the School Science and Mathematics Association for two years, 1926 and 1927, a director of NCTM from 1936 to 1939 and one of its vice-presidents for the period 1939-1941. He was on the editorial board of *The Mathematics teacher*, starting with its first number, in 1921.

Recognition of Breslich's work can be seen by the fact that, for example, vol 34, 1941, of the periodical *The mathematics teacher*, was "dedicated to Professor *Ernst R. Breslich*, who served the University High School as teacher and as Chairman of the Department of Mathematics from 1904 to 1939" and vol 24, 1941, of *The high school journal* was also "dedicated to Dr. *Ernst R. Breslich* who was one of the moving spirits, and head of the mathematics department in the University [of Chicago] High School practically from its creation". Fehr (1971, p. 91), in his review of *Readings in the history of mathematics education*, edited by Bidwell and Clason (1970), laments the omission, in the book, of Breslich and other equally important educators, "giants", as he calls them.

While he was alive and active, Breslich wrote, in the *Mathematics teacher*, the obituaries of many of his colleagues at the University High School. It is a sad fact that when he died there was no obituary written for him.

The University of Chicago and the reform of high school mathematics

As seen above, Breslich had a long association with the University of Chicago, which was established in 1890, funded by John Davison Rockefeller. Its first president, William Rainey Harper, tried to shape an institution which would combine the characteristics of and American liberal arts college and of a German research university. According to Frederick Rudolph (1990), around 1900, the University of Chicago represented an epoch's ideals and had a decisive influence in shaping American university education.

The Laboratory Schools of the University of Chicago were established in 1896 and followed Dewey's ideas, specifically his belief that you learn by doing. We recall that Breslich was the director of the mathematics department of these schools for a total of 17 years.

The role played by the Laboratory Schools of the University of Chicago is stressed in (Kilpatrick 1997, p. 955):

> At the turn of the century, reformers at the University of Chicago High School […] were attempting to unify the secondary curriculum, principally by merging the year-long courses in algebra and geometry.

These reforms attempts tried to "correlate" algebra and geometry in the high school curriculum, abolishing their separate teaching. This was a long struggle, carefully studied in (Sigurdson, 1962). As mentioned by Kilpatrick (1997, p. 955):

> Reaction to what was to become known as the "Chicago movement" was swift. Conservative mathematicians in the East, most prominently David Eugene Smith, although tolerant of the brash Midwesterners tinkering with new approaches, argued that the secondary classroom was a place for pure, not applied mathematics. In particular, the mental disciplinary power of geometry, together with its aesthetic and cultural value, demanded that it be kept in a separate course.

Differently from the United States, "correlation" was successfully achieved in Brazil, thanks to the Brazilian educator Euclides Roxo, influenced by Felix Klein and Ernst Breslich. The importance of Breslich's efforts toward "correlation" are attested by Hall-Quest (1920, p. 102):

> Perhaps the most notable scheme of correlation in the field of mathematics is that evolved by Ernest R. Breslich. His three volumes on First, Second, and Third Year mathematics, respectively, demonstrate the possibilities and advantages of correlation.

Ernst Breslich's publications

Breslich wrote at least 26 periodical articles, of which 17 appeared in the School review. He was the author or co-author of 13 textbooks. He also wrote a collection of three books on the teaching of high school mathematics:

- The teaching of mathematics in secondary schools Technique. 1930.
- Problems in teaching secondary school mathematics. 1931.
- The administration of mathematics in secondary schools. 1933.

He was the author of chapters in four yearbooks of the National Council of Teachers of Mathematics, and of several books on specific subjects, among them The slide rule and Diagrams in three dimensions for solid geometry. These books are related to Breslich's belief in the importance of the mathematics laboratory, a collection of mathematical instruments and models, to be used actively by the students.

According to the author, Correlated mathematics for junior colleges, of 1919, written to follow Third year mathematics, could be used as a fourth year textbook or in a junior college. In this book, Breslich defends again correlation of the fields of school mathematics, mainly of algebra and geometry: "As in the first three books of the series, the fundamental principle of the course is to associate closely mathematical topics which are naturally related to each other". On the other hand, he wrote books which do not seem to be related to the ideas he defended about correlation, like Trigonometry with tables for use in high schools and junior colleges, written jointly with Charles A. Stone, in 1928. In 1938, he authored the collection Purposeful mathematics, published by Laidlaw Brothers, divided in

- [v. 1] Algebra – first course;
- [v. 2] Plane geometry;
- [v. 3] Algebra – second course;
- [v. 4] Solid geometry.

His most used textbooks were the ones he wrote for junior, middle and senior high school, a total of nine books, starting with First year mathematics through Ninth year mathematics, and which had several editions and reprints. Their very titles, like, for example, First year mathematics, or Junior mathematics, Senior mathematics are statements defending the unification of school mathematics. The appendix to this paper has a hopefully complete list of Breslich's books, with its editions, re-editions and reprints. As the fusion movement lost its power, it seems that Breslich wrote more and more traditional books.

The Laboratory School of the University of Chicago, as a result of its attempts to reform the high school mathematics curriculum published, in 1906, First year mathematics for secondary schools, written by George William Myers, William R. Wickes, Harris F. MacNeish, Ernst R. Breslich and Ernest A. Wreidt. Myers is listed as professor of the teaching of mathematics and astronomy, College of Education, the University of Chicago. The other four co-authors are listed as instructors of mathematics in the University High School of the University of Chicago. The following year, 1907, it was published by the University of Chicago Press. It is interesting to note that in its first edition the book had less than 200 ages, and in a later edition (1915) it went up to 343 pages.

The fourth edition of the book was published in 1915, and it has an editorial preface, signed by Eliakim H. Moore, George W. Myers and Charles H. Judd. In it, they describe the process of the book creation. According to them, starting in 1903, the Laboratory Schools of the University started testing a program for the reorganization of high school mathematics, stressing "correlation", "fusion", of the several areas of school mathematics, worked out in "conferences of the teachers of mathematics of the college and the high school of the School of Education". The resulting program was revised, mimeographed and used in the 1904-5 school year. It was again revised, remimeographed and used in 1905-6. After further revisions, it was published, in book form, by the University High School, in 1906. In 1907, a small manual, Geometric exercises for algebraic solution, was published. It supplemented a standard geometry text which was then used by second year students. First year mathematics was further revised and expanded, with two new authors included: Arnold Dresden and Ernest L. Caldwell.

In the 1915 edition, Breslich is the only author. The editors justify this, stating that (Editor´s preface, p. vii)

> While both books are the natural outgrowth of the experiment begun twelve years ago, Mr. Breslich has entirely recast and rewritten the texts. His earnest and untiring work on the experiment from the outset has peculiarly fitted him for the task. Authorial credit for the present form of the material is entirely due to him.

Reviewing the fourth edition of First year mathematics, David Eugene Smith stated that

> The central idea of the work seems to be to select those features of secondary mathematics which are easily within the reach of beginners, postponing the consideration of the more difficult ones to a later period.
>
> …
>
> In the pursuit of this idea the author proposes to treat of algebra and geometry at the same time. … He also proposes to consider those "subjects in which practical values are most clearly exhibited," to introduce a certain amount of trigonometry, to avoid "formalism in mode of presentation," and to give the student a "broader mathematical preparation." (Smith, 1915, p. 136).

Smith is very critical of Breslich's attempt at the "fusion" of the several fields of school mathematics:

> Of course the book can be successfully taught; that is true of any book, provided the right teacher is available. But that a book with what seems to be a forced fusion of essentially different branches of a

> science, based solely upon the theory of ease of presentation, which theory does not seem to have been carried out – that such a book can be generally successful can hardly be expected. (Smith, 1915, p. 136).

Also, S. E. Slocum criticizes Breslich's Second year mathematics:

> This book is one of the numerous recent attempts to correlate elementary algebra, geometry, and trigonometry for purposes of instruction. The idea of correlation is of course one of great possibilities, but to be successfully realized it must be based on some central and unifying principle, such, for instance, as the function concept advocated by Klein. A careful examination of the present work fails to reveal any such principle of selection or arrangement, and leaves the impression of a haphazard collection of unrelated topics.
>
> …
>
> The modern demand for economy of time in education will eventually lead to correlation of mathematics in which the central idea will be to cultivate in the pupil the habit of mathematical thought and exact expression, to give him equal facility in the application of algebraic and geometric methods, and to make the subject matter vital by modern applications and interpretation. The outline given above seems to indicate that the opportunity to accomplish these results has been almost entirely overlooked in the present work, as the arrangement has no advantage over similar material selected at random from standard texts. (Slocum, 1917, p. 328)

Slocum is right when he mentions that Breslich did not choose the notion of function as the organizing and integrating concept in his presentation of high school mathematics, but, slowly, Breslich's ideas change. Without abandoning his belief in "correlation" in secondary school mathematics he adopts more and more Klein's ideas. In 1921, in the report of the discussions of the Symposium of the national committee on junior high school, he stresses the importance of mathematics and of functional relations in mathematics:

> As the primary purpose of the teaching of secondary mathematics the committee states "the development of powers of understanding and analyzing the interdependence of quantities and spatial relations which are necessary to a better appreciation of the progress of civilization and a better understanding of life and the universe about us and to develop those habits of thinking which will make these powers effective in the life of the individual. (Breslich, 1921, p. 23).

He also writes that "intuitive geometry is to be taken up early", followed by an "introduction to demonstrative geometry".

The committee believed that the new organization it proposed

> [S]hould accomplish the aims and purposes set up by the committee more easily than is possible with separate courses in arithmetic, algebra and geometry. Topics could then be put in psychological order with the result that pupils would gain from the study of mathematics the broadest possibilities. (Breslich, 1921, p. 25)

In May 1933, Breslich gave a talk at the meeting of the New York Society for the Experimental Study of Education (Breslich, 1933). He proposed that the "The curriculum must be adapted to the changes in the social order" and described the reform movement at the beginning of the 20th century and Felix Klein's role in it. He mentioned that Klein stressed two great pedagogical principles: algebra and geometry should be integrated, using the unifying power of the function concept, and that psychology should determine curricular organization. He also mentioned that several mathematicians widely applauded committees had already voiced the opinion that some kind of integration was desirable (Breslich, 1933, p. 327).

Breslich also mentioned the need to adapt mathematics to the social needs of the epoch:

> It is necessary to recognize that the pupils may be classified in two groups: a large group which has been increasing in the last decades and in which are found those who dislike deductive geometry because they cannot understand it; a second group, decreasing from year to year, which consists of pupils who study geometry because they enjoy it, and in many cases also because they plan to prepare for future work, in which a knowledge of geometry is essential, as in engineering.
>
> Deductive geometry as it is now organized serves the second group in an admirable way. […]. Perhaps the time has come to build a second course in geometry which will appeal to the interest and which will serve the needs of the first group as well as Euclid's geometry has served the second for centuries […].
>
> Such a course would continue the good work in geometry that has been started in junior high-school mathematics. […] The method of study should be inductive and experimental, although some deductive reasoning would have a place. Geometric constructions, drawing, and designing should receive much attention. A major objective would be training in space and spatial relationships. While learning the principles of plane geometry the pupil would make frequent excursions into three-dimensional space. (Breslich, 1933, p. 331)

These quotations, chosen among many from Breslich's publications, show that he really adopted Klein's ideas about the teaching of mathematics.

Breslich's influence in Brazil

Breslich influenced the teaching of mathematics in Brazil through Euclides Roxo, whose role in the Brazilian mathematics education reform of the late 20ties and early 40ties of the 20th century has been studies by many researchers, among them Carvalho (2003; 2006), Dassie (2001), Rocha (2001), Valente (2003) and Werneck (2003).

Roxo was a mathematics teacher in Colégio Pedro II, which was established in 1837 as a model institution for secondary education, and its director in the period 1925-1935. Because of his official position, he participated in the educational debates in Brazil during the 20ties and 30ties of the past century, and was aware of the need of reform. Besides, he had been a teacher in a normal school, at a time when these schools were important as a source of innovation.[3]

Roxo, aware of the international reform movement of mathematics education, and the head of an important institution in a period of great changes in Brazilian society, saw the need to reform the mathematics curriculum. In 1929 he achieved this for Pedro II. His proposals were incorporated practically "in totum" in the nationwide reform of 1931, decreed by Francisco Campos, minister of education, and which reorganized secondary education in Brazil (Werneck, 2003).

Aware of the need of reform, familiar with IMUK, and with Klein's proposals for mathematics education, Roxo vigorously adopted these ideas, in which he found a clear cut proposal, defended by a prestigious mathematician. His adoption of Klein's ideas can easily be seen in the preface of the book he wrote for the reform he was able to have approved in Pedro II in 1929. According to Miorim, this reform "represented a radical change in the mathematics curriculum" and it contained "all the modernizing ideas defended in the international movement of reform of the teaching of mathematics" (Miorim, 1998, p. 91) since the beginning of the 20th century.

Klein's ideas furnished Roxo the general principles for the reform. But a new curriculum needs new textbooks. And so Roxo decided to write proper textbooks. Financial considerations may also have played a role, because the textbooks used in Colégio Pedro II were assured of a great dissemination through the country. [4,5]

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[3] For the important role of these schools in the reform movement in Brazil see (Dassie, 2008), an important reevaluation of the reform movement in Brazil, till 2008 seen in a rather simplistic light.

[4] Roxo was violently attacked because He was the director of Pedro II and wrote the textbook to be used in the school.

Roxo liked Breslich's textbooks, since they followed the model he, Roxo, proposed, of a unified secondary school mathematics. Maybe pressed by the need of speed, before others had the time to write textbooks for the new curricula, or impressed by the fact that Breslich was a teacher at an important institution of a "more civilized country",[6] or because he did not feel himself capable of incorporating Klein's ideas into a textbook, the fact is that Roxo decided to follow the model of Breslich's books.

We shall not discuss here the objections to the new mathematics curriculum, and will deal only with the reactions to the fact that he used heavily Breslich's books to write his own collection.[7] The purpose of these attacks was to disqualify Roxo's competence, and hence weaken his influence with the Ministry of Education. Indeed, we see that Roxo retreats gradually, and by 1942, the year of the second important educational reform, many of his ideas were no more accepted.

Following the new programs, Euclides Roxo started writing his textbook series *Curso de Matemática Elementar*, which was to cover all secondary school years, but of which only the first three volumes were written independently by Roxo (1929). The preface of the first volume states clearly Roxo's debt to Breslich:

> This textbook [Breslich's] revised and modified for 25 years, following the results of its use and the students reactions, had his definitive form written by Ernst Breslich, one of the teachers mentioned above, and adopted in many secondary schools in the United States of America. Many other textbooks have been published in that country, following the modern guidelines, but the most successful ones are exactly those that have adopted Breslich's model. Indeed, it seems to us that no other successfully harmonized all the tendencies of the great reform.
>
> Because of this we did not hesitate to adopt it as the model of our modest work. Thus, chapters IV, V, VII, VIII, IX X and XIII, in whcich it is tried to present the first notions of algebra based on geometry's concrete support, followed Breslich, with the necesssry adaptations to our culture and our students. (Roxo, 1929, pp. 12-13, my translation).

Roxo was criticized both for integrating arithmetic, algebra and geometry and for choosing Breslich as a model. His (Roxo's) textbooks

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[5] Dassie (2008) shows that Roxo's books were not the only ones to propose a new approach to school mathematics.

[6] Expression used by Roxo when he defended his ideas in several newspapers articles (Dassie, 2001; Rocha 2001).

[7] For a concise presentation and references to further research, see Carvalho (2006).

were very different from previous textbooks adopted at Colégio Pedro II. Books like *Lições de aritmética*, written by Roxo (1925) in the twenties, and which is a disguised translation of Tannery's *Leçons d'arithmétique* (1894), or *Lições de álgebra*, by Almeida Lisboa (1911), one of Roxo's stauchier critics and his colleague at Pedro II, could not be used anymore. Lisboa, after attacking the new programs, says

> The books in which Mr Roxo expounds his program are too childish. Their practical applications are fictitious and useless. In these books one finds no trace of the proof of a single theorem; there are only imperfect and gross experiments and measurements. There is no deductive reasoning, characteristic of a mathematical proof. There are wrong or imprecise notions. Everything useful for the intellectual development of the student is abolished. (Lisboa, 1930, my translation)

Defending himself, Roxo repeats again and again that he followed well-known "authorities":

> As I have already stated, in my first preface, my books are compilations and adaptations of foreign works. Both in these books and in the articles I have been publishing in this newspaper the only characteristic that deserves praise is that nothing is mine, everything comes from others.
>
> I am constantly saying this, with many quotations, risking to be accused of erudite exhibitionism. I am more and more convinced that this was necessary, because my critics try to brand me as "innovator", ""discoverer", "inventor" of new methods. Some, more ignorant, even claim that I have invented a new mathematics.
>
> I did not invent, do not want to invent, do not ever intend to invent anything.
>
> And more, to destroy the objection that these books have nothing to do with Klein's ideas, […] I quoted Klein's statement in which he mentions to the German teachers the beginnings of the reform movement in the University of Chicago, stating that he believed this movement was analogous to the one he was fighting for in Germany. (Roxo, 1931, my translation)

Many of the problems and examples of Roxo's book were indeed copied or slightly changed versions of Breslich. Roxo was also criticized for trying to introduce new terms, to replace traditionally adopted ones in Brazil, like *rectangular block* for right parallelepiped, among others. Few of his attempts succeeded.

As shown by Dassie (2008), Roxo's books were not the only ones that followed the ideas preached by Klein and IMUK, but were very influential. The mandatory national programs instituted by the 1931 reform defined the contents of all secondary school textbooks, but there was considerable leeway as to the manner of presentation of the contents of secondary school mathematics. The presentation chosen by Roxo owes much to Breslich.

There were competing textbook series written by members of the faculty of Pedro II, who eventually agreed to write a joint textbook collection, of which Roxo was one of the authors, and to which he brought the experience he had acquired with Breslich. Later on, he also participated in another widely used textbook series, which had many, many editions, well into the 50ties of the 20th century. Thus, it seems safe to say that Breslich really influenced mathematics education in Brazil.

Final remarks

The main conception introduced by Roxo in the reforms he influenced, in 1931 and in 1942, was the unification of secondary school mathematics, that is, the "fusion" or "correlation" of the various branches of school mathematics. At first, Roxo tried to really integrate these branches, following Breslich's books. In the second reform, in 1942, he had already lost much space, and had to retreat to a more conventional position. For example, he was forced to postpone the introduction of the function concept to the last year of the curriculum. But he staunchly refused to accept the adoption of several textbooks for each school year, one for each field of school mathematics. He was able to preserve his initial idea of one single mathematics book per school year. So, this heritage of the American correlation movement, which became known to Roxo through Breslich, was successful in Brazil, and has been preserved till now.

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Werneck, Arlete Petry Terra (2003). *Euclides Roxo e a reforma Francisco Campos: A gênese do primeiro programa de ensino de matemática brasileiro*. Dissertação de mestrado. São Paulo, Brazil: PUCSP.

Appendix - Partial bibliographic data of Ernst R. Breslich

Sources: The Library of Congress on line catalog, accessed January, 15, 2011. The University of Chicago Library catalog, accessed may, 25, 2011.

1906 – Myers, George William et all., First *year mathematics for secondary schools*. Chicago. Related Names: Wickes, William Rockwell; Breslich, Ernst Rudolf; Caldwell, Ernest L; MacNeish, Harris Franklin; Wreidt, Ernest August. Description: xv, 181p, 2cm.

1907 – Myers, George William et al. *First year mathematics for secondary schools*. Chicago: University of Chicago press. Related Names: Wickes, William Rockwell; Breslich, Ernst Rudolf; Caldwell, Ernest L; MacNeish, Harris Franklin; Wreidt, Ernest August; Dresden, Arnold. Description: xii, 305 p. diagrs. 20 cm. First edition privately printed October, 1906.

1907 – Myers, George William et al. *Geometric exercises for algebraic solution; second year mathematics for secondary schools*, by George William Myers and William R. Wickes, Ernest A. Wreidt, Ernest R. Breslich. Chicago, University of Chicago Press, 1907. Description: 71 p. diagrs. 20 cm. Series: School of education manuals. Secondary texts

1910 – Myers, George William et al. Second *year mathematics for secondary schools*. Chicago: The University of Chicago press. Related Names: Wickes, William Rockwell. Breslich, Ernst Rudolf. Wreidt, Ernest August. Caldwell, Ernest Leroy. Mathews, Robert M. Dresden, Arnold. Description: xiv, 282 p. illus., diagrs. 20 cm. Notes: IV.

1911 – Myers, George William et al. *Teacher's manual for first year mathematics*. Chicago, Ill.: The University of Chicago press. Related Names: Ernest L. Caldwell, Robert M. Mathews, William D. Reeve. Description: xi, 164 p. 20 cm. Series: The University of Chicago. Mathematical series, E.H. Moore, general ed. School of Education texts and manuals, G.W. Myers, ed. University of Chicago mathematical series. School of Education texts and manuals (University of Chicago)

1912 – Myers, George William and William R. Wickes, Ernst R. Breslich, Ernest L. Caldwell, Ernest A. Wreidt, Arnold Dresden, Robert M. Mathews, William D. Reeve. *The point of view in "Second-year mathematics for secondary schools*," Chicago, Ill.: The University of Chicago press. Description: [11] p. 20 cm.

1915 – Breslich, Ernst Rudolph. *First year mathematics for secondary schools*. Chicago: University of Chicago press. Description: xxiv, 344 p. front., illus., ports., diagrs., 21 cm. Notes: On verso of t. p.: First edition privately printed October 1906. Fourth edition September 1915. Series: The University of Chicago. Mathematical series.

1916 – Breslich, Ernst Rudolph et al. *Second year mathematics for secondary schools*. Chicago, Ill., The University of Chicago press. Related Names: Myers, George William. Description: xx, 348 p. front., illus., plates, ports., diagrs. 20 cm. Notes: Biographies on verso of ports. "This book by its copyright purports to be a second edition of a former text of the same title by other authors" (George W. Myers, and others) cf. Editor's pref. Series: The University of Chicago. Mathematical series. E.H. Moore, general editor. The School of Education. Texts and manuals. G.W. Myers, editor. University of Chicago mathematical series. School of Education texts and manuals (University of Chicago).

1917 – Breslich, Ernst Rudolph. *Third year mathematics for secondary schools*. Chicago, Ill.: The University of Chicago Press. Description: xviii, 369 p. front., illus., ports., diagrs. 20 cm.Series: The University of Chicago. Mathematical series. The school of education. Texts and manuals.

1917 – Breslich, Ernst Rudolph. *Third year mathematics for secondary schools*, with Logarithmic and trigonometric tables and mathematical formulas. Chicago, Ill.: The University of Chicago Press. Description: xviii, 351 p., 2 l., 3125 p. illus., ports. (incl. front.) diagrs. 20 cm. Notes: Biography on verso of each portrait. Series: University of Chicago. Mathematical series School of Education. Texts and manuals University of Chicago mathematical series. School of Education texts and manuals (University of Chicago)

1917 – Breslich, Ernst Rudolph. *Teacher's guide with answers to Second-year mathematics for secondary schools*. Chicago, Ill.: The University of Chicago press. Description: 48 p. 20 cm.

1917 – Breslich, Ernst Rudolph. *Logarithmic and trigonometric tables and mathematical formulas*. Chicago, Ill.: The University of Chicago press. Description: xvii, 118 p. 20 cm. Series: The University of Chicago. Mathematical series. The School of Education. Texts and manuals

1919 – Breslich, Ernst Rudolph. *Correlated mathematics for junior colleges*. Chicago, Ill.: The University of Chicago press [1919]. Description: xiii, 301 p. front., pl., ports., diagrs. 21 cm. Series: The University of Chicago mathematical series. E.H. Moore, general editor. The School of Education. Texts and manuals, G.W. Myers, editor University of Chicago mathematical series. School of Education texts and manuals (University of Chicago

1923 – Breslich, Ernst Rudolph. *Second – year mathematics for secondary schools*. The University of Chicago Press. Description: xx, 348 p. front., illus., plates, ports., diagrs. 20 cm.Series: The University of Chicago. Mathematical series. E.H. Moore, general editor. The school of education. Texts and manuals. G.W. Myers, editor.

1923 – Breslich, Ernst Rudolph. *Correlated mathematics for junior colleges*. Chicago, Ill.: The University of Chicago Press. Description: xiii, 301 p.: front., pl., ports., diagrs.; 20 cm. Subjects: Mathematics. Series: The University of Chicago mathematical series The School of Education. Texts and manuals University of Chicago mathematical series. Texts and manuals (University of Chicago. School of Education).

1923 – Breslich, Ernst Rudolph. *First-year mathematics for secondary schools*. Chicago: University of Chicago Press, 1923. Description: xxiv, 345 p.; 20 cm. Series: The University of Chicago. Mathematical series. E.H. Moore, general editor. The school of education. Texts and manuals. G.W. Myers, editor.

1924 – Breslich, Ernst Rudolph. *Third year mathematics for secondary schools*. Chicago, Ill.: The University of Chicago press. Description: xviii, 369 p. front., illus., ports., diagrs. 20 cm. Subjects:

1924 – Breslich, Ernst Rudolph. *Junior mathematics*. New York: The Macmillan company, 1924. Description: v. front., illus., diagrs. 19 cm.

1925 – Breslich, Ernst Rudolph. *Junior mathematics*. New York: The Macmillan company, 1925. Description: v. front., illus., diagrs. 20 cm.

1926 – Breslich, Ernst Rudolph. *A critical examination and experimental reconstruction of secondary school mathematics*. Chicago. Description: viii, 251 l. diagrs. 29 cm.

1927 – Breslich, Erns Rudolf and W. C. Reavis. *Diagnostic test in the fundamental operations of arithmetic and in problem solving*. Forms A and B (For Grades VII to XII) Chicago, Ill.: University of Chicago Press.

1928 – Breslich, Ernst Rudolf and Charles A. Stone. *Classroom procedures test for teachers of mathematics*. Chicago Ill.: The University of Chicago Press.

1928 – Breslich, Ernst Rudolf and Charles A. Stone. *Classroom procedure test in mathematics for rating methods of teaching in junior and senior high schools*. Form A. Preliminary edition.

1928 – Breslich, Ernst Rudolph. *Senior mathematics*, [Bk.1]. Chicago, Ill.: The University of Chicago press. Description: v. illus., ports., diagrs. 20 cm. Series: The University of Chicago mathematical series, E. H. Moore, general editor. The School of Education texts and manuals, G. W. Myers, editor.

1928 – Breslich, Ernst Rudolph and Charles A. Stone. *Trigonometry with tables*: for use in senior high schools and junior colleges. Chicago, Ill.: The University of Chicago Press, 1928. Description: xii, 176, 122 p.: ill., ports.; 20 cm. Notes: "Logarithmic and trigonometric tables and mathematical formulas" (122 p. at end) published separately in 1917. Includes index.

1928 – Breslich, E. R. *Developing functional thinking in secondary school mathematics.* In J. R. Clark & W. D. Reeve (Eds.), *Selected Topics in the Teaching of Mathematics* (National Council of Teachers of

Mathematics third yearbook, pp. 42- 56). New York: Teachers' College-Columbia University.

1928 – Breslich, E. R. and Charles A. Stone. *Directions for administering classroom procedure test in mathematics for rating teachers of mathematics in junior and senior high schools*. form A, preliminary edition. Chicago, Ill.: The University of Chicago Press, 1928. Description: 4 p.; 23 cm.

1929 – Breslich, Ernst Rudolph. *Seventh year mathematics*. New York: The Macmillan company, 1929. Description: xi, 284 p. illus., diagrs. 20 cm,

1929 – Breslich, Ernst Rudolph. *Solid geometry*. Chicago, Ill.: The University of Chicago press. Description: xxiv, 157 p. illus., diagrs. 20 cm. Series: The University of Chicago mathematical series, E.H. Moore, general editor. The School of Education. Texts and manuals, G.W. Myers, editor University of Chicago mathematical series. School of Education texts and manuals (University of Chicago).

1929 – Breslich, Ernst Rudolph and Charles A. Stone. *The slide rule*. Chicago, Ill.: The University of Chicago press. Description: vii, 35 p. illus., diagr. 23 cm.

1930 – Breslich, Ernst Rudolph. *Ninth year mathematics*. New York: The Macmillan company. Description: ix, 319 p. illus., diagrs. 20 cm

1930 – Breslich, Ernst Rudolph. *Teachers' manual to seventh year mathematics*. New York. Description: covertitle, 31 p. 19 cm.

1930 – Breslich, Ernst Rudolph. *Eighth year mathematics*. New York, The Macmillan company. Description: ix, 296 p. illus., diagrs. 20 cm. Published in 1925 under title: Junior mathematics, Book two.

1930 – Breslich, Ernst Rudolph. *The teaching of mathematics in secondary schools - Technique*. Chicago: The University of Chicago Press. Description: vii, 239 p. illus., diagrs. 23 cm. Notes: "The first part of a series of books on the teaching of mathematics in the secondary schools." Pref. Includes bibliographies.

1930 – Breslich, Ernst Rudolf. *Algebra survey test*. Form A and B. Bloomington, Ill.: Public School Publication Co.

1931 – Breslich, Ernst Rudolph. *Teachers' manual to ninth year mathematics*. New York: The Macmillan company. Description: 44 p. 19 cm.

1931 – Breslich, Ernst Rudolph. *Problems in teaching secondary school mathematics*. Chicago, Ill.: The University of Chicago press. Description: vii, 348 p. illus., diagrs. 23 cm. Notes: The second part of a study of the teaching of mathematics in grades VIIXII. The first volume was published in 1930 under title: The teaching of mathematics in secondary schools, volume I. Technique. cf. Pref. Includes bibliographies.

1932 – Breslich, Ernst Rudolf. Measuring the development of functional thinking in algebra. *Seventh Yearbook of the National Council of Teachers of Mathematics*, chapter V. New York: Columbia University.
1933 – Breslich, Ernst Rudolph. *The administration of mathematics in secondary schools*. Chicago, Ill.: The University of Chicago Press. Description: vii, 407 p. illus., diagrs. 23 cm. Notes: The third of a series of three volumes devoted to discussions of problems related to the teaching of mathematics in secondary schools. The first volume was published in 1930 under title: The teaching of mathematics in secondary schools, volume I, Technique; the second, published in 1931 under title: Problems in teaching secondary school mathematics. cf. Pref. Bibliography at end of each chapter except chap. XII.
1933 – Breslich, Ernst Rudolph. *Solid geometry*. Chicago, Ill.: The University of Chicago press. Description: xxiv, 157 p. illus., diagrs. 20 cm. Series: University of Chicago mathematical series.
1934 – Breslich, Ernst Rudolph and Charles A. Stone. *Trigonometry with tables*: for use in senior high schools and junior colleges. Chicago, Ill.: The University of Chicago Press, 1928. Description: xii, 176, 122 p.: ill., ports.; 20 cm. Notes: "Logarithmic and trigonometric tables and mathematical formulas" (122 p. at end) published separately in 1917. Includes index.
1935 – Breslich, Ernst Rudolph. *High school mathematics*; solid geometry and related subjects. Chicago, New York [etc.]: Laidlaw brothers. Description: x, 171 p. illus., diagrs. 20 cm.
1936 – Breslich, Ernst Rudolph. *Diagrams in three dimensions for solid geometry*. Chicago: The Orthovis company. Description: 1 p. l., 15 diagr. 24 x 22 cm.
1936 – Chicago, University of. University extension division. Home study dept. *Mathematics 301; advanced problems in teaching mathematics in secondary schools*. Chicago: The University of Chicago. Related Names: Breslich, Ernst Rudolph. Description: 1 p.l., 29numb. l. 29 x 22 cm.
1936 – Judd, Charles Hubbard and Breslich, Ernst Rudolph; McCallister, James Maurice; Tyler, Ralph Winfred. *Education as cultivation of the higher mental processes*. New York: The Macmillan Company, 1936. Joint authors Names:. Description: vii, 206 p. incl. tables, diagrs. 20 cm.
1938 – Breslich, Ernst Rudolph. *Excursion in mathematics*. Chicago: The Orthovis company. Description: 4 p. l., 47, [1] p. incl. front., illus., diagrs. 23 x 21 cm. Notes: Two orthoscopes in pocket mounted on liningpaper. "The book has developed from experiments devised to train pupils in spatial imagination. It aims to supplement and

strengthen the teaching of geometry in the secondary school from the seventh grade on." Author's note.

1938 – Breslich, Ernst Rudolf. *Supervised study as a means of providing supplementary individual instruction.* Thirteenth Yearbook of the National Council of Teachers of Mathematics, chapter V. New York: Columbia University, 1938. Part I, pp. 32-72.

1938-1939 – Breslich, Ernst Rudolph. *Purposeful mathematics.* Chicago, New York [etc.]: Laidlaw brothers. Description: 4 v. illus., diagrs. 20 cm.

1939 – Breslich, Ernst Rudolph. *Algebra an interesting language*. Chicago: The Orthovis company. Description: vii, 70 p. incl. front., illus., diagrs. 24 x 21 cm.

1939 – Breslich, Ernst Rudolph. *The technique of teaching secondary schools mathematics*. Chicago: The University of Chicago Press. Description: vii, 230 p. illus., diagrs. 23 cm.

1943 – Breslich, Ernst Rudolph. *Purposeful mathematics.* Illustrations by M. F. Iserman. Chicago, New York [etc.]: Laidlaw brothers. Description: 4 v. illus., diagrs. 20 cm.

1950 – Ernest R. Breslich. *Mathematics*. Central Association of Science and Mathematics Teachers, Oak Park, Ill., pp. 39-79.

A controversy about geometry textbooks in Norway 1835–36

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Abstract

Towards the end of the 18th century, a great effort was done to establish mathematics as a school subject in the higher education in Norway, and a school reform that was introduced around year 1800 lead to proper teaching in mathematics. The mathematical community in Norway at that time was small, and all the participants did necessarily become significant members of the society. This was a time of considerable development in the subject of mathematics, and this also influenced the debate about mathematics education.

BERNT MICHAEL HOLMBOE (1795-1850) was teacher at Christiania Kathedralskole from 1818 till 1826, and after that he was professor at the University of Christiania until his death in 1850. Holmboe wrote textbooks in Arithmetics (1825, 1844, 1850, 1855, 1860), Geometry (1827, 1833, 1851, 1857), Stereometry (1833, 1859), Trigonometry (1834) and Higher Mathematics (1849). These were the textbooks in mathematics that were predominantly used in the learned schools in Norway between 1825 and 1860, a decade after Holmboe's death. He was probably one of the most influential persons in the development of school mathematics in the first half of the 19th century in Norway. His way of presenting the subject matter was in many ways very traditional, and they were challenged by his colleague and former mentor, Christopher Hansteen.

CHRISTOPHER HANSTEEN (1784-1873) became teacher in applied mathematics at the university in June 1814, and he was professor from March 1816 till he retired for medical reasons in 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern light, meteorology, astronomy, mechanics, etc. He received international recognition after an expedition to Siberia in 1828-30 to study the geomagnetism. In 1835, Hansteen wrote a textbook in geometry where he challenged the traditional Euclidean geometry, and he introduced the subject matter in a very «un-Euclidean» way.

The controversy that broke out between Holmboe and Hansteen was very bitter, and it was rooted in whether one in mathematics education should present the subject matter in a traditional Euclidean way or not. The polemics that followed in the newspapers has later been called the «dispute about parallelism». The core of it was whether one in mathematics education should - as in the case of Hansteen - let utilitarian considerations overrule logical deduction and theoretical thinking. Both Holmboe and Hansteen published pamphlets where they justified their views. I will in this paper first present the textbooks by Holmboe and Hansteen, and then focus on the dispute between them. The newspaper polemic took place in «Morgenbladet» from December 1835 till January 1836, and in «Den Constitutionelle» from June till September 1836. By an analysis of these newspapers, and the two pamphlets, I hope to throw some light on the didactical debate, and certain features of the development of mathematics education, in the first half of the 19th century in Norway.

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education.* Caparica: UIED.

Introduction

Cathedral schools[1] were schools from the medieval time that were connected to cathedrals, and they were meant to give a theological education to future priests. All cathedral schools were turned into *Latin schools*, or *grammar schools*,[2] when the reformation was introduced in Norway in 1539, and it was mandatory for every town to have a one. The new Latin schools, together with the old cathedral schools, constituted the so-called *learned schools*. Most of these Latin schools were, however, of a very poor quality, so in reality, the higher education - preceding the university – in 1814, was only four cathedral schools with a total of 200 pupils, in addition to some that had private tuition. The schools were referred to either by their names like *Stavanger Latin School* or *Christiania Cathedral School*, or by for instance *Christiania Learned School*[3] – Christiania[4] was the name of Oslo 1624–1924. Towards the end of the 18th century, a great effort was done to establish mathematics as a school subject in the higher education in Norway, and a school reform that was introduced around year 1800 lead to proper teaching in mathematics. The mathematical community in Norway at that time was small. This was a time of considerable development in the subject of mathematics, and this also influenced the debate about mathematics education. By a governmental decree in 1809, the pupils started at the learned schools at the age of 9–10 years, and the duration was normally eight years consisting of four two-year grades, and each day at school was seven hours – four before noon and three after. The university qualifying examination[5] was arranged by the university. The learned schools gave a classic education, and a higher education in scientific subjects at the same standard as the learned schools could be achieved at the Military Academy. This school admitted pupils from the age of 12–14. Several intermediate schools[6] were established in smaller towns after 1814, and they were learned schools without the upper two-year grade, see (Andersen, 1914).

The textbooks and their authors

Bernt Michael Holmboe (1795–1850) was born in southern Norway as a son of a vicar of the Church. He is mostly known as being the teacher of

[1] Katedralskoler.

[2] Latinskoler.

[3] Christiania Lærde Skole.

[4] The spelling was changed to *Kristiania* in 1877 in the land register, and in 1897 by the municipal authorities (Kunnskapsforlaget, 2006). The spelling will therefore vary.

[5] examen atrium.

[6] Middelskoler.

Niels Henrik Abel (1802–1829), when he was at Christiania Cathedral School. Bernt Michael Holmboe became a student at the university in 1814, and in 1815 he became Christopher Hansteen's assistant at his astronomical observatory. After completing his exams in mathematics, he worked from 1818 till 1826 as teacher at Christiania Cathedral School, then as a lecturer at the university from 1826 till 1834, and after that as a professor. Holmboe was an influential person in the development of mathematics education in the first half of the 19th century. He was the third person to be appointed professor in mathematics at the new university in Christiania. Holmboe was a mathematician at heart, and still a young man when he started teaching in 1818. It is said that his teaching was more «lively» and enthusiastic than what his students were used to. He gave them exercises and assignments out of the ordinary, and he caught their attention (Christiansen, 2009). Holmboe wrote – with a few exceptions – all the textbooks in mathematics that were used in the learned schools in Norway between 1825 and approximately 1860.

The following table shows an overview over Holmboe's textbooks, their Norwegian titles and their various editions.

Table 1. Holmboe's textbooks

TITLE	EDITION	YEAR	EDITED BY	PUBLISHER
Lærebog i Mathematiken Første Deel, Inneholdende Indledning til Mathematiken samt Begyndelsesgrundene til Arithmetiken (Holmboe, 1825)	1st 2nd 3rd 4th 5th	1825 1844 1850 1855 1860		Jacob Lehman J.Lehmans Enke J. Chr. Abelsted R. Hviids Enke R. Hviids Enke
Lærebog i Mathematiken Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien (Holmboe, 1827)	1st 2nd 3rd 4th	1827 1833 1851 1857	 Jens Odén Jens Odén	Jacob Lehman Jac. C. Abelsted R. Hviids Enke J.W. Cappelen
Stereometrie (Holmboe, 1833)	1st 2nd	1833 1859	 C. A. Bjerknes	C. L. Rosbaum J. Chr. Abelsted
Plan- og sphærisk Trigonometrie (Holmboe, 1834)	1st	1834		C. L. Rosbaum
Lærebog i den høiere Mathematik Første Deel (Holmboe, 1849)	1st	1849		Chr. Grøndahl

The textbook in basic geometry (Holmboe, 1827) starts with several definitions of basic concepts. The very first definition describes *geometry* as a science about the *coherent magnitudes*. Coherent magnitudes are the space with all available dimensions and time. According to Solvang (2001), Holmboe's way of organizing the subject matter was influenced by Adrien-Marie Legendre's (1752–1833) introduction to geometry (Legendre, 1819). The geometry of Legendre is constructed mainly the same way as in Euclid's *Elements*, and starts with a long list of what he calls *explanations*, similar to what Euclid calls *definitions* (Euclid, 1956).

The first definition in Legendre (1819) defines *geometry* as *a science which has for its objects the measure of extension. Extension has three dimensions, length, breadth, and thickness*. With reference to classification of coherent magnitudes in space and time, Holmboe classifies geometry in two parts:

1. *The real geometry* defined by the relations between the various magnitudes in space, without considering their changes in time.
2. *Mechanics*, defined by the changes the magnitudes goes through in time. All changes on a magnitude through time are called *motion*, and it is conditioned by *force*.

It is postulated that the space stretches indefinitely.[7] (Holmboe, 1827)

Holmboe advises the teacher to show moderation in the review of proofs, and to show examples using *numbers* before the examination of the proof. This practical advice contradicts the structure of his textbooks, which is strictly Euclidean. There are few exercises and numerical examples, and the notion of *construction* means to *elucidate* the concept, and not to use compass and ruler.

Holmboe does *not* give any detailed instructions on how to use ruler and compass in this book, nor does he mention geometric locus, but writes about *elucidative*[8] objects, magnitudes and concepts. His idea may be that the mathematics teaching shall educate the students with respect to formal logic, by encouraging them to *think* and *conclude*. There was a present debate about the use of Euclidean ideas in textbooks in geometry, and when Christopher Hansteen published his textbook in geometry

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[7] Geometri er en Videnskab om de *sammenhængende Størrelser*. Sammenhengende Størrelser ere Rummet med enhver deri forekommende Udstrækning og Tiden. Med Hensyn til de sammenhængende Størrelsers Inddeling i Rum og Tid, inddeles Geometrien i 2 Dele. (1) *Den egentlige Geometri*, der bestemmer de i Rummet forekommende Størrelsers Forhold til hinanden uden Hensyn til deres Forandring i Tiden. (2) *Mekanik*, der bestemmer de Forandringer, som Størrelserne undergaae i Tiden. Anm. Enhver Forandring, som en Størrelse i Tiden undergaaer, kalles *Bevægelse*, hvis betingelse kaldes *Kraft*. *Fordringssætning*. Rummet maa tænkes udstrakt i det Uendelige. (Holmboe, 1827, p. 1)

[8] anskueliggjørende.

(Hansteen 1835), it was evidently a controversial issue, and an attack on the Euclidean textbooks (Piene 1937; Solvang 2001).

Christopher Hansteen (1784–1873) was born in Christiania on September 26, 1784. He was first a law student in Copenhagen, but became interested in the natural sciences when he met the physicist H. C. Ørsted. He became a teacher in applied mathematics at the university in Christiania in June 1814, and he was professor from March 1816 till he retired for medical reasons in 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern light, meteorology, astronomy, mechanics, etc. He received international recognition after an expedition to Siberia in 1828–30 to study the geomagnetism. Hansteen mved with his family and servants into the new Observatory in Christiania in 1833.

In 1835, Christopher Hansteen published a textbook in basic geometry (Hansteen, 1835), which in many ways challenged Holmboe's textbooks. Hansteen's book was 278 pages, which was a lot more than what was expected of a textbook in elementary geometry. The author was intentionally trying to tear down the walls that existed between the classical geometry on one side, and the newer analytical geometry and the infinitesimal geometry on the other. The basis of the textbook is *real life*, with references to artifacts like corkscrews, stove pipes and hourglasses. The presentation of the subject matter is very unlike Euclid's *Elements*. The style is narrative, sometimes very lengthy, and there are many numerical examples (Piene, 1937). Hansteen tried to expand Euclid's definition of straight lines and of parallel lines, and Euclid's parallel postulate (Euclid, 1956). Holmboe, however, was firmly in agreement with Legendre's understanding of parallelism, and he believed that he could prove Euclid's parallel postulate.

Hansteen's textbook was published in one edition only, but one reason may be that it contained much subject matter outside the school curriculum.

The controversy

Holmboe's textbooks were more or less controlling the market regarding textbooks in mathematics in the first half of the 19th century. Hansteen's textbook challenged Holmboe's textbooks, and was the cause of a bitter controversy between the two professors in mathematics.

A newspaper polemic between Holmboe and Hansteen about Hansteen's textbook in geometry took place in *Morgenbladet* from

December 1835 till January 1836, and in *Den Constitutionelle* from June till September 1836.[9]

The core of the debate that followed was whether one in mathematics education should let utilitarian considerations overrule logical deduction and theoretical thinking. Hansteen declared that *proofs* should not be used in the elementary teaching before it was necessary for the students. This, he said, invited the students to memorizing without understanding. To this, Holmboe replied that you either have to prove all or nothing, as half a proof is worse than no proof. None of them probably knew the understanding of the theory of parallels, as it recently had been presented by Lobatschewsky and Bolyai, strongly influenced by Gauss (Holst, 1911). There were attempts in the 18th century to prove Euclid's parallel postulate by assuming it was false, and trying to produce a contradiction. Nikolai Lobatschewsky from Russia and Jànos Bolyai from Hungary published independently of each other in the 1820s, the ideas of a non-Euclidean geometry – a geometry where Euclid's parallel postulate is not valid. The works of Lobatschewsky and Bolyai did, however, not draw much attention before the 1860s (Katz, 2004). Lobatschewsky defines parallel lines by stating that

> All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes – *cutting* and *not-cutting*. The *boundary lines* of the one and other class of those lines will be called **parallel** to the given line. (Katz, 2004, p. 480).

The polemics between Holmboe and Hansteen has later been called the "dispute about parallelism" and they both published booklets where they justified their views (Hansteen, 1836; Holmboe, 1836).

The main article on the 5th of December, 1835, was written by Holmboe and called "On Professor Hansteen's new understanding of parallel lines".[10] It was a review of Hansteen's textbook and it was very critical to Hansteen's definitions of straight and parallel lines. Ten days later there was an unsigned article titled "Concerning Professor B. Holmboe's article in Morgenbladet: 'On Professor Hansteen's new

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[9] See *Morgenbladet* (1835) and *Den Constitutionelle* (1836).

The newspaper *Morgenbladet* was established in 1819, and was until 1857 a substantial voice for the opposition against the establishment, both literary and political. It was also the first daily newspaper in Norway, and it exists now as a weekly newspaper with a liberal, radical and intellectual profile. *Den Constitutionelle* existed as a daily newspaper in Norway between 1836 and 1847. The idea was to establish a newspaper on a considerably higher intellectual level than *Morgenbladet*. *Den Constitutionelle* made high demands on the journalistic content, and it introduced daily editorials (Kunnskapsforlaget, 2006).

[10] Om Professor Hansteens nye Parallellære.

understanding of parallel lines"'.[11] The author praised Holmboe for his "touch of thoroughness", but he continued that it was too much to expect from a man who had too long occupied himself with obsolete knowledge, to be an impartial judge of new knowledge. Hansteen's signed reply to Holmboe's article was published on the 18th of December. He stated that Holmboe had reviewed his textbook in a very unseemly manner, and that Holmboe considered Hansteen's textbook dangerous, and teachers should be warned against it, so that young people not are led astray from the rigour of pure and orthodox geometry, into heresy and delusion. A short declaration from Hansteen appeared a week later, where he admitted that he probably never would agree with Holmboe about how a good mathematics textbook should be, and that he will publish a booklet the following week. Then there was a short notice signed by Hansteen, dated 18th of January 1836, titled "To the owners of my textbook in geometry",[12] where he admitted that some explanations in his textbook may be simplified. He had therefore made some new pages that by the end of the week would be available at the publisher, free of charge, to the owners of the book.

There was an unsigned paragraph in *Den Constitutionelle* on the 15th of June, 1836, informing that a professor Jürgensen of Copenhagen had written a review of Hansteen's textbook in the *Monthly Journal for Literature*.[13] This review took no part in the controversy, but asserted the intention of making Hansteen's textbook known in Denmark. Three weeks later there was an article signed Hansteen, titled «*On the teaching of mathematics in the schools*»,[14] where he indicated that the reviewer had been unfortunate with his review. Holmboe now rejoined the fray. In an article he opposed Hansteen by asserting that Hansteen claimed that the only controversies that had been proposed against his textbook in geometry was mainly the question if it is allowed to define a concept before one can prove its possibilities. Holmboe wrote that this is not the case. Hansteen now wrote a long and final article, titled "*Farewell to Professor Holmboe*".[15] Hansteen ended his article with an anecdotal remark about Frederick II of Prussia, complaining over the difficulties of being at war with the Russians – "not only did you have to shoot them, you also had to knock them over with your rifle butt", meaning that you not only had to kill them once, you had to kill them twice. Hansteen concluded that he would leave Holmboe

[11] Angående Professor B. Holmboes i Morgenbladet No. 339, 1835, indrykkede Stykke: "Om Professor Hansteens nye Parallellære".

[12] Til Eierne af min Lærebog i Geometrie.

[13] Maanedsskrift for Literatur.

[14] Om den mathematiske Underviisning i Skolerne.

[15] Afsked til Professor Holmboe.

standing upright until he got tired – he would not have the trouble of knocking him down. This was the last newspaper article from Hansteen in this matter. Holmboe replied that he was surprised that Hansteen continued this polemics, even though he long time ago said that he would not. Holmboe also asserted that Hansteen had not read his booklet published after the controversy in *Morgenbladet*. (*Den Constitutionelle*, 1836; *Morgenbladet*, 1835)

Summary of the controversy

Both Holmboe and Hansteen published booklets where they justified their views. The core of the debate was whether one in mathematics education should let utilitarian considerations overrule logical deduction and theoretical thinking. Hansteen declared that proofs should not be used in the elementary teaching before it was necessary for the students. This, he said, invited the students to memorize without understanding.

Hansteen wrote a booklet (Hansteen, 1836) titled "*Investigation of Mr. Professor B. Holmboe's review of my Plane Geometry, Morgenbladet no. 339, 5th of Dec. 1835*"[16], and it is dated 26th of December 1835. It is, in other words, written towards the end of the period the polemics was active in *Morgenbladet*. Holmboe's name appears only in the title, later he is only referred to "*the reviewer*". In addition to defending his own textbook, Hansteen also criticizes Holmboe's arguments in the review, and he attacks Holmboe's textbook in geometry (Holmboe, 1827).

Hansteen's booklet is organized in five sections, labeled *A* till *E* where he focuses on five complaints from Holmboe's review.

- **A.** *Absence of contingency proof.* [17] Holmboe's complaint is that Hansteen uses the attributes of lines and surfaces *before* he defines them. Hansteen starts his textbook by classifying lines as *homogeneous*[18] and *heterogeneous.*[19] Hansteen blames Holmboe for not respecting authorities like Newton and Laplace, and he is attacking the definitions of basic concepts in Holmboe's textbook in geometry. Hansteen justifies his presentation of the subject matter by the fact that his book has been used for half a year at *Christiania Kathedralskole*.
- **B.** *Definition of a straight line*. Hansteen is accused of not using accurate descriptions and terms, and he argues by the fact that the textbook is written for children, and their only previous knowledge is their language,

[16] Belysning af Hr. Professor B. Holmboes Anmeldelse af min Plangeometrie, Morgenbladet No. 339, 5 Dec. 1835.

[17] Forsømmelse af Muelighetsbeviset.

[18] Eensartede.

[19] Ueensartede.

and names of concepts from their everyday life. Therefore one has to use a language that stimulates the imagination.

C. *«A circle is a circle»*. A vital error, according to Holmboe, is that he states that there exist only two homogeneous lines in a plane, *the straight line* and *the circle*, at a stage where it is unfounded.

D. *Theory of parallelism*. The definition of parallel lines in Hansteen's textbook states that *a line parallel to another has the characteristics that it cuts equal parts of its perpendiculars*. This relates to *straight* as well as *curved* lines, and it follows that they will never cross no matter how long you extend them.[20] This definition is, according to Holmboe, not generally correct, as parallel *curved* lines may cross one another.

E. *Euclidean definition of parallel lines*. Hansteen states that it is better for a concept to be defined by a *positive* property than by a *negative* one, and parallel lines are by Euclid defined by a property that lies beyond our experience, and it refers our minds towards the infinite. He also attacks Holmboe's statement that «*to construct is to elucidate the specified concepts of the definition of a magnitude*»,[21] and he finds it paradoxical that thorough knowledge of geometry does not assume the use of compass and ruler. How may such a *mental* construction elucidate the shape of a curved line, if it is defined by an equation between its coordinates, he asks. He also claims to have met students that didn't know one end of a compass from the other. Holmboe calls the use of compass and ruler an *insignificant requirement*[22] which should not be included in a textbook, and he claims that he has not found these instruments mentioned in textbooks by Lacroix, Legendre, Kästner, Wolff, or Vega. Only the textbooks by Hansteen and Thomas Bugge mention the use of compass and ruler.

Towards the end of his booklet, Hansteen recommends that a new edition of Lindrup's textbook[23] should be made, if one wants easily understood textbooks in arithmetics and geometry that does not frighten students away from studies in mathematics.

Holmboe responded by writing a booklet titled «Retort provoked by Mr. Professor Hansteen's enlightenment of my review of his textbook in geometry, containing: 1) Defense of the review containing proofs collected by a continued review of his textbook. 2) Refutation of his attack

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[20] Den almindelige Charakter for en Linie, som er parallel med en anden, er altsaa: At den overalt affskjærer ligestore Stykker af dennes Normaler; hvoraf altsaa følger for alleslags parallele Linier, saavel rette som krumme, at de, i hvor langt de end forlænges, aldrig kunne skjære hinanden.

[21] at construere er at anskue det ved en Størrelses Definition fastsatte Begreb.

[22] uvæsentlig fordring.

[23] The Danish teacher of mathematics, Hans Christian Linderup (1763–1809) published a basic textbook in 1807.

on my textbook in mathematics»,[24] and this was dated the 8th of March 1836 (Holmboe, 1836). It was written in the period between the two polemics in Morgenbladet and Den Constitutionelle. Throughout the booklet, Hansteen is referred to as «the author». Holmboe's booklet is structured in the same five sections as Hansteen's, and section D is – not surprisingly – the most comprehensive. Holmboe shows a wide knowledge of the subject matter by quoting Klügel's definition of curved parallel lines from 1763, in addition to the textbook Theorie des lignes courbes by Lacroix. The latter does not call curved lines parallel. Holmboe admits that Hansteen is correct in his objection against Euclid's definition of parallel lines, that it declares a property that is beyond all experience, in the sense that the definition appears before it is proven that two straight lines in a plane could have such a location that they will never cross if they are prolonged indefinitely. Holmboe is very clear in adding that Hansteen's theory of parallelism is in obvious conflict with the existing theory, which states that a curved line at a certain point is parallel to another curved line at a certain point, only if the tangents through each of the two points are parallel. The better part of Holmboe's booklet is a defense against the attacks made by Hansteen on his textbooks, and Holmboe is constantly referring to Legendre and his definitions.

Some concluding remarks

The first half of the 19th century was in many ways a breaking point for the higher education in mathematics in Norway. The position of mathematics as a school subject was strengthened through school reforms at the turn of the century, and the first university was established in Norway in 1811. Bernt Michael Holmboe's textbooks in mathematics were the ones that were predominantly used in the learned schools at that time. His textbooks were, as we have seen, not without opposition. The core of this opposition was the use of *proofs* in elementary mathematics, and whether the introduction of geometry should be in a traditional Euclidean way, using logical deductions and theoretical thinking – as in the case of Holmboe – versus a more «informal» way using everyday language and terms.

Holmboe's complaint is that Hansteen uses the characteristics of lines and surfaces before he defines them. Hansteen is also accused of not using accurate descriptions and terms, and he replies by the fact that the

[24] Gjenmæle fremkaldt ved Hr. Professor Hansteens Belysning af min Anmeldelse af hans Lærebog i Geometrien, indeholdende: 1) Forsvar for anmeldelsen med Beviser hentede ved en fortsat Recention over hans Lærebog. 2) Gjendrivelse af hans Angreb paa min Lærebog i Mathematiken.

textbook is written for children, and their only previous knowledge is their language, and names of concepts from their everyday life. Therefore one has to use a language that stimulates their imagination. Hansteen also uses a definition of parallel lines which – according to Holmboe – is not generally correct.

Finally – a curiosity. The first textbook by Holmboe that was published after the conflict with Hansteen was the second edition of the textbook in arithmetic in 1844 and on the reverse page of the title page, the following signed declaration is printed[25]

> **No. 926**
>
> The second edition of the present textbook's 1st part, or the arithmetic, is printed in 1050 copies. Each copy has a specific number, in such a way that the copies are labeled with the numbers in their order from 1 to 1050. If a copy is not numbered in this manner, and if the reverse side of the title page does not contain this declaration signed by the author, then that copy is illegal, and will be dealt with in accordance with the existing legal provisions.
>
> *B.M. Holmboe*

and this is the only publication by Holmboe that contains such a declaration.

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[25] **No. 926.** Af nærværende Lærebogs 1ste Deels eller Arithmetikens andet Oplag er trykt 1050 Exemplarer. Hvert Exemplar har sit særskilte Nummer, saaledes at Exemplarerne ere nummererede med Tallene efter deres Orden fra 1 til 1050. Saafremt noget ikke saaledes nummereret Exemplar, og hvis Titelblad paa Bagsiden ikke er forsynet med nærværende af Forfatteren underskrevne Erklæring, maatte forefindes, er samme ulovligt, og vil blive behandlet overeensstemmende med de for Eftertryk gjældende Lovbestemmelser. *B.M. Holmboe* (sign.)

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Geometry textbooks in the Dars-i-Niẓāmī educational reform in 18th century India

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Abstract

This paper focuses on a curricular reform proposed by Mawlana Niẓāmuddīn – a reform that ultimately influenced much of Muslim India through its implementation by graduates of the Firangi Mahall in Lucknow. This curriculum included an expanded list of readings in logic, philosophy, and the mathematical sciences. The mathematical portion of the curriculum mandated reading (1) the Khulāṣat al-Ḥisāb of Bahā' al-Dīn al-'Āmilī and (2) the commentary of Muḥammed Barakat on book I of the Taḥrīr (Redaction) of Euclid's Elements by Naṣīr al-Dīn al-Ṭūsī, as well as several introductory texts in mathematical astronomy. The Dars-i-Niẓāmī reform was ultimately unsuccessful in its effort to combat foreign influences on the Islamic community, partly because it turned too strongly toward traditional scholarship, rather than incorporate any of the newer aspects of learning into its program. At the same time, the inclusion of Barakat's commentary in the Islamic madrasa curriculum marked a step toward institutionalizing geometry (and mathematics) instruction in a way that it had not been before.

Introduction

The Dars-i-Niẓāmī was a proposed curricular change, part of a broader debate within the Indian Islamic community about how the community could and should protect its religious and social identity in response to continued foreign influence and intervention. The political and cultural power of the Delhi Sultanate was waning and European (especially British) interference in the internal affairs of the subcontinent posed new challenges to traditional Islamic culture. Education was central to the transmission of religious and cultural identity from one generation to the next. In this paper, I survey the institutional and curricular changes that were taking place within Islamic education. Then I consider what these changes meant for mathematics education and particularly the teaching of geometry in Indian institutions – what were the motivations for teaching geometry and what impact did these motives have on the content of geometry textbooks or the way it was being taught?

Contrasting two Indian geometry textbooks in Arabic

To introduce the topic of geometry education, let us consider two nineteenth century Arabic textbooks printed in India. The first of these textbooks was published by the Calcutta School-Book Society in 1824. It was edited by Thomas Thomason (1774-1829), a British missionary and expert in oriental languages. In his cover letter to the Society (Thomason 1820, 234), he wrote

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education.* Caparica: UIED .

> The manuscript here offered to the society is ... a revised copy of the great work ... stripped of its appendages, that is, of all that is not Euclid's ... . My chief business has been to present ... what is Euclid's ... to guard the accuracy of the text, and exhibit the work in as clear and perspicuous a form as possible, for the greater assistance of native students.

The great work to which Thomason refers is Euclid's *Elements*, in the form of a *Taḥrīr* or reworking by Muslim polymath, Naṣīr al-Dīn al-Ṭūsī (1201-1274). Al-Ṭūsī's treatise had for several centuries been the foundation for advanced study in mathematics in Islamic societies. In addition to putting the Arabic text into a more streamlined form, al-Ṭūsī had also added historical and mathematical notes, along with alternative demonstrations, many of which were borrowed without ascription from a tenth century mathematician, Ibn al-Haytham (De Young 2009). These notes were the "appendages" that Thomason had removed in his edition of the text.

Some of the colonialist values of Thomason and the Calcutta School-Book Society are already evident in the basic architecture of the title page (Figure 1). As in most modern printed books, the title is placed first on the page and is given more prominence than the author's name. A traditional Arabic manuscript has no title page but author and title information is often provided in the opening section of the treatise. Neither particularly stand out from the text, and it is not unusual that neither author nor formal title to be given. In the CSBS edition, the entire title page is in Arabic, with the exception of the logo of the Society, which consists of elegant copperplate script initials – visually reminding any reader that the book is printed by an organization that is fundamentally European in name as well as in outlook. And in the publication data at the bottom of the page, the date of printing (1824) is given using the European Gregorian rather than the Islamic Hijra, calendar, once more reminding readers that this work is the product of a fundamentally Eurocentric enterprise. The textbook, although cloaked in Eastern forms, was intended to instill Western or European values and thus "assist native students" to become more like the advanced Westerners who increasingly dominated the life of the subcontinent.

Next, consider another geometry textboook, the commentary by Muḥammed Barakat from Allahabad (fl. last half of 18th century) on book I of al-Ṭūsī's *Taḥrīr* (Rahman, 1982, 407; Sezgin, 1974, 113). The treatise, written early in the 18th century, was lithographed in Lucknow in 1290 / 1873. The architecture of its title page (Fig. 1) is typical of 19th century Lucknow publishers (Scheglova, 1999, 15). The double borders of intertwined floral motifs surround three main panels of text. The top panel contains a pious invocation, the round central panel contains basic author and title data (in traditional fashion, we must read up from the central lines to find the title information and read down from the central lines to find author information), while the bottom panel mentions the press and printer responsible for the treatise. Its internal architecture is also very traditional, using a *ḥāshiyya* (marginalia) format. The primary text (in this case, book I of al-Ṭūsī's *Taḥrīr*) is written inside the double ruled border in large script and with widely spaced lines, probably to facilitate student note-taking (De Young, 1986). The comments are written in the margins in smaller script using lines that slant about 45º relative to the main text. Each comment is keyed to a specific point in the main text using a number or other symbol (Fig. 2). One of the primary aims of the commentary is to fill in or explain the logical steps in each demonstration. Thus rather than remove comments, the author adds comments and explanations to the existing text.

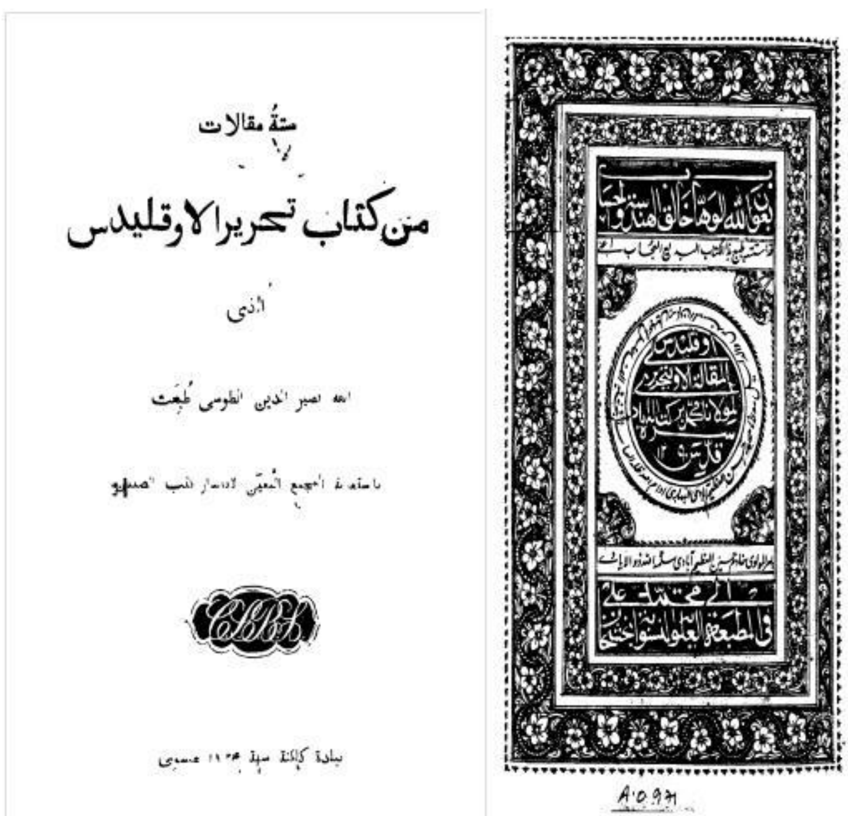

Fig. 1. Left, the title page of the Calcutta 1824 edition of Euclid. Right, the title page from the printed edition of Barakat's commentary on book I of the *Elements*. Digital images courtesy www.al-mostafa.com.

Fig. 2. Contrasting internal architecture used in the two textbooks to present *Elements* I, 22. Left, a page from the Calcutta 1824 edition of Euclid. Right, a page from Barakat's commentary on book I of the *Elements* which uses a moretraditional visual style. Digital images courtesy www.al-mostafa.com

Although originally intended as an aid to students of mathematics, the treatise was later incorporated into a more formal educational curriculum, the Dars-i-Niẓāmī, and became a textbook in a sense different from the original intention of its author. The treatise, both in form and in content, is much more consistent with traditional Islamic education in the mathematical sciences. Within the new academic curriculum, its aim was to reinforce rather than supplant traditional Eastern and Islamic values.

Thus we see already from the printed title pages themselves essential features of the outlook and motives of the publishers. The Calcutta edition of Euclid was intended to present a "pure" Euclid – as understood by nineteenth century British missionaries and emissaries to India – for Arabic speakers in India. The Lucknow lithograph, on the other hand, was aimed at students beginning their study of the traditional mathematical canon. It presents to an Indian audience a traditional commentary composed by an Indian savant. Thus it not surprising that it adopts a much more traditional architecture.

Contexts of the textbooks

In this section, I use the term "textbook" to refer to any treatise prescribed for use in a pedagogical context, whatever the original intention of its author might have been. The term evokes various understandings of what a textbook should be or what it should contain. Often we think of a textbook as a systematic overview of a field, simplified in presentation for the benefit of the non-initiate, and often accompanied not only be extensive explanation but also by exercises or learning activities to reinforce the teaching of the textbook.

Both these nineteenth century geometry textbooks begin from the text of al-Ṭūsī's *Taḥrīr*. This treatise was completed by al-Ṭūsī in 648 / 1250. It was composed as an introduction to mathematics for those wishing to devote themselves to the mathematical sciences and especially to the intricacies of mathematical astronomy and cosmology. These students would have been mature scholars, not juveniles or novices, although they might have had little formal academic training in mathematics. Instruction would have been in the form of reading a treatise, either individually or in a small group of students, with an experienced teacher-mathematician who would explain the intricacies of the text and who would, when satisfied that the student had mastered the text, give the student a permit (*ijāza*) certifying competence to teach the treatise.

The Calcutta edition, although based on this traditional introductory textbook, was intended for a different kind of beginning student. The textbook included the basic definitions and demonstrations of the propositions from the first six of the thirteen books of the *Elements*. It thus mirrors the elementary instruction in geometry provided in British grammar schools at this time, while books XI and XII were often included in university curricula (Ackerberg-Hastings, 2002, 65). But the textbook has had all the mathematical notes of al-Ṭūsī removed. This draconian editing suggests that the intended audience is considerably different from that envisioned by al-Ṭūsī. The British editor had in mind students who, although Indian, would be studying within the confines of an educational system modelled on that of Britain. No doubt he would have agreed with Macaulay's dictum (1957, 729):

> At present it is impossible for us, with our limited means, to attempt to educate the body of the [Indian] people. We must at present do our best to form a class who may be interpreters between us and the millions whom we govern; a class of persons, Indian in blood and colour, but English in taste, in opinions, in morals, and in intellect.

Geometry had been a traditional part of British education, but by the mid 18th century it had often degenerated into mere rote memorization of results. Then the pendulum began to swing, partly under the stimulus of the Scottish Common Sense Philosophy, which emphasized mathematics that embodied a clear connection to objects in the real world. Mathematics formed the basis for mathematical natural philosophy (epitomized in Newton's *Principia*) and mathematical logic, exemplified in the logic of geometrical proofs, was taken as the model for all true knowledge (Olson, 1971). If, as Brougham suggested in his speech to the House of Commons (1828), "education makes a people easy to lead but difficult to drive; easy to govern but impossible to enslave," then mathematics and especially geometry was important component of that education, training people to reason to right results. Knowledge of geometry was no less important to the colonies than to England itself and Thomason presented his textbook in that light.

The treatise of Muḥammad Barakat also takes its beginning from the treatise of al-Ṭūsī, although only the first book of the treatise. This commentary retains the full text of al-Ṭūsī, while adding many amplifying and explanatory notes. These comments either report the commentary of earlier authors or, more commonly, explicate the logic of Euclid's demonstrations. In addition to such explanations, there are careful correlations and frequent references pointing the student back to results of earlier propositions, etc. One feels that the intended reader is perhaps a not very bright student with some background in philosophy or logic who is reading a mathematical text for the first time and needs considerable help in understanding its content, rhetorical structure, and argumentation. It includes explanations that in a modern mathematics classroom might be made verbally but would scarcely need to be written down for students.

Barakat's treatise became a part of a reformed curriculum proposed by Niẓāmuddīn al-Sehalvī (d. 1748). His family had migrated into India from Iran more than two centuries earlier and brought with it a long tradition of educational activity. Niẓāmuddīn's father, Quṭbuddīn, a widely recognized teacher, had been killed in a dispute over some property. The emperor, Aurangzeb, when he heard this news, arranged to give the family the abandoned home of a Dutch merchant in Lucknow. This estate, known as Farangi Mahall, became an important centre of Muslim learning in India until the end of the nineteenth century. Students who came to study often went out inspired to teach and spread the Farangi Mahall vision throughout the subcontinent, starting new schools wherever they went.

The curriculum proposed by Niẓāmuddīn was not originally intended to describe instruction in a formal educational institution (*madrasa*) but was more akin to a suggested reading list. Niẓāmuddīn made his proposal in response to the curricular proposal made by Shah Walliullah of Delhi. Both proposals were intended to address the declining position of the Delhi Sultanate and the Indian Islamic community. The response of Walliullah was to focus the attention of the educational process still more on the traditional Islamic subjects, such as reading and interpreting the *Qur'ān*, studying the life and sayings of the Prophet Muḥammad, Islamic jurisprudence, etc. Niẓāmuddīn's vision was almost the opposite. Islamic scholars needed to be more broadly trained, not narrowly focused on religion, and they needed to be able to think independently and respond to new challenges, rather than focus on memorization with minimal understanding of the texts being studied. So he proposed to decrease the amount of specifically religious instruction and instead sought out basic treatises that would expose students to the fundamentals of various subject areas (including especially philosophy and natural sciences) and that would inculcate skills permitting them to continue independent learning. Niẓāmuddīn thus hoped to provide better and more widely knowledgeable leaders for the Islamic community. Also, because traditional education focused on memorizing many texts, it required a long preparation time and students were already middle aged or older when they finally completed their arduous studies. The result had been a lack of energetic and trained leadership in the Islamic community which had, in turn, contributed to the decline of political and social authority within the community during the eighteenth century. Niẓāmuddīn hoped that his proposal would bring adequately trained leaders to the community while still young enough to be able to provide long-term leadership.

The basic outlines of the curriculum have been given in several places (Mujeeb 1985, 407-408; Desai 1978, 14-15). In the mathematical sciences, there were five treatises, three of which were concerned with mathematical astronomy and cosmology:

1. *Khulāṣat al-Ḥisāb* of Bahā' al-Dīn al-ʻĀmilī
2. *Taḥrīr Kitāb Uqlīdis*, the commentary on Book I by M. Barakat
3. *Tashrīḥ al-Aflāk* of Bahā' al-Dīn al-ʻĀmilī
4. *Risāla dar ʻilm al-Hay'a* of ʻAlī Qushjī
5. *Sharḥ al-Mulakhkhaṣ fī al-Hay'a al-Basīṭa* of ʻAlī al-Jurjānī

The commentary of Muḥammad Barakat was not written with the intention of creating a textbook for use in Islamic madrasa courses. But as members of the family of Niẓāmuddīn and their students spread throughout India, setting up schools wherever they went, the meaning of the original Dars-i-Niẓāmī began to change and took on more and more the character of a formal curriculum. It is important to remember that the commentary did not appear in print until 1873, more than a century after Niẓāmuddīn first made his proposals for a reform of the curriculum. Prior to the time of printing there had been manuscript copies, of course, but these often show few signs of extensive pedagogical usage. The printing of the commentary coincides historically with the beginning of a period of increasing international political activism in defence of Islam by the members of the Farangi Mahall and the Indian Islamic community (Özcan, 1997). Whether the two events are indeed related must be determined by further study, but their confluence is intriguing. Perhaps both events reflect a desire to reinforce the uniquely Islamic characteristics of the community in response to rising pressure for greater social conformity from non-Islamic groups. It was also a time when the Farangi Mahall model of education was increasingly being challenged by other institutions – Deoband and Nadvat al-Ulema, for example, where the Dars-i-Niẓāmī was criticized for replacing religious subjects with more secular studies (a criticism derived ultimately from Shah Walliullah and his curricular proposals). As a result of these criticisms and competition from such institutions, the philosophical and secular content of the Dars-i-Niẓāmī was slowly but surely eroded and replaced by more traditional religious studies.

Intriguing as this hypothesis initially seems, there are some problems with it. First, one would expect that if the commentary of Muḥammad Barakat is being printed as part of a renewed emphasis on the Dars-i-Niẓāmī that the other mathematical treatises, and especially the *Khulāṣat al-Ḥisāb* would also be included in such a printing program. But a search through the WorldCat internet site has not revealed any printing of the *Khulāṣat al-Ḥisāb* in India during this period. Thus there is still a great deal that is unknown or unclear about the history of this curriculum and its actual role in the Islamic community of India. Most of the sources that could reveal that history are in Urdu, rather than Arabic or Persian, and so require linguistic skills beyond my own. But even though we are at present somewhat unsure of what might have inspired the printing of this treatise, it seems clear that the printers must have expected to reap some financial reward from their effort. The only probable way this might occur is to publish a textbook which would have some guarantee of sales.

The print edition of Muḥammad Barakat's commentary

We have already described the essential architectural features of this printed treatise. In this section, we present some features of the content of the work. It is important to note that the printed edition is not exactly the same as the manuscript tradition of the commentary. At the moment, we cannot confidently claim that these features are unique to the printed edition, nor can we usually reach a definite conclusion why these changes were introduced.

The first characteristic to which I would draw attention is the diagrams. Geometry has long included diagrams along with text. And even though the Euclidean propositions have traditional diagrams associated with them, there has always been a degree of freedom to adapt these traditional diagrams or to introduce new diagram conventions. Muḥammad Barakat's commentary was based on the text of al-Ṭūsī, to which Barakat added his own commentary. As we have already noted, in the printed edition, the architecture uses a *ḥāshiyya* format. Thus we should expect that the text and diagrams printed in the main body reflect the text of al-Ṭūsī. As we begin our survey, we note that Barakat's printed edition contains several diagrams to illustrate the concepts of the definitions of book I (parts of a circle, classes of triangles, classes of quadrilaterals, and parallel lines).

Only two manuscript copies of the commentary are currently available to me. Only one (Aligarh, Maulana Azad University Library, Arabic Ms. 1297) has been given diagrams. These diagrams are not entirely the same as those in the printed edition. For example, it includes a set of diagrams illustrating classes of non-rectilinear angles, while the printed edition does not. (These non-rectilinear angles are identical to those diagrammed in the Pseudo-Ṭūsī manuscripts and printed editions.) The diagrams illustrating classes of triangles and quadrilaterals are also nearly identical. Discussion and diagrams of the circle and its parts follow rather than precede triangles and quadrilaterals and the diagrams used are again nearly identical to those found in the Pseudo-Ṭūsī tradition. Similar diagrams illustrating the definitions also appear in the Persian translation of al-Ṭūsī's classic work by Quṭb al-Dīn al-Shīrāzī (De Young 2007). Diagrams for the propositions are placed in the margins of the manuscript and are similar in form to those in the printed edition, except that they are more strongly over-specified (Saito, 2006, 82).

The oldest known manuscript copy of al-Ṭūsī's treatise, copied in 656 AH, only a decade after the work was written and well within the lifetime of the author, does not have diagrams to illustrate the definitions. In fact, none of the half-dozen manuscript copies of al-Ṭūsī's work available for immediate inspection contain diagrams to illustrate the definitions. Both manuscript and printed copies of the so-called Pseudo-Ṭūsī version contain similar diagrams (with the exception of the diagram of parallel lines) among those used to illustrate entities defined at the beginning of book I. Does this observation indicate that over time elements from this latter treatise (composed some 50 years after al-Ṭūsī wrote his version) infiltrated the Ṭūsī tradition? It is still too early to know for certain.

Barakat informs us of the sources on which he constructed his commentary, unlike al-Ṭūsī, who did not mention his sources. Among the sources cited are Shams al-Dīn Muḥammad ibn Aḥmad al-Khafrī (d. 958 / 1551) (Saliba 1994) and "commentators on *Ashkāl al-Ta'sīs*" (the most influential being Qāḍīzāde al-Rūmī, who died about 1436) (De Young 2001). As a student of mathematics, Barakat has read the important scholarship on Euclidean mathematics, including recent authorities. And his reliance on more modern authors implies that he was not interested simply in preserving the ancient tradition but wanted to broaden his discussion. The use of broad scholarship might well have made the treatise more appealing for use in the Dars-i-Niẓāmī curricular reforms.

Barakat's comments frequently focus on the rhetorical structure and the internal logic of the mathematical argumentation. For example, in the enunciation of proposition 22 ("We want to construct a triangle, each of whose sides is equal to one of three given lines, any two of them together being longer than the remaining [one]") Barakat feels it necessary to explain that "the remaining [one]" is the third side or third line and similarly he explains that "them" refers back to the three given lines. This is only one example among many of what Netz (1998, 263-5) termed "vertical pedantry," an unnecessary explaining of what should be immediately obvious to an intelligent reader. If we imagine a reader who has had no experience with mathematical discourse but is already somewhat familiar with logical debates, one could understand why such explications might be helpful.

The demonstration of proposition 22 begins by stating the objects to be used in the proof: "let there be lines A, B, G and let ED be a line bounded at the side of D only." The first step of Euclid's demonstration is to "mark off from it [i.e., line ED] DZ equal to A." Almost any student who has read the text to this point would probably know how to do this, but Barakat inserts a reference to proposition 3, where it was demonstrated how to mark off a line segment equal to a given line, should the reader have forgotten the procedure. Similarly, when we are instructed to draw a circle with center at Z and distance (or radius) equal to ZD, the commentator provides an internal reference to the third postulate of Euclid which tells us that we are allowed to construct a circle with a given center and a given radius. Again, the purpose seems to be to round out the logic, making sure that no gaps are left.

Al-Ṭūsī had included two unusually lengthy insertions in his version of book I. The first consisted of eight propositions following *Elements* I, 28 by which he believed he could demonstrate the so-called parallel lines postulate of Euclid (Jaouiche 1986, 99-106). The second is an exploration of the various possible placements of the squares on the sides of the right triangle in proposition I, 47, the so-called Pythagorean Theorem. This discussion, in manuscripts of al-Ṭūsī's treatise, takes up nearly a quarter of the volume of book I. But in the printed edition of Barakat's commentary, it takes up a good third of the volume, as measured in terms of number of pages. But this measure is somewhat misleading without further qualification. The sheer volume of comment provided is, in this section, too voluminous to be placed in the margin and the printer has had to resort to the creation of artificial margins – creating additional margin space by subdividing the space within the double-ruled borders into two additional columns / "margins".

The treatise concludes with a brief (pages 95-96) Persian selection from Quṭb al-Dīn al-Shīrāzī's translation of al-Ṭūsī's *Taḥrīr*. This selection appeared at the end of book I in his translation and comprised a diagram that combined all the diagrams of book I into one rather complicated composite figure. The text accompanying this figure explained which lines should be taken in order to construct the diagram of each proposition. This selection is explicitly attributed to al-Shīrāzī by the publishers. Its inclusion as a summary to the commentary seems to underline the pedagogical intent of the editors (De Young, 2012).

Such “pedantic” features and an almost pathological focus on what often seem to today’s reader to be the most inconsequential topics, often repel modern historians of mathematics. Such discussions, when combined with reports of comments from earlier authors, produce a treatise that, while not necessarily inspiring in its level of mathematical innovation, was apparently considered helpful to students (especially younger or intellectually immature students) who were beginning their study of Euclid’s classic. And in that light we can perhaps more easily understand why the commentary might have appealed to Niẓāmuddīn in proposing his reading list / curriculum. If students thoroughly understand the reasoning used in Euclid’s first book, they will be able to go on to read other more advanced mathematical works more or less independently, which was the primary goal of the original curricular proposal.

Conclusion

Despite the apparent intentions of the Dars-i-Niẓāmī to counteract the influence of foreign cultural forces on the Indian Islamic community, its backward-looking stance, taking its inspiration from al-Ṭūsī’s canonical treatise rather than embracing the style and content of the newer mathematics, doomed it to ultimate failure. Although Niẓāmuddīn correctly realized that a more broadly trained religious elite was necessary to respond effectively to the allure of foreign culture and modern European knowledge, his proposed curriculum sought this training through increasing exposure to the traditional approaches to the mathematics and its logical structures rather than through a blending these classic approaches with features of modern mathematics or attempting to adapt the modern mathematics to the needs of the Islamic community.

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Martha Dantas at Centre International d'Études Pédagogiques (Sèvres, 1953): a contribution for the history of Mathematics Education in Brazil

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Abstract

Martha Dantas was a Professor of Didactics of Mathematics at the Faculty of Philosophy of Bahia from 1952 to 1968. Her contributions to Mathematics Education in Brazil is well known since she was honored as President of the First National Meeting of Mathematics Education, held in São Paulo (1987), when the Brazilian Mathematics Education Society was founded. Indeed, she was a very active promoter of key events in Mathematics Education in Brazil. She organized the First National Congress of Secondary Mathematics Teaching (Salvador, 1955), a pioneer professional meeting of the area in Brazil and Latin America. Some years later, from 1965 to 1968, she coordinated the Section of Mathematics of the Centre for Teaching of Sciences of Bahia, which conducted experiments on geometry teaching using geometric transformations and published two collections of mathematics textbooks for primary and secondary education. In 1953, she visited educational institutions in Belgium, England and France, including the Centre International d'Études Pédagogiques in Sèvres, where she interacted with Edmée Hatinguais, Lucienne Félix e Marceline Dionot. The activities promoted by the Association des Professeurs de Mathématiques de l'Enseignement Secondaire Public (APMESP) and the activities carried on by the teachers of the Nouvelles Classes introduced by Gustave Monod's Reform greatly impressed her. This text will explore the implications of these meetings for her later professional trajectory and propose some contributions for Brazilian Mathematics Education's historiography.

Introducing Martha Dantas

This text will share some results of our researches on history of mathematics education in Bahia, Brazil, where we are carrying on the project "Modernization of school mathematics in educational institutions at Bahia (1942-1976)" with ten students of the Program of Science Teaching, Philosophy and History of Science at the University of Bahia.

Martha Dantas was one of the protagonists of this history. She was born from a typical aristocratic family of landowners with active participation in oligarchic power relations that ruled Bahia along many time.

Almost always studying in private religious schools, she finished the Normal Course of Ursulines School for girls and obtained a Primary Teacher degree in 1941. In 1942, she was approved on the official public exam and was appointed as a teacher of a State Public Primary School.

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The following three years, as well as teaching, she continued studying at night in the State Secondary School of Bahia, because it was necessary doing a scientific preparatory course to candidate for the Faculty of Philosophy, where she planned to study mathematics. In fact, she was admitted in 1945 and obtained a bachelor degree in 1947, after assisting courses of Calculus, Algebra, Geometry, Mechanics and Physics, which were offered at the Polytechnic School, because her teachers were engineers. One year later, in 1948, after assisting courses on education – Sociology of Education, Cognitive Psychology, Biological Foundations of Education, General Didactic, Education Management, and Applied Didactics of Mathematics, she obtained a licence degree and became able to teach mathematics in secondary schools. So, in 1949, she was approved on the official exam and was appointed as a mathematics teacher of the same State Public Secondary School where she had studied three years before. This same year, Isaías Alves, head of the Faculty of Philosophy, invited her to be the principal of the Faculty's School of Application, and three years later, in 1952, he made another invitation for her to become the Faculty's Professor of Didactics of Mathematics, see (Dias, 2001).

Fig. 1. Martha Dantas. Opening session of the First Congress, 1955

Before telling you something that happened during her trip to Europe in 1953, I will justify why highlighting Martha Dantas, anticipating some other aspects of her professional trajectory. In fact, as soon as she returned from Europe, Martha Dantas organized the National Congress on Secondary Teaching of Mathematics, held in Salvador, Bahia, in 1955, a very important reference for history of Mathematics Education in Brazil. This was the first of five congresses held until 1966, when the sequence was interrupted because of the dictatorship. In 1987, when the sequence of congresses returned, Martha Dantas was distinguished as President of Honour of the First National Meeting of Mathematics Education, when the creation of Brazilian Society of Mathematics Education was prepared, see (Encontro ..., 1988).

Last, but not at least, during the second half of the sixties, Martha Dantas became coordinator of Scientific Section of Mathematics of the Centre of Teaching of Sciences of the University of Bahia, which conducted experiments on Geometry teaching using geometric transformations, also in-service training for Mathematics teachers, and published two collections of Mathematics textbooks for two different levels of secondary education, see (Freire & Dias, 2010).

Fig. 2. From right to left: Osvaldo Sangiorgi, Martha Dantas, Jairo Bezerra and Martha Blauth (1962)

Martha Dantas at Sèvres

So, let us return to 1953 and go with Martha Dantas to the Centre International d'Études Pédagogiques at Sèvres, France. Because Martha Dantas was fluent in French, English and German, she could receive news about Mathematics teaching from European magazines and journals, so that she was up to date on innovations and changes of the beginning of the 1950's, when the traditional teaching was replaced by "new math". So, as soon as she became Professor of Didactic of Mathematics at the Faculty of Philosophy, she asked authorization from the university authorities, to travel to Europe during one year, and to visit educational institutions in Belgium, England and France. She wrote a report when she returned to Bahia:

> The reform is founded on the most advanced knowledge in Psychology and Pedagogy. The *Association des Professeurs* has meetings at Sèvres and coordinates the researches (…) the results and findings are published in *Cahiers Pédagogiques* (…) The most important are the periods of training that teachers take there in Sèvres (…) These coordination between teachers of each discipline and between teachers of different disciplines are very important, so that the goals of secondary school can be reached; it seems to me that this takes the major attention of French teachers and educators of International Centre of Sèvres, at this moment.
>
> [...]
>
> During the last days of October 1953, I was invited to take part at the training period of the teachers for the "Nouvelles Classes". During three days, getting in touch with teachers from everywhere in France remembered me our problems, and I suffered because nothing has been done to resolve them [...]
>
> […] the causes of failure in geometry teaching are not due to the (subject) matter, but because Mathematics teachers in Brazil are not prepared, programs are not adequate, and the work is not coordinated at all.
>
> I am sure that only coordinated efforts of teachers will improve mathematics teaching. So, I proposed a national meeting for mathematics secondary teachers of Brazil, in Salvador, September 1955. Sure, this will open new routes for mathematics teaching in our country, see (Dantas, 1954).

At Sèvres, she interacted with Edmée Hatinguais, head of the Centre, Lucienne Félix e Marceline Dionot, both mathematics teachers. The activities promoted by the *Association des Professeurs de Mathématiques de l'Enseignement Secondaire Public* (APMEP) and the activities carried on by the teachers of the *Nouvelles Classes* introduced by Gustave Monod's Reform impressed her greatly.

According to Renauld d'Enfert (2010), after the *Libération* (1944-1960), the number of mathematics teachers at the *lycées* doubled and their expectations about educational reforms – Langevin-Wallon Commission (1944-1947) and following reform projects – were published in the Bulletin d'APMEP. They demanded the development of scientific teaching in general and of mathematics teaching particularly, and APMEP was their spokesman. The association organized meetings and surveys that stressed the generally dominant discourse during the fifties about the need of scientists and technicians for economic development and modernization of France.

APMEP agreed with the international mathematics teaching reform promoted by OECD and UNESCO. Its bulletin reported news and activities from ICMI and CIEAEM, where French mathematics teachers like Lucienne Felix e Marceline Dionot particpated. In 1955, members of APMEP took part at the international meeting of mathematics teaching promoted by the International Center of Sèvres.

D'Enfert stated that the actions of APMEP were not only institutional, but that was also promoted new subjects – new math - and methods for secondary Mathematics teaching. Collaborating with the Mathematical Society of France, they promoted courses and conferences of university professors that were published in its bulletin. Bourbaki had of course a great influence on all of this.

Another aspect stressed by D'Enfert was the strong link between this new math movement of APMEP and that one for the introduction of active pedagogy or active methods of teaching in secondary school, whose main focus was the International Center of Sèvres and its *Cahiers pédagogiques*. Some teacher members of APMEP were also teachers of the *classes nouvelles,* created in 1945 by Gustave Monod, a French Government educational authority that were renamed later in 1951 as "classes pilots":

> Instrument de la réforme: les « lycées-pilotes »
>
> Gustave Monod souhaite aussi créer un nouveau type d'établissement: le « lycée- pilote » (…) Plusieurs de ces lycées-pilotes accueillent les « classes nouvelles » puis les « classes pilotes » mises en place par Charles Brunold, directeur de l'enseignement secondaire à partir de 1951. L'expérience des « classes nouvelles » est, en effet, élargie à l'enseignement du second degré par le circulaire du 30 mai 1952. « Un lycée-pilote est un établissement qui, grâce à l'adhésion de tout le personnel formant une équipe, fonctionne comme un laboratoire permanent d'expérience où s'éprouvent des méthodes plus actives au service de la formation pédagogique des maîtres du second degré 12. » Le lycée de Sèvres devient le modèle de référence des lycées-pilotes; il

est le confluent des recherches des autres établissements, see (Lecoq & Lederlé, 2009).

Final remarks

It is common to say that the first step of the Modern Mathematics Movement in Brazil was done in 1960, when Osvaldo Sangiorgi returned from his trip to USA and founded GEEM – Grupo de Estudos do Ensino da Matemática (Study Group for Mathematics Teaching). After this first step, the modernization of teaching and the introduction of modern Mathematics would have begun. It is also common to analyze this historical process according to some old fashioned models of spreading of culture or science from the North to the South, from the European centers of developed civilization to underdeveloped southern peripheries. Now, it seems for us that something can be changed in such a history of Mathematics Education. The challenge is not to find another first step, nor to show that France or USA were not centers of this historical process.

Our claim is not to present Martha Dantas and her trip as the first step of mathematical teaching modernization, putting her in the place of Osvaldo Sangiorgi, because, after all, both played protagonist papers in this history. For, on the other hand, our purpose is not to present a new and extraordinary theory or model to explain the dynamic of international, regional and local nets and chains of communications and interactions that carried on plans or projects of curricular reforms, that introduced new and modern subjects in syllabus or textbooks, that effected teacher training and actualization for new methods and theories of teaching and learning. Because there some approaches for cultural interchange and communication, for a worldwide process of diffusion of science and of appropriating science locally were already enabled.

History of science, cultural history, history of education are disciplines in the neighborhood of the history of mathematics education that had already focused on historical processes like this, involved with a set of institutional and individual protagonists, collective, private and corporative interests, educational, scientific and politics aspects, so that history of mathematical education should dialogue and learn with these previous multidisciplinary experiences.

So, first of all, it seems to us, there is good evidence, that there were not only one or two geographical centers that promoted worldwide diffusion of modern Mathematics and its new culture of teaching, materialized on textbooks, programs, syllabus, or theoretically represented by the new trends of research on teaching and learning of mathematics.

Particularly, in Brazil, because of its proper geographical, political and cultural features, there were a set of groups and institutions, some of them private, like GEEM of Osvaldo Sangiorgi, another public and governmental, like Institute of Mathematics and Physics of Bahia or CECIBA – Centro de Ensino de Ciências da Bahia (Center for the Teaching of Science of Bahia), where Martha Dantas developed didactic experiences and teacher training during the sixties.

Of course, these centers and their leaderships communicated with each other in different ways. Travels and travellers were a very important part of the process. Following and analyzing Martha Dantas's personal and professional trajectory, from the early years until her maturity, her ideas, actions and work, we have found out several and different influences that she suffered and assimilated, first of all, catholic conservative modernization and scientific cognitive psychology from Isaías Alves, one of her first pedagogical masters, that confronted in Bahia and Brazil with liberal pedagogical ideas of Anísio Teixeira, one of the champions of the modernization and democratization of educational systems in Brazil and Bahia. These were the first, local and more ancient influences that she experienced.

When Martha Dantas traveled to Europe, particularly, to France and Sèvres, she experienced other influences from the Center of Sèvres, from APMEP and Bourbaki, she got in touch with modern mathematics, *lycées pilotes, classes nouvelles,* and *classes pilotes* and things like these.

All these influences can be identified, but she had a particular form of appropriating them, carrying on the professional and political organization of mathematics teachers, promoting and participating at all six scientific and professional meetings held in Brazil from 1955 to 1966, developing didactic experiences with "group super-visioned study" at the School of Application and other second degree public and private schools, or producing and testing textbooks for geometry teaching with algebraic transformations at the Center of Teaching of Sciences, from 1965 to 1968.

After all, she was not the final target. Hundreds of Mathematics teachers around Bahia learned with her and from her, transformed, all or part of these influences that circulated around schools of small, medium and big cities of Bahia, where she herself or her alumni traveled spreading modern mathematics and its new methodological and theoretical trends for teaching, see (Dias, 2008).

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Doing math or learning to count? Primary school mathematics confronting the democratization of access to secondary education in France, 1945-1985

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Abstract

In the decades following the Second World War, the reform projects and later the reforms that aimed to widen access to secondary education called into question the nature of the content, methods and aims of primary school mathematics teaching. Until the start of the 1960s, a first, rather traditional approach focused on basic arithmetic skills to ensure pupils' success in secondary mathematics classes. A second, more innovative approach appeared in the 1960s as part of the "Modern Mathematics" movement, advocating at the same time both modernization of the content and renovation of the methods. This paper studies the respective proposals and arguments formulated by the protagonists, and the French Ministry of National Education's changes in position between 1945 and the 1980s.

Introduction

After the Second World War, the French project to democratize access to secondary education – and more generally access to 'middle school' for students aged 11-15 years –, called into question the nature of the content, methods and aims of primary school mathematics teaching. On one hand, until the beginning of the 1960s, the dominant trend was to focus on basic mathematical skills (calculation and problem solving for everyday life) and to reinstate learning by rote, viewed as necessary to ensure pupils' success in secondary mathematics classes. On the other hand, the "Modern Mathematics" movement – which expanded in the 1960s at the same time as the opening up of the access to 'middle school' deeply modified the goals of primary education – placed the emphasis not on the utilitarian and technical function of primary mathematical education but on understanding and reasoning (and on the use of active methods). The aim was to introduce pupils to 'real' mathematics: in primary and secondary school, pupils had to '*do*' mathematics as mathematicians did, and thus discover the fundamental concepts and structures of modern mathematics.

This paper studies the respective proposals and arguments formulated by the protagonists, and the French Ministry of National Education's

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changes in position between 1945 and the 1980s. It shows that the process of democratizing access to secondary education during this period, and more generally structural reforms of the educational system, had many and far reaching implications for the teaching of mathematics not only in secondary schools but also in primary schools[1].

Primary education in the 1950s: acquiring the "basic skills" in preparation for secondary education

In France, the period immediately after the Second World War was marked by a succession of projects (more than ten) for the general reform of the educational system, which were implemented in the reform carried out in 1959 by the Minister of Education Jean Berthoin. These diverse projects and the later 1959 reform came at a time of strong growth in post-primary education. The principal concern of the 1959 Berthoin reform was the removal of the separation between the two educational networks – primary education for the working classes and secondary education for the middle and upper classes – which had prevailed since the XIX^th^ century, and the creation of a "middle school", open to all students aged from 11 to 15, as part of secondary education within an extension of compulsory education[2].

Primary schooling was directly concerned by the creation of this "middle school" for all in 1959. Higher primary education was inserted into secondary education and primary education was so reduced to pupils aged 6-11: it then became a gateway to secondary education for which it had to prepare and to whose demands it had to adapt. While it became less necessary to give pupils the practical knowledge-base necessary to "get on in life", primary education had however "to build solid, long-lasting foundations for the entire edifice of schooling." (MEN 1960, p. 3109)[3]

The reorganization of the educational system, planned then put in place in 1959, had important consequences for the content and purpose of the various primary school subjects. The various reform projects which followed during the 1940s-1950s made the acquisition of sound, instrumental knowledge in French and numeracy, an indispensable

[1] This study is part of the collective research project REDISCOL – "Réformer les disciplines scolaires: acteurs, contenus, enjeux, dynamiques, 1950s-1980s" – funded by the Agence nationale de la recherche (ANR). The English translation from the French text was done by Richard Kennedy.

[2] The 1959 reforms established compulsory education for all children from 6 to 16 years old born after 1953. This obligation previously only applied to children aged from 6 to 14 years old.

[3] "d'établir les fondations solides et durables de tout l'édifice scolaire".

prerequisite for a longer period of schooling in secondary education. The Algiers Plan, released by the provisional Government of the French Republic in 1944 envisaged a lightening of the arithmetic and geometry programs and "to maintain and reinforce the admirable traditions of care and precision in the areas of writing, spelling and elementary arithmetic." (Decaunes & Cavalier 1962, pp. 262-263)[4] In the 1950s, several reform proposals foresaw that elementary education had to ensure "the acquisition of basic knowledge and skills." (Ibid., p. 331 & 350)[5] This wording was also included in the January 6, 1959 decree reforming the educational system (Ibid., p. 403).

This priority given to basic learning was implemented as early as 1945[6]. The time given to arithmetic in primary schools was increased 30% in average[7] and the new program put an emphasis on studying numbers, practicing operations and arithmetical techniques. The new Ministerial directives asserted the desire to give primary education back "its old simplicity and effectiveness with respect to the acquisition of the basic skills" (MEN 1945, p. 91)[8]: the encyclopedic nature of the pre-war program and the use of active methods which were then in favor was implicitly criticized.

The 1959 Berthoin reform again played the card of refocusing on core subjects: a good command of French and arithmetic was considered necessary to be able to follow the secondary program. Teachers were invited to "build solid, long-lasting foundations for the entire schooling." A return to learning "by heart" was recommended, so that the pupils "do not hesitate over the meaning of an arithmetic operation, that they don't make mistakes due to a less than perfect knowledge of their tables" (MEN 1960, p. 3109)[9]. Michel Lebettre, who was in charge of primary education at the Ministry of National Education explained: "My first duty is to provide good students for secondary schools who have the basic skills" (Lebettre 1960, p. 10)[10].

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[4] "de maintenir et renforcer les admirables traditions de soin et de scrupule dans le domaine de l'écriture, de l'orthographe, du calcul élémentaire".

[5] "l'acquisition des connaissances et des mécanismes de base".

[6] The Vichy government had initiated this movement during the Second World War, but for different reasons.

[7] In 1945, the weekly time allocated for arithmetic was so defined: *cours préparatoire* (6-7 years old): 3h3/4; *cours élémentaire* (7-9 years old): 3h3/4; *cours moyen* (9-11 years old): 5h.

[8] "sa simplicité et son efficacité anciennes en ce qui concerne l'acquisition des mécanismes fondamentaux".

[9] "n'hésitent pas sur le sens d'une opération arithmétique, qu'ils ne commettent pas des erreurs dues à une connaissance imparfaite des tables".

[10] "Mon premier devoir, envers les lycées et collèges [secondaires], est de fournir aux classes de sixième des élèves valables, possédant les mécanismes de base".

This refocusing on the essential subjects and "basic" skills was coupled with a desire to lighten primary school programs. In 1945, the official directives affirmed "the willingness for a close relationship between school mathematics and the necessities of life" (MEN 1945, p. 102)[11]: in the late 1950s and early 1960s, the practical nature of the arithmetic program seemed less justified in that further education became the norm for a large majority of pupils. By deferring certain topics to a later stage which was allowed by the greater access to secondary education, it was possible to lighten the program in such a way that primary school teachers could concentrate more on the basic arithmetical skills (Fouchet 1963, p. 7). Cuts in the primary school arithmetic program seemed all the more necessary as certain topics, and particularly that of *cours moyen* (9-11 years old), were judged too ambitious given the intellectual maturity of the pupils. A 1960 report pointed out "parts of the arithmetic program which seem difficult for pupils of less than 11 years old to assimilate"[12], such as the rule of three (the removal of which was requested), the calculation of percentages and selling prices, fractions, and even geometric constructions (Gal 1960). Echoing the findings of various reports in the early 1960s – program poorly assimilated, premature study of certain principles – the Ministry decided in 1964 to make certain parts of the program for the *cours moyen* (9-11 years old) optional: those relating to 'practical' knowledge (simple interest, short-term investment, etc.) or whose learning did not appear necessary at this stage of the curriculum. In return, the teachers were to "ensure [...] that all the topics whose study is required are fully assimilated."(MEN 1964, p. 1795)[13]

The time for questioning

The measures taken by the Ministry of National Education in the first half of the 1960s did not call into question primary mathematics teaching fundamentally. The adjustments to the new institutional context under the Berthoin reform were only marginal changes: the content, though reduced, and the methods remained practically unchanged. General Inspector Marius Beulaygue – responsible for mathematics teaching in primary schools – highlighted: "Interesting as they are, these lightenings are not sufficient to make the arithmetic taught in primary schools a solid

[11] "la volonté d'une relation étroite entre les mathématiques de l'école et les nécessités de la vie".

[12] "les parties du programme de calcul qui semblent difficiles à assimiler par des enfants de moins de 11 ans".

[13] "veiller […] à ce que toutes les notions dont l'étude est obligatoire soient parfaitement assimilées".

basis for subsequent mathematical teaching" (Beulaygue 1962)[14]. As the program cuts were intended to refocus the teaching onto the basic skills, they did not resolve the question of continuity, in terms of content, method and also spirit, between primary school "arithmetic" program and secondary school "mathematics" program. Compared to the latter which had been reformed between 1957 and 1960, the 1945 primary school program was judged obsolete and mathematically lax (APMEP 1965, p. 522; Lasalmonie 1966, p. 10). Inspector General Marius Beulaygue pointed out the mathematically limited scope and the conceptual weakness of the topics listed in the primary school arithmetic program. Not only were the study of some topics isolated from their mathematical environment (fractions were not studied as extensions of whole numbers, the rule of three was not seen as a particular use of the properties of proportional numbers), but also arithmetic teaching, wanted to be resolutely practical, did not lead children towards the abstraction that prevailed in secondary education:

> Throughout its primary schooling, the pupil defines, calculates and operates on concrete numbers. Only they can be cited. Numbers are like prisoners, held with balls and chains: 5 marbles, 7 apples, if it's not two: 5 francs per Kg; 53 quintals per hectare. On entering secondary school they escape from their bonds. Abstract numbers appear as if from nowhere, and often in abundance. Can't we then, from time to time in CM2[15], remove some of the chains, in order to note, despite their abstract form, some properties that are worth paying this price. (Beulaygue 1962)[16]

This criticism on primary school arithmetic teaching, mathematical in nature, was not limited solely to the primary world. Secondary school mathematics teachers, who received the schoolchildren when they had finished primary school – and who were going to receive more and more –, were equally interested in making the primary and secondary programs more coherent. They wanted primary school not only ensure the

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[14] "Pour intéressants qu'ils soient, ces allègements ne sauraient suffire à faire du calcul tel qu'il est donné à l'école primaire, l'assise solide de l'enseignement mathématique ultérieur".

[15] CM2 : *cours moyen 2e année*, the highest grade in primary school (10-11 years old).

[16] "Tout au long de sa scolarité primaire, l'élève définit, calcule, opère sur des nombres concrets. Seuls ils ont droit de cité. Le nombre va comme un captif, un boulet au pied : 5 billes, 7 pommes ; si ce n'est deux : 5 F par Kg, 53 quintaux à l'hectare. Que l'on entre en classe de sixième [secondaire] et cette servitude s'évanouit. Sans la moindre lettre de créance, le nombre abstrait s'installe et souvent sans mesure. Ne peut-on dès lors, au CM2, libérer, épisodiquement, le nombre de ses chaînes, afin de noter, au passage, en dépit de sa forme abstraite, quelques propriétés dont chacun reconnaît le prix".

acquisition of the operational skills necessary to follow the secondary school classes properly, but also be a true "initiation into mathematics" in line with the prevailing spirit of the discipline. For that purpose, the *Association des professeurs de mathématiques de l'enseignement public* (APMEP), made up mostly of secondary school teachers, established a committee whose role was "to be concerned with preparatory mathematics teaching as actually taught in primary schools" (APMEP 1962, p. 361)[17]: "This preoccupation with continuity in teaching […] cannot leave any of us mathematics teachers indifferent, because mathematics teaching has a compelling requirement for its continuity to be as perfect as possible" (APMEP 1963, p. 404)[18]. As will be seen in the following sections, the attention given to good correlation between primary and secondary teaching quickly gave way to a more general challenge to the 1945 primary school program and directives, and consideration of the introduction of "modern mathematics" in primary schools.

From "arithmetic" to "mathematics"

The question of introducing modern mathematics into primary education took shape towards the middle of the 1960s. The thinking that gave rise to it was part of a broader, international movement for the reform of mathematics teaching (Gispert 2010; Gispert & Schubring 2011). In France, the APMEP was the spearhead of this movement[19]. The APMEP considered that the modernization of secondary mathematics teaching could only come to fruition if students were properly prepared to receive it. The association campaigned for "better coordination of the reforms in all grades and in particular a reasonable modernization of elementary class programs" (Walusinski 1965, p. 372)[20]. It set itself the task of developing "a comprehensive plan for teaching mathematics from pre-school to university propaedeutic classes inclusive" (Walusinski 1964, p. 402)[21] and proposed, from 1965, draft programs for preschool and primary schools. At the same time, the *Institut pédagogique national* (IPN),

[17] "s'inquiéter de l'enseignement préparatoire aux mathématiques tel qu'il est effectivement donné dans l'enseignement du premier degré".

[18] "Cette préoccupation de la continuité de l'enseignement […] ne saurait laisser indifférent aucun d'entre nous, professeurs de mathématiques, parce que l'enseignement mathématique exige impérieusement une continuité aussi parfaite que possible".

[19] On APMEP's action for the introduction of "modern mathematics" in the 1950s, see d'Enfert (2010).

[20] "une meilleure coordination des réformes dans toutes les classes et en particulier une modernisation raisonnable des programmes des classes élémentaires".

[21] "un plan d'ensemble d'enseignement des mathématiques de l'école maternelle comprise aux propédeutiques comprises".

which was under the control of the Ministry of National Education, led a discussion centered on the introduction of "modern mathematics" into primary schools. It launched an inquiry on this matter and then developed experiments based on conceptions supported by the *Commission internationale pour l'étude et l'amélioration de l'enseignement mathématique* (CIEAEM) founded in 1952 by mathematicians, philosophers and psychologists[22].

The objective for these "modernizers" was to make the teaching of the discipline open to contemporary mathematics, and more particularly to modern algebra which brought into play the notion of structure. According to them, modern mathematics was an essential element in modern human culture because of the privileged role it plays in understanding the contemporary world. They justified its introduction, even in primary education, by the link they established between the construction of mathematical structures and the development of mental structures in children as shown by the developmental psychologist Jean Piaget. The fact that the relationship between mathematical structures and mental structures had been, in the 1950s, one of the first areas of research for CIEAEM is particularly emblematic of the role given to this notion of structure (Bernet & Jacquet 1998). Thus the mathematician Gustave Choquet could declare "after all, a mathematician is a child who grew up and the mathematical structures which appear basic to him come from the construction of mental structures that develop spontaneously in children." (Choquet 1961, p. 365)[23]

This "modern" conception of the discipline appeared to be a possible response, or even *the* response to criticism made of the 1945 arithmetic programs. Modern mathematics was thus considered as an effective tool to break with their concrete and practical aspects and to put an end to their inconsistencies. The aim was no longer to communicate immediately-usable arithmetical techniques to pupils, but to build with them the mathematical concepts. In particular, the notions of structure and relationship allowed the unification of mathematical subjects previously taught in an unconnected way: "The number and its properties

[22] In the 1950s, the CIEAEM included psychologist Jean Piaget, mathematician and philosopher Ferdinand Gonseth and mathematicians Gustave Choquet, Jean Dieudonné and André Lichnerowicz. See Gispert (2010), Gispert & Schubring (2011).

[23] "après tout, le mathématicien est un enfant qui a grandi et que les structures mathématiques qui lui paraissent fondamentales, proviennent de l'élaboration des structures mentales qui se développent spontanément chez l'enfant".

are no more, in this context, than a specific case of a general logic which must be constructed of and by action." (Legrand 1967, p. 4)[24]

The modernizers also campaigned for active learning, built on children's inventive and abstractive capacities. This pedagogical renewal was considered a necessary condition for the modernization of the content. The *Charte de Chambery*, which was an APMEP action plan for the reform of mathematics teaching asserted that "the introduction of new content in mathematics education will be ineffective and even harmful, if it is not accompanied by appropriate pedagogy: active, open and as undogmatic as possible, drawing on group work and on the children's imagination" (APMEP 1968, p. 7)[25]. In that way, pupils could really 'do' the 'true' mathematics, that is to say discover the fundamental concepts of modern mathematics as mathematicians do. Reciprocally, modern mathematics was considered as a tool for pedagogical reform: they had to provide the means of doing away with all the dogmatism of traditional arithmetic teaching, such as learning rules by memorization, and solving stereotypical problems far from the children's own concerns and interests.

The modernizers' rationale – and especially those who had taken part in experiments conducted since 1964 by the IPN – was largely fueled by the ideas of Caleb Gattegno and Zoltan P. Dienes, two disciples of Jean Piaget. Their ideas were based on the conviction that young children have the taste for research and were capable of creativity. So that mathematics could be taught – and learned – in a non-traditional way: by using games, by using innovative teaching aids like Cuisenaire rods (also called "numbers in color") and Dienes' logic and multibase blocks, and more generally through the study of mathematical "situations". Moreover, experiments in primary schools seemed to confirm the pedagogical renewal a modern approach to mathematics could offer. Commenting on eight years of research "on the abstraction of mathematical concepts for children from 6 to 11 years old"[26], Nicole Picard, a head researcher at IPN, concluded:

> what appears important to me, is the discovery that, by learning through doing, nearly all children are capable of invention and that in this regard mathematics is quite unique in being the only discipline where you can be a creator and at the same time, an objective judge of

[24] "Le nombre et ses propriétés ne sont plus, dans cette perspective, que les cas particuliers d'une logique générale qui doit se construire de et par l'action".

[25] "l'introduction d'un nouveau contenu dans l'enseignement des mathématiques sera inopérante, voire néfaste, si elle ne s'accompagne d'une pédagogie appropriée : active, ouverte, la moins dogmatique possible, faisant appel au travail par groupe et à l'imagination des enfants".

[26] "sur l'abstraction de concepts mathématiques pour enfants de 6 à 11 ans".

> your creation. A discipline whereby, if you do the research, you can realize that you are someone intelligent and capable of autonomous thought. (Picard 1973, p. 12)[27]

The emergence of modern mathematics therefore disrupted the way the reform of primary mathematics teaching was initially envisaged. The priorities were now reversed: while at the beginning of the 1960s the priority was principally the acquisition by pupils of the basic knowledge and skills necessary for their secondary schooling, the discipline's strictly educative dimension, its participation in the development of the personality and character were increasingly favored. The acquisition of arithmetical techniques (numeracy, operations, etc.) was not abandoned however: on the contrary, according to the modernizers, this new approach could reinforce the acquisition of these techniques.

The new 1970 "mathematics" program

These conceptions were taken up in 1969 by a ministerial commission charged with reforming mathematics teaching in primary and secondary schools under the chairmanship of mathematician André Lichnerowicz. This resulted in the publication, in 1970, of a new "mathematics" program – and not "arithmetic" as in 1945 – for primary schools, as well as new directives.

The changes concerned, above all, the approach to the mathematical content. The dual influence of structural mathematics and Jean Piaget's developmental psychology clearly appeared: the new program focused on the properties of mathematical objects and the relationship between them, and the learning was planned to be appropriate to the child's stage of development. In *cours préparatoire* (6-7 years old), the emphasis was put on the concept of number, based on the principle of equivalent sets, learning to count was not based on the metric system – the study of which was reduced to a minimum – but on activities focusing on grouping objects. It was no longer a question of operating on concrete numbers (considered as sets of objects), but on the numbers themselves. As written in the 1970 directive: "Sentences such as: 8 apples + 7 apples = 15 apples belongs, in fact, neither to the language of mathematics nor everyday language"

[27] "ce qui me semble important, c'est la découverte sur le tas que la quasi totalité des enfants est capable d'invention, que les mathématiques constituent sur ce plan une discipline tout à fait exceptionnelle où l'on peut être à la fois créateur et juge objectif de sa création. Une discipline par laquelle, si on y fait de la recherche, on peut prendre conscience que l'on est quelqu'un d'intelligent, capable d'autonomie de pensée".

(MEN 1970, p. 355)[28]. In *cours élémentaire* (7-9 years old) and *cours moyen* (9-11 years old), prominence was given to studying the properties of the four operations. The rule of three and percentages gave way to more general concepts of numerical relationship (represented by number tables) and proportionality, the study of which preceded that of fractions, considered as operators. Geometry teaching was also concerned: studying the properties of figures was favored, and localization exercises on lines and grids were introduced.

The new directives justified these changes through the lengthening of the period of compulsory schooling and the democratization of access to secondary education, as well as by the progress in "mathematical thought", that is, the development of modern mathematics:

> Mathematics teaching in primary school should now meet the requirements that come from extended compulsory education and progress in contemporary mathematical thought.
>
> It is a question of ensuring that this teaching makes an effective contribution to the intellectual development of all 6 to 11 year olds so they enter into secondary education with the best chances of success.
>
> The goal of such teaching is no longer primarily to prepare pupils for their daily and professional life by making them learn a catalogue of problem-solving techniques suggested by 'everyday life', but to ensure they have the correct approach and a real understanding of the mathematical principles linked to these techniques. (MEN 1970, p. 349)[29]

As we can see, a new objective, mathematical in nature, was added to the institutional objective of extending compulsory education already mobilized since the beginning of the 1960s. This radically modified the aims of the mathematical education given. The "contemporary progress in

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[28] "Les phrases telles que : 8 pommes + 7 pommes = 15 pommes n'appartiennent en fait, ni au langage mathématique, ni au langage usuel".

[29] "L'enseignement mathématique à l'école élémentaire veut répondre désormais aux impératifs qui découlent d'une scolarité obligatoire prolongée et de l'évolution contemporaine de la pensée mathématique.

Il s'agit dès lors de faire en sorte que cet enseignement contribue efficacement au meilleur développement intellectuel de tous les enfants de six à onze ans afin qu'ils entrent dans le second degré avec les meilleures chances de succès.

L'ambition d'un tel enseignement n'est donc plus essentiellement de préparer les élèves à la vie active et professionnelle en leur faisant acquérir des techniques de résolution de problèmes catalogués et suggérés par la 'vie courante', mais bien de leur assurer une approche correcte et une compréhension réelle des notions mathématiques liées à ces techniques".

mathematical thought"[30] evoked by the 1970 directives concerned a priori the content, but the methods used in learning and teaching mathematics were also concerned. To prepare pupils for secondary mathematics teaching, which had itself been reformed, the aim was no longer to favor systematic exercises and learning "by heart", but to give way to understanding of concepts and the "command of usable, fertile, mathematical thought"[31] through active learning. The acquisition of techniques would be made, you could say, naturally: " The usual techniques for operations […] will be acquired all the better if, instead of learning in a purely mechanical way, the children discover them themselves through the synthesis of many, varied, real experiments." (MEN 1970, p. 360)[32]

Between challenge and consolidation

The 1970 mathematics program rapidly became the subject of much criticism, which was part of a more general – and widely publicized – denunciation of the "modern mathematics" reform as a whole. Some of this criticism was radical: by emphasizing understanding and reasoning instead of the acquisition of automatic reflexes (notably multiplication tables), the modernization program would lead to forgetting that "the mission of primary schools is to teach counting in our good old decimal system" (Turner 1973, p. 65)[33]. The press went as far as attributing the suicide of a school teacher, in January 1972, to the fact that he was "very affected by the modern teaching of mathematics in primary school" (Le Monde, January 1972 reprinted in Hélayel 2004, p. 91)[34]. Other criticism pointed out the extreme emphasis accorded by primary school teachers to set vocabulary and symbols, and calculation in bases other than 10. The quasi-systematic use of exercices books (instead of textbooks) promoted by modernizers to encourage pupil autonomy was also denounced (Duma 1973).

The reform program conducted by Minister of Education René Haby in the second half of the 1970s was the first response to this wave of criticism. Appointed in May 1974, René Haby immediately launched a

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[30] "évolution contemporaine de la pensée mathématique".

[31] "maîtrise d'une pensée mathématique disponible et féconde".

[32] "Les techniques usuelles concernant les opérations […] seront d'autant mieux acquises que les enfants, au lieu de les apprendre de façon purement mécanique, les auront découvertes par eux-mêmes comme synthèses d'expériences effectivement réalisées, nombreuses et variées".

[33] "la mission de l'école primaire est d'apprendre à compter dans notre bon vieux système décimal".

[34] "très affecté par l'enseignement moderne des mathématiques à l'école élémentaire".

comprehensive reform which at the same time affected the organization of the educational system – the most emblematic measure being the creation of the "*collège unique*" – and teaching content. For primary education, the 11 July 1975 law clearly stated the priority given to instrumental learning: French and mathematics were place in first position in the ranking of the various subjects of "primary training". A 1976 decree further provided that "primary training provides the everyday use […] of arithmetic and simple mathematical operations." (MEN 1977, p. 4577)[35]

Actually, René Haby wanted to restore the learning of techniques and operational skills the 1970 mathematics program had been accused of neglecting. He felt that modern principles could usefully contribute to the training of the mind, but should neither reduce nor undermine the part played by the acquisition of the skills, necessary for good academic achievement in secondary school, nor take the place of arithmetical knowledge useful in everyday life, as this declaration attests:

> It must not be forgotten that the 'reading, writing and arithmetic' of our ancestors remains a valid formula in primary education, and the acquisition of skills truly anchored in the mind is one of the objectives of primary school. […] With respect to the acquisition of basic skills and the training of the logical mind, I cannot but agree with the teaching of modern mathematics at the pre-school and primary levels. But where will this teaching lead us? One can, of course, link operations such as adding fractions or division to considerations of modern mathematics; I wonder if this relationship is not based on a certain intellectual sophistication. This largely axiomatic approach to arithmetical techniques seems to me to be putting the cart before the horse. […] the acquisition of automatic reflexes and techniques, whether reading or arithmetical, remain, in my opinion fundamental. The household shopping will not be done with a handheld computer in a hurry. (Haby 1974)[36]

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[35] "la formation primaire assure la pratique courante […] du calcul et des opérations simples des mathématiques".

[36] "Il ne faut pas oublier que le "lire, écrire, compter" de nos ancêtres reste une formule valable au niveau de l'école élémentaire, et que l'acquisition de mécanismes véritablement ancrés dans l'esprit est un des objectifs de l'école primaire […] En ce qui concerne l'acquisition des mécanismes de base, la formation de l'esprit logique, je ne peux que souscrire à l'enseignement des mathématiques modernes aux niveaux pré-scolaire et primaire. Mais jusqu'où cet enseignement doit-il aller ? L'on peut, bien entendu, rattacher les opérations comme l'addition de fractions ou la division à des considérations de mathématiques modernes ; je me demande si cette mise en relation ne repose pas sur une certaine sophistication intellectuelle. L'approche plus ou moins axiomatique des techniques de calcul me paraît ressembler à la mise de la charrue avant les bœufs. […] l'acquisition d'automatismes et de techniques, que ce soient la lecture ou le calcul, reste à mon avis

New mathematics programs for primary schools were therefore published between 1977 and 1980. At the same time, the discipline's weekly time allocation increased again: already at 5 hours (out of the 27 hours available per week) in all primary grades in 1969, it now reached 6 hours[37]. However, the break with the 1970 program was not as pronounced as the Minister had initially wished: far from being a return to the traditional arithmetic, the 1977-1980 reform reinforced the 1970 one as much as it modulated it.

On one hand, the new programs insisted strongly on the acquisition of operational techniques and their maintenance throughout primary schooling, on the memorization of tables, addition and multiplication, as well as the practice of mental arithmetic – which did not exclude the use of electronic calculators in the *cours moyen* (9-11 year old), The metric system took prominence in measurement activities which were themselves more diverse. On the other hand, many modern approaches were retained, such as ranking exercises to address the concept of number, handling of bases other than 10 for learning numeration, and even the use of numerical functions to introduce proportionality. Basic notions of sets and properties of operations where not, however, the subject of formal study.

The mathematical education was not limited solely to the acquisition of techniques: it also had to participate in the "training in logical thought"[38] and, in the *cours moyen*, take the pupils to a "first level of abstraction"[39]. References to active learning were numerous. Group work was recommended, as well as the study of "problem situations" in order to introduce new mathematical knowledge, to reinforce prior learning, but also to foster pupils' attitudes to reasoned research. Including mathematics in scientific or hands-on activities was also advocated, each supporting the other.

A more significant change happened in 1985 with Minister of National Education Jean-Pierre Chevènement. He wanted to reconnect with the educational values of the Third Republic – according to him, the foremost mission of school is to instruct – and remove from teachers the "pedagogical inspiration"[40] of previous directives (Barret 1988, p. 183). The "modern" content and approach was reduced in favor of a more classic vision of mathematics teaching and most of the references to set

fondamentale. Ce n'est pas demain que les ménagères feront leur marché avec un ordinateur de poche". See also (Haby, 1975).

[37] Which corresponds to a total hourly increase of 40% compared to 1945.

[38] "formation de la pensée logique".

[39] "premier niveau d'abstraction".

[40] "l'inspiration pédagogiste".

theory disappeared. Subsequently, successive reforms (1995, 2002, 2008) swung in greater and lesser degrees between the desire to (re) center mathematics teaching on the basic skills and that of making primary school a place where "mathematics and its models really begin." (MEN 2002, p. 40)[41]

Conclusion

In the decades following the Second World War, the reform projects and later the reforms that aimed to widen access to secondary education asked the various school subjects to question their contribution to the movement to democratize education. In response, two distinct approaches of primary mathematics teaching appeared. A first, rather traditional approach dominated until the start of the 1960s: considering that repetition and memorization were the best ways to establish sustainable knowledge, it placed basic arithmetic skills at the forefront and did not challenge intrinsically the teaching methods that had prevailed until then. A second, more innovative approach appeared in the 1960s: at the same time advocating both modernization of the content and renovation of the methods, it favored the discovery of mathematical structures and the understanding of concepts rather than learning arithmetical techniques, and aimed at the development of thinking and the fulfillment of the child.

These two different approaches were however consistent with two concepts of the overall mathematics curriculum. The first maintained a clean break between primary school, the place for concrete teaching where pupils learned to count and do basic operations, and secondary school, where they started mathematics and were progressively lead towards abstraction. The second instead wanted to establish continuity between the primary and secondary curriculums: rather than considering the acquisition of basic arithmetic skills as a prerequisite to 'mathematics' learning, it proposed that pupils could 'do' mathematics from primary school, which translated into the APMEP slogan "From pre-school to university". Since the end of the 1970s, mathematics teaching in primary schools has wavered between these two concepts, which remain thus in opposition.

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[41] "commencent véritablement les mathématiques et leurs modèles".

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The teaching of mathematics in the Jesuit *Ratio studiorum*

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Abstract

The Society of Jesus considered the importance that mathematics should have in its academic syllabus in a cultural context where serious discussions about their veracity and usefulness were taking place among philosophers and mathematicians. Those who favoured the valuation of those disciplines, as J. Nadal and B. Torres did, carried out several proposals to that aim, but criticism of their detractors and the lack of qualified teachers prevented these proposals to be put in practice at the colleges. When the Jesuits were deciding on their education system, C. Clavius advanced new suggestions with the intention of promoting the study and teaching of mathematics, but, even though some of them were included in the two first drafts of the Ratio studiorum*, they disappeared in the final version due to the opposition of the majority of the Society members. Therefore, mathematics remained a secondary subject in the education system of the Society of Jesus and not even exams were held.*

Introduction

An investigation carried out within the project "Spanish science in the 16th and the 17th centuries" of the Fundación Canaria Orotava de Historia de la Ciencia, focused my interest on the Spanish Jesuits Jeronimo Nadal and Baltasar Torres and their role in the inclusion of mathematics in the *Ratio studiorum* philosophy course.

According to common opinions among the historians of science, and also within the Society of Jesus, Christopher Clavius was the man who, thanks to his influence, obtained the inclusion in the *Ratio studiorum* of an important mathematics syllabus, succeeding also in the Jesuits' subsequent appreciation of this discipline in their colleges.[1]

[1] "It was Clavius who by his defence of mathematics within the context of Jesuit educational goals and by creating a mathematical school at the Collegio Romano where most the society´s scientist, was principally responsible for establishing Jesuit policy and eventual achievements in the mathematical sciences". "The influence of Clavius is evident in the first Jesuit "Ratio Studiorum" of 1586 and in the definitive version of 1599". "Both outlined a full programme of philosophical and mathematical studies". (Crombie, 1977, p. 65; 67). "Le projet élaboré par Clavius permet clairement, au niveau du Collegio Romano, d'envisager les mathématiques comme une discipline à part entière qui n´entretiendrait plus avec la philosophie les rapports de subordination hérités du modèle médiéval". (Romano, 1999, p. 132). "…the greatest legacy of Clavius was his influence on the definitive version of the *Ratio Studiorum* which led to the inclusion of mathematics as a standard subject taught in Jesuit schools". (Smolarski, 2002, p. 260).

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education.* Caparica: UIED.

However, I came to the conclusion that Clavius had not this impact on the *Ratio studiorum* and that Jesuits did not value the teaching of this discipline, since they did not accept the proposals of those who tried to include an extensive mathematical syllabus in their education system.

Given that this conclusion was contrary to common views, I thought it would be convenient submitting it to a scientific revision. Then I found out it had been already defended by the professors A. Krayer and G. Schubring.[2] Bearing in mind their researches and the guidance of professor Schubring, I researched the issue with new means. This paper contains the results of this work.

Cultural context: discussions about the certitude and utility of mathematics

During the sixteenth century there was an important rebirth of mathematics promoted by the Humanism, mainly in Italy (see Rose, 1975). Humanists, in their attempt to recover ancient knowledge, rediscovered the Pythagorean and Platonic schools that, as everyone knows, give a special worth to mathematics; they recovered important ancient mathematical texts that had been lost until that moment; they improved the translations of some well-known works as Euclid's and Archimedes', that had been translated incomprehensibly; and they valued the teaching of that discipline at the colleges as a methodological alternative to Aristotelian logic (see Gilbert, 1960).

This new situation triggered important discussions about the foundations and methods of scientific knowledge that set philosophers and mathematicians at odds, the last ones willing to raise the social

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[2] "Clavius hatte mit seinem Versuch, die Bedeutung der Mathematik in der Gessellschaft Jesu zu stärken, letztlich, zumindest auf der institutionellen Ebene, wenig Erfolg". (Krayer, 1991, p. 41). "È anche vero che Christoforo Clavio, insigne matematico e astronomo jesuita, nonché professore di matematica presso il Collegio Romano, riuscì a inserire estesi regolamenti relativi a tale materia nel progetto della Commissione per il nuovo ordinamento centrale degli studi, i quali però, a causa delle resistenze opposte da vari parti, non furono accolti nella *Ratio studiorum*. La matematica rimase così una materia d'insegnamento marginale per la quale non esistevano norme in merito alla qualificazione necessaria per l'insegnamento (mentre quelle per l'insegnamento della filosofia, per esempio, erano piuttosto dettagliate). (Schubring, 2002, p. 369). "Although it is well known that it was the Humanist movement which achieved the introduction of chairs for mathematics to the European universities since the turn from the fifteenth to the sixteenth century, and that the Protestants continued this policy, […]. In reality, Jesuits taking over humanistically reformed universities succeeded in suppressing the mathematical chairs and reducing mathematics teaching to a few month´s at secondary school level in the colleges". (Schubring, 2003, p. 1073).

consideration of their discipline and, therefore, the prestige of their profession (see Carugo, 1984; Biagioli, 1989; Pace, 1993).

These oppositions between philosophers and mathematicians also happened in the Society of Jesus, even in the Roman College. In fact, the Spanish philosopher Benito Perera, in his famous work "*De communibus omnium rerum naturalium principiis et affectionibus*", asserts that mathematics does not reach the highest level of certitude, so it is not a science strictly speaking. Perera does not accept the intermediate ontological character of mathematics and rejects they could serve to know Nature, given that physics studies substances and natural bodies and mathematics studies accidents (of quantity and condition) (Pace, 1993, pp. 75-120).

On the other hand, the German mathematician Cristopher Clavius not only defended the scientific nature of mathematics but he also attacked the proofs of natural philosophy and metaphysics because, more than a science, he thinks it is a conjecture due to the quantity and diversity of philosophers' opinions. Clavius supports, against Perera's theories, the usefulness and need of mathematics to understand the other parts of philosophy, especially natural philosophy (Kessler, 1995, pp. 285-308).

The initial approach: mathematics at the service of theology

The Paduan Jesuits, who did not have organized studies in their house, sent the students to listen the philosophy lessons at the university (Aixalá, 2001, Vol. I p. 682; Sauvé & Codina, 2001, Vol. II, p. 1203). In 1546, some standards were then developed following the approach of Ignatius of Loyola to guide these students. They said that, among others, the students had to attend for two years and a half to logic, natural philosophy, metaphysics, mathematics and moral philosophy lessons (*Constitutiones Scholasticorum Societatis Iesu Patavii*; Lukacs, 1965-1992, Vol. I, p. 11).

According to the preparatory texts of the *Constitutions* of the Society of Jesus (wrote from 1547 to 1556), the aim of frequenting those courses was to study arts and natural sciences, among them mathematics, because they "prepare minds for theology and they are useful for its perfect understanding and use, and in case they could help to achieve the order's purposes" (*De scientiis quae tradendae sunt in universitatibus Societatis*; Lukács, 1965-1992, Vol. I, p. 282).

Jeronimo Nadal's proposals

Two years later, in 1548, the Society opened at Messina, in the island of Sicily (Italy), the first college aimed at the education of non Jesuit students. The Spanish Jeronimo Nadal was appointed chancellor (see Ruiz Jurado, 1979). In that year, Nadal drew up the *Constitutions* of the College of Messina, introducing mathematics lessons in the philosophy course. There, he orders the philosophy teacher to teach this discipline reading first some Euclid's books, and then Oronce Fine's practical arithmetic and the sphere or cosmography, Sotoeffler's astrolabe and Peuerbach's astronomy (*Constitutiones Collegii Messanensis*; Lukács, 1965-1992, Vol. I, p. 26).

In 1552, he wrote the new syllabus as a rule for the whole Society. On it we find, for the first time, the figure of a specialized mathematics teacher and a detailed method of teaching this discipline that modified the one proposed in 1548 for the College of Messina, because of the inclusion of theoretical music and perspective. All the philosophy students had to attend the mathematical subjects for three years (*De studii generalis dispositione et ordine*; Lukács, 1965-1992, Vol. I, p. 143). Nadal's proposal is that philosophy be studied in four years.

Second year philosophy students, in the first lesson, studied Euclid, some practical arithmetic and the principles of astronomy. Third year philosophy students, in the second lesson, were taught speculative music and perspective. Fourth year philosophy students, in the third lesson, had to attend astronomy, starting with Peuerbach's theory of planets and following, if it was possible, with the reading of Ptolemy, Johannes Müller (Regiomontanus), Alfonsine Tables, the astrolabe study, etc. (*Ibidem,* pp. 148-149).

Other proposals about the teaching of mathematics in the Society

Nadal's interest for mathematics was stronger than the one shown by other members of the Order. So there is nothing unusual in the successive Jesuit proposals that reduced more and more the mathematical studies syllabus.

Indeed, Martin de Olave, who drew up in 1553 a treatise about the lessons and exercises to be used in the Society of Jesus' universities, reduces the teaching of mathematics to two and a half years. On it he proposes that a professor should teach it and the good and useful part of astrology (*Ordo lectionum et exercitationum in universitatibus Societatis Iesu;* Lukács, 1965-1992, Vol. I, pp. 166, 176, 177).

In 1558, it was proposed for the Roman College a new syllabus according to which the arts course should be taught in two and a half years. In this syllabus, the teaching of mathematics was reduced to two years and three months. During the first fifteen months there should be 2 months of arithmetic, 4 of geometry, 3 of sphere and 3 of astrolabe. And in the following twelve months: 4 months of theory of planets, 3 months of almanac, 3 months of perspective and 2 months of clocks (*Ordo studiorum Collegii Romani 1558*; Lukács, 1965-1992, Vol. II, p, 15).

Baltasar Torres' proposals for the Roman College

The Roman College, the main educational institution of the Society of Jesus, was founded in 1551. In 1553, the Spanish Jesuit Baltasar Torres, was appointed the first professor of mathematics. While he worked as a teacher in Rome he wrote, between 1557 and 1560, two proposals about the teaching of mathematics in the Society's colleges. Those are the first documents aimed exclusively to this purpose.

In his first proposal, in agreement with Nadal, he asks again for an extended mathematical syllabus and for a mathematical course of three years. During the first year, students should learn practical arithmetic and the first three books of Euclid's *Elements*; during the second, sphere and geography; and throughout the third, theory of planets, astrolabe and perspective. He also requests, once the course was over, the teaching of the fourth, the fifth, the sixth and the eleventh of Euclid's books and a review of other mathematical problems (*Ordo lectionis matheseos in Collegio Romano*. Lukács, 1965-1992, Vol. II, p. 434).

On the second proposal, even though he reduces the teaching of mathematics to just two years, he introduces other two innovations related to the students who were more interested in learning mathematics and with the textbooks that should be used to teach them. He suggests that the most outstanding students on this discipline, should be taught, privately, additional lessons about astronomic issues using textbooks as, for example, Maurolico's, more up-to-date than those they used to read in that time (*Ibidem)*.

Other less ambitious proposals about the teaching of mathematics

Torres' proposals, like Nadal's ones, were also considered excessive because, very soon, other proposals, that had in common their purpose of not giving the teaching of mathematics the importance suggested by these two Spanish Jesuits, were drawn up.

The first of these new proposals, elaborated in 1566 to be used in the Roman College, even though it maintains an extended syllabus for mathematical studies (the first six books of Euclid, arithmetic, sphere, cosmography, astrology, theory of planets, Alfonsine tables, perspective and clocks), it restricts their teaching to the second year of the philosophy class, that means, to those who studied physics or natural philosophy, proposing that just sometimes and with dispensation it could be taught to the dialecticians, the first year students (*Gubernatio Collegii Romani*. Lukács, 1965-1992, Vol. II, p. 179).

In a new document, written between 1565 and 1570 by the professors of the Roman College and directed by the Spanish Jesuit Diego de Ledesma with the aim to use it for the syllabus of every Society's university, just some recommendations were given about the teaching of mathematics: not to abandon their teaching; to have some extraordinary lectures; to explain mathematics, if it is possible, also in private schools; to make every student learn about sphere and cosmography; to combine their studies with philosophy and theology lessons; to make those who, by wish of the superior, are preparing for teaching liberal arts know them better (*De artium liberalium studiis*. Lukács, 1965-1992, Vol. II, p. 254).

Difficulties of teaching mathematics lessons

The teaching of mathematics at the beginning of the Society's history also had serious difficulties due to the lack of professors qualified to carry it out.

Nadal himself, when he was appointed Jesuit college visitor, could check that mathematics was hardly taught. Let's see some evidences taken from the instructions he left after his visits. In 1561, in his notes about Coimbra, he admitted that the teaching of mathematics was not guaranteed and that, if there were some lessons, they were only for second-year and third-year students during half an hour, at most (*Instructiones Conimbricae de cursu artium datae*. Lukács, 1965-1992, Vol. III, p. 61).

In 1566, in his notes about Vienna, he accepted the suspension of the mathematics lesson, because it seemed to be convenient to make the philosophy course shorter, of two and a half years (*Instructiones Viennae datae*. Lukács, 1965-1992, Vol. III, p. 116).

In 1568, about France, he admitted that in Paris, in view of the circumstances, it was not useful, by the moment, to organize mathematics lessons because there was no qualified teacher (*Instructiones Praeposito provinciae Franciae datae*. Lukács, 1965-1992, Vol. III, p. 163).

So, actually, whatever the mentioned proposals said, the truth is that, due to the low esteem that most of the Jesuits showed for this discipline and the lack of specialized professors, during the first years of the Society, mathematics was taught only in a few colleges.[3]

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[3] "En vida de Ignacio de Loyola, las matemáticas se enseñaban en los colegios jesuitas de Mesina (1548), Roma (1553) y Tournon (1556) y, hacia 1590, también en Padua, Douai y Pont-à-Mousson. Sin embargo, en muchos colegios jesuitas no había cursos de matemáticas, y en la mayoría ningún profesor especial para ellas hasta comienzos del siglo

Even the professors who taught mathematics lessons, complained about it. A Spanish professor from the college of Vienna wrote this to Nadal:

> There are not many enthusiasts of this study. I myself think it is not very convenient to the Society's institute, and because I was almost alone in the work with this science, which really has, from my point of view, a lot of vain and useless aspects, I have felt the wish of dedicating myself to ordinary studies and leave mathematics aside[4].

The first proposals of Christopher Clavius

The German mathematician Christopher Clavius, during his stay at the Roman College, wrote some documents for the Society to give, finally, more importance to the study and teaching of mathematics (see Cosentino, 1971; Gatto, 2006). Let's analyze now what he wrote before the first version of the *Ratio studiorum*.

In the first document, written in 1581, he presents a very detailed syllabus for mathematical studies with three different levels of specialization. The aim of the first level was to educate experts; the second level was created for those who didn't need a perfect knowledge of mathematics, and the third, for all the students. This last level, described by Clavius himself as "very brief", should last two years. On the first year, they would be taught the first six books of Euclid, practical arithmetic, cosmography and the computus, the use of the geometric square and the astronomic quadrant, perspective and clocks. On the second year, they would read the eleventh and the twelfth books of Euclid, the treatise of sines, geography, the structure and use of the astrolabe, the theory of planets, measuring and squaring the circle according to Archimedes, algebraic rules and the precepts for measuring figures (*P. Christophorus Clavius, S. I. Ordo servandus in addiscendis disciplinis mathematicis.* Lukács, 1965-1992, Vol. VII, pp. 110-115).

None proposal of this syllabus for mathematical studies, not even the third level, appeared in any of the versions of the *Ratio studiorum*.

The second document, written in 1582, is used by Clavius to convince the Jesuits of the need of promoting the teaching of mathematics in the Society's colleges. The paper begins denouncing the state of neglect

XVII, debido sobre todo a la falta de jesuitas competentes en este campo"(Ziggelaar, 2001, Vol. I. p. 203).

[4] "Peu de gens sont affectionnés à cette étude. Moi-même je pensais qu'elle convenait peu à l'institut de la Compagnie, et comme je me voyais presque seul occupé à cette science, où vraiment à mon avis, il y a beaucoup de choses vaines et inutiles, j'étais pressé du désir de m'appliquer aux études ordinaires, en laissant de côté des mathématiques." (Dainville, 1978, p. 324).

suffered by this discipline, blaming the philosophers who don't know and despise them. Then he adds that it is essential for the students to understand that mathematics is useful and necessary to appreciate correctly the rest of the philosophy and an ornament for all the arts; and that there is so much affinity between mathematics and natural philosophy that, without helping each other, they could not keep their dignity. He ends suggesting that, once the philosophy course is over, the students who aspire to graduate should pass an exam of mathematics and that in that examination a mathematics teacher, together with the philosophy teachers, should participate (*P. Christophorus Clavius, S. I. Modus quo disciplinae mathematicae in scholis Societatis possent promoveri.* Lukács, 1965-1999, Vol. VII, pp. 115-117).

Clavius' demanding syllabus presented in his first proposal and the defense of the usefulness and necessity of mathematics to properly understand the rest of the philosophy explained in the second proposal, clearly indicate that the German Jesuit gave this discipline a higher value than Ignatius of Loyola and most Jesuits did, and that it was, as we have already seen, to prepare minds for the study of theology.

The first version of the *Ratio studiorum*

The first version of the *Ratio studiorum*, written in 1586, brings together some of the ideas that Clavius had expounded in the second of his documents. First of all, there is a chapter about mathematics that is a real apology of it. In this chapter, according to the *Constitutions*, it is stated that its study would be convenient to reach the aims proposed by the Society, however, it also sets that this is not just because without it the academies would lack ornament, but, above all, because the other sciences need it, since poets, historians, politicians, physicists, metaphysicians, theologians and many others depend on mathematics (*De mathematicis.* Lukács, 1965-1992, Vol. V, p. 109).

Due to the importance given to mathematics, many proposals appear with a double purpose: organizing its teaching in the colleges and guaranteeing the training of the professors of this discipline. Regarding the first one, it admits that two mathematics professors are needed in the Roman College: one to teach a brief mathematics syllabus to the logic students, it means, the first-year students of philosophy - which would deal with Euclid's *Elements* and geography or cosmography- and another one to explain to the physics students the part of the mathematics compendium written by the Prof. Clavius (*Ibidem*).

Besides, according to Clavius' proposals, too, another professor is requested for teaching mathematics more widely, for three years, to eight or ten students, duly chosen and coming from all the Order's provinces, so that, once they had gained a superior education in that academy of mathematics, they could return to their provinces of origin and teach it where and when necessary (*Ibidem*, p. 110).

Criticism against the first version of the *Ratio studiorum*

As soon as the members of the Society got to know the chapter of the first *Ratio studiorum* about mathematics, intense criticism began. The professors from the Roman College argued they did not like philosophy students to learn mathematics for one year and a half, because one year would be enough (*Iudicia patrum in Provinciis deputatorum ad examinandum Rationis Studiorum (1586) tractatum qui inscribitur: "De mathematicis disciplinis"*. Lukács, 1965-1992, Vol. VI, p. 293).

The professors of the province of Milan asked the mathematics lessons to be taught only to the second-year philosophy students, because during that course they could learn the essential basis: the first three books of Euclid's *Elements*, cosmography, astrolabe and arithmetic (*Ibidem)*.

The Spanish professors of the province of Toledo did not accept that there were two mathematics professors, setting out that in the most important universities there were just one; they also declared that one professor would be enough to teach mathematics, because there were other things more necessary and more useful to do in the Society (*Ibidem*, p. 294).

The well-known theologian and historian Juan de Mariana, repeating the same idea, and almost with the same words of the professors from Toledo, he also rejected that there were two mathematics professors, because in Paris, Alcala and Salamanca there were only one; he asked likewise the reason of this new obligation when there were other more important and useful affairs (*Ibidem*, p. 295).

The second version of the *Ratio studiorum*

The second version of the *Ratio studiorum*, written in 1591, refers to mathematical studies in two sections: "Rules of the Provincial" and "Rules of the Professor of Mathematics".

In the "Rules of the Provincial", even though the intention of creating an academy of mathematics is kept, its aim would not be the education of professors, but the explanation of mathematics given twice a day by a specialist, following Prof. Clavius' compendium, to the students in order to improve their knowledge (*Regulae provincialis. De mathematicis.* Lukács, 1965-1992, Vol. V, p. 236).

The idea of teaching mathematics to logic students did not succeed, undoubtedly due to the critics, and the extraordinary course is left for the second-year philosophy students, to those called physicists, who should listen, for forty five minutes approximately and before lunch, lessons about Euclid's *Elements* and, after two months, some geography, cosmography or what they like (*Ibidem*).

It is accepted that, wherever it is possible, one teacher in two different hours or two teachers in the same hour will teach, following Prof. Clavius' compendium curriculum, public mathematics lessons for two years: all along the first one, to physicists and during the second, to metaphysicians, that is, third-year philosophy students. There is, however, a warning, only those who have the superiors' permission can attend those lessons (*Ibidem*).

Eventually, considering all the complaints that the German Jesuit had included in his second document with regard to the philosophy teachers' attitude, it is ordered to the superiors to avoid them damaging the dignity of mathematics (*Ibidem*).

In the "Rules of the Professor of Mathematics", all the aforementioned was repeated, adding that once or twice a month a mathematical problem should be discussed in a great meeting between philosophers and theologians; and, that one Saturday every month, the main questions explained during that time must be publicly repeated (*Regulae professoris mathematicae*. Lukács, 1965-1992, Vol. V, p. 284-285).

More critics to the second version of the *Ratio studiorum*

The cuts in the second draft of the *Ratio studiorum* to the teaching of mathematics were not considered sufficient by those who thought that this discipline was given too much importance in the Society's syllabus. Here, I present some critics attributed to Spanish Jesuits.

Toledo's province fathers said it was not possible to teach so many mathematics lessons as requested, nor were there people ready to do it, and therefore it was enough with two or three teachers for the whole province (*Patrum hispanorum de ratione studiorum (1591) observationes*. Lukács, 1965-1992, Vol. VII, p. 147).

To the Jesuits of the province of Castile it was not convenient to include mathematics lessons in the art course, because it would be done to the detriment of natural philosophy and metaphysics knowledge, which were much more necessary for the theology studies. They also warned that, as mathematics in Spain had barely use, there would not be an important profit in the future, so they thought the teaching of some cosmography and some of the principles of that discipline would be enough (*Ibidem*, p. 165).

The fathers from the province of Andalusia, who raised many observations to this version of the *Ratio studiorum*, briefly expressed that the mathematics academy could not be established and that they would consider if it was possible teaching that discipline (*Ibidem*, p. 175).

New proposals by Clavius

The German Jesuit, being conscious of the critics received by his proposals to empower the teaching of mathematics and the difficulties to find professors to read it in the Society's colleges, wrote two other documents aimed to make, at least, some Jesuits study it.

In the first one, written in 1593, Clavius insists on the convenience of choosing some students who were specially good at mathematics and willing to teach it (*P. Chistophorus Clavius, S. I. De re mathematica instructio*. Lukács, 1965-1992, Vol. VII, p. 118).

In the second one, written in 1594, he suggests creating four academies –one of eloquence, another one of Greek, another of Hebrew and one of mathematics– where ten specialists would be trained in every discipline. This was a way to guarantee the promotion of the study of mathematics within the Society (*P. Christophorus Clavius, S. I. Oratio de modo promovendi in Societate studia linguarum, politioresque litteras ac mathematicas*. Lukács, 1965-1992, Vol. VII, p. 121).

Mathematics in the definitive *Ratio studiorum*

The definitive *Ratio studiorum*, according to those who were against the appreciation of the teaching of mathematics, reduced even more the study of that discipline: only for students of physics and only for one year.

In fact, in this version, as in the previous one, mathematics is treated in the two mentioned sections. In the first one, "Rules of the Provincial", it is just said that second-year philosophy students would listen mathematics lessons for forty five minutes and, if any of them were suitable or prone to this study, they would receive private lessons after the course (*Regulae provincialis. De mathematicae auditores et tempus*. Lukács, 1965-1992), Vol. V, p. 362).

In the "Rules of the Professor of Mathematics" it is also said that the lessons which physics students would listen for forty five minutes would be about Euclid's *Elements* and, after reflecting on them for two months, they would be taught some geography, cosmography or what they liked to listen (*Regulae professoris mathematicae. Qui authores, quo tempore, quibus explicandi*. Lukács, 1965-1992, Vol. V, p. 402). Finally, it is repeated, slightly changed, what had been already explained in the previous version of the *Ratio studiorum* about the discussions and the public repetitions: every month or every two months, a great meeting would have place with philosophers and theologians to solve any important mathematical problem (*Regulae professoris mathematicae. Problema*. Lukács, 1965-1992, Vol. V, p. 402), and once a month, usually on Saturday, instead the prelesson, the main questions explained in the meeting would be publicly repeated (*Regulae professoris mathematicae. Repetitio*. Lukács, 1965-1992, Vol. V, p. 402).

All what had been referred to Clavius and to his proposals and which appeared in the previous drafts, were no longer present in this one.

It is also important to remark that in the "Rules of the Rector" he is asked to establish the Greek and Hebrew academies; and in the "Rules of the Academy" it is warned that there would be one for theologians and philosophers, another one for rhetoricians and humanists, and one for grammarians. However, nothing is said about the mathematics academy.

Conclusion

We have seen that, just like it happened outside the Society of Jesus, also inside it, during the second half of the 16^{th} century there was a clash between those who supported the educational value of mathematics and those who wanted to keep it in a marginal status.

We have also seen that Jeronimo Nadal, convinced of the value of this discipline, drew up an ambitious mathematical studies syllabus to be taught in the College of Messina. Some years later, he expanded his proposal with the intention of imposing it in the whole Society. According to this Spanish Jesuit, mathematics should be taught for three years to the philosophy students by expert teachers. Soon after, Baltasar Torres, following Nadal's ideas, wrote some new mathematics syllabus for the Roman College, also suggesting that the most promising students were taught supplementary lessons using more up-to-date books than those they often used. However, the lack of mathematics teachers and the hostility of those who believed that the teaching of this subject should not be considered so important caused that these studies were still left as marginal or even nonexistent in most of the Society's colleges.[5]

Finally, we have seen that Clavius, close to Nadal's and Torres' conceptions, insisted on the idea of valuing the teaching of mathematics when the Society was developing its education system, but, even though some of his proposals were included in the first two versions of the *Ratio studiorum*, they disappeared completely in the definitive version. Indeed, in the last version we find no mention of the German Jesuit or even of his proposals, which had been included in the previous ones: nothing is said about the usefulness and need of mathematics, nothing about the

[5] "In quei primi anni di vita della Compagnia le cattedre di matematica erano pochissime; in Italia, fino al 1590, quelle di Roma e Messina; in Francia nessuna fino al 1592; si hanno notizie di un possibile insegnamento di matematica a Praga ("Ex mathematicis etiam lectio adiungetur, siquidem librorum et auditorum commoditas ita feret") nel 1556, e a Colonia dell'insegnamento di elementi di matematica, senza però un apposito professore, dai programmi del 1576/77 e 1578/79. A Coimbra risultarebbe, da un documento datato 1561/62, una lezione quotidiana di matematica di una mezz'ora 'al modo di Roma'". (Cosentino, 1970, p. 197).

advantages that other sciences get from it, nothing about the selection of specialized and authoritative teachers, nothing about the creation of a mathematics academy to train professors or to support talented students, nothing about how much philosophy professors should know about it, nothing about the mathematics exam the student should take for achieving a degree, etc.

It is not justified, therefore, to say that Clavius has decisively influenced with his activities and ideas the definitive version of the *Ratio studiorum* nor, consequently, to attribute him the Jesuits' decision of including the teaching of mathematics in their syllabus. It would be more accurate to award it to Nadal and Torres, because they were the first to promote it and parts of their proposals were finally accepted.

And it is not justified either to say that the Jesuits conferred a special importance to the teaching of mathematics in the *Ratio studiorum*. They accepted teaching lessons of this discipline in the colleges of the Society, because it was established in the *Constitutions*, but they were against any attempt of enhancing them. For this reason they did neither accept any of the syllabi for an extended study of mathematics proposed by Clavius, nor the one by Nadal, that asked for a three-year mathematics course for the philosophy students; not even the one by Torres, which reduced it to just two years. For the Society it was enough that mathematics was taught for some parts of one year to physics or natural philosophy students.

So, the ideas that prevailed in the definitive *Ratio studiorum* were not those of the mathematicians who, according to humanists, expected to turn it into a main discipline. The ones that succeeded were those of the Society's philosophers and theologians who, as good scholastics, were in favour of keeping it as a secondary subject in which it was not even necessary to be examined.[6]

It is not odd, therefore, that the Jesuits' General Superior wrote to the Paris Provincial, the 11th September 1656:

> I have heard, with an immense shame, that mathematics and Hebrew are so neglected by our members that, if it was necessary substituting

[6] "In tutte le tre versioni della *Ratio* non si fa cenno a esami di matematica, che costituiva uno dei punti del programa di Clavio. Questo si spiega col fatto che, facendo delle matemetiche una materia d'esame, non sarebbe più stata una materia complementare, e si sarebbe dovuto istituire la relativa cattedra in tutti i collegi in cui vi fosse insegnamento superiore. Si sa invece che non fu mai così; solo i collegi maggiori istituirono la cattedra di matematica".(Cosentino, 1970, p. 212).

the elderly teachers, we would not find almost anyone to teach these subjects.[7]

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[7] "Ce n'est pas sans une incroyable peine, écrit le Père général au provincial de Paris, le 11 septembre 1656, que j'ai appris que les mathématiques et l'hébreu étaient à ce point negligés par les nôtres, qu'il ne se trouverait presque personne por les enseigner s'il fallait remplacer les professeurs âgés" (Dainville, 1978, p. 332).

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Secondary teachers in the unified Italy: a group portrait with a zoom

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Abstract

After the unification of Italy in 1861 one of the main problems was the creation of a system of instruction to plan curricula and programs, teacher education, textbooks. Mathematicians were very active in this construction and some of them also had important official roles in the government.

In this context important initiatives for the professional development of mathematics teachers were carried out: publication of good mathematics textbooks and books on mathematical culture for teaching, and the foundation of journals and associations. Mathematics teachers occupied a leading position in these actions. In this concern a question arises: "Who were the teachers animating these important initiatives?" In this paper I present some elements that help to answer this question by outlining some general characteristics of the mathematics teacher profession of those times and by illustrating the way the profession was lived by the founder of an important journal for mathematics teachers.

Introduction

Massimo D'Azeglio, a politician very active in the period preparing Italian unification (that happened in 1861), wrote in his book of memories *I miei ricordi* that, after having built the new nation of Italy, it was necessary to build the Italians ("*purtroppo s'è fatta l'Italia, ma non si fanno gli italiani*"). I deem that teachers deserve a particular place among people who played a role in achieving the objective of "building Italians". Their contribution not only resides in their primary task of teaching, but also in the fact that, due to the system of recruitment based on a national *concorso* (competition), they often obtained their post in school far from their home town. This mobility, unusual in Italian society (then as now), promoted contacts among people with different customs, dialects, and backgrounds in the various regions. A real concrete contribution to the process of communication in the country was given by the foundation of professional journals and associations, and by the organization of local and national congresses.

To grasp some aspects of the Italian school practice I follow Schubring's claim that studies focused on teachers help "gaining access to the historical reality, the everyday life of teaching" (2006, p. 675). I consider some general characteristics of the mathematics teaching profession in the period of transition from the unification to 1923, when the reform proposed and carried out by the neo-idealist philosopher Giovanni Gentile, then Minister of Public Education, launched the new

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curricula in force (with some variations) in all the twentieth century. To highlight some shared beliefs and values of those times I focus on the action of an outstanding mathematics teacher.

After the unification one of the main enterprises of the new government was to set a national system of instruction. The overall structure of this system – from primary schools through university – was the Casati Law, promulgated in 1859 to reorganize public instruction in Piedmont and Lombardy, and gradually extended to the whole country. In 1867 the Coppino Act introduced important changes in mathematics teaching, mainly conceived by the outstanding mathematician Luigi Cremona. Another important mathematician, Francesco Brioschi, was a component of *Consiglio superiore della Pubblica Istruzione* (The Higher Council of Public Instruction). These presences evidence the interest of the mathematicians in mathematics teaching as well as the respect of politicians for the mathematicians. Their action may be seen as the continuation of the participation of the scientists in the movement known as Italian *Risorgimento* that fostered the birth of the country, see (Bottazzini, 1981; Giacardi, 2012).

Fig. 1. Italy before unification. The dates in the map indicate the years in which the various states were annexed to the new nation (Weech, 1945, p. 787)

In the second half of nineteenth century Italian mathematical research lived a period of exceptional vitality, see (Cerruti, 1908; Segre, 1891; Tricomi, 1962; Volterra, 1909), and reached an international reputation witnessed by the acceptance of the Italian language in mathematical journals and later in the quadrennial International Congresses of Mathematicians (ICMs). This flourishing was paralleled by a great ferment in the school mathematics milieu and by the emergence of a group of noteworthy mathematics teachers. The level of professionalization reached by them is evidenced by the following initiatives carried out quite early after the establishment of the new system of education:

> In the years 1874 to 1885 the first non-ephemeral journal addressed to secondary mathematics teachers (*Rivista di Matematica Elementare*) was issued. In 1886 a new journal addressed to secondary mathematics teachers, *Periodico di Matematica* (still existing with the name *Periodico di Matematiche*) was founded.
>
> In 1895 the national association of mathematics teachers *Mathesis* (still existing, after several changes in names and goals) was founded.
>
> In 1867/68 the production of new mathematical textbooks began.

The ferment in the school mathematics milieu was not only the continuation of the spirit that fostered the construction of Italy and the reflection of the ferment in mathematical research, but it also mirrored the Zeitgeist in society. The second half of nineteenth century saw the emergence of social phenomena such as the interest in associating and solidarity, which inspired trade unions, political parties addressed to meeting the needs of less affluent people, organizations for mutual aid. The idea of internationalization and communication pervading many aspects of society endowed Italian mathematics first, and later mathematics education, with an international dimension that involved also some secondary teachers.

Sources for studying mathematics teachers of the past

There is an intrinsic difficulty in studying the teachers of the past, since usually there are no personal archives dedicated to them. The documents concerning them should be stored in the archives of the Italian schools where they taught, but usually these schools are reluctant to open their archives, or, worst, the documents have disappeared.

In this paper the sources for identifying interesting chief characters in the story of the mathematics teacher profession are of a different nature. Some secondary teachers wrote remarkable textbooks. Other teachers contributed to journals dedicated to mathematics teaching as editors or authors of articles, reviews and letters. In these journals teachers who get awards, distinctions and other official acknowledgments are mentioned. The lists of the members of the teacher association *Mathesis*, being mainly composed of secondary mathematics teachers, records those people who showed a particular interest in their profession. The reports of the regional and national meetings of *Mathesis* mark out the teachers active in the debates on curricula and on teaching methods, see (Furinghetti, 2002). A further partial source is the collection of short biographical notes on 371 Italian mathematicians who died in the period January 1, 1861 - December 31, 1960 (the first century of unified Italy) written by the mathematician Francesco Tricomi, see (Tricomi, 1962). The author qualified as "mathematicians" people who were authors of publications related to mathematical research or made some attempts to research promising future developments, though the authors' early death or their personal stories stopped the continuation. These criteria for the compilation of the biographical notes were generous and included mathematicians of different ranks, which range from internationally known scientists to mathematics teachers wishing to cultivate their mathematical culture. It is interesting for the study presented in this paper that about 100 of the "mathematicians" mentioned in Tricomi's collection had been secondary teachers for about 10 years or for their entire career. This remarkable presence of teachers deserving Tricomi's qualification of mathematician is due to different factors. Firstly, for some of them teaching in school was the easiest opportunity of earning a salary while doing research for entering an academic career. This has been the case, for instance, of Gaetano Scorza who, before becoming an important mathematician, taught in several schools. He wrote two of the 11 Italian reports for the survey carried out by the *International Commission on the Teaching of Mathematics*, the predecessor of the present-day ICMI, see (Scorza, 1911; 1912). Later on he was appointed as a vice-president of this Commission. During his brilliant academic career he maintained a live interest in mathematics education, see Livia Giacardi's cameo in (Furinghetti & Giacardi, 2008). Other mathematicians of Tricomi's list remained secondary teachers all life long, but thanks to their research were appointed as lecturers in University. Usually this happened because they earned *libera docenza*, a special qualification which no longer exists, delivered on the basis of an oral examination and scientific publications. *Libera docenza* was a kind of acknowledgement that the holder could teach in university. Other mathematicians of Tricomi's collection were secondary teachers without any appointment in University. Some of them, as some of their colleagues also lecturers in university, were asked by university mathematicians to collaborate in collective works aimed at improving mathematical content knowledge of their colleagues, contributed to didactic journals and wrote textbooks.

Of course, there are interesting teachers who not appear in Tricomi's list. Some wrote didactic papers and showed a high level of professionalism. Contrary to what happens today, at those times outstanding teachers (of mathematics and of other disciplines) had official acknowledgements from the Ministry of Education or the King: honors, promotions for distinguished merit, upgrade of the salary, inscription in the secondary teachers' roll of honour. Prizes were endowed for the best works concerning didactics. Esteemed teachers were appointed as Inspectors in schools or principals of schools.

Aspects of mathematics teacher profession in the years 1861-1923

It is not an easy task to grasp the general characteristics of the profession of mathematics teachers in the past. Luckily, as told in the previous section, beside the ordinary teachers for whom we have lost the records, there have been teachers whose presence is recorded in some way, because they were members of associations, subscribers to journals, active participants in the debates and conferences, authors of papers and books, or researchers in mathematics. In the following the profession of mathematics teacher in the past will be analyzed according to some characterizing dimensions.

Political and social dimension

The period of transition considered is a long interval of years (1861-1923) in which the Italian society slowly evolved. In the case of mathematics teachers, indeed, we may distinguish two stages: a stage of construction, and a stage of settlement. In the first period the mission and vision of teachers was still imbued by the ideals of patriotism that inspired the fights for the independence of the *Risorgimento*. A representative of this period is Aureliano Faifofer (1843-1909), a teacher famous for his highly appreciated textbooks on algebra and geometry that were also translated into foreign languages. In the foreword of a posthumous edition of one of his textbooks, (Faifofer, 1926), the editor emphasizes that he was a man of widest culture and high *patriotism* and that the patriotism was expressed by him through his authorship of good textbooks aimed at substituting the foreign textbooks used before the unification.

The mathematics teachers' involvement in socio-political issues continued after the unification with different objectives of those of the *Risorgimento*, in line with the changed spirit of the times. Just to quote some cases: among the 26 women (including Maria Montessori) who signed the petition for female suffrage published in the journal *Vita* on March 11 1906 there was the mathematics teacher Lia Predella Longhi.

Later on, when the newspaper *Il Mondo* (May 1, 1925) published the *Manifesto degli intellettuali antifascisti* (Manifesto of the Italian antifascist intellectuals) the mathematics teacher Alessandro Padoa was among those who signed it

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Giusta quanto si era precedentemente deliberato, ieri si riaprì, sotto la presidenza del prof. Padoa, la seduta, iniziando la discussione sull'ordine del giorno Boselli concordato con modificazioni e aggiunte del prof. Padoa.

Il prof. Gnocchi di Oneglia rileva nella relazione Gasti due punti essenziali, il danno igienico dell'orario troppo gravoso e la necessità di aprire nuove vie alla attività della donna. Spiega come tale scuola è venuta aggravandosi di programmi e di orari, e insiste sui danni igienici

Fig. 2. A newspaper reporting on a regional congress of secondary teachers (June 1, 1909)

The political dimension concerned not only the nation's affairs, but also the professional status of teachers. The review Masoni (1897) of a book written by a mathematics teacher mentions the many faults of the training and conditions of work of the Italian teachers. The national association of mathematics teachers *Mathesis* was founded in 1895 (Avviso, 1896) with the aim of improving school and teachers' preparation both from the scientific and didactic point of view. In 1901 the professional association of teachers FNISM (*Federazione Nazionale Insegnanti Scuola Media*) was founded with the aim of protecting economic and legal rights of teachers of different school levels and disciplines. It is remarkable that, more than in the present days, the newspapers used to pay attention to school events, as shown in the article where a regional congress of secondary teachers was reported and commented on in a Genoese newspaper, see Fig. 2.

Some events in the mathematics teacher milieu mirrored the changes in society. An example is the increasing presence of women as workers in various environments. As for school, at the beginning women taught in primary school; afterwards they taught also in secondary schools, even with some limitations for their career. When the list of the 113 charter members was published (Elenco dei soci fondatori dell'Associazione Mathesis, 1896) no women were present, but in the same year a female

(Abigaille Chelotti, R. *Scuola normale femminile* of Leghorn) was one of the eight new members (Nuovi soci, 1896). In the list of the members of Mathesis in 1914 the female members are 37 of 467 members[1], see (Elenco dei soci, 1914). In the analogous list of 1922 the females are 233 of 913 members, see (Elenco dei soci, 1922). Furinghetti (2002) reports that women were active members of the local branches of *Mathesis*, where they put forwards the problem of the laws discriminating against women.

National and international dimension

Communication is a further divide between the first and the second stage of the period of transition. In the obituary of Faifofer, Ciamberlini (1909), an outstanding teacher author of textbooks and important didactic papers, wrote that he and Faifofer corresponded for a long time and discussed key questions on geometry and on methods of teaching, but they never met in person. A decisive step in establishing an actual national communication was the creation of mathematics journals for teachers and for secondary students, see (Furinghetti & Somaglia, 1992). The meetings organized by *Mathesis* accelerated the exchanges of opinions and contacts.

Although the social status of secondary teacher made difficult contacts with foreign people, the community of Italian teachers attained an international dimension evidenced by many factors. The Italian didactic journals on mathematics used to mention foreign works, authors, and events. Foreign contributors were very rare in these journals because the readership was mainly national and the focus was on issues linked to the national situation of mathematics education. On the other hand, even *L'Enseignement Mathématique*, a journal with a marked international scope, has a relatively small number of foreign contributors, among whom there were Italian teachers (Cristoforo Alasia, Rodolfo Bettazzi, Giacomo Candido, Margherita Peyroleri).

Some teachers translated foreign works into Italian. For example, Faifofer[2] translated the third edition of the book by Lejeune Peter Gustav Dirichlet with the appendixes by Richard Dedekind, see (Dirichlet, 1881), and Francesco Edoardo Luigi Giudice[3] translated into Italian an important work for teacher training by Felix Klein, see (Klein, 1896).

It is remarkable that a few Italian mathematics teachers attended some ICMs since the first in Zurich. The first woman who graduated in

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[1] These numbers are approximated since a few members are mentioned only with the initials of their first names.

[2] See (Furinghetti & Giacardi, 2012) for biographical notes on Faifofer and Giudice.

[3] The translation was promoted by Gino Loria, as shown in the letter by Loria to Felix Klein (Genoa 22.7.1895, *Niedersächsische Staats-und Universitätsbibliothek*, F. Klein 10)

mathematics in unified Italy, Iginia Massarini, attended two ICMs (1897 and 1908) and the International Congress of Psychology (Rome, 1905). The journal *Il Bollettino di Matematica* reports that at least 43 Italian secondary teachers attended the ICM in Rome, see (La Direzione, 1908). Sometimes mathematics teachers presented communications in the section of the congresses devoted to mathematics education and also in the mathematical sections. This is the case of Padoa who had an important role in the congress of Paris (1900) and Cambridge (1912) because his studies on logic, see (Ferrera, Furinghetti, & Ortica, 2010; Borga, Fenaroli, & Garibaldi, 2008).

The creation of the *International Commission on the Teaching of Mathematics* in 1908 contributed an international dimension to the Italian community of mathematics teachers. The bulletin of *Mathesis*, which usually consisted in few pages, often inserted in *Periodico di Matematica*, was issued from 1909 to 1920 as a separate journal with the title *Bollettino della "Mathesis" Società italiana di matematica*, see (Furinghetti, 2002). This journal, especially under the editorship of Guido Castelnuovo who was member-at-large of ICMI from 1913 to 1920 and vice-president from 1928 to 1932, informed the members of the Association about the international events promoted by the Commission and published the 11 Italian reports prepared for the ICMI survey on the teaching of mathematics in various countries. Six authors of these reports were secondary teachers when they wrote their contributions. Before, the *Bollettino di Matematica* (see below) accurately reported on the article 'Reformes à accomplir' appeared in *L'Enseignement Mathématique* in 1905 which launched the idea of the future ICMI, see (Intorno alle riforme …, 1906). In the same issue results on the inquiry about the way of working of mathematicians were reported.

The Italian mathematics teaching milieu was looking abroad and also an interest was being shown from abroad. One of the most relevant aspects of the international dimension of mathematics teaching in Italy is the fame reached by some textbooks, especially in geometry. The production of new Italian textbooks started in 1867/1868 with the edition of Euclid's *Elements* – written by the outstanding mathematicians Enrico Betti and Francesco Brioschi. Soon the production of textbooks written by mathematicians or schoolteachers began to flourish. Their value is illustrated by Alpinolo Natucci, a teacher who used them, see (Natucci, 1967). Some of these textbooks were translated into foreign languages. Smith (1900, p. 304) claims that "Italy has produced some excellent works on elementary geometry; indeed, in some features it has been the leader". This opinion was shared by Klein as well, in his essay on the teaching in Italy, *Der Unterricht in Italien* (Klein, 1925-1933, II). Thomas Heath wrote: "The Italians, whose great services to elementary geometry are more than once emphasised in this work [his book on Euclid] …" (1956, p. 113). He quotes the contribution to the elaboration, interpretation, and development of Euclidean geometry given in some textbooks, notably those by Federigo Enriques and Ugo Amaldi, Giuseppe Ingrami, and Giuseppe Veronese.

GLI

ELEMENTI D'EUCLIDE

CON NOTE,
AGGIUNTE ED ESERCIZI

AD USO DE' GINNASI E DE' LICEI

ENRICO BETTI E FRANCESCO BRIOSCHI.

FIRENZE.
SUCCESSORI LE MONNIER.
1867.

Fig. 3. The front cover of Euclid's *Elements* edited by Betti and Brioschi

Foreign journals published notes and reviews on Italian didactic works. Halsted (1902) commented on the birth of the new journal *Le Matematiche Pure e Applicate* edited by Alasia with the following sentence "when a new star comes out in the skies, thither turns the observing eye." (p. 383). Books written by the teacher Gaetano Fazzari, a pioneer in considering history of mathematics as a didactical tool for teaching, were reviewed by Smith (1907)[4] and Karpinski (1918).

Smith and Goldziher (1912) prepared a booklet for the ICM 1912 in Cambridge (UK) which lists 1849 works dedicated to mathematics teaching from different countries. The criteria for the choice are not clearly stated by the authors, then the information provided by this work is incomplete, anyway we note that there are teachers among the 49 Italian authors cited in the booklet.

Managerial dimension of mathematics teachers

Usually in the period under consideration the journals dedicated to mathematics teaching belonged to their editors, then their foundation involved economic efforts and managerial capacities. Beside some academic professors also a few courageous teachers undertook this

[4] Natucci (1939) reports that the book reviewed by Smith was translated into Russian by S. Galascin of Rostow in 1923.

initiative. Though not all succeeded in keeping alive the journal they had founded, the list reported below is interesting for grasping the enthusiasm and the wish of action of these teachers.

- Giovanni Massa (Alba, May 9, 1850 - Milan, April 8, 1918) founded in 1874 *Rivista di Matematica Elementare* in Alba. The journal survived until 1885.
- Alberto Cavezzali (Reggio Emilia, February 20, 1848 - Bergamo, October 29, 1922) founded in Novara *Il Piccolo Pitagora* issued in 1883 and 1884.
- Gaetano Fazzari (Tropea, October 7, 1856 - Messina, July 13, 1935) founded *Il Pitagora* (addressed to students) in Avellino, issued from 1894 to 1919.
- Vincenzo Giriodi (?) founded in 1897 *La Palestra Scientifica* in Turin. The journal had an ephemeral life.
- Pietro Caminati (Genoa, April 7, 1837 - ?) founded in Foggia *Il Tartaglia. Periodico di Scienze Fisico-Matematiche Elementari per gli Alunni delle Scuole secondarie pubblicato per cura del Prof. Ing. Pietro Caminati* (addressed to students), issued in 1898 and 1899.
- Alberto Conti (see below) founded in Bologna *Il Bollettino di Matematiche e di Scienze Fisiche e Naturali* and *Il Bollettino di Matematica*.
- Cristoforo Alasia-De Quesada (Sassari, November 21, 1869 – Albenga, November, 19, 1918) founded in Città di Castello *Le Matematiche Pure e Applicate. Periodico Mensile di Matematiche Pure ed Applicate, Superiori ed Elementari, ad Uso dell'Istruzione Media e Superiore*, which was issued in 1901 and 1902. Among the collaborators there were teachers and mathematicians from foreign countries.

Cultural dimension of teachers' mathematical background

In the period considered in this paper many changes have been carried out in teacher recruitment and training, see (Furinghetti & Giacardi, 2012). A certain homogeneity in teachers' background was created in 1906 by legislation regarding the legal status of teachers: only those who had won a *concorso* could teach, and the admission to the competition required a university degree, see (Furinghetti & Giacardi, 2012). In 1914 mathematics teachers were given the same legal status of teachers of Italian, rectifying the inequality in favour of the latter that had existed since the Casati Law.

Thanks to the degree in mathematics or in other scientific and technological domains it may be assumed that mathematics teachers had substantial subject matter knowledge. As evidenced by Tricomi's list some of them kept contacts with academic milieu, were informed about the development of research and did also original research in mathematics. As told before, some mathematics teachers were also lecturers in university, wrote textbooks, contributed to journals of mathematical research and to didactic journals, and gained *Libera docenza*.

The role of university mathematicians in mathematics teacher education had been relevant. Firstly they contributed to the organization of Italian mathematics education through their appointment in important political positions in the government and their high level of teaching and researching in universities. Moreover they authored important textbooks, were inspector of the Ministry of Education in schools, were examiners in the national competitions for assigning chairs in schools, trainers for prospective teachers. Often they kept contacts with their university students after their degree and advised the career of teaching to those they considered having a marked attitude for this profession.

The good textbooks mentioned before contributed to strengthen the subject matter content knowledge of teachers. The collective work edited by Enriques (1900 and 1912-1914) gathered a wide range of topics belonging to the domain of elementary mathematics and became a cultural reference for generation of mathematics teachers. It was translated into German on the initiative of Klein. A project for a similar work, the *Enciclopedia delle matematiche elementar*i, was launched by Berzolari and Bonola (1909), but was completed only in the 1940s. In the journals for mathematics teachers issues central to teaching were treated. This happened for example for the theory of equivalence, see (Faifofer, 1886).

About the other side of the coin, that is pedagogical matter knowledge, the teacher professionalization has been more complex and the role of mathematicians was controversial. The full story is outlined in (Furinghetti & Giacardi, 2012). In 1875 the Minister of Public Education Ruggero Bonghi established the *Scuole di Magistero* (university level institutes for teacher training) aimed at training future secondary school teachers of the various disciplines and delivering a diploma which constituted the qualification for teaching. Not all Italian universities organized the *Scuole di Magistero* and, when organized, their impact on teacher education was very limited for many reasons. Firstly, the professors who taught in the *Scuole* were the same ones who taught university courses for the disciplinary degree and then they were often unprepared to address questions about pedagogy and method. Moreover the active contact of the prospective teachers with the classroom, which was an important point of the *Scuole*'s mission, was neglected. Supporting structures (libraries, laboratories, etc.) and teaching materials were practically non-existent, the number of assigned course hours was inadequate, and there was scant funding. The

Scuole di Magistero underwent successive modifications until 1920, when they were abolished by the minister Benedetto Croce[5].

Together with the national mathematics programmes, teacher education and the *Scuole di Magistero* have been the main objects of discussions among teachers and mathematicians, especially after the creation of *Mathesis*, see (Nastasi, 2002). In the debate two main positions may be identified. One position advocated a special university curriculum for prospective mathematics teachers with the first two years common with those wishing to become mathematicians and two years special for the teaching career. The other position proposed to keep the same curriculum for all mathematics students, but advocated making more efficient the *Scuole di Magistero*. In his paper complaining about the closing of the *Scuole di Magistero* Gino Loria (1921) advocated that prospective teachers have some contacts with the classroom in the period of training and that they become acquainted of specific problems concerning the teaching and learning of mathematics.

One main point of friction among mathematicians and secondary teachers was the role attributed to issues concerning the study of didactical problems in the school milieu and, in relation with this, in the *Scuole di Magistero* the balance between subject matter content knowledge and pedagogical content knowledge. In the review of the first edition of Max Simon's book Loria (1896) stressed the scant inclination of Italians towards studies in the field of didactics. In reviewing the second edition of Simon's book Loria (1910) noted that things had been radically changed and that Italy had joined the international movement aimed at improving mathematics teaching. Books addressed to a pedagogical view such as *L'initiation mathématique* by Charles-Ange Laisant, translated by Giulio Lazzeri and reviewed by Conti (1906), were known and appreciated. In a similar line books written by Italian mathematics teachers were published, see (Leoni, 1915) and (Ciamberlini, 1920). Leoni's work, awarded with a prize from the *Accademia dei Lincei*, raised interest in the school milieu, as shown by the reviews (Conti, 1914-1915; Loria, 1915: Scarpis, 1915). Nevertheless it is clear that pedagogical reflection was not always considered in the way some teachers wished. Scarpis (1915) polemically advocated that the national commissions examining the candidates for chairs in schools be so courageous to evaluate in the same way the works aimed at improving teaching and those of mathematical research which, he says, have a mere aesthetic value and satisfy only egoistic needs of the authors. These words by the esteemed teacher Umberto Scarpis once

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[5] For information about legislative measures concerning the Scuole di Magistero see http://www.subalpinamathesis.unito.it/storiains/uk/training.php

again stress the existence of a dichotomy in the conception of teachers' knowledge for teaching of those times.

Alberto Conti

In the big portrait group of mathematics teachers on the period 1861-1920 there are personages who best represent some of the special aspects of this community. This is the case of Alberto Conti, the teacher considered in this section: he combines good mathematical knowledge, managerial abilities, commitment in communication, awareness of the mathematics teaching problems, social sense in promoting professionalization, and fruitful interaction with university mathematicians.

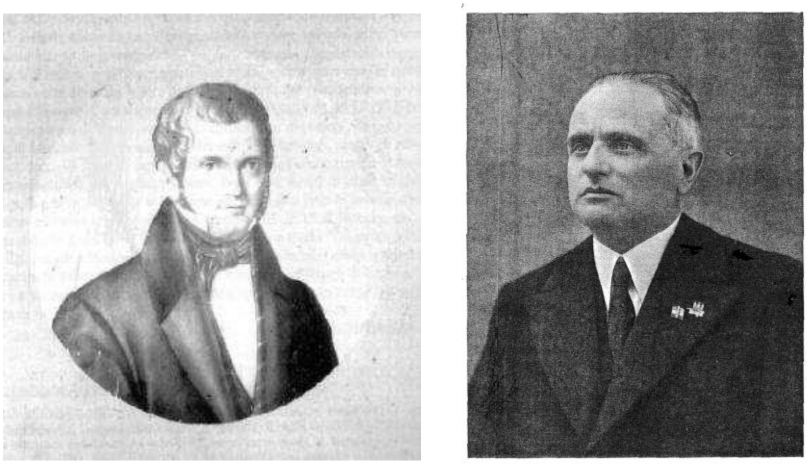

Fig. 4. Alberto Conti in two periods of his life

The two photos in Fig. 4 epitomize the period of transition between the two epochs which characterize the years considered in this paper: the left photo shows a young man dressed in nineteenth century style, the right photo shows a mature man dressed in contemporary clothing.

Conti was born in Florence (December 3, 1873) and died there (October 18, 1940). He studied in the prestigious *Scuola Normale Superiore*[6] in Pisa and graduated in mathematics in 1895 with a dissertation on the

[6] See (Furinghetti & Giacardi, 2012) for information on this institution. The name *Scuola Normale* indicates a school for training primary teachers, but the Pisa *Scuola Normale Superiore* was born with the aim of training secondary school teachers and developed as a school training researchers in mathematics.

theory of connections.[7] Among his professors there were prominent Italian mathematicians of the period (Eugenio Bertini, Luigi Bianchi, Ulisse Dini). He taught for a few months in *Scuola Normale Superiore*, afterwards he began the career of a secondary mathematics teacher that he followed all his life. As a consequence of the Gentile Reform that associated the teaching of mathematics and physic in secondary school, from 1926-27 he taught these two disciplines. Conti is an example of how teachers have been a unifying element in the newborn country, since, before going back to his home town (Florence) in 1920, he taught in a village close to Macerata, in Belluno, Bologna, Rome. Other events of his life, notably the foundation of didactic journals, contributed to establish contacts in the country.

The short biography (Conti, 1936) mentions honors, Ministry awards for his textbooks, promotions and upgrade of the salary for special merits. His publications appeared in journals for mathematics teaching, in proceedings of the meetings of *Mathesis* and FNISM, in school journals and bulletins. In his autobiography Conti (1936) lists 18 works and mentions many others: teaching materials (textbooks, …), didactic articles and surveys of the state of art of mathematics teaching, inquiries on textbooks, papers and interventions about school and university policy (recruitment, training, …) in congresses of *Mathesis* and FNISM. Enriques asked him to collaborate at the collective work *Questioni* … mentioned before. He delivered plenary talks in national congresses. He was interested in the political aspects of his profession and published also articles in Italian newspapers. He was vice-president of FNISM in 1902-1904 and president of the Roman branch of it in 1910.

While he remained a schoolteacher all his life, Conti was in contact with mathematicians and also wrote papers in collaboration with them, see (Enriques, Severi, & Conti, 1904). The letter published by Salmeri (2012)[8] shows that he corresponded with Vito Volterra asking him for a contribution to *Il Bollettino di Matematica* to commemorate Dini. He participated in the ICM of Rome (1908) and of Bologna (1928). In Rome he gave a contribution to the fourth section *Questioni filosofiche, storiche, didattiche* (Philosophical, historical and didactical questions), see (Conti, 1909). In the closing plenary section of Saturday, April 11 he presented the proposal for establishing an international association of

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[7] For further biographical notes see (Conti, 1936; Direzione, 1940; Rossi, 1983; Salmeri, 2012).

[8] The letter (2 pages) was written in Ozzano dell'Emilia, July 21, 1907. Is kept at *Accademia Nazionale dei Lincei, Fondo Volterra* (box 3, Series 1, Fasc. 324). The transcription is by Pietro Nastasi.

mathematicians[9] (Castelnuovo, 1909, p. 33). The proposal was approved, though we know that only in 1920 was the International Mathematical Union founded and after its dissolution in 1932 was reborn formally in 1951. In 1909 Enriques appointed Conti as a member of the Italian Delegation of the *International Commission on the Teaching of Mathematics*. For this Commission he authored the reports on mathematics teaching in primary teachers training schools (*Scuole Normali*) and in infant and primary schools, see (Conti, 1911a; 1911b).

Conti's main enterprise has been the founding and the editorship of two journals. The first, *Il Bollettino di Matematiche e di Scienze Fisiche e Naturali. Giornale per la Coltura dei Maestri delle Scuole Elementari e degli Alunni delle Scuole Normali*, was issued from the school year 1899-1900 until 1917. When the journal was founded Conti was a young teacher in a *Scuola Normale*, then he was aware of the problems of training primary teachers. Moreover, the new association *Mathesis* in its first national congress of 1898 had proposed the creation of a journal of mathematics for primary teachers, see (Conti, 1900), without realizing the project. The journal dealt with themes of mathematics and sciences in a way suitable to prospective and in-service primary teachers. It is likely that it is the first of this type in Italy. The second journal, *Il Bollettino di Matematica. Giornale Scientifico-Didattico per l'Incremento degli Studi Matematici nelle Scuole Medie*, was first issued in 1902 and it is still published, with changes in the editorial line, under the new title of *Archimede* taken in 1949. The two World Wars stopped the publication for some periods. In presenting the journal Conti (1902) claims that his intention was not to have a new intermediate (*speciale* was the Italian word used for this type of journals) journal, but a journal with a clear pedagogical orientation, whose ideas could easily be used in the classroom. The contributions on elementary mathematics related to school programs were the core of *Il Bollettino*, but an important aim was to inform about events that concerned school, programmes, didactic publications, teacher status, and the community of mathematicians. The readers' letters and reviews were published. Most contributors were secondary teachers, and a few university professors. From 1922, after the publication of Loria's *Bollettino*, had ended, the journal hosted a section edited by Loria entitled *Sezione Storico-Bibliografica*.

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[9] Il Prof. CONTI presenta la seguente proposta:
"Il Congresso fa voto che all'Ordine del giorno del prossimo Congresso sia posta la costituzione di un'Associazione internazionale dei Matematici".
La proposta è approvata.

Conclusions

Fig. 5. Watercolor by Aurelio Craffonara dedicated to the Physics teacher Sereno Rumi

The previous outline of some impressive elements that characterized the profession of mathematics teachers in the period 1861-1923 stresses the intertwining of scientific, ethical and civil values put into effect by mathematicians and mathematics teachers in the construction of the school system in unified Italy. Only some outstanding mathematics teachers left traces accessible to contemporary historians, while the action of their anonymous colleagues remains buried under the dust of time. Nevertheless the Zeitgeist concerning mathematics school world should emerge from the portrait group presented in this paper.

Some characteristics of teachers' condition concerned teachers of all disciplines: consideration by the institutions, respect by the students, social values, and often, very good disciplinary background. All that may be epitomized by the dedication in the watercolor on parchment by the painter Aurelio Craffonara (1875-1945) found in the archives of the Technical Institute *Vittorio Emanuele II* of Genoa, see Fig. 5. This parchment was presented in May 1897 to the teacher of Physics Sereno Antonio Rumi by his students of the third course of this Institute (then qualified as *Regio*, that is Royal). The occasion was the cross of honour given to Rumi by the King, see Rumi`s portrait in Fig. 6. The dedication in the parchment says that this honour, as an acknowledgment of the merit and learning of the teacher, is a good sign for the future of the school and the country. Are similar recognitions still present in school today?

Fig. 6. A class of the Regio Istituto Tecnico "Vittorio Emanuele II" in the school year 1908-09. In the 11th oval there is the portrait of Giudice, one of the founders of Mathesis.

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The emergence of the idea of the mathematics laboratory at the turn of the twentieth century

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Abstract

The idea of offering students spaces where they could carry out activities spontaneously and constructively, develop their own individuality, and socialise, frequently appears in the studies of psychologists and educators at the turn of the twentieth century. Examples of this are found in the works of the American John Dewey, whose vision on education is related to the pragmatism of Charles S. Peirce and William James; the German Georg Kerschensteiner, an advocate of the Arbeitsschule, or 'work-school'; the Belgian Ovide Decroly; the Swiss Edouard Claparède and Adolphe Ferrière; the French Alfred Binet, one of the principal promoters of the 'active school'; and the Italian physician and educator Maria Montessori, among others.

All these scholars were especially interested in the formation of children during the first years of their lives, and mathematics is not always mentioned in their reflections, but the idea of a school-laboratory spread also among mathematicians, who extended it to secondary schools.

In my paper, after briefly mentioning the points of view of some of the educators who were active at the turn of the twentieth century and either had an interest in mathematics or were in contact with scientific circles (Dewey, Kerschensteiner, Wells), I will discuss the contributions of the mathematicians John Perry, Eliakim Hastings Moore, Émile Borel and, Felix Klein, and then focus on Giovanni Vailati's 'school as laboratory'. By comparing the various models of mathematics laboratories proposed, I will try to make clear the most significant differences between them, and their innovative aspects.

The school-laboratory according to well-known educators interested in mathematics

John Dewey (1859-1952) can rightly be considered the father of the active school and a source of inspiration for a large number of educators of the first half of the 1900s. Believing that the school of his day was anachronistic, passive, anti-psychological and antisocial, he proposed instead an active school that was centred, not on teachers or on books, but on the activity of the students, organised in a social kind of work. Knowledge, therefore, was not to be provided ready-made, but rather presented in the form of problems, and was to spring from the personal research of the student. Because the traditional classroom was inadequate for this kind of teaching, he believed that it was necessary to transfer the educational process into laboratories, libraries, playgrounds, workshops and kitchens, where work itself would transform school into an in-embryo community. In 1896 Dewey founded an 'experimental school' in Chicago based on these educational ideals and attempted to interact with

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mathematicians as well, in particular with Eliakim Hastings Moore and George B. Halsted. In his article 'The psychological and the logical in teaching Geometry' (1903), he says that in the practise of teaching psychological aspects must be taken into account as well, and that it is thus necessary to begin with concrete reality and ordinary experience, and present the practical applications of mathematics in such a way as to arrive gradually at logical rigour.

Among the European educators influenced by Dewey was Georg Kerschensteiner (1854-1932). A teacher of mathematics and physics in gymnasia for many years, and later a school inspector in Munich, he was a social educator and promoter of the *Arbeitsschule*, or 'work-school' (Simons, 1966). He believed that in order to reform schools it was not so much necessary to broaden programs or increase the number of hours as it was to transform schools into laboratories for practical exercises, where the student could learn to use knowledge and acquire a sense of social duty. The importance he attributed to manual work and practical activity goes beyond the acquisition of abilities and skills; rather, it is connected to the capacity for carrying out an activity responsibly and autonomously: manual labour disjoined from intellectual effort is merely mechanical, and thus from the point of view of education its essential characteristics are autonomous planning and realisation combined with the possibility for self-analysis. According to Kerschensteiner, the main aim of education should be civic education (*staatsbürgerliche Erziehung*). Having completed his mathematical studies at university, he was particularly aware of the problems related to teaching of the sciences, and in 1914 dedicated the short book *Wesen und Wert des naturwissenschaftlichen Unterrichts* to these problems. Here he maintained that the study of sciences was valuable for its ability both to train the mind to follow a logical and precise process of thought, and to increase the students' power of observation, where observation meant the combination of perception with thought. For this reason he campaigned vigorously for the introduction of laboratories and practical works in science teaching (see also Wolff, 1937, p. 97 and Simons, 1966, pp. 79-81).

Herbert George Wells (1866-1946), although mainly known as an author of science fiction, had scientific training in zoology and biology and also wrote many articles of a pedagogical and social nature. In his book *Mankind in the Making* (1904), Wells criticises English schools, in particular the programs, which were redundant or lacking in what truly made it possible for students to understand the society they live in, and the textbooks, inadequate for an active teaching (p. 226). According to Wells, schools should ideally be connected to public libraries (p. 213) and the actual lessons should be alternated with sessions dedicated to individual activities such as reading, painting, and play, intended as a 'spontaneous activity that involves the imagination' (p. 235). In his book he also cites laboratory-style teaching of mathematics proposed at the time by the mathematician John Perry, but he is frank about the difficulties of putting it into practise: because this kind of teaching requires such great commitment and planning on the part of the teacher, as long as there were no adequate textbooks, it remained practically impossible (pp. 224-225). Wells's book was reviewed by the Italian mathematician Giovanni Vailati (1906) and probably contributed to directing him towards his conception of a laboratory-type teaching of mathematics.

The laboratory for mathematics teaching in the international context

The idea of a laboratory for mathematics was introduced by John Perry (1850-1920), a professor of mechanics and mathematics at the Royal College of Science in London from 1896. In fact he maintained that mathematics had to be taught 'as any other physical science is taught, … with experiment and common-sense reasoning' and proposed a new teaching method that he called 'Practical Mathematics': 'The most essential idea in the method of study called Practical Mathematics is that the student should become familiar with things before he is asked to reason about them' (Perry, 1913, p. 21). Before concentrating on theorems and proofs, the student should become acquainted with the concepts by means of experiments and measurements using squared paper, data gathering, drawing, graphic methods, and relationships with physics and other sciences.

In working out his method, Perry was inspired by the methods used in the kindergartens, which, under the influence of Pestalozzi and Froebel, were based on activity and on 'hand and eye training' (Price, 1986, p. 109-114) and he was stimulated by the discernment of the failure of traditional teaching with respect to the average student:

> Academic methods of teaching Mathematics succeed with about five per cent of all students, the small minority who are fond of abstract reasoning: they fail altogether with the average student (Perry 1913, p. VII).

> So we now teach all boys what is called mathematical philosophy, that we may catch in our net the one demigod, the one pure mathematician, and we do our best to ruin all the others (Perry 1902, p. 6).

According to Perry, the cause of this failure was the English system of separate examinations that induced teachers to teach the various subjects in 'water-tight compartments', as well as the tendency to place too much importance on the abstract aspects of mathematics and on the 'laboursaving rules', neglecting the fundamental principles and concepts (Perry, 1913, p. X). He believed that the aim of teaching was not that of 'producing finished products either at school or the university' but rather that teachers 'ought to try to produce learners' (Perry, 1909, p. 11). Perry himself first began to use this new type of approach in an English public school, later in Japan (1875-1879) and then he developed a syllabus of practical mathematics for engineers at the Finsbury Technical College, where he was appointed professor of Mechanical Engineering in 1882. In 1899 he was able to convince the Board of Education to adopt it for science classes and in 1901 at Glasgow he communicated the results at a meeting of the British Association for the Advancement of Science, giving rise to a lively debate (Brock & Price, 1980; Price 1986). He gathered the various talks in a small volume published the following year, *Discussion on the teaching of mathematics*, in which he first outlined what he considered to be the purposes and usefulness of mathematics:

> In producing the higher emotions and giving mental pleasure. … In producing logical ways of thinking. … In the aid given by mathematical weapons in the study of physical science. … In passing examinations. The only form that has not been neglected. The only form really recognised by teachers. In giving men mental tools as easy to use as their legs or arms … In making men in any profession of applied science feel that they know the principles on which it is founded and according to which it is being developed … (Perry, 1902, pp. 4-5).

Then he reiterated his criticisms of the English methods of teaching, and illustrated the programs of his courses both in elementary and advanced mathematics (pp. 25-32).[1]

In arithmetic emphasis was placed on decimals rather than fractions and on approximations; in algebra, on the comprehension and manipulation of the formulas as well as on the variations in the value of certain expressions with the varying of the values of the variables that appear in them; in geometry the Euclidean method was completely abandoned, replaced by a treatment based on measurement and drawing with the freedom to use arithmetic and algebraic methods. In the advanced course elements of trigonometry, three-dimensional geometry, calculus as well as vector methods were introduced.

Among the numerous comments, I limit myself to citing the one by David E. Smith who, while basically in agreement with Perry's point of view, indicated the problems that must be faced in order to put the

[1] See also (Howson 1982, pp. 148-149). The appendix on pages 222-224 gives Perry's 1900 proposals for a mathematics syllabus.

proposed reform into practice: new textbooks, teacher training, and the modifications of examinations (Perry, 1902, p. 90-91).[2]

In 1913 Perry published his best known book, *Elementary Practical Mathematics*, which was intended as a guide for teachers, with many carefully chosen exercises to pose to students. Perry begins with topics in arithmetic before going on to topics and problems in algebra, geometry, physics and calculus. In fact, according to him, the method of Practical Mathematics can be used at all levels of teaching as long as the presentation of subjects remains tied to real phenomena and concrete problems. The treatment of the subjects reflects a laboratory-like approach: the starting point is generally a practical problem; numerical data is gathered and interpreted; squared paper is used to tabulate the observations, solve equations graphically, represent functions, find the slope of the tangent to a curve; instructions are given for the construction of a slide rule, and its use, and so forth. Above all care is taken to provide a unified vision of mathematics, linking algebra, geometry and trigonometry, and to show how useful mathematical instruments are in addressing problems of physics and engineering. In particular, with regard to geometry, Perry criticises the Euclidean method and suggests that: practical experimentation and measuring with squared paper be carried out before rational geometry; that the experimental geometry be flanked by some deductive reasoning; that greater emphasis be given to solid geometry; that trigonometric functions be used in the study of geometry; and that more attention be paid to applications.[3]

Here and there Perry also provides remarks on methodology and advice for teachers,[4] noting the difficulties and most frequent errors on the part of the students, and the reasons for them.

Many of the problems addressed by Perry are similar to those proposed today in teaching experiments involving the use of graphic-symbolic calculators, with a strong use of numerical data, but the final object is different. The method that he proposed is based on problem solving, and on a transversal approach to mathematics highly concentrated on procedures. His text presents a mathematics to be ‘practised’ and not to be formulated in a theory. This constitutes the originality of the method, but is at the same time also its limit.

It is noteworthy that pioneers in mathematical education in England such as Charles Godfrey, Benchara Branford, Percy Nunn and William D.

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[2] See also (Smith, 1913), where he underlines how the American school is aimed at the masses (p. 3).

[3] See, for example, (Perry 1902, p. 102).

[4] See, for example, (Perry 1913, pp. 21, 25, 32, 51-52).

Eggar stressed a heuristic and experimental approach to mathematics and the importance of a close correlation with other sciences. For example Eggar in his book *Practical exercise in Geometry* writes:

> This book is an attempt to adapt the experimental method to the teaching of Geometry in schools. The main object of this method, sometime called "heuristic", is to make the student think for himself, to give him something to do with his hands for which the brain must be called in as a fellow-worker. The plan has been tried with success in the laboratory, and it seems to be equally well-suited to the Mathematical class-room (Eggar, 1903, p. V).

In the volume *The teaching of algebra*, Nunn – a mathematician with strong interests in philosophy and other sciences, and professor of education at London University from 1913 – underlines the twofold purpose in mathematics teaching: to enable pupil to understand the importance of mathematics as 'an instrument of material conquests and of social organization', and to accustom him to appreciate 'the value and significance of an ordered system of mathematical ideas' (Nunn, 1914, p. 17). Concerning the teaching of algebra he maintains that this subject should be introduced 'as a symbolic language specially adapted for making concise statements of a numerical kind about matters with which he is already more o less familiar' (p. 18). Moreover pupils should be made to perceive from the very beginning that formulae refer to realities beyond themselves. As Perry did, also Nunn invited teachers to attract pupils' attention to the connexion between variables so that they can be gradually arrive to the study of functions, and to highlight the links among the various sectors of mathematics and between mathematics and other sciences. The work of Perry was one of his point of reference (Nunn, 1914, p. VII, 24, 311, 556).

Independent of his actual influence on technical education in England, Perry's movement favoured the dissemination of the idea of a laboratory-style teaching of mathematics for students of all types, and more generally the statement of some fundamental principles: greater democracy in education, greater consideration for what is useful in real life, greater attention to pedagogical aspects. The influence of his ideas can be perceived above all in the teaching of geometry: more space was given to experimental work, laboratories were set up in many schools, and there appeared many textbooks oriented in this direction (Price, 1986, p. 124-130). In 1908 the London County inspector Benchara Branford underlined the increasing attention paid to the experimental and graphical side of mathematics and the emergence of 'mathematical laboratories, well stocked with clay, cardboard, wire, wooden, metal, and other models and material, and apparatus for the investigation of form, mensuration and movement' (Branford, 1908, p. VIII). In his speech at the International Congress of Mathematicians held in Cambridge in 1912, the mathematics educator Charles Godfrey reported the results concerning a questionnaire on the use of intuitive and experimental methods in English schools: in particular graphical representation of functions, graphical study of statistic, graphical statics, estimation of area by means of squared paper were adopted by public schools (pupils from 12 to 18 years old) in a percentage which varied from 90 per cent to 98 per cent (Godfrey, 1913).

Perry's movement aroused interest not only in Europe, but in America as well. In a 1902 lecture, Eliakim Hastings Moore, then president of the American Mathematical Society, gave his celebrated talk *On the foundations of mathematics* (Moore, 1903) in which he invited teachers' associations to concern themselves with secondary education, underlined the defects of a compartmentalised teaching too focussed on aspects that were theoretical and abstract, and expressed his hopes for an integrated teaching of pure and applied mathematics, citing Perry's experimental teaching as an example.

An entire section of his talk was dedicated to the 'laboratory method' in mathematics, which he compared to the physics laboratory, highlighting its advantages for teaching. According to Moore, this method is the only one capable of making young people understand that 'mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions' (Moore, 1903, pp. 417-420). He believed that laboratory teaching, which must be characterised by a practical approach – that is, one that is 'computational, or graphical or experimental' (p. 419) – has the following advantages: it allows the student to understand the importance of a theorem and creates in him the desire for a formal proof (p. 419); it stimulates his personal research; it permits individual work as well as work in groups, where the teacher is at once a member of the group and the leader. One of Moore's recommendations for making the method work is to present only interesting experiments: for example, in the laboratory for physics, the mere explanation of how the instruments are used is not interesting; it is better to pose problems whose solutions involve the use of those instruments, so that the student learns 'the use of the instruments as a matter of course, and not as a matter of difficulty' (p. 418).

In his conclusion he states that in his opinion the laboratory method for secondary teaching of mathematics and physics 'is the best method of instruction for students in general, and for students expecting to specialize in pure mathematics, in pure physics, in mathematical physics or astronomy, or in any branch of engineering' (p. 420).

Moore's program was taken up by several mathematicians who specialised in pedagogy and didactics, including Jacob William Albert

Young, a professor of mathematical pedagogy at the University of Chicago. He dedicated an entire chapter of his book *The teaching of mathematics in the elementary and the secondary school* (1906) to Perry's movement, devoting ample space to experimental method and to the mathematics laboratory. Young also went into detail about the equipment that a good mathematics laboratory should have: geometrical models, surveying instruments, scales, pendulums, levels, barometers and thermometers. Besides he believed that there should be a well-stocked library with a good collection of textbooks, workbooks and various kinds of tables, as well as books on the history of mathematics, recreational mathematics and journals about teaching. He thus underlined the importance of the laboratory as a physical place where students could work under the guidance of the teacher or assistant.

In 1902 in France, the reform of secondary teaching known as *humanités scientifiques*, introduced infinitesimal analysis in secondary schools, and also emphasized the importance of a concrete teaching method that takes account of relationships to the real world (Belhoste, Gispert, & Hulin 1996; Gispert 2009). Presiding over the commission to oversee the revision of the programs was the mathematician Gaston Darboux, but many other illustrious scholars also contributed. In particular, Émile Borel encouraged teachers to introduce 'more of life and a sense of reality into our mathematics teaching' and suggested creating an *atelier mathématique*, a 'mathematical workshop', where students could personally build models, take measurements, and so forth with the aim of 'bringing not only students but also teachers, but above all the mind of the public, to a more exact idea of what mathematics is and the role it actually plays in modern life' (Borel, 1904 (1967), p. 14). Borel's view is made clear in his 1905 handbook for geometry (Borel, 1905), in which the practical and intuitive aspects are amply emphasised. His aim was to 'write a more concrete geometry, where considerations of symmetry, of displacement are invoked as often as possible' and 'substitute more and more the dynamic study of the phenomena in place of their static study' (Borel, 1905, p. V, VII). The book opens with an introduction to the use of straightedge and compass, in which applications are skilfully coordinated with theory; among the exercises proposed there are some of a practical nature that involve symmetries, the use of instruments, etc. There is no rigid division between plane and solid geometry; the topics introduced include, for example, tiling the plane (pp. 111-113), approximations (pp. 280-281) and, in the complements notions regarding the conic sections and other curves, the approximate calculation of areas and land surveying (pp. 353-375). However, the idea of the mathematics laboratory he proposed was rather limited. He writes:

> … in my opinion, the ideal mathematics laboratory would be, for example, a carpentry workshop; the laboratory assistant would be a carpenter who in small institutions would come only a few hours per week, while in the large schools he would almost always be present. Under the guidance of the professor of mathematics and following his instructions, the students, aided and advised by the laboratory assistant, would work in small groups to build models or simple devises (Borel, 1904 (1967), pp. 15-16).

It was this kind of mathematics laboratory, the *Laboratoire d'enseignement mathématique*, that Borel created together with Jules Tannery at the École Normale Supérieure, aimed at training future teachers for a laboratory-type teaching. Here models in either wood or cardboard, wire and cork were conceived and built for teaching geometry and mechanics. The didactic uses of other instruments such as mechanical linkages, pantographs, inversors, calculating machines, and instruments for geodesy and land surveying were also taught. Furthermore, the laboratory was to be equipped with a library where future teachers could find the principal French publications on mathematics teaching, the most important pedagogical journals, and the scholastic handbooks of various countries (Châtelet, 1909).

In Germany, beginning in the 1890s, Felix Klein had begun to formulate his famous program for the reform of mathematics teaching which redefined the relationship between secondary schools and universities. It was first given formal expression, although with some compromises, in the *Meraner Lehrplan* of 1905 (Bericht, 1905; Klein, 1907, pp. 208-219), developed by the *Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte.* It addressed not only mathematics teaching but also that of physics and the other natural sciences. Klein's principal innovation was the introduction of 'functional thinking' (*funktionales Denken*) in secondary teaching, but other aspects, such as the importance of applications, the use of geometric models, the relationship to real problems, the connections with physics teaching, and the value of experiments were also underlined (pp. 550-553). For physics teaching, the *Meraner Lehrplan* highlighted the necessity of outfitting suitable work spaces (*Arbeitsräume*) where adequately trained teachers and their assistants could work and experiment alongside the students.

With regard to the teaching of geometry, and mathematics in general, the following aspects were emphasised: the strengthening of spatial intuition (p. 543); the use of straightedge and compass, drawing, measuring (p. 547); the consideration of geometrical configurations as dynamic objects (p. 548); the strengthening of the use of graphic representations; providing room for applications (p. 549); making use of models; the coordination of planimetry and stereometry; making mention of the historical and philosophical point of view (p. 550). The importance of using instruments and models in mathematics teaching also emerges from the books on elementary mathematics from an advanced standpoint, which Klein intended expressly for the training of mathematics teachers. The volume on geometry (Klein, 1925-1933 II; Klein, 2004) includes an introduction to various instruments such as Peaucellier's inversor, an instrument for creating affine transformations, and others. Klein observed:

> Instead of mentioning further details, I should like here to sound a general warning against neglecting the *actual practical demonstration* when such instruments are considered in illustration of a theory. The pure mathematician is often too prone to do so. Such neglect is just as unjustifiably one-sided as is the opposite extreme of the mechanician who, without taking an interest in the theory, loses himself in details of construction. Applied mathematics should supply here a bond of union (1925-1933, II; 2004, p. 15).[5]

Klein was also involved in the reorganisation and modernisation of the *Modellkammer* in Göttingen for educational purposes, in particular to aid in *Raumanschauung* (spatial intuition) (Schubring 2010) and, together with P. Treutlein, at the congress of the International Commission on the Teaching of Mathematics (later ICMI) held in Brussels in 1910 he presented the use of models in secondary and university teaching to develop geometrical intuition (Giacardi, 2008).

An important occasion for international comparison and contrast of teaching experiences in the field of mathematics in various countries was the fourth International Congress of Mathematicians held in Rome in 1908, which led to the founding of ICMI, with Klein as the first president. The session dedicated to teaching was quite rich: it was organised by Vailati, as archival documents show. Some of the numerous reports contain explicit or implicit references to the idea of laboratory: the importance of measuring, graphical representations, and the use of instruments for practical teaching is highlighted by Smith and Vailati, and the term 'laboratory work in mathematics' is introduced by Godfrey.

A picture of the international situation is found in the report prepared by Smith (1912) on the enquiry promoted by ICMI in 1911 on 'Intuition and experiment in mathematical teaching in the secondary schools', which concerned students from 10 to 19 years of age.[6] He presented a general

[5] For more on this, see (Bartolini Bussi et al., 2010) and (Schubring, 2010).

[6] Some of the topics discussed in this report had already appeared in an unpublished 1906 letter by Smith to Gino Loria. See the website http://www.icmihistory.unito.it/19081910.php.

outline of the situation in the various countries, from which it emerged that, in the teaching of mathematics in secondary schools, recourse to intuition and to practical experiments was generally more greatly prized in Austria, Germany and Switzerland, than in Great Britain, France and the United States (Smith, 1912, p. 512), and that the most debated subjects were above all the teaching of geometry, and whether or not to introduce the concept of function. Continuing on, he focused his attention on particular aspects of an 'active' teaching of mathematics: 'Measuring and Estimating' (geodetic, astronomical measurements, triangulations, etc.), 'Geometric Drawing and Graphic Representation' (teaching of descriptive geometry), 'Graphic Methods' (representation of functions on graph paper, graphic statics, evaluation of surfaces with the aid of graph paper, etc.), 'Numerical Computation' (use of tables, of the slide rule, methods of approximate calculation, etc.) (Giacardi, 2008). The aim of this report, as Smith himself made clear, was to illustrate data concerning the various countries and the efforts made to apply mathematics to real life, and not to offer solutions to problems or recommendations for the future. Nevertheless, it suggested questions which indicated a line of research and study for people concerned with education, such as: What simple, inexpensive instruments could be used to increase the interest in the early stages of mathematics teaching? What can be done to make the inductive phase of geometry teaching more real and interesting without weakening the deductive side? Could drawing and geometric constructions be used more adequately in the teaching of geometry? What should schools do to take advantage of the decreasing prices of the calculating machine?

The laboratory for mathematics according to Vailati

In Italy it was Giovanni Vailati who proposed the idea of 'school as a laboratory'. A mathematician, philosopher and educator, Vailati was a member of the Peano School. He earned his degree in engineering and then in mathematics in Torino. After having taught for several years at the University of Torino as an assistant or in open courses, in 1899 he left Torino and began teaching in secondary schools of various Italian towns and cities. His commitment to education found various expressions, but his most important contribution concerned the design of the programs of mathematics in the context of work performed for the Royal Commission for the reform of secondary schools, which he carried about from 1905 until his death in 1909. Elsewhere I have illustrated Vailati's contributions to the work of the Royal Commission (Giacardi, 2010), so here I will concentrate on his vision of the mathematics laboratory.

In the past it has been underlined that the idea of the 'school as laboratory' proposed by Vailati was analogous to the teaching experiments undertaken by the experts in pedagogy (Arzarello, 1987, p. 35), but on the basis of the scant evidence found in the published works and correspondence (*Epistolario*) it is difficult to say with certainty at what extent they were an actual reference.[7] To be sure, Vailati was very familiar with the work of Charles Sanders Peirce and William James and knew some of their papers that were expressly dedicated to questions of education,[8] while Dewey is mentioned only in passing in his writings.[9] He was also in contact with Claparède,[10] and owned a copy of Binet's *La psychologie du raisonnement* (Paris: Alcan, 1886). In his correspondence and writings he makes no references to Decroly nor even to Montessori, whose work in education began after her participation in the Italian national congress on pedagogy, which took place in Torino in 1898.[11] Even though at that time Vailati still lived in Torino, he is not listed among those who attended the congress, although the names of other members of the Peano School – Rodolfo Bettazzi, Alessandro Padoa and Giovanni Vacca – appear.

The idea of a school-laboratory proposed by Vailati appears explicitly in his review of the book of the educator Maria Begey, *Del lavoro manuale educativo* (1901), and in his brief paper 'Idee pedagogiche di H. G. Wells' (1906), and afterwards emerges in the mathematics programs that he devised for the Royal Commission and in the related instructions on methodology. However, Vailati never presented a systematic and complete exposition of his ideas. His thoughts regarding the field of pedagogy and education are in large measure found in marginal, fragmentary observations in the vast and heterogeneous collection of his writings, and are mostly contained in the innumerable reviews.

In order to understand the originality of Vailati's thinking in education, it is important to frame it within his particular vision of the function of

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7 Only a detailed, complete examination of the vast quantity of material conserved in the Vailati archive (*Fondo Vailati*, Biblioteca di Filosofia, Università di Milano), in particular in the *Notes*, which I have only explored in part, could provide a definitive answer. See (Ronchetti, 1998).

8 See for example (Vailati, 1905c), where Vailati shows that he knows W. JAMES, *Talks to teachers on psychology and to students on some of life's ideals* (1899).

9 See *Scritti*, I, p. 202, 210.

10 See *Epistolario,* p. 231 and (Busino, 1972) where Vailati's letters to Claparède are transcribed.

11 See (Molineri & Alesio, 1899). Montessori's talk on the education of 'degenerate youths' appears on pp. 122-124. See also Maria Montessori, *Il Metodo della pedagogia scientifica applicato all'educazione infantile nelle Case dei Bambini* (1909), where there appears the idea of an active school and the use of specially prepared materials; of particular interest for the teaching of mathematics are the much later works *Psico-Aritmética* (Barcelona, 1934, trad. it. 1971) and *Psico-Geometría* (Barcelona, 1934).

mathematics and its teaching, a vision in which different motives and needs converge. The relationship with Peano and his School led to his solid mastery of mathematical logic, his ideas about deductive and systematic rigour, and to his reflections on language combined with a deep interest in education and the history of mathematics, and a genuine desire to democratise knowledge. The pragmatism of Peirce also influenced Vailati, who saw pragmatism as an instrument in the struggle against senseless problems and against metaphysics; in particular, he made his own the operative and functional criteria for giving meaning to the propositions, that is, he believed that their meaning depends on the consequences that can be drawn from these propositions. Intertwined with pragmatism are positivistic requirements: the idea of a scientific *humanitas*, an appreciation of the applied knowledge, the founding teaching on a positive knowledge of man (biology, psychology), in the constant awareness that the cognitive process proceeds from facts to abstraction. Underlying all of this is the Herbartian assumption that the aim of teaching is the formation of character.

The fact that Vailati was interested in psychology leads us to think that this influenced his vision of mathematics teaching. He took part in three international congresses in psychology (Munich 1896, Paris 1900, and Rome 1905) and from his writings it emerges that he was above all interested in questions of method in psychology, and in the applications of psychology in the studies of art, literature and anthropology. In his talk given at the congress in Rome he dealt with the 'psychology of intellectual operations' (today we would say cognitive psychology), a topic already present in *The* w*ill to believe* by the pragmatist philosopher James, who rehabilitated the constructive and anticipatory activities of the human mind against those that are purely receptive and classificatory.[12] However, there was no explicit reflection on education, although, as we will see later, the influence of modern psychology can be found in his considering general concepts (including those of science) as mere instruments that make it possible to order, classify, use the raw material of experiences (Vailati, 1905b, p. 280).

In order to address problems connected to mathematics teaching, Vailati took into consideration the programs and educational organisation of other European countries, as well as the movements for school reform of Klein in Germany, Perry in England, and Darboux and Borel in France.[13] The organisation of the session dedicated to teaching of the fourth International Congress of Mathematicians (Rome, 1908) helped

[12] See (Vailati, 1905d); for more on this see (Sava 2006).
[13] See (Vailati, 1910) and (*Fondo Vailati*, Cartella 41, fasc. 346; Cartella 31, fasc. 272).

him to build his international contacts. Moreover, his teaching experience in various secondary schools in northern and southern Italy had allowed Vailati to see first-hand the shortcomings and defects of Italian schools. It was precisely the desire to remedy these that guided him in formulating his proposals for reform.

In his opinion, an improvement of the teaching of certain subjects, both scientific and not, could have been possible if teaching were organised in the form of a *laboratory*, thus eliminating the *frontal character* and *verbalism* of the traditional lesson. For Vailati, the 'school as laboratory' must not, however, be conceived in the reductive sense of a laboratory for scientific experiments, but 'as a place where the student is given the means to train himself, under the guidance and advice of the teacher, to experiment and resolve questions, to … test himself in the face of obstacles and difficulties aimed at provoking his perspicacity and cultivating his initiative' (Vailati 1906, p. 292). Maieutic lessons, hands-on work and games as suitable aids for learning, the operative experimental method, the unitary vision of mathematics, the right balance between rigour and intuition, the use of the history of mathematics, are the salient aspects of Vailati's vision of mathematics teaching, and implement what he calls the 'school as laboratory'.

Maieutic lessons, hands-on work and games

According to Vailati, one of the major causes of the ill functioning of secondary schools is the deplorable habit of conceiving teaching as a lecture where the student can't do anything but listen, to then be interrogated 'for purposes of diagnosis' (Vailati 1905c, p. 287), that is, to make sure that he has understood and memorised all that he has heard. In contrast, the kind of method more suitable for educational purposes is the maieutic or Socratic, which allows teachers to guide their students towards the discovery of mathematical truths, while at the same time stimulating enquiry and reflection. Further, to arouse the student's attention, it may be useful to exploit moments of play during the process of learning, which far from 'diminishing the dignity of the science of mathematics' (Vailati, 1899, p. 261), instead increases its attraction. Manual activity, appropriately directed and not aimed at learning a trade, can serve to 'practise the various skills of observation, discernment, attention, and judgment' (Vailati, 1901, p. 265) and constitutes an excellent antidote to the common misconception that one knows something simply because one has learned certain words.

It is clear that in this kind of teaching, the mode of examination must change as well. It is not by asking the student to define the concepts verbally that the teacher can grasp his level of comprehension, but rather by verifying that he is capable of applying them:

> In fact, there is no other point on which there is such a jarring contrast between the educational procedures ordinarily followed and the fundamental tendency of modern psychology to regard general concepts as simple instruments (*Denkmittel*), having no other role than of making it possible for us to order, classify, fashion for determined purposes, the raw material of particular experiences. In accordance with this view, not knowing how to apply a concept … is equivalent to not possessing the concept itself at all, regardless of the ability one has on the other hand to repeat the words that presume to define it or explain it. (Vailati, 1905b, p. 280).

The operative experimental method

One of the cardinal points on which Vailati's proposals were based was the conviction that since the process of learning moves from the concrete to the abstract, pupils should never be forced to 'learn theories before knowing the facts to which they refer' (Vailati, 1899, p. 261). On the contrary, they should show that they know *how to do things*, not merely *how to repeat things*. Therefore a mathematics teaching which takes these premises into account should adopt an approach that is experimental and active. The usefulness of this method can be perceived particularly in the teaching of geometry where drawing, the construction of the figures, the recourse to squared paper, to scales, etc., can aid the student in the process of learning. Vailati wrote:

> Guiding and pushing the student to procure for himself, by means of experiment and, in particular, with recourse to the instruments of drawing, the greatest possible number of real cognitions about how to construct the figures and about their properties – above all not 'intuitive' –, is on the other hand the best way to create in him the desire and need to understand 'how' and 'why' such properties exist, and to predispose him to think of learning and the search for the deductive connections between [these properties] as interesting, as well as the arguments that lead him to recognise each of them as a consequence of the other. (Vailati, 1907, p. 305).

Thus we see that in Vailati's vision of the 'school as laboratory', the 'operative' moment must be followed in a second phase of learning, by the search for 'deductive connections' and the arrangement of the knowledge acquired into a theory. The passage between the two types of teaching, experimental-operative and rational, has to be done gradually, 'applying first of all deductive reasoning, not to demonstrate propositions that students already find quite obvious … but rather to use these propositions to arrive at others which they do not yet know' (Vailati, 1909,

p. 485). In this way the deductive procedure will also appear as an instrument for discovery.

For each area of mathematics, Vailati then encourages teachers to stimulate the students' creativity by providing them with several proofs of the most significant propositions to show them how they can arrive at the same conclusion by different routes, or even by using different mathematical instruments. For example, in the notebook relative to the classes held in 1901-1904 at the Technical Institute in Como, Vailati addresses the problem of finding the sum of the first *n* odd natural numbers, of the squares of the first *n* natural numbers, and then of the cubes, presenting various kinds of proofs: direct, with the aid of graphic visualisations, and by induction.[14]

Further, Vailati maintains that as far as possible, the statements of theorems should be presented as problems,

> 'thus, for example, the Pythagorean theorem could be advantageously presented as the solution to the problem of finding a square whose area is equal to the sum of the areas of two given squares, or as an answer to the problem of how to construct, on the ground, a right angle when the only instrument available is a cord that can be divided for example, into twelve equal parts'. (Vailati, 1910, p. 38).

A unified vision of mathematics and knowledge

In Vailati's view, teachers should lead pupils to perceive the unity of the various branches of mathematics as soon as possible, making evident the close connections between arithmetic, algebra and geometry, in order to accustom them to addressing a single problem with various methods, choosing each time the most appropriate one. For instance, in the lecture given in 1908 in Rome during the International Congress of Mathematicians, Vailati provides the following example of connecting arithmetic and geometry:

> Think, for example, how much easier it would be for the student to recognise the meaning and the significance of a proposition like this one: that 'the geometric mean of two numbers can never exceed their arithmetic mean', when they are made to see that, in a circle whose diameter is the sum of two segments, the second is represented by the radius, and the other instead by half of a chord. (Vailati 1909, p. 487).

The connection that can be established between arithmetic and algebra is of particular use from the point of view of didactics, inasmuch as the

[14] Giovanni Vailati, *Appunti per Lezioni, Istituto Tecnico*, Como 1901-1904 (*Fondo Vailati,* Cartella 38, fasc. 340).

teaching of arithmetic is suitable, from the very beginning, to pave the way for the teaching of algebra. The aim is to lead him to view algebra as simply a new form of language that is much more precise than ordinary language and capable of reducing questions or problems that seem at first complicated to forms that are so simple that they require almost no mental effort to solve.[15].

Similarly, according to Vailati one of the most efficient means for preparing students to understand the significance and usefulness of formulas is to accustom them from the very beginning 'to recognise the necessary and sufficient conditions so that a given algebraic expression, a given equation, a given identity, can be interpreted as expressing, respectively, a construction, a problem, a theorem of geometry'. (Vailati, 1910, p. 57).

Vailati was not only convinced that the students ought to be offered a unified vision of mathematics, but he believed that it was fundamental for teaching to transmit a unified vision of knowledge, establishing a dialogue between humanistic culture and scientific culture. This objective could be reached by means of the historical method. Applied as much to the sciences as to the study of Latin and Greek, according to Vailati, this method can also take on a didactic function because it is particularly suited to 'avoiding pedantry'[16] and 'to rendering the teaching more fruitful … more efficacious, and altogether more attractive' (Vailati 1897, p. 10). It also constitutes a good antidote to all forms of dogmatism. The teacher can help young people approach the history of science by means of commentated readings of passages from the classics of science. Vailati himself read and commented on passages from Euclid's Elements to his students, as can be seen from his class notes.[17] To make this kind of teaching possible, he also deemed it desirable that schools be equipped with well-organised libraries, containing not only textbooks, but clear and concise works of popular science, books providing an orientation to study, editions of the works of great authors, encyclopaedias, etc. (Vailati, 1906, pp. 293-294).

The dialectic between rigour and intuition

In his writings Vailati appears to want to avoid any clear contrast between 'intuition' and 'rigour', and in particular in the article (Vailati, 1907), in which he illustrates the new programs for mathematics, he

[15] See (Vailati 1910, p. 40) and G. Vailati to G. Vacca, Crema, 20 July 1902, in *Epistolario,* p. 207.

[16] See G. Vailati to G. Vacca, 25 May 1901, in *Epistolario*, p. 187.

[17] See Vailati, *Lib. V* (*Fondo Vailati,* Cartella 38, fasc. 340).

observes that the application of new research on the foundations of elementary geometry in education had made it evident that 'rigour or the logical correctness of a proof is not something that depends on the number or the quality of the assumptions or hypotheses which are used in it, but rather depends on the way in which these are applied'. What is important is that 'each hypothesis, or assumption which … is used be clearly recognised, and formulated explicitly' (Vailati, 1907, pp. 305-306), and the only indispensable requirement for the rigour of the proof is that the postulates be compatible among themselves. Only when the students have acquired a greater degree of maturity, will they be shown if and by how many the number of postulates can be reduced.

Far from discouraging geometric intuition, Vailati believed that the good teacher should 'discipline and refine intuition' in order to avoid the errors that can arise from a 'rash and instinctive trust in it' (Vailati, 1904, p. 268). Instead, in his review of Halsted's textbook on rational geometry, based on Hilbert's work on the foundations, Vailati pointed out the educational hazards which can derive from 'the concern for guaranteeing the absolute rigour and perfect logical coherence of the proofs, purging them of any intuitive suggestion' (Vailati, 1905a, p. 289). Therefore in the practice of teaching it is necessary to reach a balance between intuition and rigour.

Furthermore, Vailati maintained that deduction had to be assigned a role more extensive than that generally attributed to it. The metaphors that represent deduction as a process aimed at 'extracting' from the premises what is already contained in them tend to diminish its importance with respect to the other process of reasoning and of research (see Vailati, 1898, p. 25). According to Vailati, deduction can also have heuristic value and efficaciousness; in fact beginning with premises that are only hypothetical can serve to develop ideal constructions to be compared with reality, so that the premises and consequences can provide confirmation of each other in a 'reciprocal check' and 'mutual support' (Vailati, 1898, p. 42).[18]

Vailati's reflections, together with the notes of the lessons he taught in secondary schools, are illustrative of a mathematics laboratory intended in a broader sense that the various meanings we mentioned earlier: it is a laboratory that involves people (students and teachers), structures (classrooms, equipment, instruments), working methods, experimentation and commented readings, but it also involves the search for new deductions and proofs, and for different ways of interpreting a single

[18] See also the letter from Vailati to F. Brentano, Como 16 April 1904 in *Epistolario*, p. 305.

result. It is a place where the work is done with both the hands and the mind, beginning with the problems; a place where the student becomes accustomed to using concrete objects and instruments to measure, but also to 'take his own measure', to communicate his own hypotheses, propose new solutions, new proofs, in close alliance between the experimental and theoretical aspects. In the meantime, all of this contributes to the formation of character.

Conclusions

In 1909 the reform project of the Royal Commission on which Vailati had served was published, and in May of that same year Vailati passed away. A few months later, during the congress of the Associazione Mathesis, an Italian association for mathematics teachers, held in Padua in 1909, the distinguished scholar of algebraic geometry Guido Castelnuovo, speaking on the work of the newly created International Commission on the Teaching of Mathematics, spoke in praise of the reform proposals developed by Vailati and suggested that teachers put the general lines into practice at once in their classes (Castelnuovo, 1909, p. 3). Outside of Italy Vailati's project was seen as innovative and following the footsteps of the reforming proposals of Klein. In 1910 Florian Cajori wrote:

> Under the leadership of Loria and Vailati there is a movement afoot that favours greater emphasis upon intuition, the introduction of some modern geometrical notions, the fusion of geometry with arithmetic, and the concession to the demands for practical applications made by this age of industrial development. In fact, Italy is entering upon a reform much like that of Germany and France (Cajori, 1910, p. 192).[19]

Various factors prevented the mathematics laboratory proposed by Vailati from becoming widespread in practice, or from being realised in textbooks. First of all, the reform set forth by the Royal Commission was never carried through (Giacardi, 2009). Second, Vailati, unlike Perry, never wrote a systematic exposition, his ideas are scattered throughout his writings, and his premature death prevented any further developments. Third, laboratory method-inspired textbooks were never published in Italy even though some authors paid attention to the geometrical constructions and use of the instruments, to experiments with folded or cut paper, sand, or small models in geometry, or made use of squared paper to introduce the concept of function.[20] Fourth, the discussions of the various sections of the Mathesis Association show, that not all mathematicians in Italy

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[19] See also (Lietzmann, 1908, p. 181).

[20] See for example the textbook by Federigo Enriques and Ugo Amaldi, *Nozioni di matematica ad uso dei licei moderni* (Bologna: Zanichelli, 1914-1915).

shared Vailati's approach to teaching of mathematics (Giacardi 2009, pp. 17-20). Ultimately, his efforts would have been in any case nullified by the Gentile Reform of 1923, which made the humanities the cultural axis of national life in Italy, and especially of education.

Nevertheless, his portrayal of the 'school as laboratory' remains significant even today, and in recent times has been taken up again in the mathematics curricula proposed by the Italian Commission for Mathematics Teaching, in which we read:

> The mathematics *laboratory* is not a physical space outside the classroom, but is rather a structured set of activities aimed at constructing the *meanings* of mathematical objects. Thus, the laboratory involves people (students and teachers), structures (classrooms, instruments, organisation of spaces and times), and ideas (projects, plans for educational activities, experimentation). (Matematica 2003, p. 28).

The mathematics laboratory therefore has a significance that extends beyond both the carpentry laboratory proposed by Borel, and Perry's Practical Mathematics, based on problem solving and a transversal approach to mathematics that was highly concentrated on procedures, and beyond even the laboratory method proposed by Moore, which concentrated above all on 'computational, or graphical or experimental' aspects or, as Klein suggested, on the use of models and mathematical machines, or as we would say, technological tools for visualisation.

It is, as Vailati had hoped, a methodology based on problem solving, conjecture and argumentation, but whose ultimate goal is that of arriving to the construction of meanings and to a theoretical systemisation of mathematics.

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The rise and fall of Reality Geometry - a contribution to the history of geometry in Danish schools 1900-1960

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Abstract

Denmark experienced much discussion on the proper way to teach geometry in schools in the early 20th century. Euclid's Elements had already been challenged for some decades. We describe two new developments, the "Experience Method" of T. Bonnesen and the "Reality Geometry" of Hjelmslev. In his vision of Reality Geometry in 1916 he indicated that geometry was only the beginning and that the long term aim would be to get rid of the recently discovered "pathological" functions in analysis. This kind of geometry was highly recommended by some of the most respected teachers, but other teachers reacted against the idea that a tangent and a circle in reality touched each other in a small interval instead of one single point. The beauty of geometry seemed to disappear in such messy theorems. In the reform of mathematics education after 1960 only little is left of this discussion, although Reality Geometry in its golden days was considered closer to the "Truth" than abstract Euclidean Geometry.

Introduction

Like most countries Denmark experienced much discussion on the proper way to teach geometry in schools in the early 20th century. Euclid's *Elements* were challenged already in the latter half of the 19th century, although the Dane J. L. Heiberg had finalized a very accurate Greek edition in 1886, translated into Danish from 1897 by Thyra Eibe (1866-1955). She also made a modern adaptation for schools (1908), taking advantage of the short cuts made possible by allowing rigid motions in proofs, and her book was widely used[1] until 1950.

However already J.J. Rousseau claimed in 1762 that his *Émile* could and would learn geometry by geometrical observation and experiments:

> Make the diagrams accurate; combine them, place them one upon another, examine their relations, and you will discover the whole of elementary geometry by proceeding from one observation to another, without using either definitions or problems, or any form of demonstration other than simple superposition. For my part, I do not even pretend to teach Émile geometry; he shall teach it to me. I will look for relations, and he shall discover them. (Rousseau 2009, p. 86)

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[1] See Table 1.

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education.* Caparica: UIED.

This kind of approach was also important among many leading mathematics teachers in Denmark from as early as 1858, when the influential mathematician, politician and educator Adolph Steen in his book on solid geometry emphasized observation of solid objects as the natural starting point. In 1881 Poul la Cour introduced everyday language in geometry in his teaching and textbooks at the Folk High School in Askov e.g. by calling a "right-angled right-standing parallelepiped" a "box". (Hansen 2002, p 107)

However it was after 1900, when a reform of the educational system including mathematics for a wider and younger group of students was discussed and decided upon (see Hansen 2009), that experiment and observation became the main pedagogical idea for a number of leading mathematicians and textbook writers in Copenhagen.

Poul Heegaard (1871-1941) gave in his book *Visual solid geometry*[2] from 1900 everyday definitions of a straight line and was the first to include the practical definition used in the steel industry when they wanted to produce rulers in steel. To make the prototype you have to make three rulers at the same time: "Let's call them A, B and C. First you compare the edges of A and B, by joining them and hold it up towards the light to see where there are curved pieces that have to be filed into shape" (Heegaard, 1900, p. 2). Having done this with A and B in such a way that they fit nicely together even when they are displaced in relation to each other and turned around, one could think that they must be perfectly straight rulers, but they might be circle arcs, and that is why they both have to fit together with the third ruler C as well as with each other. But then finally they must represent straight lines, according to Heegaard.

His textbook also included a pop-up triangle and suggested many experiments with paper, scissors and rubber bands. No wonder that he, ten years later when he was Education Officer in the Ministry of Education, had a correspondence with the officer responsible for sloyd (educational woodwork), where he expressed his appreciation of sloyd as an important school subject to develop many mathematical abilities in the students and not least geometry. Heegaard soon became professor of mathematics at Copenhagen University and later twenty years at Oslo University. (Hansen 2002, p. 108)

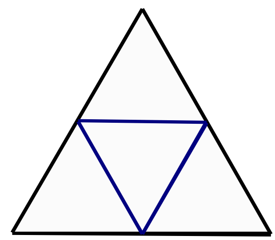
The experience method by Bonnesen

The former actor at the Royal Theatre in Copenhagen, dr. phil. Tommy Bonnesen (1873-1935), realized that his method, as it appeared in

[2] Titles of books and papers are here usually given in English translation. The original Danish titles can be found in References.

the textbook *Geometri for Mellemskolen* from 1904, was a discovery method, but he preferred the term "Experience Method", which meant that the student could experience many of the results in geometry, and obtain a rich basis, before the mode was switched to a deductive sequence of proofs.

He outlined his method by commenting on this theorem: *The ratio of the area of two similar triangles is the square of the ratio of the sides.*



According to Bonnesen the proof of this theorem is so long that the substance of the theorem will be forgotten by the student before the proof is finished. Instead this theorem should from the very beginning let the student imagine and see a big triangle divided into smaller congruent triangles:

> He should see this so often, that a smaller triangle is contained 4 times in a triangle with sides of double length, that this picture instantly appears in his eyes, when he needs the theorem. The best result is obtained if we let the student find this theorem on his own in simple cases such that it appears with the clarity and certainty that only occurs by self discovery. When this has happened we pose more difficult assignments from which it becomes clear that a general proof is needed, and only then we make that proof. (Bonnesen 1906, p. 1)

Bonnesen is well aware that many teachers use similar approaches, i.e. discovery methods. However he emphasizes that he sees not only a method but also a goal in letting the student get acquainted with geometrical facts in almost the same way as is used in Physics. (ibid., p. 3)

> Whatever aim is put for Geometry, whether emphasis is put on practical application or it is supposed to support the development of the brain[3], visual geometric lucidity and geometric experience must teach the students the theorems of Geometry in such a way, that they realize the real, physically demonstrable truths behind the theorems. (ibid., p. 10)

We know that Bonnesen's Experience Method has been successfully tried by other teachers than Bonnesen himself. But his textbook only came in one edition. According to one teacher – who had in fact taught

[3] Later in the article Bonnesen speaks ironically about this brain effect: "Luckily for the Maths teacher, the alleged effects are of such fine and spiritual nature, that it only with great difficulty is possible to evaluate whether they are attained or not, and even more difficult to know if they are easier obtained by mathematical than by other means."(ibid., p. 11)

Bonnesen's own class when Bonnesen had become principal of the school – everything went well as long as the students observed and experimented. However, the students acquired an inclination to solve geometrical problems by measurement alone and thus "reduced Geometry to a series of experiments" (Petersen 1908, p. 44). According to Bonnesen's pedagogical philosophy, the students should change to a stringent deductive mode half way through his book, and that seemed to be the weak point that prevented Bonnesen's method from becoming wide spread. Bonnesen himself ended his career as professor in geometry at the Technical University in Copenhagen 1917-35.

Fig. 1. Copenhagen mathematicians 1919. The person at the far left is Bonnesen, at the far right is J.L.W.V. Jensen (the telephone specialist and spare time mathematician known from Jensen-inequality for convex functions) and next to him Hjelmslev who invented Reality Geometry

The Reality Geometry of Hjelmslev

Johannes Trolle Hjelmslev (1873-1950) became professor in descriptive geometry at the Technical University in 1903 and later from 1917 to 1943 professor in Mathematics at Copenhagen University. He would agree on the pedagogical advantage of Bonnesen's method, but the matter was much more serious because – according to Hjelmslev – Euclid had built on sand and his results did not agree with reality. Geometrical constructions and drawings were dominant subjects in geometry and not least in descriptive geometry. Hjelmslev considered these precise drawings to be the reality in geometry. In this reality it was obvious for everybody who had eyes that a tangent to a circle would coincide with the circle periphery on an interval, a small one, but clearly more than a single point. And the axiom stating that you could draw one and only one straight line through two distinct points was a very poor description of the real drawing process, especially when the two points were very close to each other. He presented his thoughts on "Reality Geometry" - in Danish "Virkelighedsgeometri" - in two papers titled respectively "On the foundation of practical Geometry" and "The natural foundation of Geometry", (Hjelmslev, 1913 and 1917) and later in a lecture at Hamburg University printed in a paper titled "Die natürliche Geometrie" (Hjelmslev, 1923). From this it was clear that geometry was only the beginning and that the long term aim would be to get rid of the recently discovered "pathological" functions in analysis[4], which may be the reason why his textbook *Elementary geometry I-IV*, published from 1916 to 1923, finished by a volume on Calculus.

Elementary geometry 1916-23

In his 1916-paper Hjelmslev states that he only needs one axiom, "the existence of ruled paper". In his textbook *Elementary geometry I* (Hjelmslev 1916) this is more elaborated and extended to three dimensions. The leading idea from the practical world is bricks or building blocks. They can be densely packed, and two such blocks fit together in many ways, six by translation only, but then you can turn one block 180 degrees and it will still fit nicely on top of another. However *Elementary geometry I* is a Mathematics textbook, so Hjelmslev must define a building block or *normal box*, as he calls it.

To do this he first of all defines a plane very much like we saw Heegaard define a ruler using the actual production process in the steel industry. So to make a plane you have to make three at the same time. By putting a very thin layer of red powder on one plane, and putting another on top, you can see where there are bulges or cavities. Then you file and grind until all bulges and cavities disappear, such that finally all three planes fit perfectly


Normalkilen.

5. Ved en Kile vil vi forstaa et Legeme, der har to nøjagtigt tildannede plane Flader, der støder sammen langs en Kant. Hvorledes Legemets Overflade ellers er beskaffen, er ligegyldigt. To Kiler siges at være Nabokiler, naar de kan lægges saaledes, at de udfylder hinanden, d. e. de hviler begge paa en Plan, idet de hver har en Flade liggende i denne Plan, medens de slutter tæt til hinanden langs de Flader, som ikke ligger i Planen (se Fig. 3). Er Planernes indbyrdes Stilling den samme i begge Kiler, har man to rette Kiler eller Normalkiler (Fig. 4). Men hvorledes kan man kontrollere, om denne Egenskab virkelig er til Stede? Svaret

Fig. 3.

Fig. 4.

[4] He points to Peano's Curve (1890), mapping the unit interval onto the unit square as one of the extreme"pathological-mathematical" objects. (Hjelmslev 1913, p. 46)

together pair wise. That is Hjelmslev's definition of a plane. He can now make a sequence of definitions (the figures and are from Hjelmslev's 1916-book):

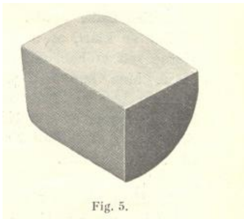


Fig. 5.

- A *wedge* (Fig. 3) is bounded by two planes, which meet at an *edge*.
- If two wedges resting on the same plane fit perfectly together (Fig. 4) we call them *neighboring wedges*.

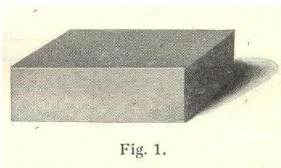
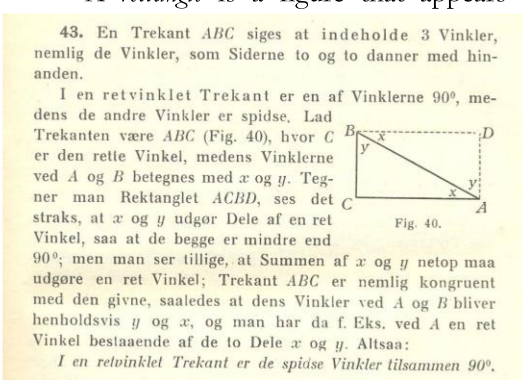


Fig. 1.

- If two neighboring wedges fit with the same third wedge, we shall call them *normal wedges*, and the corresponding planes are called *perpendicular*.
- A *straight line* is what you draw along the edge of the normal wedge.
- The *normal corner* (Fig. 5) is a normal wedge along all three edges leading to the vertex of the corner. The three edges are called *perpendicular* to each other, and each edge is perpendicular to the plane containing the two others.
- A polyhedron like Fig. 1 where alle 8 corners are normal corners is called a *normal box*. It can be made in all sizes.
- A *rectangle* is a figure that appears as side of a normal box. Consequently rectangles can be made in all sizes.
- Two lines are called *parallel* if they are perpendicular on the same third line. Hence four edges of the normal box are parallel (three times) and the opposite sides of a rectangle are parallel.



43. En Trekant *ABC* siges at indeholde 3 Vinkler, nemlig de Vinkler, som Siderne to og to danner med hinanden.

I en retvinklet Trekant er en af Vinklerne 90°, medens de andre Vinkler er spidse. Lad Trekanten være *ABC* (Fig. 40), hvor *C* er den rette Vinkel, medens Vinklerne ved *A* og *B* betegnes med x og y. Tegner man Rektanglet *ACBD*, ses det straks, at x og y udgør Dele af en ret Vinkel, saa at de begge er mindre end 90°; men man ser tillige, at Summen af x og y netop maa udgøre en ret Vinkel; Trekant *ABC* er nemlig kongruent med den givne, saaledes at dens Vinkler ved *A* og *B* bliver henholdsvis y og x, og man har da f. Eks. ved *A* en ret Vinkel bestaaende af de to Dele x og y. Altsaa:

I en retvinklet Trekant er de spidse Vinkler tilsammen 90°.

Fig. 40.

Based on this foundation (ibid., p. 9-23) it is now possible to prove new propositions, like the one about the angle sum of a triangle.

Hjelmslev starts by proving the statement for a right triangle in paragraph 43. As rectangles come in all sizes, triangle *ABC* can fit tightly with its three vertices coinciding with three of the vertices of the rectangle. By definition a rectangle turned 180 degrees will cover itself. Thus triangle ABD is congruent with ABC, and it becomes obvious that $x + y = 90$ and $A+B+C = 180$.

The only possible criticism here is that it is taken as obvious that line segment AB falls on itself, when the rectangle is turned. This claim seems to call on the fundamental axiom of two points determining one and only one line. Certainly an argument for this can be made from Hjelmslevs definition of a straight line as the edge of a normal wedge, but he is – except for a few lines about rulers on page 16 - not explicit about it in *Elementary geometry I*, although he seems to use this axiom of unicity often to cover all the geometry in the syllabus for the Junior Secondary School.

I mention this detail because he in his 1913 plan of action wrote: "We have made no demand that a plane or a line can be extended infinitely. Moreover: We have made no claim that 2 points determine a straight line. And this claim is, taken to its extreme consequences, one of the worst demands ever made in Geometry, because it can lead to serious faults." (Hjelmslev, 1913, p. 46). A probable explanation of this possible conflict can be that Hjelmslev both had a philosophical (ontological and epistemological) and a pedagogical agenda that on certain points differ slightly. But in general they seem to be in agreement, like in this final quote from the 1913 article:

> Some may find that it is too easy to finish Geometry in a hurry when we do it like this, finding such things experimentally that Euclid had to reach by laborious reasoning from his axioms. My first answer to the persons is: "Well then! Why do we need proofs for things that often are more obvious than the axioms that the proofs build on? Let's experiment as far as possible and then use proofs*: There are lots of theorems that need proof*. (Hjelmslev 1913, p. 44)

The progression in Elementary geometry II-IV

Elementary geometry II is dedicated to solid geometry or more precisely stereometry, Euler's theorem, projections, spherical geometry, conic sections and finally a little about coordinates. With *Elementary geometry I-II* Hjelmslev had made his presentation of a natural curriculum in practical elementary geometry. (Hjelmslev, 1919, Preface).

In *Elementary geometry III*, a geometry isomorphic to abstract Euclidean geometry is constructed. To do this the real numbers $\Re$ are introduced as possibly infinite decimal representations in chapter 1, which ends with definitions of number sequences, limits, functions and continuity. In the following chapter on the theory of geometrical measurement, numbers are used to measure objects from Reality Geometry by a process called *fixing*, i.e. choosing a number by applying a measuring tool on the geometrical object. It is emphasized that the feeling, that there is a precise number that should be attached to the object if our tools and senses were perfect, is a false feeling (Hjelmslev, 1921, p. 73). Depending on the field of observation and the tool used there are several equally correct numbers and fixing is choosing one of these. Applying this terminology now let us restate the theorem on the angle sum of a triangle: It is possible to fix the angles such that $A+B+C = 180$.

It is also possible to fix straight lines by using the coordinate system $\Re x \Re$. If for a open line segment *OP*, *O* can be fixed by (0,0) and P can be fixed by (*a*,*b*), then the OP can be fixed by the set of coordinates $(x,y)=(ma,mb)$, where $0< m<1$.

Following this line of thought it is possible in the third chapter to construct the analytical plane with points, lines, distances and other objects from Reality Geometry, but now in an ideal abstract edition isomorphic to classical Euclidean geometry, where the axiom of unicity holds and a circle has only one point in common with its tangent. In Reality Geometry things might look at bit more messy and only becomes nice by suitable fixing. But it is the real observable truth, whereas the analytical geometry is only a model that by suitable fixing can help os find results in Reality Geometry (ibid., p. 180).

This concludes the main development in *Elementary geometry III* but it also treats theoretical construction by ruler and compass, trigonometric functions, analytical conical sections and area measurement.

Elementary geometry IV is 98% ordinary calculus but Hjelmslev maintains in the preface that it is geometry. It grew historically from geometry and it ends in the reality of geometry: "A theory with the objective of – not substituting things with numbers – but to describe them with the assistance of numbers, must always return to the things themselves. Therefore also this book is a book on Geometry." (Hjelmslev, 1923, Preface) and on the last pages this leads an observation like "In the geometry of reality there are no other converging sequences that those that finally becomes constant", which again dissolves the old paradoxes of Zeno of Elea (ibid., p. 87). And there is a similar observation that the differential dy/dx in reality can be represented by a geometrical ratio $\Delta y/\Delta x$ such that below a certain magnitude of Δx, this ratio can be considered as constant. After these observations Hjelmslev can conclude that "Thus only very special classes of functions have our interest in modelling the reality. We have already mentioned that only functions $f(x)$ that are two times differentiable and with limitations on $f''(x)$ can be considered." (ibid., p. 90). He certainly leaves Peano's Curve out by this choice.

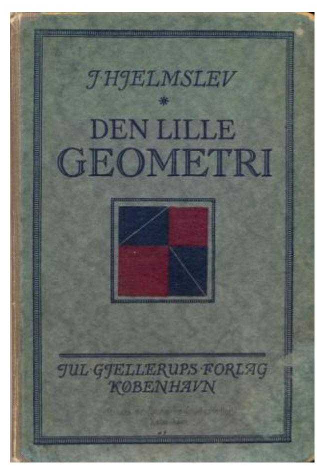
The fate of Reality Geometry in Danish schools

Hjelmslev's Reality Geometry was for three decennia considered to be the only consequent alternative to the textbooks that were faithful to the main core in Euclid's Elements. For the practical use of his book in Junior Secondary School it of course was a problem that it was in four volumes and that even *Elementary geometry I* was ambitious compared to the official standards. That is probably why Hjelmslev in 1926 - in cooperation with a teacher, Vera Andersen - produced a more elementary edition "The small Geometry", finally giving the improved classical geometry books some competition, also with younger students.

In 1926 the later new doctor of mathematics and later professor in geometry Fr. Fabricius-Bjerre (1903-84) wrote a small thesis on the position of mathematics in Secondary School, where he came out in favour of Reality Geometry at least in Junior Secondary School:

> It is the strength of Reality Geometry that it teaches children facts related to real things. Neither the teacher nor the students have to compromise with the truth. Every statement can be controlled and rejected if it does not correspond with actual facts.
>
> However it is the weakness of Reality Geometry that its results do not appear in the same elegant form as the results in abstract geometry. The limited validity of its theorems is a hindrance of formal simplicity. This is a natural consequence of Reality Geometry being a real and not a formal science. (Fabricius-Bjerre 1926, p. 88)

He gives a short presentation of all the possible contemporary textbooks in geometry in Junior Secondary School, and finds that only Eibe's book very faithful to Euclid's Elements, a middle group (Foldberg and Jul. Petersen) are more independent but still inspired by Euclid. Pihl & Rasmussen's book belongs to the same group but he made a book dominated by geometric 'experiments' and exercises. Bonnesen on the other hand is far more dominated by physical reality and Hjelmslev is of course put at the end of the spectre near natural experimental science (ibid., p. 85).



Fabricius-Bjerre himself prefers Reality Geometry because it is built on the foundation of "objective truth", but he realizes that this transition almost to physical science will deter many teachers from Reality Geometry. In this he was in agreement with other reviewers of geometry books (e.g. Andersen (1930)).

In one of the few copies of *Elementary geometry* still existing in Danish libraries a reader has written ‘nonsense’ next to Fig. 34, which illustrates the fact that a tangent shares not only a point but a small segment with its circle. This is probably more than a coincidence and even today mathematics teachers could react like this, because the abstract geometry survived and still survives as a closer to the “Truth” than other geometries like Reality Geometry. In this respect it is interesting that many reviewers of Hjelmslev’s book in fact agreed that his geometry was closer to the “Truth” than abstract Euclidean geometry. Fabricius-Bjerre thought that Euclidean geometry was more beautiful but not the truth. Unfortunately he ended his thesis by writing “*Truth - not truth*, is not the only decisive thing in the school”. This called for an immediate response in *Matematisk Tidsskrift* by the fierce advocate of Reality Geometry, Vilh. A. C. Jensen: “My many years as a teacher have given me the clear conviction that Reality Geometry neither is too difficult nor is considered too simple by the students”. And should it turn out that the modified abstract geometry is easier “I will maintain that if this should be at the expense of the truth, then *easy - not easy* is not the only decisive thing in the school” (Jensen, 1928, p. 6). Jensen later, in 1940, published his own textbook for geometry in Junior Secondary School and it might have been the best textbooks in this tradition with fine reviews and appearing in three editions in the 1940s. But as it can be seen from Table 1 the pioneers of the Experience Method and of Reality Geometry, Bonnesen and Hjelmslev, did not sell well, when we look at the number of editions/impressions of their books compared to the other authors of geometry books for the Junior Secondary School.

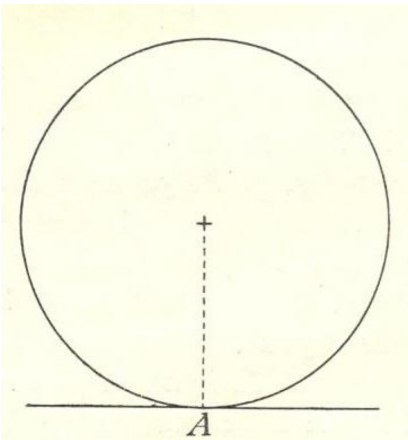


Fig. 34.

However ‘*selling - not selling* is not the only decisive thing’ for an author or educator. Although few teachers did use their books in their teaching, Bonnesen and Hjelmslev are important as frames of references in the development of geometry in Danish schools. For a short while during the start of New Math in the 1960s it is still possible to find references to Reality Geometry, e.g. in the first New Math geometry written by Bent Christiansen, where the distinction between the reality of drawings and abstract geometry is the natural starting point before geometry is absorbed in set theory, relations and functions. In the fifty years since then geometry has become more realistic and applied, but also more eclectic. That is however another story from Postmodern times, while our story was about how late Modern times tried to interpret and develop geometry as a school subject.

Table 1. Danish textbooks in geometry

Textbooks in geometry for Junior Secondary School published (first edition) 1903-1931			
Authors	First edition	Last edition	Number[5] of editions
Bonnesen, T.	1904	1904	1
Foldberg, P.T.	1904	1937	14
Petersen, Julius	1905	1953	7
Eibe, T.	1908	1950	9
Pihl & Rasmussen	1914	1931	4
Hjelmslev, J.	1926	1926	1

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Andersen, Carl Christian (1930). Anmeldelse af "Mellemskolens ny geometri", *Matematisk Tidsskrift A*, 60-62.

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Bonnesen, Tommy (1904). *Geometri for Mellemskolen.* Copenhagen: Gyldendalske Boghandel. (year of publication is not printed in the book)

Bohr, Harald (1964). Johannes Hjelmslev in memoriam, 1873–1950. *Acta Mathematica 83* (1), viii-ix.

Christiansen, Bent (1967). *Geometri for realafdelingen, arbejdshæfte - elevhæfte – lærerhæfte* (3 volumes). Copenhagen: Danmarks Skoleradio.

Eibe, Thyra (1908). *Geometri for Mellemskolen med 530 Opgaver*. Copenhagen: Gyldendal.

Fabricius-Bjerre, Frederik (1927). Matematikkens stilling i den højere skole fra 1850 til vore dage. Kursusarbejde ved prøven i teoretisk pædagogik. *Matematisk Tidsskrift A*, 57-104.

Foldberg, Peder Thomsen (1904). *Geometri for Mellemskolen, med en Samling af 240 opgaver*. Copenhagen: Schubothe (later editions Gyldendal).

[5] This number is found by counting the editions existing in The Royal Library, which receives two copies of all books published in Denmark. As to Julius Petersen (1839-1910) it must be mentioned that he wrote many similar textbooks in geometry 1870-1902. He was however outlived by his system of textbooks for half a century – with different new co-authors.

Hansen, Hans Christian. (2002). *Fra forstandens slibesten til borgerens værktøj. Matematik og regning i folkets skole 1737-1958*. Aalborg: Centre for Educational Development in University Science, University of Aalborg.

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Hjelmslev, Johannes (1919). *Elementær Geometri. Anden Bog*. Copenhagen: Gjellerup.

Hjelmslev, Johannes (1921). *Elementær Geometri. Tredje Bog*. Copenhagen: Gjellerup.

Hjelmslev, Johannes (1923). *Elementær Geometri. Fjerde Bog*. Copenhagen: Gjellerup.

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Jensen, Vilh. A.C. (1927). Virkelighedsgeometri-Abstraktionsgeometri. *Matematisk Tidsskrift A*., 1-10.

Jensen, Vilh. A.C. (1940).*Geometri for mellemskolen*.Copenhagen: Arnold Busk.

Petersen, C. (1908). Min undervisning i Geometri efter Dr. Bonnesens Lærebog for Mellemskolen. *Matematisk Tidsskrift*, 43-46.

Petersen, Julius (1905). *Lærebog i Plangeometri for Mellemskolen*. Copenhagen: Schønberg.

Pihl, Hans Jensen, & Rasmussen, Hans (1914). *Geometriske Øvelser*. Volume I: Geometriens Begyndelsesgrunde, Volume II: Geometri for Mellemskolen. Copenhagen: Gyldendal.

Rousseau, Jean-Jacques (2009). *Émile*. The Project Gutenberg E-Book of Émile, by Jean Jacques Rousseau. Original: Rousseau, Jean-Jacques (1862). *Émile ou de l'éducation.* Amsterdam: Chez Jean Néaulme, Libarire.

Steen, Adolph (1858). Oversigt over Hovedformerne i Rummet som Indledning til Geometrien. Copenhagen.

Supervising and monitoring: How the work of mathematics teachers was checked and assessed in the Soviet Union between the late 1930s and the 1950s

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Abstract

The belief that the work of mathematics teachers, in order to be accountable, must be systematically checked and monitored is becoming increasingly widespread. In the Soviet Union, "supervision and monitoring" (the expression usually employed in descriptions of the work of school administrators) was regarded as the foundation of all pedagogical activity; for this reason, the history of Russian (Soviet) mathematics education offers examples of ways in which such systematic and comprehensive monitoring was implemented. This paper focuses specifically on the assessment of mathematics lessons both within the school and by inspectors from outside. It also considers how educators conceived of the ideal lesson and what deviations from this ideal schema existed. The paper is based mainly on archival materials—transcripts of school meetings and meetings of organizations responsible for monitoring the work of teachers, transcripts of various conferences, as well as all kinds of accounts and reports concerning actual monitoring.

Introduction

The period between the late 1930s and the 1950s is usually seen as quite a prosperous time for the development of Soviet (Russian) mathematics education (Kolyagin, 2001), and indeed certain accomplishments are obvious. A crucial method for shaping education in general and mathematics education in particular during these Stalinist years was seen to consist in the meticulous regulation of teaching activity and the no less meticulous observation and monitoring of this activity: "supervision and monitoring," as this aspect of administrative control was officially designated in the Soviet Union.

The concrete ways in which this supervision and monitoring were implemented constitute an interesting object of study for several reasons. The surviving documents make it possible to clarify certain methodological issues, in particular, what demands were imposed on mathematics lessons, how these demands were met, and how the Russian tradition of conducting mathematics lessons took shape overall. On the other hand, studying the demands and, more broadly, the relations between people in schools of this time makes it possible better to understand the life of society during this period. Finally, without implying that history offers patterns of development that must necessarily be repeated in the future, it nonetheless seems natural today—when

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education.* Caparica: UIED.

increasing attention is being paid to monitoring and assessment, as, for example, in the United States—to turn to the history of how mathematics teachers were monitored and evaluated.

This paper represents a continuation of the studies Karp (2009, 2010).

Who evaluated teachers, how, and how often

Teachers were systematically evaluated in schools. For example, the following facts about a school were presented at a methodological conference (IUU, 1954a):

> 5 to 10 of a teacher's classes are observed in a row and the observations are written down by the principal and vice principal in a separate notebook (50 pages), kept for every teacher individually. Also recorded in these notebooks, in addition to notes about observed lessons, are the results of tests and independent assignments, the results of oral exams administered by the school's principal, assessments of the state of the students' knowledge, assessments of the state of the students' notebooks ... At the end of the inspection, the teacher's work and all of its aspects are discussed in a meeting either with the principal or with the subject committee. Conclusions and suggestions are recorded in the same notebook. After a period of time, an inspection is made to make sure that the suggestions are being carried out. Every teacher is inspected twice a year. At the end of every half-year, an overall evaluation of the teacher's work during the preceding half-year is recorded and then read out at a staff meeting; at the end of the year, these evaluations are used to compile a performance report for the teacher (pp. 49-50).

Consequently, a school's principal and vice principal were required to observe a very great number of classes. Inspectors, for example, specifically noted that the vice principal in one school "observed only 44 classes in two and a half months" (LenGorONO, 1952, p. 98). Mathematics classes also had to be observed by other mathematics teachers, either in the company of administrators or independently, and above all by the head of the mathematics subject commission. The results of such observations were discussed at the commission's meetings, and also from time to time at school staff meetings and party organization meetings.

In addition, teachers were assessed by inspectors from district school boards, and during large-scale school inspections by officials from city school boards as well, and most importantly by special district and city mathematics supervisors. For example, employees of the Leningrad (St. Petersburg) Teachers' Continuing Education Institute (IUU)—that is, mathematics supervisors—reported in 1952 that each of them had observed approximately 30-50 classes over the course of the year, and at the end of the meeting acknowledged that their work had been completely unsatisfactory and insufficient (IUU, 1952). The assessments given by the school board inspector and the subject specialist were not always identical. Already at the dawn of Stalinist schools, one mathematics supervisor noted:

> I look at a test, which happens to be the test of an excellent student, and I find mistakes in it. What will the inspector do? He will take this work, which appears brilliant on the outside, and say: this teacher is outstanding. And he will give the teacher a few more classes to teach (LenGorONO, 1934, p.13).

It was believed that all teachers needed to be monitored and comprehensively assessed, although naturally teachers who had many failing students attracted special attention. The struggle against the failing rate in mathematics (which was quite high) was seen as a problem of prime importance, although manifestations of "rotten liberalism" and grade inflation were also decisively censured (Karp, 2010).

Mathematics at Communist party meetings

Surviving transcripts of Communist party meetings reveal how teachers' work in schools was monitored (the topics discussed at such meetings were by no means limited to ideological questions, but included all current issues—effectively, these meetings mirrored faculty meetings). For example, in November 1953, at Leningrad's School # 189, the state of instruction in mathematics was specifically examined at a party meeting (School #189, 1953).

The reason for heightened attention to the teaching of mathematics lay in the fact that this school had one of the lowest pass rates in the district (mainly, students failed mathematics and the Russian language). The head of the subject committee (mathematics department) reported that the committee was studying the work of teachers who had no left-back students in their classes and also conducting consultations on specific topics and analyzing observed classes (p. 54). Poor preparation in grades 5-7 was cited as the cause for the low pass rate, although the causes of this poor preparation itself were not mentioned. It was reported that "all measures [had been] taken with a view to liquidating poor performance," namely, testing was being implemented, particularly intensive efforts were being made to work with weak pupils, students' notebooks were being regularly reviewed, visual aids were being widely used, and so forth.

The vice principal of the school gave a generally positive assessment of mathematics classes at the school: "demands on students are high, attention is being paid to students' speech and the formatting of materials." She noted the difficulty of the mathematics curriculum, complained about the insufficient work done by students at home, which was explained by lack of monitoring at home, and urged her listeners to take steps to prevent students from copying one another's homework assignments.

During the discussion, participants complained about the lack of continuity in education, named two teachers who lacked adequate education themselves, lamented that weak students were called on too rarely, that teachers observed one another's classes too rarely, and that open classes were too few in number. One of the presenters commented that classroom visits "should be made not only by the school administration, but also by the party organization, the local labor union committee, and the subject committee. Our goal is to establish discipline—that will improve the pass rate" (p. 57).

Note the strident tone of this and similar discussions. For example, the transcripts preserve the following comment from a presenter: "[The teacher] does not work in the way demanded of a Soviet school teacher. Over the past two years [she] has let her work slide to an incredible degree. [She] herself is spreading illiteracy in the classroom. Don't forget that you, as a teacher, have been given the responsibility of teaching children—but your attitude toward your work is one of negligence." Others continued: "The party organization cannot consider such work conscientious. You have broken away from the collective" (p. 10).

As a result of the discussion, the following decisions were made:

1. When tests are administered in grades 6-7, teachers from higher grades must make sure that students have assimilated the necessary materials.
2. In lower grades, a policy of having teachers observe one another's classes must be implemented.
3. The concrete causes of poor performance in mathematics in each student must be identified.
4. Open classes must be made a more widespread practice.
5. The administration must be required, in putting together schedules for the next school year, to preserve continuity of mathematics teachers (p. 59)

(The last objective likely meant that it was recommended, for example, to have a teacher who taught mathematics in sixth grade to teach mathematics in seventh grade as well.)

Several years previously, more detailed methodological techniques aimed at improving student performance in mathematics had been proposed at a party meeting at the same school:

1. Identify all poorly performing students, identify those sections in which students exhibit poor performance.

2. Keep an evaluation notebook for each person in each class. If a teacher knows a student's lacunae, the teacher will see clearly how to correct that student's failings (by giving individual homework assignments, checking them). Determine students' shortcomings by testing them.
3. Take measures to ensure that students remain alert; ask questions more frequently.
4. Send students homework assignments when they are sick and not in school.
5. Hold individual consultations with students (School # 189, 1949, p. 28).

A party meeting at another school (School # 105, 1954) passed resolutions that were pretty similar to those passed in School # 189: assess lessons, assess homework, assess supplementary classes, monitor the work of the subject committee. There were, however, also certain differences: it was decided, on the one hand, to add extra time for mathematics in certain classes and also to start after-school math clubs; and, on the other hand, to make broader use of the Young Pioneer and Komsomol (Communist Youth) organizations in the struggle to improve student performance.

Assessing mathematics classes

Monitoring teachers, as we have already seen, involved a great deal. Supervisors analyzed both test results and how these results were evaluated by the teacher; they examined how and how regularly students' notebooks were evaluated; naturally, they talked about the number of failing students and took note of the number of additional classes that the teacher held for poorly performing students; nor did they overlook the general, "nonmathematical" demands that a teacher had to meet (Karp, 2010). Even so, observing classes was regarded as a crucial means of assessing a teacher's work. The mathematics supervisors who visited classes were often quite critical. For example, at a meeting of the IUU mathematics department, one of the employees expressed the following opinion (IUU, 1952, p. 41):

> Very few teachers have control over their students and involve them in active work. Unsatisfactory lessons reveal teachers' lack of preparedness and unconscientious attitude toward their work. The students know nothing. The work of the teachers displays very little creativity; in the best cases, one finds a conscientious attitude toward work, but the knowledge of the material itself is quite mediocre.

Her colleague agreed with this view:

> There are instances when classes are well organized and well activated, but the necessary knowledge is lacking. Students are questioned poorly. There is no opportunity to locate the mistake made by a student. New material is presented in a lecture format. It is rarely introduced in an analytical manner.

A third colleague was only marginally milder in her judgments:

> Some of the classes observed were quite active, revealing a grasp of the subject. But there were a number of classes that were not even bad on the surface, but revealed no ability to make sense of the materials—there was too much routine memorization. Teachers ... are illiterate in the area of subject knowledge.

At an inspectors' conference (IUU, 1954a), the following typical shortcomings of lessons were listed:

1. Insufficient attention to the conceptual side of the lesson.
2. Inability to plan and coordinate the separate parts of the lesson in time.
3. Insufficient activation of the knowledge assimilation and reinforcement process.
4. Insufficient attention to teaching the students independent work skills.
5. Inadequate attention to repeating what has been covered and to visual models in teaching.
6. Inability correctly to determine students' knowledge through questioning.
7. Lack of attention to students who are falling behind in class.
8. Gap between theory and practice, poor implementation of the principles of polytechnic education in the teaching process (p. 18).

How common such shortcomings were, however, was not reported at the conference. As for official reports, they on the contrary often mentioned constant improvements. The following passage, for example, appears in the report for 1949 (IUU, 1949):

> In the first place, we must note significant shifts in the ideology of teaching. In preparing a lesson, teachers give serious thought to the lesson's content, plan, and methodology, orienting themselves toward active techniques, conscious comprehension, the development of logical thinking, the cultivation of attention, perseverance, mental agility (p. 10).

All this, however, did not prevent the report's authors—who noted that lessons were usually constructed in accordance with the schema of (1) checking homework assignments, (2) presenting new knowledge, (3) reinforcing this knowledge, and (4) assigning homework for the next lesson—from concluding that "in most cases, the lessons observed were of average quality" (p. 13).

Specific comments made by those who observed classes are noteworthy. For example, one supervisor, who observed an eighth-grade lesson devoted to operations involving radicals, raised the following objections: the teacher did not check to see that students had completed their homework assignments, did not go over a test that the students had taken previously, gave very simple problems to the students called up to the blackboard, paid no attention to the students' mathematical speech—for example, one student, talking about the division of powers of the same number, said that "when powers of the same number are divided, the base remains the same while the exponents of the powers are subtracted" (which is incorrect, as the observer notes, because such a formulation leaves it unclear what must be subtracted from what). Mathematics supervisors from the Teachers' Continuing Education Institute (IUU, 1954b, p. 7) pointed out such shortcomings as carelessness in the use of mathematical language, mistakes in notation, imprecision in solving trigonometric problems, formal explanations (what this means is not always clear from the documents), etc.

On the other hand, the lessons of a teacher who achieves impressive successes (as evidence for such successes the observers cite the fact that her students regularly win mathematics Olympiads) are described as follows:

> The lesson is constructed in accordance with a plan, all information is conveyed to the students in a thoroughly thought-out and systematic manner. Work in class is highly intensive; everyone is interested. Her students never allow themselves to respond in a chorus. (LenGorONO, 1936, p. 41)

The same document describes a technique used in higher grades by another good teacher:

> Say a student is proving an assigned theorem: neither the teacher nor the students usually interrupt him, except in those cases when a mistake must be corrected in order to continue with the exposition. After the student has completed the proof of the theorem, the teacher asks the class to make corrections to this proof; the students correct the formulation of the theorem, and then the teacher proposes that they make certain additions: the students themselves make the additions. And only after the students have made these corrections and additions does the teacher herself correct and augment the student's answer with her comments. This technique, although it somewhat slows down the pace of the work, is unquestionably beneficial to developing the students' mathematical speech. (p. 47)

The observers sometimes described the lessons they liked in their entirety, as was the case with the following lesson in arithmetic:

> Homework assignments were checked for five minutes. The solution of the assigned problem was checked. To this end, five students were called up to the blackboard; each student formulated the question posed in the problem and explained how he would solve it. This procedure demonstrated that the assignment had been completed by all of the students in the class; they understood it well and explained it willingly, judging by the number of raised hands and the children's flawless responses.
>
> For going over new material, the teacher had prepared notations on the blackboard, leaving only a quarter of the blackboard open, with the remaining three quarters covered by sheets of paper, which were removed as needed; at the end of the presentation of the topic, the whole blackboard was open, which allowed the teacher to review the whole topic in the order in which it had been analyzed in class. (p.42)

As far as can be judged from the description given by the observer, presented on the blackboard in sequence were different series of problems: first, multiplication by integers; next, multiplication by fractions, including mixed fractions, with students being asked to indicate whether the product was less than or greater than its factors; and finally, students had to determine a factor based on the product. The observer notes: "To a greater or lesser extent, the whole class took part in the discussion; then the students worked independently; then they checked their work." Some problems were solved orally. Then a general overview of what had taken place during the lesson was once more given, the homework assignment was explained, and the class even had time to examine a word problem—and "the notations [were] thought-out, the students offered intelligent explanations for every step that they took ... the problem was solved with explanations" (p. 43).

Discussion and conclusion

It is difficult to assess how effective all these measures of control were. Shortcomings and deficiencies, some of which we have listed, were noted by the observers again and again, although sometimes, as we have seen, improvements were also noted. The number of failing students did not noticeably decrease until the onset of a new age, which subsequently came to be known as the "period of stagnation," when not even lip service was any longer paid to the need to fight against liberalism in giving grades, and pass rates rose to almost 100%. The ideal of an intensive lesson, packed with diverse mathematical materials and employing a plethora of pedagogical techniques, which had been promoted by supervisors, indeed came to be shared by many (Karp and Zvavich, 2011); but here again it may be said that this ideal was promoted less in the context of monitoring and observation than through various kinds of teacher education.

It is likely that systematic monitoring led teachers to take what was said in teacher education classes more seriously; and the fact that the pass rate did not fall while assignments became more and more difficult (and they did become more and more difficult—Karp, 2010) must be regarded as an accomplishment. On the other hand, one might give some thought to the price that was paid for such accomplishments, in particular, the frequent public humiliations that teachers often went through.

The improvement of student performance was considered a task of paramount importance for schools, but the monitoring of the work of the teacher was by no means limited to an assessment of the number of failing students or even to the examination of how this number changed over time. First of all, supervisors also paid attention to the number of particularly successful students (for example, winners of Olympiads); and second of all, and most importantly, supervisors looked specifically at how a teacher taught, evaluating the educational process not "in general," but specifically with reference to the subject—comments about errors in the subject, as can be seen from the passages cited above, were very frequent; in fact, this was the primary focus of attention.

Again, as the documents show, not all supervisors were sufficiently qualified to assess the teaching of mathematics. An intensive lesson that combines mathematical depth with pedagogical subtlety and variety can be genuinely evaluated only by those who are at home both in school-level mathematics and in the specific methodology of its teaching. An inspector "in general" could focus only on superficial aspects, sometimes insisting on formal and even meaningless requirements. On the other hand, the specialist mathematics supervisor was by no means always eager to engage in monitoring, particularly in the context of the rigid and often inhumane system that existed at the time. Although at official meetings the employees of the Teachers' Continuing Education Institute did self-critically acknowledge the need to observe more lessons, in reality they clearly preferred to teach teachers through lectures and other types of classes, and not to blame them for poorly conducted lessons. As a result, a problem typical of a system based on monitoring and supervision arose: someone had to monitor those doing the monitoring and force them to monitor the teachers. The system of "supervision and monitoring" threatened to collapse if any step in it grew weak or if the punishment dealt out to those who deviated from its requirements became less rigid or even less cruel (which is what eventually happened).

But even the successes of the system of monitoring, when it first appeared, were made possible only by the fact that, on the one hand, achieving genuine mathematical knowledge was clearly established as a priority—successes in other subjects were far more modest (Karp, 2010); and on the other hand, because definite methodological and human resources had in fact been accumulated—in particular, there was a large number of sufficiently educated observers and supervisors.

It bears repeating, though, that the danger of making things meaningless was always great. Students' exams, for example, rather quickly began to expand to enormous lengths, since the quite reasonable requirement that all mathematical operations be justified led students to "justify" the steps in their solutions in more and more verbose fashion, upon the instruction of teachers who feared that otherwise the supervisors who re-checked their work would remain dissatisfied (Karp, 2010).

The Russian collective memory of education during the period 1930s-1950s remains mixed—on the one hand, it is said that things worked well during those years because teachers' work was monitored and supervised; on the other hand, the figure of the inspector who brought the teacher to tears has also, it appears, not been forgotten. It may be useful for educators in other countries, too, to give some more thought to this experience, particularly in those countries where there is more and more talk of assessment and evaluation of teachers.

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U.S. mathematicians and the new math movement

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Abstract

In conventional accounts, the new math movement in the United States, which lasted from the mid 1950s to the mid 1970s, was initiated and led by prominent research mathematicians largely supported by their colleagues. Recently, revisionists have claimed that the mathematicians were "a small band," "not highly respected," and the "wrong mathematicians." Further, the accomplishments of those mathematicians are said to have been undermined by the "professional education bureaucracy." The mathematics community was indeed divided over the new math. But the fault line dividing U.S. mathematicians was not so much a product of the number, quality, or influence of those leading the movement as it was a reflection of the community's divergent views on the nature of mathematics and how it should be taught and learned.

Introduction

In many countries of the world from the mid 1950s to the mid 1970s, and later in some countries, various attempts to change school mathematics were made under the label "modern mathematics" or "the new math." These efforts arose from various sources and took many forms, but they tended to have in common the aim of bringing school mathematics closer to the mathematics being taught in the university (Kilpatrick, 1997/2009). In some countries, research mathematicians played a leading role; in others, it was groups of teachers, usually secondary teachers; and in still others, it was both. Sometimes, the leadership was in the hands of the ministry of education; at other times, organizations of mathematicians, teachers, or both were involved.

The new math in the United States

In the United States, the new math can be traced to 1951, when a committee of University of Illinois faculty in engineering, mathematics, and education reported that beginning engineering students at the university needed increased preparation in high school mathematics (UICSM Project Staff, 1956/1970). The next year, recognizing that the problem went beyond engineering and extended throughout the state, another committee (the University of Illinois Committee on School Mathematics [UICSM]) with much the same composition but including representatives from the University High School, the laboratory school of the College of Education, began the development of a secondary school mathematics curriculum. The resulting materials, a complete curriculum for Grades 9 through 12 supported by the Carnegie Corporation and the

U.S. Office of Education, were guided by two principles: the language used should be precise and the student should have the opportunity to discover generalizations (Beberman, 1958). The mathematician who was most influential in shaping the UICSM curriculum was Herbert E. Vaughn, a specialist in logic, which helps explain why the materials gave so much attention to set theory and formal logic.

A second major effort began in 1955, when the College Entrance Examination Board (CEEB) appointed a Commission on Mathematics, chaired by Albert W. Tucker of Princeton University, whose charge was to study the mathematics needed by students entering the university. The Board's examiners were concerned about a growing gulf between their examinations and the mathematics being taught in some college preparatory programs as well as about the low levels of mathematical understanding and poor attitudes toward mathematics on the part of many high school graduates (Meder, 1959; Wooton, 1965, p. 8). The commission's influential final report was accompanied by a series of appendices detailing a proposed secondary mathematics curriculum (Commission on Mathematics, 1959a; 1959b).

> The commission recommended the introduction of new topics such as logic, modern algebra, probability, and statistics so that secondary school curricula would better reflect important new facets of pure and applied mathematics. It also proposed the combination of previously separate courses in plane and solid geometry and in advanced algebra and trigonometry so that students could proceed more quickly to the frontiers of mathematics and meet the national need for a sophisticated scientific workforce. (Fey & Graeber, 2003, p. 524)

Two conferences of mathematicians held in early 1958 led to the establishment of the School Mathematics Study Group (SMSG) financed by the National Science Foundation and under the leadership of Edward G. Begle, then of Yale University (Wooton, 1965, pp. 9–16). The SMSG began by writing textbooks for the junior and senior high school grades and then turned its attention to the elementary school grades. It also produced an extensive collection of books and films for teachers as well as monographs for students (Begle, 1971, 1973). Ultimately, the SMSG became the largest and best known of the U.S. mathematics curriculum development projects.

A variety of other projects also involving mathematicians, mathematics educators, and teachers were formed to reform school mathematics. For example,

> the National Council of Teachers of Mathematics set up its own curriculum committee, The Secondary School Curriculum Committee, which came out with its recommendations in an article in the May

> 1959 issue of *The Mathematics Teacher*. Many other groups, such as the Ball State Project, the University of Maryland Mathematics Project, the Minnesota School Science and Mathematics Center, and the Greater Cleveland Mathematics Program, were soon formed and began their work. (Kline, 1973, pp. 16–17).

An account of subsequent projects and activities undertaken in the United States during what might be called the new math era can be found in the report of the National Advisory Committee on Mathematical Education (NACOME, 1975), which recommended "that the term 'new math' be limited in its use to describe the multitude of mathematics education concerns and developments of the period 1955–1975" (p. 137).

Mathematicians involved

The SMSG began when Richard Brauer of Harvard University, the President of the American Mathematical Society (AMS), named a committee of eight mathematicians to carry out the recommendations of the two conferences in 1958 that a writing session be held that summer to prepare a syllabus for a model secondary school mathematics curriculum beginning with the seventh grade and that a series of monographs on mathematical topics for students be prepared and published (Wooton, 1965, p. 11). The committee members were as follows:

A. Adrian Albert, University of Chicago
E. G. Begle, Yale University
Lipman Bers, New York University
Albert E. Meder, Jr., Rutgers University
G. Baley Price, University of Kansas
Henry Van Engen, University of Wisconsin
Raymond L. Wilder, University of Michigan
Samuel S. Wilks, Princeton University

Yale had indicated its willingness to accommodate the project, and since Begle had just stepped down as Secretary of the AMS, a position that had put him in contact with mathematicians across the country, he was an obvious choice to direct the SMSG, which he did throughout its life from 1958 to 1972. One of his first actions was to obtain a grant from the National Science Foundation to support the work of SMSG. After receiving the grant, the committee appointed a 26-member advisory committee for SMSG that was geographically representative and a wide cross-section of the U.S. mathematical community, including "persons from colleges and high schools, both public and private, and from school districts of various types and sizes" (Wooton, 1965, p. 15).

Many U.S. research mathematicians were involved in the work of the SMSG and other projects—far too many to enumerate here. The 27-page appendix to Wooton's (1965) history of the SMSG lists the participants in many SMSG activities, and anyone familiar with 20th-century mathematics in the United States will recognize the many prominent mathematicians in those lists, people who were members of the National Academy of Sciences or officers not only in the AMS and the Mathematical Association of America, but also in the International Commission on Mathematical Instruction.

Some U.S. projects made use of distinguished mathematicians from outside the country. In particular, two so-called second generation projects—the Secondary School Mathematics Curriculum Improvement Study (SSMCIS) and the Comprehensive School Mathematics Program (CSMP) that began in the mid-1960s (Fey & Graeber, 2003, p. 537) involved foreign mathematicians at their conferences and on their writing teams. For example, a listing of the contributors to Courses 1 to 6 of *Unified Modern Mathematics* (SSMCIS, 1973, p. 23) includes Gustave Choquet (France), Ray Cleveland (Canada), Lennart Råde (Sweden), André Revuz (France), Willy Servais (Belgium), and Hans-Georg Steiner (Germany). Råde and Steiner also served on the advisory board for CSMP, and Georges and Frédérique Papy (Belgium) and Arthur Engel (Germany) contributed to the writing of CSMP material (Institute for Mathematics and Computer Science, 2007, pp. 11–18).

Conventional accounts

In his contribution to the report *Change in mathematics education since the late 1950's–Ideas and realisation* (Freudenthal, 1978), James Fey claimed that in the U.S. new math projects, the collaboration of mathematicians, teacher educators, and teachers was an especially notable feature. He saw the collaboration as productive:

> Much of the stimulus for curriculum development came from criticisms by the professional mathematics community, and many fine mathematicians devoted tremendous energy to production of new and better school programs. In many cases this activity led mathematicians and teachers to a new and healthy respect for each other's challenges and abilities. (Fey, 1978, p. 347)

An outsider to mathematics education, the literary critic and social commentator Benjamin DeMott (1962), professor of English at Amherst College, undertook an analysis of the controversies (he called them "the math wars") that were surrounding the U.S. new math reform efforts. He clearly admired the talents of the reformers. In particular, he noted that the mathematicians contributing to the School Mathematics Study Group project

> chose not to isolate themselves in tight graduate school hives; they were willing to expose their studies and their present modes of thought to the eyes and understandings of humbler teachers and humbler students; they have not hesitated to accept responsibilities that other professionals too often cynically delegate to publishers' drummers or third-rate academic minds. Even if they had failed in the task they undertook, they would for these reasons deserve praise and honor rather than abuse. As it is, they have a claim to a tolerably high rank among the exemplary intellectuals of this age. (pp. 166–167)

Although not everyone lauded the work of the research mathematicians who contributed to the new math, no doubt was expressed in contemporary accounts regarding their numbers or their qualifications.

Revisionist accounts

In 1995, two decades after the SMSG ended its work, Ralph Raimi, a professor of mathematics at the University of Rochester, retired and began to turn his attention to the new math. Before his retirement, he had published a few articles on education at the college level, but he had not written on school mathematics. On his Web page, he put a link to drafts for what he called "a history of the new math" (Raimi, 2006). The following is quoted from his proposed introduction to that history:

> The "new math" of the 1950s was the creation of a combination of mathematicians and school-level mathematics teachers, each group giving credibility to the enterprise that neither group alone might have been able to command. Yet by 1958 the mathematicians had assumed the lead.
>
> Thus, for the only time in the history of public education in the United States, mathematicians, as distinguished from professors of mathematics education, and from school teachers of mathematics, and from professors of education, were given the opportunity, via enormous public expenditures, to influence the schools. For ten or fifteen years their efforts were spectacular—even given a name ("The New Math")—and then they were gone, back to their universities and their theorems, leaving the field, after about 1975, to the educators who had been there before, to the colleges of education, and to the professional educational bureaucracies of the States and school districts.
>
> The new directors of school mathematics education were, after 1975, much the same as the old, who had had command of the schools and their curricula before 1958, and were for good reason called, "The

> Professional Education Bureaucracy" or "PEB"[1] by William L. Duren, a mathematician and Dean at Virginia, and prominent in advocacy of the "new math" reforms of the 1960s.

In Raimi's view, then, whatever the mathematicians had accomplished during the new math had been undermined by the education bureaucracy, who presumably had been waiting the whole time to take back the reins of power.

Another view was offered by Wilfried Schmid, professor of mathematics at Harvard University, whose interest in mathematics education arose in 1999 after he was disturbed by the experiences of his second-grade daughter in her mathematics class (Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmid, & Schaar, 2005, p. 1057). Schmid was heavily involved in the drafting of the Massachusetts Mathematics Curriculum Framework in 2000 and later served on the National Mathematics Advisory Panel of the U.S. Department of Education. In an article in the *Harvard Crimson* about recent "math wars," Schmid (2000) offered his own analysis of what had happened during the new math era. It had involved only a few mathematicians:

> The current fight echoes an earlier argument, over the "New Math" of the Sixties and Seventies. Then, as now, the old ways were thought to have failed. A *small band of mathematicians* [emphasis added] proposed shifting the emphasis towards a deeper understanding of mathematical concepts, though on a much more abstract level than today's reformers. Math educators took up the cause, but over time, most mathematicians and parents became unhappy with the results. What had gone wrong? Preoccupied with "understanding", the "New Math" reformers had neglected computational skills.

A more severe finding comes from another outsider to mathematics education, Suzanne Wilson, a University Distinguished Professor at Michigan State University, chair of the Department of Teacher Education, and director of the College of Education's Center for the Scholarship of Teaching. In her book *California Dreaming* (Wilson, 2003), which reports on two decades of efforts to reform school mathematics in California, she explains why the earlier reforms of the new math era had failed:

> Some concerns were voiced by mathematicians. . . . They worried about the mathematical knowledge of some New Math leaders: not everyone was a mathematician, and some of the mathematicians who participated were *not highly respected* [emphasis added]. …

[1] Duren, 1988, p. 404.

> Some mathematicians would argue that the mathematics was inappropriate or that the *wrong mathematicians* [emphasis added] were involved. (pp. 14–16)

Divisions in the community

From the beginning of the new math era in the United States, the community of mathematicians was divided on the direction that reforms were taking. Fey (1978) noted that "by 1973 concern about the character and effectiveness of resulting mathematics programs made Morris Kline's book, *Why Johnny can't add,* a best seller among books on American education" (p. 339). Already in October 1958, however, Kline—in an article based on a presentation to the National Council of Teachers of Mathematics (NCTM)—was arguing against what he understood to be the position of the CEEB Commission on Mathematics; namely, that traditional mathematics was outmoded and that the abstract should be taught before the concrete. In a reply to Kline, Albert Meder (1958), who served on the Commission on Mathematics, the AMS Committee of Eight that set up SMSG, and the first SMSG Advisory Committee, argued that Kline had misinterpreted the position of the Commission and should wait until the final report appeared before attacking it.

In 1962, a memorandum "On the mathematics curriculum of the High School" was published that was authored by Lipman Bers and Morris Kline of New York University and George Pólya and Max Schiffer of Stanford University (Roberts, 2004). It was signed by them and an additional 61 U.S. and Canadian mathematicians. The memorandum reiterated many of the points that Kline had made in earlier articles, arguing that the curriculum should cater to the needs of all students and not just those who might become mathematicians, that students should not be introduced to abstractions prematurely, that they should see the links between mathematics and the other sciences, that intuitions and conjectures should come before formal proof, that wherever possible mathematical ideas should be introduced as they had arisen genetically, and that traditional school mathematics should not be entirely replaced by so-called modern mathematics.

Although SMSG was not mentioned by name in the memorandum, it was clearly the target. Responding to the memorandum, Begle (1962a, 1962b) essentially agreed with the guidelines it proposed for the high school mathematics curriculum, claiming that the SMSG textbooks largely reflected the spirit of those guidelines and asking the authors for specific suggestions and criticisms. Many of the signers of the memorandum were applied mathematicians, and a few—such as Bers, Pólya, R. Creighton Buck of the University of Wisconsin, and Henry Pollak of Bell Laboratories—had been involved in one or more SMSG activities. Pollak has indicated (Roberts, 2004, p. 1063; personal communication, September 9, 2011) that he signed the memorandum to indicate that he thought the SMSG was following the guidelines and, in fact, that the SMSG materials would prepare students in and for something broader than the usual applied mathematics—what might be called *applicable* mathematics.

It is important to note that despite the clear divisions in the U.S. community of mathematicians, the participants were unfailingly polite—at least in public—and did not disparage one another. In the 1958 exchange between Kline and Meder, Kline refers to the "high-mindedness, liberality, and true leadership" (p. 418) of Howard Fehr, of the Commission on Mathematics, in inviting him to address the NCTM, and Meder thanks Kline and the journal editors for their courtesy in giving him an opportunity to reply "and the evidence it constitutes that Professor Kline desires the debate to be conducted on a high level of intellectual integrity" (p. 428). In the 1962 exchange concerning the memorandum, Bers, Kline, Pólya, and Schiffer end by wishing success to those working on the new curriculum, and Begle begins his reply as follows:

> I am sure that the large number of individuals who, through SMSG, have worked to strengthen the school mathematics program will be delighted that such distinguished mathematicians as Professors Bers, Kline, Polya, and Schiffer have stated guidelines for curriculum improvement so much in harmony with their own views. (1962a, p. 425; 1962b, p. 195)

Mathematicians and school mathematics

It seems reasonable to expect that mathematicians will differ in their opinions about what school mathematics should be. Agnieszka Wojciechowska (1989) argued that there can be no canon of school mathematics:

> Mathematics develops constantly and it often happens that changes take place rapidly, so that it is difficult for the experts to come to a unanimous opinion on what is the most important part of contemporary mathematics, and what part of it should be introduced to common education. Educators are doomed to some second-hand opinions following mathematicians, especially well-known figures in mathematics. And, usually, a working mathematician is open to the conviction that the centre of this science lies somewhere near his own specialism. (p. 154)

School mathematics will inevitably be torn between those mathematicians who argue that because no one can predict what mathematics students will need, they should be equipped with the academic skills and understanding that will allow them to learn the mathematics they need when they need it, and those who argue that students learn mathematics best when they are using it to model situations and solve practical problems. A reasonable path would appear to lie somewhere in between and will always be shifting.

The many U.S. mathematicians who worked on new math curriculum development projects may have been too optimistic regarding the average student's attraction to formalism and rigor in school mathematics and the average teacher's ability to take on a radically different program, but those who objected to that approach were never able to devise a coherent alternative based on realistic applications of mathematics. Whatever one's view of the new math movement in the United States, one needs to recognize, as Robert B. Davis (2003, pp. 623–627) pointed out, that it took many diverse and even irreconcilable forms, affected relatively few schools, and was sometimes quite successful.

Such recognition does not appear in recent discussions of the involvement of U.S. mathematicians in reform efforts. The revisionist histories cited above are essentially unanimous in branding the new math a failure while simultaneously lauding the efforts of research mathematicians over the past two or three decades to improve school mathematics. Anthony Ralston (2004) is an exception. He characterizes many of those recent efforts as arrogant, uninformed, petty, and ultimately counterproductive. He contrasts what he considers the knowledgeable and restrained approach of the mathematicians who signed the 1962 memorandum ("On the mathematics curriculum") with what he sees as the uninformed, political endorsement by several hundred research mathematicians of an open letter to the U.S. Secretary of Education published as an advertisement in the *Washington Post* in 1999 (Ralston, 2004, p. 405). He concludes,

> The research mathematics community, through its hubris, has by and large contributed—and continues to contribute—to a worsening situation in school mathematics in the U.S., a situation that shows no signs of getting much better in the foreseeable future. The lesson of the New Math has not been learned. (p. 410)

The revisionist views of how research mathematicians in the United States contributed, or failed to contribute, to the new math movement half a century ago undoubtedly arise from many sources. Some of those contentious views may be prompted by the general decline in the civility of public discourse in the country over the last generation. At least a few, however, may well arise from a defensive response to the hypothetical question, "If research mathematicians were responsible for the failed new math, why should we listen to your proposals today?" The response:

"Only a few research mathematicians were involved, they weren't prominent or well respected, and anyway the professional education bureaucracy was in charge." In the revised history of the new math movement, we apparently stand on the shoulders not of giants but of midgets.

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Mathematics education for poor orphans in the Dutch Republic, 1754-1810

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Abstract

A millionaire in the 18th century Dutch Republic, the baroness of Renswoude, initiated an unusual educational experiment. She bequeathed her capital to three orphanages in different towns, under condition that they should select talented orphaned boys to educate them in a technical or artistic profession. Mathematics was the core subject of this education. Each orphanage established a Foundation, the Fundatie van Renswoude. The three Foundations formulated General Regulations as a common basis. In this paper the formal curriculum in Utrecht, its implementation and results are described. Though the starting conditions in Utrecht were poor, the Foundation was successful; quite a few students acquired good positions in their professions. Some curricular aspects which seem to have contributed to the success of the Foundation are discussed.

Introduction

In the past ten years in the Netherlands the debate on the content of new mathematics curricula for secondary education has at times been very emotional. During the process some decisions were taken of which the rationale was not always clear. The question arose in which way mathematics curricula were determined in the past. Which factors and which people influenced the design and implementation of new mathematics curricula? Are there lessons to be learned from those procedures? These questions were the starting point for a research into the history of Dutch mathematics curricula.

Several researchers on the history of mathematics education mention one or more important influences on the mathematics curriculum in a particular country during a specific period. Bjarnadóttir (2009) describes the influence of a single teacher on the curriculum in Iceland. Schubring (2010) draws attention to the positive effect of coherence through consultation between groups of participants (teachers, school directors, representatives of government). Prytz (2009) discusses the influence on the curriculum of the debate of a group of educational professionals with a PhD in mathematics. External influences, such as from international conferences, may be noticeable (Giacardi, 2009; Matos, 2009). Ideas about the learning of mathematics and/or about the aims of mathematics education are mentioned (Hansen, 2009; Kilpatrick, 2009; Krüger, 2010). However, when analysing the development and implementation of a curriculum, not just one or two but several factors and persons appear to influence the intended and the implemented curriculum. It might be

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

possible to find factors which repeatedly were an important influence on mathematics curricula in a country throughout different periods.

Research question and outline of research

This paper forms a part of a research which describes and analyses mathematics curricula in Dutch education from different centuries, starting in the early 17th century. However the present paper is restricted to a curriculum in the 18th century. So the research question for this paper is

Which factors and actors influenced the mathematics curriculum of the 'Fundatie van Renswoude' in the 18th century Dutch Republic?

The research discussed in this paper concerns the curriculum of the *Fundatie van Renswoude* in Utrecht from 1756 - 1810. This Foundation was established in 1756, with the aim to educate poor orphaned boys in a technical or artistic profession. In 1810 the Dutch Republic was taken over by the French administration and the financial and social circumstances changed dramatically. During the 18th and early 19th century mathematical subjects formed the core of the institute's curriculum. The archives in the town of Utrecht possess many documents of the Foundation. To structure the information the domains described by Goodlad, Klein & Tye (1979) and Van den Akker (2003) are used. These curriculum domains are:

- the *intended* curriculum, the ideal of persons who initiated the curriculum and the translation of the ideal into the formal curriculum;
- the *implemented* curriculum, interpretation and implementation of the formal curriculum, by teachers and through teaching materials;
- the *attained* curriculum, success (or lack of success) in relation to the students.

When researching influences on the content of mathematics curricula one should realize that usually the mathematics curriculum is part of a more extensive curriculum, depending on the aims of the educational institution involved. In the 18th century in the Dutch Republic mathematics curricula outside universities were usually part of vocational training or professional education (Alberts, Atzema & Van Maanen, 1999; Beckers, 2005; Boekholt & Booy, 1987; Smid, 1997).

Mathematics and mathematics education in the Dutch Republic

During the 18th century there were major developments in mathematics in Europe, especially in calculus, analysis, mechanics and probability theory. There was a fruitful interaction between mathematical development and application of mathematics to practical situations (Alberts, Atzema & Van Maanen, 1999; Grattan-Guinness, 2000). Subjects which at present belong to physics, such as optics and mechanics, were considered to be part of mathematics. In the Dutch Republic professors in Mathematics and Philosophy of Nature, Willem Jacob van 's Gravesande, Pieter van Musschenbroek and Johan Lulofs, emphasized application of mathematics to practical situations such as water management, engineering and navigation techniques (Israel, 2008; Siegenbeek van Heukelom-Lamme, 1941). The publication of an increasing number of journals and of books on mathematical subjects and the emergence of scientific societies in the second half of the 18th century were important for the dispersal of mathematical developments and technological innovations. They facilitated exchange between academic mathematicians and mathematical practitioners (Van Maanen, 2006). Societies such as the *Hollandsche Maatschappij der Wetenschappen* (Dutch Society of Sciences, HMW) and the *Wiskundig Genootschap* (Mathematical Society, WG) published treatments on mathematical and scientific subjects which were of practical importance to Dutch society and its economy. Some of these societies were active in education, either in science and mathematics (*Mathesis Scientiarum Genitrix*) or in general education (*Maatschappij tot Nut van 't Algemeen*). Religion remained an important factor in the Dutch society, but there was not necessarily a contradiction between religion and believe in progress through mathematics and science. God's work as revealed by science, with mathematics as the basis for science, was a form of Enlightenment which was very popular in the Dutch Republic (Jacob, 2001). However, though mathematics and science were considered very important for professions such as navigation, architecture, the technical weapons, shipbuilding and water management, there was a serious shortage of mathematically trained professionals. The main reason for this was the absence of national or regional institutions for mathematical education. Nearly all school education was a matter of local councils and private enterprise. Local councils provided for primary education and also for Latin schools (grammar schools), which prepared students for university. It was mainly left to individuals to offer mathematical training for any profession (Beckers, 2005; Boekholt & Booy, 1987; Dodde, 1991). At the same time the ideas of Enlightenment on the value of education, also for poor people, became more widespread among the Dutch. As a consequence initiatives of wealthy individuals, of learned societies and of local government became more common during the 18th century (Dodde, 1991; Muller & Zandvliet, 1987; Roberts, 2010, Smid, 1997). An early and extraordinary example of such an initiative is the legacy which formed the start of the *Fundaties van Renswoude*.

The Fundaties van Renswoude

Fig. 1. Maria Duyst van Voorhout, ca. 1686

One of the wealthiest women in the Dutch republic in the first half of the 18th century was Maria Duyst van Voorhout, baroness of Renswoude (1662 – 1754). She and her husband, Frederik Adriaan, baron of Renswoude, belonged to the political and social elite of the country (Fig. 1). They were well read and both were interested in science. She was evidently convinced of the value of education, also for the poor; she liberally financed instruction for poor children, of which the numbers grew rapidly during the 18th century (HUA 754, inv. 20.12, 21.4, 22.5). She survived her husband by 16 years and there were no surviving descendants (Gaemers, 2004; Langenbach, 1991).

The intended curriculum: the ideal

On 11 March 1749 Maria Duyst van Voorhout signed her testament in which she left her capital to three orphanages, in the towns of Delft, Den Haag and Utrecht, under condition that the most promising boys should be selected to receive an education in technical or artistic professions. Apart from her ideas about the value of education for the poor and the importance of mathematics, she may have had additional reasons for her unusual testament. The sometimes devastating floods which had been taking place from about 1726 onwards, also in the region where she lived, must have made an impression on her as well. In her testament she stipulated that "*The income shall be used to select some of the most talented and suitable boys, at least 15 years old, to set them apart from the orphanage in order to teach them Mathematics, Drawing or Painting Art, Sculpture or Stone Cutting, practices in building dykes to protect our Country against floods or similar Liberal Arts….*"

Mathematics, but also (technical) drawing, were indispensable competencies for technical professions, such as water management, surveying and architecture. Maria Duyst van Voorhout stipulated that the selected boys should be taught separate from the other orphans. She also required separation of her capital from that of the orphanages involved, financial accountability and regular consultation between the governors of the three institutes and her executors (HUA 771, inv. 1).

The intended curriculum: the formal curriculum

Maria Duyst van Voorhout died on 26 April 1754. There were some legal problems concerning the execution of her will, but within two years, on 5 April 1756, it was clear that each of the three orphanages received about half a million Dutch guilders, an enormous capital by the standards of that time (HUA 771, inv. 3). At each orphanage a foundation was created, the *Fundatie van Renswoude*, hereafter called the Foundation. Between May 1754 and April 1756 the executors of the testament had worked with the regents of the three orphanages to formulate General Regulations, which would ensure that all three Foundations fulfilled the wishes of the testatrix in a similar way (HUA 771, inv. 5). The final text of these General Regulations was signed by representatives of the regents of the three orphanages and the three executors on 17 May 1756 (Fig. 2). Among other things the regents of the three orphanages agreed on practical curriculum aspects, such as

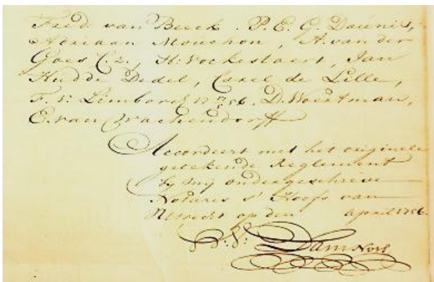

Fig. 2. Signatures General Regulations

- learning environment and structure of the course: theory in the first years, followed by an increasing amount of practical training;
- aims: training for painter, sculptor, stonemason, builder of mills, locks, sluices and dykes, shipbuilder, architect (civilian and military), navigator, instrument maker (mechanical, astronomical and clockworks), chirurgeon[1] and such others as “the Governors will deem useful”
- subjects to teach: reading, writing, arithmetic, drawing, principles of mathematics and if necessary French and English language;
- accountability: of governors, teachers and Foundation personnel:
- assessment: admission test and regular attainment tests;
- harmonisation through consultation between the three Foundations, at least once a year during a meeting of regents and executors.

[1] A chirurgeon (Dutch: chirurgijn) in the 17^{th} and 18^{th} century gave practical medical care, originally all care which involved drawing blood. Chirurgeons usually were not university educated and thus had a lower status than a medical doctor with a university degree. They formed guilds, through which they trained their own apprentices. The word ‘surgeon’ is derived from chirurgeon. .

The Foundations in Delft en Den Haag took in the first students in 1756, but in Utrecht the start was five years later, due to insufficient basic skills of all children in the orphanage and to lack of suitable housing. The orphanage in Utrecht took in orphaned children who were refused by other local orphanages and children whose parents could not take care of them or who were not allowed to keep their children. It was overcrowded and had very limited funding. In 1756 out of 47 boys in the right age group for admission into the Foundation only 12 could read and write a little, the others couldn't. Learning arithmetic followed on learning to read and write, so none of these boys showed any proficiency in arithmetic. The regents in Utrecht appointed a schoolmaster to teach the boys in the orphanage reading, writing and arithmetic (HUA 771, inv. 8, inv. 37). In Den Haag and Delft the orphanages had sufficient room to separate the first Foundation students from the other orphans (Gaemers, 2004). In Utrecht however the orphanage was already overcrowded, so the regents started the construction of a new, spacious house, next to the orphanage (Fig. 3).

Fig. 3. Foundation house Utrecht, 1789

The General Regulations formed the basis for other documents of the formal curriculum: the Instruction for the mathematics teacher and regulations for other personnel. In the Instruction of 1761 some specific mathematical topics were mentioned: "*General Mathematics, Military and Civilian Architecture, Surveying, Etc.*" (HUA 771, inv. 8, inv. 37). The content of those topics, the method of instruction, the order of topics and similar matters were not specified, that was left to the expert: the 'mathematician' as the mathematics teacher also was called. Other duties of the mathematics teacher, mentioned in the Instruction, were advising the regents on which students to admit and on the choice of profession for each student. He also had to examine each student twice a year in the presence of the regents.

The decisions on admission and dismissal of students had to be approved by the executors. The executors also were present once a year at the so-called 'grand examination' when the more advanced students showed their competency in mathematical subjects. This grand exam took place during the yearly meeting of regents of all three Foundations and the executors (HUA 771, inv. 8, 9). Usually the mathematicians of the Foundations in Delft, Den Haag and Utrecht were present as well, so this was an excellent opportunity to exchange experiences and ideas. Especially the mathematician of Delft and Utrecht exchanged materials and information during the first period (HUA 771, inv. 8).

The mathematics teacher

The mathematics teacher was an influential actor in the implementation of the intended curriculum. He had, as instructor of the main subject, many responsibilities, so the regents took great care to find a mathematician with an excellent reputation in his field. They consulted their colleagues in Delft, where a young but very able and enthusiastic graduate from Leiden University had excellent results with his students and they inquired within their network (Booy & Engel, 1987). There was one person whose name came up most often and who was praised highly (HUA 771, inv. 8). That was Laurens Praalder (1711 – 1793), a very knowledgeable, versatile and respected mathematical practitioner, though not university educated. He started his career as a teacher of mathematics in a village in the Western part of the Republic, an area with many ports and also land reclamation. Praalder acquired a good reputation, based on his knowledge of general mathematics, of mathematics in surveying, both theory and practice, and of mathematics for navigation (Graafhuis, 1961; Langenbach, 1996). He was a good example of mathematical practitioners who were instrumental in the dispersal of mathematics and in the interaction with mathematical developers (Van Maanen, 2006). In 1751 Praalder was appointed as mathematician and examiner at the Naval School in Rotterdam. In 1755 he became a member of the scientific society HMW. In 1761 he was asked for the position of mathematician of the Foundation in Utrecht. In 1773 he was instrumental in setting up a new scientific society, which still exists today, the *Provinciaal Utrechts Genootschap* (Provincial Utrecht Society, PUG).

Fig. 4. Map of Utrecht, by students of the Foundation, 1778

Implementation of the curriculum in Utrecht

For more than 30 years, from September 1761 until December 1792 Laurens Praalder taught the students of the Foundation in Utrecht the theory of mathematical sciences and some practical applications. He was a very central figure in the implementation of the curriculum. During his years as the mathematician of the Foundation he not only taught theory of mathematical subjects, he also performed modern physical experiments, did practical work with all students and practised surveying techniques with more advanced students (Fig. 4). He used his extensive network to find apprenticeships for students and in general to promote their interests. He advised the regents on which books and instruments to acquire. He weekly attended the lessons in the orphanage in order to discuss with the schoolmaster which boys might be suitable for education in the Foundation (HUA 771, inv. 8- 12, inv. 37, inv. 38).

The course was structured in three phases (Gaemers, 2004).

1. Theory and practical exercises in mathematical subjects, drawing, French language, religion etc. Exams took place twice a year. The duration of this phase usually was two years.
2. Start of practical work as apprentice, continuation of theory, some specialisation in subjects. The duration depended on the chosen profession.
3. Final years of apprenticeship, often out of town, sometimes abroad. The learning of theory continued, but depended on the profession. The duration varied, depending on when the student acquired a decent position in his profession.

TheGeneral Regulations allowed for 25 to 30 students at a time, but in reality there were never more than 15 students in Utrecht, see Fig. 5 (HUA 771, inv. 63). It was the same in the other two Foundations (Gaemers, 2004). An important reason for limitation of the number of students was the lack of suitable boys in the orphanage in some years. Another major reason was the cost of this type of education, which was much higher than the Governors had estimated beforehand.

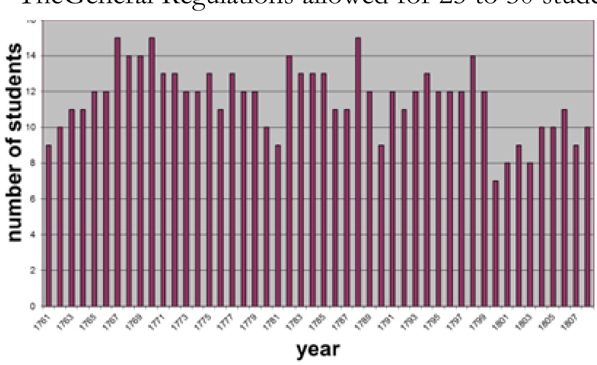

Fig. 5. Number of students 1761-1808

During the first phase teaching might have taken place in small groups, but overall the teaching was individual, though the students studied in the same room and could help each other. There were lessons on six days of the week; the remaining hours of those days were spent on homework. However the Foundation offered much more than lessons in school subjects, the students learned correct manners, reading newspapers, conducting a civilised conversation, writing letters, etc. In short, they learned to conduct themselves properly at the level of the middle class in the Dutch society. That was useful for their future career, competency in mathematics was not sufficient, they needed language skills and social skills as well.

In Utrecht the first nine students entered the Foundation home in October 1761(HUA 771, inv. 8). During the following years, until 1810, the number of students admitted each year varied between zero and five. In phase one the students had about 32 hours of lessons, of which 20 hours on mathematical subjects, 8 hours drawing and the remaining hours were dedicated to religion, geography, history and writing. The remainder of the time outside meals was spent on homework and practical exercises (Gaemers, 2004). The mathematical subjects for all students in phase one were arithmetic, plane geometry, solid geometry, algebra and basic principles of surveying. The majority of the students went on to study trigonometry, surveying theory and some practice of surveying. In phase two and three they might study more algebra and other specialized subjects, depending on their future profession (HUA 771, inv. 8 – 13, 82 – 101, 105 – 152). For some professions, such as chirurgeon, no more mathematical knowledge was necessary after phase one.

Materials for learning

The students had various learning materials at their disposal: their own notes of the teacher's lessons, books for personal use, books available in the study rooms, mathematical and physical instruments, models made by students and maps, globes, etc. The collection of manuscripts of the Mathematical Society in the University Library of Amsterdam contains notes of a student, Jan Mentz, on lessons in trigonometry and surveying (Fig. 6). It gives some idea of the relative high quality of the mathematics taught (UBA IV-H-4). Compared to books on the same subjects which were available at the time, the treatment in these notes is more general, there is more attention

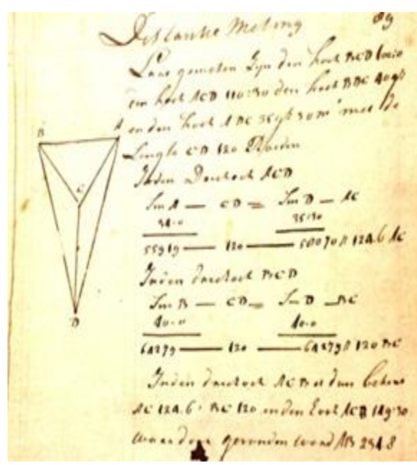

Fig. 6. A student's notes on surveying

to mathematical reasoning. For geometry each student received Euclid (parts I to VI, XI and XII) in the edition of Warius, an often reprinted publication in the Dutch Republic. For algebra the mathematics teacher and governors tried different authors, but after some years chose a simple book, *Algebra of Stelkonst*, by Venema; this contained a lot of exercises to practice skills. For surveying students used a book on surveying practice, *Werkdadige Meetkonst* by Morgenster & Knoop, which at that time was highly appreciated by the practitioners of mathematics (Van Maanen, 2006). For navigation students used mainly a well-known book: De Vries. *Schatkamer ofte Konst der Stuurlieden (*HUA 771, inv. 8, inv. 37, inv. 75, inv. 76).

There was opportunity for differentiation as well, the students had the possibility and probably were encouraged to try different authors on the same subject. For example on algebra, apart from Venema, the book which every student used, titles from different authors were bought, in Dutch translation and usually more than one copy. Some examples of books which are mentioned in the accounts and / or inventories are


DE
ALGEBRA
GEMAKKELYK GEMAAKT:
OF DE
BEGINSELEN
DER
STELKUNDE
NATHANAEL HAMMOND
AANHANGSEL,
ISAAK NEWTON
Te AMSTERDAM,
By ISAAK TIRION,
MDCCLVI

Fig. 7. Title page of N. Hammond *Algebra*

Clairaut, *Gronden der Algebra (Principles of Algebra);* Euler, *Algebra;* De Graaf, *Inleiding tot de Algebra (Introduction to Algebra);* 's Gravesande, *Algebra;* Halke, *Over de Algebra* and Hammond, *Over de Algebra* (The elements of Algebra), Fig. 7. There were many books available on other mathematical and physics subjects, such as optics by Smith, perspective by Philips, military science by Belidor and subjects such as geography. In the year accounts (1761 – 1810) at least 110 different titles are mentioned, for some years with the name of the student for which they were bought (HUA 771, inv. 75, inv. 76, inv. 82 – 101, inv. 105 – 152). In the accounts books are indicated by author or some words of the title or author and subject. Usually the full title and author were determined with help of the catalogue of Bierens de Haan (1883, 1965). Advanced students used books in French, German or English language as is evident from their own inventories or the accounts. All students had French lessons; there were lessons in English, German or Latin language for those who needed these languages.

Laurens Praalder could choose the books he wanted for his students and many of those books were written with the purpose of self-tuition, as is evident from the style and the text in the preface. However he still wrote his own teaching texts, for all the subjects in the first and some in the second phase. After his retirement the regents paid for the copying of 19 different texts, in order to make them available to the successor of Praalder. In the resolutions of the regents are mentioned: four texts on algebra, eight texts on Euclid (book I to VI, XI and XII), one text on trigonometry (Fig. 6), one on mechanics, one on wine-gauging, one on sundials, one text on transformation of figures and two texts on mathematics, the topics of those two are not specified (HUA 771, inv. 13, inv. 38, inv. 91.

The attained curriculum: students and their results

The archives of the Foundation in Utrecht contain lists of all students admitted, with information on age, specialization, etc. See Table 1 and 2.

Table 1. Statistics on students, 1761 - 1808

Number of students	71
Age on admission	12 - 18
Average age on admission	15
Length of stay, in years	1 – 17
Average length of stay	8,2

Though the age on admission of students varied, over the years the average age at which they entered the Foundation turned out to be 15, as the baroness had intended (Table 1). Of the 71 students admitted during this period, the first one to finish his education was Jacobus van der Meer, who became wine gauger of the city of Utrecht in 1767. The last one in this group was Willem Brouwer, who left the Foundation in 1814 with mathematics as his profession; he became an army engineer. This is fairly exemplary of some of the changes which took place during the period researched; wine gauger disappeared as a chosen profession, perhaps because there was not much employment opportunity. From about 1780 a career as an engineer in the army became more common, also for students who had studied for mathematician. The majority of the students in the Foundation completed their education, in Utrecht 52 (73%). Table 2 gives information on the specialization of these students during 1761 – 1808 (HUA 771, inv. 63).

Table 2. Professional education of students

Chirurgeon /medical doctor	10	Instrument maker	2
Stonemason	1	Wine gauger	2
Art painter	2	Navigator	4
Engraver	4	Carpenter/builder of mills and locks	7
Sculptor	2	Water management	7
Organ builder	1	Army engineer	3
Watchmaker	4	Mathematician	3

Mathematician was the most general option; such a student might become a schoolmaster, an engineer in the army, a wine gauger, a surveyor or a combination of these and similar occupations.

Of the students who did not complete their education, seven died while in the care of the Foundation, four were sent away, and eight left for other reasons. Students of this last group often managed to earn a decent living in some technical or semi-medical profession. Many of the students did quite well in life. For instance the first student to become a military engineer, Dirk Kuijper, entered the Foundation in 1779, when he was 13. He was in the orphanage since his third year (Langenbach, 1991). As he was the first to enter the army to become an engineer, he experienced some problems with the regents, who were not accustomed to a frequent change of abode of their students. His superiors, such as Carel Diederik du Moulin, thought well of his knowledge and abilities and advised him on which books to study. The Foundation always paid for the books. In 1789 he became junior teacher at one of the first Dutch Military Academies, in Breda; in 1795 he was appointed director of the Military Academy in Groningen (HUA 771, inv. 12, inv. 38).

Quite a few students went into water management (Table 2). Jan Wormerus Raven, in the Foundation from 1769 – 1778, became inspector-general of water management and mayor of Sas van Gent (HUA 771, inv. 9 - 11, inv. 63). Nearly 30 years later, in different political and social circumstances, Dirk Mentz, who was a Foundation student from 1798 – 1808, had a similar career, he became chief-inspector of water management (HUA 771, inv.13, inv. 39, NL-HaNA 2.16.06). There are many more examples of successful students, which will be discussed in a future publication.

Similar curricula

Later in the century there were similar initiatives. In 1786 at the Burgerweeshuis (Orphanage for Citizens) in Rotterdam a Mathematical College was established. Boys were taught surveying, navigation and mechanics. The mathematics teacher was a well-known mathematician, Jacob Florijn (1751 – 1818). He also was mathematician and examiner at the Naval School in Rotterdam, as Laurens Praalder had been earlier. One of his students at the Mathematical College was Andrew Munro, who would become architect of the town of Rotterdam. Though the College offered a sound mathematical education, had good teachers and its students were successful, it was closed in 1797, due to financial problems at the orphanage (Muller & Zandvliet, 1987; Pouls, 1997).

Also in 1786, in Leiden, a mathematical society with the motto "*Mathesis Scientiarum Genitrix*" offered to teach a small number of orphans of the Roman Catholic orphanage mathematics and science. The subjects on the timetable were very similar to those taught in the Foundation. It seems that this enterprise was less successful than the Foundation concerning the career of orphans. The educational programme was restricted to lessons, at first by members of the Society and by students from Leiden University, from 1802 on by teachers (Roberts, 2010). There was no social and cultural education combined with the school lessons, as was the case in the Foundation. There are no known cases of students who did well in their career after leaving the orphanage.

Institutes which might have a similar curriculum were some of the so-called 'French schools'. These private schools offered several subjects, sometimes mathematical, depending on the skills of the teacher-owner; a relatively high fee had to be paid. The quality of the schools was variable, as was the duration of their existence (Boekholt & Booy, 1987; Dodde, 1991, Smid, 1997).

The first Dutch Military Schools were established in 1789, with a mathematics curriculum that somewhat resembled that of the Foundations (Janssen, 1989).

Discussion and conclusion

Maria Duyst van Voorhout was of the opinion that there was potential for professional development through education among the poorer part of the population and that mathematical sciences were extremely important for the future of the country. She combined these ideas, formulated goals and conditions and provided financial means to fulfil some of her ideal. As a consequence three Foundations were established in order to provide a professional education for selected boys from the orphanage, with the mathematics curriculum as the main constituent of the institute's curriculum. Was this enterprise successful? Which factors and actors contributed to the success or failure?

Success in this case means that during a longer period boys from the orphanage were selected, received an education for technical or art professions, secured a good position in society in those professions and in this way contributed to the welfare and defence of the Dutch nation. In Utrecht the starting conditions were not good, but 71 boys were admitted from 1761 – 1810. The other two Foundations had only slightly more students, though they started five years earlier (Gaemers, 2004). In Utrecht seven students died, of the 64 students who survived 93% managed to acquire a position in their field. Some were very successful, others were in a middle position, but overall they did much better than if they had remained in the orphanage. As mentioned earlier, attempts to offer a mathematics curriculum to orphans in Leiden and Rotterdam were less successful. These initiatives as well as comments from outsiders such as the Director-general of the Technical Weapons, Du Moulin, are an indication that the curriculum of the Foundations served as an example of high quality professional education. Which factors may have contributed to the success of curriculum? A few factors which seem to have been important and are mentioned by other researchers are discussed briefly.

- The mathematics teacher: his qualities and his role
- Goals, aims, content and teaching methods
- Harmonisation, coherence

The first *mathematics teacher*, Laurens Praalder, was chosen carefully, based on his reputation as a mathematician with excellent knowledge of mathematics and its applications and his reputation as a teacher. He showed himself a very good teacher with an open mind for new developments in his field and with an extensive network which he used to promote the interests of the students. He was appreciated in the community of mathematical practitioners and had good contacts with university mathematicians in Utrecht. This is a remarkable difference with the situation described by Bjarnadóttir (2009) for the Icelandic mathematics teacher Gunnlaugsson, half a century later. He also was a good mathematician, he was university educated and a highly appreciated teacher, but as a mathematician and teacher he worked in isolation, more like a pioneer than Praalder was.

Because of the *goals* and *aims* of the Foundation, mathematical sciences formed the core of the curriculum. The *content* of the mathematics curriculum and the *teaching methods* were mainly left to the mathematics teacher. This differs from the formal curriculum of the Engineering school at Leiden University in the 17th century. For that curriculum not only the aims (to train military engineers) but also the content and the teaching methods were prescribed (Krüger, 2010). Kilpatrick (2009) shows the influence on the mathematics curriculum in the USA of the movement towards a vocational directed curriculum (the social efficiency curriculum), in the early 20th century. He states that as a result the mathematics curriculum has as its goal to equip students to become productive members of the society. That in a way was the goal of the *Fundatie van Renswoude*. However in Utrecht and probably also in the Foundations in Delft and Den Haag students not only were taught procedures and algorithms, they also learned why these procedures were mathematically sound and they were encouraged to study the topics more in depth if they had an inclination to do so. This seems to indicate a difference with 20th century vocational training. The selection of the students differed, another difference is the conviction in the 18th century example that students would be able to get far, provided they were given the right environment and were taught well.

Harmonisation was important at all levels; starting with the meeting once a year of the governors of the three institutions. This meeting was one of the conditions in the testament and the executors made sure the regents acted on this. In the Dutch Republic it was very unusual for an educational institution to have a regular and formal exchange with similar institutions (Boekholt & Booy, 1987). However it was effective, as practices which worked well in one institution were implemented in one or both of the others, as becomes clear from some of the entries in the resolutions of regents (HUA 771, 8-13). Within the Foundation in Utrecht harmonisation was encouraged as well: the mathematics teacher and other teaching personnel were required to keep each other informed about the students on a daily basis. The same was true for the mathematics teacher and the caretaker of the home. We may assume this helped to create a coherent educational environment for the students. Schubring (2010) describes a more extensive example of some harmonisation between mathematics curricula of Gymnasia in Westfalen, in the first half of the 19th century, through consultation of the main actors involved. The professional debate on geometry instruction in Sweden, discussed by Prytz (2009) could in a sense be seen as lacking in harmonisation. According to Prytz the practice of teachers differed from what was considered important in the professional debate and written in the textbooks. There seems to have been a lack of harmonisation between theory and teaching practices in this case, possibly resulting in low scores in the final exams.

In this limited description of the Foundation in the 18th century other factors, which are not always obvious in curriculum research turned out to be rather important as well. Examples of those are the following.

- Financial means and accountability, which enabled the building of a good home, the daily care of the students, paying the teachers well, buying books and instruments, paying the work masters to take on apprentices, etc.
- Learning environment: a safe physical environment, good basic provisions such as clothes, food and attention to hygiene, rooms for lessons and for practice and supervision on homework.
- A curriculum embedded in social and cultural education in order to profit from the mathematics learned and to find a position in accordance with one's level of professional education. A good mathematics curriculum, sufficient time, supervision on homework and a very good mathematics teacher would not have been sufficient to fulfil the goals of the institute.

The quality of the learning environment; the financial means which were available and the allocation of those means; the social and cultural education in which the mathematics curriculum was embedded are examples of factors which did not influence the content of the mathematics curriculum directly, but which were nevertheless important for the quality of implementation and the likelihood of reaching the goals of the intended curriculum. It might be worthwhile to take these factors into account when researching the history of mathematics education.

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The proposals of the School of Peano on the rational teaching of Geometry

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Abstract

Between 1890 and 1940, the School of Peano formulated many proposals for the renewal of the teaching of mathematics, showing a sensitivity regarding the pedagogical tasks and the syllabuses. In this context we intend to analyze the reflections of some members of this team about the introduction of the results of their research activity on the foundations of geometry into education, illustrate the ways used to train teachers in this field, and examine the debates that the 'rational teaching' of geometry ignited. An examination of the textbooks by G. Ingrami and A. Pensa will make explicit the epistemological assumptions, elaborated by Pieri, Peano and Padoa, on the possibility of presenting maths in classroom as an hypothetical-deductive abstract system and the use of the logical symbolism.

Introduction

Between the late 19th century and the 1920s, many of the best Italian mathematicians were particularly aware of logic-foundational issues, sometimes spurred by their own scientific activity, at other times by interests of a philosophical or methodological nature. It is in this context that we may consider the reflections of members of the School of Giuseppe Peano[1] on the so-called *rational teaching* of Geometry, aimed at renovating its traditional treatment from elementary school to University.

First of all it is necessary to determine what is being alluded to with the expression *rational teaching*. In a general sense, it refers to a didactic praxis which substantially uses the output of the foundational research. More precisely, it is articulated along two branches: *hypothetical-deductive* and *axiomatic-deductive* teaching. The two trends, although coinciding in many aspects, greatly differ in relation to the nature of mathematical objects. In fact, according to the hypothetical-deductive approach, geometrical concepts must be viewed and presented in classroom *à la* Hilbert, i.e. as mere names attributed to classes of abstract objects, and without any reference to their physical substratum. On the contrary, according to the axiomatic-deductive method, opportunely revitalized by some positivist educationalists such as Carlo Cattaneo, Roberto Ardigò, Aristide Gabelli and Andrea Angiulli, the fundamental geometric ideas should always

[1] The characteristics of the School of Peano (its members, the role of Peano as a *Maestro*, the choice of research themes, …) are problems currently debated in historiography. Cf. (Luciano & Roero, 2012; Roero, 2010).

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

maintain a concrete connotation and they must be transposed in teaching activities through adequate interpretations and 'metaphors', derived from the real world.

Once this distinction is made, we can say that the majority of Peano's collaborators dedicated themselves to examining how to adapt the advanced research on the foundations of mathematics to textbooks for elementary, middle and secondary schools and how to insert it into teacher training, showing only occasionally an interest in the psychological aspects linked to these themes. Strangely enough, the strong theoretical engagement on the part of the School of Peano in the field of hypothetic-deductive teaching did not correspond to an analogous commitment to school publishing. Indeed, almost none of the authors of manuals of rational geometry published in Italy belonged, strictly speaking, to the School of Peano, even though they maintained varying levels of relationship with some of its members.

The institutional context

Around 1890, large sectors of the Italian mathematical community agreed on the advisability of illustrating in Universities and *Scuole di Magistero* (schools for teacher education, see Roero 1999, Furinghetti & Giacardi, 2012) the results of research carried out in the foundations of mathematics from the scientific, critical and pedagogical point of view. The main objective was to train up teachers for middle schools who were aware of the importance of these studies in order to deal with elementary mathematics in the classroom in a more clear and precise way. There soon arose, however, a divergence of opinions regarding both the expediency of establishing formal courses in this discipline, and on the contents and outlines that should characterize them. On one side there was the School of Peano, whose members maintained that logic and foundations ought to be strictly related to each other, and that the presentation of such studies necessarily involved the use of ideographic symbolism. On the opposing side was the School of Algebraic Geometry, whose members believed that the principles of mathematics ought to be addressed in connection with the so-called *elementary mathematics from an advanced standpoint*, and taking into account the physiological and cognitive aspects, in the wake of Felix Klein and Henri Poincaré's influences.

In spite of the emergence of signs of hostility, in 1897 Luigi Certo presented for the first time a proposal to create chairs in Foundations of Mathematics in the second biennium of Universities, and to teach the simplest and most essential notions of logic (conceived as an introduction to the mathematics) in secondary schools, technical institutes and normal schools.[2] In particular, he invited the colleagues to take as a starting point the volumes by Wilhelm Killing *Einführung in die Grundlagen der Geometrie* (Paderborn, Schöningh, 1893, 1898) and the works by Hermann Grassmann, Hermann Helmholtz, Bernhard Riemann, Adrien-Marie Legendre, Jules Hoüel, Felix Klein, Moritz Pasch and Giuseppe Peano. This idea was then presented and received with enthusiasm at the first national congress of the Mathesis Association, held in Turin, in the "citadel where these methods were championed so pertinaciously and valiantly".[3]

In the months that followed, within the local sessions of the Mathesis, perplexities emerged about Certo's proposal concerning mathematical logic, although his suggestion of establishing courses in foundations enjoyed a general consensus (Certo 1899, pp. 114-116). In spite of the insistence of the Peanian followers, the project to introduce a formal teaching of logic was firmly rejected during the next congress of the Mathesis, held in Livorno. On the contrary, Giulio Pittarelli, who had taken over from Certo as a speaker, continued to underline the expediency of devoting some of the lectures in the *Scuole di Magistero* to questions pertaining the primitives ideas and postulates of the science, taking Federigo Enriques' *Questioni riguardanti la Geometria Elementare* (Bologna, Zanichelli, 1900) as a model, see (Pittarelli 1902, p. 163)

Meanwhile, the debate was growing on the idea of introducing courses in *Metodologia Matematica* (*Mathematical Methodology*) or *Matematiche Elementari da un Punto di Vista Superiore* (*Elementary Mathematics from an Advanced Standpoint*), which were expected to include elements of logic and considerations on the foundations of mathematics, along with didactic issues and the history of mathematics.[4] Regarding this question, the majority of Italian mathematicians, whether of the Peanian *entourage* or not, recommended tackling questions such as the deductive method, the partial arbitrariness in the choice of primitive concepts and postulates of a theory, the various axiomatic systems for the different branches of geometry, etc.

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[2] Estratto del verbale dell'adunanza tenuta in Palermo nel giorno 27 febbraio 1897, fra i soci di "Mathesis", professori Certo, Pepoli e Rozzolino. *Boll. Mathesis* (1897-98), *2*, p. 9.

[3] (Certo, 1899, p. 116): "nella cittadella dove quei metodi sono così pertinacemente e con tanto valore propugnati".

[4] Among the dozens of interventions on the introduction of foundational research in the teachers training we mention: (Pincherle, 1903, p. 47, p. 49; Loria, 1906; Pittarelli, 1908, p. 35, p. 36, pp. 38-39; Loria & Padoa, 1909; Padoa, 1909, pp. 110-111; Pincherle, 1911, p. 11, p. 13; Fano, 1922, pp. 107-110).

Furthermore, a suitable education on these themes was judged to be important also for the training of teachers in kindergartens and elementary schools. In this case, the cultural background to be imparted had to be especially calibrated according to the exigencies of this kind of educators, charged with the difficult task of teaching young pupils (3-10 years old) the first concepts of mathematics, the very same concepts whose treatment naturally implied many delicate foundational questions.[5] In this context, authors such as Rodolfo Bettazzi, concentrated on naive mathematical education identifying a typical Achilles' heel of pre-school teaching of geometry in the linguistic aspects, wherein the foundational research itself had focused attention, see (Bettazzi 1939, pp. 72-77).

In the absence of courses specifically dedicated to the principles of geometry, there is no shortage of mathematicians, like Corrado Segre, who hinted at these themes in their university lectures of Higher Geometry and even scholars, like Mario Pieri, who entirely restructured his courses in Projective Geometry after his studies on foundations.[6] Furthermore, it can't be forget the long lasting experience of Giuseppe Veronese at the University and *Scuola di Magistero* of Padua, beginning at the end of the century and producing the well known volume *Fondamenti di Geometria a più dimensioni e a più specie di unità rettilinee, esposti in forma elementare* (Padova, Tip. del Seminario, 1891).[7]

In the 1920s, the institution of *Matematiche complementari* chairs provided the ideal context for the presentation of critical research, crystallizing a situation of substantial differences from place to place. Looking at that period we can see an authentic proliferation of courses entitled *Fondamenti di Geometria*, held by Luigi Berzolari, Enea Bortolotti, Gaspare Mignosi, Giuseppe Marletta and Alfredo Perna in the Universities of Pavia, Bologna, Palermo, Catania, and Rome.[8] Logic ideography combined with the axiomatic treatment of arithmetic and geometry were integral parts of the lectures of *Matematiche complementari* given in Turin by Peano and in

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[5] Cf. (Bettazzi & Burali-Forti, 1899; Conti, 1912, pp. 121-123, pp. 125-126, pp. 137-138; Bisson-Minio, 1910, p. 98).

[6] Cf. (Segre, C. (1916-17). Vedute superiori sulla Geometria elementare, Quad. 30, pp. 7-27. In Giacardi (Ed.) 2002 and Pieri M. (1902-03). *Lezioni di Geometria Superiore dettate dal Prof. Mario Pieri nella R. Università di Catania nell'a.a. 1902-03*. Catania: Bianca, lithographic print, Dipartimento di Matematica dell'Università di Pavia, Archive L. Brusotti). On Pieri's teaching at the Universities of Turin and Parma cf. also Marchisotto, 2010, pp. 340-342, pp. 350-353.

[7] Veronese held courses specifically entitled *Fondamenti* or *Principi di Geometria* in 1910-11, 1912-14, 1916-17 cf. *L'Enseignement Mathématique*, *12*, 1910, p. 343; *14*, 1912, p. 334; *15*, 1913, p. 357; *18*, 1916, p. 362.

[8] *L'Enseignement Mathématique*, *25*, 1926, p. 131; *26*, 1927, p. 148, p. 149; *29*, 1930, p. 168; *31*, 1932, p. 128, p. 129.

Milan by Ugo Cassina,[9] and were often chosen as the subject of degree dissertations (*tesi di laurea*) by their students.[10] Also some algebraic geometers approached these questions, dedicating time to non-Euclidean and non-Archimedean geometries, the concepts of elementary geometry from the point of view of groups of transformation, according to the *Erlangen Program* etc. Open to international influence, they repeatedly discussed the problems related to the transposition of foundational research into education with Felix Klein and David Hilbert, on the occasion of their visits to Göttingen and during trips by their German colleagues to Italy.[11]

The lectures and seminars on logic and foundations

The teaching carried out in Universities and *Scuole di magistero* was flanked by a great number of initiatives that fell halfway between propedeutic education and high-level popularization: the series of lectures and seminars on logic and foundations given by Cesare Burali-Forti, Alessandro Padoa, Giovanni Vacca and Michele Cipolla in Italy and abroad, which were widely distributed in lithographic or printed form, see (Luciano 2009 and 2010). In this context, the principal figure was Padoa, who gave series of seminars in Brussels (1898), Pavia (1899), Rome (1900), Geneva (1910) and Genoa (1932-36), which were imprinted by a single approach, outlined in the paper *Logica matematica e matematica elementare* (Padoa, 1902) not by chance defined as the "manifesto of Italian logicians".[12] Convinced that "only a few misunderstandings stand today between Mathematical Logic and the eminent place that it seems to merit among the most important manifestations of human thought",[13] his intention was to combine the teaching of ideographic formalism with that

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[9] *L'Enseignement Mathématique*, *27*, 1928, p. 153; *28*, 1929, p. 321; *29*, 1930, p. 168, p. 169; *30*, 1931, p. 151; *31*, 1932, p. 129.

[10] For example the Peano's student Cesarina Boccalatte presented a degree thesis entitled *La geometria basata sulle idee di punto e angolo retto*, where she illustrated an axiomatic system for elementary geometry based on the primitive ideas of point and right angle, taking her cue from Pasch, Pieri and Peano's works. The dissertation was published in *Atti della R. Accademia delle Scienze di Torino*, *64*, 1928-29, pp. 47-55.

[11] These relationships with Hilbert and Klein, (closer than those maintained by the Peano School), were also behind the different importance given to the presentation of Hilbert's *Grundlagen der Geometrie* in Castelnuovo and Enriques' courses in Rome and Bologna, and in the lectures given by Segre in Turin. Cf. (Giacardi, 2003; Gario, 2006, pp. 254-258; Giacardi, 2012; Luciano & Roero, 2012).

[12] G. Vailati to G. Vacca, 19 February 1902, c. 1r-v, *Peano-Vacca Archive*, Turin.

[13] (Padoa, 1902, p. 186): "soltanto alcuni equivoci contrastino oggi ancora alla Logica matematica il posto eminente che sembra spettarle fra le più importanti manifestazioni dell'umano pensiero".

of foundational themes. So, for example, in the first section of his Belgian lectures, Padoa introduced the meaning and syntax of the principal Peanian symbols and, in the second part, commented on a *selecta* of essays on the foundations of geometry by Peano and Pieri, see (Padoa, 1898, pp. 78-79).

In spite of the strong legacy of Peano's views, these series of lectures also displayed elements of originality: indeed, until 1910, research in and teaching of logic and the foundations proceeded on parallel tracks, and the best of the results were anticipated or re-examined in such didactic contexts. For example, in a series of *Conferenze su l'Algebra e la Geometria quali teorie deduttive*, given in Rome in 1900, Padoa developed the problem of absolute independence of a system of primitive propositions in an axiomatic theory of arithmetic and geometry. Just a few months later, the same topic would have been the subject of his celebrated lectures at the International Congresses of Philosophy and Mathematicians held in Paris.[14]

The warm reception by the public of these teaching initiatives confirmed the faith of the Peano School in the rapid establishment of their scientific style. However, the commitment of these scholars was not sufficient to clear the hurdle constituted by the hostility of many colleagues with regard to foundational research, which was denied the stamp of originality. This seems evident, if we take into consideration the various bureaucratic difficulties that Burali-Forti or Padoa had to face, in order to attain a habilitation (*libera docenza*) in Logic or Foundations of Mathematics, see (Borga, Fenaroli & Garibaldi 2010, pp. 283-285).

In addition to university and para-university teaching, the members of the School of Peano were engaged in putting together a set of editorial initiatives aimed at the communication of their studies. Noteworthy among these is the *Rivista di Matematica* (*RdM*) directed by Peano in the years 1891-1906. The editorial policy of the journal, conceived as essentially didactic, gradually placed the accent on the ideographical address to the point that the *Rivista* ended up as a journal of mathematical symbolism, with the majority of articles being compiled in Peanian language. It was in relation to this journal that the School of Peano began to devise the idea of inserting the output of scientific activity in logic and foundations into middle teaching and textbooks. On the other hand, it was precisely in the terrain of these questions that the *Rivista* offered to

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[14] (Padoa, 1900, pp. 18-20). The interchange between scientific and didactic activity was continuous so much so that it is precisely at the origins also of the numerous works by Pieri and Padoa on Euclidean and neutral geometry that followed one another in the years 1898-1900. Cf. (Luciano, 2012, pp. 49-52).

consolidate a collaboration between university professors and teachers in upper and lower level secondary schools. The foundations of mathematics, in fact, had to be initially studied for purely scientific purposes, but in order to be transmitted into educational practice, they had to be investigated by teachers themselves, being the only figures who could get a feedback of the reactions of students to this new methodology of teaching. In the words of Pieri:

> Reconciling the needs of Schools with the ideality of the deductive method is such an undertaking that, if it ever comes to be, it will only be thanks to the work and fatigue of many.[15]

The epistemological frame of the rational teaching of geometry

The overall initiatives outlined so far led to the maturation of a composite framework of why, where and how far it was reasonable and useful to promote the rational teaching of geometry. More precisely, the reflections related to the transposition of foundations touched on 1) the criteria for choosing primitive concepts and propositions; 2) the varying degrees of concern to avoid implied admissions; 3) the advisability of mentioning the problems of independence, completeness and coherence of axiomatic systems; 4) the different ways of schematising language, between the opposite poles of everyday expressions and the symbolism of mathematical logic and 5) the importance given to the adherence to physical or psychological reality *vs.* the logical-deductive structure.

Faced by these problems, we can say - without trivializing excessively the positions of the single authors - that a general consensus was shared within the Peano School as far as many aspects were concerned,[16] first of all a propos the ascertainment that the first aim of teaching is to develop and promote “the practice of reasoning with exactness; that is, the sure knowledge of the logical relationship between principle and consequence: in short, the art or capacity of correctly arguing and concluding”.[17] Equally shared by Bettazzi, Burali-Forti, Padoa, Peano, Pieri and Vailati was a lucid examination of the defects in the way geometry was presented in

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[15] (Pieri, 1899, p. 181): “conciliare i bisogni della Scuola con le idealità del metodo deduttivo è tale un’impresa, da non poter maturare, se mai, che per opera di molti e a fatica”.

[16] Cf. for example (Bettazzi, 1891; Peano, 1894, pp. 51-55; Burali-Forti, 1898; Bettazzi, 1899; Pieri, 1898-99; Pieri, 1901; Vailati, 1907a; Vailati, 1907b; Pieri, 1908; Peano, 1910; Padoa, 1910).

[17] (Pieri, 1908, p. 447): “la pratica del ragionare con esattezza, vale a dire la cognizione sicura dei rapporti logici di principio e conseguenza: insomma l’arte o la facoltà di rettamente argomentare e concludere”.

classrooms.[18] The solution to these problems, according to the School of Peano, was to be looked for in clever exploitation of foundational studies in oral teaching and textbooks. This practical role rather justified the interest in this type of scientific activity and provided proof of its usefulness. At the basis of hypothetical-deductive teaching of geometry there was thus a set of contents that the Peanians unanimously identified in these terms: the model was provided by the Peano system based on the primitive ideas of point and segment and on seventeen postulates, see (Peano, 1894). However, the knowledge of other systems – for example the *point-motion* and *point-distance* systems of Pieri – was recommended both in oral teaching and as a theoretical basis for the editing of textbooks. Finally, the greatest attention was given, by the members of the Peano's School, in describing how to apply the critical studies to the selection of logical and mathematical vocabularies for the textbooks:

> The logical introduction of our scientific treatise on Elementary Mathematics ought to be formed of the lists of the logical symbols and of the logical propositions that will be used therein. These two lists, provided respectively by the Logical Ideography and by Mathematical Logic, complete each other by turns: one is the vocabulary, the other the grammar […]. The list of the undefined mathematical symbols and of the unproven mathematical propositions compose the first chapter, the premise of the entire deductive theory.[19]

While the collaborators of the School of Peano were alike in sharing these scientific and educational opinions on the rational teaching of geometry, the divergences – or at least – differences in opinions about other issues were certainly not lacking.

First of all, there was debate concerning the *concrete* ways in which the foundational contents should be transmitted. Indeed, all of Peano's collaborators agreed on the fact that an axiomatic approach constituted the only way to bridge the gap between pre-university teaching - which *could* be intuitive - and university teaching - which *must* instead be formal

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[18] (Pieri, 1901, p. 377): "Les conséquences n'y découlent pas toujours des prémisses par la Logique pure: les arguments d'*évidence* (ou, comme on dit à présent, d'*intuition*) se dissimulent derrière les syllogismes les mieux ajustés, ou même sont invoqués ouvertement. Les notions primitives y sont plus nombreuses qu'il n'est besoin; etc.".

[19] (Padoa, 1902, pp. 194-195): "Quindi, l'*Introduzione logica* del nostro trattato scientifico di Matematica elementare dovrebbe esser formata dagli elenchi dei *simboli logici* e delle *proposizioni logiche* di cui sarà fatto uso. Questi due elenchi, forniti rispettivamente dall'*Ideografia logica* e dalla *Logica matematica*, si completano a vicenda: l'uno è il *vocabolario*, l'altro è la *grammatica* […]. Gli elenchi dei simboli matematici non definiti e delle proposizioni non dimostrate formano il Primo capitolo, la premessa dell'intera teoria deduttiva".

and hypothetical-deductive. However, the majority of these scholars usually limited themselves to observing that 'many parts of their works' could be used to advantage, even if it not be suitable for adoption in all details in teaching.

The use of symbols constituted the first object of discussion. According to Peano, in fact, one of the features of rational teaching was singled out with the purpose of presenting to the students the symbols and algorithms of logic that best rendered common procedures of proof rigorous, without debasing the ideography to mere tachigraphy. In this regard the translation of Euclid's *Elements* into ideographical language took on a special significance. With it the Peano School entered into the debates in Italy over the use of Euclid in secondary school teaching.

Strongly persuaded that it was above all "in the field of teaching that logic can demonstrate its brilliant simplicity"[20], as early as 1898 Peano maintained that it was possible to impart an axiomatic and symbolic teaching since middle schools, through the use of textbooks modeled on the *Formulaire*. Strangely enough, even though recommended as a guide to the compilation of textbooks, the *Formulaire* did not include the majority of results on the foundations of geometry, as opposed to the situation regarding the foundations of arithmetic.

At their turn, many of his collaborators frequently underlined their recourse to ideography, recognising in it a valuable tool "in virtue of the intellectual skills which [it is] capable of teaching and promoting, and also for certain of its evocative capacities, which often point the way to observations and investigations that would otherwise go unseen"[21].

This notwithstanding, scholars like Peano or Pieri were perfectly aware of the risks connected to a completely formal treatment of mathematics, and they warned the teachers against an indiscriminate introduction of symbols in classrooms.[22]

The most delicate element of the rational teaching was, however, that related to the different opinions about the epistemological status of geometrical objects.[23] Among the figures who participated most actively in

[20] (Peano, 1919, p. 960): "è nel campo dell'insegnamento che la Logica manifesta al meglio la sua fulgida semplicità".

[21] (Pieri, 1898-99, p. 177): "in virtù degli abiti intellettuali, che i metodi e le dottrine di questa scienza si manifestan capaci di educare e promuovere, ed anche per certa loro facoltà suggestiva, che guida spesso ad osservazioni e ricerche non curate altrimenti".

[22] Cf. (G. Peano to E. Catalan, 25 January 1892, in Jongmans 1981, pp. 307-308 and Pieri 1903, p. 293).

[23] Also the on-going research in mathematics education deal with the problem of "the contrast between the abstract nature of mathematical objects, which are usually seen as having no perceptual existence, and their representations, which are tangible and upon

these discussions were Pieri who, in effect, is one of the mathematicians of the School of Peano that more warmly defended the *hypothetical-deductive* teaching of geometry.[24] His positions were the fruit of several works of a scientific nature – namely, his famous publications on the foundations of projective and Euclidean geometries – and were consolidated in relation to his teaching experience in the technical schools in Livorno and Pisa.

Pieri's starting point, taken from Edmond Goblot and Filippo Masci's papers,[25] was the natural evolution of *all* sciences, *in primis* of geometry, towards the structure of a hypothetical-deductive system, corresponding to the evolution of teaching in a rational-abstract direction.

More precisely, for Pieri, the propositions at the base of geometry correlate primitive concepts that are not associated with any meaning. As a consequence, the notions of point and space are not unequivocally determined *a priori*; rather, they are characterised by formal conditions, freely imposed, and are resolved in the set of all the interpretations that they satisfy, subject only to the constraint of consistency. Maintaining that the postulates of geometry are nothing other than Euclidean forms of the intuitive concept of space, means remaining tied to a rigid and subjective representation. Consequently, Pieri's ideal was that of presenting Euclidean geometry in classrooms as an abstract speculative system, a doctrine or a science "of all that is capable of being figured or represented".[26] In this type of teaching, geometrical objects must be introduced as pure creations of the spirit, the postulates as simple acts of our will, i.e. "artefacts of the mind and truths by definition"[27], arbitrary in so far as their ordering is determined - according to the pragmatist philosophical view - by the deductive end that the author sets for himself. As a consequence, concluded Pieri, a mind educated in general ideas and supported by a discreet faculty for abstraction "becomes capable of perceiving not only the abstract logical meaning, but also the nexus of the various propositions and their deductive outcomes"[28], appealing only to the properties that the axioms confer on primitive notions and referring only to the definitions of the various objects.

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which subjects' activities can develop in a concrete way" (Arzarello, Bosch, Gascón, & Sabena, 2008, p. 179).

24 An other member of the Peanian team who partially agreed with Pieri on these questions was Padoa. Cf. (Padoa, 1902, pp. 194-200; Ferrera, Furinghetti, & Ortica, 2010, pp. 387-404).

25 Cf. (Goblot, 1898 ; Masci, 1885).

26 (Pieri, 1908, p. 447): "scienza di tutto ciò che è figurabile ovvero rappresentabile".

27 (Pieri, 1901, p. 373): "scelte dello spirito o verità di definizione".

28 (Pieri, 1908, p. 447): "diventa infatti capace di percepire sia il senso logico astratto sia il nesso delle varie proposizioni e le loro veci deduttive".

It is easy to understand that this conviction was a significant point of rift from other mathematicians such as Peano, or Cassina who, in accordance with the dictates of positivistic pedagogy of that time (from the *concrete* to the *abstract)*, held the physical interpretation of geometrical objects and the practical or experimental nature of the postulates to be indispensable.[29]

Pieri himself, however, shared Enriques's interest towards the psychological foundations of geometry,[30] and admitted that the selection of primitive ideas should not only be the most appropriate from the point of view of logic, but also from that of perception, and confessed that there was no reasonable way, in schools, to avoid presenting geometry *also* as a "mathematical physics of the extended bodies":[31] a guise that history, educational traditions and the results of cognitive research necessarily conferred on it. Attentive to the criticisms that emerged during the debates on rigour and intuition, Pieri could then recommend to teachers that a mote of dust, or the hole made by the point of a needle in a sheet of paper, could be usefully exploited to provide an image of a point. And again, systems of rigid sticks arranged in simple structures, or threads attached to a frame, could lend themselves to experimental verifications of axioms.

By the way, resorting to concrete representations of fundamental objects did not mean remaining silent regarding the hypothetical-deductive connotation of geometry, and in fact Pieri himself provided some suggestions for setting up school activities, describing how the first mathematics lecture in a secondary school might begin.[32]

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[29] Cf. (Peano, 1894, p. 54, p. 75; Cassina, 1961, pp. 197-201, p. 203).

[30] Pieri knew the scheme for a positive gnoseology worked out by Enriques in the Bolognese period, and significantly opened one of his papers referring to the conclusions of the search carried out by Enriques in the wake of Herbert Spencer, George Romanes, George H. Lewes, Wilhelm Wundt and Victor Henri, saying (Pieri, 1908, p. 345): "All in all, it would seem that the primordial constructive elements which are most evidently involved in the creation of tactile-muscular space, are not notions of line and plane, but rather of distance and thus of circles and spheres"; "Tutto sommato, parrebbe che gli elementi costruttivi primordiali, che più spicciamente intervengono a formare lo spazio tattile-muscolare, non siano le nozioni della retta e del piano, ma sì della distanza e quindi dei cerchi e delle sfere".

[31] (Pieri, 1901, p. 377): "fisica matematica dell'estensione".

[32] (Pieri, 1908, p. 447): "The first time the teacher ought to address his disciples thus: Allow me the truth of these primitive propositions; and I will lead you step by step by means of successive deductions, to having to recognize the truth of all the other geometric propositions. The axioms are like the seeds of all geometric truths: but the fruits of these do not grow from the seeds if they are not fertilized by reason. In this way can be grounded, for example, Geometry and Algebra; in brief, in what consists the deductive process, which informs all of pure mathematics"; "La prima volta il Maestro così parli ai

The textbooks of Ingrami and Pensa

An examination of some textbooks that attempted the transposition of results of logical-foundational research into teaching according to the suggestions of the School of Peano makes it possible to see how the appeals for collaboration between the world of research and that of schools, launched by the *RdM*, translated into concrete terms.[33]

The first example is constituted by the *Elementi di Geometria per le scuole secondarie superiori* (Bologna, Tip. Cenerelli, 1899) published by Giuseppe Ingrami, who did not attend himself Peano's lectures, but all the same decided to apply the rational approach, after having autonomously studied Peano and Veronese's works. In this textbook the rigorous deductive treatment began with three primitive concepts (point, segment and a congruence relation between two segments) and constituted the first application of the reflections of Pasch, Peano and Pieri in a teaching context. "Fruit of long meditations and a careful and intelligent study of the most recent works on the foundations of geometry",[34] is particularly the generation of geometrical entities. Taking the point and segment as primitive concepts, the line is defined as the set of infinite points formed by a segment and its extension; the plane and space are generated by projecting the point towards the perimeter of a triangle and a tetrahedron respectively, by means of radii whose origins are inside this triangle and this tetrahedron, etc.. Ingrami's textbook received an excellent review in the *RdM*, where Pieri concluded:

> While (didactically speaking) Elementary Geometry does not for now show signs of that degree of hypothetical and frankly deductive science that we so much admire in Arithmetic, nevertheless this work of Prof. Ingrami, in which some aims speculatively hoped for by a few, and relatively recently, have begun to be put into practice in concrete

discepoli: Concedetemi la verità di codeste proposizioni primitive; ed io vi conduco man mano, per via di successive deduzioni, a dover riconoscere la verità di tutte le altre proposizioni geometriche. Gli assiomi son come il seme di tutte le verità geometriche: ma i germi di queste non si svolgon da quelli, se non sian fecondati dal raziocinio. A questo modo s'istituisce, p. es., la Geometria e l'Aritmetica; in ciò consiste sommariamente il processo deduttivo, che informa tutta quanta la Matematica pura".

[33] Many other Italian textbooks of Geometry used the output of the research on the foundations, but they were published by authors such as F. Enriques and U. Amaldi, outside the Peano's School or according different kinds of approach. Cf. (Natucci, 1967; Mammana, 2000, pp. 240-251; Giacardi, 2004, pp. CVIII-CXIX; Giacardi, 2006, pp. 592-594; Menghini & Cannizzaro, 2006).

[34] (Palatini, 1900, p. 85): "frutto di lunghe meditazioni e di uno studio accurato e intelligente dei più recenti lavori sui fondamenti della geometria".

> and practical ways, is already an excellent sign, and a sure presage of new and increasingly greater steps forward in that direction.[35]

Another book that is emblematic of the 'modern wave of logic' in the teaching of geometry was Angelo Pensa's *Elementi di Geometria ad uso delle Scuole secondarie inferiori* (Torino, Petrini-Gallizio, 1912). This work, published on the recommendation of Burali-Forti by a former student of Peano in the courses of Infinitesimal calculus at the Turin University, assumes as its scientific underpinnings the paper by Pieri *La Geometria elementare istituita sulle nozioni di punto e sfera*. In place of the concept of sphere, however, is the more common and intuitive idea of distance, rather, the material one of a cord stretched between two points, or that given by the compass. The treatment is distinguished by its simplicity and at no point is it dry or boring, thanks to the fact that "the concern for the logic of the whole resides solely in the mind of the author"[36]. Physical interpretations are provided for all of the primitive ideas, renouncing those pseudo-definitions that mathematical logic had unmasked as vicious circles and inserting in the text an evocative set of illustrations. Moreover, making the suggestions of Pieri and Peano his own, Pensa courageously suppressed the majority of proofs, and substituted them with experimental justifications, obtained through superposition. All else aside, according to Pieri, Burali-Forti and Peano, this is the best form of transposition of Pieri's system, since its complete justification is not possible even in upper-level secondary school.

The debates on rational geometry

The concept of teaching maintained by the Peano School gave rise to a long series of debates. Generally, it might be said that the detractors thought that an excessive use of the axiomatic method, matched with the new logical language, would mask the natural paths of geometric reasoning, lead to mechanicalness in learning, in addition to obscuring the connections to the applications of geometry to the physical and natural sciences. In essence, it was feared that rational teaching of geometry would be suitable only for students who were exceptionally brilliant or exceptionally mediocre. In some sense, these criticisms are similar to those

[35] (Pieri, 1899, p. 182): "se (didatticamente parlando) la Geometria elementare non accenna per ora a quel grado di scienza ipotetica e schiettamente deduttiva, che tanto ammiriamo nell'Aritmetica, non di meno quest'opera del prof. Ingrami, dove alcuni propositi vagheggiati speculativamente da pochi, e da non molto tempo, cominciano ad attuarsi in forma concreta e pratica, è già un ottimo pegno, e un affidamento sicuro di nuovi e sempre maggiori progressi su quella via".

[36] (Moglia, 1912, p. 195): "la preoccupazione logica dell'insieme risiede soltanto nella mente dell'autore".

aimed in the 1960s at textbooks compiled with a view to the so-called Modern Mathematics and imprinted with the concepts of the Bourbakis.

Furthermore, a serious problem was posed by the fact that the majority of textbooks of rational geometry required constant and complex epistemic-cognitive mediation, which could not be left up to the good intentions of the teachers, but necessitated adequate teacher training. Testimony gathered in the experiments regarding the use in class of this type of textbooks effectively show a wide range of different behaviours.

There were teachers who considered these works as a basis for preparing their lectures, but offering the relevant contents to the students only after adaptations and simplifications made by themselves without any specific technical competence.[37] Others, instead, adopted them in a traditional way, although they misrepresented, in some sense, Peano's advices. This was the case of Michele De Franchis who, in his textbook *Geometria elementare ad uso dei Licei e dei Ginnasi superiori* (Milano, Sandron, 1909) presented geometry as a rational science founded on experimental bases, following an exclusively axiomatic treatment, based on the primitive objects of point and segment, motion being defined as an affinity. He appreciated the foundational approach and inserted, for the first time in a geometry textbook, a short accounts of mathematical logic. However, De Franchis rejected the ideography, holding it to be an educational impediment and, by doing so, he risked rendering logic almost as an artificial and useless appendage to his manual, even more so because the symbols explained in the first chapter were then never used in the rest of the book.

Despite a decent editorial success, the textbooks of rational geometry aroused huge criticism - even in the ranks of the Peano School - from the

[37] Cf. (Nannei, 1904, p. 24): "I was once told, for example, that a colleague in an Italian technical school explained arithmetic and geometry using Peano's logical symbols. I don't dispute the method, although, for all that I am an admirer of the professor of the University of Turin and his work […] I doubt that future merchants, salesmen, or even future students who would come out of that school, would profit much from it. But I was also told that the teacher did not use a textbook. And I feel a shudder of compassion when I think of how the students at home must have cursed that poor teacher, when they couldn't make head or tails of all those symbols!"; "Mi fu detto una volta, per es., che un collega di una scuola tecnica italiana spiegava l'aritmetica e la geometria usando i simboli logici del Peano. Io non discuto il metodo, benché, per quanto ammiratore del professore dell'università torinese e dell'opera sua […] dubito che i futuri commercianti, futuri commessi, o anche futuri studenti d'Istituto che saranno usciti da quella scuola, abbiano potuto trarne molto profitto. Ma mi si disse anche che quel professore non adoprava libro di testo. E io sento un brivido di compassione a pensare alle benedizioni che avran mandato da casa gli alunni a quel povero insegnante, quando non riuscivano a raccapezzarsi fra tutti quei simboli!".

moment of the publication of the *Elementi di Geometria ad uso dei Licei e degli istituti tecnici (1° biennio)* by Giuseppe Veronese and Paolo Gazzaniga (Padova, Drucker, 1897).[38] Even more problematic was the reception of Pensa's textbook, which despite eleven successive reprints, was strongly opposed by the author's colleagues in Turin and by inspectors for the Ministry, who went so far as to prohibit its adoption, see (T. Boggio to G. Peano, 1 July 1912, in Roero 2010, p. 120). The climax of these disputes arrived in 1913 when Paolo Ricaldone stated, in a lecture to the Piedmont session of the Mathesis Association: "Some books are informed by the principles of pure logic. Even while admitting that mathematical logic has in certain cases a beneficial effect, the lecturer is contrary to adopting in middle schools books informed by them."[39]

This type of vicissitude, along with a series of quite heated debates due to a complex web of scientific reasoning and academic quarrels, led to a contraposition and eventually to a rupture between the Schools of Peano and Segre. The different ways of considering the research activity strongly conditioned the approach of the two Schools towards educational problems. What carried weight was not so much the component 'foundations', as much as the divergent conceptions of logic, see (Enriques 1906, pp. 69-78; 1921, p. 8; 1938, p. 188; 1942, pp. 65-67).

The methodological instruction supplying the Italian *curricula* well reflects the tensions within the Italian scientific world as regards the advisability of embodying in the mathematics at the pre-university level the 'philosophical' reflexions on the foundations, see (Vita 1986, pp. 16-17, pp. 22-27, p. 29, pp. 40-43, p. 49, pp. 64-65, pp. 70-72, p. 75, pp. 78-79, p. 80).

On the other hand, this kind of debate was not confined to within the Italian public. It's just the case to remember, in this respect, that in his report of the inquiry on rigour and intuition in secondary teaching commissioned by the ICMI, Henri Fehr stated that not a single country had adopted in a systematic way the entirely logical teaching method of

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[38] Cf. (Padoa, 1899, pp. 3-22; F. Klein to M. Pieri, 31 March 1897, M. Pieri to F. Klein, 9 Avril 1897, in Luciano & Roero, 2012, pp. 188-190).

[39] *Bollettino della Mathesis*, *5*, 1913, p. 49: "Alcuni libri sono informati ai principii della logica pura. Pur ammettendo che la logica matematica abbia in certi casi un'azione benefica, il conferenziere, prof. Ricaldone, è contrario ad adottare nelle scuole medie libri ad essa informati". As regards this debate, which involved Peano, Pieri, Burali-Forti, Sebastiano Catania, Alpinolo Natucci, Giacomo Bellacchi, Francesco Gerbaldi, Francesco Giudice, Enrico Nannei, Michele Cipolla, Giuseppe Marletta, Giuseppe Sforza, Guido Castelnuovo and Gaetano Scorza, cf. (Mammana & Tazzioli, 2001, pp. 223-232; Luciano, 2009).

Peano, David Hilbert, and George B. Halsted, which had instead been tried out only by isolated teachers, see (Fehr, 1911, pp. 462-464).

The hypothetical-deductive teaching after 1910

After 1910, the School of Peano made a serious self-criticism of the proposals regarding the rational teaching of mathematics. An overall re-assessment became indispensable following the outcome of debates on rigor and intuition, and the ascertainment of the difficulties faced by the teachers who had experienced the hypothetical-deductive approach. It was even further necessary in the light of some extra-mathematical circumstances, such as the clash in the Turin Faculty of Sciences between Peano and Segre in March of 1910, see (Luciano & Roero 2008, pp. 65-68, pp. 135-143). However, the pedagogical context too had changed in the meanwhile, and Masci's reflections were now recognized as outdated in some popular volumes on the teaching of mathematics, such as those by Carlo Leoni (1915, pp. 178-221) and Jacob William Albert Young (1924, pp. 236-253, pp. 345-349). In this period indeed, and still more in the Twenties and Thirties, we can observe an authentic flourishing of criticism concerning the use of logic-foundational studies in education, and a sharp defense of intuition, also on the part of scholars such as Vincenzo Cavallaro (1928, pp. 80-81) and Alpinolo Natucci (1928, p. 269), who had been fervent supporters of the Peanian trend.

The School of Peano continued to play an active part in teacher training, through the establishment of the *Conferenze Matematiche Torinesi* and, to confirm the cultural influence which this team exerted, we point out the ample space given to the logic and foundations of geometry, in accordance with Peano and his collaborators' views, in a series of books dedicated to prospective teachers, such as the *Questioni riguardanti le Matematiche Elementari* and the *Enciclopedia delle Matematiche Elementari*.[40]

On the other hand, with reference to publishing, textbooks of geometry which properly adopted the rational approach disappeared almost completely, with a few exceptions such as the manuals by Giuseppe Marletta (1911), Piero Benedetti and Carlo Rosati (1924), and the appendix to Francesco Severi's *Elementi di Geometria* (1926, pp. 175-184).

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[40] Cf. Enriques, F. (Ed.) (1924-1927). *Questioni riguardanti le matematiche elementari*. Bologna: Zanichelli, I, p. 46, pp. 87-89, p. 114, p. 212; Berzolari, L., Gigli, D., & Vivanti, G. (Eds.) (1929-1949). *Enciclopedia delle Matematiche Elementari*. Milano: Hoepli, 1_1, pp. 1-79, 2_1, pp. 3-118; 3_2, pp. 800-802, pp. 892-893, pp. 900-902, pp. 924-927, pp. 954-958, p. 968, pp. 977-1014.

Irrespective of the results (more or less successful) of the efforts of the School of Peano in the field of education, there remains the fact that the contribution of these scholars was by no means insignificant in the evolution of Italian teaching. In this sense, the question of the transposition of the foundations can provide an effective key for illustrating the practices of shared creation, socialisation and transmission of mathematical knowledge typical of the illustrious Schools of Peano and Segre.

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Mathematics teaching and learning in the late 1970s in Portugal: intentions and implementations

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Abstract

By the late 1970s, on the wake of the 1974 democratic revolution, the Portuguese school system has grown very rapidly and is undergoing major changes in its structure. At the same time, modern mathematics reform for middle and upper grades of the secondary school gradually adopted from the late 1960s is confronted with the practicalities of its implementation: textbooks, teacher education, didactical strategies, evaluation, etc.

A series of 16 reports from the Ministry of Education written between 1977 and 1981 under the leadership of Swedish specialists aimed at characterizing the situation and some reports addressed specifically mathematics teaching and learning. This paper will study the implementation of mathematics curricula in Portugal in the late 1970s, in particular the programs, the mathematics classes, students' learning.

Introduction

This work studies school culture of the late 1970s in Portugal, especially as it relates to the conditions of teaching and learning of mathematics in secondary schools as they appear in a coherent set of technical reports from the Project for the Evaluation of Unified Secondary Schooling published by the Ministry of Education between 1977 and 1981.

One of the consequences of the revolution of 1974 that ended the Portuguese dictatorial regime was the elimination of the distinction between two educational tracks and the creation of the Unified Secondary Schooling beginning at 7th grade (7º ano unificado) in 1975. By 1974, there was a formal centralized education system based on legislation in the late 40s, and several sub-systems, all of them "experimental". Portuguese students entered school with six years of age and went through four years of primary schooling followed by two years at preparatory schools after which they could pursue studies either at Liceus oriented to the universities, or at technical schools oriented to the work market or polytechnic institutes. The elimination of this distinction in 1975 was viewed as a means to balance educational opportunities for all students.

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

The Project for the Evaluation of Unified Secondary Schooling

In May 1976, when the first year of the unification reform was nearing the end, the Ministry of Education commissioned the evaluation of its implementation (Freitas, 1979). In the following years this assessment was extended to the next two years of unified schooling (the 8th and 9th grades and the Portuguese Ministry of Education established a cooperation agreement with Sweden. From January 1977, a major project designed to evaluate this process of unification of the two cycles, the *Project for the Evaluation of Unified Secondary Schooling* began[1]. By 1979 the Project had monitored the generalization of the unified schooling until the 9th grade and later published a total of 16 reports, 15 of which refer to empirical research by collecting data from students, teachers, parents, members of Executive Councils, and businesses. Various methods are used in particular questionnaires, tests, interviews, among others, and in parallel with analysis of large samples of students and teachers data, particular schools.

As mentioned, the evaluation of the Unified Schooling starts in 1975/76 with a comprehensive study of its implementation. In the second year the Project began to think on the need to assess the appropriateness of curricula to the age level of students as well as their degree of apprehension of the new curricula. In this second phase the discipline of mathematics was the subject of specific studies (it was, in fact, the only subject worthy of an in-depth study), mainly because it is, in the words of researchers, a discipline "easy to handle, both with regard to the hierarchy of objectives in content as the preparation of tests" (Catela & Kilborn, 1979, p. 2[2]). Thus, from the 16 studies produced five refer specifically to teaching and learning of mathematics between 1975 and 1979 (Catela, 1978a, 1980; Catela & Kilborn, 1979; Leal & Fägerlind, 1981; Leal & Kilborn, 1981). Other reports, though not focused on mathematics, also contain data on this subject.

Under the Luso-Swedish agreement, Wiggo Kilborn, "consultant and advisor of the activities inherent to the study of Portuguese unified secondary school mathematics curriculum" (Catela & Kilborn, 1979, p. 3), and Ingemar Fägerlin participated in the implementation of these studies focused on mathematics and worked in collaboration with Maria Emília Catela, science teacher and later with Leonor Cunha Leal, teacher of mathematics, both participating full time in this project.

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[1] The term *Project* will be used in the text.
[2] All translations are by the author.

Mathematics was thus the subject of intensive investigations focusing on the conditions of teaching and learning. These studies assume greater importance today, as they provide an opportunity of studying the conditions in which the reform of modern mathematics was applied. We can, therefore historically study the functioning of mathematics classes, some dimensions of the mathematics curriculum and, finally, some elements on the learning of mathematics in particular of the typical themes of modern mathematics.

Conditions for teaching mathematics classes

Although not containing descriptions of mathematics classes, the Evaluation Project gives some partial information on how they were conducted during the late 1970s. From the responses to a questionnaire sent to the Executive Councils of public and private schools where the 7th year was taught in 1975/76, we see that mathematics classes began late in many schools (Freitas, 1979, p. 12). Only 17% of the schools had some mathematics classes starting in the beginning of the school year[3] ($n = 307$). 20% of schools had mathematics classes starting in January, and there was still a significant number of schools with classes (13%) starting in May. This panorama is similar in other disciplines.

In 1976/77 this problem was further investigated with a sample of 29 schools from the northeastern interior and 48 from the Lisbon district (the sample accounts for 80% of schools with grade 7) (Mendonça, 1980). The Executive Councils indicated delays in the beginning of classes in mathematics and other disciplines, although apparently there has been a small improvement. Failures of placement of teachers are appointed by the Executive Councils as one of the reasons for this situation. In October, on the district of Lisbon, only half (56%) of the schools had started mathematics classes (31% only started in November). In the northeast the outlook is even worse, with only 14% of schools started classes in October (31% in November and 24% in December). Note that in 7% of sampled schools in the district of Lisbon and 21% in the northeast there were no mathematics classes for 7th graders throughout the school year (p. 18). In the following year the situation was similar Ingemar Fägerlin participated in the implementation of these studies focused on mathematics and worked in collaboration with Maria Emília Catela, science teacher and later with Leonor Cunha Leal, teacher of mathematics, both participating full time in this project.

The system for teacher placement was complex and centralized, but the main reason for this situation relates to the shortage of teachers which

[3] Beginning of October 1975.

implied the absence of a stable teaching staff in schools. In the case of mathematics, professionals from other fields (engineering, university students, and even high-school students from the schools themselves) were performing teaching duties in schools. The results of the policy of limiting the access to teacher training from the 1950s (Matos, 2009) are now patent. In the academic year 1975/76 most of the schools (163 of 307) indicated that the number of teachers of mathematics was insufficient (Freitas, 1979, pp. 16, 18). Only 24% of the teachers who taught 7th grade schools had educational qualification (p. 17) and two thirds were under 31 years (p. 21). For teachers of mathematics only 43% had a degree (not necessarily for teaching mathematics). In short, many teachers are missing and among those few had the necessary scientific and pedagogical skills.

The program of modern mathematics

Modern Mathematics started in Portugal from the middle 1960s (Matos, 1989, 2009). From 1965 TV lessons of modern mathematics for grades 5th and 6th were nationally broadcasted and the cycle for the same grades had, since its beginnings in 1968, a program following the precepts of modern mathematics. The 7th through 9th grades either in Liceus or in the technical schools had programs of modern mathematics since 1971 and grades 10th and 11th of Liceus were in transition from Classical to Modern Mathematics.

After the decision to unify the Liceus and technical schools tracks in 1975, new mathematics programs were designed by Leonor Filipe, Alfredo Osório dos Anjos, and Francelino Gomes (Catela & Kilborn, 1979). The first two came from Liceus and the third from a technical school. The last two had an extensive experience in teacher education. These new programs were very close to the contents of the now extinct course from the Liceus. Essentially, the programs of Liceus were extended to the technical schools.

In 1977, to counter widespread students' failure and the impossibility for many teachers to teach all the required topics, new mathematics programs were published. Table 1 presents the topics of the mandatory mathematics programs of 1977.

The new 9th grade program includes a repetition of 8th grade topics (essentially geometry). *Minimal Programs* — a smaller list of topics that all teachers should teach — were also proposed. Teachers rapidly interpreted that the topics not included in these minimal programs would be "optional". In 1980, even these shorter programs were replaced with new programs, almost all of them explicitly providing for the review of issues in previous years.

Table 1. Mathematics programs of 1977, grades 7th through 9th

7th grade	1) Language. 2) Rational numbers and their operations. 3) Equations of the first degree and problem solving. 4) Binary relations. 5) Applications and functions. 6) Geometric transformations. 7) Equal triangles.
8th grade	1) Problems — Equations and systems of equations. 2) Set of real numbers (including the Pythagorean Theorem). 3) Homoteties in the plane. 4) Similarities in the plan.
9th grade	1) Order relations in R (including inequalities). 2) Geometry of the plan. 3) Extension of the concept of power. 4) Radicals. 5) Problems and equations of the second degree. 6) Similarities of the plan (from the 8th grade). 7) Trigonometry (from the 8th grade). 8) Space geometry.

Modern mathematics was justified in the discourse of Portuguese educators for psychological reasons and for what was perceived as its closeness to the development of mathematics as a science (Matos, 2009). The match between the three parent bourbakist structures and Piagetian intelligence operative structures reinforced the merits of the new ideas. It was hoped that the new programs were psychologically simpler and mathematically more robust than previous ones.

Apparently what happened was the opposite. The successive reduction program reveals either a generalized implicit resistance to the application of new programs from the teachers, or the practical impossibility of its implementation. The consequence was that whole parts of the programs would no longer be taught in Portuguese schools. The main victim was geometry systematically left to the end of the school year and subsequently dropped. But perhaps more importantly, the adoption of a terminology strange to students, and especially to the teachers, made it more difficult to teach and learn mathematics. Only in 1989 new programs will depart from modern mathematics and adopt a problem solving approach to mathematics (Matos, 2011).

Project's appreciation of the mathematics curricula

The Project assessed mathematics curriculum and as for the prescribed curriculum, reports are generally very critical:

> By analyzing the curricula for the 7th, 8th and 9th grades it appears that they are curricula typical of the first generation of modern mathematics; they show, moreover, a keener interest in teaching pure mathematics that will make the students good mathematicians. (Catela & Kilborn, 1979, p. 41)

The reports mention several times the bad results the new approaches produced in foreign countries. It is possible to note here the strong influence of the opinions of Swedish experts Ingemar Fägerlin and Wiggo Kilborn. Adopting simultaneously a pedagogical and a critical tone the reports explain that in other countries the initial enthusiasm was replaced by a disenchantment caused by learning difficulties, motivation, and major limitations of the students in solving even simple problems. These opinions are supported by references to the three ICME held recently (in 1976) and to international studies.

> To what extent is modern mathematics necessary, both in Portugal and elsewhere, to the child's education or profession in the future. The answer to this question is also one of the main reasons many countries have changed their [modern] mathematics curricula, using other alternatives according to the capabilities and needs of children. (Catela & Kilborn, 1979, p. 42).

The curriculum presented to teachers through textbooks was also appreciated by the Project. Limitation to unique textbooks for each grade adopted in 1948 and revoked in the beginning of the 1970s considerably restricted the range of available textbooks. Although the regime of a unique book was gradually falling into disuse, only from 1976 an alternative collection of textbooks becomes available. However, the reports do not mention it, and treat the first collection of books as if they were the only *de facto* books.

Shortages in the use and access to textbooks are reported (Catela & Kilborn, 1979). For example, although in the year 1975/76 new programs had been published, new manuals did not immediately accompany them. Even in 1977/78 the project documents some problems accessing books. In general, students could only acquire the book in December, at the end of the 1st school period.

The content of these books draws strong criticism from the evaluators (Catela & Kilborn, 1979). Firstly, they argue that books are almost replicas of the programs, which can pose problems for students and teachers. They indicate that programs use concepts and sequences very different from regular teaching, so that the structure of "pure" logic of the books does not facilitate their understanding by teachers who are not comfortable with meanings and assumptions on which these concepts and sequences are based. The structure of the books is also questioned. Students are often required to go through five or more pages of text before having the opportunity to solve tasks.

A second aspect is addressed. For the authors of the report, the Portuguese books, unlike those of other countries, do not accommodate an individualized learning, because they do not contain diagnostic tests allowing the verification of knowledge by the students. On the other hand, there are very few exercises at the end of each section that allow students to assess understanding before moving on. It also points out the linear structure of the books. Students only have a single contact with each topic, there are no ways to compensate later for a more hasty study.

> For example, students have only contact the Pythagorean theorem once and only once, instead of going in small steps in order to pass from one step to the next only when the first is understood. (Catela & Kilborn, 1979, p. 45).

Learning outcomes

Mathematics learning was studied by the Project. The design used longitudinal testing (three comparable tests, five questions each) through the years 1977/78 and 1978/79. Each question should encompass three different levels of difficulty and so each had three items and it was expected that 75% of students responded correctly to the first, 50% to the second and 25% the third. It was therefore expected that students obtain an average score of 50% in each test question. Almost every question privileged knowledge of algorithmic or algebraic nature, in which the greater complexity corresponds to replacing integers by fractions or by additions and subtractions multiplications and divisions, or incorporate powers.

The first two tests, M-I with topics from 7th grade and M-II with 8th grade contents, were prepared by Maria Emília Catela, Wiggo Kilborn and the authors of the new programs (Catela & Kilborn, 1979). The tests were previously verified: the test for the 7th grade was passed on the 8th and the test for the 8th grade in the 9th. The results of the students were very weak. However it was decided to keep M-Iand change some details of M-II (which now became M-II/2). The third test, M-III, with contents of the 9th year, was apparently prepared by the same team, and tested on students in 10th grade and after some minor changes became test M-III/2 (Catela, 1980).

For example, question 2 from MI, passed in 7th and 8th grades, was:

2. Compute the value of each of the expressions:

2.1. $(1 - 3x)/(x - 2)$ for $x = -3$
2.2. $(2 - xy)/ (x + y)$ for $x = 2, y = -1$
2.1. $(x^3 – 3x^2)$ for $x = -1/2$

The average score of this question was 3.9% among 7th graders and 18.7% among the 8th (Catela & Kilborn, 1979) well off the expected percentage of 50% for each grade.

The design had some methodological flaws, mainly related to sampling and the instruments chosen. The results were comprehensively examined elsewhere (Ponte, Matos, & Abrantes, 1998). Overall results are extremely low. The performances are much lower than expected, even for the contents of the 7th grade tested on the 8th. On average, students in the 7th grade get a rating of 13%, and the 8th achieve 24% in the contents of 7th grade and 25% in the 8th. In average, students in 9th grade score 29% (Catela, 1980).

Discriminating by topic, it appears that 7th grade results are particularly low (about 5%) in matters involving algebraic expressions and solving equations and the 8th in solving equations of the second degree. The higher scores appear in numeric expressions for 7th grade (27%), and in operations with polynomials (39%) and solving systems of equations (36%) for the 8th, still far from the average of 50% expected by the authors programs. No distinct difference is apparent between the results of the group in the district of Lisbon and the Northeast. As for the 9th grade, the results are also very low. Scores for similarity of triangles (10%), second-degree equations (15%) and powers (29%) are the most difficult topics.

Why these results? Another report (Catela, 1980) reflects on the poor performance of students pointing the excessive difficulty of the tests. Referring to the initial tests that were tried, she states

> These trial tests were constructed by the authors of the programs which are also teachers. The level of the tests M-I and M-II (1st [trial] version) shows, therefore, the notion that [these] teachers have of the level of their own classes, and it was ultimately proved that this notion contains higher expectations than the actual knowledge of students. (Catela, 1980, p. 24)

Searching for explanations for this mismatch, the author concludes:

> One reason for the high expectation on the part of authors [of the programs] (and perhaps teachers in general) may be the fact that modern mathematics, whose concepts have guided the current

> programs is considered easier for students than conventional mathematics. In fact, when mathematics was introduced in secondary programs in several countries there has been no investigation of how students would accept it in terms of learning, and it was immediately assumed that Modern Mathematics was actually simpler. However, experience has shown otherwise. (Catela, 1980, pp. 24-5)

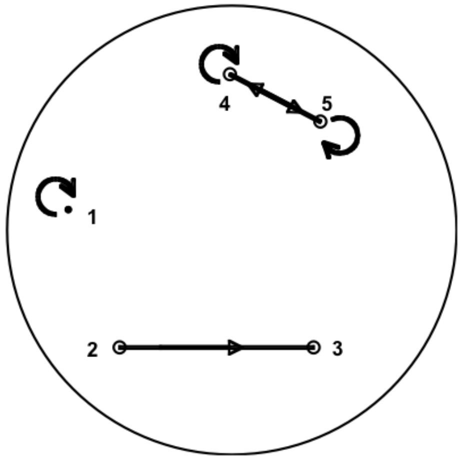
The Project includes another study that is not directly related to the introduction of the unified school (Leal & Kilborn, 1981). This work had the purpose of assessing basic mathematics computing in secondary school. It used a test with several variations of the four arithmetic operations and was applied in 1978/79. This study follows the international movement known as "back to basics", assessing the "basic knowledge for elementary mathematical calculation, such knowledge provided to students by the end of fourth grade students" (p. 24), a movement that appears to please particularly Kilborn.

The results of this last work have also been studied in (Ponte, Matos, & Abrantes, 1998), and show major flaws of 4th graders in subtraction, multiplication and division. There was also a decrease of correct responses in the 6th grade students. Students repeating the 7th generally had the lowest scores. When compared with similar studies in Sweden and Norway, the results of Portuguese students are superior in the subtraction and similar with respect to multiplication and division of whole numbers.

Learning modern mathematics

The answers to the questions of the tests involving modern mathematics would be of particular importance since they would allow us to have some insight about the quality of learning of these new topics. Unfortunately, there is only one such question, question 5 of the test M-I that deals with binary relationships and conditions. The question and its three items that the researchers expected had increasing complexity were:

> 5.1. Given the sets
> A = {0, 1, 3} and B = {-1, 1, 2} and the condition $x + y < 1$, indicate the ordered pairs of the relationship defined from A to B.
>
> 5.2. Consider the binary relationship defined in the set {1, 2, 3, 4, 5} and represented by the diagram. Indicate the missing pairs for the relation to be reflexive.



5.3. Represent in extension the classes in which the set {11, 18, 21, 28, 37, 31} is divided by the relationship defined by the condition "if it has the same unit figure of y".[4]

From the available copies of the report it is difficult to read the number of correct answers per class for each item. But in one 7th grade class, which has all numerals legible, the following percentages of correct answers were obtained: 5.1 - 0%, 5.2 - 0%, 5.3 - 6.7 %. It is also possible to know the average percentage of correct answers per class, as computed by the Project. For example, the mean percentage of correct answers for the previous class was 2.2% which corresponds to the average of three percentages (Catela & Kilborn, 1979, p. 21). For purposes of this article, these percentages means were grouped at 10% intervals and the data obtained are presented in table 2.

Performance in this question about binary relations is very weak. For example, sixteen 7th grade classes (64%) and twelve 8th grade classes (55%) had an average percentage of correct answers less that 10%. According to the expectations of the authors of the test, the average percentages should be 50%. It is, however, from 8.8% in 7th grade and 9.7% in the 8th, which, at the same time correspond to the lowest percentage of correct responses of the test.

The aggregation of the correct answers to the three items just give us a partial view. Looking at the disaggregated maps (Catela & Kilborn, 1979, pp. 21, 22), we detect many classes in which few or no students (0 or 1) answered correctly, and some (few) classes with slightly better results but

[4] The answers were: 5.1: (0, -1), (1, -1); 5.2: (2, 2), (3,3); 5.3: {11, 21, 31}, {18, 28}, {37}.

still very afar from the expected results. In the Northeast region, in particular, the classes showed a high number of null responses (33 in 57, compared to 28 in 84 in the region of Lisbon). The best results (over 30% average of correct answers) are obtained in three groups of schools in the city of Lisbon which had teachers well acquainted with the reform.

Table 2. Number of classes with correct answers to question 5 of test M-I per grade

Average percentage of correct answers	Number of classes	%
7th grade		
0 a 10 %	16	64
10 a 20 %	4	16
20 a 30 %	3	12
30 a 40 %	2	8
40 a 50 %	0	0
Total	25	
8th grade		
0 a 10 %	12	55
10 a 20 %	8	36
20 a 30 %	1	5
30 a 40 %	1	5
40 a 50 %	0	0
Total	22	

What are the reasons behind such poor results? The M-I test was developed in cooperation with the very authors of the program and that there was a pre-testing. We should also reject the hypothesis that students had difficulties computing these items. These difficulties were clearly present in the items related to algebra or fractional numbers, but the answer question 5 only requires a correct linguistic interpretation.

The poor performance of students on the issue of binary relations seems to be due to two factors: shortcomings in the teaching process and intrinsic difficulty. In other words, it is very likely that, not having learned binary relations during their initial scientific formation, teachers tended not to teach this subject or teach it in inappropriately. But even the classes taught by teachers conversant with the new ideas, as those from Lisbon, performed poorly. And even these teachers seemed not being able to teach in such a way that learning would remain as students went from the 7th to the 8th grade, contrary to what happens in other "classical" items. Binary relations seem indeed to have an intrinsic difficulty of their own.

This is stated in the report that concludes that this topic is not appropriate for these students.

These results are coherent with a previous study about the performance of 6th grade students on topics of modern mathematics in the national examinations of 1972 (Matos, 2005). There it seemed clear that the introduction of terminology characteristic of modern mathematics, although superficially seized by students, kept them from answering some mathematically simple questions. This study also suggests problems with teacher preparation and the inadequacy of certain topics to the mental age of the students.

Conclusions

When these reports were conducted, the debate in Portugal about the teaching and learning of mathematics was very limited (Matos, 2011). The regime that ended in 1974 limited for nearly half a century the democratic functioning of organizations in which social forces could meet. No associations of teachers were allowed and the Portuguese Mathematical Society had a reduced activity since the expulsion of many of its leaders in the late 1940s. As a consequence, this reports had no effect on actual teaching of mathematics.

Together with this the picture of the spread of mathematics educators have the vision given by the reports of a rapidly changing educational system. On the one hand, the sustained growth of the school population since the 50 was not accompanied by the creation of infrastructures and an adequate training of human resources. Therefore we have simultaneously overcrowded schools and very young teachers, many of them (the majority in some areas of the country) without adequate scientific and pedagogical training.

The landscape of learning mathematics is disappointing. Schools changing sharply (integrating Liceus and technical schools, rapid expansion of the school network), classes starting with delays of months, entire sections of the program that are not taught, general shortage of teachers with adequate training, new programs, mathematics approached by a radically different philosophy appreciating the more formal aspects of mathematics combine to produce poor quality learning. Although many of these problems are now outdated, this is good not to forget the past to better understand the current situation.

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Rational approaches to arithmetic and real numbers in the Italian textbooks and programmes for the classical and the scientific curricula

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Abstract

Rational arithmetic was traditional in the Italian Classical Lycée, considered in some way parallel to rational geometry, and was maintained also in the Modern Lycée. The comments to the programmes before 1923 suggest anyway a not too rational approach to arithmetic and real numbers; on the other hand, textbooks seem to shift towards a rational treatment of these topics, also for the Technical Institutes.

With the birth of the Scientific Lycée in 1923 rational arithmetic disappeared from the programmes, while the introduction of real numbers proved to be still a problem, which became more evident looking at the frequent rearrangements of this part of the programmes in the successive years. On the one hand real numbers had to be introduced geometrically, and this required a detailed introduction of ratios of magnitudes and proportions; on the other hand there was the need to introduce quite early geometry of coordinates, which required real numbers.

Rational arithmetic

Rational arithmetic was introduced in the Upper Gymnasium (corresponding to the first two years of the Gymnasium-Lycée, that is for 14 -15 years old pupils in upper secondary classical instruction) in 1867, when the *Elements* of Euclid became the official geometry textbook for this school (Maraschini & Menghini, 1992).

Rational arithmetic refers to the part of algebra that deals mainly with the properties of integers and of rational numbers exposed with theorems derived from axioms and definitions, in parallel to the rational teaching of geometry.

The aim was to show that all mathematics, not only geometry, is a deductive science.

In that period the scientific instruction was covered by the Technical Institutes, which had various sections. In the first two years of the Technical Institute, that were common to all the sections, the teaching of mathematics had 6 hours per week. Moreover the *physical mathematical section* covered two further years with 5 hours mathematics each and allowed the entrance to the courses for engineers; this section can thus been considered the scientific alternative to the Classical Lycée.

In the programmes for the Technical Institutes the term "rational" doesn't appear. The subject was "Arithmetic and algebra", but we find –

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in programmes of that period (in 1876 ad 1891) - the word “theorem” and then “theory” (of prime numbers, of fractions) as it was in the Gymnasium before 1867.

In fact most books of *rational arithmetic* were devoted both to the first two years of the Technical Institute and to the Gymnasium-Lycée until the 20th century.

The differences among the teachings in the Upper Gymnasium and in the Technical Institute were not as much a question of the type of school but rather of the period and the author. In fact there was an increase of rigor until the disappearance of rational arithmetic in 1923 (which would be then limited to the Normal School, for the future teachers of the elementary school), while a part of arithmetic (mainly factorization and the theory of prime numbers) would be shifted up to the last years of the new scientific Lycée, becoming optional.

The definition of an integer number and of sum of numbers

Let us now compare the presentations of rational arithmetic in different textbooks. The first question concerns the definition of a number and of the sum of numbers. Different terms were used.

For instance in a text by Faifofer (1883, pp. 5-19), written in the 1870s, that had many further editions, we find:

- Number is a word that expresses exactly a multitude of things seen as parts of a whole.
- Sum is the arithmetic operation by means of which from numbers that represent parts one obtains a number that represents the total[1]

The latter property is explained through examples. The book doesn’t contain axioms.

In 1881 the study of rational arithmetic was abolished and also the teaching of rational geometry was postponed to the Lycée, but in 1884 both were again introduced. Therefore Ricotti (1887, pp. 1-4) states that his text is written according to the programmes of 1884. Among his definitions:

- We consider a whole of elements [...] Each element is a concrete unity. An integer number is a collection of unities.
- 0 is called an insignificant figure [numeral].

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[1] • Numero è una parola che esprime con esattezza una moltitudine di cose che si pensano come parti di un tutto.

• L’Addizione è l’operazione aritmetica per mezzo della quale dai numeri, che rappresentano le parti, si ricava il numero che rappresenta il totale.

- To sum means to add [in the sense of to join] the second number (or quantity) to the first[2]

The axioms are expressed in words, and correspond to following properties for integer positive numbers:

1. if $a = b$ and $c = b$ then $a = c$
2. $a + b > a$; $a + b > b$
3. A sum doesn't change if we change the order of the numbers
4. If $a = b$ then $a + c = b + c$ and $a \cdot c = b \cdot c$
5. The analogous property is then stated in the case $a > b$.

Arzelà and Ingrami (1893, pp. 1-5) present a geometric approach to the definition of number:

> We count successive segments on a line [...]
> Postulate 1: Infinite elements exist called unity that we denote by 1
> Postulate 2: All these elements are equal
> Postulate 3: With this unity we can count and generate infinite different elements that we call numbers
> Postulate 4: the operation of counting doesn't depend from the order with which we count the unities
> The numbers generated by counting constitute the series of the numbers:
> 1, 1 + 1, 1 + 1 + 1, ... (translated by the author).

Addition is then introduced with reference to the counting of unities.

In a further edition of his text, Faifofer (1906, pp. 4-14) introduces a more formal approach, even if his text still doesn't contain axioms. He explains that the word number cannot be defined, being a *fundamental* concept:

- Number (quantity) is a quality of a heap (of books).[3]

He then considers a collection of objects. If we can put two collections in a 1-1 correspondence, then their numbers are equal.

- A number is an aggregation of unities.[4]
- The sum of two numbers is the number made of all the unities that make up the two summands (translated by the author).

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[2] • Si immagini un complesso di elementi [...] Ciascun elemento prende il nome di unità concreta.

• La cifra 0 prende il nome di cifra insignificativa.

• Addizionare o sommare due numeri [...] significa aggiungere al primo numero il secondo, o alla prima quantità la seconda.

[3] Codesta qualità [del mucchio], si dice il numero (la quantità) di quei libri.

[4] • Un numero è un aggregato di unità.

• La somma di due numeri è il numero composto con tutte le unità che compongono i due addendi.

We note that the concept of 1-1 correspondence between sets arises, and such the cardinal aspect of the number.

A text by Catania (1908) introduces Peano's arithmetic, with its symbolism, first in the Technical Institute and then in the Classical Lycée.

In the same year the text by Martini Zuccagni (1908, pp. 5-14) for Technical Institutes appeared. It presents following definitions:

- A category is the set of all objects with the same name (homogeneous objects).[5]

Then the author considers groups of objects and a 1-1 correspondence between two groups.

- Postulate: If two groups of objects are in a 1-1 correspondence, then they can be put in a 1-1 correspondence whatever order we chose within the group.[6]
- The sum of two groups is obtained by adding to the first group all the elements of the second group.
- Unity is an abstract object represented with one.
- A natural number is a group of unities. The succession of natural numbers is: 1, 1 + 1, 1 + 1 + 1, ... (translated by the author).

In the book we find also a geometric reference to the comparison of magnitudes and the symmetric and transitive property of the equality of magnitudes.

Gigli (1914/21, pp. 12-17) speaks in his text of collection of objects and of correspondence between the objects, where

- a collection A is less than, equal to or greater than a collection B (translated by the author).
- a number is a character of the collection when abstracting from species, time and place of its elements.[7]

The fundamental properties of numbers a, b and c are expressed in words and correspond to:

- the sum $a + b$ exists
- a is equal, less, or greater than b; $a = a$; if $a = b$ then $b = a$;

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[5] • L'insieme di tutti gli oggetti che possono essere indicati col medesimo nome si chiama categoria [...] Gli oggetti si dicono omogenei.

[6] • Postulato. Se due gruppi di oggetti [...] sono in corrispondenza biunivoca debbono sempre potersi porre in corrispondenza biunivoca comunque si cambi la disposizione degli oggetti nei singoli gruppi.

[7] Astraendo dalla specie degli elementi che compongono una collezione, dal loro collocamento e dal tempo [...], è nella collezione ancora un carattere, e cioè il numero degli elementi che la costituiscono.

- if $a > b$ then $b < a$; if $a > b$, then c exists such that $a = b + c$
- $a + b = b + a$
- $(a + b) + c = (a + c) + b$
- [...]
- the unity exists; if n is a number then $n + 1$ is its successive number.

Severi (1931) criticized those forms of false rigor that correspond only to circular definitions. In fact many equivalent words were used to try to define a number and the sum of numbers, as “to add”, “to join”, “to unite”, “to aggregate”, ...

The commutative and associative properties

Let us see how the various texts formulated the commutative and the associative property. We start from the last book mentioned, the one by Gigli, and from its axioms. The formulation of the commutative property is the correct “rational” one and is an axiom (Gigli, 1921, p.16).

The associative property is a

Theorem: $(a+b)+c = a+(b+c)$ [this too corresponds to the correct formulation].

The proof is based on following steps:

- $(a + b) + c = (b + a) + c$ (commutative law)
- $(b + a) + c = (b + c) + a$ (axiom stated previously in the text)
- $(b + c) + a = a + (b + c)$ (commutative law).

This (together with Catania’s) is the highest level of rationality, it is quite a university level.

Most books are not so rigorous with the commutative and associative property, giving more generic formulations as:

- The result of a sum is the same if we change the order of the numbers
- We can substitute to a sum of numbers their value. Or also $(a+b)+c = a+b+c$.

Faifofer doesn’t even mention the terms commutative and associative.

The distributive property

Let’s see the proof of the distributive property by Faifofer. This book has a very practical approach and doesn’t use letters (it is the only case).

> Theorem. If you have to multiply a sum by a number, you can instead multiply any term by this number and sum the products.
> Proof. Let the sum (14 + 7 + 10 + 602) be multiplied by 3.

We will write (14 + 7 + 10 + 602)3. [...]. Let us write the numbers and repeat them as many times as the units of the multiplier.

14	7	10	602
14	7	10	602
14	7	10	602

If we have to add all these numbers, we have two options:

- to sum up all the numbers in a line and multiply the sum by the number of lines
- to multiply each term in a line by 3 and to sum the partial products.
- This is denoted by the expression:
- $14 \cdot 3 + 7 \cdot 3 + 10 \cdot 3 + 602 \cdot 3$ (Faifofer, 1883, p. 45, translated by the author).

The proofs of Faifofer are practical, convincing, and accustom the pupil to visualization and mental calculation. They correspond to what David Tall calls "generic proof" (Tall, 1979), that is a proof made on a particular example, which has anyway a general character.

The exercises contained in the book of Faifofer don't ask to apply a rule, but rather require a proof:

- Prove that the product of two numbers decreases if you add 1 to the biggest and subtract 1 from the lowest.
- If I take four consecutive numbers, the product of the means is 2 + the product of the extremes (Faifofer, 1883, p. 54, translated by the author).

These proofs are not supposed to be made with letters, as Faifofer doesn't use letters in his text; in this case the use of letters could be considered anyway appropriate.

De Franchis (1912) uses induction, without stating it, to prove the distributive property:

We must prove that $(b + c) \times a = b \times a + c \times a$.
In fact, for $a = 0$ the theorem is true, as:
$(b + c) \times 0 = 0 = 0 + 0 = b \times 0 + c \times 0$.

(the facts that $a \times 0 = 0$ and $a \times 1 = a$ are "given" previously in order to prove in general that $a \times (b + 1) = ab + a$, which is obvious given the definition of multiplication when b $\neq$ 0 and 1)

Now we state that, if the theorem is true for a value a, it is true also for $a + 1$, that is:
$(b + c) \times (a + 1) = b \times (a + 1) + c \times (a + 1)$.
In fact
$(b + c) \times (a + 1) = (b + c) \times a + b + c$ [see above].

> In our hypothesis $(b + c) \times a = b \times a + c \times a$
> so
> $(b + c) \times (a + 1) = b \times a + c \times a + b + c = (b \times a + b) + c \times a + c$;
> but also
> $b \times (a + 1) + c \times (a + 1) = (b \times a + b) + c \times a + c$.
> In the end
> $(b + c) \times (a + 1) = b \times (a + 1) + c \times (a + 1)$.
> So, if the theorem is true for a, it is true also for its successive. But the theorem is true for 0, so it is true for 1, then for 2 a.s.o. (De Franchis, 1912, pp. 34-35, translated by the author).

Proofs by induction can be found very often in the text by De Franchis, but also in other books (Catania, Arzelà & Ingrami, ...).

De Franchis presents also another proof, based on the definition of multiplication:

> $(b + c) \times a$ means $(b + c) + (b + c) + (b + c) + (b + c) +$... a times. Due to the commutative property this becomes: $b + b + b$ ... a times $+ c + c + c$ ... a times (De Franchis, 1912, pp. 35-36).

This proof probably helped to clear, but it remained at a theoretical level. This is anyway the proof that we find most in the other books.

The proofs by de Franchis are correct theoretical proofs. His book doesn't present exercises. And doesn't contain numbers! Most books use mainly letters, but there are anyway some examples with numbers.

A geometric approach is used by all the authors when introducing fractions. Also in this case we can find different levels of rigor:

Faifofer (1883, p. 141) only affirms that considering magnitudes it becomes sometimes necessary to divide the unity into a certain number of parts: "1/number"; and then to consider a certain number of these parts: "n/m"...

This is an example of simple introduction, but we have also different presentations, as the very long presentation of De Franchis (1912, pp. 120-129). It contains (as in Gigli, 1914):

- homogeneous magnitudes, comparison and sum of magnitudes; multiple of a magnitude mA; the related theorems are always proven with geometrical explanations:

- $(m + n)A = mA + nA$; $m(nA) = mnA$
- If $A = B$ and $m = n$ then $mA = nB$ and vice versa;

Then the submultiple of a magnitude A/n is introduced, and further theorems follow:

- $nA/n = A$;
- if $A = B$ and $m = n$ then $A/m = B/n$ and vice versa...;
- $mA/n = m(A/n)$

The proof of the latter theorem is based on the following steps:

$n(mA/n) = mA$; $n[m(A/n)] = nm(A/n) = mn(A/n) = mA$

After these theorems De Franchis can give his definition:

> we write mA/n as $(m/n)A$. The symbol m/n is called fraction ...

The fraction is in this case an operator. In some books we find at this point also the Axiom of Archimedes.

Theorems as the distributive property are then repeated also for the fractions.

Faifofer explains that algebra has instead to do with *general* properties, in which the letters can assume all values, including negative values (Faifofer 1897, p. 5). In fact we don't find negative values in the development of rational arithmetic.

It is clear that such a rational approach to arithmetic is at a too high a level. Already in 1911, Scarpis (1911, p. 25) wrote:

> given the strict theoretical development of the teaching of rational arithmetic, the pupils enter the Lycée with very little training in the fundamental rules of calculations, particularly of fractions ... (translated by the author).

Many mathematicians asked to abolish this topic introducing only some properties of integers and prime numbers.

After the abolishment of rational arithmetic in 1923, Severi (1931) explained that algebra could in fact be understood also without rational arithmetic, basing only on the middle school experience.

The approach to real numbers

Real numbers were never mentioned explicitly in the programmes of the first two common years of the Technical Institutes, nor in those for the physical-mathematical section. We find anyway in the first two years always mention of square-roots and logarithms, sometimes of incommensurables (in 1976 and in the ph.m.s. in 1885), limits (1891 and ph.m.s. in 1885), irrational numbers (1891).

In the classical Lycée they were explicitly mentioned only in 1911, in the second year of the Lycée.

But we find also (in the second year of the Lycée and in the first (!) year of the Technical Institute the (Theory of) Proportions of magnitudes (T.I. 1891, class. L. 1911, in the class. L. 1867 Book V of Euclid), needed to treat similarities and which can lead to irrational numbers.

Real numbers could in fact be introduced in different ways, and were often treated twice:

- in geometry we have the Euclidean approach to proportions (comparison of ratios between magnitudes) and the incommensurables, more or less the contents of Book V and X of Euclid. This was a typical content for the Lycée, but we could find it also in Technical Institutes, as there were textbooks devoted to both kinds of school (as Lazzari & Bassani, 1891).

The programmes for the Technical Institute seemed to suggest a tacit introduction, with mention of square-roots, of the way to calculate them (there is for instance a simple introduction in Faifofers's book of Arithmetic).

But in the books of Algebra, even in Faifofer, (1897), we find the approach by means of the so called *Classi Contigue* (something similar to the idea of nested intervals), which were treated also in the courses of algebra for the classical Lycée. The approach was very complicated in most books, in Gigli (1914/21, III, p. 1) as well as in Faifofer (1897, p. 81), both devoted in general to secondary schools. The presentation was mainly geometric and referred to sequences of segments[8]:

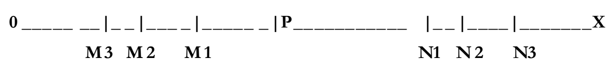

Fig. 1. Segments in *Classi Contigue*

The class A of segments OM_k, is less than the class B of segments ON_k. The difference between two segments OM_k and ON_k can be less than any given segment (Fig. 1).

After the definition of *Classi Contigue* all the texts present the operations between them, their properties, the concepts of an upper or inferior limit, of a separator and then the definition of an irrational number. A real number is defined as the separator of classes of segments commensurable with a given segment U.

[8] G. Chrystal (1889/1920), in his text of Algebra "for the higher classes of secondary school and for colleges", writes "The greater refinement and rigour of modern mathematics [...] have led mathematicians [to give] à priori an abstract definition of irrational real quantity and building thereon a purely arithmetic theory. There are three distinct methods, commonly spoken of as the theories of Weierstrass, Dedekind and Cantor. A mixture of the two last, although perhaps not the most elegant method of exposition, appears to us best suited to bring the issues clearly before the mind of a beginner". In fact, the method exposed by Chrystal corresponds to the method of Classi Contigue, even if not expressed in geometrical terms (part II, p. 98).

Chrystal warns that his book is not for beginners (see also Rogers, 2011), but the level was anyway too high for Scottish standards.

These approaches were probably due to the enthusiasm for the recent definitions of real numbers. As the teaching in school had to be rigorous, these rigorous systematizations couldn't be ignored.

We don't find in those years the algebraic introduction via Dedekind sections. The *Classi Contigue* are mostly presented in a geometrical language.

A difference between technical and classical instruction could be found in the introduction of logarithms via geometric progressions, because this was clearly suggested in the programmes for the Technical Institutes, and – only once in 1904 - for the classical instruction.

Here we present the introduction of Catania (1908, p.176), which followed the programmes for the Classical Lycée of 1904:

> Let's consider the two progressions (the first geometric with q integer number, the second arithmetic, whose terms are the exponents of the first progression):
>
> ....; q^{-3}; q^{-2}; q^{-1}; 1; q; q^2; q^3; ...
> and
> ....; -3; -2; -1; 0; 1; 2; 3; ...
>
> The terms of the second progression are called the logarithms of the corresponding terms of the first progression.... (translated by the author).

After the formulation of their properties, logarithms tacitly become real numbers.

The book of Catania contains anyway also the *Classi Contigue*.

The cited reform of 1904 was a first step towards a Modern Lycée. It allowed choosing, in the last two years of the Classical Lycée, between Greek and mathematics. But the reform was opposed by mathematicians themselves as contrasting the educational value of the Lycée. In particular, mathematics teachers of Mathesis protested against the proposed programmes, which – in order to anticipate certain topics to the first year of the Lycée – became too similar to those of the Technical Institutes (Consiglio Direttivo, 1909). They considered scientifically and didactically wrong

- to treat the equations of the second degree and square roots without having treated irrational numbers
- to derive the theory of logarithms from the geometric progressions
- to treat the theory of measure before that of irrational numbers (translated by the author).

This means that nothing can be done if pupils haven't a complete cognition of the basic topics, in particular of irrational numbers. This became in the successive years the main obstacle to an introduction in the first years of upper secondary schools of analytical geometry and of functions.

The problem is as described by Perry in 1900 (see also Howson 1982, p.147):

> Why not ... let a boy jump over all Euclidean philosophy of geometry and assume even the forty-seventh proposition of the first book of Euclid [Pythagoras theorem] to be true? Why not let him replace the second and fifth books of Euclid by a page of simple algebra, and give him much of the sixth book as axiomatic? ... The present rules of the game are really a little too absurd. A difficult vector subject like geometry must be studied before algebra. Simple exercises on squared paper ... must not be approached until one has wasted years on higher algebra and trigonometry and geometrical conics, because they belong to the subject of coordinate geometry. It is assumed that it is not until after coordinate geometry is thoroughly studied that a man can take in the idea which underlies the calculus, an idea which is possessed by every young boy with absolute accuracy.

A non-traditional position can be recognized in the suggestions of Guido Castelnuovo for the *Liceo Moderno*, created in 1913. The first three years of this new Lycée were in fact common to the Classical Lycée, and thus presented a rational teaching of arithmetic and geometry. Only in the last two years there was something new. A relevant topic was that of approximation, used to introduce incommensurable magnitudes and irrational numbers, but also – in the last year – to introduce definite integrals. Functions and infinitesimal calculus were introduced for the first time in Italian programmes in this kind of school.

As to real numbers, Castelnuovo suggested, in the explanations annexed to the programmes (Castelnuovo, 1912 & 1913), the following path:

Approximate measures (with measuring instruments); operations with decimal numbers (Castelnuovo invites the teacher to reason about a limited number of numerical examples); comparison between approximate and exact measures; the problem of the existence of a common measure; irrational numbers "about which the teacher will say what is strictly necessary to understand the concept, with only few hints on the operations among them".

But this approach did not enter in the Italian habits. Still in 1931 Severi wrote (Severi, 1931):

> After the inquiries of ICMI we know that [...] in no country real numbers are treated in such a complete way as in Italy. Everywhere only a few hints necessary to applications were given. The introduction of modern mathematics was considered a priority with respect to methods (translated by the author).

Mathematicians as Vailati, Castelnuovo and Severi suggested connecting real numbers to the measure of geometric objects. Every other Italian mathematician would agree that the approach had to be geometric: but what means geometric? The way of Castelnuovo (to measure objects), or the way of Euclid, with proportions and magnitudes? Or the *Classi Contigue*? Most books chose the two latter ways to be more rigorous, even if the programmes were absolutely not so explicit. And geometry of coordinates as such was not introduced before theory of measure.

Another problem was the complete absence of the concept of function. In most books we find a reference to the exponential equation or function before treating logarithms, but that is all.

The geometric approach can be an obstacle to the concept of function, as it has been to the concept of a variable (Schubring, 2005, p. 19), but also a certain way to treat arithmetic without the idea of substitution (of numbers to letters) can be an obstacle.

The approach to the idea of function by means of a gradual substitution of numbers in formulae is well explained in the Meraner Lehrpläne of 1905 for the different classes (see Klein & Schimmack, 1907, 208-220)[9]:

> Quarta - Meaning of formulae and their evaluation by substitution of special values.
> Untertertia - Continuation of exercises in evaluation of formulae, introducing negative quantities and continually insisting on the funtional character of the resulting variations in the quantities.
> Obertertia - Dependence of an expression upon one of its own variables. Graphic representation of simple linear functions and the use of this method in the solution of equations.
> Untersekunda - Consideration of the variation of a quadratic expression in one variable due to the changes in that variable with graphical illustration (translated in Price, 1911)

Also Perry (1900) gives an analogous advice in his proposal for a new school syllabus:

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[9] Quarta- Deutung vorgelegter Buchstabenausdrücke und Auswertung solcher Ausdrücke durch Einsetzung bestimmter Zahlenwerte.

Untertertia - Fortsetzung der Übungen in Auswertung von Buchstabenausdrücken unter Heranziehung der negativen Größen und steter Betonung des funktionalen Charakters der auftretenden Größenveränderungen.

Obertertia - Abhängigkeit eines Größenausdrucks von einer in ihm auftretenden Variablen. Graphische Darstellung einfacher linearer Funktionen und Benutzung dieser Darstellung zur Auflösung von Gleichungen.

Untersekunda - Betrachtung des von einer Variablen abhängigen quadratischen Ausdrucks in seiner dadurch bedingten Veränderlichkeit unter graphischer Darstellung.

> [Practical Mathematics. Elementary stage] Algebra: To understand any formula so as to be able to use it if numerical values are given for the various quantities ... The determination of the numerical values of constants in equations of known form, when particular values of the variables are given. The meaning of the expression 'A varies as B'.

Real numbers in the Scientific Lycée from 1923

We will now look at the further changes that happened in the programmes with reference to the real numbers. We will limit the observation to the programmes for the Scientific Lycée, which was born with the Gentile reform of 1923 (Marchi & Menghini, 2011).

In 1923 the programmes spoke of "absolute and relative real numbers". There was no particular suggestion, but real numbers were immediately followed by infinitesimal calculus and, afterwards, by their application to the measure of lengths and surfaces.

In 1925 and 1933 real numbers were connected again to the measure of geometrical objects, in 1925 an earlier approach to functions seemed to be suggested.

In 1936 real numbers were introduced as decimal numbers and were followed by the introduction of plane coordinates and not by their application to measure.

This latter reformulation is interesting. It was with no doubt a suggestion to avoid complicated geometric presentations of real numbers and to start soon with the Cartesian plane. Note that Severi, who still belonged to the Council of Public Instruction, never participated in the meetings of the commission, and the programmes were presented by the physicist Enrico Fermi. This probably explains the more practical character of these programmes.

In 1937 there seemed to be a correction, as decimals were "ratios between magnitudes", which can be commensurable or incommensurable. Continuity was also mentioned explicitly.

The changes to the programmes made by the Allied Commission in 1944 were very small. They seem to correspond to the programmes of 1936, with the slight changes suggested in 1937. As to real numbers, they didn't make a choice between the different formulations and simply wrote "concept of a real number".

But this doesn't necessarily mean that the teachers were free to choose, because, as said before, the textbooks mainly went on with a rational approach – which was mainly geometric – in terms of theory of magnitudes, proportions and *Classi Contigue*. After the 1950s also the approach by means of Dedekind sections entered in Italian schools, often substituting the *Classi Contigue*.

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The notion of method in 19th century French geometry teaching: three textbooks

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Abstract

The notion of method is referred to in numerous 19th century French geometry textbooks. The three most important textbooks under this consideration of the method are presented here. Their respective approaches turn out to be very different and thus show various facets of the notion.

Introduction

The notion of method is mentioned in several geometry textbooks of the French 19th century. It seems that this notion of method has been subject to profound modifications along the century. Three geometry textbooks could be found that were explicitly dedicated to the notion of method itself and that developed the subject at length. Those three are studied here, and show as many different conceptions of the notion of method: first an elaborate analysis, then a mere list of particular procedures and finally a structured set of prescriptive procedures.

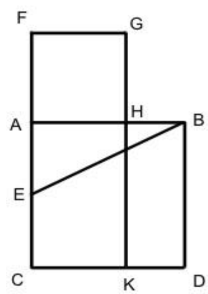
Most of the 19th century geometry textbooks mentioning the question of the method responded to the same preoccupations. The authors being most of the time teachers themselves were concerned with their students' comprehension and consequently intended to provide them with efficient tools for the resolution of geometry problems. And for instance, the importance of problems in geometry teaching had been increasing all along 19th century, and some textbooks included up to thousand of problems in the second half of the century. This development was accompanied by modifications of the very notion of problem, which evolved from a graphic construction to any kind of training exercise (Moussard, 2006).

The notion of method sends back to the scientific revolution of 17th century, a time when it knew profound transformations. "When defining method as an art of invention, 17th century made it a thought process instead of a manner of exposition, a way to acquire knowledge instead of a purpose, an intellectual exercise which can be undergone by anyone

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

instead of a scholar exercise"[1]. Such conceptions were present in 19th century textbooks dealing with method. In geometry in particular, they oppose methodical inventions to Euclidean demonstrations. Descartes reproached to the latter, in the *Règles pour la direction de l'esprit,* that they don't show to the intelligence why a proposition is true and how it has been discovered.

Barbin (2006) gives as an example Euclid's proposition 11 of book II. "The matter is to split in a point *H* a given line *AB* in such way that the rectangle with sides *HB* and *BD*, equal to *AB*, be equal to the square with side *AH*. Euclid gave in plain view the construction: to construct the square of side *AB*, *E* middle of *AC*, line *EB*, *EF* equal to *EB*, th
e square of side *AF*, then *H* is the requested point. Then he proved that the rectangle is definitely equal to the square out of a series of rigorous deductions […] The reader is compelled to ascertain that the result under his eyes is true, but he does not know how was discovered the construction of point *H*. Descartes is unsatisfied: he wants to know the process that enabled to invent it".



The method of Descartes has lead in mathematics to the creation and development of analytic geometry, whereas synthetic geometry designates geometry without the help of the resolution of equations. Though, Descartes' arguments of the *Discours de la méthode* may be revisited outside analytic geometry. Theses arguments reduce to four precepts: evidence, analysis, order, and enumeration. Evidence consists in "never receive anything as true, that I would not know it evidently being such". Analysis consists in "dividing each difficulty I would examine, in as many fragments as could be and would be required in order to solve it easier".

[1] "En définissant la méthode comme un art d'inventer, le 17e siècle en fait un processus de pensée et non un procédé d'exposition, un moyen d'acquérir du savoir et non une fin, un exercice intellectuel qui est accessible à chacun et non un exercice d'école". (Barbin, 2006, p. 35)

Order consists in "leading with order my thoughts, starting with the simplest and easiest to know objects, so as to raise gradually up to more composed knowledge". And finally enumeration consists in doing "reviews so detailed that I am assured not to forget anything"[2]. We will see how these features of the notion of method appear clearly in the textbooks studied here.

The concern for method has a particular importance in 19th century geometry. As wrote Kline, "in the early nineteenth century several great mathematicians decided that synthetic geometry had been unfairly and unwisely neglected and made a positive effort to revive and extend that approach" (Kline, p. 834). For mathematicians, this revival of synthetic geometry is made fruitful by a focus on method as an invention and generalisation process. Poncelet (1822) wrote in the *Traité des propriétés projectives des figures*: "I tried, above all, to refine the method of demonstrating and discovering in simple geometry"[3]. This intention of refining geometrical method, as will be studied in this paper, will be shared by several secondary geometry textbooks authors of the rest of nineteenth century.

Georges Ritt: a general method in synthetic geometry

Georges Ritt (1801–1864) was a departmental primary schools inspector when he published in 1842 his *Problèmes de géométrie et de trigonométrie rectiligne et sphérique avec les solutions*. He had a successful career and became in 1852 a general inspector for the ministry of public instruction. Among the variety of mathematics textbooks he published, this particular one is intended for the preparation to the various examinations placed at the end of secondary school, mainly the entry to the government schools and the baccalaureate. The book had a certain editorial success, as it was reedited nine times until 1894.

The *Problèmes de géométrie* are not only a wide collection of 675 pure geometry problems, much more than the maximum of 150 problems that could be found in the most provided contemporaneous textbooks. In fact, this textbook is above all dealing with the notion of method. It opens on a long and detailed presentation of "the method to be followed for the solution of geometry problems". This method was supposed, together with a great training, to enable any student to solve "the most difficult problems". Straightaway, Ritt presented his method as an "analytic method" in the following sense:

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[2] Descartes, *Discours de la méthode*, quoted in Pimbe, 2011, p. 24.

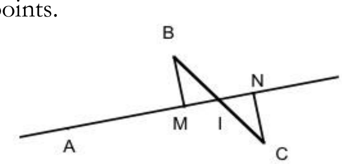
[3] Poncelet, 1822, preface p. vi : « j'ai cherché, avant tout, à perfectionner la méthode de démontrer et de découvrir en simple géométrie ».

> As the method that proceeds from the known to the unknown through inconspicuous degrees, and that stems from the simplest of intellectual faculties, can only have constituted the whole science when it still was an irregular collection of individual observations and experiments, is it not natural to have recourse to it when it comes to reach the knowledge of a new truth with the help of the established ones? Yet this method is nothing else but analysis, which, breaking the considerate object down to its simplest elements, makes the mutual relations between these basic ideas known, and then reconstructs them, the opposite way, in their initial unity[4].

Three precepts of Descartes' *Discours de la méthode* are referred to here: evidence, analysis and order, though without any reference to analytic geometry. According to Ritt, analysis is the method that has constituted the whole science, and it is natural to have recourse to it to reach new truths from established ones. Yet Ritt noted that analysis was in most textbooks used for the demonstration of theorems, but not generally enough for the resolution of problems, meaning here construction problems. His textbook's purpose was precisely to remedy this situation by implementing an analysis for the resolution of construction problems.

This analytic method is inferred from the observation of a few common problems' solutions exposed in the beginning of the textbook. Ritt presented their methodic solution, compared with their synthetic solution. The first problem exposed is:

PROBLEM 1. To draw through a given point a straight line that passes at the same distance of two given points.



Synthetic method. Through the given point A, and the point I, midpoint of the line that links the two given points, draw line IM: it satisfies the question. […]

[4] "Si la méthode qui procède du connu à l'inconnu par des degrés insensibles, et qui dérive de la plus simple des facultés intellectuelles, a seule pu former l'ensemble de la science alors qu'elle n'était qu'un assemblage irrégulier d'observations ou d'expériences individuelles, n'est-il pas naturel d'y avoir recours lorsqu'il s'agit d'arriver à la connaissance d'un vérité nouvelle à l'aide des vérités déjà acquises ? Or cette méthode n'est autre chose que l'analyse, qui, décomposant jusqu'à ses plus simples éléments l'objet soumis à son examen, fait connaître les relations mutuelles de ces idées élémentaires entre elles, et les reconstruit ensuite, par une marche opposée, dans leur ensemble primitif".

Analytic method. Suppose the requested line ANM be drawn; through each of the given points if a perpendicular is dropped to this line, the two perpendiculars BM, CN, will be equal from the given, and parallel from the construction, and the two triangles BIM, CIN, will be equal, having a side in common, adjacent to two equal angles: therefore the hypotenuses are equal, and the requested line goes through the midpoint I of the line that links the given points.

This attention paid to the method on a few particular solutions leads to its formulation in four points[5]:

1° *Hypothèses et constructions préparatoires;*

2° *Examen des relations entre les données et les inconnues du problème;*

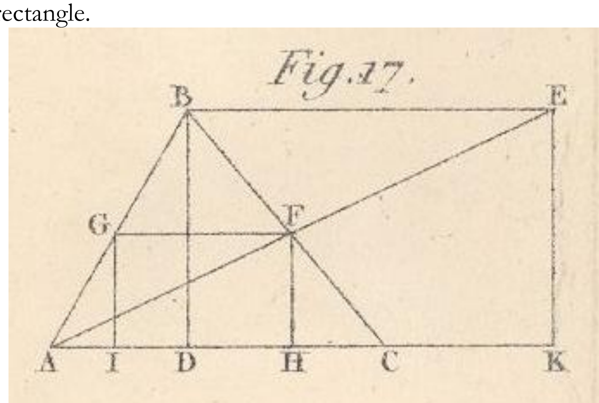
3° *Solutions et construction finale;*

4° *Démonstration.*

A quoi il faut ajouter une cinquième partie, très important, *la discussion.*

And the immediate continuation of the textbook is the illustration and commentary of these five steps on many particular problems. Here is one example:

PROBLEM 17. To inscribe in a given triangle a rectangle of a given kind, i.e. similar to a given rectangle.



The resolution starts with the problem considered as solved. The first step consists in drawing a few preliminary lines "in order to bring out the relations between the data and the unknowns". The data mean here the given figure, i.e. triangle *ABC*, and the unknown the lines to be drawn, i.e.

[5] 1° *Hypothesis and preliminary constructions*; 2° *Consideration of the relations between the data and the unknowns of the problem*; 3° *Solution and final constructions*; 4° *Demonstration*; To what should be added a very important fifth part, *the discussion.*

rectangle *GFHI*. The present figure is completed with lines *BD* and *AE* and then rectangle *BEKD*. The second step consists in realizing that the two rectangles *BEKD* and *GFHI* are similar, which constitutes "the relation between the data and the unknown of the problem". Third step is the effective construction of rectangles *BEKD* and *GFHI* successively. Fourth step is the synthetic proof that the rectangle *GFHI* is a solution to the problem.

As can be noticed with this example, Ritt showed to the reader the way he had followed to find the solution of a problem, in accordance with the outline of his general method. Nevertheless, the weak point of this method, from the point of view of generality, is the double choice of the "preliminary constructions" to be drawn and of the theorems to be referred to. All along the textbook, the author described these preliminary constructions as "immediate" and "natural", and the propositions to be used as "presenting themselves to the mind". To uphold such arguments of evidence, Ritt considered that some truths are so close that the mind slips "naturally" from one to another. In fact, he developed an elaborated conception of the corpus of geometrical truths as an organized and structured one. The base is constituted with easy constructions, combined together to form successively more complex ones. In this architecture, the analytic method to solve a problem makes sense as a backward route in the organization of geometrical truths:

> One will try to bring the problem back to a simpler one, the latter to another even simpler, and so forth, until one comes to a last problem the solution of which presents itself immediately; and one will conceive easily that such must be the approach of analysis, if one considers that the conditions that make a question complicated can only have been added successively to other simpler conditions, until that, after the combination of the various properties successively found, it has lead to complex questions of a higher order. (Ritt, 1842, p. 51).

Consequently, Ritt's textbook is innovative as a large collection of geometry problems presented for once independently from the theorems, and thought of as a structured set. But above all, it dedicates a first row importance to the question of the method. Ritt's intention is the description within synthetic geometry of an analytic method as general as the analytic geometry method that has recourse to algebra and equation solving.

Frère Gabriel Marie: the identification of a diversity of methods

Frère Gabriel Marie (1834–1913) was a member of the congregation of the Christian Schools Brothers and as such a productive author of secondary school science textbooks. His 1882 textbook *Exercices de géométrie comprenant l'exposé des méthodes géométriques et 2000 questions résolues* has a similar structure as Ritt's: it contains a wide first part about methodology, and a second one that is a list of 2000 exercises presented together with their solutions. These 2000 questions are of three types: they are either theorems to demonstrate, geometric loci to find or construction problems to solve.

The notion of method is totally different from Ritt's. If Marie referred to Ritt, saying that he took from him the idea of presenting a methodology for the solving of the problems, he nevertheless seemed not to consider Ritt's book as a proper answer to this challenge, as he wrote in (Marie, 1920, foreword) that "no French author, we believe, had yet published any methodology for the teaching of elementary geometry". Certainly, Marie, as Ritt, recognized analysis as the very method of discovery:

> To deal with a graphic problem by analysis, it is considered as solved; then the relations between the data and the unknowns are taken into consideration, and consequences are deduced until attested results are reached.[6]

But instead of providing a single general method, Marie classified within analysis several of what he calls particular methods. The six particular methods presented in the textbook are called: "geometric loci", "use of auxiliary figures", "transformation of the figures", "discussion and extension", "algebraic methods", and "maxima and minima". Each one of them is presented together with many examples. Problems are consequently, at least in the first part of the book dealing with methods, grouped under the criterion of the particular method used for their solution.

The so-called "geometric loci" method is described in the following way: "not considering one of the *datum* of the proposed question, one finds a line containing the requested point; then, taking the neglected condition, but not considering another *datum*, one obtains another line to which belongs the requested point; therefore, the intersection of the two

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[6] "Pour traiter par l'analyse un problème graphique, on le suppose résolu ; puis on considère les rapports des données et des inconnues, et l'on en déduit des conséquences jusqu'à ce qu'on arrive à des résultats connus". (Marie, 1920, p. 15)

geometric loci provides the requested point" (Marie, 1920, p. 15). Here is an example of this method:

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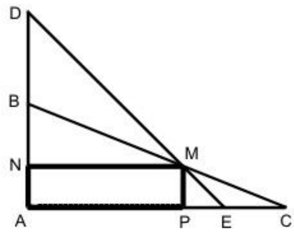
99. Through a point of the hypotenuse of a right-angled triangle, draw [an inscribed rectangle such that] the perimeter of the rectangle [be] of given length 2*p*.



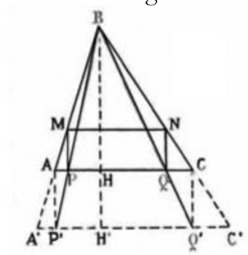
Point *M* must belong to the locus of the points which distances to the perpendicular lines *AB* and *AC* have sum *p*. Such locus is line *DE* where *AD* = *AE* = *p*. Second locus is here obviously line *BC*.

The second method presented is the "use of auxiliary figures". Marie wrote that it is "not much a method as a process which use is required by most questions to be treated" . Here can be seen how much Marie's discourse is different from Ritt's. Marie did not consider the use of auxiliary figures as a proper method, whereas it constitutes the first step of Ritt's analysis. Ritt tried to convince the reader that the auxiliary lines to be drawn are "natural" and "present themselves instantly to the mind". On the contrary, Marie described particular methods within which he can tell what kind of auxiliary figures should be drawn. This is clear in the previous example where line *DE* is drawn as a particular geometric locus obtained when "not considering one of the *datum* of the proposed question", for instance the condition that *M* be lying on line *BC*.

Next method is "the so-called transformation of the figures method" and "consists in replacing a given figure by a simpler figure". Let's consider for example the following problem:

With this example, Marie identified a group of problems that can be solved with this particular method: "*Similarity* is the natural method, for one wants a figure of a determined shape". When a figure of determined shape is to be constructed, the "transformation of the figure" method should be attempted, which means the construction of a simpler and similar figure. This idea, mentioned in some places in Marie's textbook, that there would be "natural" links between a method and a type of problems is fully developed by Petersen as will be seen in the next chapter.

209. In a triangle *ABC*, to inscribe a rectangle similar to a given rectangle"



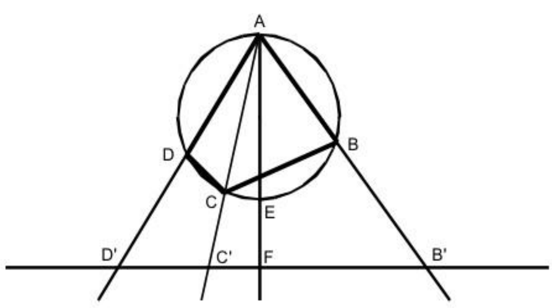
The figure to construct is replaced by another one, easier to construct, and providing finally the solution to the initial problem. "One should construct on *AC* a rectangle similar to the given rectangle, and join apex *B* to points *P'* and *Q'*; then raise the perpendiculars *PM*, *QN* […]"

Another example of this "transformation of the figures" method can be given in order to illustrate its efficiency. The theorem of Ptolemy is proved in a very elegant way:

226. In all inscribed quadrilateral, the product of diagonals equals the sum of the products of opposite sides"
"A is taken as a centre of inversion so that the circle and line B'D' are inverse one of another with power of inversion k^2.
Therefore from equality D'B' = D'C' + C'B' derives

$$DB.\frac{k^2}{AD.AB} = DC.\frac{k^2}{AD.AC} + CB.\frac{k^2}{AC.AB}$$

and then DB.AC = DC.AB + CB.AD"



"Discussion and extension" constitutes rather a method for the discovering of new geometric truths from established ones. Marie refers to the G*éométrie de position* from Carnot : "From one single figure well studied derive a mass of questions one encounters in most geometry exercises collections; but these various theorems, being presented independently one from another, let scarcely suspect their common origin” [7].

“Algebraic method” consists in solving equations. It is very common at the time, and many textbooks are specifically dedicated to it. Marie’s approach is not different except that it is presented as a mere particular method and it is not given any first role importance.

“Maxima and minima” is a method proposed for the resolution of the problems concerning the search of maxima and minima without having recourse to the previous method, i.e. to the resolution of equations. Though less fruitful than the algebraic method, this method provides “very simple” solutions. The idea is mostly the application of a few principles about operations on magnitudes. Here are the first of these principles and one of its applications:

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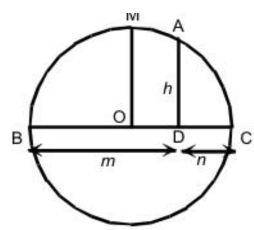
343. The product of two factors, which sum is constant, is maximum when the two factors are equal.



The proof given is geometric. It says that “on BC, constant sum of the two linear factors, *m* and *n*, let’s draw a half circumference. We have: $m.n = h^2$, therefore the maximum of the product occurs when factors *BO*, *OC* are equal”.

An application of this first of eight principles is the following problem:

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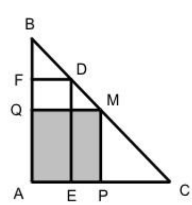
349. In an isosceles right-angled triangle, to inscribe the rectangle of maximum surface.



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[7] (Marie, 1920, p. 142): “D’une seule figure bien étudiée dérivent une foule de questions qu’on rencontre dans la plupart des recueils d’exercices géométriques ; mais ces divers théorèmes, étant présentés indépendamment les uns des autres, ne laissent guère soupçonner leur commune origine”.

"For any point of the base of an isosceles triangle, the sum of the perpendiculars dropped to the other sides is constant; therefore $MP + MQ = DE + DF$ therefore, from first principle, rectangle *MPAQ* is maximum when sides *MP*, *MQ* are equal".

Marie's work has thus consisted in classifying thousands of solutions of problems in order to identify general ideas common to whole groups of solutions, and then in characterizing these general ideas with the description of "particular methods": "the exposition of the methods enables to associate thousands of various exercises to a few principle types" (Marie, 1920, foreword, p. iv). Doing this, he reorganized the corpus of problems under the criterion of the method. This order of the method is respected in the first part's exposition of the various methods only, and is referred to when necessary in the second part's list of 2 000 exercises. Those are grouped in the order of the eight books of the *Elements*.

Besides, the search for a single general method valid for all the problems is replaced by the exposition of a variety of distinct particular methods, having each its corresponding field of geometry propositions. Marie moderated himself his achievement when saying that "particular methods only have a relative value; the great art is to know how to use them properly, according to the nature of the question to be treated" (Marie, p. 161). This "great art" was achieved by Julius Petersen, at least concerning the construction problems.

Julius Petersen: methods deduced from the properties of transformations

Julius Petersen (1839-1910) was a Danish mathematician. His PhD dealt precisely with constructions problems with ruler and compass, a subject he appreciated as challenging. He was not yet famous when he wrote in 1866 the first edition of the *Methods and theories for the solution of problems of geometrical constructions* and still had to teach mathematics as a living. Later, he will become a famous mathematician and get himself involved in education politics. His textbook was translated in French in 1880, and Marie referred to it for some demonstrations, though he considered Petersen's textbook rather different from his own, as it had "its own originality, either in the classification and exposition of the methods or in the choice of the questions studied". Though Petersen was not French, the large success of this translation in France explains why his textbook is taken into account in this paper.

As the title indicates Petersen's textbook focuses on constructions problems with ruler and compass. The author acknowledged that the field of problems taken into account is circumscribed. He explained the reason of such a choice saying that constructions problems "have hardly penetrated schools" although they "sharpen the faculty of observation and combination and provide the mind with clearness and logic". Petersen pretended to develop the mathematical skills of the students by the practice of constructions problems. "The object I have mainly in mind is method" he wrote in the foreword of the *Méthodes et theories*. The textbook is indeed a collection of methods, and the presentation of each method is followed by a long list of problems to be solved with it. One important difference here with the previous textbook is that each method is described once for all and this is considered to be sufficient for the resolution of all the following problems. Neither solutions nor figures are given, barely a few indications for the most challenging of the 400 problems.

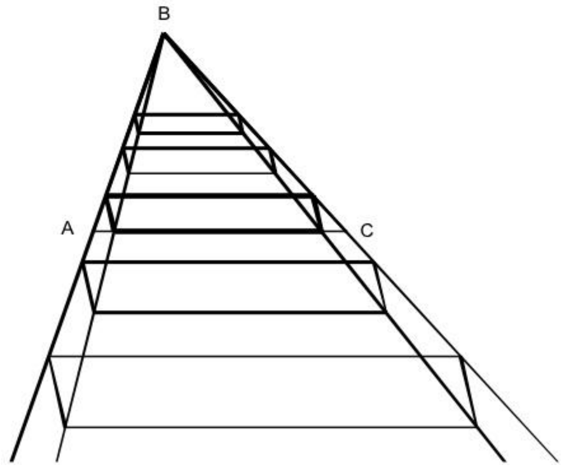
Like Ritt and Marie, Petersen described reasoning by analysis, in the sense of reducing a problem to another one easier or already known, and of decomposing a problem in several pieces of difficulty. More precisely, the textbook is divided in two chapters called "geometric loci" and "transformation of the figures". The first one presents several versions of the geometric loci method, and the second one exposes in various ways how to distort the given figure to bring back the problem to a situation solved in the first part when this one was not directly applicable.

The description of the "geometric loci" method is at first similar to what we saw in Marie's textbook: "one will consider one of the conditions imposed to the figure as inexistent, and one will try to find the geometric loci of the points of the figure, made in that way undetermined" (Petersen, 1880, p. 4). This method asks to know a certain number of geometric loci, and Petersen, exactly like Marie, started with the description of several basic ones that will be used for the resolution of the problems to come.

To face the problems up to then unsolved in the textbook, Petersen describes three more types of geometric loci obtained with particular relations between the figures called "multiplication of the curves", "method of similarity" and "inverse figures".

Let's illustrate with an example the "method of similarity". It applies, wrote Petersen, when "each time that leaving out one of the conditions required, one gets a system of similar figures". Then "one figure of the system leads easily to the solution, for the geometric loci of all points of the figure are straight lines through the center of similarity" (Petersen, 1880, p. 28). The solution of the following problem may be compared with Ritt's and Marie's solutions presented above in the case of the inscription of a rectangle:

156. In a given triangle, to inscribe a parallelogram similar to a given parallelogram



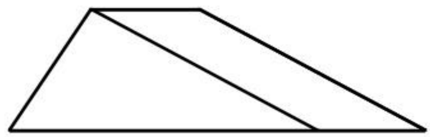
The resolution apparently expected by Petersen is to "leave out one of the conditions required", in this case the condition that the parallelogram should lie on line *AC*. One gets a system of similar parallelograms, represented on the figure, and the construction of anyone of them "leads easily to the solution" by drawing the lines that join up the points of the figure to the center *B* of similarity.

Obviously, the solution is based on a property of similarity rather than on properties of the present figures, a triangle and a parallelogram. The property of similarity used here is that the center is in a line together with any point of the figure and its homologous point. The particularity of the figures is absent from the description of this "similarity method". And the fact that Petersen did not provide any specific indications for the resolution of the various problems corresponding to one single method highlights that the particularity of the figures is not to be taken into consideration. The properties used for the constructions of the solutions are above all those of similarity here, and not properties of the particular figures.

In the second chapter of the textbook, Petersen considered the problems that could not be solved with the methods described so far. In such cases, from the requested figure should be deduced another one "the given elements of which are gathered in such manner that the construction can be done". In other words, it consists in displacing a part of the figure in order to assemble the given elements. Petersen considered three types of displacements: "the parallel translation", "the turning over" and "the displacement by rotation".

Let's illustrate with an example the "parallel translation". It is used "to move one closer to another the given elements, bringing some of the lines of the figure in new positions, parallel to the primitive positions". Here is an application of this method:

259. To construct a trapezium, knowing its four sides.

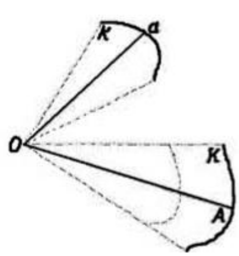


Petersen provides (for once) an explication: "Let's bring one of the non parallel sides all the way to the other one; we give birth to a triangle, the three sides of which are known" (Petersen, p. 52).

Here Petersen did not content himself with the drawing of the line parallel to one side through another apex. He formulated the construction in terms of a displacement, for instance a "parallel translation". This has two consequences. First, it presents the solution to a particular problem as an application of a more general idea that would consist in displacing a line of the figure in order to constitute a figure that can be constructed with the given elements. Secondly, the solution has recourse to the properties of the displacement itself – parallelism and equality of the two lines – and not of the particular figure.

The third displacement, the "displacement by rotation" is in fact treated apart in a specific third chapter called "theory of the rotation", a title explained by the new material and extensive development it contains. A rotation is described in Petersen's textbook in the following way:

> If from a given point O are drawn lines up to the points of a given curve *k*; and if these lines turn of an angle *v* around O, when at the same time they increase in a given ratio *f*, a new curve K is obtained as the geometric locus of the extremities of the lines that have turned (Petersen, p. 70).



As can be seen, Petersen presented this transformation as a relation between figures, though it is defined without any reference to any particular figure. The next few pages are dedicated to the establishment of various properties of this relation. Petersen proved for example that:

> Given three similar sets *A*, *B* and *C* and a point *O* that is the centre of rotation of *A* and *B* and at the same time of *B* and *C* […] and three homologous points *a*, *b* and *c* of these three systems […] the shape of triangle *abc* is constant, and for this reason is called *fundamental triangle* (Petersen, 1880, p. 79).

Reciprocally, it is proven that the shape of such a triangle with one apex one each system *A*, *B* and *C* characterizes one single centre of rotation. This result can be applied to three lines, considered as similar figures, with a given point on each line, considered as homologous points. The property says that there is one and only one centre of rotation common to the three pairs of lines associating the homologous points one to another. Besides, such a centre is characterized by the shape of the "fundamental triangle" constituted by the three given points. Now this property provides a solution to the following:

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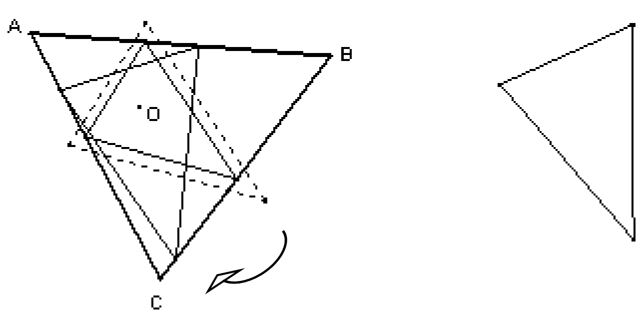
377. In a given triangle *ABC*, to inscribe another triangle congruent to a given triangle.

As a beginning, a first triangle of the given shape is inscribed in triangle *ABC*, using the same method as problem 156 (see above) for a parallelogram. Then, an immediate consequence of the property cited above says that all similar triangles inscribed in triangle *ABC* have a common centre of rotation (*O* on the figure). Consequently, the first inscribed triangle is increased in the proper ratio to be congruent to the given one, and then turned around point *O* until its apices fall on the sides of triangle *ABC*.



Petersen's *Méthodes et theories* are exclusively a method book. They are totally structured by the exposition of the methods and do not refer at all to *Elements* or any other kind of lessons, unlike Ritt's and Marie's textbooks. Petersen reached a high degree of generality by making the most of the properties of relations between figures, like similarity and inversion, and of displacements in the plane, like translation and turning over. The last chapter about rotation shows obviously that Petersen concentrated on the properties of transformations as powerful tools to solve geometrical constructions problems.

Conclusion

We have seen through the three textbooks surveyed in this paper that the importance taken by problem solving in French geometry secondary teaching in 19th century has lead to take into consideration the question of the method. The editorial success of these works shows undoubtedly a widespread concern of 19th century geometry teachers for method.

The three authors reviewed base their notion of method on a Cartesian analysis, in the sense of the *Discours de la méthode,* though within synthetic geometry. Furthermore, their construction process of a method can be compared with Descartes' own process:

> He [Descartes] elaborated the rules of a general method tackling various problems, i.e. starting from particular inventions, but this method may enable him to solve new particular problems[8].

This is explicitly what Ritt, Marie and Petersen have done. They intended to extract from a large set of problems' solutions the keys for the identification of one ore several methods.

Ritt constituted a wide and independent corpus of geometry problems. He claimed that analysis is the proper method to solve a problem, and described this analysis with many details so to make it as natural and easy as possible, in order to convince the reader of its generality. Marie spotted various particular methods and associated the solutions of the problems together with one of these particular methods. Petersen, as well, described a collection of methods. These methods structure completely the presentation of the problems, from the more simple to the more elaborated. They reach a high degree of generality by using the properties of transformations.

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[8] "Il [Descartes] a élaboré les règles d'une méthode générale en s'essayant à résoudre des problèmes divers, donc à partir d'inventions particulières, mais cette méthode peut lui permettre de résoudre de nouveaux problèmes particuliers". (Barbin, 2006, p. 43)

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Historical research in mathematics education: The case of the metric system in Spain

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Abstract

In this paper, we highlight some issues concerning the history of mathematics in mathematics teaching and some ideas on historical research in mathematics education and its contribution to these educational processes. This background forms a prelude to the presentation of the historical study we are currently conducting on the introduction of the Metric System in the Spanish education system in the second half of the nineteenth century and its implications for the teaching of weights and measures in the mathematics curriculum in Spain.

Introduction

For several decades, the role of historical research in mathematics education has become a subject of interest for researchers in this field. Connecting history to processes in the didactics of mathematics also attracts mathematicians, historians of mathematics and teachers in this area.

This interest has led, among other issues, to attention to and analysis of problems, methods, techniques, and concepts that form part of the teaching of mathematics as part of various historical studies (Anacona, 2003; Baumgart, 1993; Carrillo, 2005; Fauvel & van Maanen, 2000; Furinghetti, 2004; Gómez, 2003; Rico, 2003; Ruíz, 2003; Schubring, 2011; Sierra, 1997; Siu & Tzanakis, 2008).

We see the increased attention to history from the perspective of the teaching of mathematics in the growing number of studies that encourage the explicit inclusion of the history in teaching and its progressive value for the learning of mathematics.

Researchers such as Arcavi and Isoda (2007), Bagni, Furinghetti, and Spagnolo (2004), Fauvel (1991), Furinghetti (1997), and Radford (1997) characterize the positioning, modes and ways of including history explicitly in programs for the teaching—and thus ultimately the learning—of mathematics in different levels of mathematics education. In Spain, we find this tendency in authors such as De Guzmán (2007), González (1991), Montesinos (2000), Puig (2003), and Sierra (2000).

Firstly we will discuss the relationship between history of mathematics and mathematics education, its implementation—history of

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mathematics—in the teaching mathematical knowledge, and the role of historical research from a didactic perspective. Based on this foundation, we will present the purpose of and advances to be gained from a historical investigation of the study of curriculum reform carried out because the introduction of the Metric System in the Spanish education system in the nineteenth century.

History of mathematics and mathematics education

From a specific perspective, Maz, Torralbo, and Rico (2006) have emphasized the relationship between the history of mathematics and mathematics education through three considerations. First, they stress this relationship as the object of research to understand and interpret the way in which advances in mathematical knowledge have been incorporated into the teaching of this discipline; and how philosophical, political, social, and economic currents and cultural changes have influenced the way in which such knowledge has been presented and disseminated in society by means of the education system.

A second topic is the contribution that the history of mathematics education makes by revealing academic and cultural processes and processes of social cognition that give meaning to the mathematics concepts and structures that we know today.

Finally, these authors stress the contextualization of concepts, its use for curricular interdisciplinarity and motivation in the learning of mathematics. Figure 1 shows the connection between these three considerations. The horizontal connections indicate the possible uses that including history might have in educational practice. The vertical relationships focus on research as a method for studying the scope of this integration into the teaching and learning of mathematics.

From the latter perspective, historical research should gather evidence on the mode of discovery, contextualization and comprehension of mathematical concepts from history and provide explanations of how they were achieved.

This research enables us to verify the relationship between mathematics and mathematics education and different areas of knowledge, provides contexts that show these interrelationships and their practical treatment as a whole, and gives examples to foster their use so that they act as motivating elements in educational practice.

The use of history in the teaching of mathematics

One of the most striking phases in teaching activity consists of choosing strategies for putting the curriculum into practice. As Katz

(1997) points out, the teaching of mathematics requires good strategies for developing the class.

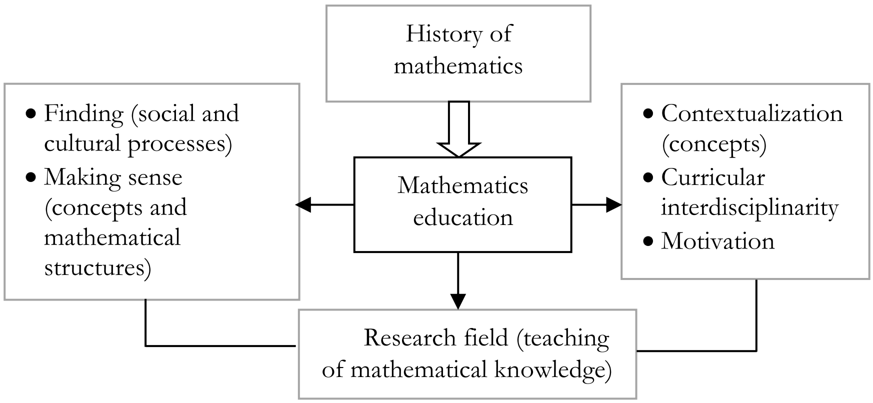

Figure 1. Including history in mathematics education

From a general perspective, in recent decades, the use of history has begun to gain in appeal as a strategy for teaching mathematics. Particularly in Spain, school learning of mathematics, as part of the minimum training for a common education of mathematics students, has focused on choosing the mathematics contents that must contribute to two fundamental goals of this learning: their utility for various areas of everyday life and their contribution to intellectual training that stimulates students' cognitive capacities (Ministerio de Educación y Ciencia [MEC], 2006). Within this framework, the Spanish curriculum for compulsory education stresses the historical and cultural dimensions of mathematics, both in the way of understanding mathematical knowledge and in their contribution to the development of students' basic competences (MEC, 2007).

It is the teacher's job to achieve these goals; his or her labour is crucial, since the teacher is the main agent for giving concrete form to the curriculum (Rico, 1997a). The teacher's practical activity is oriented, among other issues, to the selection, development, implementation and evaluation of teaching strategies for the students' acquisition, comprehension and application of specific mathematics knowledge.

The teacher finds in history a means of appropriating mathematical knowledge, ordering the presentation of topics, discovering obstacles, identifying difficulties, and detecting the errors that mathematicians themselves have made and the way of appreciating mathematics as human activity. Through strategies, activities and tasks proposed by their teacher, students can see history as an element that motivates them to understand the genesis of the concepts and problems that humanity has had to face (Sierra, 1997).

With history, it is possible to avoid a mechanical and automatic interpretation of the way in which mathematics is included in our knowledge. History facilitates the incorporation of teachers into a process of sensitization that enables them to identify their students' difficulties in understanding mathematical knowledge (Arcavi, 1991).

History in the mathematics curriculum

From the curricular perspective, we can characterize the incorporation of history in the teaching of mathematics as part of the socio-cultural approach to the organization of the mathematics curriculum for school. Rico (1997a) proposes four dimensions through which to organize the levels of curricular reflection. These dimensions consider specific ways of understanding knowledge, interpreting learning, putting teaching into practice, and evaluating the utility and mastery of the learning achieved. The dimensions focus on cultural/conceptual, cognitive, ethical and educational, and social issues in the curriculum.

Within this structural and systemic framework, we stress the historical consideration of mathematical knowledge and its relevance, since:

> *[This knowledge]* must be considered an integral part of culture, socially constructed and determined, influenced by the different educational needs of mathematics and the political and moral, general and specific connotations related to the mathematical education of students. *(p. 389)*

History becomes a key element in curricular design and development, an indispensable component of didactic processes for school mathematics.

As part of these processes, the mathematics teacher must, when planning a mathematics class, a topic or didactic unit, consider a series of additional elements of knowledge that enable him/her to design learning tasks, put them into practice and evaluate them.

To this end, Rico (1997b) stresses ways of tackling mathematics contents differently from their formal consideration, that is, by characterizing curriculum organizers. The following are examples of organizers:

- Errors and difficulties; problems or obstacles to learning.
- Diversity of representations and modelling.
- Phenomenology of knowledge.
- Manipulative materials and resources.
- Historical evolution of each mathematical field or concept.

History as a curriculum organizer enables us to point out specific historical moments in which the development of mathematical knowledge displays some unique qualities or plays an interesting and significant role in the history of humanity.

As to integrating history into the teaching of mathematics, Farmaki, Klaudatos, and Paschos (2004) stress that, in educational practice, history helps the student to understand that the errors, doubts, and diversity of perspectives involved in tackling a problem and the controversies that arise during the definition of mathematical concepts are an integral part of the development of this science. Furinghetti and Somaglia (1998) underscore the history of mathematics in establishing a process of interdisciplinarity in the school.

Other authors focus on the activities considered as tasks that are useful in helping students to benefit from the use of the history of mathematics in teaching. Among these, we would stress reading original documents, which enable one to study the nature of mathematical activity in its most varied aspects, such as the analysis of the role of problems, proofs, conjectures, evidence and error in the construction of mathematical knowledge. Reading ancient documents also enables us to access the epistemological and philosophical concepts embedded in these texts, to analyze the scientific, philosophical, cultural, and social contexts in which mathematical knowledge has developed, and to point out the cultural aspects of this knowledge from an interdisciplinary perspective (Barbin, 1991; Sierra, 1997).

Research in the integration of history in mathematics education

The role of history in school mathematics is not trivial. Integrating history into the teaching of mathematics does not mean collecting various events, anecdotes or biographies and making them an isolated, self-contained part of the class's development. "The value of historical knowledge does not consist of having a battery of curious stories and anecdotes to entertain our students and provide a break along the way (De Guzmán, 2007, p. 33)."

De Guzmán argues that integrating history into mathematics education supports the achievement of objectives such as: making the origin of mathematical notions clear; framing ideas and problems temporally and spatially according to each period of mathematical development; distinguishing the problems of each period and their subsequent evolution and current state, and showing the connections between mathematics and other sciences that have led to important knowledge. These goals highlight a series of orientations for the professor that underscore the didactic meaning of history for the teaching of mathematics.

Reaching this point requires specific historical studies that provide the information needed to organize and highlight historical aspects of the contents of school mathematics and to contribute to a constructive and motivating discussion of their origins, particularities, development, and application. It even includes tackling issues that belong to the didactics of concepts and structures relevant to the mathematics curriculum.

Historical research provides interpretations of specific situations considered problematic in the area of education, whose meaning could not be accessed in any other way. It promotes understanding the origins of specific practices, rules and institutions in the education system of a country; describing the rise and development of different educational theories; and providing an understanding of the relationship between politics and education, institution and society, and teachers and students.

In mathematics education, historical research has been an object of analysis in various research studies. In Spain, we would point out studies like that by González and Sierra (2003), which present the method of historical research in the didactics of mathematical analysis; by Gómez (2003), who systematizes historical research in the Didactics of Mathematics; and Bruno and Martinón (2000), who examine school mathematics for secondary teaching in Spain in the twentieth century. On the historical evolution of specific mathematics topics, Sierra, González, and López (1999) hold a prominent place for tackling the historical evolution of the concept of the limit of a function in texts for the two most recent college-preparatory high school curricula (Bachillerato and the course for orientation to the university [COU]). Recently, Maz (2005) studied the treatment of negative numbers in Spain in the 18th and 19th centuries; Picado (2009, 2012) has worked on the treatment of the Metric System in Spanish mathematics texts in the 19th century, focusing on the period from 1849 to 1892, and López (2011) on the teacher's training in Arithmetic and Algebra through textbooks. All these studies focus on historical aspects of teaching and learning the concepts involved and they integrate historical research into mathematics education.

These considerations show historical research as a labour with a sense of utility and significant application in mathematics education, currently characterized by its attempts to clarify problems through the study of materials from other time periods. The idiosyncrasies of historical research and its utility in education studies, especially in the didactics of mathematics and the contributions that it provides to the community of educators and to educational processes, have enabled researchers like González and Sierra (2003) to consider historical research in education as a current that is "attractive, passionate, and fruitful" (p. 109) in the field of the didactics of mathematics.

The study

Historical research provides an in-depth analysis of mathematical concepts and structures over time. Our interest in it stems from the contribution that this analysis can make to the teaching of different mathematics contents.

We focus our study on one mathematical structure and its teaching, one social medium, and one specific period: the Metric System (MS) in the Spanish education system in the second half of the 19th century. Our historical investigation has a dual purpose. Firstly, we wish to investigate the curriculum reform that occurred as a product of adopting a new system of measurement and its inclusion in the Spanish educational system. Secondly, we seek to decipher the concepts and their interpretation as components of a mathematical structure, the MS; its forms of representation and the situations used in presenting it; and the goals and competences, difficulties and errors, tasks and materials proposed as part of the teaching methodology.

The introduction of the MS in the middle of the 19th century involved a considerable change in Spanish society in the areas of science, commerce, administration, education, and society. This change affected Spaniards' ideas and practices of measurement and quantification, causing a transformation in the concepts, usages, and customs that had predominated up to that time.

Introducing the new units of weight and measure as compulsory required the development of a process that included formulating and publicizing new legal regulations, organizing and disseminating the knowledge that structured the new system of measurement, and systematizing and regulating the rules for its application. The universalization and dissemination of the new system was carried out as a political decision by the Government and articulated through the public administrations of the period. This process took place over several decades, in particular the period 1849-1892, defined by the laws of weights and measures put into effect July 19, 1849 and July 8, 1892 for the adoption of the MS and the compulsory use of the new weights and measures, respectively.

Within this general framework, our study focuses on the curriculum change initiated by Article 11 of the Law of July 19, 1849, one of the instruments for the dissemination of the new measurement system in Spanish society through the education system. From this perspective, the Spanish education system had to respond to the requirements of legal orders and regulations involved in putting into practice the regulations stemming from the 1849 Law of Weights and Measures:

> In all public and private schools in which arithmetic is taught or should be taught or in any other area of mathematics, it will be compulsory to teach the legal system of weights and measures and its scientific nomenclature, starting the first of January, 1852; the Government is authorized to close these establishments if they do not comply with this regulation. *(Ministerio de Comercio, Instrucción y Obras Públicas [MCIOP], 1868, p. 2)*

The political demands to optimize the resources for the introduction of the MS in the teaching of mathematics lead to a change in the mathematics curriculum of the time. Specifically, research examines the treatment of the MS in the Spanish educational system in the period 1849-1892 and the didactic characteristics of mathematics texts for primary and secondary education and for the training of teachers, as well as curricular documents that specify the political-educational regulations and the curriculum changes to carry out the introduction of the new measurement system in Spain.

From the perspectives of Howson, Keitel, and Kilpatrick (1981) and of Rico, Sánchez, and Llinares (1997), curriculum reforms involve a change in the defined goals, content, teaching methods and means of evaluation. That is, in the period considered, this change should reflect changes in the learning expectations not only of the MS, but also of Arithmetic in general; in the contents of the documents chosen for the teaching of mathematics and the relation of the new units to other blocks of mathematics contents. This should produce a change in the ways of organizing how this material is taught and how the new topic is tackled; it should in turn reformulate or consider suggestions for evaluating the new learning.

But political currents are not usually the only forces that exert pressure on a curriculum change in the teaching of mathematics. Other forces can influence and drive a curriculum change (Rico, Sánchez, & Llinares, 1997). These forces may originate in social groups or movements, in mathematics, education and desires for innovation. At the same time, one finds forms of resistance that work against the transformation of the mathematics curriculum, especially due to values, changes in power relations, practical questions, or psychological motives.

Isabel II's enactment in Spain of the Law of July 19, 1849 marks the most intense form of political pressure, which itself responds to the urgent need for a unification of the weights and measurements due to commercial and trade abuses that affected most Spaniards. Adopting the meter as the only and fundamental unit of the new system, from which the other units for measuring the size of surface and volume and the related measures in weight and capacity are derived, had the effect of bringing clarity, simplicity, and efficacy to commercial transactions that would influence the search for solutions to the artificial and ineffective character of measure derived from the prior multiplicity of measurements in Spain. Further, this change meant an advance in the development of mathematics as an applied discipline and in the ways of presenting knowledge.

There was also a desire to improve relations in international trade: the unification of weights and measures became a way to achieve this goal, as it served to modernize Spain's contribution to the scientific, mathematical and commercial advance of Europe.

From the mathematical point of view, the close relationship between the MS and the Decimal System (DS) had a marked influence on the organization of the mathematics curriculum. Mastery of the DS, as a numerical system used in society and on which, at the time, the teaching of arithmetic was grounded, facilitated the introduction of the DS. Moreover, the study of decimal fractions took priority over the general study of fractions.

As to the barriers that a curriculum reform might have encountered on introducing the MS in teaching plans, we should point out the strong popular support of the population for the old measurements, which were called the measurements of Castile. This support constituted one of the most difficult obstacles to overcome in adopting the new weights and measures in Spain. In spite of the need and the demand for a unification of measurements that would eradicate the abuses and difficulties in the commercial arena, it was not easy to put the use of unknown weights and measures into practice. The reform would have to involve a change in the common ideas and practices of merchants and citizens.

From another point of view, the unification of weights and measures signified a loss of power, of the prerogatives and privileges that some local agents maintained to preserve commercial gains derived from changes in units from one territory to another. Similarly, the adversaries of the change attempted to impede the dissemination of the advantages of the new system, creating an obstacle for formulating the curriculum reform and putting it into action.

We must also mention the obstacles to teaching the reform and putting it into practice for teachers and professors, since teachers lack both sufficient preparation to teach the new system of weights and measures and the school documents and materials appropriate for implementing the reform.

We should also mention psychological factors as grounds for rejection, since the use of a new system of weights and measures could cause citizens to feel insecurity and fear, normal feelings in the face of something new and unknown.

The result of the forces discussed above, both for and against the change, shows how the educational system of the period contributed to the implementation of the new policy of weights and measures. The panorama described indicates that we are primarily interested in investigating the way in which this curricular change occurred as a result of introducing the MS into the education system in Spain.

Mathematics texts as historical documents

Studying this curriculum reform cannot be achieved without using texts—primary sources—that provide the information necessary for our study and that constitute the means of putting it into practice during the period. As Vea (1995) proposes:

> The information that one wishes to have about instruction at a given historic moment can be obtained with considerable precision through the textbooks used.
>
> The importance of complementing the educational information with the contents reflected in the materials used for teaching is greater, if possible, because the specific programmes of the different subjects are not shown clearly in the different legislative orders. *(p. 33)*

A change in the programmes for mathematics was established, but the details of its organization—objectives, contents, methods of teaching, and evaluation—are not made clear in the documents that went into effect—laws, royal decrees or orders, and study programmes. Some of the latter simply indicate the inclusion of the MS as part of the contents of arithmetic for primary and secondary instruction. There were textbooks—documents commonly used in the classroom—that show these details, namely the inclusion in the arithmetic contents of the new weights and measures, multiples and divisors, conversion tables, procedures for the correct use of the new terms and units, and examples and exercises for everyday use in which the new weights and units were to be applied. Therefore, the textbooks at this stage of the 19th century play a relevant role as a means of establishing, describing, presenting, and publicizing the curriculum or a relevant part of it.

To summarize, our problem is based on two questions: What treatment was granted to the MS in the Spanish education system in the period from 1849 to 1892, and what didactic characteristics did mathematics textbooks have as documents for carrying out the curriculum reform proposed in this education system in light of the adoption of a new system of weights and measures.

Methodology

We then turned to a selection of mathematics textbooks as curriculum documents used for the teaching of the MS in establishments for primary and secondary education and Normal Schools (school for training teachers) published in the period 1849-1892.

This methodological orientation leads us to analyze the textbooks from a mathematical perspective to identify, describe, and characterize the conceptual structure, representations, and situations that frame the MS as a mathematical structure. This analysis also involves studying the expectations, limitations, opportunities, materials, and tasks for the learning proposed to tackle the research didactically.

The content analysis focuses on the technique of Didactic Analysis, a procedure useful for exploring, understanding in greater depth, and working with the different meanings of school mathematics knowledge in the design, putting into practice, and evaluation of didactic activities from the local level of the curriculum (Gómez, 2002; Rico, 1992). This technique is organized into a cycle composed of single issues that give rise to four different analyses: analysis of the mathematics content, cognitive analysis, instruction analysis, and action analysis, all of which are tied to the dimensions of the curriculum discussed above. For this study, we emphasize the three first analyses.

The study of the meaning of the mathematics contents is taken from the proposal by Rico, Marín, Lupiáñez, and Gómez (2008). The study is accomplished under three dimensions of meaning:

- Conceptual Structure, understood as an organized system of concepts and procedures that considers the three levels of conceptual knowledge: facts, concepts, and structures; and the three levels of procedural knowledge: skills, reasoning, and strategies. The descriptions of the concepts in which they are justified, their interrelations, and the mathematical structure are given particular attention.
- Systems of Representation, related to the ways of making perceptible the different notions and properties that constitute a specific mathematics concept and show its different facets.
- Phenomenology, which determines the situations and contexts in which the contents to be studied are presented. That is, those social, natural, or formal phenomena whose abstraction gives rise to a mathematical structure.

We implement the content analysis in the historical studies of mathematics education (Cohen & Manion, 2002; Fernández Cano & Rico, 1992), emphasizing its use as a technical tool in the analysis of old mathematics texts (Picado & Rico, 2011a).

The cognitive analysis attempts to recognize the objectives and competences, the difficulties and errors, and the tasks planned or proposed or for teaching a specific content (Lupiáñez, 2009). The instruction analysis studies the design, selection, and sequencing of the tasks as components of a didactic unit—the text—and including issues of classroom management, such as the use of materials and didactic resources and evaluation (Lupiáñez, 2009).

Preliminary study

In 2009, before the present study, we established a preliminary approach to this problem. On this earlier occasion, we explored the treatment of the MS in Spain from 1849 to 1892, the period that defines the promulgation in Spain of the two laws on weights and measures that mandate the adoption and the compulsory use of the system, respectively.

This initial study enabled us to establish three historical phases in this period, with which we can locate and characterize the mathematics textbooks used in disseminating the metric-decimal units in the different social spheres in Spain during the period, such as the administration, commerce, education, science, and technology. Other results stress a series of styles defined for these texts and their authors and list some questions on the evolution of the way of presenting the metric system and its components in mathematics texts—conceptual structure, representations, and contexts—from its adoption in 1849 to its compulsory use at the end of the 19th century.

We have derived two additional studies from this analysis and the corresponding selection of texts. The first has enabled us to characterize this process of dissemination of the MS in some texts published in Cuba, Puerto Rico, and the Philippines, Spanish colonies overseas in the second half of the 19th century. The results of this study form part of the presentations at the 13th Inter-American Conference on Mathematics Education (Picado & Rico, 2011b). The other study analyzes two texts for the dissemination and teaching of the MS in Spain: "Elementary Memorandum on the new decimal weights and measures, grounded in nature" ("Memoria elemental sobre los nuevos pesos y medidas decimales, fundados en la naturaleza") by Gabriel Ciscar y Ciscar, published in 1800 and "Explanation of the decimal system or French Metric System" ("Explicación del sistema decimal ó métrico francés") by José Mariano Vallejo y Ortega in 1840. This analysis forms part of the papers presented at the First Ibero-American Congress of the History of Mathematics Education (Picado, 2011).

Advances

The current study focuses on three perspectives: mathematical, educational, and historical. The mathematical approach tackles the research problem from the concepts that converge in establishing and organizing the MS. The educational approach enables us to focus on the Spanish curriculum and the didactic processes of a specific period. The history connects the mathematical and the educational, while also supporting the use of a specific research method. All of this constitutes the essential foundation of our study: research into the history of mathematics education in Spain. The bibliography consulted has provided data for contextualizing and describing the historical antecedents of the research.

As part of the methodology, and to fulfil the study's purpose systematically, we have proposed a series of objectives for organizing the phases of selection and analysis for the study.

The phase of selecting the sources began with the search for preliminary studies on the use of texts for teaching mathematics in primary and secondary instruction and the training of teachers in Normal Schools in the 19th century. The lists of texts from these studies and the definition of the selection criteria for representative texts throughout the period established and our rigorous pursuit of the purpose of the study have provided a preliminary list of 114 textbooks. We are currently identifying and locating these sources in Spanish libraries and archives centres for future review and final selection of the texts to be analyzed.

Once we have finalized our choice of texts, we will analyze them. The analysis will use a series of categories for extracting the data required by establishing statements and relationships concerning the process of introducing the MS into the Spanish education system, its inclusion in school mathematics textbooks, and the evolution of the way of organizing the units of weights and measures in this system, emphasizing the idiosyncrasies in the concepts, representations, situations, objectives, difficulties and errors, and the tasks and materials presented for teaching and learning them in the curriculum documents.

Final comment

Researching the history of a mathematics concept or structure requires exploring its definition, representation, and use in common practices in the context and environments in which it is generated and developed and in which it evolves and is consolidated. This means studying the progress and regressions in its adoption; how it is disseminated, transmitted, taught and learned.

The research in progress seeks to clarify various questions on the introduction of the MS in the Spanish education system, the curriculum change and its inclusion in school mathematics texts for three educational levels: primary instruction, secondary instruction, and the training of teachers in Normal Schools. The study will enable the establishment of justified opinions, judgments, and evidence that characterize this period in mathematics education in Spain, focusing on one of the most important mathematics contents and areas of knowledge in the history of the study of measurement systems worldwide: the MS.

The outcomes of the study could become a resource of data to be used in the future to enrich the process of teaching and learning the MS in mathematics education in Spain, and the methodology used could be adapted to other contexts to generate historical results that can be implemented in the mathematics curriculum in other countries. But, fundamentally, this study could demonstrate the social, scientific, political, educational, and cultural influence of human groups in establishing, developing, and transmitting mathematics knowledge. These are facts, situations, and events that constitute a key piece in presenting and discussing the processes of emergence and consolidation of mathematics in the area of education.

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On a missed attempt to introduce history of science in curricula. An analysis of Paul Tannery's program for the secondary school

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Abstract

In 1891, History of science nearly managed to find its place in French lycées. Indeed, while reshaping curricula, Elie Rabier, Head of the Secondary School Department, asked Paul Tannery to write a program for that new teaching. The Ministry soon abandoned his plan. However, Tannery is offered at the same moment by the influent Ernest Lavisse to write the chapters on the history of science for his General History from the 4th Century to the Present. He found there the opportunity to develop his program on history of science, in a frame that had been directly aimed at teachers. With these two complementary texts, the teaching program of and the chapters on science for the General History of Lavisse, we want to provide a historical perspective on the teaching of history of science and mathematics, at the turn of the twentieth century. Focusing on mathematical topics, our goal is to lighten the interpretation of history and the practices of teaching that Tannery suggested to implement in secondary schools.

Introduction

The opportunity to introduce history of science in secondary school has been discussed for years; it has also and often failed to reach and maintain a position in curricula[1]. In this paper our goal is precisely to come back on an early but missed attempt to introduce history of mathematics and science in French secondary curricula. This happens in the last decade of the XIXth century.

In July 1891, the Ministry of Public Instruction engaged an important reorganization in the French secondary school. It especially defined a 'modern course' whose objectives were, by founding a real "elementary mathematics class," to balance the classical course and its "philosophy class" (Belhoste, 1990). Though the expression "history of science" had been convoked in previous curricula, a real effort seemed to happen then :

> For this reason of general education of the mind, the teacher will not neglect the history of science. The student sometimes gets less benefits in the presentation of a truth than in the history of its discovery. Through his initiatives, his doubts, and even his mistakes as much as by his successes, the working scientific genius provides a highly

[1] An overview of that issue, in the French context, can be found in (Hulin, 1984).

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

> suggestive and moral teaching to young people. Finally, the science teacher can link his lessons to those of the professors of letters, history and philosophy. Using as brilliant examples and while exposing the evolution and laws of nature, he will outline the laws and progress of the human mind. Thus, he offers his own contribution to the teaching of history and humanities[2].

An even greater step seemed to be taken some months later. In the early 1892, while reshaping curricula, Élie Rabier, Head of the Secondary School Department (Verneuil, 2006), was given back the draft of a program he had requested to implement History of Science in the upper class of the modern course. Indeed, Rabier was not thinking over a history-based alternative science teaching; in other words, he did not try to reaffirm the distinction that Auguste Comte had made between a dogmatic study of science and a historical one[3]. Rabier needed for pupils a really independent teaching in History of Science, that is, a somewhat counterpart in the modern course, to the teaching of Philosophy in the classical course. Taught in contemporary science, pupils would have been also given some milestones and train of thought on its development.

To prepare this program, Rabier had required Paul Tannery[4], a state engineer, known to be a or 'the' French specialist in History of science, at this very moment: being preparing a critical edition of Diophantus, he was also working on that of Fermat, and ending his third book on Greek science, his *Researches on Ancient Astronomy*.

Eventually pushed away, the proposal has been preserved and even posthumously published (Tannery, 1907). Moreover, if he had not managed to impose his *Program* in the classroom, Tannery had found

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[2] General instruction for the sciences teaching in the modern course. Original text : "Pour cette même raison d'éducation générale de l'esprit, le professeur ne négligera pas non plus l'histoire de la science. L'élève a parfois moins à retirer de l'exposé d'une vérité que de l'historique de sa découverte. Le génié scientifique en travail fournit à la jeunesse, par ses initiatives, ses doutes, ses erreurs même, autant que par ses succès, un enseignement éminemment suggestif et moral. Par là enfin le professeur de sciences peut relier ses leçons à celles des professeurs de lettres, d'histoire, de philosophie. Tout en exposant les lois et l'évolution de la nature, il fait connaître, comme eux et par des exemples non moins éclatants, les lois et les progrès de l'esprit humain ; il collabore à sa manière à l'enseignement de l'histoire et des humanités", (Belhoste 1995, p. 539-540).

[3] In the second lesson of his *Positive Philosophy*: "Every science may be exhibited under two methods or procedures, the Historical and the Dogmatic. […] By the first method knowledge is presented in the same order in which it was actually obtained by the human mind, together with the way in which it was obtained. By the second, the system of ideas is presented as it might be conceived of at this day, by a mind which, duly prepared and placed at the right point of view, should begin to reconstitute the science as a whole…"

[4] On Paul Tannery and his role in the development of history of science, see (Pineau, 2010).

another way to develop it. Indeed, Ernest Lavisse[5], a French influential historian in this moment, offered him to write chapters on science for his *General History from the Fourth Century to the Present*[6]; a book that Lavisse aimed directly at secondary teachers, for he assumed their need for "good handbooks" covering history up to the end of XIXth century[7].

With these two complementary texts, the *Program* of and the chapters on science for the Lavisse's *General History*, we are given the opportunity to study how, a century ago, History of Science – and especially History of Mathematics – has been thought in relation with secondary education. In the next pages, we will consider some of the main epistemological and historical options supported by Tannery, and we will try also to clear the pedagogical practice he was expecting.

Elements for a pedagogy in history of science and mathematics

More than a foreword to the contents of his *Program*, Tannery wrote a long preliminary discourse, delivering *Advice and Directions* to the teacher. This introduction doubtlessly do not claim to present a complete pedagogy in history of science and mathematics, and sounds close to the General Instruction, that had been published in 1891 (see above). However, it offers some first interesting elements on Tannery's views.

As a first evidence, the balance between occurrences of the terms[8] 'students' (3) and 'teacher' (9) in that text deeply favors the second. The teacher stands in the center of a traditional disposal, the lecture, that reinforces him in his role of unique holder and provider of knowledge. According to Tannery, the teacher has to 'show, define, trace, make clear, expose, provide insight, develop, illustrate', or again "maintain the attention" and "choose and propose a detailed study". On the other hand, pupils are only listeners, worded in the passive. Indeed, the suggested pedagogy exclusively relies on the teacher, intended to "develop as much

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[5] In his *Realms of Memory*, Pierre Nora has precisely devoted a study to Ernest Lavisse 'The Nation's teacher' and his in the construction of a national consciousness (Nora, 1997). On Lavisse's pedagogy, see (Bruter, 1995).

[6] The whole work by Lavisse is under reference (Lavisse & Rambaud, 1893), in which Tannery's contribution is spread along the twelve volumes: III, 5-II (244-262); IV, 8 (306-324); V, 11 (450-491); VI, 10 (394-429); VII, 15 (726-762); IX, 11 (361-392); X, 20 (733-767); XI,25 (940-966); XII, 18 (557-580). To only read Tannery's contribution, see its reprint in his collected works. From now and in the next pages, we will refer to this reprint (Tannery, 1893), unless otherwise specified.

[7] Actually, the program for History in 1890's curricula had been written by Lavisse himself, who insisted on the importance to study history up to the present. See for instance (Ministère de l'Instruction publique, 1890, p. 47-88).

[8] Or their pronouns.

as possible in pupils some general ideas"; it is on him not to forget that "too short historical information often only leave false ideas in pupils' minds" or again "to make pupils feel the need to...", (Tannery, 1907, p. 386-387).

Centered on the teacher, the previous list of actions insists on the speaking dimension of his job. Anyway, some indication on his preliminary abilities as holder and provider of knowledge may be inferred from Tannery' work. In this, the teacher has to provide general statements on the whole progress of science and that of each discipline. But, he must also be able to complete a detailed study on topics he would have especially chosen to illustrate these general statements. By the expression of 'detailed study', Tannery exhorts the teacher to deal as completely as possible with any chosen particular issue, and warns him not to confound a detailed review with an empty nomenclature. Hence, the previous advice refer less to the completeness of historical facts and data, than to the depth of the historical analysis. The concepts of 'knowledge' and 'taught knowledge', which specialists in Education Studies have been trying to tell apart for some decades, are still strongly coincident, in Tannery's mind. However, this may be an attempt to legitimize the teaching of history of science, but an attempt aimed directly at teachers, the firsts to convince for its introduction in curricula[9].

What about the actual position of this prospective teacher in history of science: does he already teach mathematics, science, or rather lecture history, or philosophy? The previous General Instruction took it for granted that History of science should take place in the science teaching. But here, the silence of Tannery sounds like hesitation. It suggests one of the most likely arguments that leads Rabier to abandon his idea to introduce history of science in the class of 'elementary mathematics'. Without a special training, few teachers at the end of the XIXth century, if Tannery himself, should have been able to review, with as many details, ancient mathematical works on conics, Paracelsius' Alchemy, circulation of the blood by Harvey or XVIIIth century controversies on the figure of the Earth, all topics in the suggested program.

[9] Hence, the end of the *Advice and Directions* is interesting to note. Indeed, from the issue of teaching, Tannery suddenly goes to the question of historiography: "[the teacher] will not forget that the historical study of sciences must not trace only the progress of human mind in the pursuit of truth, and has to remember its errors. He will not forget that the historical study of sciences must not trace only the progress of human mind in the pursuit of truth, and has to remember its errors; the healthy appreciation of these errors is the only way to achieve the understanding of the true importance of sciences".

Which history of science and mathematics for the Lycée?

From now, our goal is to cast light on the global interpretation of history defended by Tannery, in his *Program* and contribution to Lavisse's work.

Actually, a difference occurs between these texts and concerns periodization. Following the usual three-term school year, the *Program* divides history in three main periods: Antiquity first; then Middle Ages to the first half of the XVII^th^ century; the third and last term is devoted to the study of sciences up to recent times. Tannery's contribution to Lavisse *General History* is chronologically ordered too. But first, it lacks any review on ancient science; indeed, Lavisse has evinced Antiquity from his plan. Then, and above all, Tannery's chapters follow a ten-term unusual periodization; but actually this only relies on the global structure of Lavisse's whole work, and does not really reveal any of Tannery's preconceptions on the History of science and mathematics[10].

To summarize the following ideas, either in the *Program*, or in his contribution to the *General History*, the historiographical option is clear. Not only do mathematics and sciences progress, but this movement is first oriented towards our present state in the sense of rationality increase; then it happens in a European-centered frame, with different phases that will sound familiar to us, post-kuhnian readers.

Moreover, in *Advice and Directions*, Tannery insisted on the idea that the teacher has "to show the rational connection that has bound the evolution of any science, either with that of others, or with that of civilization in general". But, this guideline is balanced from the start, as he precises "the different sciences will be succesively considered, in the following order: arithmetics and geometry, mechanics, astronomy, physics, chemistry, natural history" (Tannery, 1907, p. 387) – that is more or less A. Comte's classification of sciences. Then focusing on mathematical topics of the *Program*[11] do not cause to our study any real loss in generality; on the contrary, this will even help us highlight some limitation in this program.

In Tannery's words, 'civilization' obviously refers to the European one. Though he assumes that Ancient people (that is by Egyptians and Chaldeans) used to get practical knowledge, he defends the 'Greek miracle' above all. To him, "Ancient Greece did create Theoretical

[10] In the beginning of the chapter devoted to sciences from 1559 to 1648, Tannery warns his reader that this period could be divided in two. The first half of the period had no well-marked character, whereas a decisive movement happens in the first decades of the XVIIth century, that is the birth of the modern thought. (Tannery, 1893, p. 146).

[11] In the Annex below, we have collected the mathematical topics from the *Program* and chapters of Lavisse's *General History*.

science, as we now conceive it" (Tannery, 1893, p. 124). Elsewhere, he wrote "the formation of the concept of science is the work of one only people, the Hellenic people that communicated it to the others" (Tannery, 1906, p. 221). Thus, Chinese or Indian mathematics are not present in the *Program*; and Tannery considers that mathematics had been completely elaborated in Europe, first by Greeks, and later in countries under Latin influence.

Blamed for their lack of originality, mathematics in medieval Islam get little favor from him, and are only mentioned for their role in "the transmission of Greek science, to Latin west" (Tannery, 1907, p. 389).

A few years later, Duhem would refurbish Latin Medieval science. But in the last decade of the XIXth century, Tannery kept sounding severe to an epoch when, he explains in the *General History* (Tannery, 1893, pp. 125-127), 'science does not yet exist', despite some "rare genies" in mathematics, Fibonacci, Oresme, or Chuquet. But these, he adds, were isolated and they "surpassed their contemporaries so far they could not to establish a school". From what he calls the "Barbarian ages" to the XVth century, arithmetics and geometry are only *practical knowledge*, not yet theoretical sciences:

> In the Middle Ages, the arithmetical theory did not appear as a deductive science at all; that was a sequence of propositions, easy to check afterwards or simply based on an insufficient induction, but whose study relied on memory more than on reasoning. This can also be said of problem-solving: there was a collection of rules, each standing for a special case; the benefit of general methods has not yet been felt and the power of invention not aroused[12].

From then, Tannery's thought sounds before his time, very closed to a kuhnian interpretation of history. If he, of course, does not explicitly refer to notions like paradigm, scientific revolution, normal science, Tannery gives us the main ingredients. Following the medieval period, and its several different but also isolated theories, the Renaissance times and XVIIth century sat up the paradigm of modern mathematics. He considers that this moment has lead to "a definitive return to Greek sources and a reawakening of trends towards pure science" (Tannery, 1907, p. 389).

[12] (Tannery, 1893, p. 126). Original text: "La théorie de l'arithmétique n'apparaissait donc nullement au Moyen Age comme une science déductive; c'était une suite de propositions faciles à vérifier a posteriori ou simplement fondée sur une induction insuffisante, mais dans l'étude desquelles la mémoire jouait en tous cas un plus grand rôle que le raisonnement. Il en était de même pour la solution des problèmes (toujours numériques): on avait une collection de règles valant chacune pour un type particulier ; on ne soupçonnait pas l'intérêt des méthodes générales ; la faculté d'invention n'était pas mise en éveil".

And he adds that, "even before it has fully assimilated the Greek mathematical books, once preserved by Byzantines, the Modern mind asserts itself and departs from Ancient knowledge, unveiling news ideas and its own field of exploration" (Tannery, 1893, p. 146). To him, that innovative period covering both XVI^th and XVII^th centuries is precisely related to some great figures of mathematicians - first Cardano and Tartaglia, then Viete and Descartes, and eventually Leibniz, Huygens and Newton – who established scientific theories on "foundations hereafter unwavering".

Then comes the XVIII^th century and what sounds *normal science*, or more exactly in Tannery's words:

> Any creative time, such as the XVII^th century, is followed by a period when the principles, once discovered by innovative genies, are normally developed. During this period, individual efforts diverge rather than focus towards a common goal; all directions have to be explored, in order to recognize how far can lead the new way to use. The development end can be noticed within attempts to coordinate all the results; this work is done by other genies, different but as powerful as the innovators used to be[13].

Actually, Tannery has tried to give a general frame to his previous description: first the many analytical consequences that are implied in the first part of the XVIII^th century by Leibniz and Newton conceptions; and then the effort of synthesis achieved by Lagrange, Laplace, however leading to some moment of crisis in 'normal science'. Dealing especially with physics, Kuhn associates crisis with discoveries unexplained in on-going theories. Considering the history of mathematics, Tannery suggests another form of crisis, happening as a backlash of the synthesis of the whole knowledge. He found an evidence of it, in the mathematical weariness that Lagrange had experienced, believing for a while to have explored analysis up to its furthest consequences. Tannery notices:

> Meeting the spirit of research, these attempts of coordination also imply consequences; either a break in the scientific progress, as seen in Antiquity, or a major change in direction, when it fortunately happens

[13] (Tannery, 1893, p. 295). Original text: "À toute époque créatrice, comme le XVII^e siècle, succède une période où se déroulent normalement les conséquences des principes dus aux génies novateurs. Pendant cette période, les efforts individuels divergent plutôt qu'ils ne se concentrent vers un même but ; car il s'agit de reconnaître, dans toutes les directions, jusqu'où peut conduire le nouveau moyen à employer. Le terme du développement est marqué par des tentatives de coordination de l'ensemble des résultats acquis, travail qui réclame des génies d'une autre nature, mais d'une puissance au moins égale à celle des novateurs proprement dits".

> in the same time, as it occurs at the end of XVIIIth century and the beginning of the XIXth century: a new renovating thrust[14].

To end his chronological overview, Tannery tries to cast light on that renewal happening in the XIXth century: first noting an evident trend towards physical applications, he soon recognizes another phenomenon in recent mathematics (and science), that is a domain of knowlege dividing in more and more specialized and independent fields; however a domain searching for a new unity (paradigm) with a new special interest on foundations.

Have we given an accurate picture of the history of mathematics, which Tannery is expecting to be taught in secondary school? Actually not, but we can get a better one, by observing, for instance, the complete absence of any really technical content in his contribution to Lavisse's *General History*, or again the large place given to portraits of great scientists. A discussion on the program limitations can help understanding this historiographical option.

A simple reading through the *Program* contents reveals evident lacks. The most significant one concerns the invention and early development of infinitesimal calculus in the XVIIth century. Indeed, The *Program*[15] does not mention Cavalieri, Descartes, or Fermat methods to get tangents or quadratures; but, then Leibniz and Newton inventions and controversies are only left a very short place: that is a double page in the *General History*; worse, Leibniz is not mentioned in the *Program* whereas Newton is… but for his only great physical theory. This introduces a real difficulty, when Tannery then overviews 18th century mathematics, with the point of view, we have remembered earlier. Surely, not a heterodox historiographical position, it reveals, however, limitations in this outsized ambition to fit a full overview of the history of science within a curriculum.

Indeed, Tannery's *Program* and contribution to Lavisse's work share the same difficulty to deal with a history implying mathematical notions, which students have often even not heard about. This issue becomes particularly pressing for the recent times, and Tannery considers that "it becomes difficult to appreciate, without special studies, the progress in mathematics since 1815" (Tannery, 1893, p. 336). Actually, it occurs far earlier, as we have just got an evidence, when we have noticed the unusual

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14 (Tannery, 1893, p. 295). Original text: "Ces coordinations satisfaisant l'esprit de recherche, ont à leur tour pour conséquences, soit un arrêt du progrès scientifique, comme on l'a vu dans l'antiquité, soit un changement de direction générale, quand il se produit heureusement en même temps, ainsi que cela est arrivé à la fin du XVIII^{e} siècle et au commencement du XIX^{e} siècle, une nouvelle poussée rénovatrice".

15 Here is a difference between the *Program* and the chapters in Lavisse's *General History*. See below.

short summary on the invention and early developments of infinitesimal calculus. Indeed, in the 1890's secondary teachers were not supposed to give a large introduction to calculus[16]; hence, Tannery's embarrassment or even certain omission, when he comes to Leibniz and Newton mathematical inventions. Aimed at teachers more than at pupils, his chapters for the *General History* do not need to fit exactly with mathematics in curricula. Therefore, Tannery offers a short summary on the controversy between Leibniz and Newton, though keeping away any technical analysis. What may strike, when we remind of Tannery's advice to the teacher to avoid either any empty nomenclature or too short historical indications.

This failure is more likely to happen when it comes to deal with recent history. Indeed, Tannery assumes that mathematics "used to take what seems to us a relatively simple way, because it was based on a few guiding ideas" (Tannery, 1893, p. 336), so that it still seems possible to give an accurate picture of it. But, he then considers that, from the beginning of XIXth century, mathematics seem to experience a phenomenon of move forward, with a profusion of new special results occurring from and in many directions. The *Program* contents may delude an inattentive reader, but in the *General History*, Tannery hardly manages to set his work beyond a factual nomenclature[17]. An evidence of this is the sudden use of the expression "elements on…" among the issues for the last term of the school year. Tannery then confesses:

> The task of the historian becomes the hardest; everyone cannot be given the part which is due to him, unless getting into infinite detail. In the next few pages, I could not do more than briefly reporting some of the major advances and some of the new ways that have been opened to researchers[18].

---

[16] That is, according to the program published in 1891 :"Representative curve for a function - notion of derivative. The derivative is the angular coefficient of the tangent. Variation of the following functions :

$$y = ax^2 + bx + c, y = \frac{ax+b}{cx+d}, y = \frac{ax^2+bx+c}{dx^2+ex+f}.$$

For this last function, only numerical examples will be given. Note : In the study of the variation of the previous functions, Suffice it to introduce the derivative of a sum, a product and a quotient." (Belhoste, 1995, p. 345).

[17] For example, we can compare, in the Annex, the mathematical contents of chapters on science in the XVIIth century, and in the period between 1815 and 1847.

[18] (Tannery, 1893, p. 381). Original text: "La tâche de l'historien devient des plus ardues ; il est impossible de rendre à chacun la part qui lui est due, à moins d'entrer dans un détail infini. Je ne puis avoir d'autre prétention, dans les quelques pages qui suivent, que de signaler brièvement quelques uns des progrès décisifs et quelques unes des voies nouvelles qui ont été ouvertes aux chercheurs."

These words sound like acting as a serious brake on Tannery's ambition to complete an overview of the whole history of mathematics for pupils.

Before ending, a short word on the large place given to portraits of mathematicians (and generally scientists) either in the *Program* or in the chapters, an evident parallel can be made with the moral role usually found in the teaching of History at the moment. Here, Tannery explains that, by choosing interesting details in the life of "illustrious scientists", the teacher will make pupils feel the need to know them better; a clear echo to the 1890 *Instructions, Programs et Règlements* that quoted the famous chemist and politician J.-B. Dumas "Teach your students to know and revere the names of famous men who created science"(Ministère de l'instruction publique, 1890, p. 172).

Conclusion

Some years later, surveying current researches in history of mathematics and recent handbooks by Cajori, Rouse Ball and others, Tannery expressed doubts towards the real benefit for pupils of such long-term syntheses. In contrast, he suggested to offer pupils a conceptual history, focusing on the issues present in mathematics curricula. For each one, the teacher's main objective would be "to follow its evolution either in practice or theory up to its current form"(Tannery, 1900). Three more years, and Tannery gave a limited but real try to this new way to have pupils learned in history of mathematics. Pointing the benefit to give 'some historical elements' in the mathematics classroom, 1902 curricula offer Tannery a real aim for a common work with his brother Jules, mathematician and chair for science studies at the École normale supérieure. Indeed, they soon publish together a textbook (1903) including all mathematics notions for the classical (!) upper class, but also some twenty pages, covering half a dozen of historical notions connected to the previous mathematical ones; and among these, a special issue is devoted to the invention of infinitesimal calculus, a notion also included for the first time in mathematics curricula[19].

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[19] The five other historical notions concern: Origins of Algebra; the meaning of the terms analysis and synthesis by Greeks, and their geometrical algebra; Positive and negative quantities; the curves that Ancients studied; the origin of the use of coordinates to give a graphical representation of variations in phenomenons.

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Annex:

Mathematical topics in the Program in History of science and the chapters by Tannery in the General History from the IVth century to the Present

Program in History of science	Chapters in Lavisse's *General History*
Practical knowledge *that have formed the basis of pure science theories. Development of that knowledge at the various degrees of civilization - Level by Chaldean and Egyptian people.* **Emergence of pure science**, *by Greeks in the VIth century BC. Its double trend: abstract (mathematics), concrete (natural sciences in general).* **Pythagoras and his school** *Establishment of a scientific teaching - Classification in arithmetics, spherical geometry (astronomy), music, experimental discovery of numerical ratios concerned with musical scale - Progress of mathematics in the IVth century (Academy) - Importance of the pythagorean classification; Quadrivium and Trivium in the Middle Ages.* **Alexandrian period** *(from Alexander's conquests to the establishment of the Roman Empire): Protection of pure science under the ptolemies - Foundation of the Mouseion at Alexandria - Euclid: elementary geometry. Apollonius: geometry of conics. on the usefulness of Governments' support to pure theoretical research: their unexpected practical applications (conics and astronomy, other historical examples) - Archimedes, his geometrical researches and discoveries in statics - Mechanics by Ancients: Hero of Alexandria - Scientific astronomy: Hipparchus.* **Greek-roman period** *(up to Constantine): Inability of Romans in the sciences: they remain stationary - Coordination of previous works: Ptolemy.* **Period of decadence** *Practical trends in mathematics teaching*	

Program in History of science	Chapters in Lavisse's *General History*
Barbarian Ages *Preserved practical knowledge in Latin West, after the fall of the Roman Empire: surveyors, ecclesiastic calculations.* **Islamic science** *Scientific development of islamic civilization: lack of originality in this development - Its importance for the transmission of Greek Science to Latin West - Mathematics and astronomy.* **Origin of modern numerals** *their introduction in the West - Elements on the Greek and Roman numerical notations - Calculation using abacus - Gerbert.*	**Scientific knowledge up to the XIVth Century** *Default of an exact conception of science - Arithmetics and calculation - Geometry - Astronomy.*
Renaissance period *Definitive return to Greek sources and reawakening of trends towards pure science - Progress in mathematics teaching: Tartaglia, Cardano – Copernic's hypothesis, renewing the idea Aristarchus of Samoa.*	**Sciences in Europe for the first part of the XVIth century** *Mathematical sciences: arithmetics and algebra, geometry, astronomy.*
XVIIth century (first part) *Viete: Invention of modern algebra - Napier: logarithms - Kepler: his laws, and how they made Copernic's hypothesis triumph and lead to the discovery of universal gravitation.*	**Sciences in Europe from 1559 to 1648** *General overviews - Role of the different European nations - Number theory - Modern algebra: Viete - Geometry - The problem of quadratures - The problem of tangents - Astronomy - The last astrologer: Kepler.*
XVIIth century (second part) *Newton: mathematical optics.*	**Sciences in Europe from 1648 to 1715** *Academies - Scientific journals - observatories - Leibniz (1646-1716): the infinitesimal calculus - Newton (1642-1729): the universal gravitation - Huygens (1629-1695): rational Mechanics - Optics - Mathematics.*

Program in History of science	Chapters in Lavisse's *General History*
XVIIIth century *Progress in mathematics and astronomy - Elements on the nature of problems that can be solved - Clairaut and Halley's Comet - Flattening of Earth at its poles: definitive confirmation of Newton's theories - Origin of political economy: statistics.*	**Sciences in Europe from 1715 to 1788** *Leibniz heirs: the bernoullis, Euler, Lagrange - Newton's school: Taylor, Maclaurin - The French geometers: Clairaut, d'Alembert - General characteristics of the scientific movement in the XVIII*[th] *century - The encyclopedic attempt.* **Sciences in Europe from 1789 to 1814** *The transformation of scientific education: the École polytechnique and the École normale - Pure mathematics: Lagrange, Monge, Carnot, Gauss - The system of the world: Lagrange - General summary of the whole scientific movement.*
XIXth century *Elements on the current trends, increasingly abstract, in pure mathematics - New practical results: discovery of a planet by Le Verrier - Physical applications of mathematics: Fresnel, Joule.*	**Sciences in Europe from 1815 to 1847** *General overview on the progress of mathematics - Modern geometry: Poncelet, Chasles, Möbius, Steiner - Non-euclidean systems: Lobatchefski, Bolyai - Analytical geometry: Plücker, Hesse - Algebra: Hamilton, Grassmann, Galois - Analysis: Fourier, Cauchy - Function theory: Abel, Jacobi - Number theory: Lejeune-Dirichlet - Mechanics: Poinsot, Poisson, Lamé - Astronomy: Le Verrier, Bessel, Hansen.* **Sciences in Europe from 1848 to 1870** *The problem of scientific education - Mathematical sciences - Geometry: Chasles, Von Staudt, Culmann, Cremona, Riemann, Lejeune-Dirichlet, Helmoltz, Beltrami, Plücker, Clebsch - Algebra et analysis: Hankel, Peirce, Cayley, Sylvester, Hermite, Bertrand, Fuchs, Kummer, Smith, Rosenhain, Borchardt, Riemann, Dedekind.* **Sciences in Europe from 1870 to 1900** *The scientific education - Mathematical sciences - Geometry: enumerative geometry, construction of notion of the distance, theory of curves and surfaces - Algebra: continuity and set theory, universal algebra - Analysis and function theory: group theory, differential equations, general theory of functions, number theory - Mechanics and Astronomy.*

A course of mathematics (1798-1841): The American story of a British textbook

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Abstract

A course of mathematics is an English mathematics textbook first published in 1798 by Charles Hutton, professor at the Royal Military Academy in London. The document, which praises the mathematical science for its practical power, is characterized by deliberately general-interest content and a pragmatic style. Following its publication in England, the textbook was immediately and integrally sent to the United States. In 1812, a pool of New-York publishers ordered an American version of A course of mathematics from Robert Adrain, a professor whose works and contributions to scientific papers had given him some recognition. The textbook was then annotated, corrected and reorganized. With the transatlantic circulation of the texts, both the English and the American versions were updated until 1841, with each version being used as an incomplete source for the other. These reciprocal updates enlighten a slow but undeniable differentiation of the two books; the American course gains a specific identity especially with the integration of French mathematical content. The analysis of the evolutionary path of this textbook illustrates how the mathematics taught in America, previously strongly influenced by their English peers, became more independent in the first decades of the nineteenth century.

Introduction

Between 1607 and 1776, the American territory was under British domination. English values, experiences, books and institutions influenced education, in primary and secondary schools as well as in colleges. Harvard, founded in 1638 was, for example, initially a seminary modelled on Oxford and Cambridge.

The declaration of independence marked a will of resistance to the former political and cultural authority, as well as a will of progress. Until the beginning of the nineteenth century the textbooks were mostly British books or their local adaptations[1]. Some American textbooks, printed in the United States, started to emerge as a version of Euclid's *Elements* in 1784. In 1800, one could count 19 colleges which remained strongly influenced by their British peers regarding the contents of what was taught, see (Cajori, 1890), the pedagogical methods, the aim of education and the undertaken values. But the beginning of the nineteenth century witnessed a change in the way mathematics were perceived and done. Societies, journals and college mathematics departments were established.

[1] Many examples can be found as *The schoolmaster's assistant, being a compendium of arithmetic, both practical and theoretical* by Thomas Dilworth (1811).

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

In 1804, *The Correspondent*, the first mathematical journal, was published. The contents, especially in geometry, were at first linked to British mathematics. However, as the century progressed, they were increasingly influenced by French scholars in the first half of the century and by Germans in the second, see (Parshall & Rowe, 1994, Chapter 1). In this political and educational context, the adaptation of a foreign, especially English, textbook is a relevant gate to understand how mathematics in America were taught and were changing in the beginning of the nineteenth century.

A course of mathematics is an English mathematics textbook first published in 1798 by Charles Hutton, professor at the Royal Military Academy in London. The book became a best-seller, in England, where it was republished several times, and abroad where it was translated in other languages (in Arabic for example) or adapted in English-speaking countries. In the United States, the original textbook circulated in the first decade of the century, along with many other English books.

In 1812, a pool of New-York publishers ordered an American version of the *Course,* from Robert Adrain, a mathematics professor who had achieved some recognition due to his work as a teacher, his professional appointments and the papers he published in scientific journals. The textbook was then annotated, corrected and reorganized. This paper will analyse the reasons why this textbook was chosen and the nature of the adaptation(s). With the transatlantic circulation of the texts, both the English and the American versions were updated during the first half of the century. How did the modification of each version influence and inform the other and how can the evolution of the relationship between the two documents be understood? The paper argues that this double migration illustrates the development of American mathematics education at the beginning of the nineteenth century.

The American sources analysed in this paper are the 1812, 1816, 1822, 1825 and 1831 editions (respectively the first, second, third, fourth and final edition) written by Robert Adrain. The Hutton's English edition being analysed is the sixth edition (1811) that had been used by Adrain for his first American edition. The eleventh English edition, edited by Olinthus Gregory[2] in 1837, was very helpful to draw echoes between the English and American versions during almost 50 years[3].

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[2] Olinthus Gregory (1774-1841) was an English mathematician, also professor at the Royal Military Academy. He wrote many mathematics textbooks, and held, in 1818, the editor's position of *The Ladies' Diary*, a long-time mathematics journal in the English press of the eighteenth and nineteenth centuries.

[3] Nevertheless, this comparison is necessarily incomplete, due to the lack of some English editions of the text.

Charles Hutton and Robert Adrain: a portrait of two curiously similar authors

A comparison of the life and works of the authors[4], Charles Hutton (1734-1823) and Robert Adrain (1775-1843), reveals oddly similar portraits of two men involved in the development and spread of mathematics education in their respective countries.

Fig. 1. Charles Hutton

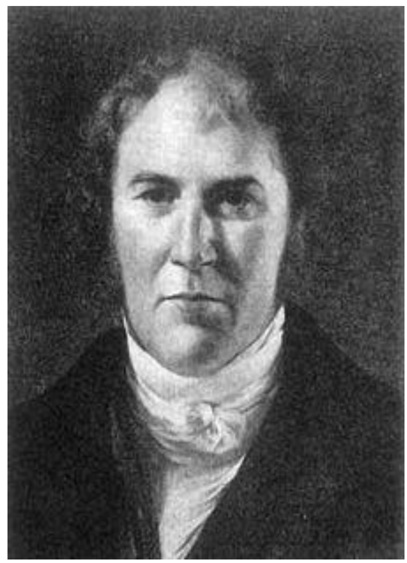


Fig. 2. Robert Adrain

Charles Hutton and Robert Adrain were each self-educated, especially in mathematics. At the age of fifteen for Adrain, and twenty for Hutton, they both became school masters. Hutton published several maths textbooks: in 1764, *The schoolmaster's guide, or a complete system of practical arithmetic*, and in 1797, *A course of mathematics: for the use of academics as well as private tuitions*. Those books were written for the help of schoolmasters. During his whole carrier, Hutton was always concerned about teacher-education. Robert Adrain was also interested in the publication of textbooks. He was responsible for the American adaptation of the *Course of mathematics* which included five editions over a period of twenty years.

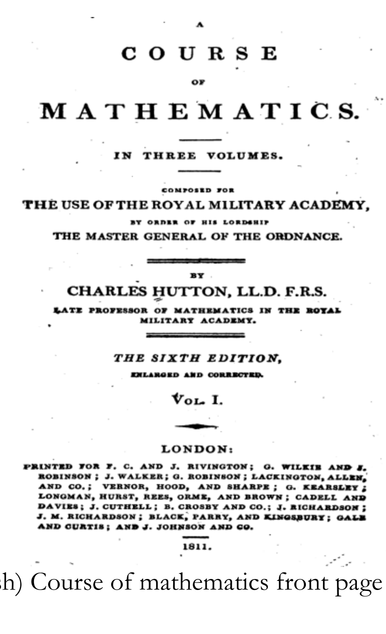
The careers of both men, as professors of university, were remarkable. Hutton was hired as a professor at the prestigious Royal Military Academy

[4] For a more complete overview see (Howson, 1984) for Hutton and (Coolidge, 1926; Hogan, 1977; Swetz, 2000) for Adrain.

of Woolwich in 1773, while Adrain held several positions at Queen's College (now Rutgers) in New Jersey, Columbia in New York and Penn between 1809 and 1834. Hutton and Adrain's activities turned also towards journal edition. In 1774, Hutton became the editor of the famous *Ladies' Diary*, the longest-running English mathematics journal ever published. In 1804 and 1825 the American author launched, as editor, *The Analyst* and *The Mathematical Diary*, the second and third American mathematics journals.

The motto of the *Course of mathematics*: useful and practical content

The British *Course of mathematics* contains three volumes and 1200 pages. The document, which praises the mathematical science for its practical power, is characterized by deliberately general-interest content and a pragmatic style. Its structure is organized in five sections: Arithmetic, Algebra, Geometry, Fluxions and Natural Philosophy. This last section is a compilation of several subjects including astronomy and mechanics. The definitions and the principles are immediately followed by many examples and exercises. At the end of each section, one can find more complex problems.


A
COURSE
OF
MATHEMATICS.

IN THREE VOLUMES.

COMPOSED FOR
THE USE OF THE ROYAL MILITARY ACADEMY,
BY ORDER OF HIS LORDSHIP
THE MASTER GENERAL OF THE ORDNANCE.

BY
CHARLES HUTTON, LL.D. F.R.S.
LATE PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY.

THE SIXTH EDITION,
ENLARGED AND CORRECTED.

VOL. I.

LONDON:
PRINTED FOR F. C. AND J. RIVINGTON; G. WILKIE AND J. ROBINSON; J. WALKER; G. ROBINSON; LACKINGTON, ALLEN, AND CO.; VERNOR, HOOD, AND SHARPE; G. KEARSLEY; LONGMAN, HURST, REES, ORME, AND BROWN; CADELL AND DAVIES; J. CUTHELL; B. CROSBY AND CO.; J. RICHARDSON; J. M. RICHARDSON; BLACK, PARRY, AND KINGSBURY; GALE AND CURTIS; AND J. JOHNSON AND CO.

1811.

Fig. 3. The (British) Course of mathematics front page (sixth edition, 1811)

The preface to the textbook explains why Hutton chose this style, approach and material:

> It [the *Course*] is not designed to hold out the expectation of an entire new mass of inventions and discoveries: but rather to collect and arrange the most useful known principles of mathematics, disposed in a convenient practical form demonstrated in a plain and concise way, and illustrated with suitable examples, rejecting what ever seemed to be matters of mere curiosity, and retaining only such parts and branches, as have a direct tendency and application to some useful purpose in life or profession." (Hutton, 1811, 1, Preface)

Using several years experience in teaching methods, he gathered in his book the mathematical knowledge he had thought useful to every student. According to Hutton, a textbook had to be as convenient and practical as possible. The importance of the demonstration was low because it was seen as pure curiosity. Indeed, mathematics were perceived as a practical branch of science useful in everyday life and work. The book was written for the help of students, but of teachers as well. The last lines of the preface contain advice, as for a user's guide: teachers could omit the footnotes, study the book starting with any of the five sections or not use every example.

New York and the order from publishers

In 1812, Robert Adrain, published his American version of the *Course*. At first glance the reader may be led to believe that this was a local edition of Hutton's text. The contents were expressed with exactly the same wording. The editions (regarding typography, mathematics symbols, images and drawings) were almost identical.

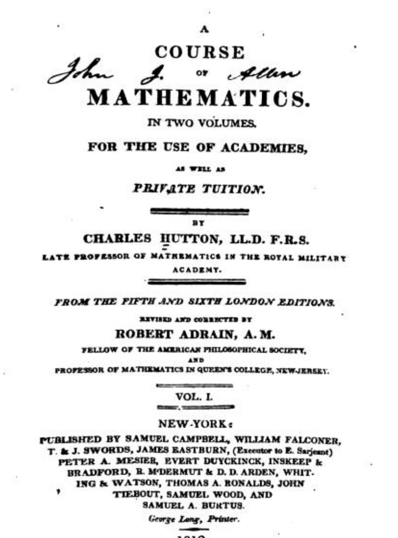
In order to understand what exactly distinguished the two books, it is necessary to know why and how this textbook was chosen to be adapted. A group of New York publishers hired Robert Adrain, to revise the pages of the *Course* and propose an American version. This is clearly established in a note before the preface:

> … they [the publishers] engaged a gentleman of acknowledged eminence to revise its pages and superintend the printing; and they confidently trust this duty has been performed with some profit to the work generally. (Adrain, 1812, 1, Advertisement)

Why did they choose this book? First, *A course of mathematics* possessed several qualities: it embraced all the branches of mathematics for "its utility to private students as well as to Colleges and other Seminaries in which Mathematical Science constitutes a branch of education" (Adrain, 1812, 1, Advertisement). It was also an explicit way to give a comprehensive mathematics textbook for the Military Academy of West

Point[5], as the original textbook was written for the training of cadets in the English Military Academy. Thirdly, the publishers fulfilled a financial purpose: earning money in a time when "an increasing taste for mathematics" (Adrain, 1812, 1, Advertisement) was seen. The last objective dealt with flattering the patriotism among future buyers, promoting a strong and independent culture from the British press:

> To gentlemen, therefore, who study this delightful science in private, and to the literary and military institutions of their country, the publishers and proprietors look for remuneration – and they feel as though they should not look in vain. An increasing taste for Mathematics Studies will produce a correspondent increase of purchasers; while the preference which an honourable patriotism gives to American editions when well executed, will receive additional activity from the super-eminence of the work itself. (Adrain, 1812, 1, Advertisement)



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ROBERT ADRAIN, A. M.
FELLOW OF THE AMERICAN PHILOSOPHICAL SOCIETY,
AND
PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, NEW-JERSEY.
VOL. I.
NEW-YORK:
PUBLISHED BY SAMUEL CAMPBELL, WILLIAM FALCONER, T. & J. SWORDS, JAMES EASTBURN, (Executor to E. Sarjeant) PETER A. MESIER, EVERT DUYCKINCK, INSKEEP & BRADFORD, R. M'DERMUT & D. D. ARDEN, WHITING & WATSON, THOMAS A. RONALDS, JOHN TIEBOUT, SAMUEL WOOD, AND SAMUEL A. BURTUS.
George Long, Printer.
1812.

Fig. 4. The (American) Course of mathematics front page (first edition, 1812)

[5] The United States Military Academy located in West Point was founded in 1802. In 1817, Sylvanus Thayer became superintendent and brought back from a European journey over a thousand textbooks, a majority of which were French. Until this date, all the textbooks used at the academy were English or their American adaptations. With the exception of the Hutton-Adrain book, the cadets would read English texts such as *Elements of Geometry* by John Playfair and Robert Simson's *Elements of the Conic Sections*. (Albree, Arney, & Rickey, 2000).

The publication of an American version of the *Course* that broke away from English influence could be seen here as primarily being driven by a strategy by publishers to distinguish their volume for marketing purposes, rather than emerging as a result of scientific or pedagogical differences between the two countries. Their motivations directly impacted the path of the discipline.

What kind of adaptation(s)?

The 1812 version introduced by Adrain in the first American edition of the course included, at first sight, few changes. But their nature revealed a new path the book was taking.

Adrian started to reorganize the structure of the text, going from three to two volumes. He did not delete any part of the text, but shrewdly inserted the paragraphs of volume three (whose material deals with "practical sciences"[6] (Adrain, 1812, 1, p. ix)) into the two first volumes. As a result, the table of contents became more relevant.

Moreover, he made some corrections to the text itself. The example of the surds (volume 1, Algebra section) is very significant. Hutton defined surds as "quantities as have no exact root" (Hutton, 1811, 1, p. 196) while Adrain proposed the following definition: "surds are such quantities that have no exact value in number" (Adrain, 1812, 1, p. 206). According to the first section, "Arithmetic treats of the nature and properties of numbers", either the "whole numbers" (the integers), either the "broken numbers, or parts of numbers" (the rational numbers) (Adrain, 1812, 1, p. 4). The author considered surds as quantities[7], not as numbers, that cannot be stated with exactness by numbers arithmetic usually deals with. In his preface, he remarked that the Hutton's previous definition lead automatically to consider "that the integer 2 is a surd, for it has not an exact root" (Adrain, 1812, 1, p. x).

Adrain's correction was meant for teachers or textbook writers, especially as he noticed that the same mistake could be found in several famous mathematics English textbooks (Bonnycastle's *Algebra* or Emerson's *Algebra*) in use in the American colleges until the 1810s. In regard to the scientific authority these authors had in American education institutions, this non-trifling change should be considered as a mark of the

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[6] Adrain referred to "practical gunnery", "motion of machines and their maximum effects", "trigonometrical surveying", etc.

[7] In the 19th century, the term quantity is very frequently used to express a moving idea between the magnitude and the number. Many examples can be found in textbooks by John Farrar, Charles Davies or Jeremiah Day, the three leading American mathematical textbooks authors in the first half of the century.

extent to which Adrain was willing to differentiate his *Course* from other highly respected texts.

A comparison with some French Algebras (Lacroix's *Algebra* for example) shows similarities with Adrain's text: "Tous les nombres entiers, qui ne sont point des quarrés parfaits, n'ont point de racine, non-seulement en nombres entiers, mais encore en nombres fractionnaires"[8] (Lacroix, 1808, p. 146). These echoes don't state the direct influence of the French textbook, but merely illustrate the closeness of the approaches.

Robert Adrain also added a few remarks and footnotes, based on his teaching experience. Take for example a note he wrote about the sign of fluxions, in the fourth section of the book. The propositions 16 to 19 handled with the product and the quotient of two fluents. Then he made clear:

> The fluxion of the algebraic quantity xy is properly $\dot{x}y + x\dot{y}$ in all cases of increase or decrease. We should always use the signs of the fluxions of algebraic expressions as those signs arise from the known rules, without considering whether the quantities increase or decrease. (Adrain, 1812, 2, p. 311).

Teachers would be sensitive to this remark about a constant difficulty in the differential calculus the learners have to deal with.

The third kind of change Adrain operated in the *Course* concerns new solutions to problems proposed by Hutton. One of the most meaningful examples is the "new method of determining the Angle contained by the chords of two sides of a Spherical Triangle" (Adrain, 1812, 2, p. 555). Hutton's method relied on a calculating approach using trigonometric equations. The solution provided a practical formula for obtaining a solution for the sought-after angle. One could blame Hutton for equalling the sinus to the angles. But it is important to reiterate the practical aspect of the *Course*: the problem was included in the "trigonometrical surveying" section (volume 2), which was only preoccupied with the measurement of fields lengths and angles, considered as very small surfaces on a sphere like the earth.

Adrain did not contest the resolution but proved and used a new theorem equalling the searched angle to an arc of an easily constructible circle. His approach, purely geometrical, marked a break with the British edition. The solution did not give an applicable but a theoretical method, also very ingenious and remarkably simple, and whose result could be generalized to any angle, small or large.

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[8] "All the whole numbers, that are not perfect squares, have no root, not only in whole numbers, but also in fractional numbers."

In the nineteenth-century-first-years England, many textbooks or supplements about the application of algebra to geometry were published for the British Military Academy in Woolwich: the *Application of algebra to geometry* by Charles Hutton, the *Treatise on conic sections and the application of algebra to geometry* by John Hymers, or *Appendix on the application of algebra to geometry* to his *Introduction to algebra* by John Bonnycastle. All these textbooks praised mathematics for its practical genius.

At the same time in France, mathematicians tried to develop new purely theoretical geometrical and algebraic methods to solve problems, see (Dhombres & Dhombres, 1989). More than thirty years later, speaking about the 1810s and 20s American mathematics education, Charles Davies would write in *The Logic and Utility of mathematics:* "The exact and beautiful methods of generalization, which distinguish the French school, were blended, with the practical methods of the English system" (Davies, 1850, Introduction). There is no strong evidence of the influence of such a theoretical trend on Adrain's work, but it is nevertheless interesting to remark the simplicity and the non-practical aspect of his approach in contrast to Hutton's demonstration.

The reference to French authors is more striking in another new solution that appeared in 1812, about the article "New method of determining the oscillation of a Variable Pendulum" (Adrain, 1812, 2, p. 556). The problem investigated the number of vibrations from the vertical made, in a given time, by a weight connected to another weigh through a pulley, and descending from the point of suspension. Adrain criticized Hutton's solution for forgetting the "accelerative tension of the thread" joining the two weights. He then took into account the vertical action of the tension of the thread to calculate the accelerative force urging the weight.

Fig. 5. Illustration of Problem 45 (Adrain, 1812, p. 536)

Another "erroneous" aspect of the British proof was pointed out by Adrain. Hutton wrote a complex motion equation that gives the fluxion of the number of vibrations, but without making clear its origin: "the stating by which he finds the fluxion of the number of vibrations, is referred to no geometrical or mechanical principle, and appears to be nothing but a mere hypothesis" (Adrain, 1812, 2, p. 556). Indeed, the fluxional equation Hutton provided included the fluxion of the number of vibrations, the accelerative force, the fluxion of the time, the fluxion of the distance between the weight and the pulley. But Hutton's statement was not justified. Before introducing the number of vibrations in the equations, Adrain reconsidered the whole approach by only dealing first with the fluxions of time and length, to describe the horizontal as well as the vertical motion.

$$\frac{\ddot{x}}{\dot{t}^2} = g - \frac{fx}{r}, \text{ and } \frac{\ddot{y}}{\dot{t}^2} = -\frac{fy}{r}.$$

Fig. 6. Motion equations[9] for Problem 45 (Adrain, 1812, 2, p. 557)

He referred his statement to "the general and well known theorem of variable motions (see Mec. Cel. B. 1, Chap. 2)" (Adrain, 1812, 2, p. 557). The French treatise *Celestial Mechanics* (*Traité de Mécanique Céleste*) was started by Pierre-Simon Laplace in 1799[10], and translated into English by the Bostonian mathematician Nathaniel Bowditch by 1829. It can be argued that Robert Adrain did read the French version. In the quoted book 1 (chapter 2), the motion of a material point submitted to three forces (P, Q, and R in the three directions of the space) is described by the following equations:

$$\frac{ddx}{dt^2} = P; \ \frac{ddy}{dt^2} = Q; \ \frac{ddz}{dt^2} = R \ \text{(Laplace, 1799, p. 21)}$$

These equations, with the exception of the third dimension, can be associated with Adrain's work. Why did he choose Laplace as a reference, whereas such principles can be found in Newton's *Philosophiae Naturalis Principia Mathematica*? The question strikes if we also point out the fluxional notations in Adrain's work, compared to the differential notations in Laplace's book. The name of the relevant chapter in the *Course* is also "Doctrine of fluxions". Despite the curiosity, interest, or respect the American author may have had toward French books such as the *Traité de mécanique celeste*, the influence of Newton remained. Nevertheless, the quotation was addressed to the readership, young American college students who had probably no idea of recent French

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[9] Adrain chose x and y as the horizontal and vertical shifting, t as the time, g as the measure of the accelerative gravity, f as the measure of the retarding force which the tension of the thread exerts on the weight, and r as the length of the thread between the weight and the pulley.

[10] The fives volumes of the piece were completed in 1825.

analytic work[11]. In this context, regarding Adrain's piece, the French author can be viewed as referential rather than really influential.

Finally, in the 1822 third edition, Adrian introduced for the first time in any general-contents textbook in America a treatise of descriptive geometry, announced in the preface:

> Besides the numerous corrections in this third American edition, there is added to the second volume an elementary treatise on *Descriptive Geometry,* in which the principles and fundamental problems are given in a simple and easy manner with a select number of useful applications, in Spherics, Conics, Sections, &c. (Adrain, 1822, 1, Preface)

This chapter was inspired by *A Treatise on Descriptive Geometry for the Use of the Cadets of the United States Military Academy* (1821), an American adaptation of Gaspard Monge's original text, from Claudius Crozet, a French teacher at the Military Academy of West Point. This is a short *vade-mecum*, written to simplify and clarify the objects and principles of Descriptive Geometry for a general-interest audience (colleges students). This introduction was probably the first English text spread in the United States, because the use of Crozet's Treatise was confined to West Point. It is a remarkable example of the introduction of new French mathematical contents.

It is possible to argue that the number of notes, corrections and remarks were trifling, in comparison to the total number of pages. Nevertheless, the American version carried its own identity. Adrain's *Course* proposed a corrected book where the structural inconsistencies were removed. Although his work was not revolutionary, the significance of the *Course* interest is not to be found in its differences with the English book. Rather, it comes from the origin of the project itself: writing a textbook by an American for Americans. This scientific production could have remained marginal, but the numerous new editions on both sides of the Atlantic would illustrate the breaking with the British edition.

Also of significance in Adrian's textbook is its inclusion of French authors. He inserted few but noticeable contributions from French mathematicians. Adrian had no ambition to import French textbooks, not even to combine or synthesize French and British contents. Before John

[11] There is no evidence of continental calculus teaching in the United States before the third decade of the century. We've analyzed Columbia curricula between 1818 and 1825, a period when Adrain was Professor of mathematics there. Starting in 1820, he introduced Lagrange and Laplace works. In 1824, the Harvard teacher John Farrar translated *Bézout's Principes de calcul différentiel et integral,* probably the first American calculus textbook with no reference of Newton's fluxions approach (Cajori, 1890).

Farrar or Charles Davies[12], Adrain preceded the great movement of translation of French textbooks the 1820s would know.

Migrations and distance across the Atlantic

The specificity of the *Course* relied on its change when circulating across the Atlantic after 1812. Like the book it was inspired on, the American version came back to England. The seventh British edition, published in London in 1819, took care of some but not all the modifications introduced by Adrain. Thus, Adrain wrote in his 1822 preface:

> In several places of the last or seventh London edition, the corrections made in the first American edition have been adopted. The definition of Surds which had been improperly given in the fifth and sixth London editions, is now correctly in the seventh. (Adrain, 1822, 1, Preface)

But Adrain was never quoted, neither in the preface neither in the text. His complements, remarks and new solutions to problems were not updated in London.

> There are in the last London edition several errors continued from the sixth edition, which had been corrected in the first American edition. Among these we may notice the demonstration given to the third theorem in Spherics. (Adrain, 1822, 1, Preface)

Indeed, Hutton kept his own algebraic solution about the "Determination the Angle contained by the chords of two sides of a Spherical Triangle".

The American version was read in London but filtered. Was this a form of censorship due to matter of mathematical background, political context, or personal egos? The original text, transformed by Adrain, came back to the hands of the first writer. It was not put away, but not fully adopted, as an incomplete reversal of the source. The following tables illustrate the circulation of the books across the Atlantic.

The audience of the book, both in America and Great Britain, reveals how their use and perception were different. It is clear that Adrain did not use the book when he taught at Columbia College between 1818 and 1825 (if we trust the curriculum he wrote every year about his course there). After 1818, when French teachers were hired, and while Superintendent Thayer was reshaping the teaching methods and courses contents, the US

[12] John Farrar, in Harvard, and Charles Davis, in West Point, leaded a huge wave of French textbooks translations during the 1820s. Their adaptations of Lacroix, Bézout, Monge or Legendre remained bestsellers during almost the whole of the 19th century.

Military Academy in West Point turned towards French textbooks[13]. The *Course* stopped being reedited in 1831 in the United States, probably because editors and publishers turned their interest to texts of national origin, and on French then German books. Also, colleges lost their interest in huge general publications, in favour of more specific and high-level textbooks. In England, the life of the book lasted longer, until the latter half of the century, even in the Royal Academy. This difference can illustrate the difficulties the British mathematics encountered in opening their field to continental mathematics during the first decades of the century. The introduction of new content (some of which can be related to French authors) within the *Course* was probably not enough for an American audience attracted by new textbook adaptations, new cultural inspirations, and proper American publications.

Table 1. English editions trajectory

English Edition	**First**	**Sixth**	**Seventh**	**Ninth**	**Eleventh**
Year	1798	1811	1820	Unknown	1837
Author	Hutton	Hutton	Hutton	Gregory	Gregory
Comments	3 volumes		Use partly the corrections of the 1st American edition	2 volumes	

Table 2. American editions trajectory

American Edition	**First**	**Second**	**Third**	**Fourth**	**Fifth**
Year	1812	1816	1818- 1822	1825	1831
Author	Adrain	Adrain	Adrain	Adrain	Adrain
Comments	2 volumes. From the 5th and 6th English editions		From the 7th English edition. Contains the Treatise of Descriptive Geometry		From the 9th English edition

Conclusion

A course of mathematics is not only an example of a textbook adaptation from one country to another. It witnesses the rule of publishing and printing on the development of national science, and illustrates also the French influence on American mathematics education at the beginning of the nineteenth century. It is a migration away from the former British influence, as the life of Adrain's book would move away, even slightly, from its source, over five editions. The link between mathematical

[13] Lacroix's, Legendre's, Monge's, Bourdon's, Boucharlat's textbooks were used. Some of them were taught in their original version, but by the end of the 1820s, the whole French textbook corpus had been translated or adapted in English.

knowledge circulation and the national political background is printed in the rule of the publishers.

With the story of the *Course*, the teaching of mathematics in America in the 1810s remained linked to the British textbooks editions or contents. But with emerging borrowings from French authors, and a critical regard towards English importations, this paper shown this was clearly a way, among other, for the discipline to stimulate itself and gain autonomy and identity in a time of change.

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Changes in the exercise of power over school mathematics in Sweden, 1930-1970

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Abstract

In connection with the major school reforms of the 1960s, the forms of course development changed. The aim of this paper is to investigate how the relationship between teachers in leading positions in course development and the common teachers changed as the forms of course development changed. The focus is on two teachers: one had a leading position before the reforms and the other during the reforms. Thus, it is a limited study; a pilot study where methods and theories are tested. Both text analysis and sociological analysis is applied. The main sources are articles in a teacher periodical, national curricula, textbooks and biographical literature. The study indicates that the new forms of course development brought a new way of reasoning to the professional debate about school mathematics.

Introduction

One of the most comprehensive reforms of the Swedish school system is the introduction of the so-called Grundskolan in the 1950s and 60s (Lindensjö & Lundgren, 2000). By this reform three different types of schools, partly parallel, were replaced by one mandatory 9-year school type: Grundskolan.[1] In connection with this reform, the forms of the development of the school subjects changed as well. In comparison with national curriculum documents before 1950, the new ones contained much more detailed descriptions of goals, contents and teaching methods (cf. *Undervisningsplan för rikets folkskolor*, 1920; Wallin & Grimlund, 1939; Skolöverstyrelsen, 1955a, 1955b, 1962). Thus, the government introduced a new kind of administrational tool in addition to national examinations.

With regard to school mathematics, the Central School Board also issued terminology books in mathematics. Among other things, they established which terms that were to be used in national exams (cf Skolöverstyrelsen, 1961).

But the changes of the 1950s and 60s were not only about new administrational tools, the mathematics courses changed extensively in connection to the introduction of new national curricula in 1962 and 1969. The most profound change took place in 1969 when concepts, symbols and expressions from set theory was supposed to permeate

[1] The three older school types were Folkskolan (year 1-7), Realskolan (year 5-9) and the lower part of the Girl school (year 5-9). See Prytz (2007) for an overview of the Swedish school system during the period 1905-1965.

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mathematics education from year 1 to the last year in the Gymnasium (Kilborn, 1977; Unenge, 1999).

The process of authoring the new mathematics course plan of Grundskolan in 1969 has been studied in detail by Kilborn (1977). So far, however, there has been no investigation of how the process of course development in mathematics changed when the curricula were made more extensive in the 1960s.

The aim of this paper is to investigate how the relationship between teachers in leading positions and the common teachers changed as the forms of course development changed. Two teachers in leading positions in the development of Swedish school mathematics have been investigated. One had a leading position before the reforms and the other during the reforms. Especially three articles about mathematics education published in 1938, 1963 and 1966 in the Swedish teacher periodical *Elementa* have been studied. The content of the communication is analysed, but also the interpersonal relation between author and reader. Due to the small number of authors and texts, the investigation should be seen as a pilot study.

Method and theory

The main material consists of an article by Carl-Erik Sjöstedt (1900-1979) from 1938 in the teacher periodical *Elementa* and two articles by Matts Håstad (1931-) from 1963 and 1966 in the same periodical.[2] Sjöstedt's article was about geometry and geometry instruction, in particular textbook design, in lower secondary schools (Realskolan). The first article of Håstad is a report from a UNESCO-conference on school mathematics; the second one is about coming changes in the mathematics courses of Grundskolan. When they wrote these articles, both were in the beginning of their careers in school mathematics, yet they had key positions in the development of school mathematics; Sjöstedt as a popular textbook author and Håstad as chairman of the group that was responsible for the production of the course plan for the new 1969 national curriculum (Prytz, 2007; Kilborn, 1977). It may appear as if Håstad had a position with more influence over school mathematics than Sjöstedt's and therefore should a comparison of their texts be deceptive. In the following section it is explained why both of them had influential positions in the development of school mathematics.

The investigated articles are relevant in relation to the aims of this study since they are authored by teachers with power that try to convince their colleagues, the common teachers, about the necessity of certain

[2] See Sjöstedt 1938 and Håstad 1963, 1966 in the list of references.

contents and methods of school mathematics. This rhetorical component makes the texts suitable for an investigation of the relation between the author and the reader. The explicit attempts of persuasion distinguish these texts from for instance curriculum documents and textbooks which are more prescriptive in character if we consider the choice of content and teaching methods.

One part of the study has a sociological approach and concerns the character of the relationship between the authors and the readers, but also the function of pedagogical texts in this relationship.[3] Dependencies between authors, readers and the state, in this case the Central School Board, have then been considered. This part of the analysis is based on Bourdieu's theory of capital and field (cf Bourdieu, 2000; Broady, 2002). I have taken up the notion symbolic capital and consider recognition as a symbolic capital; an asset that is used in transactions with other sorts of commodities. For instance, recognition is exchanged for work as people are promoted to leading positions in an organization. If people have been involved in this type of transactions and both parties give and receive something, I have considered them mutually dependent of each other. I have worked with the following questions. What did the authors achieve in mathematics education? For whom did they prove themselves capable? Who did show recognition and in what way? What was the function of educational texts in the process of receiving and giving recognition?

In order to answer these questions, the authors' backgrounds have been studied: for instance educations, book production and professional occupations and positions. The main sources in this respect have been biographical literature: the Swedish state calendar, teacher rolls and biographical dictionaries.[4] Book production has been investigated by means of the Swedish national library database LIBRIS.

The analysis of the content of the articles, but also the relationships between author and reader, is based on a method for analysing non-fictional literature presented by Hellspong & Ledin (1997). In non-fictional literature, esthetics is subordinate to other purposes. Types of non-fictional literature are professional literature, technical literature and scientific literature, but also texts produced for commercial as well as domestic ends. In this paper, texts with educational purposes are in focus. The starting point for the analysis of the articles has been the explicit

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[3] By pedagogical texts, I mean texts that have been authored for or used by teachers in connection with the planning or execution of teaching, e.g. textbooks, exams, books on teaching, but also articles in periodicals about education.

[4] More specifically Sveriges statskalender (1876-), Gejrot & Räf (1934), Sandén, Räf & Gejrot (1946), Karlsson (2004). Sjöstedt's background has been reported in another paper. Therefore, I sometimes refer to that paper.

educational statements, i.e. statements about the goals, contents and methods of mathematics education. As these statements are identified, they are arranged in types of statements or themes. The statements of a theme are about the same object or the same group of objects.

Also the assumptions that constitute the foundation for the argumentation have been investigated. This has been done by an examination of the statements against a basic structure: if statement A, then (probably, reasonably) statement T. The question I have worked with in this respect is which statements functions as arguments (A) for other statements, the theses (T)?

However, in the argumentation it is also possible to discern an author's attitude towards the readers. By an argument the author tries to convince the reader about the correctness or plausibility of a thesis. It is then relevant to study how the author, in addition to the logic of the argumentation, expresses certitude of the statements. On this point, I have focused on the statements that serve as a foundation for the rest of the argumentation in the articles since they are not motivated by means of other statements. It is essential for the author to make these basic statements appear as certain. This can be done in different ways, ways that include different attitudes towards the readers.

A statement can be presented as a fact – the statement is presented as accurate regardless of anything. Yet, a statement can be presented as non-factual without being proclaimed inaccurate. The author then puts up reservations that mitigate the certitude in some respect. This can be done by words or expressions that indicate probability, assumptions or needs. Hellspong & Ledin (1997) give the following scheme.

Table 1. Different types of factual and non-factual statements

Types of statements	**Signaled by**	**Examples**
Factual		Euclid's *Elements* is a textbook.
Non-factual: possibility	adverbials	Euclid's *Elements* is *probably/maybe* a textbook.
	auxiliary verbs	Euclid's *Elements could/might* be a textbook.
	adjectives	It is *possible/likely* that Euclid's *Elements* is a textbook.
	non-factual verbs	A person *thinks/assumes* that Euclid's *Elements* is a textbook.
Non-factual: assumption	conditional clause	*If Euclid's Elements is a textbook*, then I will eat my hat.
	hypothetical verb	*Assume/Imagine* that Euclid's *Elements* is a textbook.
Non-factual: need	auxiliary verbs	Euclid's *Elements should/must* be a textbook.
	Adjectives	It is *necessary/desirable* that Euclid's *Elements* is a textbook.

The opposite way is to underscore the certitude of a statement. Hellspong & Ledin (1997) do not provide a corresponding scheme, but they give different types of examples. One type is utterances like "I am certain about …" or "There is no doubt about …". In those cases, it is the author who guaranties the certitude of the statement. Another strategy is to refer to individuals, groups of people or institutions whose authority might enhance the certitude of the statement. On that point I have chosen to investigate which authorities the authors were referring to and how they did that.

Regarding the authors attitude towards the readers, I have considered reservations as a way to open up for further discussions; the reservations indicate other viewpoints and interpretations. In contrast, the use of underscores closes a discussion as it indicates that the statements are already discussed or agreed upon; there are no alternative viewpoints and interpretations.

The author's attitude is interesting if we want to study the relationship between an author in a position with power and the readers, who is the target of the author's attempts of persuasion. My working hypothesis has been that if an author in a position with power and the readers are mutually dependent of each other in some way, the author's argumentation opens up for further discussions; he or she does not merely present statements as undisputable facts. By a comparison of the authors' dependencies of the readers and the authors' attitudes in their articles it is possible to confirm or reject the hypothesis. However, the main point of doing this comparison is to investigate if new ways of reasoning, new assumptions and new authorities occurred in the professional debate about mathematics education when the forms for course development changed and new types of power positions and power relations were established.

Changing conditions for course development

When Sjöstedt wrote his article about geometry instruction in Realskolan in 1938, the secondary school statutes of 1933 was in force. That curriculum document was quite different from the ones that came into force in the 1960s.[5] In the statutes of 1933, descriptions of the contents of courses were brief.[6] The contents of the mathematics courses of Realskolan (year 5 to 9) were described in approximately 240 words. There were no sections on teaching methods, neither general ones nor specific ones for each subject. In the curriculum of 1962, the description

[5] This applies also for the course plans before 1933.
[6] The investigated text in this case is Wallin & Grimlund 1939.

of the purely mathematical content of the corresponding courses of Grundskolan (year 5-9) comprised about 1000 words.[7] In addition to that, there were comments about planning and teaching methods in mathematics that comprised approximately 1550 words. And in addition to that, there was a final section on how, in what order and to what extent each mathematical topic ought to be treated. This final section comprised about 3500 words. Considering this difference in the national curriculum documents, the textbook authors before 1950 had greater freedom and more power over the content of the mathematics courses. Although there were national exams before 1950, which constituted long term goals that textbooks authors could not ignore, there were still a lot of things they could influence: the ordering of topics, the extent in which each topic should be treated, the choice of symbols and terminology and the design of explanations. Moreover, before 1950 there were no competing texts about mathematics education, for instance certain books on how to teach mathematics. The first book in that genre appeared in the early 1950s (Prytz, 2007).

The geometry subject of Realskolan constitutes a good example of the freedom and power of the textbook authors. The national curricula from this period do not establish to what extent a scientific approach to geometry should be included. In fact, there were no formulations about that matter at all. Still, all textbook authors of the period, except one, sought to give their books a scientific structure with explicit axioms and definitions; every proposition was also proved by means of axioms, definition or preceding propositions. However, it should be noted that several of the authors pointed out that their textbooks in minor respects deviated from scientific principles, which they justified with educational benefits.[8]

The greater importance of the textbook authors and textbooks before 1950 was reflected in the debate in *Elementa*, the major professional periodical for mathematics and science education in the secondary schools. In the case of geometry textbooks, the textbook authors criticized each other's work. Moreover, these debates occurred in connection with other events related to textbook production. In 1922, one

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[7] The investigated text in this case is Skolöverstyrelsen 1962. Year 7-9 of Grundskolan comprised two types of courses in mathematics: general and special. The latter corresponded to the mathematics courses of Realskolan. The chapter of the 1962 curriculum that treats the mathematics courses contains common sections for both types of courses and specific sections for each type of course. I have counted the words of both the common sections and the sections that treat the special course.

[8] For an analysis of the geometry textbooks of Realskolan from this period, see (Prytz 2007).

textbook author, Henrik Petrini, took over the editorship for an older textbook. In 1924, he published an article in *Elementa* where he criticized other contemporary textbooks. This article was followed by articles where his standpoints were countered. In 1936, the first edition of Sjöstedt's textbook appeared. This textbook was later on published in several editions. In 1938, Sjöstedt's textbook and a textbook by Hjalmar Olson were criticized by Ragnar Nyhlén, a prospective textbook author. Also this article was published in *Elementa*. Nyhlén mentioned explicitly that he had selected two of the most "prominent" textbooks of that time. Also Olson, by this time an established textbook author, criticized Sjöstedt in his defence. Sjöstedt replied in a third article, which is the article investigated in this paper.[9]

In contrast there were no detailed textbook debates of this kind in *Elementa* during the period 1950-1975, even though new textbooks were issued, especially in connection with the introduction of the New Math in early 1970s. When for instance Håstad introduced an entire set of textbooks and other educational material inspired by New Math, they were not discussed in *Elementa*. The textbook was named *Hej Matematik!* (Hello Mathematics!) and together with additional materials it covered all nine years of Grundskolan.

Prytz (2009) highlights another aspect of the importance of textbooks and textbook authoring during the period 1900-1950. The common, but also unique, feature of the people who took part in the professional debate on school geometry and who reached leading positions in school mathematics, e.g. Sjöstedt, was that they were successful authors of mathematics textbooks for the secondary schools. Regarding the leading positions, Olson was editor of the periodical *Elementa*, involved in teacher education and advisor at the Ministry of Education; Sjöstedt got a high position as counsellor of education at the Central School Board in 1939. These positions were leading in the sense that Olson and Sjöstedt could influence the agenda regarding the development of school mathematics. Here we should note that success in mathematics as a science was not a common and unique feature of the people who reached leading position in school mathematics during this period. However, all of them had a Ph. D. in mathematics. As a matter of fact, Petrini was the only one that distinguished himself scientifically in mathematics, in this case through a small number of scientific publications and a position at Uppsala University during a few years. But, he did not reach a leading position in school mathematics. Note that Prytz (2009) does not demonstrate a causal connection between textbook success and leading position. We can

[9] The content of these debates are analysed in Prytz (2007).

however say that those who reached leading positions had the ability to author textbooks. This ability was recognized, both by teachers who used the textbooks to a large extent, and by the publishers who accepted their textbooks. Thus, textbook authors not only had greater freedom and more power during the period 1900-1950, some of them also occupied other leading positions.

The conditions for the development of school mathematics changed in connection with the preparation of the much more extensive course plans in the 1960s. The groups responsible for the production of the course plans became more influential since the course plans treated the content more comprehensively, but also since lengthy recommendations about teaching methods were included. The latter was not part of the curriculum of the secondary schools before 1950 and they were much shorter in the curriculum of the primary schools, Folkskolan.

If we consider the people responsible for the new course plan of 1969, Matts Håstad, Göran Holmström, Leif Hellström, Birgit Bodén and Olof Magne, they had different relations to textbooks. None of them had published a great number of textbooks before 1965. Actually, Magne and Håstad were the only two who had published textbooks before that; a couple of textbooks each. The textbook production of the group members really took off in the early 1970s, i.e. after the course plan came into force. Regarding the background of the members, all of them, except Magne, had experience of being a teacher when they entered the group. Håstad was active as mathematics teacher in a secondary school, Holmström as science teacher in secondary schools and Hellström and Bodén as primary school teachers. Magne, on the other hand, had a Ph. D. in education and educational psychology.[10] Thus, he functioned as the expert on educational science. Håstad was the only one with scientific background in mathematics. In 1959 he became Ph. L. in mathematics (a degree between Ph. D. and M. Sc.). However, he did not make a career in the science of mathematics. Still, in the group he had the most solid background in mathematics. But, he also functioned as expert on mathematics education. In the first half of the 1960s, he had been appointed member of two groups that worked for the reformation of mathematics education. Both groups developed and tried different types of educational texts, materials and teaching methods. One of the groups, The Nordic Committee for the Modernization of Mathematics Education, was leading in the introduction of New Math in the Nordic Countries. Håstad was the secretary of that group. The other group, Individualized Mathematics Education, tried ways to develop teaching methods and

[10] Education and educational psychology was a single subject.

material so that is should fit the needs of each individual student. In 1965, Håstad was appointed chairman of the group responsible for the mathematics courses to the new national curriculum of 1969. In the previous year, he got a position as advisor at the Central School Board (Kilborn, 1977; *Sveriges statskalender*).

Regarding changing conditions for course development in school mathematics, I want to point out two things.

1) Between the 1930s and 1960s the character of the leading positions changed. Both Sjöstedt and Håstad had high positions at the Central School Board with influence over school mathematics. But they also had other means of influence to their disposal as authors of educational texts; and therein lay the difference. Sjöstedt had influence as textbook author, while Håstad, initially, had influence as author of course plans and as member of various committees, only later on in the 70s as textbook author. Moreover, the position as author of course plans was not that influential in the 1930s; textbook authoring was the main means to develop school mathematics on a general level.

2) The dependencies between the state, the people in the leading positions (in this case Sjöstedt and Håstad) and the teachers were different. Let us first consider Sjöstedt's situation. Between him, as a textbook author, and the teachers, i.e. the readers of his textbooks and article in *Elementa*, there was a mutual dependency. A teacher that used a certain textbook also complied with the author's educational principles, but at the same time the teacher also gave recognition to the author. I think it is reasonable to say that recognition was given by teachers as they chose to use a textbook. But, we must not understand recognition in the sense that teachers chose the textbook that was optimal in relation to their idea of teaching; because we do not know what the teachers considered optimal. However, it is fair to say that if a teacher chose a textbook, he or she also appreciated that same textbook on the basis of some, at least, elementary standards; especially if there were always at least two different and regularly republished textbooks to choose from, which was the case during the whole period 1900-1950 (Prytz, 2007). In this sense, I think it is reasonable to talk about teachers giving recognition of skills to write and edit textbooks.

Another argument to why a symbol of recognition was linked to the teachers' choice of textbooks is that textbook design was something the debaters in *Elementa* fought about: it was a central subject in the professional debate about geometry instruction during the period 1905-1950 (Prytz, 2007). Moreover, people in influential positions were recognized as skilled textbook authors (Prytz, 2009).

Regarding Sjöstedt's situation, he got recognition as textbook author independently of the state. Thus, some of his power over school mathematics was not dependent on the state, even though he later on in the 40s and 50s made a career in Central School Board (Karlsson, 2004).

Håstad's situation was different. Between him, a course plan author, and the teachers, i.e. the readers of course plans and his articles in *Elementa*, there was a non-mutual dependency. The teachers were expected to comply with the directives that eventually were set forth in the course plan and they did not possess any means to give recognition to any of Håstad's abilities. However, in Håstad's case there was a mutual dependency between him and the state. In exchange for doing work in different committees and complying with certain principles, he was recognized as expert on mathematics education. And indeed there were principles to comply with: both Håstad and Magne left the group that was responsible for the new course plan when some of their proposals were not accepted by their superiors at the Central School Board (Kilborn, 1977).

If this model about different dependencies between authors and readers is appropriate, there ought to be differences in attitude towards the readers in the articles authored by Sjöstedt and Håstad; articles intended for teachers. This hypothesis is described in the previous section on sources, method and theory.

The content of the articles

There were similarities as well as dissimilarities in Sjöstedt's and Håstad's articles. Some of the dissimilarities are linked to the fact that Sjöstedt's article was about textbook design and that Håstad's were about on-going course development and a conference on mathematics education. Not surprisingly, Håstad covered more issues and topics. For instance, Håstad treated organisational issues, while Sjöstedt did not and Håstad discussed more mathematical concepts and topics. Therefore, Håstad's articles get a longer treatment in this section.

Let us first turn to Sjöstedt's article from 1938. Most of the statements went about the science of geometry, school geometry and how geometrical concepts ought to be understood and treated in a textbook. These were the major themes apart from textbook design. Sjöstedt discussed how geometry as a science ought to be understood, especially Hilbert's work on the foundations of geometry. He also discussed issues regarding geometry and space. His basic standpoint in these issues was that Euclidean geometry is a science about space. Moreover, a geometrical proof is not a question of pure logic; in order to work, a proof also needs a conception of space. Otherwise you cannot understand the proof. In addition to this, he discussed how geometrical objects ought to be treated in scientific geometry and in school geometry. In the latter, according to

Sjöstedt, you also have to consider educational needs, which sometimes interfere with scientific demands on language and precision; in those cases, the educational needs are superior. However, the educational needs, e.g. the needs of the students or the teachers' use of textbooks, were not explained in length. By the few cases where students were mentioned, it seems to be about language: teachers and textbook authors should avoid complex formulations whose scientific function is beyond the grasp of the students and that do not improve their understanding of the matter. These general considerations about geometry, science and education functioned as foundation for his argumentation. On the basis of them, Sjöstedt delivered standpoints, in length, about how particular propositions (axioms) and concepts (congruency and angles) ought to be treated in a textbook.

A theme that stood out in Sjöstedt's article was about how textbook authors should behave in a debate. In a short paragraph he expressed his dissatisfaction with how one of his critics had delivered his critique.

Håstad's articles contained more themes, but they were treated more briefly. The first article from 1963 was a report from a UNESCO-conference on mathematics education held in Budapest in 1962. The article contained three main themes: 1) the mathematical content of courses, 2) teaching methods and 3) the education of mathematics teachers. Regarding the first and third themes, Håstad more or less gave a list of mathematical topics and concepts that the conference suggested to be included in future courses. A subtheme of both the first and the second theme was the relation between teachers and students. For instance, the teaching had to be stimulating and individualised; the students' activity needed to be utilized and they should make their own discoveries. A second subtheme of the first and the second theme was educational research: it was supposed to enhance the content and the methods of mathematics teaching.

Throughout the article the word modern was frequently used: the content and the methods had to be modern and they had to change. The word change was not used very much, but the words reconsider, deepen, modify, remedy and alter was used more often.

Håstad's second article from 1966 was about the proposal of new courses in mathematics for Grundskolan. Also this article was more general than Sjöstedt's. It contained three main themes: A) the contents of the coming courses, B) the improvement of operational structure of the teaching and C) arguments about why the current courses and methods had to change.

Under the heading "learning matter" (lärostoff) the content of the new courses was described very briefly, mainly in terms of mathematical topics and concepts. In comparison with the current courses some topics and concepts remained as they were, some were moved to other years, some got a different treatment, some were reduced (e.g. classical geometry was reduced to a minimum) and some were brand new. Examples of the latter were the concepts set, element, union, subset and the empty set, which were to be introduced in year 1. Håstad did not mention skills and competencies in the section on "learning matter". Elsewhere, however, understanding was mentioned twice and verbal skills once.

The section about the improvement of the operational structures contains three subthemes: i) learning and development, ii) the organisation of the teaching in the classroom and iii) general organisational issues. The suggestions about learning and development were motivated by results from educational research, for instance meaning, understanding, discovery and creative activity are important when students learn mathematics and a positive emotional attitude facilitates learning. Regarding the organisation of the teaching in the classroom Håstad maintained, for instance, that students shall get used to work efficiently in smaller groups and that traditional textbooks ought to be replaced by leaflets, working cards and experimental devices. Hence, Sjöstedt and Håstad valued textbooks very differently. Observe that Håstad kept the presentation short; suggestions and arguments were not developed much beyond the points given above.

Apart from the changes, Håstad presented, in a separate section, a list of arguments as to why the changes should be implemented.

a. Societal changes required that the content must change. The following examples were given.
   - The students had a new type of background in mathematics.
   - The increased amount of data in society and new calculating machines required less knowledge of complex numerical calculations. More time may instead be spent on attaining understanding.
   - Due to developments in research, industry, business, communications and administration, more categories of personnel encounter situations in which mathematical concepts, methods and models were used.
b. The forms of teaching were insufficient and had to change. The teaching was boring and needed to be made more accessible, fun and interesting.
c. Research results showed how the changes could be carried out. Research in mathematics had made essential contributions to mathematics education, e.g. concepts from set theory. According to Håstad, these concepts had enhanced the students' possibilities to acquire the mathematics a society could be expected to demand. Several development projects, for instance in the US, but also in the

Nordic countries, showed how new courses could be constructed and taught.

d. The science courses in the upper secondary schools have changed which requires a new type of mathematical knowledge.

Note that Håstad did not develop the arguments beyond the points just given.

In a comparison of Sjöstedt's and Håstad's articles, the relation between science and education was treated by both. But whereas Sjöstedt displayed a more critical attitude towards science as he underscored the differences between scientific geometry and school geometry and that educational needs were superior to scientific demands, Håstad made no such comments; instead Håstad considered scientific mathematics as a provider of suitable concepts for school mathematics. Moreover, a recurring statement in Håstad's articles is that educational research would provide solutions for mathematics education. Results from educational research were also used to motivate suggestions about how teaching methods ought to change. Several statements of Håstad involved the relation between teaching and learning, or the teacher and the student. He pointed out that teachers had to consider various aspects in order to meet the needs of the students and to achieve efficient learning. In Sjöstedt's article, the relation between teacher and student did not stand out. The needs of the students were mentioned only a few times and concerned their ability to grasp certain formulations. In what way textbooks could be used by the teachers in the classroom or if the textbooks met other needs of the students were not discussed by Sjöstedt. Instead he based his argumentation on general assumptions about how geometrical concepts, but also proofs, are to be understood. In those questions Sjöstedt was much more explicit and he related to scientific works of mathematicians, especially Hilbert's work on the foundations of geometry. A type of statements that appeared only in Håstad's article concerned societal change and the needs of the society. The coming changes of the contents that he reported were motivated by societal changes.

The interpersonal relationship between authors and readers

Let us first consider Sjöstedt's article. Sjöstedt mainly used his own voice. The words I, me and my were used frequently. Educational statements of all sorts (also statements that functioned as the foundation for his argumentation) were accompanied by expressions like "According to my opinion …" and "I think …". When he discussed his basic assumptions and referred to scientific authorities, like Hilbert, their results were always interpreted and commented upon.[11] Hence, authorities were not merely used to underscore the certitude of a statement. Here is a typical quote where Sjöstedt discussed how Hilbert's axiomatic system ought to be understood.

> In his article, Professor N. represents several opinions about the basic concepts and axioms, opinions that I have criticized before in Z.E. I must here confine myself to emphasize some of the most important. The geometrical concepts are often conceived as non-determined; the straight line could for instance be thought of as closed (p. 22). The basic concepts are defined by the axioms (p. 22-23). These thoughts show close similarities with the so-called axiomatics represented primarily by D. HILBERT. … The main function of axiomatics ought to be formulated in the following way: To establish a system of propositions which are necessary and sufficient for the system to apply only to a certain group (resp. certain groups) of concepts. The usual epistemological interpretation of this is according to my opinion, incorrect, as recently suggested. (Sjöstedt, 1938, p. 166-167)

In this quote it is clear that it was Sjöstedt who made the statements. He also lessened the certitude of statements important for his argumentation even more by pointing at needs rather than facts by using expression like "… ought to be formulated…". Thus, Sjöstedt's way of reasoning contained a lot of reservations, if we follow the scheme of Hellspong & Ledin (1997).

However, Sjöstedt carefully explained why his assumptions were sound. But this was done without references to scientific or administrative authorities. He rather tried to show that the standpoints maintained by his critics were unrealistic in an educational perspective. Here is a typical quote.

> When discussing the basic concepts of geometry and axioms, one must, in my opinion keep sharply apart, what belongs to scientific geometry and what can fit in school geometry. Professor Nyhlén seems to me not having enough responded to this in his article in the first number of this year's volume of the periodical. It can obviously not be professor N's view, to carry out proofs of some of the axioms on magnitudes of the type he confides on pages 15, 16, 26, 27 in Realskolan. But the petitions [presentations of axioms] he investigates, however, are taken from textbooks for secondary schools. Furthermore, it is clear that in scientific geometry the organization of the whole, the proofs of a great deal of the propositions etc. ought to be conducted by other means than in school geometry. In the case of a

[11] In this case, Hilber's work on the foundations of geometry.

> textbook for Realskolan, must in my opinion, the logical point of view, never be satisfied at the expense of the educational. In this context let me mention only one example. An expression such as "however far they may be extracted" in the usual definition of parallel lines is, of course, from the logical point of view, unnecessarily! But is it unnecessary or inappropriate in a textbook? I think not. (Sjöstedt, 1938, p. 165)

Observe that Sjöstedt’s reasoning relied on the reader’s experience of teaching and textbooks. See especially the third sentence and the last three sentences in the previous quote. Thus, the certitude of his standpoints was not underscored by authorities like scientists, administrative institutions or curriculum documents, but the readers’ experience of being a teacher and user of textbooks; it was a practical teaching rationale that functioned as an authority in this case. Note that all scientists mentioned in Sjöstedt’s article (mathematicians and philosophers) were explicitly named and references were given.

Håstad proceeded in a different manner in both his articles. His voice is passive in that he presents information about what has being said by others at a conference and how the work with the new course plans was proceeding. The words I, me and my were not used in any of his articles. A great part of all the educational statements in Håstad’s articles were suggestions and they expressed possibilities or needs: the contents and methods of school mathematics ought to change due to certain needs and there were possible ways to do it. Here is a typical example.

> The teaching shall be made as stimulating as possible. A class should comprise between 15 and 25 students. Surveys should be undertaken to study the effect of an individualized education that enables each student to work at their own pace within the class. Many experiments show that it is possible to use expressions, concepts and operations from elementary set theory and functional and relational concepts from 12 years of age and even earlier. (Håstad, 1963, p. 19)

Reservations regarding the certitude of some statements were expressed as needs for further research or testing. We find an example of this in the third sentence in the previous quote.

However, the statements that functioned as basic arguments for the changes of Swedish mathematics education were mainly given as facts, especially in the second article about the coming course plan. The following quote is a typical example.

> By society's rapid development the students’ experiences of mathematics is different from just a decade ago. Mathematics courses need to be adapted to current conditions. The large amount of data that are necessary in today’s society has greatly increased the use of

> computers, calculators, slide rules and other means of numerical calculations. It therefore becomes less important to perform complex numerical calculations with great security. It is more important to make estimates in a given situation. (Håstad, 1966, p. 302)

On several occasions scientific authorities were mentioned, but then Håstad referred to anonymous groups: research had made certain experiences. Here follows a couple of examples.

> Mathematical research has not only greatly increased the amount of knowledge in our time but also made significant contributions to mathematics education. It has been shown that concepts from set theory, some simple logic concepts, functions, relations, vectors, groups, etc. have increased students' opportunities to acquire the mathematics that society can be expected to require. (Håstad, 1966, p. 303)
>
> Educational research has among other things made the following experiences:
>
> 1. Emphasis on meaning, understanding, discovery and creative activity is important in the learning of mathematics.
>
> 2. A positive emotional attitude facilitates learning.
>
> 3. For many students, it is necessary to make repeated concrete experiences before an abstraction can be experienced.
>
> 4. Learning should proceed in a pace adapted to each student. (Håstad, 1966, p. 303)

These research results were not further explained and by no means criticized. Hence, the references to authorities functioned as underscores of the basic statements. Moreover, he did not provide any references at all (scientific works included). Hence, not only did Håstad include very few formulations that indicated alternative basic assumptions and standpoints; he also made it difficult for the readers to check the proclaimed facts and form their own opinions in the matter.

In both Håstad's articles, several statements, and not only the basic ones, were also presented as standpoints of various administrative institutions, for instance UNESCO, the Central School Board and the Nordic Committee for the modernisations of mathematics education. Hence, the certitude of the statements was underscored not only by scientific, but also administrative authorities.

In a comparison of how Sjöstedt and Håstad underscored and made reservations about the certitude of their basic statements, they display different attitudes versus the readers. All the reservations in Sjöstedt's article opened up for a discussion by indicating alternative assumptions and standpoints. All the underscores in Håstad's, on the other hand, closed the discussion by leaving out alternative assumptions and standpoints.

Still, there was another kind of openness in Håstad's reasoning. But this is more to do with the generality of the statements' content, and not so much with the degree of certainty the author's formulations endowed them with. For instance, when Håstad criticized, he did not address people or identifiable groups of people, but mathematics education as a whole. Moreover, suggestions about change were often kept on a general level. The following quote is a typical example, where we also can observe the use of factual statements (sentence two and three).

> Mathematics must be made more accessible. Many students have found the teaching boring and tedious. Textbooks often have the character of collections of examples. Learning of mathematics must be made more enjoyable and applications must be selected from interesting areas. Many times, mathematical concepts can be learned in ways which are of the nature of the game and play. (Håstad, 1966, p. 302)

The meanings of these statements were not further explained. Hence, no individual or specific group of people was really criticized. And when the statements were not further explained, they allow a wide variety of interpretations. We can compare this to how Sjöstedt criticized a standpoint about the nature of geometric proofs.

> According to a widely held perception about the nature of the geometric proof, it is not necessary to realize a conception of space in connection to a geometric proof. When formulating the axioms, you make links to spatial intuition, but then your conception of space is not needed other than perhaps as support for perception. Lecturer N. aligns himself to this idea (p. 29). You then believe that geometry in fact would not have space as the object of research. On the contrary, the only real thing would be to free geometry from 'spatial intuition'. (Sjöstedt, 1938, p.168)

In this quote, the criticized standpoint of a particular person was presented and further explained. A bit later, Sjöstedt presented his own standpoint.

> But a geometric proof contains something in addition to these statements, which are certain spatial conceptions. These are not statements, and could therefore not be traced back to axioms. But they are nevertheless necessary conditions for the proof. Without realizing spatial conceptions you cannot carry out a geometrical proof. (Sjöstedt, 1938, p.168)

My point is that if we consider both the contents and the form of the statements, we can discern two types of openness. Several of Håstad's basic statements, whose certitude were underscored, allowed a greater variety of interpretations since he did not provide explicit explanations of their meaning. Sjöstedt, on the other hand, put up reservations about the certitude of his basic statements, but he explicitly presented the alternative to his own standpoint.

Closing discussion

If we just consider the content of the statements in Sjöstedt's and Håstad's articles, the differences between them confirm known changes in the Swedish educational system. We know that psychological and educational research was more involved in the planning of school reforms in the Nordic countries during the 1960s than in previous times (Oftedal Telhaug, 2006). Thus, Håstad's references to such authorities confirm such historical accounts. Håstad's broader outlook on the relation between teacher and student, but also the emphasis on the needs of the students, were in line with how educational ideals had changed in the Nordic countries (Oftedal Telhaug, 2006). Moreover, the changes of school mathematics that Håstad suggested were in line with the actual changes in the curriculum of 1969.

Another difference between the articles is that Håstad included an analysis of the relation between societal needs and the needs of the students. However, that type of reasoning had appeared in articles in *Elementa* about geometry instruction in the 1920s (Prytz, 2007). In that respect Håstad was not original. However, if we consider the conclusions of the societal analysis and the arguments they were different: Håstad put greater emphasis on the need to master modern mathematics in order to function in certain roles in working life.

The interesting differences lie in how two people in leading positions exercised power over school mathematics: the way they reasoned; how they delivered their statements; how they tried to convince their readers, i.e. the common teachers.

The investigation of the authors' way of reasoning has focused on attitudes towards the readers. I have analysed how Sjöstedt and Håstad made reservations about or underscored the certitude of statements that were fundamental for their argumentation about school mathematics. In this perspective Sjöstedt had a more open attitude towards the readers; he frequently used reservations, but also few underscores, which indicated that there were other basic assumptions and standpoints apart from his. Alternative assumptions and standpoints were also described. In this way Sjöstedt opened up for further discussions. He also gave explicit references to the scientific works he was referring to. Håstad's way of reasoning was quite the opposite: few reservations and several underscores. Hence, he did not indicate the possibility of alternative basic assumptions and standpoints. He closed further discussions. Moreover, he did not provide any references when he reported standpoints of various authorities or institutions.

Note that the openness of Sjöstedt does not mean that he refrained from proving his standpoints. Both of them did that, but the difference is that Sjöstedt indicated clear alternative views. However, if we instead of studying how the two presented certitude of their statements (the use of reservations and underscores), consider the meaning of the statements I think there was another kind of openness in Håstad's articles. His statements generally had a wider meaning than Sjöstedt's. In this way he may have opened up for interpretations.

Sjöstedt's and Håstad's choice of authorities can also be seen as a reflection of their attitudes towards the teachers. Sjöstedt sometimes based his refutation of competing standpoints on a rationale based on teaching experience; he argued that the alternative assumptions would lead to unrealistic consequences in the teaching situation. Håstad did not refer to that type of authority at all. His standpoints regarding how to carry out the teaching was based on the results of educational research or development projects. Håstad also referred to administrative authorities like the Central School Board Sjöstedt did not use that kind of authorities at all. However, both of them included the science of mathematics in their reasoning.

But there were not only differences in the choice of authorities; also their treatments of the authorities were different. Sjöstedt discussed scientific results in a more critical manner, in this case mathematical results: he discussed how they could be interpreted scientifically and should be interpreted and used in the context of school mathematics. Håstad mainly reported scientific results and administrative decisions as facts, mainly without critical comments, that he thought should be implemented; scientific mathematics provided sound content and the psychological and educational sciences provided sound principles for teaching.

So far the results of the textual analysis, but I have also investigated the dependencies between the readers (the teachers), the state and the people with influence over school mathematics (in this case Sjöstedt and Håstad). Due to the introduction of comprehensive course plans in the 1960s, authors of course plans (e.g. Håstad) got more influence over school mathematics on the behalf of textbook authors (e.g. Sjöstedt). This also changed the dependencies between the people with influence and the teachers. A textbook author was mutually dependent of the teachers in the sense that teacher gave recognition, a symbolic capital, to the authors when they chose a textbook; the teachers changed recognition against adaptation to a certain course in school mathematics. A course plan author like Håstad, is his case with little merit as textbook author, was not involved in that kind of transactions. He was rather mutually dependent of the state, more precisely the Central School Board, which had appointed him as expert in various committees. My working hypothesis has been that if an author in a position with power and the readers were mutually dependent of each other in some way, the author's argumentation opened up for further discussions. I think that my investigations confirm this hypothesis.

The significance of this study is that it indicates that the reformation of the school system and of school mathematics in the 60s brought more than a changed content, new teaching methods, new authorities and new arguments; the introduction of a comprehensive curriculum also brought a new way of reasoning about school mathematics and a new attitude towards science and administrative authorities. Kilborn (1977) does not consider this aspect when he investigates the work with the mathematics courses for the National curriculum of 1969. However, in order to be more than indications more material has to be investigated.

This study is also interesting in relation to research on the introduction of the New Math. It is commonly believed that New Math in Sweden was not only a failure, but also an isolated episode. The project was closed in the late '70s and the new ideas took over (c.f. Unenge, 1999). An alternative view is presented by Bjarnadóttir (2006). Although her study primarily concerns Iceland, it raises general questions about New Math. Bjarnadóttir (2006) argues that New Math in Iceland not only gave rise to a new understanding of concepts and notation. The reform caused other, more far-reaching changes: e.g. a broad consensus on mathematics education among teachers from different types of schools developed and differences between school types were bridged. It is possible that the changes in power relations and ways of reasoning reported in this paper were of a more far-reaching kind. These changes were perhaps also a part of the creation of consensus. However, the same changes also raise questions about the notion consensus. E.g. was consensus evenly distributed among the mathematics teachers? Håstad's way of reasoning, might have attracted some, but repelled others.

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Practical mathematics in 16th century England: Social-economic contexts and emerging ideologies in the new Common Wealth

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Abstract

Unlike most mathematics education programmes established by governments in Continental Europe in the latter 18th and early 19th century, English Mathematics Education was an entirely 'grass roots' enterprise, established by the vision of a community of people in 16th century England. They introduced a new conception of mathematics as a practical tool for aiding the solution of social, economic and political problems and building the Common Wealth of the nation by the application of mathematics to gradual advances in technology allied to opportunity and enterprise. By the end of the century England was on the threshold of new and profitable discoveries in science due to the new instruments and technologies that established a tradition, continuing through the 17th century, that came to form the basis of mathematical and technical education for England's Industrial Revolution.

Introduction

In earlier papers (Rogers, 2006; 2007) I described aspects of the work of four individuals, Robert Recorde (1510-1558), John Dee (1527-1609) and Leonard (1515-1559) and Thomas Digges (1546-1595) who were principally responsible for the promotion and foundation of the tradition of practical mathematics as an educational enterprise in 16th century England.

The following discussion of original texts and contemporary sources is intended to indicate the range and variety of epistemological approaches we find where new philosophy and new uses of mathematics come together in a context of growing scientific and social-economic awareness. A text acts as a source of information for its readers, and also a catalyst for the development of new ideas, and this paper focuses on changes in belief systems and the production and validation of knowledge in the social construction of the 'utility' of mathematics in this historical period.

The book as an agent of change

The general methodology of this paper considers a selection of available sources in terms of the way particular beliefs are received, change, and become institutionalised in a community. The investigation of the cultural contexts of belief systems, and the process of the formation,

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

assessment and validation of knowledge, has arisen from research in Social Studies, History, Anthropology, and in Philosophy of Science, and these have directly influenced the historiography of mathematics and mathematics education. Today historiography consists of a general set of approaches to the study of knowledge in cultures, regarding the development of knowledge as a collective achievement in a community. This takes into consideration local conditions, espoused and enacted beliefs of individuals, and the social, political, and economic influences on the establishment of particular kinds of knowledge. With regard to the contexts and artefacts studied here, the aim of an author of a book may be to produce an organised system of knowledge, or to establish a point of view and this assumes that their text, when read in the 'right' way provides a reader with a clearer understanding of the nature and purpose of their subject. In the 16th century the proliferation of printed works enabled an outburst of cross-fertilisation of ideas, particularly about the establishment of scientific beliefs and technical practices.

In the study of the practice of mathematics in Renaissance England, the epistemological varieties concern the views of the nature, use, and processes of formation of mathematics and its uses that were espoused by the different agents whose views depended on their ideologies, and were inevitably involved in the production of their texts. The purposes of mathematics that were perceived, and the uses to which it was put, to some extent determined the kind of mathematics that an individual was motivated to learn and to apply, and these needs were felt in different ways in particular sections of society.

England in the 16th century

From the reign of Henry Tudor to Elizabeth I, over a span of ninety-four years,[1] political, economic, and cultural forces advanced a *New Philosophy*, part inspired by the rediscovery of classical sources, and the rise of continental humanism, and part by the technical needs of economics, empire, and defence of the realm. Mathematics was regarded by different groups in English society as the foundation and inspiration for new approaches to the understanding of nature, for their own development of knowledge, and enhancement of personal power.

Henry VIII received a humanist education and attracted scholars from Europe to his court to embellish his image as ruler, impress his political rivals, and develop his military ambitions. Outstanding in his Court were Thomas More with his own artistic and scientific circle, Cardinal Thomas

[1] The Tudor period is from Henry VIII (1509 - 1547), through Edward VI (1547 - 1553), Mary 1 (1553-1558) and Elizabeth 1 (1558-1603). For details see Elton (1991).

Wolsey, later to become Chancellor, Erasmus, Hans Holbein the artist, Nicholas Kratzer the German astronomer, instrument maker, and horoligst who taught mathematics at Christ's College Oxford, and many other notable visiting scholars and craftsmen. Henry provided an environment in which the mathematical arts were favoured as much for their display as for their practical, political, and strategic use. He designed fortresses with the help of master masons from Germany and Italy, with shipwrights from Venice expanded his navy, and English artisans and craftsmen quickly adapted themselves to these new skills.

After his dispute with the Pope over a divorce from his first wife, Henry took over Church property and was responsible for the dissolution of the monastic system and the establishment of Protestantism with himself as Head of the English Church. His son, Edward VI became king at the age of nine but only lived for six more years. In that time, the Church liturgy was changed from Roman Catholic to Protestantism. Edward was succeeded by Mary I, the only child of Henry VIII. She was a Catholic and ruled as Head of the Church of England, but was unable to return England to Catholic practices. In 1554 Mary married Philip of Spain whose interests were entirely strategic and political, to bring England back the Catholic Church. However, Mary died in 1558 without issue, Elizabeth I became Queen, and England returned to Protestantism.

With so many changes of internal political and religious loyalties England was not united; the ambitions of the Scots, the Welsh and the Irish also caused domestic problems. Claims to regions of France, changing alliances, the threat of Spanish invasion, and the Dutch rebellion against Spanish domination preoccupied diplomatic and political activity, but in spite of these uncertainties, England's economic growth continued, particularly during the 50 year Elizabethan period, due to the expansion of sea power and the developing class of business and crafts people who saw opportunities in the practical applications of newly acquired technical knowledge and the advantages of trade and expansion.

The influence of Renaissance Humanism

As the works of classical philosophers became available, scholars began to look at new ideas about the way truth was established, and began to realize that there was a different way of organising their lives and challenged traditional accounts of the physical world. Renaissance Humanism became a cultural and educational reform movement developed by scholars, writers, and civic leaders. It emphasised eloquence, clarity, and practical and scientific studies, and sought to develop people capable of engaging in the civic life of their communities by persuading others to virtuous actions. It changed the methods of reasoning, spread the idea that the natural world had a rational order, and that it was man's duty to understand something of God's creation.

Humanism led to a process of investigating phenomena to establish new facts by developing the *method of induction*. An important practical aspect of investigation began to be introduced, and that was the process of *thinking with objects*. The use of early measuring instruments in astronomy had an influence on the way the results were regarded; not as static individual items, but as sequences of results, conveying changes and movement. This accelerated the refinement of the instruments themselves, seeking greater precision and sophistication. Sophisticated instruments in common use assisted the patient enquirer, and scholar-craftsmen were improving them, and inventing new devices for other tasks. In this way, thinking with objects began to challenge established theories, and the *role of mathematics as an important contributor to both theory and practical applications*, was slowly becoming established.

The power of printing

William Caxton (1422?-1491) introduced the printing press to England in 1476. In doing so, he also founded the tradition where printers supplemented their activities with authorship. As the industry grew, the presses poured out a stream of books and pamphlets on all kinds of subjects. There was a revival of interest in history, and related classical literature. The appetite for these works was voracious; printers produced both new editions of well-known works and the effect these had on both the rising middle class and the artisans was substantial. Books were the means for circulating ideas, stimulating demand, making possible mental contact and cross-fertilisation of ideas. A serious student could cover a wider and more varied body of material than was even conceivable before.

As well as text, books contained maps, plans, diagrams, tables, and allegorical or dramatic pictures, where people acquired the new skill of 'reading' and interpreting the diagrams as well as the text. Contradictions in received opinions became more visible and could no longer proceed smoothly where diverse systems of ideas were brought together and formerly separate disciplines became combined. This influenced new writing in the Arts; in Drama, Music, Poetry, and Literature, and on the presentation of English History, leading to a new vision of National Identity so that "… the great increase, [of published works] leading in the end to a unique manifestation, in the workings of the human mind and spirit which distinguished the sixteenth century in England." (Elton, 1991, p. 435)

The Common Wealth

Edmund Dudley's book *Tree of Commonwealth* (1509) initiated a political concept that was adopted by Thomas Cromwell (1465-1540) into the idea of the 'body politic' of King and people living in natural harmony and mutual dependence. It was a mixture of the national sovereignty of the king and yet a paternalistic view that the state knew what was best for its citizens. Under Cromwell's authoritarian bureaucratic regime, the Act of Supremacy (1534) established the King as ruler and head of the Church in England, yet with a responsibility towards his citizens. There became established a group of people, some connected with the government and others as private individuals, working in and thinking about the welfare of the community, many of them were printers who produced pamphlets supporting the idea, and these propagandists spread the idea of obedience to authority. The Common Wealth became the idea of the 'well-being' (physical and economic) of men, (i.e. Gentlemen) directly related to the good of all, and in particular to the betterment of the lot of common people. The good effects of this policy were to ensure that governance of the country remained relatively stable after the death of Henry (1547) and during the reigns of Edward VI and Mary, until Elizabeth came to the throne in 1558. The social and economic benefits of this policy became self-evident in the cooperation between intellectuals and practical men in improving organisational efficiency and technical production (Elton, 1991, pp. 180-192).

Robert Recorde as an educator

Robert Recorde (1510-1558) was the first person in England to realise the potential of a carefully planned programme of self-education using mathematical texts. He had considerable acquaintance with classical materials, and was clearly aware of local and continental vernacular and academic sources. Among his contemporaries was John Dee to whose extensive library he had occasional access. For much of his life, Recorde was a civil servant, responsible for assaying the coinage, (Williams, 2011) and he developed considerable mathematical skills in this occupation becoming an expert at many aspects of computation.

Recorde was an educator who passionately believed in the value of mathematics to the common man (Fauvel, 1989). His desire to help the 'unlearned' was typical of the Tudor commitment to the idea of the Common Wealth. He planned five books where the learner first had to establish the firm *Ground of Arts* (arithmetic 1543), and then take the *Pathway to Knowledge* (geometry 1551), which led to the *Gate of Knowledge* whereby he entered the *Castle of Knowledge* (astronomy 1556) to find the *Treasure*. Three of these books exist, but *The Gate* and *The Treasure*, if they were ever written, have not survived Finally, in 1557 he produced the first algebra in English, *The Whetstone of Witte*. Apart from his geometry. all instruction in his books was in dialogue form, with the Master patiently leading the Scholar step by step through a carefully designed programme, introducing new skills before harder material, and encouraging enquiry, criticism and debate in order to justify the purpose and usefulness of the new knowledge. "I have written [this book] in the form of a Dialogue, because I judge that to be the easiest way of instruction, when the scholar may ask every doubt orderly, and the master may answer to his question plainly."[2] (*Ground* 1543, p. 7.) He advocated 'true understanding' and warned against learning methods by heart. The *Ground* appeared in 47 editions (with amendments and additions by other editors) until 1699. (Denniss, 2009, p.450) Recorde wrote in southern English with many variations in spellings and used translations of Latin, French German and Dutch works. Since there were few English technical terms for the new ideas, he invented or borrowed terms for operations and objects from other languages many of which have become embedded in our language.

Recorde, Dee, Humanism and Copernicanism

Coincidentally, Recorde's first book, the *Ground of Arts* was published in the same year as Copernicus' *De Revolutionibus Orbium Coelestium.* Recorde's general approach to learning was strongly influenced by Neo-Platonism, and this is exemplified by the illustration on the title page of *The Castle of Knowledge*. Rational enquiry was the cornerstone of Recorde's approach:

> It is commonly seen that when men will receive things from elder writers, and will not examine the thing, they seem rather willing to err with their ancients for company, than to be bold to examine their works or writings. Which scrupulosity hath ingendered infinite errors in all kinds of knowledge, and in all civil administration, and in every kind of art. (Castle, 1556 p. 171).

The allegorical figures on the title page show the opposition of true Knowledge, whose security is obtained by philosophy and enquiry, with blind Ignorance, ruled by instability and chance. The illustration also hints at the opposition to the Ptolemaic World from the new Neo-Platonic ideas. Recorde indicates that the heavens may be unstable when he says. "…the true servants of God which have reposed the love and fear of God in their hearts, are never afeard of any tokens that God sendeth, but

[2] In these quotations I have changed the old English spellings for ease of communication.

rejoice to see them, and glorifiy God for them." (*Castle* 1556 f. aviii). Images like this conveyed the beliefs of the times, and the 'tokens' that God sent were demonstrated in the supernova observed by John Dee and Thomas Digges later in 1572.

Fig. 1. The Castle of Knowledge 1556 title page

Recorde refers to works by well-known authors, often correcting their errors, and while the *Castle* is an introduction to Ptolemy's version of astronomy, Copernicus' theory is mentioned, "Copernicus a man of great learning, of much experience, and of wonderful diligence in observation, hath renewed the opinion of Aristarchus ….. but because the understanding of that controversy depends upon profounder knowledge than in this Introduction may be uttered conveniently, I will let it pass till some other time." (*Castle*, 1556) Recorde did not publicly accept Copernican theory, he would have needed more evidence than was available at the time (Williams 2011 Chapter 9).

John Dee (1527-1609) made his Neo-Platonism the foundation for his scientific and astrological beliefs. He travelled in Europe visiting well-known figures and accumulated the largest personal library of anyone in England at the time (Roberts & Watson, 1990). He gained a reputation for scholarship and his opinions were widely sought. Among his scientific activities Dee prepared nautical information and instructed ships' crews on geometry and cosmography. He guarded his professional secrets closely and kept his treatises on navigation and navigational instruments in manuscript.

Dee proposed a plan to Queen Mary to establish a national library, partly to rescue the books and manuscripts dispersed in the dissolution of the monasteries; he tried again in Elizabeth's reign, but neither plea was successful. His major educational contribution was his *Mathematicall Preaeface*, written as an introduction to the first English Edition of Euclid (Dee 1570). This is a manifesto containing his statement of belief, and a vision of the nature and scope of the broad applications of mathematics. Accompanying the text, he provided an elaborate fold-out plan of how the different aspects of mathematics are related with the heading, "Here have you (according to my promise) the Groundplat [plan] of my MATHEMATICAL Preface: Annexed to Euclid (now first) published in our English tongue." The plan shows the *Sciences and Arts Mathematical* separated into two Principal categories, Arithmetic and Geometry, and some 32 Derivative categories (i.e. mathematics applied or underlying other subjects as in sciences like *Perspective, Astronomy, Music, Cosmography, Astrology, Navigation, Architecture,* etc.). Rampling (2011) provides an interesting discussion of Dee's Preface and his position in Renaissance Humanism.

He insisted there were discoverable, all-permeating numerical harmonies underlying the manifestations of the physical world, the cosmographical world, and human nature, and his defence of mathematics was conducted by appealing to the absolute certainty of its demonstrations, its apparently *a priori* source, and the doctrine that the intellect of man was, in the sphere of pure reason, a reflection of the creative intellect of God.

> All things (which from the very first original being of things, have been framed and made) do appear to be Formed by the reason of Numbers. For this was the principal example or pattern in the mine of the Creator. …. By Numbers property therefore, of us, by all possible means (to the perfection of the Science) learned, we may both wind and draw our selves into the inward and deep search and view, of all creatures distinct virtues, natures, properties and Forms. And also farther, arise, climb, ascend, and mount up (with Speculative wings) in spirit, to behold in the Glass of Creation, the Form of Forms, the Exemplar Number of all things Numerable: both visible and invisible; mortal and immortal, Corporal and Spiritual. (Dee, 1570 sis. j)

This claim that mathematics underlies everything in the physical world, and mathematical knowledge is the key to our understanding of the universe and its creator was at the basis of Dee's approach to all kinds of natural phenomena, and to his belief in the power of mathematics.

Since mathematics was used in astrology, genuine mathematicians were often looked upon with suspicion, and the terms 'Astrologer', and 'Mathematician' were often synonymous in the popular imagination. Recorde refers to Roger Bacon (1214-1294) as a *"great necromancer"* (*Pathway* 1551 sis.3v). The character of Prospero (the sorcerer) in Shakespeare's *Tempest* is modelled on Dee, and in the play *Dr. Faustus* Marlowe (1564-1593) tells how having sold his soul to the devil, "he [Faustus] became the most famous name of all the mathematicians …" While at Oxford, Dee constructed a device for a play where an actor appeared to fly. This gave rise to his reputation as a Conjuror, and *Thaumaturgicke* (one of the *mathematicall artes* listed in his plan) was the name he gave to the way that people could be amazed by devices that relied upon mechanics, or optics for their operation. Dee often used this reputation to his advantage; he could be benign, casting horoscopes for important people, including Queen Elizabeth herself, and he could use his reputation to deter idle curiosity.

A startling discovery in astronomy

In 1572 Dee observed the 'new star' often called 'Tycho Brahe's Supernova' and published *Parallacticae commentationis praxosque* (1573), an account of his observations showing the trigonometric methods he applied trying to find the distance to the star. Independently, his assistant and ward Thomas Digges wrote his account *Alae seu scalae mathematicae* (1573) that included the trigonometric theorems used to determine the star's parallax. For Dee and Digges, the fact that no parallax could be determined, showed that the new star was *outside the 'sphere of the moon'*, and they wrote in Latin to broadcast their discovery to other scholars that something was changing in the 'immutable' heavens, presenting a serious challenge to Aristotelian theory.

Following this new tradition was the astronomer John Blagrave (1561-1611) an important Tudor mathematician who designed and made various astronomical instruments and sundials. He occupied an influential position, being a protégé of William Cecil, Baron of Burleigh, the principal civil servant, political supporter and confidant of Queen Elizabeth.

In particular, Blagrave's *Mathematical Jewel,* was an instrument for enabling the simplification of astronomical calculations. This device (a combination of armillary sphere and astrolabe) is reputed to be one of the most ingenious scientific instruments, of the time.

> *… Containing in sum, A reduction of the Arts Mathematicke tending thereunto, and to diverse other good uses, from that deep difficulty, wherewith hitherto they have been sequestered and closed up as it were in several, only to the most learned kind: unto an easy methodical, plain and practical discipline lying wide open unto*

> *every ingenious practicer whence I presume, many singular inventions and notable commodities in time shall ensue and spring, even a number yet unthought of, even from the common sort of handy craftsmen and workers.* (Blagrave, 1584 folio ii)

Blagrave was one of the new generation of astronomers who began to look at the rational, Copernican universe.

Thomas Digges and the wider applications of Mathematics

Thomas Digges (1546-1595) received his early education from his father Leonard, who died when Thomas was aged fourteen. John Dee was a family friend, and became Thomas' guardian, training him in mathematics and introducing the humanist beliefs that were to be the foundation of Thomas' future reputation. Thomas translated some of Copernicus' material giving an exposition and diagram of the heliocentric theory and using his observations of the 1572 supernova to justify the system in "*A Perfect Description of the Celestial Orbs*, …" published as an appendix ("*according to the most ancient doctrine of the Pythagoreans: lately revived by Copernicus*") to a new edition of his father's almanac *Prognostication* in 1576. This book was reprinted with its Copernican universe seven times until 1605.

Thomas edited and republished many of his father's works: *A Booke Named Tectonicon*… (1556), was a manual of mensuration *"… most conducible for Surueyors, Land-meaters, Joyners, Carpenters and Masons"*. In Digges' mind the title defined his audience, and we have some evidence of who the actual readers were, when we find references to Digges' books being reprinted and read by technicians and artisans in their own publications. Taking advantage of his father's papers he produced a new edition of: *A Geometrical Practise named Pantometria, diuided into three Bookes, Longimetria, Planimetria and Stereometria,* … (length, area and volume) showing his familiarity with the latest continental sources, ("framed by Leonard Digges, Gentleman, lately finished by Thomas Digges his son") was published in 1571 where Thomas added his own *Mathematical Discourse*, a treatise on the dimensions of Platonic and Archimedean solids (*1571*: f.S4v). The combination of geometric diagrams with algebraic working in the *Discourse* shows that he was a highly competent and original mathematician. This 1571 publication went through many further editions. Leonard Digges is now regarded as the inventor of the 'Perspective Trunk', comprising of a plano-convex lens with a spherical mirror (a device mentioned in *Pantometria*) (Ronan 1992). Thomas completed his last book *Stratioticos* in 1579, on military administration that included two chapters on the 'Rule of Cos' (linear and quadratic equations). It also has a chapter on gunnery where some results from his father were developed to criticise Tartaglia's (1537, 1546) work on ballistics.

In 1582 Thomas was in charge of the rebuilding and fortification of Dover harbour. This was an enormous task, involving drawing accurate plans, a new technical skill that had developed since Henry VIII's enthusiasm for new fortifications (Shelby 1967) (Gerbino & Johnston 2009: 31-44). In order to facilitate this, Digges had to negotiate with the Privvy Council, (the highest level of government) who had charge of the treasury. He was joined in this task by others, and these practitioners learnt how to organise men, calculate quantities of materials, and present persuasive arguments based on mathematical calculations. The execution of this enterprise demonstrated the powerful position the mathematistion of craft skills had gained in society. Finally, from 1586 to 1594 Thomas served as an administrator with the English forces in the Netherlands procuring weapons and supplies.

Gunnery emerging as a mathematical science

Tartaglia's *Nova Scientia* (1537), was the first publication to identify ballistics as a new mathematical science. He gave an Aristotelian account of the trajectory of canon shot in three parts; a straight path as the ball left the canon, a circular arc as its 'weight' overcame the propelling force, and a final perpendicular section as the shot sought the centre of the earth. The gunner *"should know approximately and work out from its situation the distance of the place which he has to batter. … he should know the range … of his canon according to their elevations; knowing these things he will not err much in his shooting"* (1537 Dedication sig. Aii). In his second book *Questi et Inventioni diverse* (1546) he used more evidence from experience, and claimed that the maximum range would occur at an elevation of 45^0 because this was exactly between 0^0 and 90^0, but he does not advance the theory and is unable to say anything useful about ranges achieved for different elevations of the barrel.

In the latter sixteenth century, there were many instruments available for measuring distance and ranging the canon, and an important part of most gunnery manuals began with a section on the basics of arithmetic and geometry with simple surveying, progressing to the use of proportions. Tartaglia had already pointed out the importance of the 'method of proportionals' for measuring heights and distances, and how finding square roots was necessary for calculating the powder charge, weights of canon balls of different materials and sizes, and to solve the problems of applying crude range tables.

The importance of mathematical knowledge as well as skill in preparing for war was recognised by Leonard Digges in his *Pantometria*, completed and published in 1571 by his son Thomas. Most of the illustrations on surveying show the activity in a military context. After serving in the army in the Netherlands Thomas published *Stratioticos*, (1579), an *"Arithmetical and Military Treatise"* where the final chapter discusses *"*Certain questions in the Art of Artillery, by Mathematical Science joined with Experience …" (p.181). Here he identifies four principal aspects of the problem; Powder, Piece (the canon) Bullet and Elevation. He also considers such parameters as the *"Rarity and Density of the Air"* the direction and strength of the wind, the way the canon is loaded, the local topography, and the mounting and manufacture of the canon, and questions whether varying the weight of the bullet changes the range in direct proportion or whether the relation involves a quadratic, cubic, or quartic root, arriving at some forty questions that could be the basis of practical experiments. He suggested that the mechanics of planetary motion could also describe the trajectory and spin of the projectile, before we can bring the 'art' into a true science, (thus in thought, unifying the mechanics of heaven and earth) and asks *"Whether it be or not rather a Conical Section and different at every several Randon"* and *"Whereupon by demonstration Geometricall a Theorike may be framed that shall deliver a true and perfect description of those Helicall lines at all Angles made betweene the Horizon and the Peeces lines Diagonall."* (1579: 187-189). Digges' books were still read well into the seventeenth century showing that the 'certain rules' of mathematics (if they could be discovered) were vital to this science.

Others advertised their views on these problems and made the concerns of Digges and European writers known to English gunners. William Bourne (1535-1582) in *The Arte of Shooting in Great Ordnance* (1587) described "Considerations to be had" (1587, p.3), complaining against the gunner's reliance on instruments when measurements of distance are not accurate, and no account of the local topography was taken, and against theories by 'armchair gunners' when empirical results are more reliable. His *A Regiment for the Sea* (1574), a practical manual for seamen (much of it derived from French and Italian authors) had eleven English and three Dutch editions to 1631. He was the first to show how to plot coastal features from the ship by taking bearings using triangulation.

Robert Norton (fl 1590-1635) in *The Art of Great Artillery* (1624) considered Digges' questions in the light of practical experience and provided a helpful analysis and clearer definition of the variety of technical terms used by gunners. He also made some sensible progress on the use of range tables. Four years later, Norton's *The Gunner* (1628) appeared but offered little improvement.

At this time theory was still a matter of speculation and notwithstanding some experimental evidence, too many variables made the practical experience of gunners outweigh any theoretical advances until well into the eighteenth century. Artillery tables were drawn up to record the size and weight of shot and the amount of powder required for different types of artillery to help the gunner, but despite Digges' questions, the trajectory was still being described in Tartaglia's terms even at the end of the seventeenth century.

Defending the shores

Since the introduction of gunpowder and the newly effective manoeuverable siege cannon that came into service in the later 15th century, artists (who were also military advisers to their patrons) were employed to plan, construct, or supervise new fortifications. Henry VIII made the reorganisation of the defence of Britain a priority.

In 1532 Vincenzo Volpe, who was Henry's painter produced an imaginative representation of improvements to Dover Harbour to persuade the King to fund the project, but this was useless as a working diagram. (Volpe, 1532)

Mediaeval masons and craftsmen had traditionally worked independently making adjustments to the building design as problems arose. However, by the late 15th and early 16th centuries the Renaissance influence of classical forms had influenced architectural drafting that by now had become a recognised part of the master mason's practice, and this employed not only accuracy in plane geometrical drawing, but also an essential knowledge of surveying and calculation. Working to an agreed design was a crucial concept that needed collaboration and cooperation.

In 1539 prompted by threats from France and Spain after his divorce from Catherine of Aragon, Henry commissioned a series of artillery fortifications along the South coast. Typical of these was Deal Castle (near Dover) with a hexagonal form and squat semicircular bastions intended as gun platforms to defend against bombardment from the sea. Some thirty of these castles or forts were constructed by 1545. Henry needed specialists with experience of building forts that could be more structurally robust and able to provide crossfire for mutual defence, *star forts* incorporated new design principles with a very flat profile and more angular polygonal shapes with arrowhead bastions specifically designed to provide fire cover for each other.

Two well-known figures involved in the fortification of costal defences along the south coast of England and the Pas-de-Calais in France and were often found working on the same projects during this period: John Rogers (1533-1558) and Sir Richard Lee (1513-1575).

Richard Lee was an early example of a soldier-engineer. His first major appointment was as the surveyor of Calais in 1536 and the next years of his life were chiefly devoted to the fortification and defence of Calais. He accompanied an expedition against the Scots, as captain of pioneers, and was knighted at Leith on 11 May 1544. He returned to Calais, and thence to Boulogne, where he was involved in their defences in 1545. Later in the same year he became the Member of Parliament for Hertfordshire. Lee served in the Scottish war of 1547 as chief engineer for fortification and was occupied with the defence of the northern border from 1548 to June 1550. Queen Elizabeth made repeated use of Lee's military experience, and he was sent to report on the fortifications of Dover in October 1553. Lee was well connected, and appears to have been more of a supervisor and entrepreneur than an inspired designer. (Coros, 1982)

On the other hand, John Rogers was an important figure in the transition period between the mediaeval master mason and the diverging professions of architect and military and civil engineer. His work was clearly influenced by the scholars with antiquarian and archaeological interests who were interested in making correct copies of classical buildings. He rose from the level of a simple mason at Hampton Court to a Master Mason at Guînes (near Boulogne) in 1541. He became Henry VIII's most trusted adviser on fortification and held the chief responsibility for the extensive works at Calais and Boulogne in the 1540s and 1550s and was instrumental in replacing the rounded bastions of the early forts on the south coast with angle bastions. In 1544 he became Clerk of the Ordnance (weapons, ammunition and fortification) in England which post he held for the next 13 years. He was one of the first and most eminent examples of the professionalization of the military engineer as it separated itself from traditional building (Shelby 1967), and his achievements included some of the earliest substantial harbour works undertaken by a British civil engineer. (Chrimes & Hots, 1998). The problem of reconstructing Dover Harbour was complicated and ran on for many years until Thomas Digges took charge in 1582 and the project was completed in 1585. Stephen Johnston (1994) has an extensive and detailed discussion of the construction of this project.

The shipwright's new science

New challenges faced the shipwright in the sixteenth century; defending the country needed better and more powerful ships. *Henri Grace a Dieu* and the *Mary Rose* were English warships built during Henry VIII's reign to ambitious designs, but they required considerable modification after their construction, due to instability and lack of manoeuverability, and the *Mary Rose* famously foundered in 1545 during an engagement with the French off the Isle of Wight, while the King watched from one of his new forts.

Hitherto, shipbuilding had been regarded as the occupation of *artisans*, semi skilled workers but John Dee had recognized ship-building as one of the three branches of architecture in his work on the *Mathematical Arts* (Preface diiij verso). Traditionally, 'laying down the lines' of a ship was done in the dockyard, at the site of its construction, and the building process continued there 'from the ground up', using the experience of the shipwright, often assisted by a scale model. Any modifications, due to instability or any other apparent faults were made as work progressed, or after the ship was launched. Shipwrights were a valuable asset to an economy, and Mathew Baker (c.1530-1613) took a very different approach. James Baker, Matthew's father was one of a group of skilled shipwrights with special licences from Henry VIII to build ships and instruct others in the art in 1543. Matthew and others, as Master shipwrights, raised their status from being mere craftsmen to that of innovators and managers in a developing and strategic industry. (Johnston 1994)

Matthew set out to lay the lines by *drawing the ship on paper*. In this way he was able to use geometry and arithmetic to *design* the ship, experiment with ideas and discuss with others modifications and improvements, thus introducing a mathematical and theoretical aspect to the shipbuilding process. As exploration developed, ships traditionally designed for costal trade were not suitable for long sea voyages, neither were they suitable for attacking other sips or establishments ashore. Baker reduced the height of the castles, so the ships were easier to manoeuvre. The hulls were designed in the shape of a fish, with a full body tapering towards the stern that produced a smoother flow of water round the hull, increased their seaworthiness, and made them faster. The ships were smaller and more stable than the Spanish-designed galleons, enabling them to fire broadsides, and this became the new technique for naval warfare, replacing the need for carrying large numbers of soldiers for boarding enemy ships. (Friel, 2009).

Navigation and the quest for Empire

During the early 16th century the Spanish and Portuguese dominated sea routes, and England was denied access to India and the Spice Islands. This motivated exploration of coasts of both North America and Russia. Expeditions were dangerous, due to the lack of navigational experience and the suitability of the ships for long voyages, so improvements became imperative. In 1496 Henry VII had commissioned John Cabot (1450-1499) to lead the first in a series of missions to North America, hoping to find a route through a North West Passage for Asian trade, but this was unsuccessful. Later, in a harrowing attempt to find a North East Passage to China in 1553 Richard Challoner (?1525-1556) reached what is now Archangel in the White Sea and journeyed to Moscow. This event opened trade with Russia and the Muscovy Company was established as the first joint stock trading company in 1555.

After Queen Elizabeth came to the throne, political ambitions encouraged more aggressive interventions against the Spanish; in 1562 and 1564 Sir John Hawkins was engaged in the slave trade, carrying cargoes to the West Indies and in 1570 Sir Francis Drake started privateering excursions against the Spanish. In 1577 Drake sailed south along the coast of Brazil, attacked a Portuguese ship and *stole her charts and pilotage notes*, and passed through the Straits of Magellan. After attacking Spanish settlements in Chile and Peru, he turned North in the hope of discovering a passage to the Atlantic, but had to return home via the Cape of Good Hope, accomplishing the first circumnavigation of the globe by an Englishman[3]. There were voyages from 1576 to 1578 by Martin Frobisher (1535-1594) backed by a group of merchant adventurers to find a northwest passage, and from 1584 to 1590 expeditions to colonise North America were organsied by Sir Walter Raleigh.

Key to early developments in navigation was John Dee, who had accumulated the range of knowledge to support exploration and provided charts, instruments, and training in navigation for many voyagers. He was followed by a number of enterprising people who were able to extend the theoretical knowledge and practices of navigation such as Edward Wright (?1558 -1615) mathematician and cartographer at Cambridge University, who prepared new charts using his version of the Mercator projection and published *Certaine Errours in Navigation* (1589). In the same year he created the first world map produced in England. (Parsons & Morris 1939). Robert Norman (?1560-1596) was the influential author of *The Newe Attractive* (1581) on magnetic variation and discovered magnetic dip. Emery Molyneux (fl 1587-1605) popular maker of many globes supplied to mariners, and John Davis (1552-1609) Navigator, and author of *Seamans Secrets* (1593, 1657) declared "… what strangers may be compared with M. Thomas Digs Esquire, our Country man the great master of Archimastrie: [One of the Derivative Arts from Dee's Praeface] and for Theoricall speculations and most cunning calculations M. Dee and M. Thomas Harriott are hardly to be matched: and for mechanicall practices drawn from the Arts Mathematic our Country doth yield men of principall

[3] The first circumnavigation West to East was by Magellan's ships (1519 -1525) but Magellan was killed in the Philippines in 1521 and only one ship and 18 men returned to their starting point. Drake set sail West to East from Plymouth in 1577 and returned in 1580.

excellency as M. Emery Mullinaux for the exquisite making of Globes bodies." (Davis, 1593. Dedicatory Preface). Thomas Harriot (1560-1621), mathematician, astronomer, ethnographer and translator, who after graduating from Oxford in 1580, started in the service of Sir Walter Raleigh as mathematical tutor, used his knowledge to provide navigational advice when he accompanied the colonists to Virginia in 1585. He wrote his *A briefe and true report of the new found land of Virginia,…* about the expedition in 1588, describing the inhabitants, the countryside, and the use of many instruments including the lodestone, and 'perspective trunks' (by 1609 he was making telescopic observations of the Moon. (Ronan, 1992)) showing that he concerned himself with all outstanding matters on navigation and chart making. (Haklyut 1589-1600).

Drake's circumnavigation in 1580 produced an enormous boost to the self-image of the English people. No longer were the 'Indes and Cathay' the sole possession of the Portuguese, Spanish and the Dutch, but the route to the 'Spice Islands' lay open to trade, and this showed that English ships could potentially reach any part of the world, and as Puck says, "I'll put a girdle round about the earth in forty minutes." (Shakespeare, Midsummer Night's Dream Act 2, Scene 1)

Promoting all of these activites to the wider public was Richard Hakluyt (1552-1616), a diplomat and strong advocate of exploration. His first book, *Divers voyages touching the discoverie of America*, was published in 1582, and in 1589 he reinforced his argument emphasising the advantages to England for colonial expansion in *The Principal Navigations, Voiages, Traffiques and Discoueries of the English Nation* (1589–1600). This work promoted commerce through colonisation by stimulating interest in recently discovered parts of the world. In the preface he announced the intended publication of the first terrestrial globe made in England by Emery Molyneux. A few copies of this important work contain the first map of the Mercator Projection made in England according to the 'true principles' laid down by Edward Wright.

Into the early 17th century

Mathematical practice was clearly distinct from academic natural philosophy and contemporary scholarship. Even though a number of these innovators had academic training, they were interested in the vision of education and its useful applications and they appeared as published authors, devisers of instruments, technical advisors, and entrepreneurs quite different from craftsmen and mechanics, and the class of patrons and statesmen. The work of Recorde, Dee and Digges was typical of gentlemen committed to the idea of the Common Wealth, the well-being of men, directly related to the good of all, and in particular to the betterment of the lot of common people. The obvious applications of mathematics in Astronomy, Navigation, Surveying and Commerce had now influenced the Master Masons, Gunners, Shipwrights and others who embraced the potentialities of practical mathematics and raised themselves to a professional status that was respected by virtue not only of their craft skills, but also for their ability to communicate new ideas through technical drawing, thereby enabling more efficient working practices.

All these people had a deep faith in the potentialities of mathematics, and each expressed this faith in different ways; Recorde through his ambitious practical education programme, Dee through his erudition, mysticism and vision of mathematics as underlying Divine creation, and Digges as the entrepreneur who made the mathematical practitioner important and essential at the highest level of government. It is also notable that, along with other scholars of this time, they could communicate important scientific ideas in Latin, which was to remain the common language for international learned discourse for some time to come.

The relationship between mathematical practice and that of other skills was complex and ambiguous and the mathematicians' rhetoric of utility should not be accepted just as a simple statement of the practicality of their work. This new group of people were defining themselves according to their position in society, their beliefs about the nature and applications of mathematics in various occupations, and their individual skills in different ways:

- by identifying a position for themselves as essential to society and necessary for economic development, exploration, and military enterprise
- by their contribution to the Common Wealth in promoting mathematics as useful, yet an essentially intellectual and spiritual activity
- by refining the vision of the individual in personal improvement and changing the status of pedagogy and its practice
- as quite separate from astrologers, necromancers, and sorcerers and technically different from traditional artisans, mechanicians, artificers, and craftsmen
- and by the contexts of their applied knowledge as the originators of the an emerging professional class.

At one end of the spectrum, we have a vision of spirituality and wide claims for the *Mathematical Arts* and at the other we have the very practical application of mathematical knowledge in the invention of new instruments and their methods of application. By the beginning of the 17th Century there was a thriving community of Instrument Makers and Practitioners in London, and on the basis of common mathematical principles a whole range of astronomical, surveying and navigational instruments were invented, improved and refined for popular use. But now, the term 'Practitioner' covers not only those whose interest was principally mathematical but a range of technicians using and applying mathematics in their various occupations.

These advances in technology were attributable to the accumulation of small improvements, essentially empirical, collaborative and democratic, which were used by Francis Bacon (1561-1626) to demonstrate the manner in which intelligent application could lead to economic progress and intellectual advancement in the development of his system of Natural Philosophy. This was to have its most popular and powerful expression in his *The Advancement of Learning* (1605) and his *Novum Organon* (1620).

> Printing, gunpowder and the compass: These three have changed the whole face and state of things throughout the world; the first in literature, the second in warfare, the third in navigation; whence have followed innumerable changes, in so much that no empire, no sect, no star seems to have exerted greater power and influence in human affairs than these mechanical discoveries. (Bacon, *Novum Organum*, Book 1, Aphorism 129)

The mathematical practitioners of the 16th century had laid the theoretical and practical foundation for the development of the varieties of applied mathematics, the institutions, and teaching methods that were to develop in the following century.

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From the few to the many: Historical perspectives on who should learn mathematics

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Abstract

Today we take for granted that everybody can and should be offered the opportunity to learn and study mathematics. Here, we discuss the fact that it was not until well into the twentieth century that "mathematics for all" came to be regarded as an achievable goal. Indeed, the characteristic difference between the two essential social functions of mathematics, its utilitarian function and its capability to sharpen the mind and logical thinking, led in many cultures to opposed forms of teaching: in vocational institutions and in liberal education, where mathematics over long periods was regarded as less contributing to this function than classical languages. Mathematics proves to be, as its perennial characteristics, at the crossroads between liberal and professional training, and even to be the only school discipline with such a double orientation. Mathematics education was considered in earlier times to be important only for a few gifted individuals, but now it is accepted as a necessity for all within democratic societies that are more and more shaped by mathematical-technical applications.

In particular, methodological issues will be discussed as necessary for meaningful investigations into this key question of the history of mathematics teaching and learning. The variance of of meaning of the notion "Mathematics for All" will be analysed as well as sociological categories like the historical change from hierarchically organised societies to functionally differentiated societies. Changes in the functions of the state prove therefore to be essential to understand interrelations between qualification profiles and striving towards egalitarian structures in education.

Introduction

First of all we will have to reflect on the meaning of the notion „Mathematics for All". Actually, it proves to have a very broad spectrum of meanings. An impression is given by a contribution of Peter Gates in the Second International Handbook of Mathematics Education (2003):

> "first, there was the demand to give access to mathematics not just to some, but to all pupils, to make school mathematics beyond the primary phase compulsory" (Gates 2003, p. 33),
>
> the second meaning concerns problems of quality: in particular, overcoming causes for the restricted teaching and learning opportunities for pupils of certain minority groups defined by gender, class and ethnicity in industrialised countries,

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> and as particular problem of quality the great number of young people in non-industrialised world or developing countries without having general access to schooling or any mathematics education (33-34).

The notion emerged in the early 1980s and became prominent since ICME 5 in 1984 when there was a thematic issue with numerous contributions. The key persons featuring then this concept were Peter Damerow (Federal Republic of Germany) and Bienvenido Nebres (Philippines). It was not by accident that the notion emerged then, right within the phase when the former colonial countries had become independent and were about to establish proper school and educational systems, urged therefore to reflect upon their educational principles and conceptions and to decide how to realize them.

A wrong historical claim underlying the notion of 'Mathematics for All'

As a matter of fact, one has to remark that the notion of "Mathematics for All" was burdened with mortgage right from the beginnings and continues to burden it. This mortgage was an entirely erroneous affirmation on the teaching of mathematics in 19th century Europe. In the classical key document for the movement of 'mathematics for all', the proceedings of the theme group on this issue at ICME 5, 1984, published as UNESCO Document Series no. 20, it is repeated ever and again as starting point for the analysis of present-day mathematics teaching world-wide:

> "Traditionally, mathematics curricula were developed for an elite group of students who were expected to specialise in the subject, and to study mathematics subsequently at higher levels in a tertiary institution" (Damerow et al. 1986, p. 4).

Since it is claimed that this traditional curriculum was transferred to developing and third world countries, it is deplored that one did transfer a curriculum for forming *specialists* and thus ignoring all social and cultural conditions, in particular when now universalising the teaching of mathematics. A dichotomy is therefore constructed between "how to cater for the elite" and "how to cater for the wider group of students for whom mathematics should be grounded in real world problem solving and daily life applications" (ibid.).

Actually, it is the general tendency of this *Document* to claim such a curricular approach, focussing one-sidedly on applications, to serve the needs of the alleged non-elite groups, also called as "underrepresented groups, primarily females and minorities", as a prerequisite "to increase their enjoyment of mathematics" (ibid., p. 6). It is strange enough to thus parallelise females and minorities.

But the key claim about the 19th century negates completely what was the historical achievement of this period in particular in Germany: to have established a first realization of a 'mathematics for all', namely a general education in mathematics for all students of these secondary schools, independently of their future profession and in particular of their future studies – hence the contrary of a specialist training preparing mathematics studies at a university. And the propagation of diversified curricula according to alleged particular future professional profiles and needs of students, as practiced during the 18th century by a "Realien-Pädagogik", the philanthropism, was overcome by the program of general education for all – as a common level allowing a higher degree of subsequent specialisation (see Schubring 1991). One wonders to see resurrected the propagation of specialised curricula for all the various social groups.[1]

Moreover, the attribution of an elite status to the students of these schools is sociologically not well informed. German Gymnasia used to be socially multi-layered and were able to serve for social advancement. The UNESCO Document proves that the inspirer of these so terribly misleading assertions was Peter Damerow, from Berlin. In other papers arguing more historically, he constructs already for the 19th century Germany a conflict between an elite-oriented pure mathematics and a real-life oriented, applied mathematics in teaching. Damerow on his part proves to be inspired by Lewis Pyenson's booklet *Neohumanism and the Persistence of Pure Mathematics in Wilhelmian Germany* (1983), which – although using a lot of sources – misses all knowledge of the German context and thus arrives at grotesque misinterpretations. Pyenson's construction of an anti-reformist pure mathematics and a reform-minded movement for "real world" in mathematics teaching (Pyenson 1983, 3) at the end of the 19th century is repeated by Damerow, giving as reference a debate among German mathematics and science teachers in 1890, but where the source cited clearly shows that the debate was about the traditional formalist Euclidean geometry – and not about pure mathematics (Lorey 1938, p. 15; Damerow 1984, p. 82; Damerow & Westbury 1986, p. 23).

[1] Conceptual basis for this resurrection was the contemporaneous curriculum theory, as propagated by Saul Robinsohn (Max-Planck-Institut für Bildungsforschung Berlin) and practiced in the Cockcroft report: assessing within enterprises which qualifications they deem their employees should have and transform this into curricular structures for secondary schools.

Given all the emphasis on improving the quality of mathematics teaching it is the more astonishing that it is taken for granted in the entire movement of 'mathematics for all' that mathematics has to be one of the key pillars of school education. One speaks there of "universalisation" of primary schooling and likewise of universalisation of secondary schooling, but it is never reflected that mathematics is presupposed to constitute a basic element of that universe. Yet, it was a quite recent development that mathematics had become such a basic element.

The historical challenge

Over millennia, mathematics had been a rather marginal subject in the educational systems. Thus, the basic question, with which one is confronted when studying the evolution of how mathematics became a learning subject for all, is to understand what effected the change from a marginal to a major teaching subject. Actually, this occurred for the first time in those institutions, which are denounced as elite training schools. Understanding this developmental process will also contribute to better discuss the question of quality of teaching.

As we will see, it is by fundamental changes in the functions of state and of society that these processes were initiated. Changes in the functions of the state prove therefore to be essential to understand interrelations between qualification profiles and striving towards egalitarian structures in education.

The second important dimension results form the nature of mathematical knowledge. The perennial characteristic of the access to learning mathematics is the split into two distinct and practically dichotomic patterns: learning for a direct application in some professional context, what one might call in modern terms vocational training – and on the other hand a learning as one of the constitutive elements of the formation of the mind and of the personality, to develop one's cognitive abilities, thus what one calls today general education. It is the tension between these two different social functions of mathematics, to serve for vocational training and for general education, which determined the evolution of related institutions and the access to them. Evidently, by its very nature, vocational training used to be restricted to definite groups of a society, typically corporations.

Antiquity

It is highly revealing that the very first institutionalized forms of learning some kind of mathematics were for professional use and established by a state, for its service. Over all the millennia, these were the dominant forms of getting to know of mathematics so that the access used to be clearly restricted to small social strata. Forms allowing broader accesses to mathematics have to be searched for within the other pattern, of general education, but might have been induced by some changes on the vocational part.

The very first professional group we find in history which had to apply mathematical knowledge were the scribes. Their activity was of vital importance for the constitution of the first states, in Mesopotamia and in Egypt, and enabled a rational administration of the society. Recent research has documented and analysed in an impressive manner for the early states in Elam, Uruk, and Sumer that the emergence of mathematics and of writing are due to the same social process of establishing bookkeeping for the goods delivered by the population as a form of taxes for the maintenance of the ruling class, within the temple administration: the signs developed and used designated at the same time an object and its quantity (Nissen et al., 1993). In fact, the Sumerian word for school – *edubba* – literally means House of Tablets, where one had to learn to engrave the signs into the clay tablets.

Robson relates not only that "presumably all large administrations had to train their scribes in various numerate activities", but also clearly resumes the character of the accounting activity of the scribes:

> While many individuals and families doubtless earned their livings through personal enterprise based on trust and memory, never coming into contact with the written record, institutional economies continued to run on centrally managed quantitative models, planned and carried out by literate and numerate professionals. Literacy and numeracy thus remained overwhelmingly in the hands of bureaucrats managing large-scale quantities of land, labour, and livestock, yet also began to develop into independent genres that, by the last third of the [third] millennium, can clearly be identified as literature and mathematics (Robson 2008, p. 1).

There is even a study on the social structure of the scribes active in the Early Dynastic Suruppag period, around 2500 BCE, which proved a marked hierarchical structure, distinguishing administrators, accountants/land surveyors, scribes and scholars. The study even allowed the identification of about forty named literate and numerate officials working for the bureaucracy of Suruppag in a single year (Visicato 1995; according to Robson 2008, pp. 31-32). And it is by accounting documented on such tablets that one knows the name of the oldest person practicing mathematics! On a tablet from the palace archives, at the Syrian city of Ebla, dating from about 2350 BCE, one finds at the end: "Nammah wrote the calculation" (ibid., p. 32).

An enormous number of tablets document the type of exercises used for trainee scribes, either in *edubbas* or in training on the job. Actually, the training had to embrace both writing and calculating, given the inseparability of both. For a number of schools, one even knows the standard "curriculum"; among them was a list of standard words, a list of professions and tables with reciprocal values (ibid., pp. 41-44 & 192-198). One was even able to identify one building in Nippur, "House F", as an *edubba* and to understand the training practice there by means of its architecture (ibid., pp. 97-102).

One does not have to imagine, however, this development as a continuous one. Well to the contrary, periods of stable development used to be the exception: due to ever new invasions and occupations, in particular of Mesopotamia, by tribes and peoples accustomed to warriors' habits and values, state organisation could change abruptly and brutally.

Regarding Egypt, contrary to the traditional understanding attributing the rise of geometry simply to the annual inundations by the river Nil, the true meaning of the often repeated report by Herodot is one of a state organized activity of land surveyors: they had to measure anew the areas when inundations had changed the former demarcations – as a basis for calculating the respective taxes. Clearly, there were analogous professions of scribes, but one knows much less than for Mesopotamia as regards their training and its organization. A stele preserved in the *Musée du Louvre*, of the scribe Irtisen, datable to about 2000 BCE, shows that he had to keep his knowledge secret and that he would introduce his eldest son into this art, thus training by apprenticeship. Irtisen's specialty had been weights and measures (Schubring, 1985, p. 350).

The Greco-Roman world

It is well known that it was first in Greek city-states where pure mathematics emerged, together with a strict epistemological and social separation between practical arithmetic and its practitioners on the one hand and theoretical mathematics and philosophers on the other hand. It is also well known that first forms of a certain general education became established, yet already in a somewhat bifurcated structure: for the sons of higher social strata within the free citizens, there was some elementary schooling, including arithmetic, and thereafter the possibility to study with a focus on the rhetoric, preparing to political activity, or to follow a more philosophical-scientific education. Since these states were based on slave-holdery, the access to such forms of education was restricted to those whose parents were able to pay for it.

Contrary to the formerly highly valued scribal profession and the key role of training for it in Mesopotamia and in Egypt, the training of practitioners, like land-surveyors, were no longer state controlled or organized, but left to individual initiative or to organization by the respective professional group.

In the Roman Empire, basic features of the Greek-Hellenist educational structures were adopted and further developed. The formation of rhetoric abilities received more emphasis, for the political life, whereas philosophical-scientific qualification became more marginal. By the end of Classical Antiquity, these foci of general education became conceptualised as the seven liberal arts: the *trivium* for the rhetorical formation and the *quadrivium* for the selection of those who would continue the four mathematical disciplines: arithmetic, geometry, music/harmony, and astronomy as conceived of by Martianus Capella and Isidor of Sevilla). These liberal arts should constitute the counterpart to the traditionally less valued mechanical arts.

Contrary to Greek disdain of practical professions, however, and in concordance with the less philosophical spirit of Roman culture, the *agrimensores*, the land-surveyors, enjoyed considerable social rank, acting as a corporation, with proper teaching facilities.

Middle Ages

The various cultures and states of greater importance during this epoch confirm the structures established during the Antiquity.

On the one hand, there was China, well known as a state of high bureaucratic organization and enjoying stability over extended periods. It was the first state to introduce official and sophisticated exams for entering its administrative careers. Mathematics, although not highly valued, constituted one of the subjects for these exams. These exams became systematically organized after reforms realized by the *Sui* dynasty, by the sixth century. For the training to be examined, there was a well-structured curriculum, with programs and textbooks for each of the exam disciplines. The quota of students to be admitted and the criteria for admission were meticulously defined as well as the learning resources. This system was in practice for about 700 years. These exams are of particular interest for mathematics learning, since they led to the first official list of textbooks admitted for the preparatory training: in the year 656, the list known as the Ten Classics, comprising among others the *Jiu zhang suan shu* – the *Nine Chapters of Calculation* -, became endorsed (Martzloff 1997).

In the countries of Islamic civilisation, there was no state initiative or organisation with regard to learning and education. It was entirely by private means and organisation when there existed institutions for teaching, be it within mosques or by the *madrasa*. Mathematics, if taught there, had an auxiliary, propedeutic character (see Schubring 2000).

In the Christian Europe of the Middle Ages, there were marked ruptures with the established educational structures of the Roman Empire. The states were dominated rather by warriors' behaviour than by rational administration. The rise of the Frankish Empire facilitated a certain advancement of learning. In some schools attached to monasteries, some parts of the seven liberal arts were taught, but in view of the future career of priests; the mathematical knowledge taught there focused on the *computus*, basic astronomical knowledge to calculate the calendar for the religious holidays. Later in the Middle Ages, from the 13th century on, the first universities began to function in Western Europe; the universities used to function as corporations, i.e. without interference by the state. They attracted students, who could afford it, from various regions of Europe to study law, medicine or theology. In the preparatory Faculty of Arts, the seven liberal arts were taught to the youngsters. Contrary to the important role often attributed to the *quadrivium*, its lectures were rather marginal, delivered as "extraordinary" ones, while the *trivium* constituted the core of "ordinary" lectures (Schöner 1994).

New structural elements emerged, however, on the side of professional training. On the one hand, there was the corporation of the master-builders of the gothic cathedrals, organized in the famous *Bauhütten*, who are known to have kept their geometrical-architectural practical knowledge secret and transmitted it only internally to the corporation.

On the other hand, there emerged, spreading from 14th century Florence, the rising commercial centre of Western Europe, the *scuole d'abbaco*, reckoning schools, providing training in calculating techniques necessary for the trading commerce. Given the importance of these schools for the flourishing of their economy, several city governments – entering now as new entities into organizing education – at least supervised the functioning of these schools, attributing them thus a certain "public" character (van Egmond 1976).

Early Modern Times

In the following sections, we will concentrate on Western Europe, since the developments there became decisive for what proved to be established later on as general overall pattern.

Decisive for the future development of general education became an innovation, namely establishing secondary school types, introduced by the pedagogue Johannes Sturm in Strasbourg in the 1520s, extending preceding models of such structured teaching in some Parisian colleges. Given the inconveniences caused by youngsters studying the liberal arts within a university, he founded a Gymnasium with yearly ascending classes and providing a systematic curriculum for those teens whose parents had destined them to university studies. This innovation signified the emergence of secondary schooling, differentiating it from higher education. Following the then dominating educational conception of Humanism, the core of the curriculum at this Gymnasium was the *trivium*, with the classical languages.

This innovation became applied and generalised by the two great movements, which proved to determinate the educational structures in Western Europe – until the French Revolution, which caused a complete restructuring. The Protestant Reform, according to its general approach to suppose the population to be literate, set out to establish Gymnasia in greater towns; they, too, had to prepare its students for university studies. These secondary schools were founded, organised and financed by a number of city governments – thus, the first time that a certain level of state administration assumed direct responsibility for a type of general schooling.

The Gymnasia in Protestant regions provided, however, at first only some basic arithmetic. Only later on, in particular during the 18th century, geometry entered into the upper grades so that by the end of that century, there were already a number of Gymnasia teaching first arithmetic and then geometry in all grades. This change was brought about by a continuation of the new tendency established by the *scuole d'abbaco* in Italy, and which is best studied for Germany: in economically prospering towns, the municipalities assumed responsibilities for running reckoning schools. They organized contracting the reckoning masters ("Rechenmeister"), up to establishing contests (*concours*) when there was a vacancy, even setting up the catalogue of exam questions for these contents (Folkerts, 2003). What used to be professional, vocational training, eventually transformed in some towns into kinds of elementary schools. And during the eighteenth century, a German form of Enlightenment philosophy called for instruction in useful knowledge, challenging the monopoly of classical learning in the Gymnasia and featuring mathematics and the sciences. A first *Realschule* implementing these goals became founded in 1747, in Berlin. Due to this institutional alternative within secondary schooling, the Gymnasia had to adapt somewhat to modernize their curriculum and to increase mathematics teaching (Schubring, 1991, p. 31).

It was the Catholic Counter-Reform, which understood the challenge to establish some general education structures, too. Here, however, no state interfered with education. Secondary schooling was delivered to the Jesuit order which was the first to organise such colleges, from the 1550s on and definitely structured by the *Ratio Studiorum* of 1599, assuring uniformity of teaching in all its colleges. Courses, formerly given within the arts faculty, became integrated into the last grades of the colleges, the philosophy classes. Without an intermediate faculty, students would continue directly to theology studies, or even to law or medicine. Mathematics, understood by them as part of philosophy, was taught in a part of the second year of the philosophy classes. As empirical research into the history of schooling in France has shown, many students left the colleges before the last grades – thus without ever have learned something of mathematics (Dainville, 1986, p. 61). Prior to the philosophy classes, the *Ratio* obliged to an exclusive teaching of grammar and rhetoric. Later on, other orders entered into this domain, organising colleges – but usually without direct interference of the state.

Various states – actually mainly Catholic ones – that either suffered from inundation problems or needed navigational experts, developed initiatives for the formation of technical specialists who were able to deal with these problems. The governments created and endowed teaching positions for such formation of adult professionals in engineering, hydraulics, navigation, etc., in particular so in Portugal, in France, and in the Papal States in Italy, from the late 17th century on (Fiocca & Pepe 1985, Fiocca & Pepe 1986, Romano 1996, Baldini 2004). Typically, these teaching positions became attached to one or the other Jesuit college, so that there the practice of lecturing for short periods by generalist padres, who were substituted by others, would be changed to someone being permanently in charge of these lectures. Featuring thus some areas of applied mathematics, the states acted in a traditional way for their immediate means.

Origins of a new function for mathematics

So far, nothing in the development of school structures hinted at anything like a key role of mathematics. Surely, humanism had argued for the importance of mathematics as one of the domains of human knowledge, but it had not been strong enough to ensure mathematics a role, say comparable to language teaching in the new secondary schools. In fact, in both educational systems, mathematics enjoyed but a marginal status. Melanchthon, the principal architect of Protestant Gymnasia, argued always in favour of mathematics and its teaching in these schools, but more for pragmatic reasons than for epistemological ones. And the Jesuits, who were the first to establish secondary schools in Catholic regions, relegated mathematics epistemologically and practically mathematics to a marginal status, too, due to their reception of Aristotelian philosophy.[2]

From where then stemmed a different appreciation of mathematics, leading eventually to the status as main teaching subject for all? The

[2] See the contribution by Paradinas in this volume.

answer, which I found, might be somewhat surprising, coming from theology; however, in this period, all cultural thinking was permeated by theological reflection. The argument came from Antoine Arnauld (1612-1694), an important French philosopher and theologian: actually not from Catholic mainstream, but from a reform Catholicism: from the Jansenism. Striving among others for a Gallican Church, Jansenists and Arnauld in particular were persecuted by the Jesuits. He was the author (resp. co-author) of three important textbooks, which decisively modernized their respective areas: the Logic and the Grammar of Port-Royal and the *Nouveaux Elémens de Géométrie* (1667), which challenged the dominance of Euclid's Elements.

In the preface of his geometry textbook, Arnauld developed an argumentation why the study of geometry is necessary for achieving the way to faith and to God. One key argument of Arnauld was:

> L'étude de la géométrie [applique] l'esprit à des veritez abstraites et difficiles (Arnauld 1667; preface [7]).

The argument implied that achieving faith was not an easy or direct way, and that it was not via mediators, but by proper intellectual efforts that one had to arrive at entering into the truths of Christian faith. Studying geometry proved to constitute a privileged manner to accustom the mind to this intellectual mastery. Arnauld affirmed:

> disposer meme l'esprit à recevoir les veritez Chrestiennes avec moins d'opposition […], il semble qu'il n'y en ait gueres de plus propre que l'étude de la Geometrie.

He continued that God makes use of the geometric disposition of the mind for leading to salvation:

> pour nous faire entrer dans l'amour & dans la pratique des veritez qui conduisent au salut (Arnauld 1667, préface [4]).

Arnauld's geometry textbook became widely disseminated and several times reedited. His reflections and justifications about the epistemology of mathematics became additionally even more disseminated and already transposed to more worldly matters by the order of Oratorians. In their teaching, they focused on mathematics, and they published several of likewise modernising mathematics textbooks. The group around the philosopher and theologian Malebranche, the leading group of the Oratorian order, contributed decisively to the reform of the Paris Academy of Sciences in 1696, which resulted in restructuring it according to classes and thus achieving for the first time a mathematical class, devoted to research and counselling in mathematics (Robinet 1967).

What had begun as a theological discussion and argumentation, became eventually secularized: it was Enlightenment thinking in France, which emphasised rationalism as dominant epistemology and elevated mathematics to the leading discipline, capable of promoting social and scientific progress. Well known are the seminal contributions by Diderot and d'Alembert to propagate these new functions of mathematics (see Magalhães Gomes 2008). The secularised conception is particularly seizable with Condillac: for him, the sciences were but languages and algebra provided the common language for them. Condorcet had a decisive role in transforming the new conceptions into the educational structures after the Revolution. Within higher education, mathematics became together with chemistry – likewise understood as a language – the two basic components.

Already before, the military schools, founded from the middle of the 18th century on, were the first to implement mathematics as dominant discipline, as proposed by rationalism.

A first universalising of instruction: elementary schooling

While rationalism remained at first restricted to Enlightenment in France, Enlightenment occasioned in several European states – Protestant as well as Catholic - initiatives of the state for education, which transcended for the first time the hitherto dominant focus on professional finalities.

In fact, it were primary, or elementary, schools, which were founded since the turn from the 17th to the 18th century. As the example of Prussia shows where such a measure was introduced in 1717, the intention was to have compulsory schooling for the entire population. Teaching subjects should be the three 'R': reading, writing, and basic reckoning, accompanied by religion. Evidently, there were too many practical obstacles for realising the noble aims generally; the effectiveness was limited, in particular due to the feudal structures.

On the other hand, having introduced this as at first a quantitative measure, it would eventually enhance to focus on improving the quality of teaching in these schools: better teachers, better material, better school equipment, access for all, independent of agrarian work of children for the family. In fact, already by the middle of the 18th century, so-called normal schools were founded in various of the states – for instance in Austria – for training teachers for the elementary schools.

The French Revolution: - the true turning point from the Few to the Many

The French Revolution enabled the decisive step for realising an education for all. In fact, founding a civic society, its utmost rationale became *égalité* – equality of all citizens. By now, the state changed its functions profoundly: society no longer being hierarchically organised like in the feudal period, but according to functional relations. A key such system became the educational system, now organised by the state itself for assuring its new task to provide equality via education for all citizens. Characteristically, the states now used to establish a proper ministry of instruction or of education, within their government. Likewise characteristic is that it was now the state that became responsible for instituting teacher training also for secondary schools – what had been the big deficit of pre-modern times.

One must be aware, however, that this constitutes but an ideal picture. In social reality, equality used to be far from being realised – in particular due to social obstacles regarding access to education.

Beyond primary schooling, states by now instituted and organised secondary schools as general education. And it is highly revealing that within these school systems mathematics became a major teaching subject for all children who got access to these schools.

The French secondary schools, first named *lycées*, realized the concept of general education as exposed by the minister Roederer in 1802, upon presenting the organising law.

In the first organisation of the *lycées*, there was a strong role of mathematics, in a somewhat bifurcated system: after the first year at the *lycée*, where the teacher of Latin gave basic instruction in arithmetic, too, the students entered the system of parallel teaching of Latin and of mathematics, in either subject it were six "classes", of half a year each. In the next reform, of 1809, the bifurcation system was abolished and the role of mathematics somewhat weakened (Schubring 1985).

A decisively different conceptualization of general education became conceptualised in Prussia, as part of the comprehensive social, political, and educational reforms enacted after the political catastrophe of the 1806 defeat when fighting against Napoleon.

In fact, there was a direct relationship between social and educational reforms: the corporations, which as closed social subsets had controlled access and performance of the various practical professions ("Zunftzwang") were abolished and the access to professions opened for everyone. And as a consequence, general education became established as the primacy in schooling and training. Only after having pursued general education, the youth should specialize and qualify for professions.

The neo-humanist conception of general education envisaged the formation and development of cognitive abilities by several major teaching subjects representing the "Organism" of human knowledge. Thus, contrary to the just two disciplines, Latin and mathematics in Napoleonic France, three constitutive elements of general education became the core of the curriculum in the reformed *Gymnasia*, from 1810 on: classical languages, mathematics and the sciences, and history and geography. Mathematical qualifications, hitherto essentially conceived of to be of relevance for certain professional careers, were now regarded as constituting a key dimension for cognitive development of anybody (Schubring 1991).

Assessing the common pattern to elevate mathematics to a major teaching subject, the big question arises: hitherto, when there had been initiatives by a state concerning mathematics, it had rather been for assuring proper services for some of its functions, emphasising thus the utilitarian role of mathematics. From now on, it is the state, which introduces mathematics into general education and even tends to universalise this function.[3]

How can this be explained? A hypothesis is the following: what used to be sufficient for the needs of the state in earlier periods of social development, namely directly applied mathematical qualifications, affords under the new conditions of a civic society a much broader anchorage and a more diversified distribution of mathematical qualifications within the entire population.

Modernisation of state and education in other cultures

The pattern, which emerged with the first states becoming modernised after the French Revolution, proves being confirmed by numerous other cases all around the world, even independently of their cultural orientation: states striving for establishing civic societies and thus transforming profoundly their character and functioning, focus on founding a public school system and providing them a firm position for the teaching of mathematics.

A revealing such case was Egypt: decisive step there for modernization was the establishment of a public school stem by the state and the founding of a ministry of education. Founded in 1837, the *Diwān al-madāris* was an administration considered as the Egyptian ministry of education The *Diwān al-madāris* was parallel to and independent of the

[3] For studies on this process in Italy, France, Germany and the Netherlands see: Giacardi 2008, Gispert 2008, Schubring 1989; 2008, Smid 1997 and the documentation by Belhoste on France (1995).

traditional system managed by the *Ulemas*, which the viceroy Muhammed Ali was not strong enough to replace completely. It consisted of *madāris ibtidā'iyya* (State primary schools) established in Cairo and in the provinces, *madāris tajh"ziyya* (preparatory schools) and special graduate schools. Each level of study was organized so as to provide qualified students for the next level. It was an inverted pyramidal system grown from the need for qualified students to pursue higher education in special schools (Abdeljaouad 2012).

In Japan, before the opening to the West in 1867, schools in the feudal Edo period were run by each Samurai clan and reserved to the children of their proper clan. With the Meiji Era, the government became centralised and a Ministry of Education became established. Among the first means were: abolishment of the private feudal schools; creation of a system of public elementary schools all over the country, with arithmetic as one of the major teaching subjects; normal schools for teacher training and many more systematic measures (Matsubara & Kusumoto 1986).

In China, analogously, the modernisation of state and society initiated since the end of the 19th century, entailed the establishment of a state-school system and the founding of a Ministry of Education, in 1906 (Chan & Siu 2012).

In all of these historical cases, mathematics became instituted as a major teaching subject.

Even in the USA, where one distrusts the state and prefers a weak state, state authorities are those who organise schooling and education – though in general not central ones, but local or regional ones. And the curriculum uses to be dependent on pressures by the local public.

A counterexample confirms furthermore the general pattern: it is England where there had been practically no state interference in education throughout the 19th century. And there, mathematics used to be a marginal teaching subject in secondary schools.

Universalising schooling: from primary to secondary, and equality for girls

The process of universalising schooling in general and the teaching of mathematics in particular is continuing – in industrialised and in developing countries. After primary schooling has been universalised – for Western countries basically already during the 19th century, and for developing countries quite recently, now secondary schooling is becoming universalised, in the sense of extending the age limit of compulsory schooling.

But one has to be aware that a true universalism can only then become effective when primary and secondary school levels became consecutive.

Such a consecutive structural pattern constitutes the essential pre-condition for egalitarian access to education; its emergence and realisation is not well researched, however. It seems not to be well reflected and researched upon that a fundamental pre-condition for realizing ‘mathematics for all’ consists in removing the two socially segregated pillars of primary schooling for the lower classes and secondary schooling, with a proper preparatory education for the upper classes. Only a truly consecutive system, as first conceived of by Condorcet in his plan of 1792 and as next instance planned by the neo-humanists in Prussia in the 1810s, provides the necessary structure. Since such a change affects most deeply rooted social functioning, it is evident that specific circumstances are necessary for achieving it.

The, to my knowledge, first such realization where boys from all social classes had to attend the same primary schools occurred in Denmark, in 1903 (Hansen, 2009, p. 19).[4] Besides revolutionary Russia, the next case was Germany where the November Revolution of 1918 led to abolishing the segregation and establishing in 1920 a consecutive system. Other states seem to have followed only in the later period of social expansion - France, for instance, only in 1959/1963 (Gispert & Schubring, 2011, p. 91).]

Another key issue, which is not yet systematically researched: when became the separated schooling for girls abolished in the various countries? Strangely enough, there use to be reports for some states when secondary schools for girls have been established. But there use to be no studies about when and how these merged with the schools formerly reserved for boys. A rough estimation goes like this: such schools for girls were founded during the second half of the 19th century or at the turn to the 20th century (see Tobies 1992; Thanailaki 2009). Merging with boys’ schools happened during the phase of expansion of the educational system, thus about the 1960s. While the mathematics program in the separate schools for girls used to be a reduced one, supposing that girls are less capable to understand mathematics, the program upon merging the schools became identical – the former claim of less intellectual capacities disappeared, apparently even without any debate, as completely outdated, and thus giving equal access to learning mathematics to both genders. Revealing how these processes had come about would evidently present precious research!

[4] Hansen does not tell what effected this change.

Universalising schooling: which curriculum for all?

But after solving these more quantitative aspects of extending secondary schooling for all and thus including the entire age cohort, there emerges ever again the qualitative aspect. In particular: which type of mathematics curriculum is adapted for the new type of mass education?[5]

It even proved to constitute an unsolved problem: which school mathematical contents and which gradation of them should be taught for the schools within a democratised society and an educational system characterised by massification? Even the reform movement of the so-called modern mathematics was not prepared to give an answer, although claiming to conceive of a 'mathematics for all', as the case of France shows where the Lichnerowicz Commission was established, in 1966, to realize an extremely demanding and ambitious reform project.

The *procès verbaux* of the meetings of the Commission Lichnérowicz show the objective difficulty in conceiving a common program for the different streaming within the secondary school, which was functioning then as a comprehensive school, uniting *filières courtes*, with less years of school attendance and leading to vocational life, and *filières longues*, leading to university studies. For the fourth and the third grades, common to all the diverse curricular directions, the Commission had planned to teach the same contents, and according to the same spirit and methodology – conceiving this exclusively from the logic of the a curriculum for those who would continue to university studies. Nobody in the Commission had experience and imaginativeness of thinking in more diversified terms. Evoking, in a meeting of the commission, the question of the curricular reform for these *filières courtes*, one of the members resumed the matter, saying: "Do we have to teach obsolete mathematics to less clever children?" (D'Enfert & Gispert 2011, pp. 11-12).

Both enhanced by the PISA concept of "numeracy" and by attempts to cope with the consequences of massification for the syllabi, there is at present the tendency to over-emphasize the utilitarian aspect in the bipolar functioning of school mathematics (see Hansen, 2009, pp. 35-37). The crude differentiation of the mathematics syllabus in the Dutch secondary schools, for example, according to conceptions of Freudenthal, into a course A, for those supposed later not entering into a profession which would "need" mathematics and into a course B for those later

[5] A particular aspect is constituted by discussions, in the wake of the TIMSS studies, whether there are cultural factors enhancing special qualities of access to mathematics: TIMSS and PISA seemed to show that East Asian students outperform students from Western countries; see Leung 2006 and my review of the East-West-ICMI-Study (Schubring 2007).

entering into a mathematics-loaded profession re-woke earlier dichotomies, which are particularly dubious since all professional careers make nowadays use one way or the other of mathematics, albeit in a not always perceivable manner, and since one should be able today to be flexible in one's professional career.

The developing countries adopted, after overcoming the colonial system and becoming independent in the 1960s and 1970s, the same functioning of the state as developed since the Enlightenment and the French Revolution, organising hence an analogous universalising educational system, with – in particular – mathematics as major teaching subject. In this sense, the dissemination of the state function has entailed the adoption of a teaching of mathematics for all.

Yet, the crucial issue of the relation between the goals of general education and of vocational/professional goals is quite open still and is at the core of improving the quality of access to mathematics teaching.

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The first International reform movement and its failure in the Netherlands

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Abstract

In the early years of the 20th century an international reform movement on mathematics education originated in Germany and France, which influenced math teaching in many European countries. One of the countries in which this movement had relatively little impact was The Netherlands. Although several attempts were made to incorporate calculus into the (exam) program of the gymnasia and the HBS (the Dutch Realschule), these attempts were not very successful. It was not until 1958, that calculus was included in the exam program of these schools. Concerning geometry, attempts for reform were even less successful. The only major reform was the introduction of analytical geometry in the program of the gymnasia in 1918. The geometry program for the HBS remained unchanged until 1958, when descriptive geometry was replaced by analytical geometry. It was not before 1968, when also a major reform of Dutch secondary education as a whole was established, that under the influence of the New Math movement mathematics teaching underwent a drastic reform. In this paper we will discuss the failed attempts for reform and the underlying reasons for this failure.

Introduction

Recently Hélène Gispert and Gert Schubring published an article with the title Social, Structural and Conceptual Changes in Mathematics Teaching: Reform Process in France and Germany over Twentieth Century and the International Dynamics, see (Gispert & Schubring, 2011). In the first part of their article, they discuss the reform movement initiated by Felix Klein, and the great reform of secondary education in France in 1902. On page two, they write "In this period [1900-1914] a common reflection and cooperation on reform in mathematics curricula were developed all over Europe". Obviously, common reflection and cooperation on reform does not mean that this international reform movement[1] was equally successful in all European countries. One of the countries in which this movement was relatively unsuccessful, was The Netherlands. Functions and their graphs were introduced slowly in the algebra program, and it was not before 1958 that one of the purposes of the reform, the introduction of calculus in secondary education, was realised fully in The Netherlands. The only major change in geometry teaching took place in 1918, when analytical geometry was introduced on

[1] The reform movement of the early years of the 20th century is usually called the First International Reform Movement, the second one being the reform movement that originated in the sixties and seventies of the last century

Bjarnadóttir, K., Furinghetti, F., Matos, J. M., & Schubring, G. (Eds.) (2012). *"Dig where you stand" 2. Proceedings of the second International Conference on the History of Mathematics Education*. Caparica: UIED.

the gymnasia. In 1958, on the HBS, the Dutch variant of the German Realschule, descriptive geometry was replaced by analytical geometry. Plane geometry for the lower classes remained largely unchanged until 1968. Then a major reform of the whole building of secondary education in The Netherlands was achieved, and this opportunity was seized for a drastic reform of Dutch math teaching.

Although the reform movement was not very successful, that does not mean that no attempts were made for reform. There were three attempts to introduce calculus, and there were fierce debates about geometry reform. In this article we will discuss these attempts and debates. Furthermore we will provide an explanation why all this remained relatively unsuccessful. For this aim, we will use the theoretical framework of the article of Gispert and Schubring, demonstrating that the circumstances and conditions that made reform successful in Germany and France were not met in The Netherlands.

The social, political and educational scene

The Dutch system for secondary education had, in a lengthy and complicated process, evolved during the 19th century and contained two, strictly separated parts. First, the gymnasia, intended for what was called “the learned class”, those who wanted to go to the university. According to the law, the gymnasia were a part of the “higher education”, regulated by the same law as the universities.

Of course classical languages were most important there, but since 1876, when the gymnasia were modernised, one could also take an exam in which mathematics and the sciences played an important role.

Second, there was the more modern type of schools, where modern languages, mathematics and sciences were dominating. These were called the “Higher Burgher Schools” and can be compared best with the German *Realschule*. From their name it is already clear that they were also intended for a well defined class of the population: those who should fulfil the leading positions in modern industry and trading business. The law maker of 1863, when the HBS was established, had not intended that the graduates of these schools should receive additional formal education, and certainly not that they should go to the university. If some extra education was needed, they could go to the Polytechnic school in Delft, an institute then considered as belonging to secondary education, not to higher education.

But law makers can not see into the future, and things turned out differently. The Delft polytechnic school soon developed into a technical university like the German *Technische Hochschule*. Around 1900, a substantial part of the graduates of the HBS went to Delft. Moreover, it soon became clear that the HBS offered also an excellent preparation for studying sciences and mathematics at the universities. Famous scientists like Heike Kamerlingh Onnes, Hendrik Antoon Lorentz, Jacobus Henricus van ‘t Hoff and many others attended the HBS, not the gymnasium. But, to enter the university, HBS graduates had to pass extra exams in Latin and Greek – a useless barrier for many talented students.

Already around the turn of the century, it was clear this system of secondary education, based on class divisions, was outdated, and during many years in the first decennia a State committee was studying on a better system. But just as in the 19th century, the Dutch government had great difficulties to take any decision in educational matters. Since the enlargement of the vote, Dutch governments were coalitions from different parties, with usually different views on education. Since it proved difficult to reach a compromise, the result was the status quo. Furthermore, much of its energy was absorbed by what was called the *School Battle*, a battle about the financial equalisation of schools maintained by the State or by religious groups. When this battle was at last resolved in 1917, the central government wanted to keep away from the internal affairs of the schools as much as possible. It was not before 1968 that the totally outdated laws on education from the mid 19th century were replaced by a new one. Only for the polytechnic school in Delft things went faster: it became a Technical University with the right to grant PhD’s already in 1905.

The first attempt to introduce calculus on the HBS

Although The Netherlands were slow in picking up modernisation, the country was of course not isolated from the rest of Europe. The reform in France and the ideas of Felix Klein were welcomed by at least two math teachers, who started a movement to introduce elements of it in The Netherlands. They were Cornelis Adrianus Cikot and Franciscus Johannes Vaes, math teachers on the HBS in ‘s Hertogenbosch and Rotterdam. They had something in common: they were educated as engineers, not as mathematicians. Cikot was a military engineer; Vaes was a mechanical engineer from Delft. Perhaps by this background, Cikot and Vaes had a broader view on the teaching of mathematics than most of their colleagues. While for most of them the idea of math as “gymnastic for the mind” was still dominating, Cikot and Vaes had an open mind for the applications of mathematics in physics, mechanics and engineering. They were also well informed about the developments in Germany and France. Vaes founded a journal especially for math teachers, the *Wiskundig Tijdschrift*, in which he propagated his ideas, and wrote an interesting textbook in which he, on an intuitive basis and using many practical applications, introduced differential and integral calculus for the HBS, see (Vaes, 1907). Cikot played a major role in the attempts to convince their colleagues to support their ideas.

In those years there were no special Unions for math teachers only. Teachers of the gymnasia and HBS were organised in two separate general Unions for teachers. Cikot held presentations for the math branch on meetings of the Union for HBS teachers, propagating Klein's ideas, and talking about the experiments and experiences Vaes and he had gained with teaching differential and integral calculus in their classes.

Vaes and Cikot thought that discussions that took place in those years about the course length of the HBS offered an extra opportunity for their proposals. The HBS had a five years course, but the possibility of a six years course was then discussed. They persuaded their Union to set up a committee to develop math curricula including calculus, in two varieties: one for a five and one for a six years course. When taking it to a vote, the majority of the math teachers supported the idea of including calculus in a six years course, but rejected it for a five years course. Most math teachers did not want to give up much of the traditional teaching stuff and they did not believe that it was possible to teach calculus within the five years course. The government however decided to stick to the five years course, and when Vaes and Cikot in a final attempt in 1908 presented their proposal for calculus in a five years course to the general meeting of the teachers Union, it was clearly rejected. It was the end of their attempts.

Reform at the gymnasia

No reform at the HBS then, but, maybe surprisingly, there came some kind of reform on the gymnasia. In 1915, the Union of teachers at the gymnasia appointed a committee to develop proposals for a new curriculum, including mathematics. The committee delivered its report in 1918. Concerning algebra, some outdated topics could be removed, and some time should be devoted to graphical representations. For the math/science branch, also a little bit of analytical geometry could be taught. The committee did not want to prescribe the teaching of differential and integral calculus; it suggested that a teacher, if he had a good class and enough time, could teach something to this class on a voluntary basis. On the whole, the report was not completely conservative, but at least very cautious concerning reform.

But when in 1919 the government issued the official new curriculum, it turned out more progressive than the report. Concerning geometry, a complete course of analytical geometry including conic sections was prescribed in the math science branch. Even more interesting, it said that also differential and integral calculus should be taught on that branch.

Most likely, the driving force behind this surprising development was the inspector for the gymnasia, Cornelis Johannes Vinkesteyn.

Vinkesteyn was not a mathematician, but he had been one of the authors of the Dutch report on mathematics teaching for the great ICMI survey of 1910. In that report, Vinkesteyn already showed that he was not an opponent of reform. Moreover, in 1917, graduates of the HBS were allowed to study math, sciences and medicine on the universities, without having to do extra exams in Latin and Greek. That implied that the HBS, with its five years course, became now a serious rival for the gymnasium with its six years course. There was a general fear that talented youth with interest in math/science, would avoid the gymnasia and go to the HBS. One of the things the gymnasia could do was to offer a more modern program which gave a better preparation for the university.

The math teachers at the gymnasia however rejected the idea that calculus should be treated as a normal part of the curriculum. They considered the teaching of calculus to be such a novelty, that is should be handled with utmost care and asked for complete freedom for the teachers how to go about it. Due to their pressure calculus was not included into the written part of the exam, and it was up to the teacher whether or not he wanted to pose questions about it on the oral part.

If this had been a measure for a few years, so that experience could be obtained with the new subject, this could have been a wise decision, but that was not the case. This situation, that calculus had its place in the curriculum but *de facto* not in the exams, endured until 1958. It must be added that the State, although since 1917 it subsidized all schools that met a set of legal standards, could not prescribe the curriculum for non-State gymnasia. The teaching on these schools had to be influenced by the final exams organized by the State, and on these exams no questions on calculus could be asked. The introduction of calculus in the curriculum of the gymnasia was in practice much less important than it seemed on paper.

A new attempt for the HBS

For the HBS, prospects for reform seemed gloomy. In 1918, a committee created by the teachers Union to study the problem of the overloaded math program, proposed only minor changes and advised against the introduction of calculus. No wonder, the committee was chaired by Elibert Jensema who in the years of Vaes and Cikot was already a strong opponent of their proposals. In 1920 Jensema was even appointed as inspector, which made reform even more unlikely.

But that was too pessimistic. A new initiative came from another inspector, the colleague of Jensema for the HBS, Gerrit Bolkestein, who was much more in favour of reform[2]. Although he was not a mathematician, he realised that something should be done about the outdated math curriculum. He and Vinkesteyn overruled Jensema, and a new committee was created, with the task to formulate proposals for a new math curriculum for the HBS.

The new committee consisted of four persons and was chaired by Hermanus Johannes Elisa Beth, but the most important member was its secretary: Eduard Jan Dijksterhuis. He is well known as historian of mathematics and physics, mainly by his masterwork *The mechanisation of the world picture.* Most of his professional life, he was a HBS math teacher and he played an important role in the didactical discussions in The Netherlands during the interbellum.

The proposals of the committee included the introduction of calculus on the HBS, but for different reasons as for instance Klein, or Vaes and Cikot. Dijksterhuis considered calculus as an "incomparable step forward" in the cultural history of mankind and in his opinion; every well educated citizen should have knowledge of it. Apart from theoretical mechanics, Dijksterhuis was not very interested in applications or the practical use of calculus. There is not any indication in his didactical work that he was influenced by the Reform movement or other examples from abroad. His vision on math teaching was in fact a 19th century one: math should be taught as a mean of selecting and education, not for its applications in science and industry. According to his convictions on math teaching in general, also calculus should be taught as rigorous as possible.

The reactions to the report, especially to the proposal to introduce calculus, were mixed. The just founded association of math teachers on the HBS supported the idea, but the association of HBS headmasters rejected it out of fear of overloading. There were some positive reactions from the universities, but more important however was the fierce resistance of the math department of the Technical University in Delft. In those years, they formed an extremely conservative group, having little contact with the engineering departments in Delft. The committee had proposed to reduce the teaching of skills to make place for calculus, and this aroused their anger. The math professors in Delft wanted to keep the HBS as it had been in their youth, and to keep their teaching in Delft as it was now.

While the mathematics community had mixed opinions, it was easy for inspector Jensema to ignore the report. Nothing happened. Jensema retired in 1934, and in his place Jan van Andel, a member of the

[2] There were three inspectors: one for the gymnasia, always a philologist, two for the HBS; one from the math/science field, one from the modern languages field

committee was appointed as inspector. It seemed as if this opened new possibilities, but Van Andel was not a strong figure and the result was disappointing. In 1937, calculus was incorporated in the curriculum of the State schools (less than a half of all schools), but the exam program was not adapted to the new curriculum. It was the same unsatisfying situation as on the gymnasia. It took more than 20 years before, in 1958, calculus was made part of the final exam.

Debates about geometry teaching

The international reform movement was not only about the introduction of calculus into the teaching of algebra. Renewal of geometry teaching was also an important topic in the reform movement in the beginning of the 20th century. The ideas of Klein to base plane geometry on transformations are well known, but introductory courses in geometry, based on intuition and self-activity, were also important elements of the reform movement.

While the reform of the teaching of algebra was already problematic, reform of geometry teaching proved even more difficult. As mentioned before, in 1919 analytic geometry was introduced on the gymnasia, most likely due to the pressure of inspector Vinkesteyn, but that was in fact the only change in geometry teaching until 1958. The idea to base plane geometry on transformation groups was met with complete silence. It was not before the late fifties, that a group of teachers picked up the idea and in cooperation with Adrianus Dingeman de Groot from the University of Amsterdam conducted a series of experiments to compare the results of traditional geometry teaching with that based on transformations. Before transformation geometry could be implemented however, the new math wave swept away all traditional plane geometry.

Concerning introductory courses based on intuition and self activity, there were at least some scattered endeavours by individual teachers to develop such a course. One of them was Willem Reindersma, a teacher at a progressive school in The Hague, where the climate was favourable for change. Reindersma was a participant of a mathematical-didactical seminar, attended by teachers, mathematicians, physicists and pedagogues who interested in renewal of Dutch math teaching that was founded around 1913 by Tatyana Ehrenfest-Afanassjewa. She was born in Kiev, had studied physics and mathematics in Petersburg and in Göttingen with Felix Klein and David Hilbert, where she met Paul Ehrenfest, who in 1912 was appointed as the successor of Lorentz at the Leyden University. She was deeply interested in the teaching of mathematics and she and her group became for decennia the focus for all who had hopes that eventually reform of Dutch math teaching could be accomplished.

In 1924 Ehrenfest-Afanassjewa published a brochure propagating an intuitive introduction to geometry, see (Ehrenfest-Afanassjewa, 1924). Her brochure aroused the anger of Dijksterhuis, who concerning geometry was clearly conservative. Dijksterhuis wrote an article against Ehrenfests brochure, defending traditional geometry, to be taught as rigorous as possible, even claiming that the teaching of this type of geometry had a beneficial influence on the characters of the pupils. So it is no surprise that the report of Beth-Dijksterhuis committee a few years later contained no proposal for reform on geometry teaching.

Reform at last

In 1958, at last, a modest reform was achieved. Concerning geometry, an intuitive introduction was now prescribed – although in practice that had not much importance. Descriptive geometry was abolished at the HBS, and replaced by analytical geometry. Functions and graphical representations became important elements for the lower classes, and differential and integral calculus became a part of the final exam. The curricula and exams of the HBS and the gymnasia were from now on the same. The reform of 1958 was the result of a report of the math teachers Union, but that report was provoked by a report of a group of professional mathematicians, pedagogues, didacticians and teachers. That was a group that had evolved from the discussion group around Ehrenfest-Afanassjewa, a group now chaired by Hans Freudenthal. At that time Freudenthal and his group were by most teachers still seen as dangerous outsiders, and the report of the teachers Union and the reform of 1958 was also a way of keeping the reform that now was inevitable, within limited boundaries

Later on, in the sixties, the government stepped into the business of modernizing the total educational system, and the combination of the government and a new class of curriculum developers, didacticians, etc. realized a total renewal the mathematics teaching in The Netherlands. The teachers themselves and their Unions lost the initiative and could only, with more or less enthusiasm, follow the developments. The report by the teachers Union that had led to the reform of 1958 was the last report of this kind.

What went wrong?

We have seen three attempts to introduce calculus into secondary education. The first one failed, the other two achieved some success but the final and decisive step, the incorporation into the exam program, did not happen. Concerning geometry, there were heated discussions, but those who wanted reform had little influence. What went wrong?

To analyse this, we will use the theoretical framework from the article (Gispert & Schubring, 2011). They distinguish three spheres of actors: the sphere of the experts, that is for this period the sphere of the university mathematicians, next the sphere of the decision makers, that is the sphere of politics, and finally the sphere of the professionals, that is the teachers.

In France, the great 1902 reform was the result of a comprehensive enquiry, initiated by the parliament in 1899. The essential question in the enquiry was about the modernisation of the country. Of course, concerning mathematics the answers of the mathematics community were not unambiguous, but when the government decided to an ambitious reform, it was supported by eminent mathematicians like Émile Borel and others. One could say that the success of the reform in France was due to the cooperation between the decision makers and the experts, while the professionals were hardly in favour of the reform. In the interbellum, when political climate was more conservative, the cooperation between the politicians and the experts broke down and, in fact, politicians and professionals cooperated in reducing the pre war reforms.

In Germany, Klein succeeded in "forging an extraordinarily broad and powerful alliance of teachers, scientist and engineers" to support his ideas. It must be added however, that for instance the support of the professionals was far from unanimous. The *Förderverein* for instance opposed to at least parts of Kleins reform agenda. But in general, Kleins alliance was successful in realizing reform – which eventually of course required the cooperation of the decision makers.

In The Netherlands, initiatives from politicians such as in France could be hardly expected. The government realised that the educational system was outdated, but the discussions endured for years and finally sanded. Two times, an individual representative of the political sphere, the inspectors Vinkesteyn and Bolkestein, provided an opportunity for reform. But when the support from the other spheres, experts and professionals, was minimal, or even declining, they were not in the position to push their individual initiative further forward and it sanded half way. There was no real sense of urgency toward modernisation with the government.

Within the sphere of the experts, there was not much interest for reform in math teaching. The first Dutch ICMI committee contained two math professors from Delft, and only one math professor from the, certainly in those years, more prestigious general universities. Generally speaking, the math professors in the first decennia at the Dutch universities formed a rather conservative group. When the Delft math professor Johannes Georg Rutgers in 1934 held a speech on the occasion of becoming rector of the University, he not only launched a fierce attack against the proposals of Dijksterhuis to introduce calculus, but he unequivocally glorified the 19th century class based society, when people knew their place in the society. His opposition against reform in math teaching was just one element in his strong conservative view of life. The only exemption in this conservative company was Gerrit Mannoury, professor in Amsterdam, a man of strong Marxist political convictions who played a role in the Dutch communist party. He rejected the idea of the "formative value" of math teaching and considered the way math was taught as useless for most children. His ideas, both political and on education, were so much divergent from the general opinions, that he had little influence. It might be added that in the early years of the 20th century, unlike as in France or Germany, there weren't Dutch mathematicians anyhow with a high international reputation who could have influenced politics. Later on, such a man could have been Luitzen Egbertus Jan Brouwer, and if he had raised his voice, it could have had some impact. But Brouwer wasn't at all interested in teaching, he even complained about the obligation to lecture a few hours a week at the university. There was hardly any support, let alone any initiative, to be expected from the sphere of the experts.

So, reform had to come from the sphere of the professionals but that is, as a rule, rather unlikely. School systems behave like Newtonian bodies: if no external forces are exerted, they stay at rest. The no doubt most influential figure in the sphere of the professionals before World War II was Piet Wijdenes. He had been a math teacher but his schoolbooks were so successful that he was able to stop teaching and earned his living by his books. He was the founding editor of the journal *Euclides*, in which he promoted his mainly conservative ideas. He was in favour of the proposals of Dijksterhuis, but he strongly opposed all other proposals or attempts for reform, especially on geometry.

The last opportunity for reform, for the HBS in 1926, was perhaps the best chance. The report was written by professionals and supported by the recently founded math teacher Union. It could be clear beforehand however, that it would not be easily to achieve implementation. It would be crucial to formulate proposals that were clearly feasible within a five years course, to profit from the experiences in other countries or on the Dutch gymnasia, and to be secure of the support of Delft Technical University.

None of this was the case. Only once in the report the phrase "functional reasoning" seems to refer to Kleins reform movement, on the whole the report seems to be written in an educational isolation. The program was so ambitious that it was easy to criticise. It stressed on theoretical understanding and historical and cultural aspects, not quite the aspects that created enthusiasm in Delft. To gain support from Delft would have been difficult anyhow, but it might have been possible to gain support from engineering circles if another approach had been chosen. But Dijksterhuis wanted to keep math teaching as pure as possible, and abhorred the idea of cooperation with physicists or engineers. In those same years he was engaged in a heated debate with physicists on the teaching of mechanics on the HBS. This topic was taught by math teachers, on an axiomatic basis, purely theoretical without any experimental aspects. Leading physicists wanted to incorporate mechanics within the physics teaching and Dijksterhuis was leading in the resistance of the math teachers against their attempts. An alliance between physicists and mathematicians to promote the modernisation of math teaching was unthinkable in those years.

So, one has to conclude that in none of the three spheres of actors there was a feeling that reform was really urgent or even important. Attempts for reform came mainly from outsiders as Vaes and Cikot, who were engineers by training, or Vinkesteyn and Bolkestein, who weren't mathematicians at all. Ehrenfest-Afanassjewa was a very capable mathematician (and, may be added, physicist), but being an immigrant from Russia she was obvious an outsider too. The only attempt which came from leading circles within the sphere of the professionals, the proposal of the Beth-Dijksterhuis committee, was a rather problematic one. Moreover, their proposal to teach calculus on the HBS was an isolated element in a mainly conservative report.

When in the sixties at last renewal was to have its way, it was mainly under the pressure of another outsider: Freudenthal. But Freudenthal was not merely an outsider in the world of professionals, he was after World War II also the leading figure in the sphere of the experts and he did not hesitate to use this prominent position.

A key concept in the article of Gispert and Schubring is *international modernisation*, arguing that the math reform was part of this process. France and Germany, they say, were leading in this process. No doubt, The Netherlands was at the rear. The Netherlands had no ambition to compete with other European countries on modernity. An important Dutch historian once described the Netherlands before the Second World War as "a conservative country". The failure in math reform adds just another illustration to this.

In the sixties and seventies, The Netherlands underwent a rapid transformation from a conservative country into a progressive one. It is not a coincidence that just in those years Freudenthal and his supporters could seize the opportunity to bring about their far reaching reform. The transformation of the outdated Dutch math teaching into one in which the idea of *realistic mathematics education* was so clearly dominant, is another illustration of the coherence of mathematics education with the more general social and cultural context.

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North American influence in mathematics teachers' education for primary school in Brazil

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Abstract

The goal of this article is to analyze U.S. influence in mathematics teacher education in primary education in Brazil, from the analysis of current teaching courses for teachers. The period considered in the study included the late nineteenth century to the early decades of the twentieth century. It will show that the influences of the U.S. linked directly to the circulation of cultural pedagogical model of teaching math intuitively. And more, this movement contributed to changing the school culture of schools of teacher education. There was a shift from a culture of high school mathematics to establishing a culture of teacher training.

Initial thoughts

In the final years of the Monarchy, in the late nineteenth century - after 1870, an economic boom occurred in Brazil as a result of coffee productivity. After accumulating of huge wealth, Sao Paulo took the national development lead. Along with that wealth and because of the new Republic regime, education became the main subject of politicians. Since then a common consensus was established that education would conduct the country to a higher degree, closer to the developed countries. As there were a lot of resources, a great movement towards the reform of the Brazilian educational system began. The first actions were taken originally in Sao Paulo and they had as reference the schools that were considered great ones: the North American private schools which were run by Protestants and used to educate the children of the Sao Paulo cream society. These schools were considered innovative as they applied a serial studies system, an instructional program divided into school years, known as grades or forms. And furthermore, they used to follow an international education movement which worked with intuitive pedagogy based on the Pestalozzi teachings.

The Protestant missions in Brazil, a predominant Catholic country, have been longstanding but the first school of these missions was founded only in 1869, under the name of International College at the city of Campinas, Sao Paulo State. We know that it was the first Protestant school in South America (Venancio Filho, 1940, p. 14). In 1870, the North American School was founded in Sao Paulo Capital. It became then the model adopted by Republican politicians that proposed an educational

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reform from the 1890s, one year after the proclamation of the Republic. Influenced by this North American school, started in São Paulo a new way of organizing primary schools: the school groups - these primary schools are composed of isolated schools units that form together groups, offering each one four educational school grades. Later on, these school groups spread out through Brazil. Thus, the first school years in Brazil had as reference curricular grade of the North American Protestant schools, their syllabus, pedagogic books and organization serial form, plus the intuitive method considered the best way of teaching different school disciplines.

The mathematics of primary education and the U.S. influence in Brazil

Treating mathematics in primary education, in terms of didactic books in Brazil and its characterization from the late nineteenth century, leads us to consider that the influence of the United States configuring a culture of mathematics teaching is different from that of secondary education.

In an abbreviate way but representative, it is possible to say that the treatment and selection of mathematics content to be taught in early school years, until the late nineteenth century, used in a great measure a reduction process. The explanation is that: the books originally directed to secondary school are taken and from these books they proceed in order to cut and reduce content, staying with initial ideas for development of works in primary school. This practice was many times reiterated in the production of the first works, the first authors in Brazil of Brazilian didactic books.

This procedure began to be modified and a space was given to another type of production with the creation of the so-called School Groups. And with them, it is possible to observe the U.S. influences in discourses for pedagogical practices of primary teachers, especially those founded in didactic books for the initial school years. The School Groups represent a new paradigm of teaching organization. Created in São Paulo State in 1893, those groups spread out as a model of primary education throughout the country, over the early decades of the twentieth century, representing results of Republicans actions in Sao Paulo, supported by the economic role that Sao Paulo State acquired in the final decades of the nineteenth century and that remains until today.

The actions taken by reformers from Sao Paulo in benefit of primary education on 1890s included interference in pedagogical practices. The emphasis on vocational training was highlighted. The actions of the reformers that we can characterize as guidelines for teaching practices aimed at the teachers who were already working in primary schools.

Rosa de Souza highlighted the priority given to the adoption of works in schools that were written by professors involved with teaching renovation in Sao Paulo. Accordingly, Oscar Thompson, Arnaldo Barreto, João Kopke, Roca Dordal and René Barreto did stand out as educators among many others (Souza, 1998, p. 232).

Modifying the teaching in schools of Sao Paulo was urgent. Pedagogical journals, books and teaching materials were the means to carry out the transformation of public education while the curricular structure was maintained similar to that of the secondary school.

It is interesting to note that at least since the time of the second foundation of the Escola Normal de São Paulo (Sao Paulo Normal School) in 1875, the references for mathematics education are directly or indirectly linked to French works. Classic works of French authors such as Bézout, Legendre and Lacroix parameterize the mathematics of the secondary school and of training teaching from the early decades of the nineteenth century, were also used in the teaching for normal school. They were often taken indirectly, composing manuals for elaboration of programs and books to use at the mathematics formation of teachers at normal schools.

Adding to that, the usual training of mathematics teachers at that time was at engineering schools. This characterizes what may be called the permanence of mathematic school culture of secondary course in the teaching at normal schools.

Immediately preceding the creation of normal primary schools (teacher training schools for primary school teachers), the teaching of mathematics took on new bases, and new international authors were considered authorities to define methods and mathematics content for formation of primary school teachers in Sao Paulo. The discussion on the mathematics teaching was refined theoretically, especially from the United States new referrals. Before that however, one should mention the first theoretical mark of the U.S., which was present since the reforms of the 1890s. You can find it in speeches, legislation for education, pedagogical journals and didactics books of arithmetic teaching. It refers to Parker, name that appears on these distributed materials with pedagogical guidelines for teachers at primary schools.

Francis Wayland Parker (1837-1902), according to Lawrence Cremin (1961), is one of the pioneers of the *progressive movement in American education*. And yet, by the same author, in the words of John Dewey, Parker is the "father of progressive education" (p. 129). Also according to Cremin, within his pedagogical activities, Parker had the opportunity, after receiving a family inheritance, of travelling to Europe and being in contact with the theoretical development of pedagogical studies. After knowing what was news in European works, talking about teaching of first words, he decided to fund and promote similar actions in the U.S. His ideas and curricular innovations were successful, especially since 1883, when Parker became the director of the Normal School of Cook County in Chicago. In this new environment, the educator formalized its pedagogical proposals that come from elements of Pestalozzi, Froebel and Herbart (Montagutelli, 2000, p. 161). That year he published "Talks on Teaching" and in 1894, "Talks on Pedagogics". The latter book, Cremin (1961, p. 134) considers that is possibly the first North American treaty about pedagogy to gain international renown.

The proposals on the teaching of mathematics, defended by education reformers of Sao Paulo, have in the name of Parker a guarantee of change and breaking with a model considered outdated in mathematics teaching by memorization, verbiage and the logical organization of contents to teach. This respect and admiration for the North-American, from the point of view of mathematics teaching, is evident in the reiterated indication of use of the so-called *Parker´s Letters* – a set of prints for learning the four fundamental operations of arithmetic.

Many years later, further changes occurred, possibly after the visit of the director of the Normal School, Oscar Thompson, in 1904, to Missouri, United States, in terms of didactic and pedagogical references that guide teaching and teachers' training in Sao Paulo.

In a text entitled "Exhibition of Oscar Thompson in St. Louis (1904): the exhibit showing 'machinery for making machines' the Professor Mirian Jorge Warde analyzes the presence of this educator in the United States. Warde (2002) informs that[1]:

> Oscar Thompson traveled to St. Louis Exposition with Horace Lane, at the time Director of the Escola Americana de São Paulo (American School of Sao Paulo). Lane had become long time ago consultant to the leaders of Sao Paulo public education for educational affairs of diverse orer. It was through his indication that Marcia Browne, North-American educator, has arrived to the direction of the Escolas Modelos de São Paulo (Sao Paulo Model Schools). Along with Browne, Thompson began his professional career, becoming her deputy. [...] Lane not only disclosed the Exhibition to be held in St. Louis, encouraging educators from Sao Paulo to attend to it as she prepared with Oscar Thompson and Carlos Reis a memoir about teaching in São Paulo to be published at the Exhibition. Speaking to the Associação Beneficente do Professorado Paulista (the Benevolent Association Professorship in Sao Paulo), in 1903, Mr. Lane called those present: 'You Misters/Mrs. should go to the United States. Many

[1] In this paper the translations are by the author.

> of the problems studied with dedication, to be solved here, are already in a full implementation there'[2].

Therefore, on the occasion of his visit to the United States, present at the Exhibition, the educator had contact with the technical production and related bibliography on primary education of that country.

On his return, he brought in his luggage works by United States authors that became reference for didactic production of mathematics teaching in Brazil. One of the Brazilian authors to consider the text from the United States is Professor René Barreto.

René de Oliveira Barreto was born in Campinas, Sao Paulo State, on 30th July 1872. He studied at the International College, a school founded by North American Presbyterian pastors. Barreto received education influenced by the pedagogy of reformers of the U.S. education system in the late nineteenth century (Gomes, 1908, p. 14). He qualified at the Normal School of Sao Paulo Capital in 1895. After his qualification, he was named Complementary School teacher. For several years he served as School Inspector. Later, he became professor of pedagogy and psychology at the Normal Secondary School (Please note that at the time referred, school teachers used to be qualified in Brazil via "Normal Secondary Schools" which were professional courses after what is nowadays called secondary school). Shortly before his death, Barreto began publishing a series of math books (Valente, 2011).

In his works, Barreto informs that he seeks to fill the lacuna mentioned in the sequences of Thompson actions presenting his didactic book. It also informs how he organized the text from foreign didactic literature, "especially American one". He concludes by observing that his work is a result of a compilation of authors. Thus, the new references to the teaching of mathematics are explicit as:

> The way to distribute the subject, the advices and comments - I took them mostly from the teachers Hall, Wentworth and William Milne,

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[2] Oscar Thompson viajou para a Exposição de St. Louis com Horace Lane, então diretor da Escola Americana de São Paulo. Lane havia se tornado, há tempo, consultor dos dirigentes paulistas da instrução pública para assuntos educacionais de diversa ordem. Foi através de sua indicação que Márcia Browne, educadora norte-americana, chegou à direção de Escolas Modelos de São Paulo. Com ela, Thompson iniciou a sua carreira profissional, tornando-se o seu substituto. [...] Lane não só divulgou a Exposição a ser realizada em St. Louis, estimulando os educadores paulistas a comparecerem a ela como preparou com Oscar Thompson e Carlos Reis uma memória sobre o ensino paulista a ser divulgada na Exposição. Falando à Associação Beneficente do Professorado Paulista, em 1903, Mr. Lane conclamou os presentes: 'Os senhores deveriam ir aos Estados Unidos. Muitos dos problemas que dedicadamente estudam para resolver aqui, lá já se acham em completa execução' (Warde, 2002, p. 450).

> with their remarkable works. My personal contribution, what was born from my own experience and observation is certainly the smallest part and magnitude (Barreto, 1912)[3].

It is a complex task to characterize the changes in methodological terms that occurred in mathematical teaching proposals with the appropriation of works from the United States.

The appropriation made by René Barreto, it seems, imply a need of convincing the teacher that he should not get mathematical contents of arithmetic and segment them through an arithmetic logic. The graduation of teaching must have another meaning. It is necessary to follow the analytical method: from the whole to the parts, no longer to follow the path of each mathematic element to the whole of arithmetic to be taught. Creating the situations of dialogues with students, from the use of numbers to counting of objects, the formal writing is delayed to the maximum. Moreover, the arithmetic is no longer alone, dealing with issues exclusively of numbering.

In situations and in dialogue geometry and design were presented and compounded what René Barreto greatly named of "Mathematica".

In summary, the United States influence in didactic works, which were dedicated to the teaching of mathematics at elementary school, emphasizes an ingredient apart from discussions related to selection and organization of mathematical content in primary school classrooms, a teaching methodology. In the symbiosis between elementary mathematical content with the first rising influence of an emergent psychology there is a new mathematics for the primary school. No, it is not a reduction of the one of secondary school. It is a proper knowledge for children, understanding context of early decades of the twentieth century (Valente, 2011).

Final considerations

Inheritances and references for the teaching of mathematics at primary schools in Brazil, as intended to be shown, reveal aspects that may justify how school cultures were built up differently for the primary school and the secondary school in Brazil. And these cultures do not distinguish themselves simply by addressing the specific mathematical content of each grade level.

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[3] A maneira de distribuir a matéria, os conselhos e as observações – tomei-os em grande parte aos professores Hall, Wentworth e William Milne, em suas notáveis obras. Meu contingente pessoal, aquilo que nasceu de minha própria experiência e observação é, certamente, a parte de menor extensão e monta (Barreto, 1912, p. 6).

It is possible to say that the secondary school is not different from the primary, only because it teaches mathematical content of a higher level of mathematical complexity. The distinction between them, however, is not given in mathematical terms. The distinction historically constructed is given in terms of different teachings.

In the case of the secondary school, remotely created from the medieval universities, inheritor of a higher education culture where pedagogical practices are centered on the content, the speech and the explanation of mathematical contents is the teacher's task. However, the primary school, also seen in long-term perspective, had its origin on the interest of the Catholic Church and the formation of faithful people, giving them the status of practice of writing and reading, especially for the catechism. In this sense, the pedagogical task is to deal with how to teach: how to write and read. And, in the case of mathematics, to consider the teaching of arithmetic and geometry as ingredients of everyday life.

In specific terms of the mathematics, of education in mathematics, the connection with this ancient tradition and origin is made by the constitution of different cultures that teach this knowledge. From the secondary school side, the contents from the French books origin do guide the whole organization and selection of mathematics at schools. In the case of elementary school, the methodological concerns on "how to teach" do generate ruptures with the organization of logical mathematics content teaching, approaching this study on teaching with how the development of a child happens, no longer seen as an adult in miniature.

This rupture of origin - the separation of content and methodology - can be explained considering the differences between culture of secondary school (content used as a drive to education) and culture of primary school (methods to teach writing and reading exercises suitable for children). In the first case - culture of secondary school - they teach contents considered necessary to reach higher education courses. These contents are given by Colegio Pedro II (Pedro II College), a reference for secondary school in Brazil. The mathematics teaching programs include arithmetic, algebra and geometry courses. These are courses based on lessons from the French mathematics manuals. In the case of the culture of primary school they are worried with popular teaching and extension of fundamental teaching to the majority of Brazilian people. And it is in São Paulo that this expansion begins with the spread of the teaching model given to school groups.

In synthesis, it is possible to say that mathematics teaching for primary schools in Brazil has direct influence from the United States. In a different way, the organization of Colegio Pedro II in Rio de Janeiro follows a model didactic-pedagogic with the structure of French lyceums. In the Brazilian primary schools, transformed into school groups following the

American protestant school model, they use the serial courses and intuitive pedagogy adapted from the didactics of American authors.

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Defining modern mathematics: Willy Servais (1913-1979) and mathematical curriculum reform in Belgium

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Abstract

The New Math reform which swept Europe in the 1960's was in a significant way influenced by the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques. Since its foundation in 1952 the CIEAEM held annual meetings where mathematicians, logicians and psychologists discussed the direction of the modernization process. Several Belgian mathematicians played a prominent role in the CIEAEM, in particular Willy Servais and Georges Papy. In particular, Papy has been recognized as a leading, if not uncontested, architect of the new mathematical curriculum. Much less is known about Willy Servais, who for more than twenty years acted as secretary of CIEAEM. In this paper we retrace the career of Servais against the background of the mathematical curriculum reform in Belgium. We reconstruct his views on the modernization of the mathematical curriculum, his work on mathematical models and his concern about the cultural role of mathematics in the modern world. Our analysis shows that the need for an abstract, unified mathematics (as expressed in Papy's work) was not a dominant theme in the early 1950´s debates in Belgium. Much more attention was given to the creation of teaching aids and the introduction of possible new topics such as probability theory, statistics or electrical technology. We further draw attention to the wider issues involved in the reform and the divergent views of mathematicians and school psychologists in Belgium.

Introduction

The New Math movement, or 'Modern Mathematics' as it was commonly called in Europe, refers to a rather brief, but dramatic and influential change in the way mathematics was taught in the U.S., in various European countries, and in some other parts of the developed world, from the end of the 1960's (Moon, 1986; Stanic & Kilpatrick, 1992; Walmsley, 2003). The main feature of the movement was the introduction of new teaching contents, materials and practices in order to amend the generally perceived poor state of mathematics teaching after World War II. New Math emphasized insight in mathematical structure (rather than computational skill), often but not exclusively through the study of abstract concepts like sets, relations, graphs, algebraic structures, number bases other than 10, etc. Other characteristic changes were the replacement of traditional synthetic geometry based on Euclid by an algebraic approach, and the introduction of calculus through the concepts of continuity and limits, strictly defined in a topological environment. The

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New Math movement affected in various degrees mathematics in both primary and secondary schools, and often implied the replacement of Euclidean geometry and calculation skills.

The implementation of the New Math curricula generated a lot of controversy. In the U.S., Morris Kline became an outspoken opponent of the new curriculum, warning that it would have a negative impact on mathematics teaching by its overemphasis on abstract concepts and by its neglect of practical applications (Kline, 1973). In Europe, the situation was more complex. Although the New Math movement adhered to a common core, its implementation depended on national cultures and local educational systems. In 1975, Willy Servais presented an extensive overview of the various national traditions and reform movements of mathematical teaching in continental Europe (Servais, 1975). In France, Belgium, the Netherlands, West and East Germany, Switzerland, Luxemburg, Spain, Italy, Poland and Hungary, new mathematical curricula had indeed been implemented in secondary schools, with, as their main common features, the replacement of synthetic geometry by an algebraic approach and the use of set theory. But Servais also noted many divergences between these national developments. The aims of the reform were not always the same, ranging from bringing the content of school mathematics closer to the current scientific level of the field, to renewing old fashioned teaching methods. In some countries the reform of mathematics was restricted to a limited number of experimental classes, driven by highly motivated individuals; in others the curriculum was strictly determined by a central authority, leaving little room for teachers' own initiative. Servais (1975) concluded that "continental Europe seems more homogeneous than it actually is. In the evolution of their mathematics education some countries have been bold, even rash. Others are advancing more cautiously, more patiently, more deeply" (p. 55). In this state of "permanent confusion", Servais concluded, "we must have confidence."

This diversity in the actual implementation of New Math curricula stands in sharp contrast to the very centralized and coherent preparation of the reform on an international scale. In particular, two organizations served as focal points for discussions and debates among a few dozen mathematicians, psychologists and educational scientists, who were responsible for giving the New Math movement its central orientation. The *International Commission on Mathematical Instruction* (ICMI), founded in 1908, convened the famous Royaumont conference (1959), chaired by its president Marshall Stone and supported by the *Organisation for European Economic Co-operation* (OEEC, later OECD). The proceedings of the conference were published in 1961 (OEEC, 1961), and became the first manifesto of the New Math reform in Europe (Furinghetti, Menghini, Arzarello, & Giacardi, 2008). But previously to the Royaumont conference, the debate had primarily been developed within the small group of members of the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM) / *International Commission for the Study and Improvement of Mathematics Teaching*. The CIEAEM was founded in 1952 on the initiative of Caleb Gattegno, a tireless and versatile researcher and reformer of mathematical teaching methods (Bernet & Jaquet, 1998; Félix, 1985). Gattegno gathered a group of mathematicians, logicians, psychologists and mathematics teachers, who in annual meetings explored the various aspects of the modernization of mathematics teaching. The Royaumont Conference was in effect the outcome of the work carried out by the CIEAEM.

The historiography of the New Math movement in Europe is still in its infancy. Only a few authors have attempted to describe in some detail the curriculum reform in their country (Charlot, 1984; De Bock, Janssens, & Verschaffel, 2004; Matos, 2009; Noël, 1993). Others have focused on the main architects of the reform, either in the form of a tribute or as a commemorative text. Most of these accounts have been written by people directly involved or closely connected to the events described. Recently, historians have begun to reassess the reform from a critical distance, carefully retracing the various proposals and controversies which united and divided the small community of reformers (Bjarnadóttir, 2006; Coray, Furinghetti, Gispert, Hodgson, & Schubring, 2003; La Bastide-Van Gemert, 2006). These studies highlight the importance of differentiating between various approaches and doctrines, and of understanding the epistemological, educational and cultural context of the debates.

This paper will focus on the Belgian contributions to the CIEAEM, in particular on the work of Willy Servais, who served as its secretary from 1956 until 1979. Belgium played quite an important role in the elaboration and implementation of the curricular reform movement at the European and international level. The pivotal role of the Brussels mathematician Georges Papy is generally acknowledged, but at the same time it has tended to overshadow the contributions of his fellow countrymen, which were by no means negligible and did not coincide with Papy's emphasis on the teaching (from an early age on) of abstract mathematical structures derived from algebra and topology. Papy was irrefutably a main actor in the reform movement, but to understand the reception of the New Math reform in Belgium and abroad, it is necessary to look beyond Papy, and to evaluate his position against the background of a much broader spectrum of contributors.

A career in mathematical education

As in many other European countries, debates on the improvement of education flared up in Belgium during the aftermath of World War II. One of the earliest initiatives was the foundation of the *Comité d'Initiatives pour la Rénovation de l'Enseignement en Belgique* (CIREB) in 1945 by a group of supporters of the school system developed by Ovide Decroly. This *Comité*, under the presidency of the physicist Frans van den Dungen, consisted mainly of teachers of the *École Decroly* and professors of the *Université libre de Bruxelles*, among them Paul Libois, whose wife Lucie was director of the school. As professor of geometry and senator for the Belgian Communist Party, Libois was an important intellectual voice, strongly pleading for educational reform (Schandevyl, 1999). At the *École Decroly*, he experimented with new approaches to mathematics teaching on which he would later report during the annual conferences of CIEAEM. In (Libois, 1963) he dated his first attempts in this direction to "almost thirty years ago" (p. 56). In line with the views of Decroly (Libois, 1971), Libois started from the global, implicit notions of the pupils, as they were formed in real life. He also used shadows to show how geometrical figures could be transformed, thus enabling the students to grasp intuitively the idea of transformation. It is difficult to assess the real importance of Libois in the ensuing discussions on the modernization of mathematics. But it seems probable that Libois was at the centre of a group of mathematicians at the *Université libre de Bruxelles* who would become major proponents of the New Math movement in Belgium. In particular, Frédérique Lenger, the future wife of Georges Papy, worked as an assistant with Libois from 1947 until 1950, teaching at the same time at the *École Decroly*.

Noël (2002) has argued that the debates on mathematical reform in Belgium after World War II should also be seen in the context of the emancipation of the mathematical and scientific sections in secondary schools, at a time when elite students were still largely directed towards humanistic studies. The introduction of the new mathematics, which, at least in the mind of the reformers, raised the intrinsic value of the courses and brought them nearer to the academic standards of the field, may have given mathematics a status comparable to the classical courses on Latin and Greek. That Latin-Greek remained for a long time the elite orientation in Catholic schools, whereas state schools were more inclined to emphasize science and mathematics, may also have played a role in the position taken by the New Math reform in the turbulent political atmosphere in Belgium in the 1950's. Although the documents show no political agenda and indeed point to a non-exclusive participation of teachers from both state and Catholic schools, it may not be a mere coincidence that enthusiasm for the New Math reform first originated in the circles around the *Université libre de Bruxelles*, a university devoted to secular and freethinking commitment.

A key personality in the math education reform debates in Belgium from the early 1950's is Willy Servais. Unfortunately, there is hardly any reliable information on the early years and education of Servais. Félix (1985) provides some biographical elements based on a document written by Servais' wife Renée. Other information can be found in obituaries or commemorative texts, often without mentioning sources. Many uncertainties remain. Born in Nivelles on 1 February 1913, Servais studied at the *Athenée Royal* (secondary school) of his home town (Paulus, 2009), and went on to study mathematics at the *Université libre de Bruxelles*. He graduated and also obtained his teacher's certificate in 1936 (Paulus, 2009 erroneously mentions 1963). In 1937 he started to teach at the *Athenée du Centre* in Morlanwelz, a secondary school for boys with a secular, anti-clerical background. Félix (1985) mentions that besides mathematics, Servais had a broad interest in poetry, literature and painting. Servais was also president of an unnamed freethinkers' society (Gaulin, 1979).

At the outbreak of the war, Servais was enlisted in the Belgian army as an officer and deported with the majority of Belgian officers to the German prisoner of war camp Oflag III B, situated in Tibor (Cibórz) near the current German-Polish border. During his time as a prisoner, he acted as a mathematics teacher for his fellow prisoners, who wished to prepare for their university exams. He also organized small seminars on advanced mathematics and gave lectures. The extent of intellectual activities in Oflag III B can indeed be imagined by reading the paper by Edgard Vandekerkhove (1941), who as a prisoner of war in Tibor constructed two rudimentary telescopes from simple magnifiers and ordinary glasses, and who was kept informed of other astronomical observations through correspondence with a Belgian astronomer. With the help of the Red Cross, Servais was able to obtain several works on mathematics, logic and methodology, in particular by the Swiss psychologists and philosophers Jean Piaget and Ferdinand Gonseth. He also studied the first books of Bourbaki. Christine Keitel (2005) stated that Servais met Hans Freudenthal while in a German labour camp, but this is improbable as Freudenthal was only in 1944 and for a very short time imprisoned at the labour camp in Havelte (the Netherlands) (La Bastide-Van Gemert, 2006). It shows, however, how the story of mathematical foundations being studied during the war in prison camps had become part of the collective memory.

Upon his return to Belgium in 1945, Servais resumed his work in Morlanwelz. In 1958 he was appointed *préfet des études*. At a lecture given at Neuchâtel in 1961, Servais was introduced as the prefect of the *Athenée* and professor at the *Université libre de Bruxelles* (Séance, 1961, p. 192). He is further described as a specialist in mathematical logic and author of important works in that field. As far we know, Servais did not publish papers on mathematical logic and was never a professor at the University of Brussels, although Félix (1985) and Gaulin (1979) mention that he lectured at the *Université de Mons*, an affiliation which is confirmed (Unesco, 1973). Actually, Servais taught since 1951 a course on logic at the *Institut Supérieur de Pédagogie* at Morlanwelz, a teacher training school, which in 1965 was incorporated in the *Université de Mons*.

Servais entered the international arena in 1951, when he was invited to attend the second meeting of Gattegno's group, which took place in the Belgian municipality of Keerbergen. Servais rapidly acquired a central position in that group, collaborating closely with Gattegno in the organization of the meetings. In 1952, Servais was one of the founding members of the CIEAEM. Four years later, he was nominated as its secretary, a position he held until 1979. Bernet and Jaquet (1998) describe him as the "soul of the meetings" of the CIEAEM, providing "brilliant syntheses" at the end (p. 8). Servais' amiable character allowed him to remain untouched by the many controversies which divided the international community. Freudenthal wrote that "never in my life, with people I met, was friendship and profound disagreement more closely knitted than in my relation with Servais" (quoted in La Bastide-Van Gemert, 2006, p. 280).

In 1953, Servais founded the *Société belge des Professeurs de Mathématiques* (SBPM), of which he himself acted as president until 1969. The SBPM brought together a few hundred mathematics teachers from both linguistic communities (Dutch and French) and from all school types (state schools and Catholic schools). It started its own journal *Mathematica & Paedagogia*. Servais served on the editorial board and was secretary from 1961 until his death (Miewis, 2003). The aims of the journal were reflected in the many papers by teachers on examples of successful didactical approaches, the presentation of teaching aids and the exploration of new applications of mathematics in physics or statistics. It also contained reports on and papers from CIEAEM meetings, making sure that the spirit of reform and reflection was transferred to the Belgian community of mathematics teachers.

In 1959 Servais was invited speaker at the OEEC conference, held from 23 November until 4 December at the *Centre Culturel de Royaumont* in Asnière-sur-Oise (France), where he presented his views on the new curriculum to be constructed. Subsequently, Servais and Libois were appointed as international experts to prepare a modern syllabus in secondary school mathematics for students who were strong in scientific studies. The syllabus was adopted at the conference in Dubrovnik (1960), and widely disseminated (OECD, 1961). Servais became recognized internationally as one of the main experts on mathematics education and was often invited to lecture and to present reports at conferences. In 1966, Servais inaugurated a training course for teachers in Montréal and Sherbrooke. He also became a member of the first editorial board of *Educational Studies in Mathematics* (founded by Freudenthal, its first volume appearing in 1968). Servais contributed to the debates in ICMI, wrote articles for ICMI's Unesco volumes on the teaching of mathematics and attended its new conference series ICME. A major achievement was the publication with Tamás Varga of *Teaching School Mathematics* (1971), which provided an often consulted guide to the field of mathematics education. Servais suddenly died in Budapest on 25 August 1979, only a few days after attending the CIEAEM meeting in Veszprem (Hungary).

The Papy era

During the 1950´s, Servais started to collaborate with Frédérique Lenger, who was then teaching mathematics at the *Lycée royal* (secondary school for girls) in Arlon. As was quite common at the time, their interest was directed towards the use of mathematical models and concrete materials that could be used to stimulate students (and not only the best students) to discover mathematical structures in their everyday life (Lenger, 1953; Lenger & Servais, 1956). Much of Servais' early work was concerned with the problem of how to help children to make the transition from intuitive experiences towards abstract understanding. In 1955, the SBMP organized a special conference on the topic "Sources concrètes et intuitives de la mathématique", in particular with regard to the conceptualization of space. The use of cardboard models, light projections, Meccano constructions, geoplans, films, electrical circuits and the famous Cuisenaire rods, promoted by no one less than Caleb Cattegno himself, seemed to be the missing link between intuition and abstraction. These views were reflected in the structure of Servais' textbook on plane geometry (Servais & Jeronnez, 1959), following a path from an intuitive approach towards abstract and finally theoretical geometry.

The same inspiration was at the origin of Servais' attempt with Lenger to construct a new experimental program for mathematics teaching at the training school for nursery school teachers. As these teachers had very little first-hand knowledge of mathematics, the new program aimed at providing them with a number of everyday situations in which mathematical problems could be approached in a dynamical way. The curriculum was tested in schools in Arlon and Liège. Subsequently, they sought the advice of Georges Papy, professor of mathematics at the *Université libre de Bruxelles*. Their collaboration soon led to a deeper involvement of Papy in the debates on the modernization of mathematics teaching. Papy was interested in the pedagogical problems of teaching mathematics to young children. He devised a simple system based on Venn diagrams, arrow-graphs and colour conventions to bring out the mathematical structures underlying the real world situations. In 1959 he exposed his ideas during a meeting of the SBPM in Arlon.

Lenger married Papy in 1960 and moved to Brussels, where she worked as a teacher at the *Institut supérieur pédagogique "Berkendael"* in Uccle. Papy himself would also teach at the Froebel training school of the same institution. In 1961 Papy founded the *Centre Belge de Pédagogie de la Mathématique* (CBPM), in which leading Belgian mathematicians and authors of textbooks were brought together (Vázquez, 2008). The *Centre* was to work out a new curriculum for the first years of secondary schools, which with the support of the Belgian government, was tested in a number of secondary schools during the year 1961-1962. In 1961 Papy published a first draft of his new mathematics curriculum for the first three years of secondary school (Miewis, 2003; Papy, 1961). In the following years, the curriculum was modified. Papy himself was not involved in the final version which was submitted to the government. In 1964, the Belgian minister of education Henri Janne made the new curriculum optional for the first three years in secondary (state) schools. After some further negotiations involving several commissions of university mathematicians, teachers and school inspectors, and under a new minister, it finally became obligatory for the first year in all secondary schools from 1 September 1968. Every year the program was extended for the next year of the curriculum. In 1976, the reform was completed by the introduction of a new program for primary education in Catholic schools and two years later also in state schools.

The reform put an end to the traditional division of mathematics courses in arithmetic, geometry, and algebra. The foundation of all mathematics was set theory, from which the exposition of other topics was deduced in a purely logical way. The main features of the curriculum were the unity of mathematics, the progressive introduction of fundamental mathematical structures, the use of (pseudo)concrete situations, the student's ability to construct mathematical demonstrations, and a basic familiarity with logical concepts (Noël, 1993). The curriculum aimed to attain a mathematics that was abstract, structural and algebraic. In the following years, Papy published several volumes of his textbook *Mathématique moderne* (Papy, 1964-), which served as the basis for the preparation of school manuals.

The success of Papy's *Centre* in creating and implementing the new curriculum overshadowed other initiatives. In 1961 the city of Brussels created a *bureau* to prepare the modernization of mathematics teaching. Paul Libois developed a program based on the teaching of geometry, which was used in the *École Decroly*, but his views were not included in the official reform. There was also a reaction from teachers and inspectors involved in schools for technical education, who deplored the loss of geometrical representations and the emphasis on logic and abstract concepts (Smet, Vannecke, & Baeten, 2002, p. 488-492). The new curriculum isolated mathematics from other courses as technical drawing, where the understanding of spatial forms was required. The opposition was headed by *Matec*, an organization of mathematics teachers in technical schools. But their protest was in vain. Technical schools had to adopt the new curriculum at most one year after the obligation had been imposed on other schools. *Matec* dissolved after 1969.

Although the new curriculum was prepared by a small group of mathematicians, its success depended on the adequate training of secondary school teachers. From 1960 until 1968, the SBPM and later the CBPM organized special three days workshops in Arlon, where hundreds of teachers studied the new concepts. Apart from these workshops, the CBPM organized weekly study groups in some fifteen cities. It is estimated that some 3000 mathematics teachers were involved in these groups. After the new program was made obligatory, the government itself attempted to organize some form of obligatory training for teachers, but it was not a success and was soon abandoned (Noël, 1993). From 1968 until 1979, the CBPM published a journal *Nico* (a reference to Nicolas Bourbaki), in which didactical approaches to implement the new syllabus were explored. But Papy himself, together with his wife, turned his attention increasingly towards mathematics teaching at the primary school level, the development of a minicomputer to be used in the classroom and research on the mathematics education of young children. When in the late 1970's protesting voices were heard that asked for a reconsideration of some of the most extreme aspects of the New Math, Papy did not enter the debate.

The international renown of Papy reached its apogee in the 1960's. Papy succeeded in consolidating his position in the CIEAEM, of which he became vice-president in 1960, and president from 1963 until 1970. But his dogmatic, and according to some, unpleasant of even insulting behaviour, alienated many of the members. In 1970 he was forced to step down as president and he left the commission (Félix, 1985). Servais, who had assisted Papy for many years and at an early date had welcomed his views, did not leave with him.

Between content and methodology

During the 1960's Servais showed himself a loyal supporter of the curriculum changes proposed by Georges and Frédérique Papy. He wrote with enthusiasm about the choice of set theory as the foundation of mathematical education, and in particular lauded Papy's use of Venn diagrams and arrow-graphs, which were, in his opinion, ideally suited to the mind of young children. He also endorsed their attempts to bring the New Math to primary schools. According to Servais, the development of mathematical understanding in students was based on the experiences and notions acquired at an earlier age (Servais, 1959). It was therefore necessary to introduce ideas and elements for reflective activities "as soon as they can be made accessible to the child. It can be done in kindergarten." (Servais, 1968b, p. 797). But at a closer look, some differences emerge, perhaps obfuscated during the Papy era of the 1960's. Although Servais never directly criticised Papy, it is clear that his vision of the modernization of mathematics did not entirely coincide with that of Papy. In 1975, when discussing the New Math reform in various European countries, he expressed some reservations as to what had been done. He warned that, although "reform fights to get rid of the ancient and to improve radically and once and for all ... it may lead us to build a new static stage and to deprive us of still valuable parts of our patrimony. What we want is to up-date teaching of mathematics both in content and in method and to keep it alive as a permanent activity" (Servais, 1975, p. 55). Already in 1964, Servais warned: "Let us make no mistake: any syllabus, however sensible and balanced it is, can degenerate into mere dogma in the hands of a dogmatic teacher" (Servais & Varga, 1971, p. 219). For Servais, every reform was but a temporary step, which needs to be continuously evaluated and improved in a next phase. Any curriculum, including the one created by Papy, could not be left unchanged forever.

Servais' views on mathematical education were greatly influenced by the work of the Swiss mathematician and philosopher Ferdinand Gonseth. Gonseth's theory of idoneism postulated an epistemology based on an endless dialectical series of experiences and representations of objects. Knowledge about the world could only be attained in small steps which had to be reiterated. Even logic was not inborn or predetermined by the internal structure of the mind, but acquired through a chain of interactions with the world. Applied to mathematics, Gonseth pointed to the role of the scheme, a mental image or sketch of reality, which was not a faithful or 'true' representation of the objects but an abstract structure reflecting our experience of the objects. Servais often mentioned Gonseth in his writings and wrote several papers on his work (Servais, 1957; 1970b).

Servais in particular emphasized the concept of a *pédagogie ouverte*, an open approach to the learning process. For this reason, he was critical of the use Euclid as a model for geometry courses to young learners. It was a book written for adults, aiming at building a stable and prestigious mathematical structure, but hardly suited to stimulate the mental activity of young children. Servais favoured a more active approach. "If we want our mathematics education to be a learning event, rather than a drill, it is indispensable that we should make ample space for the mathematizing activity of the student," by which he meant an "internal experience and an active intuition" (Servais, 1957, pp. 209 and 212). But the stage of intuition should be followed by a further step towards abstraction, which would allow understanding of the mathematical structures behind the experience.

> From the point of logic, mathematics is evolving towards an axiomatic structure. At that stage, the intuitive support is reduced to the nominal form of axioms and rules of deductive logic. A mathematical culture should introduce [the student] to these questions. This initiation may be more or less advanced, but in a dialectical pedagogy, it can never be reduced to the dogmatic imposition of a ready-made system of axioms. The role of the system will be better understood if the axioms are obtained after an 'inductive synthesis' and if they are subsequently tested with regard to their deductive scope. […] All should be done to set the abstract mathematical model free from the intuitive relations, so that its structure can serve as the rational skeleton ("ossature") of other sciences and technology. (Servais, 1957, pp. 212-213).

The practical application of this pedagogical program was provided by the "pedagogy of situations" which was developed by Caleb Gattegno. The idea was to confront the student with an intriguing phenomenon or object, that would stimulate a sustained investigation by the student.

> The true involvement of students in mathematical work can only be assured by an adequate motivation at their level: pleasure of playing or of competition, interest for application, satisfaction of the appetite for discovery, the affirmation of themselves, a taste for mathematics itself. In order to learn mathematics in an active manner, it is best to present to the students a situation to be mathematized. So today's didactic is based, as far as possible, on mathematical initiations to situations easy to approach at the basic level and sufficiently interesting and problematic to create and sustain investigations by the students. They learn by experience to schematicize (*sic*), to untangle the structures, to define, to demonstrate, to apply themselves instead of listening to and memorizing ready-made results. (Servais, 1968b, p. 798)

It may seem that this sequence of mental activities would in the end lead to the most abstract understanding of mathematical structures, but for Servais this was not the ultimate goal. The same sequence could be found at different levels of mathematical knowledge, and it had to be repeated over and over again for every new step in the learning process.

As in the open philosophy of Gonseth, any acquired knowledge was always the starting point of new investigations.

A direct consequence of this pedagogical view was Servais' work on concrete models. These models were not simply toys nor every-day situations, but stylized objects which could provide a concrete form to mental activities and which allow for active manipulation aiming at a better understanding of mathematical ideas and structures. Typical examples of concrete models were the Cuisenaire rods, the Dienes multibase arithmetic blocks, the geoplan, mathematical films and projections, etc. Also the Venn diagrams and arrow graphs utilized by Papy, were for Servais good examples of concrete models, since they acted as an intermediate step between actual classroom situations (putting a rope around objects) and mental images (the concept of set). Apart from being useful, a concrete model had to be appealing. Servais himself experimented from the late 1940's with electrical circuits to study the properties of conjunction and disjunction (Servais, 1969b). The repeated and investigative manipulations would lead to mental representations which would become independent of the model. "Every perception or action derived from the concrete duplicates itself in mental imagery; this becomes structured and can then be recalled in its own right" (Servais, 1970a).

The same pedagogy allowed Servais to interpret and appreciate the role of applications of mathematics. Any application was a test for one's knowledge and an opportunity to learn. The inclusion of applications was from the start an important argument to legitimate the reform. Once the basic structural order was acquired in mathematics, it was easy to apply them to the understanding of other sciences. Servais dreamed of a better coordination and a dialogue between school programs for mathematics and physics. "Without physics and the other sciences, mathematics could be reduced to a formal game […]. Without mathematics, physics would regress to the level of an at most qualitative phenomenological description" (Servais, 1966, p. 187). But Servais was aware that this was not easily done in practice. It depended too much on the willingness of the teachers, and he conceded that the mathematics teacher may not know enough physics, and vice versa. He had more hopes for the integration of probability theory and statistics in the curriculum, topics which could also be based on set theory and which had an enormous impact on modern natural and human sciences. But these topics were not accepted as essential parts of the reform (Servais & Varga, 1971).

Servais' pedagogical views do not seem to have changed much over the years, but from 1958 on, he became increasingly involved in the preparation of a new mathematical curriculum. In his first attempts with Lenger, he already included the basic concepts concerning sets, relations, elementary functions and topology. Over the next years, Servais collaborated with Papy and the CBPM to work out a comprehensive syllabus for the new curriculum. In 1964, Servais was able to present the basic guidelines of the new syllabus. The topics could be arranged, depending on the needs of the course, in either a logical, a practical or a psychological order. But, in any case, as a result of an improved understanding of mathematical ideas and theories, mathematics had recovered its unity through set theory. Servais was convinced that this basic unity provided the solution to the pedagogy that he had in mind: as the goal of mathematics teaching was to activate the mind of the child towards grasping the mathematical structures in the world around him, it was necessary to define these structures and to make them the backbone of the whole syllabus.

> Teaching should proceed in the light of these findings and, using set theory as a basis, should build up a more unified construction, structured by homogeneous modern ideas. It should do this not only to present an authentic, albeit elementary, image of the science of mathematics, but also to develop the psychological ability to use mathematics as a tool in a broader, more deliberate and more effective way. (Servais & Varga, 1971, p. 217)

Although the preceding text was first published in 1964 in a OECD document, Servais left it unchanged in the publication in 1971. He also added a detailed syllabus based on several sources, including the work done at the CBPM. But Servais was careful to point out that any syllabus had to be implemented with great caution. The course could be adjusted to conditions in each particular country, but he added, "mathematical education of this kind depends not so much on the syllabus as on teaching method, for it is only good teaching that can make a syllabus meaningful" (Servais & Varga, 1971, p. 219).

Servais always maintained an open attitude towards the syllabus, and showed great willingness to compare and integrate the proposals originating from various countries. But he did not change his pedagogical commitment to his *pédagogie ouverte*. He summarized his views on the teaching of mathematics in 1967, during an ICMI conference organized by Hans Freudenthal in Utrecht on "How to teach mathematics as to useful". He sketched the process of mathematisation and the pedagogy of stimulating situations and the progressive genetic axiomatisation, for which he drew attention to what had been achieved in the Belgian reform (Servais, 1968a, p. 45). At the end he summed up the basic conditions for any reform of mathematics.

> Let us formulate at least some general wishes: (1) that one teaches first and foremost the essential core of mathematics, 'its marrow

> substance'; (2) that one offers to every child the opportunity to acquire early on a structural thinking, as conscious, extended and organized as his intelligence will allow, (3) that one makes the pupils aware of the formal and real beauty which originates from the functional order of mathematics. (Servais, 1968a, p. 53)

In the end it was the beauty of mathematics and its crucial role for the emancipation of the human mind, which motivated Servais and many of his fellow mathematicians. As Servais (1954, p. 89) wrote in the first editorial of *Mathematica & Paedagogia*,

> Our time marks the beginning of the mathematical era. […] This fact, whatever the reactions, the opinions and the judgments it may provoke, increases the responsibility of every teacher, who, no matter on which level, teaches mathematics. […] If it befits to be worthy of a mathematical tradition, it is also important to allow the mathematization [of the world] to come. As much as it is true that he who devotes his life to teaching, accepts a mission of a world gone-by to build a world being born. The responsibility towards the future is greater than loyalty towards the past.

It is probably impossible to fully understand the motives and the efforts of the early reformers without taking this moral commitment into account. Whether legitimated by the fundamental advances in higher mathematics, or by the growing number of applications of mathematics in other sciences, the basic view was that mathematical thinking was a necessary tool for any modern citizen, and that it was the responsibility of mathematicians to educate their fellow citizens. Mathematics, Servais wrote, was "an activity through which structures were created that allowed people to grasp reality" (Servais, 1968a, p. 53). The modern world could only be understood and lived in through an understanding of modern mathematics.

Teachers, education researchers and mathematicians

The New Math movement was motivated by a desire to improve the existing mathematical instruction in schools. It was generally recognized and often repeated that for many students (and their parents) mathematics at school had become a nightmare. Of course, Servais (1957) observed, there were some very talented students who entered the realm of mathematics with great ease, but so many others only picked up just enough to pass their final exams. The result is that many adults, too, are disgusted by mathematics, and, as Servais undoubtedly knew from his own experience in the *Athenée*, "the fear of mathematics is more hereditary than the gift for mathematics" (p. 208). Many teachers, he concluded, felt that mathematics was only reserved for a small intellectual elite, and they were satisfied to make life difficult for the others.

But was there a real problem? The Belgian reformers, including Servais, hardly ever mentioned empirical information on the current situation of mathematics teaching. Actually, according to international standards of that time, the quality of the teaching of mathematics in Belgian schools was not particularly problematic. In a comparative UNESCO study of the achievement of 13 year old students in 12 European countries, carried out in 1959-1961, Belgium stood out with the highest scores for mathematics, before France and Switzerland. The scores were "particularly high for items which require reasoning and the use of concepts" (Foshay, Thorndike, Hotyat, Pidgeon, & Walker, 1962, p. 49). Since the 1930´s, Belgian primary education (ages 6-12) was considered a highly-esteemed model of child-centred pedagogy, as introduced by pedagogues such as Ovide Decroly. It was intensively studied and monitored by school psychologists, who created empirical tests to measure the level attained by the pupils at different stages of their development. In particular with regard to mathematics, Fernand Hotyat, the director of the *Institut Supérieur de Pédagogie* in Morlanwelz, where Servais taught logic, had made detailed studies of levels of mathematical understanding and conceptual development (Hotyat, 1936, 1948, 1952). He concluded that generalized and symbolic reasoning was usually not attained by pupils before the age of 13. Younger pupils would still return to a more practical level to solve analogous problems with a trial and error approach (Hotyat, 1961). This would suggest that an abstract understanding of mathematics could only start in secondary school. Apparently, these empirical findings were ignored by the debating teachers in Servais' society. Servais and others maintained, based on their own professional experience, that it was possible to start abstract mathematical education from a very early age on.

The lack of collaboration between the mathematicians and the educational researchers can be understood by looking at the widely diverging interests of both groups. Whereas mathematicians studied the content and structure of mathematics and its reflection in the psychological development of children, the educationalists attempted to measure achievement and more in particular, the intellectual and social determinants of low achievers and the characteristics of learning difficulties. In the 1950's, most empirical researches in Belgium were directed towards professional orientation, the length of obligatory schooling, the use and effect of homework, the social determinants of school achievements, etc. Furthermore, although there was a body of research on mathematical instruction, it was in the first place concerned with primary schools and arithmetical skills. But it was exactly the emphasis on these skills in teaching that the reformers were attempting to replace by abstract mathematical understanding.

The relevance of empirical studies to the reforming debates of the mathematicians was not immediately clear. For the educational researchers, the problem was to improve the efficiency of the existing teaching methods, and the understanding of the causes for failure. They often preferred to study the way children assimilated a doctrine that was taught to them, or concentrated on the mental operations of students in solving definite problems. Hotyat himself based his research on the somewhat outdated theory of the French logician Edmond Goblot, who put the nature of mathematical reasoning in the correct application of previously established rules to mental situations. This enabled the empirical study of performance and efficiency of teaching methods but had little to say on the desirability and relevance of the skills or knowledge acquired. In particular at the time when Servais and others were focusing on the selection of topics and basic structures to be included in the curriculum, they had little interest in learning about the shortcomings of the old curriculum.

Hotyat showed himself rather critical of the "audacious experiment" of Papy, aimed at introducing set theory and intuitive topological notions in several teachers' training colleges, but he saw a common ground between teachers and educational researchers in the attention of teachers to the development of teaching aids and in the shared aim to increase the efficiency in the transformation process of intuitive understanding to abstract reasoning. The efforts of teachers regarding innovating didactical approaches lacked the control of objective measurement, necessary to build a truly experimental science. Their personal experiences led to a form of "experiential" (or experience-based) psychology, but did not contribute to "experimental psychology". Their proposals remained mere hypotheses as long as they were not supported by objective verification.

Conclusion

The career of Willy Servais spanned almost three decades. Therefore, he may be taken to be a privileged witness of the early development of research in mathematics education, particularly marked by the New Math reform of the 1960's. During this period, the reform movement went through several phases (Bernet & Jaquet, 1998). From World War II until the Royaumont conference, the meetings of CIEAEM were dominated by foundational issues, confronting debates on mathematics, educational psychology and teaching methods. Although we are poorly informed about Servais' mathematical background, it is clear that his interest in these debates was primarily directed towards the formulation of active forms of teaching. Experimenting in his classroom with electrical circuits to explain the laws of logic, he discovered the work of Gonseth and his open approach to scientific methodology. Only rarely did he mention Choquet or Dieudonné, the leading mathematicians in CIEAEM, in his papers. His primary concern was always the invention of "stimulating situations" for the learner, by devising concrete models adapted to particular mathematical topics.

By the end of the 1950's, Servais became involved in the creation of an actual syllabus. His first attempts in collaboration with Lenger were positively received and he continued to work on this for several years. In this he was probably overwhelmed by the imposing personality of Georges Papy, who translated the Bourbaki view of a unified mathematics into a comprehensive syllabus. Although we need to know more about these crucial years of collaboration, it appears that Servais enthusiastically supported Papy, but without losing sight of his own pedagogical interest. Committed to a *pédagogie ouverte*, he appreciated the work of others and tried to integrate it into his own. This reflected the increased international outlook of the reform movement as exemplified by CIEAEM being absorbed by OECD and projects, and by the more prominent role of ICMI (which included U.S. mathematicians) in the field.

In the 1970's the reform movement was well under way in a number of countries. It was clear that not all countries followed the same Bourbaki line. The discussion on the 'ideal' syllabus lost much of its importance, and was replaced by a renewed interest in pedagogical modernization. Servais had no difficulty in adapting to the new agenda. Although his views, inspired by Gonseth, were by now superseded by a new generation of professional specialists in mathematics education, he remained a highly esteemed representative of the modernization movement. It remains to be seen how his ideas were received and whether they contributed to the development of modern pedagogical views.

Finally, the figure of Servais offers a particularly interesting case because of his long-lasting involvement in both the international and the national movement. In Belgium the debate on the New Math reform was effectively monopolized by Georges Papy, Frédérique Lenger and Willy Servais. Once the basic choices had been made, all effort was directed towards the diffusion of the new curriculum and the training of teachers. There was also but little contact with educational experts. But on the international level, debates remained much more open and diverse. It is not known how much of these debates filtered through to the Belgian teachers' community, or how much of the Belgian experience contributed to the positions taken by international educational researchers. Servais was probably right when he observed that Continental Europe did seem more homogenous than it really was.

Note. All translations were made by the authors.

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The mathematics teaching in the first years after the 1772 Reform of the University of Coimbra

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Abstract

Frei Alexandre de Gouveia (1751-1808) studied Mathematics and Physics at the University of Coimbra in the first years after the big Reform of the University of Coimbra that created the new Faculties of Mathematics and Sciences. He was not a student of Anastácio da Cunha but he was a student of Monteiro da Rocha, Miguel Ciera, Miguel Franzini and João António Dallabela.

Frei Alexandre de Gouveia was a franciscan that came from a very modest family but his studies were paid by Frei Manuel do Cenáculo, Bishop of Beja, Archbishop de Évora, President of the Royal Board for Censorship and President of the Junta that proposed the Reform of the University of Coimbra, honorary member of the Royal Academy of Sciences of Lisbon.

As a consequence of this fact, Frei Alexandre de Gouveia wrote regularly to his benefactor telling him details about his studies. In these letters he tells something quite unique about the academic life in the University, including details of the classes, of what and how he studied, the relations with the professors, the examinations (he gives details about the scientific content of the examinations). So we can understand something of what happening with the teaching and whether the main goals of the Reform were really being put into practice.

Frei Alexandre de Gouveia was nominated Bishop of Beijing after getting his Ph D in Mathematics and stayed in Beijing from 1782 till 1808, when he died. In Beijing he was vice-president of the Mathematics Tribunal of the Astronomical Observatory of Beijing and a mathematics teacher at the Imperial Academy of Beijing. Frei Alexandre de Gouveia was chosen for this job exactly because he had training in Mathematics and Astronomy.

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Forerunners of the field: Mathematics faculty and the academic climate for mathematics education in America during the 1890s

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Abstract

In the United States during the 1890s, first steps were taken toward establishing a field of mathematics education with the development of new courses devoted to preparation for teaching secondary school mathematics. What was the academic climate in which those first steps arose? The views of faculty who taught mathematics at the secondary and post-secondary levels certainly influenced that climate. How did mathematics faculty of the time view their subject, the teaching of their subject, and their students? With the passing of more than 100 years, it would seem a most difficult task to determine today, in a reliable way, the views of late 19th century teachers; however, clues to those views do exist in reports of contemporaneous studies. This paper will examine the findings of such studies, with particular focus on a study conducted circa 1890 by the U. S. Bureau of Education. Approximately 400 mathematics faculty from universities, colleges, normal schools (for teacher training), and secondary schools, located in more than 35 states and territories, responded to the Bureau's inquiries. The report contains some unexpected questions and intriguing answers. For example, university and college faculty were asked whether males or females showed greater aptitude for mathematics. Normal school faculty were asked to provide information on their own preparation for teaching mathematics and to state what other subjects they may have taught. Secondary school faculty were asked to suggest reforms that were needed in the teaching of arithmetic. Data from this and other studies of the era can contribute to the construction of a profile for the academic climate from which the field of mathematics education was to emerge in the United States.

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Descriptive Geometry and the Polytechnic Schools - the origin and spread of an idea

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Abstract

This study aims to describe two phenomena in the history of mathematics education – the spread of Descriptive Geometry as a mathematical subject in various countries around the world, and the modelling of mathematics education on the principles of the French *École Polytechnique*.

Descriptive Geometry was delivered into the world through newly founded *Écoles de la Republique* (*Normale* and *Polytechnique* in the first instance). The esteem in which it was held by these schools, as well as Monge's role in both the political and educational revolution that France was going through at the time, propelled the technique into the role of a 'revolutionary' subject. This study looks further into the reasons for such state of affairs. From this position Descriptive Geometry was also spread around the world by those societies which took up the social model of the bourgeoisie and its educational principles. Through modelling the transformation of their own educational structures upon those of the revolutionary France, these societies found Descriptive Geometry an important part of the new curriculum, and the translations and elaborations of the original textbook became important part of the development of the history of mathematics education in these societies.

The tracing of some of the translations of Monge's (1746-1818) most famous work will be discussed in this paper/presentation, as will the comparisons between the different interpretations of certain crucial concepts present in the technique of descriptive geometry. An example is the understanding of the process of 'rabatting', which refers to pulling the plane of projection to the plane of drawing – a process which brought, to English language at least, a new word to its mathematical vocabulary.

The countries and translations which will certainly be examined in more detail are English speaking countries, Russia and other countries of Central and Eastern Europe, the Balkans, as well as the general overview of the spread of the technique and the modelling of the teaching after the example of *École Polytechnique* through examination of developments in some African and Asian countries. Whilst wide in scope, there will be opportunities to delve into deeper detail on examples of a few national

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schools of mathematics and the role Descriptive Geometry played in their development.

The organization of the mathematics teaching in Europe in the 1870s according to Jules Houël (1823-1886)

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Abstract

Jules Houël (1823 -1886) studied in the *École Normale Supérieure* (1843-1846), passed the *Agrégation des sciences mathématiques et physiques* in 1847 and completed two Ph.D. in 1855 (one in mechanics, one in astronomy). He taught mathematics in different high schools from 1846 to 1859 and then in the Faculty of Bordeaux from 1859 until his death in 1886.

He is not well-known although he had three major influences:

- As a pedagogue. His 'autographed' lessons have had a great success and have been edited by Gauthier-Villars. The most important ones are *Théorie élémentaire des quantités complexes* (1867-1874; four parts: from definition and history of complex numbers to the Cauchy-Riemann's theory of the complex functions and Hamilton's theory of the quaternions) and *Cours de calcul infinitésimal* (1878-1881; four parts: from elementary real analysis to elliptic functions).
- As a translator of many important mathematicians. He translated from German (Riemann, Lejeune-Dirichlet, Clebsch & Gordan, Baltzer, …), from Russian (Lobatchefsky, Imschentsky, …), from Italian (Bellavitis, Battaglini, Beltrami, …), from Swedish (Mittag-Leffler, Dillner, …), from Norwegian (Bjerknes), from Hungarian (Bolyai). In particular, Jules Houël was the first one to translate and diffuse the Riemann theories and the non-Euclidean geometry in France.
- As a chief editor of the *Bulletin des Sciences Mathématiques et Astronomiques* (its director was the mathematician Jean-Gaston Darboux), scientific journal created under impulse of the École Pratique des Hautes Études (whose president was Chasles) in order to propose a synthesis of the European contemporary mathematics to French mathematicians.

Jules Houël was connected to many European mathematicians and used to ask them about the organization in their own country. So Jules Houël was really well documented about the different organizations of mathematics teaching in Europe (1860-86).

The present study considers, in particular, some letters from or to Jules Houël to precise his knowledge and his point of view about the organization of the mathematics teaching in Europe in the second part of the 19th century.

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Lessons in mathematics and astronomy in the court of Emperor Kangxi

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Abstract

Between the first transmission of European science and mathematics into China at the beginning of the 17th century to the westernization of science and mathematics during the "Self-strengthening" movement in the mid 19th century there went on inside the Imperial Palace since 1688 lessons in mathematics and astronomy Emperor Kangxi and subsequently his princes took from the Jesuits. This led to the establishment of an Office of Mathematics at Mengyangzhai (Studio for the Cultivation of the Youth) in 1713. This episode of learning mathematics and astronomy would be better understood in a political context along with its educational context. The learning experience of this group, though confined to a rather special class, would still shed light on the future evolvement in the "modernization" of mathematics education in China since the late 19th century.

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