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Proceedings of the Third International Conference
on the History of Mathematics Education
September 25–28, 2013, at Department of Education,
Uppsala University, Sweden

Editors:
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Introduction

From 25 to 28 September 2013 the third International Conference on the History of Mathematics Education (ICHME-3) was held at the Department of Education, Uppsala University, Sweden. The department also sponsored the conference financially.

The local organizer was Johan Prytz. The Scientific Program Committee was composed by Kristín Bjarnadóttir (University of Iceland), Fulvia Furinghetti (University of Genoa, Italy), Johan Prytz (Uppsala University), Gert Schubring (Universität Bielefeld, Germany/ Universidade Federal do Rio de Janeiro, Brazil).

Altogether there were 35 participants from 13 countries, 31 contributions were presented. After processing by peer reviews, 26 papers are published in these Proceedings. They may be categorized according to the following thematic dimensions:

Ideas, people and movements
Kristín Bjarnadóttir, Elisabete Búrigo, João Bosco Carvalho Pitombeira, Dirk De Bock and Geert Vanpaemel, Livia Giacardi and Alice Tealdi, Jeremy Kilpatrick

Transmission of ideas
Nerida F. Ellerton and McKenzie A. (Ken) Clements, Thomas Preveraud

Teacher education
Henrike Allmendinger, Mária Almeida, Marta Menghini, Gert Schubring.

Geometry and textbooks
Andreas Christiansen, Regina Manso De Almeida, Frédéric Métin, Isabel María Sánchez and Maria Teresa González Astudillo

Textbooks – changes and origins
Sara Confalonieri, Alexander Karp, Desirée Kröger.

Curriculum and reforms
Evelyne Barbin, Kajsa Bråting, Jenneke Krüger, Johan Prytz, Hervé Renaud, Leo Rogers, Ana Santiago and María Teresa González Astudillo.

The abstracts of five more papers presented at the Conference are included by the end of this volume.

To emphasize the continuity of the project behind the conference on research in the History of Mathematics Education held in Uppsala, the volume containing the proceedings keeps the original title of the first conference, i.e. "*Dig where you stand*" (followed by *3*, which is the number of the conference). This sentence is the English title of the book *Gräv där du står* (1978) by the Swedish author Sven Lindqvist. Hansen (2009) uses it to explain what he did when took up the position as teacher in mathematics. His "Dig where you stand" approach is based on the idea that "there was important and interesting history in every workplace, and that the professional historians had neglected this local part of history writing, so you had to do it by yourself." (p. 66) We deem that "Dig where you stand" may be a suitable motto for those (historians, educators, teachers, educationalists) who wish to sensitively and deeply understanding the teaching and learning of mathematics.

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Hansen, Hans Christian (2009). From descriptive history to interpretation and explanation – a wave model for the development of mathematics education in Denmark. In K. Bjarnadóttir, F. Furinghetti, & G. Schubring (Eds.), "*Dig where you stand*". *Proceedings of the conference on "On-going research in the History of Mathematics Education* (pp. 65–78). Reykjavik: University of Iceland – School of Education.

The editors:

Kristín Bjarnadóttir, Fulvia Furinghetti, Johan Prytz, Gert Schubring

Klein's *Elementary Mathematics from a Higher Standpoint* – An analysis from a historical and didactic point of view

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Abstract

In the early 20th century, a demand arose for a course of university studies considering the special needs of future teachers. One of the well-known representatives of this movement is Felix Klein. Inter alia, he held lectures on Elementary Mathematics from a Higher Standpoint. *In the work at hand, the lecture notes are analyzed concerning the underlying intention and inner structure. The results show that Klein adheres closely to several principles, such as the principle of mathematical interconnectedness, the principle of intuition, the principle of application-orientation and the genetic method of teaching. Those principles contribute greatly to the development of Klein's higher standpoint. In addition, Klein conveys a multitude of perspectives that widen this higher standpoint. As a result, in the lectures two different orientations can be declared: Klein regards* elementary mathematics from a higher standpoint *and* higher mathematics from an elementary standpoint.

Introduction

Felix Klein pinpointed the main problem of teachers' education:

> The young university student [is] confronted with problems that did not suggest [...] the things with which he had been concerned at school. When, after finishing his course of study, he became a teacher [...] he was scarcely able to discern any connection between his task and his university mathematics [...]. (Klein 1932, p. 1)

In order to solve this problem, Klein held a series of lectures, *Elementary Mathematics from a Higher Standpoint* ("Elementarmathematik vom höheren

Standpunkte aus"*).[1] In total, three lecture notes were published: one on *arithmetic, algebra and analysis*, another on *geometry* and a last one on *precise and approximation mathematics*.[2] The third volume however aims to show the connection between approximation mathematics and pure mathematics. Klein doesn't cover questions on mathematics education in that last volume.

Klein's main task primarily in the first two volumes was to supply an overview to school mathematics to connect the different mathematical branches and to point out the connection to school mathematics (cf. Klein 1932, p. 2). In order to fulfill those tasks, Klein expected his students to have basic knowledge in different subjects of higher level mathematics, such as functions theory, number theory, differential equations:

> I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the chief fields of mathematics. I shall often talk of problems of algebra, of number theory, of function theory, etc., without being able to go into details. You must, therefore, be moderately familiar with these fields, in order to follow me. (Klein 1932, p. 1)

These days, Felix Klein's lectures are regarded as an important part of teachers' education, which naturally should be (re)established in (German) teachers' education: In the COAKTIV-study Krauss et. al. (2008) found out that a large number of students lack profound knowledge in elementary mathematics and school mathematics, when leaving university, and therefore state:

> Clearly, teachers' knowledge of the mathematical content covered in the school curriculum should be much deeper than that of their students. We conceptualized CK [content knowledge] as a deep understanding of the contents of the secondary school mathematics curriculum. It resembles the idea of 'elementary mathematics from a higher viewpoint' (in the sense of Klein, 1932). (Krauss et al. 2008, p. 876)

In 2008 IMU and ICMI commissioned a project to revisit the intent of Felix Klein when he wrote *Elementary Mathematics from a Higher Standpoint*. The aim is to write a book for secondary teachers that shows the connection of ongoing mathematical research and the senior secondary school curriculum.[3]

However, in all discussions the term *higher standpoint* is used intuitively and, without making it explicit or naming concrete arguments, Klein's lectures are assumed to have a role model function. With my PhD thesis (Allmendinger

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[1] As Kilpatrick (2014) noted, the original English translation of the title using the word "advanced" as translation for "höher" is misleading, as the term "advanced" could be interpreted as "more developed", which Klein aiming for a panoramic view had not in mind. Taking Kilpatrick's concerns into account, I will use the literal translation "higher" instead.

[2] The latter hasn't been translated into the English language. It is based on a lecture Klein held in 1901. In his last years he decided to republish it as a third part of the series on *Elementary Mathematics from a Higher Standpoint*.

[3] For more Information on the *Klein project*, visit the project's website: *www.kleinproject.org*.

2014) I attempted to help closing this academic void, by analyzing the lectures of Klein in an attempt to answer the guiding question: What is Klein's understanding of the term *higher standpoint*?

I decided to focus on the first volume of Klein's lecture notes, as the different approaches in all three volumes make it difficult to compare them directly. In the first volume (on *arithmetic, algebra and analysis*) Klein includes pedagogical remarks throughout the whole lecture, while in the second volume (on *geometry*) Klein focuses on the mathematical aspects in the first chapters and discusses pedagogical questions in a final chapter. Kilpatrick concludes, that

> the organization of the first volume allows Klein to make specific suggestions for instruction and references to textbooks and historical treatments of topics, whereas the comments in the second volume tend to be more general. (Kilpatrick 2014, p. 34)

With his concrete remarks the first volume gives us the possibility to analyze in detail, what characteristics Klein's *higher standpoint* has. However, these characteristics, which will be presented in this article, can be found in the second and third volume as well.

For the analysis of the lecture notes I used a *phenomenological approach,* like Seiffert (1970, p. 42) describes it. Such an approach analyses a historically sensible source, but it concentrates on the source itself and doesn't focus on the historical background in first place.

Additionally I integrated *didactic concepts and vocabulary* to describe and specify Klein's procedure. I am not claiming that Klein actually used those concepts consciously, but want to show the strong resemblance and coherence of Klein's ideas with today's movement towards improved mathematical university studies for teacher trainees.

As Klein directly comments on his intentions in his lecture notes, this seems to be a possible procedure to locate the characteristics. But especially with regard to an adaption of Klein's concept nowadays, it is important to understand which circumstances led Klein to construct this lecture and which premises he had to face. For example in Klein's days there was no distinction made between teacher trainees and "plain" mathematics students. Therefore the students in Klein's lectures had more background knowledge compared to students these days. So, in my PhD thesis, I embedded my analysis in its *historical context* in order to detect those intentions that might have beacon Klein in his days and that might not be of the same relevance in the present days.[4]

In this article, however, I will concentrate on my first phenomenological analysis. The results show that on the one hand Klein adheres closely to several principles, such as the principle of mathematical interconnectedness, the principle of intuition, the principle of application-orientation and the genetic

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[4] A good overview of the historical context can be found in (Schubring 2007).

method of teaching. Those principles contribute greatly to the development of Klein's *higher standpoint*. On the other hand, Klein conveys a multitude of perspectives – a mathematical, a historical and a didactic perspective –, that widen this *higher standpoint*. I will give an overview of these characteristics and specify them generically by reference to the chapter in Klein's lecture notes on *logarithmic and exponential functions* (Klein 1908, pp. 144–162).

As a result, in the lectures two different orientations can be declared: Klein regards *elementary mathematics from a higher standpoint* as well as *higher mathematics from an elementary standpoint*. In order to describe Klein's understanding of the term *higher standpoint* and the two different mentioned orientations correctly, one should take in account the counterpart – *elementary mathematics* – as well. As this term has always been used quite intuitively, just like the term *higher standpoint*, it is not possible to give a concrete definition.[5] For this article I will use a preliminary definition: Calling everything "elementary", which can be made accessible to an "averagely talented pupil" (Klein 1904, p. 9, translated H.A.), his lectures cover subjects of the established school curriculum and subjects, that according to Klein should be part of school curriculum, for example calculus (cf. Meran Curriculum 1905).

Klein's chapter on logarithmic and exponential functions

Before describing the located characteristics of Klein's *higher standpoint*, I will give a short résumé of the chapter on *logarithmic and exponential functions*. This chapter is paradigmatic and outstanding at the same time, as all characteristics I found in Klein's lecture cumulate in this chapter. Therefore, it seems appropriate to outline Klein's intentions and his proceeding.

Like in many other chapters, Klein starts by giving a short overview of the curriculum and teaching practice: Klein describes how, starting with powers of the form $a = b^c$ with c a positive integer, one extends the notion for negative, fractional and finally irrational values. The logarithm is then defined as that value c, which gives a solution to the named equation. What matters is, that he critically reflects on this procedure: To uniquely extend the values to fractional values, stipulations have to be made, that – in Klein's opinion – "appear to be quite arbitrary […] and can be made clear only with the profounder resources of function theory" (Klein 1932, p. 145).

In the second section of this chapter Klein shows a different approach to the definition of the logarithmic function by describing the historical development of the theory: The main idea Bürgi followed, when calculating his logarithmic tables, was to avoid the stipulation, by choosing a basis b close to 1.

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[5] In the beginning of the twentieth century different mathematicians aim to give a definition of elementary mathematics (e.g. Weber (1903) and Meyer and Mohrmann (1914)). (cf. Allmendinger 2014)

In this way, simply the calculation with integer valued y will lead to a table, where the distance between neighboring values of x is rather small.

Klein interrupts his historic overview to set up a differential equation, generalizing Bürgi's approach. His analysis and calculations lead to the definition of the natural logarithm as

$$\int 1/x \, dx$$

Klein concludes, that the right way to introduce the logarithmic function – in fact to introduce new functions in general – is to square known curves, and he completes his chapter on logarithmic and exponential functions with a section on the function theoretic standpoint, where "all the difficulties which we met in our earlier discussion will be fully cleared away" (Klein 1932, pp. 156f). In this last part one aspect of Klein's understanding of the *higher standpoint* becomes evident. Klein doesn't expect his students to teach this prospectively to their pupils:

> I am, to be sure, all the more desirous that the teacher shall be in full possession of all the function-theoretic connections that come up here; for the teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them. (Klein 1932, p. 162)

Klein's perspectives – A characterization of "his" higher standpoint

A mathematical perspective

One aspect of Klein's understanding of a *higher standpoint* on elementary mathematics is being capable of connecting school mathematics with higher mathematics, taught at university. It involves having *background knowledge*. Therefore higher mathematics becomes a tool to explain the contents of school mathematics. The section on *the standpoint of function theory* is a typical example: Function theory isn't part of school mathematics – neither in Klein's days nor today – but in Klein's opinion the teacher has to have basic knowledge on that subject to understand the definition of the logarithm adequately.

Furthermore Klein uses higher mathematics and its vocabulary for a *precise and significant representation* of school mathematics. In order to do so, he occasionally has to discuss up-to-date research, as in his remarks on the *logical foundations of operations with integers* (Klein 1932, pp. 10–16).

And finally, school mathematics is shown to be the *origin of research*: The search for algebraic solutions of equations is a problem that is easily accessible

to pupils and is covered in school. However to understand that an equation of fifth degree or higher isn't algebraic soluble, you have to have profound knowledge of Galois' theory.

All these examples give evidence of a mathematical perspective on the contents of mathematics classes. Klein shows how university studies are connected to mathematical school contents, in order to oppose the double discontinuity: He connects elementary mathematics with "higher" mathematics – literally discusses elementary mathematics from a *higher standpoint*. It can be assumed, that this *mathematical perspective* shows Klein's *higher standpoint* in the narrow sense of the word.

By analyzing the whole lecture notes, more aspects of Klein's *higher standpoint* can be recognized. Klein's lectures feature a steady variation of perspectives: The *mathematical perspective* is amended with a *historical perspective* and last but not least a *didactic perspective*: Both can be clearly notified in the chapter on logarithmic and exponential functions as well: Klein starts reviewing and reflecting on the current teaching practice and making suggestions on how to improve the introduction of this theme in school. So, on the one hand, he is regarding the subject virtually from a didactic perspective. On the other hand, he gives an overview of the historical development and therefore gives us an insight in his historical perspective.

A historical perspective

Klein always showed a great interest in historical developments (e.g. Klein 1926). He is said to be one of the first representatives of a historical genetic method of teaching, as shown in Schubring's (1978) work on the genetic method. Klein warrants his approach with the biogenetic fundamental law, "according to which the individual in his development goes through, in an abridged series, all the stages in the development of the species" (Klein 1932, p. 268).[6] The lectures on *Elementary Mathematics from a Higher Standpoint* can be seen as an example of Klein's understanding of this historical genetic method itself.

In Klein's opinion expressed in the following, the historical development is the "only scientific" way of teaching mathematics, so this should be supported. So he furthermore aims to provide the future teachers with the necessary background to use this method in school. The fulfillment of this task, especially the impregnation with the genetic method of teaching, requires profound knowledge of the historical development, which Klein allocates by steadily integrating historical remarks and overviews:

> An essential obstacle to the spreading of such a natural and truly scientific method of instruction is the lack of historical knowledge which so often makes

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[6] A belief that nowadays is criticized, as it suggests that every individual has to go through the same learning process (cf. Wittmann 1981, p. 133)

itself felt. In order to combat this, I have made a point of introducing historical remarks into my presentation. (Klein 1932, p. 268)

In the chapter on *logarithmic and exponential functions* you'll find one of the more rare parts of the lecture, where Klein extensively shows his understanding of a historical genetic approach. Other than that, he constantly adds historical remarks and digression, which are both rich in content and distinguished by a rather scarce depiction. They are rather sophisticated sections, which demand intensive post-processing from the students.

So the historical parts in Klein's lecture notes not just have a special meaning for mathematical education in general, but for mathematical teachers' education as well. Nickel (2013) gave a classification on how and why the integration of history of mathematics should be part of teachers' education. You can range Klein's historical perspective clearly in this suggested classification. According to this, Klein uses history of mathematics as a tool of comfort and motivation, by presenting fascinating anecdotes and as a tool to improve insightful contact with mathematics by reliving the historical development. It becomes obvious, that Klein doesn't teach history of mathematics as an autonomous learning subject matter.[7]

A didactic perspective

Now let us take a closer look at the *didactic perspective* – the standpoint of mathematical pedagogy: In the first place Klein's *higher standpoint* can be understood as a methodological one. Klein aims to help future teachers to prepare for their upcoming tasks and to provide them with the necessary overview and background, using – as described above – a mathematical and a historical perspective.

Klein's great interest in questions of mathematical education (as stated for example in (Schubring 2007; Mattheis 2000) and others), is present throughout the lectures. He was one of the main protagonists in the Meran reform, supporting and accelerating the integration of *perception of space* as well as the *prominence to the notion of function*, which culminates in the introduction of the calculus. In my analysis I was able to show that all the demands made in the Meran reform strongly influence Klein's lecture: Klein adheres closely to the principle of intuition ("Primat der Anschauung") and nearly all aspects of the notion of function, that Krüger (2000) carved out in her PhD thesis, can be detected.

Beyond that he specifically criticizes the common procedures in school: For example he reviews the way the logarithmic function is introduced in school and then analyzes the mathematical content from a historical and mathematical point of view, in order to develop an alternative that avoids the emphasized

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[7] The complete classification can be found in (Nickel, 2013).

problems. Klein conclusion is to introduce the logarithmic function as the integral of $1/x$.[8]

Finally, although Klein dedicates the implementation in the classroom to the "experienced school man" (Klein 1932, p. 156), he has concrete ideas for successful and ideal teaching methods, which he mentions in remarks throughout the whole lecture:

> I am thinking, above all, of an impregnation with the *genetic method of teaching*, of a stronger emphasis upon *space perception*, as such, and, particularly, of giving *prominence to the notion of function*, under *fusion of space perception and number perception*! (Klein 1932, p. 85)

Summarizing, from a *mathematical perspective*, the characteristics of Klein's *higher standpoint* on elementary mathematics are a high degree of abstraction, a formal technical language and a foundation of school mathematics' contents. Additionally a *historical perspective* helps to range the object of investigation in an overall context and provide knowledge on the mathematical history of development. From a *didactic perspective*, Klein promotes a reflective attitude on the school curriculum and provides possible alternatives to the current teaching practice.

Klein's principles – The manifestation of his didactic orientation

In the Chapter *concerning the modern development and the general structure of mathematics* (Klein 1932, pp. 77–92), Klein introduces two different processes of growth in the history of mathematical development (calling them direction A and direction B), "which now change places, now run side by side independent of one another, now finally mingle" (Klein 1932, p. 77). While in direction A each mathematical branch is developed separately using its own methods, direction B aims on a "*fusion of the perception of number and space*" (Klein 1932, p. 77) – mathematics is to be seen as a whole.

The education of mathematics in school and at university, in Klein's opinion should clearly be guided by direction B:

> Any movement toward reform of mathematical teaching must, therefore, press for more emphasis upon direction B. […] It is my aim that these lectures shall serve this tendency […]. (Klein 1932, p. 92)

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[8] This approach has been discussed widely. Nowadays it is often used as an example for a concept Freudenthal (1973) called *antididactical inversion,* meaning that the smoothened end product of a historical learning process becomes the point of departure in education (e.g. Kirsch 1977).

In this chapter, Klein not only expresses his attitude towards mathematics education in general, as shown in the Meran reform, but also legitimates the procedure in his lectures on *Elementary Mathematics from a Higher Standpoint* (cf. Allmendinger and Spies 2013): The main principles, which are characteristic for the favored direction B and which Klein wants future teachers to implement in their school classes, are principles Klein himself attempts to pursue: the *principle of interconnectedness*, the *principle of intuition*, the *principle of application-orientation* as well as the *genetic method of teaching.*

By applying these principles, in Klein's opinion all "will […] seem elementary and easily comprehensible" (Klein 1932, p. 223). Therefore a second orientation becomes visible: Klein not only introduces *elementary mathematics from a higher standpoint*, but also covers h*igher mathematics from an elementary standpoint.* This hypothesis can be underlined by Kirsch's aspects of simplification (cf. Kirsch 1977), as those show a striking resemblance to Klein's procedure in his lecture and his principles.

A higher standpoint – First conclusions

Klein's *Elementary Mathematics from a Higher Standpoint* can be characterized by its underlying principles and by a constant variation of different perspectives. Both – the principles and the perspectives – can contribute to a connection between school and university mathematics and therefore help to overcome the lamented *double discontinuity:* The mathematical, the historical and the didactic perspective help to restructure the *higher standpoint* on elementary mathematics. With the didactic perspective Klein shows an orientation, that distinguishes his lecture from other contemporary lectures on elementary lectures. Furthermore, the underlying principles detect an additional orientation: Klein also demonstrates *higher mathematics from an elementary standpoint.*

Toeplitz (1932) questioned whether the establishment of elementary mathematical lectures, like Klein's *Elementary Mathematics from a Higher Standpoint,* is the right way to prepare students for their future tasks. On the one hand, he criticized the selected contents. For example, in his opinion a teacher doesn't necessarily need to know the proof for the transcendence of *e.* On the other hand, Klein chooses topics that require background knowledge, which can't be provided in a lecture that attempts to give an overview of the complete school mathematics' content (cf. Toeplitz 1932, pp. 2f). Toeplitz argues that a desirable *higher standpoint* can't be taught in one single lecture, but has to be accomplished in every lecture of mathematical studies.

Nevertheless, the skills that accompany a *higher standpoint* in Toeplitz' understanding, clearly resemble the ones Klein conveys in his *Elementary Mathematics from an Advanced Standpoint.* Altogether, Klein's lectures can be understood as a paragon and can be seen as a paragon for current university

studies, although adaptions have to be made concerning the given circumstances.

> It's not the task anymore to create new thoughts, but to bring to light the right thoughts in the right way regarding the given circumstances. (Klein 1905, translated H.A.)

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The influence of New School ideas in the preparation of mathematics teachers for *liceus* in Portugal from 1930 to 1969

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Abstract

This paper addresses the formation of mathematics teachers in Portugal, tracing ideas from the New School movement (Escola Nova).The education system instituted in 1930 that lasted almost nearly forty years will be analysed. We will detail the selection of prospective teachers, describe the central elements of the system and try to understand the organization of the training course and the role of the teacher trainers. The paper is based mainly on legislation concerning the teacher education system and educational magazines, but also oral interviews.

Introduction

In 1910, the Portuguese political system became a republic deposing the monarchy. By 1926, the military overthrow of May 28th ended the period called First Republic. The Constitution of 1933 established the dictatorship of the New State (Estado Novo) that persisted until 1974.

In 1930 the Government of the military dictatorship introduced changes in the field of teacher education and a new system to become a qualified *liceu*[1] teacher was instituted. The reasons that justified this system were grounded in the legislator's belief that the pedagogical culture (cultura pedagógica) and teacher training (prática pedagógica) should operate independently, since they belonged to different places, the first was located in the universities and the latter in the *liceus*. So, Pedagogical Sciences Sections within the Faculties of Arts were established in the Universities of Lisbon and Coimbra, and two Normal *Liceus* (Liceu Normal) – one in Lisbon (Normal *Liceu* of Pedro Nunes), and one other in Coimbra (Normal *Liceu* of Dr. Júlio Henriques, later named Normal

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[1] By 1930, Portuguese students entered a mandatory 4-year primary schooling at the age of six years, after which they could attend one of the branches for the secondary schooling: the *Liceus* and the Technical Schools. The former was oriented to studies at the Universities and went through seven years, encompassing three cycles: 1st (10-11 years old), 2nd (12-14 years old) and 3rd (15-16 years old). And, the latter, was oriented to the preparation of workers. Almeida (2013) gives an overview of the Portuguese school system during the period 1930-74.

Liceu of D. João III) – were created. These Normal *Liceus* were intended to be teacher training schools for future teachers allowing for the experimentation of innovative teaching approaches and the discussion of mathematics syllabus (Decree n.º 18973, 28 October 1930).

In the early years of the First Republic, the ideas of New School or Active School were already discussed within the Society for Studies in Education (Sociedade de Estudos Pedagógicos). This Society gathered an important group of intellectuals and pedagogues that tried to contribute to structure the Portuguese pedagogical field, having as references the ideas and practices of the New School (Pintassilgo, 2007). Adolfo Lima, a pedagogue and member of the Society, writes

> the intuitive method, constantly building observation and experience, as Pestalozzi wanted, the inventive method, or heuristic, or analytical, or rational, demand that the child discovers truths from his work, i.e. that truth arises in his mind by an active process and not merely by passive magister dixit ... instead of packing the memory with words and formulae, these are naturally suggested by the observation, by induction. (…) [Method] which through a series of questions and problems previously arranged individuals are led to acquire knowledge by themselves. (Lima, 1916)

In the 1920s, contrary to what took place in most European countries, the New School ideas had not penetrated private schools or institutions in Portugal but impacted mainly in public primary schools and acquired a significant dimension in primary teachers' training institutions. Late in the 1930s, while Portuguese innovative educators were persecuted and marginalized, a nationalist pedagogy that incorporated some ideas of the New School started to emerge. After the 1930s, the pedagogical discourse besides showing a conservative and Catholic reading of New School was focused in the teaching context, especially the use of instructional methods that were in line with the New School basics (Ausejo & Matos, 2014; Palma, 2008).

Bloch (1993) stated that understanding is the word that dominates and illuminates the historical studies. In this paper we focus on the teacher education system legislated in the 1930s, addressing especially mathematics teachers and giving an overview on how the prospective teachers were selected, the organization of the training course, and the role of the teacher trainers. We are also interested in knowing more about the influence of New School ideas in the orientations for mathematics teaching, particularly in teacher education. This text does not focus on practice. It draws mainly from documents produced by mathematics teachers, teacher trainers and teacher trainees, on the subject of teacher training.

The teacher education system created in 1930 for *liceus*

The novel teacher education system for *liceus* teachers that was set up in 1930[2] consisted of two components: the pedagogical culture (cultura pedagógica), taught in the Faculty of Letters (Faculdade de Letras) of Coimbra and Lisbon Universities, and the teacher training (prática pedagógica) developed at two Normal *Liceus*. Established to grant the future teachers a suitable working environment, the Normal *Liceu* was the place where, during a 2-year training period, all through which the trainee teacher was not paid, teaching practice and other tasks related to teachers' duties were performed. In this professional experience, the teacher trainee was supervised by a teacher trainer (professor metodólogo) (Decree n.º 18973).

The curriculum of the pedagogical culture comprised five courses: Pedagogy and Didactics, History of Education, School Organization and Administration; General Psychology, Educational Psychology and Psychological Measurement; with School Hygiene (a one semester course) (Decree n.º 18973). Aiming to provide the prospective teachers with planning and management skills, psychological and philosophical aspects of teaching and learning, the emphasis of the pedagogical culture is clearly on teacher professionalism, in the sense that the knowledge it provides is not subject specific, but general to teaching. The prospective teacher usually attended these courses during the first year of his professional training period (Almeida, 2013).

To become a certified *liceu* teacher one had to submit an application to the training course at a Normal *Liceu*. With regard to mathematics teachers, a candidate could only apply if he or she had qualified in a mathematics course (four years), at a Sciences College. The reason for setting this norm was the belief that an in-depth understanding of mathematics content knowledge is essential to a good teaching performance. After applying, the first step to become a qualified *liceu* teacher was submission to a health inspection and being considered physically able. The second step, a widely more difficult one, was to pass the admission exams (exames de admissão). These examinations to select the applicants for the training course were administered at a Normal *Liceu*. They consisted of written and oral tests which required of the applicant extremely good mathematical knowledge, a good knowledge of physics and chemistry, as well as a good language (Portuguese) proficiency (Decree n.º 18973). Due to the difficulty and detail of the entrance examination, the approval rate was normally 15% to 20% of the number of applicants (Almeida, 2013; Pintassilgo, Mogarro & Henriques, 2010).

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[2] The teacher education system established by Decree n.º 18 973, 28 October 1930, amended on 22 November, was clarified and adjusted, in particular with regard to the selection process: Decree n.º 19 216, 8 January 1931; Decree n.º 19 518, 26 March 1931; Decree n.º 19 610, 17 April 1931 - Regulation of Normal *Liceus*; Decree n.º 20 741, 11 January 1932 - Secondary Education Statute; Decree n.º 24 676, 22 November 1934 - Regulation of Normal *Liceus*; Decree n.º 26 044, 13 November 1935 - amendments to Decree n.º 24 676, 22 November 1934.

For future mathematics teachers, the written admission exams consisted of two essays, one concerned the history of mathematics in relation to the mathematics curriculum of the *liceus*, and another focused on a topic of the physics and chemistry curriculum (1st and 2nd cycle). The practice test, which was also written, encompassed an algebra item and a geometry item, related to the *liceu's* mathematics curriculum. Finally, there were three oral tests: one covered the subject-matter content (mathematical knowledge), one other covered the topics of the *liceu's* mathematics curriculum and the last one covered a topic of physics and chemistry curriculum (1st and 2nd cycle). The exams were administered by a five member jury, three of them were university teachers and the other two were *liceu* teachers. To succeed in the entrance examination, the applicant had to score above 10 (scale: 0–20) at each test. Finally, the applicants that succeeded were graded by the jury. However, the applicants that succeeded the examination still were subject to numerus clausus (Decree n.º 24676, 22 November 1934). So an applicant could enrol in the first year of practical training at the Normal *Liceu* only if he was in the top four places of the applicants graded list (Almeida, 2013).

At the Normal *Liceu*, the trainees' responsibilities included: attending the teacher trainer classes, as well as their colleagues; perform classes, with pre-instructional plans and subsequent evaluation; attend pedagogical conferences (*conferências pedagógicas*), which the lecturer also attended; attend and organize field trips; evaluate students; engage in the tasks related to students' examination. During the first year of his teacher training, the trainee had to attend arts and crafts classes and physics and chemistry classes, whose were determined by the teacher trainer. Working at the school library was also a duty. At the end of each of these assignments, the trainee had to write a final report (Decree n.º 24676). The trainee was expected to be aware of the trainers' performance in the various aspects of teaching, in order acquire his skills. The trainee should become conversant in the use of instructional methods that were effective in communicating mathematical ideas, as well as to elicit and engage pupils' thinking and reasoning. The teacher training also provided future teachers with curriculum knowledge and classroom management skills (Almeida, 2013).

The trainee evaluation depended on: his attendance, punctuality, and proficiency in performing the tasks he was asked to do; his expertise in the teaching practice; his willingness to commit himself to students learning. At the end of the second year of teacher training, the legislation required that the trainee qualified in the State Exam (Exame de Estado) in order to become a certified *liceu* teacher. This Exam comprised three examinations: a) a written test, which consisted of two parts, one concerning general didactics and, the other, regarding mathematics teaching or school supervision; b) an essay (ensaio crítico), a plan on the teaching of a particular topic of the mathematics syllabus, providing selected lesson plans for documentation. This essay was discussed with a jury member, the candidate could be asked to justify his

decisions by explaining his reasoning; c) teaching of a lesson (fifty minutes) to an assigned class (Decree n.º 24676).

The teacher trainers were attentive to new approaches to teaching, trying to incorporate instructional planning in the trainee teaching practice, as well as reflecting about student interest and influence in the learning process (Rodrigues, 2003). Throughout the 2-year training period, the teacher trainer was expected to transmit a broad body of knowledge to the trainee. The aims of the teacher trainers' work were to enable the trainees to acquire a clear vision of mathematics teaching and learning goals, and to promote their willingness to be efficient in their vocation, once they started to teach at a *liceu*. The teacher trainer also prepared the trainees for some supervision tasks, like class director (Almeida, 2010; Pintassilgo & Teixeira, 2011).

In 1957, the Normal *Liceu* of D. Manuel II, in Oporto was created. At the same time, the system created in the 1930s was adapted in order to attract male candidates for teaching profession (Decree n.º 41273, 17 September 1957) and this lasted until 1969 (Almeida, 2011).

António Lopes[3] attended the teacher training course at the Normal *Liceu* of D. João III, from 1939 to 1941. This teacher underlined, in an interview, the importance of his training, by declaring that it stimulated him to reflect on practice, allowed him to achieve a very good teaching performance, prepared him for school supervision, that is, it allowed him to deal with the exigencies of future situations in his everyday work (Almeida, 2013).

The New School ideas for mathematics teaching

The New School pedagogic movement advocated the principle of active participation of an individual in his own instruction. The student must learn to think appropriately, and choose which approach is easier by means of experiment or the use models and instruments. It is up to the teacher to take actions in the classroom to put active teaching into practice. From this standpoint, aiming to highlight New School ideas for mathematics teaching relating to teacher training courses from the 1930s, we searched for articles published in Portuguese education publications and written by mathematics teacher trainers, trainees or mathematics teachers related to Normal *Liceus*. From those articles we selected the ones where we could trace the influence of New School ideas for mathematics teaching, which was noted in references to the involvement of students in learning and the teacher as the supervisor of such learning, as well as the production of materials for teaching. We will centre our attention on the productions of the Normal *Liceu* of D. João III and Normal *Liceu* of D. Manuel II. Here we will use two articles, both printed by Labor, an education magazine produced especially by and for *liceus* teachers.

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[3] António Lopes is a former mathematics teacher trainer at Normal *Liceu* of D. Manuel II.

The authors of the articles are António Augusto Lopes, a mathematics teacher and teacher trainer; and, Maria Fernanda Estrada, a trainee.

As mentioned above, one of the activities that took place during teacher training were the pedagogical conferences. The conference author and presenter was usually a trainee. Pintassilgo and Teixeira (2011) analyzed the training course of the group of mathematics teachers who began their process in the academic year 1934–35. Both papers analysed displayed part of Francisco Panaças's pedagogical conference proceedings, from which we can perceive that Panaças addressed the use of two teaching methods: the dogmatic method and the heuristic method. For geometry teaching he advocated the latter. According to Pintassilgo and Teixeira (2011), one of the *liceu* mathematics teachers attending the conference stated that the experimental method, using manual activity, is the most convenient to be followed when teaching younger students. The heuristic method was one of the most discussed in the pedagogical conferences of different disciplines and that a real difficulty that those teachers were faced with was the overcrowding in classes.

Among the studies that report on teacher education, Matos and Monteiro (2010) presented a longitudinal analysis of the papers prepared by mathematics teacher trainees at the Normal *Liceu* of Pedro Nunes between 1957 and 1969. The authors stated that few works discuss the methodologies in detail. However, some trainees studied the most appropriate pedagogical approaches. Several trainees declared support for a heuristic or active education. In a text published in Palestra, the mathematics teacher trainer at Normal *Liceu* of Pedro Nunes, Jaime Leote (1958), argued that teachers should "enjoy and encourage" (p. 37) the creative activity that students possess. He further sustained that the teacher must be an investigator and should not think that concepts that he himself took years to learn are obvious to pupils.

In 1940 an article about the Normal *Liceu* of D. João III (Liceus de Portugal, 1940) addressed mathematics teaching in that teacher training school. After mentioning that several teaching methods were used at that *liceu*, it was emphasised that the use of a modern approach to mathematics teaching was spreading, especially among the younger generation of teachers working at public and private schools. The article emphasized that to correspond to modern teaching methods it was imperative to transform the traditional classroom into a workroom. The mention of classroom transformation led to the characterization of old school and modern (active) school by means of the role of the student. In the first, the student had a passive role, absorbing information provided by the teacher; in the latter, the students worked in an environment where they had the opportunity to take part in their own learning. According to the article, experience allowed the use of active teaching methods in the *liceu*. Declaring that a proper use of the method would draw good results, it was stated that in order to apply this strategy in the classroom, the teacher had to organize a variety of exercises on a topic of the syllabus; the exercises must range from the simplest to the more complex ones. The application of the

technique was as follows: at the beginning of the lesson content should be presented in a way that allowed the student to appropriate new knowledge. Then various exercises were practiced in collective work and their resolution was written down on the blackboard; sometimes this was followed by a discussion. The teacher should help and guide his students through their work. This article also mentioned that the prospective mathematics teachers and the 1st cycle students, in Arts and Crafts classes, built their own models (Liceus de Portugal, 1940).

In an article published by *Labor*[4], António Lopes[5] (1952) discussed the applicability of the *laboratory method* in mathematics teaching, in the Portuguese *liceus*. He began by posing some questions relating to the use of this method: did the order of the syllabus topics allow its division into convenient working units? Was it possible to transform our classrooms into real laboratories, allowing the full development of the personal initiative of the students? Would one Mathematics Laboratory per *liceu* be enough to be used by all 1st grade classes, or did we need several laboratories, especially in high attendance schools? If the laboratory method prescribes individual examinations should students' assessment be changed?

Trying to establish some aspects to the use of the laboratory method, considering *liceu* organization, he focused on three items: the laboratories organization, the order of syllabus topics and the organization of a textbook.

Concerning the organisation of laboratories, António Lopes mainly addressed the necessary materials. He supported that each *liceu* should have, at least, one laboratory for mathematics

> that ought to have appropriate furniture; books; drawing, measurement and calculus tools; a set of geometric shapes and of geometric solids; containers and common objects, with geometric shapes; models suitable for theorems proof; other materials and tools needed to accomplish the works. (Lopes, 1952, p. 568)

According to António Lopes (1952), the laboratory method would be used, preferably, in the treatment of program topics for which the necessary materials might be acquired by the students. He remarked that, allied to this, some educational material could be taken by the teacher to the classroom, but not in condition of allowing the students to perform, themselves, their experiences. In short, he considers that the *liceu* reality did not allow for the use of the laboratory method; it only permitted a limited application of the method.

Referring to the order of syllabus topics, he suggested changes in sequence of the topics. On methodology, he provided some remarks and presented one example, see Table 1. Linking with the example, Lopes wrote

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[4] *Labor* was a quarterly 'journal of education, and teaching and cultural extension' published from 1926 to 1973, produced especially by and for *liceu* teachers.
[5] In 1952 António Lopes was working as a mathematics teacher at Normal *Liceu* of D. João III.

It is true that students bring from primary school some information on solid shapes, but the laboratory method requires different understanding. Assemble a solid shape from its net is useful procedure, especially in later applications, but I think that students should build solid plan, after discovering it! (Lopes, 1952, p. 570)

Table 1. Lesson plan

Lesson. ° ...

Subject: Construction of a hexagonal prism, regular.

Material: cardboard, tape, glue, scissors, compass, ruler and square tool.

Implementation plan: the solid has two hexagonal faces, equal; the other faces are rectangular, equal.

Data: the length of the side of hexagonal faces 4 cm; the length the side of the rectangular faces 7 cm.

I

a) Shape, separately, all faces of the solid and cut them; bind them with tape and assemble the solid;

b) Disassemble the solid, while maintaining the faces connected, so as to obtain the plan and noting the relative position of the faces;

c) Draw the net, bind the edges and build up the solid.

II

In the notebook-diary:

d) Jot down the solid net, by drawing it;

e) Compare, by difference, the number of edges with the sum of the number of faces and the number of faces vertices;

f) Calculate in cm², the area of the solid's surface;

g) Describe the process of building the solid, from a piece of wood.

Regarding the organization of a schoolbook, he proposed that, at least, in the 1st year of the 1st cycle, it was better to use a special notebook-diary, with work units printed, rather than a textbook.

In 1960, *Labor* published an article by Maria Fernanda Estrada, then a trainee at the Normal *Liceu* of D. Manuel II, in the school years 1957/58 and 1958/59; the article was the text of her pedagogical conference and in it we can also trace the influence of the New School movement. Fernanda Estrada discussed the use of an axiomatic approach in geometry teaching, in the 2nd cycle, stating that

> several studies and experiments have been conducted by mathematicians an psychologists and the conclusion is that it is not possible to give to young students, aged accordingly to 2nd cycle, at least until fifteen years old, a rigorous elementary Geometry axiomatic, forcing students to construct a perfect deductive Geometry. (Estrada, 1960, p. 559)

Estrada asserted the importance of allowing the students to understand the function and significance of an axiom, underlining that "giving the students a

list of axioms, without any explanation, might lead them to think they are a mere product of the geometers' mind" (Estrada, 1960, p. 560). She advocated that students should become acquainted with axioms in a natural way, appealing to their intuition; and, remarked that

> It is convenient to teach some chapters [Geometry, 3rd year] (…) without previous reference to any axiom, but pointing them out, as it is needed to consider them, and lead students to be aware of them.(…) only with the possible rigor (…) [so that] students understand the need of an axiomatic. (Estrada, 1960, p. 560)

About geometry teaching, Estrada emphasized the following

> heuristic method which will let students rediscover theorems, thus meaning to reinvent geometry and to develop pleasure and interest by its study. To achieve that goal it is necessary to recourse to things that make the students reflect and think, and will pose them real problems to solve. We indicate the use of models, films, (…)

> A model, such as a film illustrates a fact to be questioned, incites the students' intuition and comprehension, and encourages them to search for evidence supporting the observed fact. Then, when the students arrange the acquired knowledge into a proof, the request is that they express it with the best possible rigor.

> To simplify reasoning one can adopt, if considered useful, Klein's idea to colour figures and, instead of Δ [ABC] or Δ [MNP], you just say the 'yellow triangle' or 'red triangle'. (Estrada, 1960, pp. 561–562)

Afterward she stressed the use of references to items of history of geometry and features of mathematicians' lives, as a suitable aid for teaching for it permitted to arouse students' interest.

Concerning the teaching strategies, teacher trainers and trainees both made significant references to the use of materials (Lopes, 1952; Estrada 1960; Matos, 2009; Matos & Monteiro, 2010; Pintassilgo & Teixeira, 2011). António Lopes stated, in an interview, that during his prospective teacher training he followed an active method particularly in the teaching of geometry where pupils made drawings, reconstructions, measurements and thus were learning the mathematical notions through intuition and material objects, so he used simple mathematical instruments and models in class. But, this teacher stressed that their use was to be methodically coordinated with theoretical learning (Almeida, 2013).

The documents made it possible to clarify that future mathematics teachers discussed teaching reforms influenced by the New School ideas i.e. 'active' teaching approaches, during their teacher training. Lopes' discussion on the laboratory method stresses the difficulties of 'active' teaching, mainly because

of the *liceu* organization and inadequate textbooks. However, according to him, it was possible to teach a laboratory lesson, in which, the right questions, the employment of the senses, careful observation and logical thinking finally result in new knowledge, which the pupil has obtained himself. In Estrada's opinion, it an 'active' teaching of geometry was convenient. In teaching geometry, the teacher must take into account the recourse to intuition and the use of models as a way to arrive gradually to logical rigor, before concentrating on theorems and proofs.

We want to point out that the opinion of the trainees about the use of the heuristic method does in no way imply that they had applied it in their future practice. Normal *Liceus* were not the pedagogical laboratories idealized in full by some sectors of New School (Pintassilgo & Teixeira, 2011). However, it is fair to say we can recognize a predisposition for the use of a methodology based on intuition, on the use of concrete materials, and fostering conjecture and argument, aiming at the construction of meaning and a theoretical systematization of mathematics. All these perspectives had their roots in the movement.

Conclusion

The teacher education system for *liceu* teachers that was instituted in 1930 consisted of two components: pedagogical culture, taught at a University, and teacher training, developed at a Normal *Liceu*. An important feature of this system was the creation of two Normal *Liceus* as teachers' training institutions. Few candidates for the teaching training course were able to qualify at the admissions exams, but those that achieved it engaged in a 2-year training period supervised by a teacher trainer. The teacher trainers' purpose was to pass on to the trainee the essence of good teaching, by developing communication skills as a key element of the teaching practice and individual reflection, as well as, to involve him in tasks of school supervision. Within teacher training the trainee reflected on the new trends in school mathematics. The last step to become a certified teacher was to qualify in an exam that was applied at the end of the second year of teacher training.

In the materials analysed, we have noted several principles which stem from the New School (e. g.: from the simple to the complex, from the concrete to the abstract, from passive receptivity to action and personal engagement). A heuristic approach to teaching was valued, where the students were allowed the experience and interaction with manipulative material; students had the opportunity to take part in their own learning. In the learning environments the accent was on the need and interest of work, as well as on cooperation. The influence of New School ideas can be perceived above all in the teaching of geometry.

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Euclides Roxo's deductive geometry

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Abstract

This paper aims to discuss specific strategies proposed by Euclides Roxo for the deductive teaching of plane geometry at secondary school level. Between 1929 and 1931, Euclides Roxo, a Brazilian Mathematics teacher and textbook writer, published a series entitled Mathematica Elementar (equivalent to 'Elementary Mathematics') for secondary school education. The third book of Mathematica Elementar, 2ª série-II Geometria ('Elementary Mathematics, 2nd year-II Geometry) approaches the teaching of plane geometry deductively. This was the first time in Brazil that a textbook author proposed strategies for students to construct the proof of a theorem. The question this study aims to answer is – Which were these basic strategies? Roxo's deductive approach to plane geometry stemmed from the Mathematics reform movement in early 20th century to get learners to prove a theorem deductively. As it is still considered a critical feature of teaching Mathematics at present, it deserves further discussion.

Introduction

Euclides de Medeiros Guimarães Roxo (1890–1950) lived and worked in the city of Rio de Janeiro – the capital of Brazil then – in the first half of the 20th century. After graduating in Engineering, he began teaching Mathematics at Colégio Pedro II, a top-quality reference state secondary school where he was headmaster from 1925 to 1935. His expertise in Mathematics education led him to publish an innovative textbook series and to co-author other book series as well. He also published articles in journals and participated in two major educational reforms – one of which was led by Francisco Campos in 1931 and the other by Gustavo Capanema in 1942. Table 1 below shows a summary of the major educational reforms and related events in Brazil between 1837 and 1942, according to Carvalho (2006, p.72).

Table 1. Major educational reforms in Brazil in the period from 1837 to 1942 and some related events

Year	Event
1837	Establishment of the Colégio Pedro II in Rio de Janeiro
1889	Fall of the Empire and the proclamation of the Republic
1890	Creation of the Ministry of Instruction, Mail and Telegraph Curricular Reform made by Benjamin Constant, Minister for Instruction, Mail and Telegraph
1892	The Ministry of the Interior Affairs and Justice becomes responsible for Education
1925	Euclides Roxo becomes head of Pedro II
1929	Euclides Roxo reforms the mathematics curriculum at Pedro II
1930	Vargas overthrows the established government and becomes President Creation of the Ministry of Education and Health
1931	Francisco Campos organizes secondary education – the Francisco Campos Reform
1937	Vargas establishes a dictatorship
1942	Gustavo Capanema reorganizes secondary education – the Capanema Reform

Carvalho (2006, p. 72)

Roxo published his three-textbook series *Curso de Mathematica* between 1929 and 1931. The books complied with the methodological directives set forth by the Ministry of Education. He also published the book *A Mathematica na escola secundária* ('Secondary School Mathematics) in 1937. These two works provide an overview of the educational principles that influenced his work.

The first international Mathematics Curriculum Reform movement happened in early 20th century. This movement gathered Mathematics scholars and experts from other fields to rejuvenate elementary school teaching. Mathematician and reformer Felix Klein´s (1849–1925) program for the modernization of the teaching and learning of Mathematics is a reference of such movement. In Brazil, Roxo embraced the modernizing reform of mathematics teaching practices based particularly on Klein's ideas. According to Carvalho (2006, p. 73), "Klein"s ideas were the source of Roxo's reform program. Among authors as Poincaré, Jules Tanery and Boutroux, "Roxo makes extensive reference to Klein's *Elementarmathematik vom höheren Standpunkt aus* ('Elementary Mathematics from an Advanced Perspective') (1925–1928)" (ibid, 2006, p. 72).

Regarding general Mathematics instruction, Klein says,

> We, who are called the reformers would put the concept of function at the very center of the instruction, because, of all the concepts of Mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into instruction as early as possible with the constant use of the graphical method, the representation of the functional relation in the x y system, with is used today as a matter of course in every practical application of mathematics. (Klein, 1939, p.4)

Within this perspective, Roxo modernized the teaching of Mathematics. Carvalho (2006, p. 77) claims that Roxo introduced "... a genuine innovation in

the teaching of Mathematics in Brazil" and that "In his two first textbooks (Roxo, 1921 and 1930) Roxo takes Klein's advice to the letter. It should be noted that such advice had already been put into practice by Breslich in his several textbooks." (Carvalho, 2006, p. 78). An illustration of this claim is that "Roxo proves the familiar rule $(a+b)^2 = a^2 + 2ab + b^2$ geometrically, putting into practice what he preached, the "correlation" between algebra and geometry". (Carvalho, 2006, p. 77). The concept of function is conveyed by graphical representation, tables of values and analytical representations. However, since strategies for teaching plane geometry deductively are the focus of this paper, only geometric representations will be discussed herein.

Klein advised that the teaching of deductive geometry be preceded by an introductory geometry course and that deductive reasoning is gradually introduced by dealing with concrete mathematical situations and by considering learner´s intuition (Klein, 1925, p. 227; 1939, p. 191; Roxo, 1929, p. 7; 1937, pp. 244–245). Roxo introduced teaching strategies to deductive teaching of plane geometry in the third book of his series. He said that Klein and Breslich were important sources for his grading and sequencing the introduction of the strategies. Breslich (1874–1966), based on Klein's ideas, wrote textbooks which were used by several institutions, most particularly at the laboratory schools of the University of Chicago, United States, where starting in 1903, a group of teachers helped to reform the teaching of Mathematics (Carvalho, 2006, p. 75).

For the purposes of this paper, some features of Roxo's work will be initially discussed in two sections that are *The textbook series Curso de Mathematica* (1929–31) and *Strategies for teaching plane geometry deductively.* The section *Framework for teaching how to construct the proof* will be showing the importance of teaching strategies for approaching plane geometry deductively in the secondary school, given this subject's complexity and up-to-date epistemological, educational and sociological features. I conclude by acknowledging the importance of Roxo's contribution for modern teaching of mathematics in Brazil.

The textbook series *Curso de Mathematica* (1929-1931)

For centuries, several of what are now called correlated mathematics contents were introduced and dealt with in separate textbooks. Some examples are the previously published textbooks *Elementos de Geometria* (Elements of Geometry), *Elementos de Aritmética* (Elements of Arithmetics), *Elementos de Álgebra* (Elements of Algebra) and *Elements of Trigonometry* (Elements of Trigonometry). Such practice led Felix Klein to claim for an integrated approach. Textbook *Elementos de Geometria* followed a basic deductive study framework: each chapter is introduced by a set of definitions, followed by a series of theorems or problems and their respective proofs. No guidance for how to prove a theorem was provided. In Brazil, these textbooks were adopted in the curriculum at Colégio Pedro II for decades. However, in the 1930s, this perspective changed when

Roxo's book series for secondary instruction was published aiming to reform the teaching of mathematics in Brazilian secondary schools. The three-book series, published by Livraria Francisco Alves (Rio de Janeiro), were designed in compliance with the Ministry of Education directives for secondary education:

- *Curso de Mathematica Elementar*, Book 1, 1929
- *Curso de Mathematica Elementar*, Book 2, 1930,
- *Curso de Matemática, 3ª Serie. II – Geometria*, Book 3, 1931.

Figure 1. Cover of the textbook Curso de Mathematica, 3ª serie II – Geometria (Course of Mathematics, 3rd year II - Geometry).
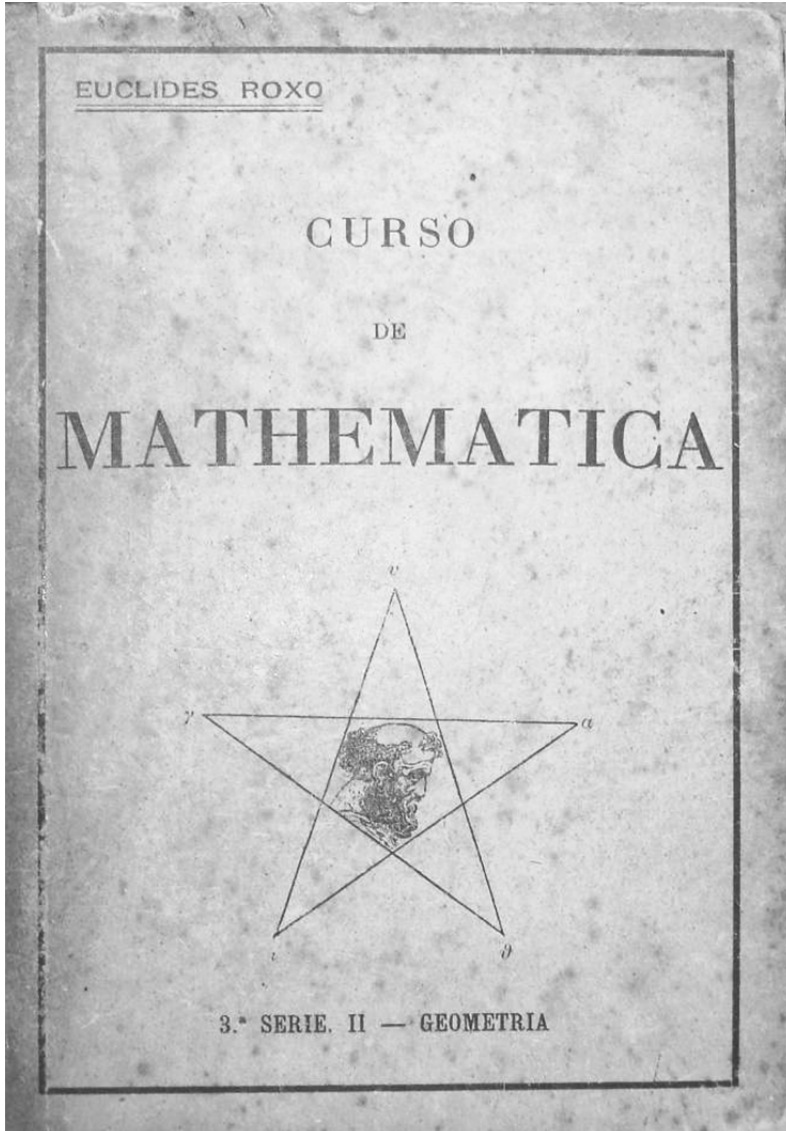
Roxo, Euclides (1931).

Roxo set new standards for the teaching of Mathematics as his series integrated arithmetics, algebra, geometry and trigonometry in addition to addressing the concept of function. Such innovation also featured teaching how to prove a theorem, although such approach was not a standard featured by textbooks (Almeida, 2008).[1] Roxo's work became a landmark in the teaching of mathematics in Brazil.[2]

Roxo proposed a series of procedures in order to teach students how to deductively deal with plane geometry. He contended that since deductive reasoning uses the laws of logics to link together true statements to arrive at a true conclusion, a correspondence can be set between deductive proofs in a chain of reasoning and the writing of proof text. The author further explored major related issues that are illustrated by Chapter I of Book 3 – *Introduction to the Formal Study of Geometry* – which was subdivided into the following themes: *Historical Background*; *A set of fundamental propositions that constitute the intuitive basis of deductive geometry; Theorems*; *Symbols and Abbreviations*; *Thales of Miletus (reading).*
So as to understand Roxo's deductive approach, I analysed the three–book series that he published. In the preface of the Book 1 Roxo discussed the Mathematics Reform Movement and its claims in order to contextualize and justify his work. Book 2 identified properties and geometric relationships, considering the students' previous knowledge, and Book 3 provided clear instructions on how to prove a theorem.

Strategies for teaching plane geometry deductively

As the three books comprised by Roxo's series were analyzed, a basic deductive framework for teaching plane geometry became evident: (i) Addressing the theme intuitively; (ii) Predicting axioms and propositions that support the proof; (iii) Strategies for writing the proof.

The first and second strategies share a conceptual baseline – they show that to prove means to validate a statement. According to Roxo, deductive geometry is characterized by the proof – a theorem is a statement that must be proven. The proof involves logical reasoning, which comprises a set of propositions, such as axioms and theorems, and which results in a piece of writing. The third strategy, strategies for writing the proof, has a procedural aspect which aims to teach learners how to perform the proof.
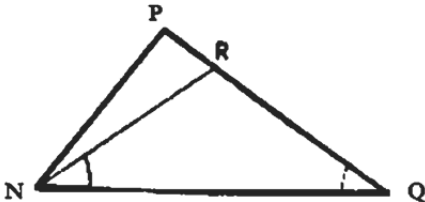
Roxo proposed a way for writing the proof by using five basic strategies. The first strategy is *working with conditional sentences – if … then*, so as to state two specifically related components of the theorem, i.e., the *hypothesis* and the *thesis statement*. It should be noted that the latter which can be plural. In short, we have: *if* the hypothesis, *then* the thesis statement(s), as shown in Figure 2 below.

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[1] Almeida, 2008.
[2] Carvalho, 2006; Valente, 2003.

The author proposed some exercises of this nature that should be directly related with the second strategy, the design of the geometric figure.

Figure 2. Working with the conditional sentence if … then [3]

Theorema. (Reciproca do precedente) Si dois angulos de um triangulo são desiguaes os lados oppostos a eles são desiguais, e ao maior ângulo se opõe um lado maior.

Translation:
Theorem. (Reciprocal of the precedent) If two angles of a triangle are unequal, the angles opposite to them are unequal, the greater angle lying opposite to the greater side

Hypothese: △ NPQ no qual se tem

∠ N > ∠ Q.
Thesis: lado PQ > lado PN.

Hypothesis: a triangle NPQ angle N > angle Q.

Thesis: side PQ > side PN.

Roxo, Euclides (1931, pp. 138–139).

However, it should be noted in Figure 4 below that designing the geometric figure requires the following procedure prior to that: "(1) Draw NR so as to have angle QNR= angle Q. Thus, Roxo introduced and guided students to use Mathematics symbols when designing the table for symbols and abbreviations (Figure 3 below).

Figure 3. Symbols and abreviations.

≡ é idêntico a — equal or equivalent
≅ é congruente a — congruent
> maior que — is greater than
< menor que — is less than
∠ ângulo — angle
∟ ângulo reto — right angle
△ triangulo — triangle

Roxo, Euclides (1931, p. 22–23).

The third strategy was *using sequential markers for deductive connection* (Figure 4 below), which consisted of naming the phases *Hypothesis* (H), *Thesis* (T), *Proof*

and *Conclusion* in the proof text, thus connecting the hypothesis to what has to be proven.

Figure 4. Sequencial marks of the deductive connections.

Roxo, Euclides (1931, p 138).

The fourth strategy aimed to elicit each deductive step of the proof based on the corresponding reason. Thus, the strategy of *constructing the two-column proof (*Figure 5) emerged. The *left column* named *Statements* comprises the deductive steps of the proof – the series of statements that connect the hypothesis and the conclusion. For each of such statements, on the *right column* named *bases or reasons*, the student must write the corresponding reason.

Roxo provided guidelines for performing a proof and warned students "If we do not promptly identify the logical sequence that links the hypothesis to the thesis statement(s), it is important to remember all the known geometric propositions that are related to the subject of the theorem and that will allow us to deductively bridge the hypothesis to the conclusion" (Roxo, 1931, p. 115).

Figure 5. Constructing the two column-format.

Roxo, Euclides (1931, p. 180).

Finally, the fifth strategy was *presenting the incomplete deductive proof.* Note that the question – *"Por que?"* (Why?) (Figure 5 above) – challenges the learner to justify the sequence. The learner must fill in the incomplete steps. This and the other procedures mentioned above can also be found in Breslich's 1916 textbook, *Second-Year Mathematics for Secondary School* (Figure 6 below), thus illustrating one of the similarities between Roxo and former. The theorem is: "Two triangles are similar, if the ratio of two sides of one equals the ratio of two sides of the other and the angles included between these sides are equal, then." (Breslich, 1916, p. 109).

Figure 6.. Strategies of *Arranging the text in two columns* (statements and reasons) and *Incomplet presentation of the deductve proof steps* according to Breslich (1916).

Breslich, Ernst (1916, p. 109).

Carvalho (2002, p. 8) says Roxo was often accused of plagiarism by others teachers and ponders why Roxo relied so heavily on Breslich. "Was he too busy to write a completely new textbook because he was headmaster at Colégio Pedro II? Or was it because he was not or did not feel capable on transposing Klein's general ideas into a workable textbook? We do not know."

The five strategies discussed above will guide deductive teaching of plane geometry under Roxo's perspective.

Figure 7. Strategies for writing a proof text.
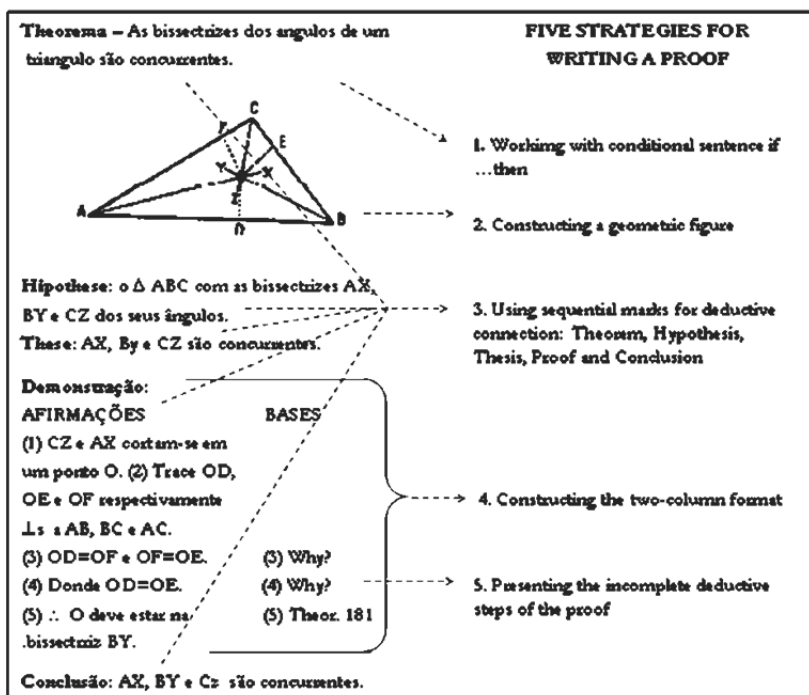
Roxo, Euclides (1931, pp. 114–115).

Translation of Figure 7: **Theorem**: The bisectors of the angles of a triangle are concurrent. **Hypothesis**: Triangle ABC with AX, BY and CZ, the bisectors of its angles. **Thesis Statement**: AX, BY and CZ are concurrent. **Proof**: Statements: (1) CZ and AX cross point O. (2) Draw OD, OE and OF parallel to AB, BC and AC respectively. (3) OD = OF and OF = OE. (4) Then OD = OE. (5) Therefore O is on the bisector BY. **Reasons**: (3) Why? (4) Why (5) Theorem 181. **Conclusion**: AX, BY and CZ are concurrent.

Framework for teaching how to construct the proof

As shown above, Roxo's deductive teaching of elementary plane geometry assumed a correspondence between the rules of logical reasoning and the writing of proof text. Consequently, emerges a proof text schematization that involves characteristic text markers, which we identify while reading the proof. The required sequence of propositions is linked to the logical status that rules it, thus relating *the hypothesis to the conclusion* and the *thesis* of the theorem. The two-column format strategy shows, on the left hand side, the *statements* and on the right hand side their corresponding *reasons*.

In Roxo's work (1931), deductive reasoning was planned to be framed over a period of three years of instruction. Figure 8 below illustrates the following theorem: If two straight lines intersect, the opposite angles are equal.

Figure 8. Interplay between books

Demonstração AFIRMAÇÕES	RAZÕES	Translation Proof Statements	Bases or reasons
(1) ∠AOC+∠AOD= um dir. ou m+n=180°.	(1) Dois angulos adjacentes com lados exteriores em linha reta são suplementares (2° anno, n° 17).	(1) Angle AOC + angle AOD = two right angles, or m + p = 180°.	(1) Two adjacent angles whose exterior sides are on the same right line are supplementary. (2nd year, n° 17)
(3) Donde ∠ AOC= ∠ BOD, ou m=n.	(3) Postulado 19. (Enunciate);	(3) Then angle AOC = angle BOD or m =n.	(3) Postulate 19. (Enunciate.)

Roxo, Euclides (1931, p. 16).

In Figure 8, *reason (1)* is referred to as *(2nd year, no. 17)*, that is, 'two adjacent angles, whose exterior sides are on the same straight line, are supplementary'. But *reason* (3) leads to *Postulate 19* and the student must state it as required. In Book 2 we find *"Supplementary angles of the same angle or equal angles are equal"* (Roxo, 1930, p. 19). This example shows that Book 2 and 3 of the textbook series bring up an interesting and relevant interplay, since in Book 3, differently from Book 2, properties previously learned and used to calculate numerical values of areas, volume and distances are recapped on and used in Book 3 to develop students' theoretical knowledge.

Roxo (1931) also discussed terms such as *definition, theorem, postulate, axiom, proof, logical reasoning.* He points out that the study of geometry and mathematics in general serves both practical and theoretical purposes, "hence the need to establish geometrical truths through rigorous logical reasoning" (p. 13). He presented a set of propositions to support the study of deductive geometry and highlights that some facts will be taken for granted while other facts must be

proven, so it is important that students are able to distinguish either case and the relevance and importance of the proof (ibid, p.14).

Roxo added that deductive teaching of geometry starts with intuitive ideas and concrete situations, and that should scaffold understanding definitions and deductive study. For Roxo, often times the simple visualization of a figure might lead to wrong conclusions and so it should be proven by experiential learning. An example of this is Figure 9 below, where segments AB and CD seem to be unequal, but actually have the same length.

Figure 9. An example of intuitive visual perception.

Roxo, Euclides, 1930, p. 36.

Another example of Roxo's dealing with visual perception and experiential learning is the exercise below. It should be noted that the proposition – the sum of the interior angles of a triangle results in 180º – was previously studied in Book 2 through numerical exercises and justified experimentally by having students use cut outs and properly position the angles of the triangle (Figure 10).

Figure 10. An example of experimentally justification.

Roxo, Euclides, 1930, p.38.

Nevertheless, propositions can be viewed as a theorem. Roxo (1930, p. 20) defined a theorem as *"any proposition we must prove.",* thus indicating a transition at the core of deductive teaching and learning because it reveals a change of epistemological status (Douady, 1991). In other words, this is the case when the statement is used as a *tool* for numerical calculation, as it bears a functional and practical application within the realm of intellectual activity. However, when presented as a theorem, it acquires the status of *object* of intellectual work, so the purpose becomes to develop students' ability to challenge and prove the validity of a proposition – or theorem.

But as the validating a theorem must abide to logical connections rules and requires a specific theoretical context, parallels must be set between deductive reasoning and argument building. In this sense, Roxo once again draws on Breslich (1985, p. 371): "It is possible to work out a detailed, definite program which will lead the pupil gradually from the method of direct observation through a period of informal reasoning to the stage of demonstrative geometry".

Currently, a number of researchers contend that a proof should meet the need for an explanation, but the explanation may vary according to its supporting arguments and to three critical considerations: the search for certainty, the search for understanding and the need for successful communication (Balacheff, 2010). Considering teaching how to construct the proof, academic validity should be addressed by providing students with meaningful mathematic activities even if such activities are not similar to those performed by professional mathematicians. Harel and Sowder (1998, p. 275 apud Balacheff, 2010, p. 130) argue that "one's proof scheme is idiosyncratic and may vary from field to field and even within mathematics itself. However, this view misses the social dimension of proof, which transcends an entirely subjective feeling of understanding." This transcendence allows the claim that a mathematics proof is "A collective knowledge which can be shared and sustainable without depending on its author(s) or circumstance(s)" (ibid, p. 131). However, collective knowledge depends on the language that is shared socially because "The level of the language will bind the level of the proof learners can produce and/or understand" (Ibid, p. 132).

The discussion above shows that Roxo's approach clearly relies on and culminates in students' ability to writing the proof text following a standard model. However, since mid 1950s, researchers have criticized this teaching approach on the basis that only the practice of logical mathematical thinking would entail students' understanding to properly select and sequence the propositions necessary for proving a theorem. We can never be sure students understand they must apply deductive reasoning rules and know how to do so only by following proof construction framework as argued by Roxo. According to Harel (2007) "proving within the *external proof schemes* class depends on a) an authority such as a teacher or a book, b) a strict appearance of the argument (for example, proof in geometry must have a two-column format)" (pp. 66–67). For Herbst, "this format set a standard for constructing and controlling proofs by both teachers and students" (2002, p. 298). And Hoyles & Healy claim that "This standard practice was to make formal deductive proof into a ritualized two-column format without regarding how it might link with the student's intuition of what might be a convincing argument" (2007, p. 82).

Conclusion

This paper has analysed an approach to plane geometry and a syllabus based on teaching students how to prove a theorem. Until the early 1930s no textbook had ever been published in Brazil about such theme. Although textbooks in the 1970s still applied a two-column format, currently in Brazil the study of plane geometry in elementary school does not follow a deductive approach. However, despite criticism, the deductive approach in plane geometry, as discussed herein concerning Roxo's unprecedented work, remains of utmost importance.

When contextualized within the time of their publishing, Roxo's innovative textbooks and approach show that deductive geometry was embraced heartily by state school education. They assumed and aimed to develop actively engaged students who would be able to construct proofs going beyond their traditional role of passively memorizing and simply reproducing theorem proofs. Additionally, the very role of textbooks changed as they aimed to teach deductive geometry. Roxo indeed contributed to the teaching of mathematics because he discussed the subject from both a teaching and a learning perspective and warned us about the need to develop and practice logical thinking based on supporting argumentative features through daily practice. This epistemological nature – the claim that students must understand that rules of logical inference must be based on mathematical justification – is argued to hold its effectiveness in contemporary mathematics.

Roxo proposed constructing proofs based on a strongly procedural feature of proving a theorem and writing the proof text. It should be noted, however, that new approaches to introduce the proof aimed to prevent proof construction from becoming a ritual devoid of meaning, due its educational importance in developing mathematical reasoning and in empowering students in the process of their own construction of mathematic knowledge. Even so, the challenge remains.

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Top-down: the Role of the *Classes Préparatoires aux Grandes Écoles* in the French teaching of descriptive geometry (1840–1910)

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Abstract

From the end of 18th century in France, there were two different higher educational systems: the faculties and the prestigious Grandes Écoles, *like the* École *polytechnique. Admissions to these two systems, even today, are different. To be admitted into the Grandes Écoles, there are competitive examinations, which are prepared in special Classes of* Lycées, *today named* Classes Préparatoires aux Grandes Écoles. *Until 1970, it was common to hear that the French mathematical curricula were built top-down, and that the curricula of the* Collèges *(11–15 years old) and the* Lycées *(15– 18 years old) were conceived as a part of the curriculum of the* École Polytechnique. *Our purpose is to examine this assertion by studying the special role played by the mathematics teachers of the* Classes Préparatoires aux Grandes Écoles *in the French mathematical system. Indeed they form a community of teachers who share the same training in the* École Normale Supérieure, *who have mathematical activities and who wrote textbooks. They form a strong social network around journals, like the* Nouvelles Annales de Mathématiques, *created in 1842, or the* Journal de mathématiques élémentaires, *created in 1877. For our purpose, it has been necessary to focus on a domain of mathematical teaching and to examine a sufficiently long period: we have chosen the teaching of descriptive geometry from 1850 to 1910.*

Introduction

My general purpose in this paper is to examine *the conditions and process of changes* in the teaching. I call *conditions of change* all the factors, which call for and favor new contents or methods in mathematical teaching, or which fight against them, like persons, places, social nets, journals and books. The *process of changes* correspond to the needs to link these factors to understand how new curricula are proposed and adopted, and how new conceptions of mathematics and

teaching lead to changes. I already investigated this problematic with the example of the teaching of conics in the period 1850–1960 (Barbin, 2012). In this present paper, I chose to study the teaching of descriptive geometry because it is the teaching *par excellence* of the École polytechnique (EP) (Belhoste, 2003, pp. 267–270) and also because it was propagated in a great part of the mathematical teaching all through one century.

In the table below I summarize the French educational system, from the upper grade of secondary schools (*Lycées*) to higher education (Prost, 1968), which is more or less the same for the period that we study. There were two different higher educational systems: the faculties (for the liberal professions) and the *Grandes Écoles* (for military and engineering professions).

Table 1. Educational system from secondary schools to higher education

Upper						
	*École polytechnique**	*École Normale supérieure***	*École Centrale des Arts et Métiers**	*Schools for military and civil engineers ** School for teachers of upper grades of *Lycées* *** Government Military schools		
	Examinations			*École militaire de Saint-Cyr****	*École navale de Brest****	Faculties
	Classes Préparatoires of state *Lycées* or private *Collèges*: *Classes de Mathématiques spéciales* (1809) *Classes de Mathématiques supérieures*					
	Baccalauréat			Examinations		
Lower	*Classes de Mathématiques élémentaires* of *Lycées* or private *Collèges* (secondary schools)					

Competitive examinations for entrance to *Grandes Écoles* (Belhoste, 2003, pp. 54–56) have to select the best students, independently of their social origins, nevertheless it led also to an elitist system (Bourdieu, 1981) (Belhoste, 2012). The examinations are prepared in special school forms, the *Classes Préparatoires aux Grandes Écoles* (CP), which are installed in two parallel establishments: the state *Lycées* and the private *Collèges* (Belhoste 2001). The upper grade of the *Lycées* before the Baccalauréat is the *Classe de mathématiques élémentaires*.

The examiners for entrance into the EP were persons who used to be former students of the EP. After their proposal by the *Conseil de perfectionnement* of the EP, they were appointed every year (for three years after 1852). This *Conseil* published a programme for the examination every year. In the middle of the century, the *École normale supérieure* (ENS) (sciences), the *École centrale des arts et métiers* (ECAM) and the Faculties adopted the same programme.

From the *École polytechnique* to the *Classes Préparatoires* (1843-1863)

With his descriptive geometry, Gaspard Monge created a science for representing the figures of space on a plane and an art useful for engineers

(Sakarovitch, 1994). The first professors (then called *instructeurs*) on descriptive geometry at the EP were Monge and Jean Nicolas Pierre Hachette from 1794 to 1816, then François Arago, Charles-François Leroy and Félix Savary (*Annuaire*, 1837, p. 111). After the first *Leçons* of Monge (1798) and of Lacroix (1795), many treatises of descriptive geometry were edited for the students of EP. The *Traité de géométrie descriptive* (1819) was written by Louis Léger Vallée, who was a former EP student and who dedicated his book to Monge, the *Traité de géométrie descriptive comprenant les applications de cette géométrie aux ombres, à la perspective et à la stéréométrie* (1822) by Hachette, whose book contained an historical account of Monge's geometry, and the *Traité de géométrie descriptive* (1830) by Étienne-Louis Lefébure de Fourcy who was a former EP student, tutor and examiner at the EP, and teacher at the Lycée Saint-Louis in this period. All these textbooks refer to Monge.

Rapidly, a new public for the teaching of descriptive geometry had been formed by the candidates to the examinations for the EP and for Government schools (*École militaire de Saint-Cyr, École navale de Brest*). Textbooks were edited for them, like the *Notions élémentaires de géométrie descriptive exigées pour l'admission aux diverses écoles du gouvernement* (1838) of M. F. Amadieu, who was a former *École militaire de Saint-Cyr* student and teacher at the Lycée of Versailles, near Paris. Another important text is the *Traité élémentaire de géométrie descriptive* (1850) of Henri Charette de Lafrémoire and Eugène Catalan. Both were former EP students and Catalan was teacher at the *Classe Préparatoire* of the Lycée Saint-Louis of Paris in 1850.

The *Cours de Géométrie descriptive* of Théodore Olivier (1843)

My history will begin with the textbook of Théodore Olivier, because it was the first occasion of a disagreement between some professors at EP and those who prepared students for the examination. Olivier was a former student and EP examiner and one of the creators in 1829 of the private *École Centrale des Arts et Métiers* (ECAM), founded to form civil engineers for industries. His textbook was written for the students of this school. It is interesting to remark that he wrote many papers on descriptive geometry. Olivier explained in the preface that he did not want to write a treatise but only a « Lesson » containing what is essential to become engineer: "by writing a Lesson, I could content myself with expounding my ideas and research on this science, while I gave all that is essential to those studies with the purpose of becoming an engineer". In the beginning of the textbook, he introduced a systematic method to decompose the teaching, which is the method of « point, line, plane »:

> As soon as we know how to represent a point, a line and a plane by the method of projections, and to solve, by the method of projections, the various problems which can be proposed on the point, the line and the plane, we will know descriptive geometry; in the sense that we will know all we need to apply the

method of projections to research the geometrical truths which allow us to prove results about figurate space; because we have to recognize that each method is more specially applied to a particular kind of question. (Olivier, 1843, p. vi, tr. E. B.).

This teaching method consists of coming from the simplest acts to the others. In chapter II, Olivier introduced the « changes of planes of projection and rotations of figures around an axis ». He justified them by writing:

In descriptive geometry, a figure drawn on the planes of projection can be very complicated, some difficulties can result from the position of the planes of projection in relation to the spatial figure that we have to represent; these last will disappear by a suitable choice of planes of projection; we also can keep the same planes and change the position of the figure, this last operation is always made by turning the figure around on an axis (Olivier, 1843, pp. 18–19, tr. E. B.).

These changes are the fundamental problems, as he wrote, and they have to be used systematically to simplify a representation. It is a method. Olivier showed how to change the vertical plane in relation to a point, to change the planes of projection in relation to a line and to lead a plane in a parallel position to one plane of projection, etc.

We learn from his Preface of the book's second edition of 1852 that this method of changes received criticisms from teachers: « when this book was edited in 1843, it was severely criticized by the teachers whose habits are upset, however little by little it became recognized that I could be right » (Olivier, 1852, p. x). He defended his method as a principle to solve problems about the point, the line and the plane:

Accordingly to these views, I had to present the principles of all the systems of projection in the First part; I had to make known some particular procedure of descriptive geometry to solve proposed problems, as changes of planes of projection; movement of rotation of a point, of a line, of a plane around an axis; transformation of a figure to another one, etc (Olivier, 1852, p. ix, tr. E. B.).

The controversy on « Olivier's method »

Indeed, the method was criticized by Émile Martelet, a teacher at the same school as Olivier who annotated in the fourth edition of the *Traité de géométrie descriptive* of Charles-François Leroy (1855). He considered that Olivier's method led to more complicated constructions and he wrote in a « Note on the changes of planes of projection and on the movements of rotation » that:

[The changes] lead to more complicated constructions than those which had been honored by the teaching of Monge and are adopted generally. But if their use lacked simplicity in practice, it has, as an exercise, the advantage of habituating the new students to read space easily, and it is for this reason that, as

> an indication of the advantage that we can obtain from this point of view, we gave some examples here (Leroy & Martelet, 1855, p. 394, tr. E. B.).

So, he admitted that the changes of planes could be good, but only as a subject for exercises. Another criticism came from a professor at EP, Jules de la Gournerie, in his *Traité de géométrie descriptive* (1860). In three pages, he attacked « Olivier's method » by explaining that the method is not suitable to applications. He added that the method is not new, that Abraham Bosse used it in 1643, and that his book was not approved. Moreover, Monge did not give any example of that in his drawings. He wrote: « I devoted one paragraph to the changes of planes of projection, because the procedure can be useful, and because knowing it is requested in the official programme » (Gournerie, 1860, p. viii). Indeed, Olivier's method appeared in the programme for the examination of entrance in Polytechnic in 1850, and this fact had been the object of a controversy.

In the second edition of his *Traité Élémentaire de géométrie descriptive* (1852), de Lafrémoire added an appendix on changes of planes of projection, where he wrote: "these last years, the auxiliary projections were recommended over much. It was believed that we had to recourse to their use in every circumstance, and in all the problems of descriptive geometry [...] A new thing needed a new name: the Method of planes of projection was invented!" (Lafrémoire & Catalan, 1852, p. 120). He said that it is true that the « method » was added in the programme for entrance in Polytechnic, but also that Olivier was a member of the Commission. We have to remark that in the same year 1852, it was decided to forbid anyone to be both examiner and author of textbooks, like it was the case with Olivier.

But in the same years, Olivier's method was promoted and adopted by many other authors of textbooks, all entitled « Elements of descriptive geometry » and intended, not for EP students, but for candidates for entrance to Polytechnic and to *Grandes Écoles*. It is the case for the *Traité élémentaire de géométrie descriptive rédigé d'après les ouvrages et les leçons de Th. Olivier* (1852) written by Henri Edouard Tresca, who was a former EP student and teacher of Mechanics in the *Conservatoire des arts et métiers* (Tresca, 1852), the *Éléments de géométrie descriptive à l'usage des aspirants aux écoles du gouvernement* (1850) written by Camille-Christophe Gerono and Eugène Cassanac (Gerono & Cassanac, 1850), the *Éléments de géométrie descriptive* (1850) written by Jacques Babinet, a former EP student and an engineer (Babinet, 1850).

From the *Classes Préparatoires* to the secondary schools (1853-1867)

We now examine the process by which descriptive geometry entered the *Classe de Mathématiques élémentaires* of the secondary school. The textbook of

descriptive geometry of Antoine Amiot is very interesting for this purpose. This textbook was edited many times from 1853 to 1863. Amiot was a former ENS student and teacher of the *Classes Préparatoires* of the Lycée Saint-Louis in Paris.

The *Leçons nouvelles de géométrie descriptive* of Amiot (1853)

Indeed, the « New lessons on descriptive geometry» were « new ». Firstly because they were intended for a new public, formed by the examination candidates for the EP, and also the ENS. That meant that now the future secondary teachers would be taught descriptive geometry. Secondly, the book was organized in short chapters, which can be delimited as lessons. It is new by its format: a little textbook of 290 pages only.

The textbook followed Olivier's textbook with five chapters on projections of the point, projections of the line, drawings of the plane, construction of drawings of a plane accordingly with given data and on the transformation of projections ». Amiot referred to the Olivier's changes of planes and called them the « transformations of projections ». He wrote:

> In every problem of descriptive geometry, the position of data in relation with the planes of projection can have a great influence on the simplicity of the solution. So, when we want to solve a question, we have to begin by researching, with regard to the planes of projection, a position of data which leads to an immediate solution, and when this position is found, to come down to the data of the question by a movement of the figure or by the change of planes of projection (Amiot, 1853, pp. 25–26).

The chapter on « transformations of projection » contains the changes of Olivier, and all the textbook used transformations systematically. Amiot introduced a special symbolism to help students: an index gave the number of transformations of a point. The « transformations » became a new subject of teaching, interesting in itself.

Descriptive geometry became a part of teaching in the *Classes de Mathématiques élémentaires* in 1867, these school forms prepared the students to the *scientific baccalauréat* and for the entrance to the St-Cyr Military and Brest Naval schools. The topics of the programme were: Projection of a point, of a line; Plane of projection; Drawing a line; Length of a line when the projections are known; Angles of a line, of a plane with the plans of projection; Method of *rabattements* (that means the process of rotation of the image plane by which an image of a point coincides with this point); Intersection of two planes, of a plane and a line; Distance between a point and a plane; Angle of two lines, of a line and a plane, of two plans; Projections of prisms, pyramids, cylinders, cones; Plane intersections of polyhedra; Notions on the method of quotes.

It is interesting to remark that the curriculum is a part of Amiot's lessons that were devoted to future teachers (to ENS students). As often happens, a book for teachers became a book for their future students. This textbook

constituted two conditions of changing teaching: it was a tool for teacher's training and a coherent model for a new curriculum.

An example is the *Éléments de géométrie descriptive pour les élèves de mathématiques élémentaires et les candidats au baccalauréat* of Charles Briot and Charles Vacquant (first ed. 1863), two former ENS students, teachers of *Classes Préparatoires* of the Lycée Saint-Louis and Lycée Henri IV. Chapter V on « Methods in descriptive geometry » contained three parts: rabattements, rotations, changes of the planes (that means Olivier's method). The chapter treated the case where the drawing needs two changes. The last chapters extended the notion of projection to a circle, a sphere, to some solids and to a general curve. It was proven that the projection of a circle on any plan is an ellipse.

Here, the projection became a subject by itself in teaching and problems, far away of the original problem of descriptive geometry. Many problems given for the entrance to Saint-Cyr concerned the notion of projection and its geometrical consequences. After seeing this textbook, we can understand how some authors proposed introducing notions of descriptive geometry in the teaching of geometry.

Notions of descriptive geometry for secondary teaching of Geometry

The *Traité de géométrie élémentaire* (1866) of Rouché and Comberousse is not a textbook, it is a very good example of a book of initiation [Barbin, 2013]. It was edited three years after the first edition of Briot and Vacquant's textbook. The authors did not want to restrict them to the official curriculum. The treatise contained historical notes, where they referred to Monge, but also to the theory of projections of Poncelet. The first edition (1866) is composed of 776 pages in one volume. No defined audience was exactly given, it could be composed of teachers and students of mathematics. Eugène Rouché was a former EP student. At this time, he was a teacher of the *Classes Préparatoires* at the Lycée Charlemagne, and then at the ECAM. In 1872, he wrote supplements for the third edition of the Olivier's textbook, published forty years after the first edition. This very rare case of a new edition of a textbook shows the importance given to this textbook for the teachers of *Classes Préparatoires*. At this period, Charles-Jules Félix de Comberousse was teacher at the Collège Chaptal. Both were EP tutors. My paper (Barbin, 2012) describes the important role of the treatise of Rouché and Comberousse for the teaching of conics.

The treatise introduced the notion of projection in the teaching of the geometry in the space, as an important geometrical object. Indeed, the definition of a projection is given to introduce the notion of angle of a plane and a line and the shorter distance of two lines. They called « projection of a point A on a plane P the foot of the perpendicular dropped from this point to the plane". They explained that « the projection of a line $ABC...$ on a plane P is

the locus of projections *a, b, c,* ... of the various points of the line » and they proved the theorem which stated that « the projection of a straight line *AB* on a plane is a straight line » (Rouché & Comberousse, 1866, p. 347). The presence of this theorem shows that projection was considered as an abstract notion, which needs proving that it could be considered as perfectly obvious.

After having given the theorems on projection, the authors extended the notions of projection and perspective by considering many kinds of projections. The curves and usual surfaces are defined and studied in the part on geometry of the space. For instance, it is proven that the intersection or a circular cone by a plane is an ellipse, a hyperbola or a parabola. Now, the notion of projection became central in a textbook of geometry.

The network of the teachers of the *Classes Préparatoires*

My purpose is to analyze the role of the network (Lemercier, 2005) of the teachers of the *Classes Préparatoires* and the relations between this network and other actors, by analyzing the group of the authors of descriptive geometry over a long period. In the tables 2, 3, 4 below I give a (non exhaustive) list of textbooks on descriptive geometry from Monge's teaching to the years 1900. For each textbook, I indicate the school where the author(s) had been formed: *École polytechnique* (EP), *École centrale des arts et Métiers* (ECAM) and *École normale supérieure* (ENS). Then, I give the functions of the authors (in the period when they wrote the book): teacher at EP, ECAM, ENS, *Classes Préparatoires* of Parisian *Lycées* or *Collèges*, many of them worked in the Lycée Saint-Louis. Finally, I present the intended audience of the textbook: students of the *Grandes Écoles*, students who prepared the admission to the *Grandes Écoles* (EP, ECAM, ENS), the admission to the Military schools (EGV), students of *Classes Préparatoires*, *Classes of Mathématiques élémentaires* and 1st grade of secondary teaching.

The authors of textbooks on descriptive geometry (1843-1907)

Until 1840, most of the authors were former EP students and taught, their readers were the students of these schools. Only some of them taught in *Classes Préparatoires*, their public were the EP students and later, the candidates to the examination for the *Grandes Écoles*. In this period, the teaching of descriptive geometry is the affair of the EP. Then, we can distinguish two great periods.

In the first period from 1843 until 1877 (table 2), many authors were former ENS students, they taught in some *Classes Préparatoires* of Parisian *lycées* or private schools (Lycée St Louis, Collège Stanislas) and their public was the students of the *Classes préparatoires* and the *Classes de Mathématiques élémentaires*. We note that, during the years 1850–1900, there were around fifteen *Lycées* around Paris and forty *Lycées* in provinces with *Classes Préparatoires*. In this

period, the textbooks of descriptive geometry were the affair of the teachers of *Classes Préparatoires*, more and more linked with the ENS.

Table 2. Textbooks on descriptive geometry (1843–1877)

1rst ed.	Title	Author(s)	Form.	Functions	Student audience
1843	*Cours de géométrie descriptive*	Théodore Olivier	EP	ECAM	ECAM
1843	*Compléments de géométrie descriptive*	Théodore Olivier	EP	ECAM	ECAM
1850	*Traité élémentaire de géométrie descriptive*	H. C. de Lafrémoire Eugène Catalan	EP EP	PC St-Louis PC Charlemagne	Entrance EP
1852	*Traité élémentaire de géométrie descriptive d'après (...) Th. Olivier*	Henri Edouard Tresca	EP	CAM	Entrance EP
1853	*Leçons nouvelles de géométrie descriptive*	Antoine Amiot		PC St-Louis PC Chaptal	*Classes Préparatoires*
1855 (4e)	*Traité de géométrie descriptive*	C.-F.-A. Leroy M.E. Martelet	EP ENS EP	ECAM	EP
1860	*Traité de géométrie descriptive*	Jules de la Gournerie	EP	EP et ECAM	EP
1863	*Éléments de géométrie descriptive à l'usage des élèves (…)*	Charles Briot Charles Vacquant	ENS	PC St-Louis MCF ENS PC St-Louis	*Mathématiques élémentaires*
1869	*Éléments de géométrie descriptive suivis de notions de géométrie cotée*	Jules Dufailly		PC Collège Stanislas	*Mathématiques élémentaires* Entrance EGV
1877	*Éléments de géométrie descriptive*	F. J. C. Frère Gabriel-Marie		Frères de la doctrine chrétienne	*Mathématiques élémentaires*

In the second period (table 3), most of the authors were former ENS students. They were teachers of *Classes Préparatoires*, mainly of the Lycée Saint-Louis. In this period, teaching of descriptive geometry was open to the first grade of secondary teaching to prepare the students for the *Classes de Mathématiques élémentaires*, to enter the Preparatory Schools or in Military schools. Now, descriptive geometry became an important subject, and not only for students of *Grandes Écoles*. This period corresponds to the Third Republic in France, which is important for the ENS (Smith, 1982).

The two periods correspond to two steps in the changing of teaching Descriptive geometry that we can read not only in the curriculum, but also in factors concerning the authors of textbooks and specially the network of *Classes Préparatoires* teachers. This network constitutes a very small Parisian world, concentrated more or less in the fifth « arrondissement » (district) of Paris, in around 2,5 km². Here, there are the EP, ENS, and the *Lycées* and *Collèges* where the authors worked, but also all the new editors of textbooks, who become more numerous to edit textbooks for students of scientific school forms, like Masson, Delagrave or Vuibert.

Table 3. Textbooks on descriptive geometry (1880–1907)

1rst ed.	Title	Author(s)	Form.	Functions	Student audience
1880	*Traité de géométrie descriptive et de géométrie côtés*	Ernest Lebon		PC Charlemagne	*Mathématiques élémentaires*
1881	*Cours de géométrie descriptive de l'École polytechnique comprenant les éléments de la géométrie cinématique*	Amédée Mannheim	EP	EP	EP
1882	*Traité de géométrie descriptive*	Adrien Javary	EP	PC St-Louis PC Louis-le Grand	Entrance EP, ENS, ECAM,
1883	*Questions de géométrie descriptive*	Étienne Jurish			Entrance EP, ECAM
1891	*Cours de géométrie descriptive : à l'usage des candidats à l'école spéciale militaire*	Ch. Brisse (2e) 1900 C.Bourlet	ENS	PC St-Louis	Entrance Military school StCyr
1901	*Cours de géométrie descriptive*	Joseph Caron		PC St-Louis	Baccalauréat Entrance EGV
1902	*Géométrie descriptive et géométrie cotée*	E. Schlesser	ENS		*Mathématiques élémentaires* Entrance EGV
1897	*Cours de géométrie descriptive*	Xavier Antomari	ENS	ENS	Entrance EP, ENS, ECAM
1905?	*Éléments de géométrie descriptive*	Henri Ferval	ENS	PC St-Louis Lycée Brest	Baccalauréat Entrance EGV
1907	*Cours de géométrie descriptive pour l'enseignement secondaire*	C. Roubaudi		PC St-Louis PC Louis-le Grand	*Mathématiques élémentaires* Entrance EGV

Journals for *Classes Préparatoires*

From the middle of the 19th century, many journals were created, intended to students who prepared for entrance to the EP and ENS. Most of them are created by teachers of *Classes Préparatoires* of Paris, like Camille Gerono, Justin Bourget, Albert Gohierre de Longchamps. One of their purposes was to propagate new mathematics and new methods, not only for the students but also for their teachers, in particular teachers in the province. We give two examples.

The first is the *Journal de mathématiques élémentaires (et spéciales)*, presented as the « Journal for the candidates of the Schools of Government and to Baccalauréat » and created in 1877 by Justin Bourget, who was teacher at the Collège Sainte-Barbe. In the first issue, Bourget wrote that "the numerous teachers who live in provinces, need to know the questions which are proposed in the examinations for the various schools, in the administrative examinations, in the sessions of *baccalauréat es sciences* of various universities" (Bourget, 1877, p. 3). In this issue, teachers can find new subjects like the determinants, the involution and the theory of inversion. The papers were written by teachers at the Lycée Saint-Louis and by teachers at the private Collège Sainte-Barbe. In 1882, this journal was replaced by the *Journal de mathématiques élémentaires et*

Journal de mathématiques spéciales, directed by Albert Gohierre de Longchamps, who was a former ENS student, teacher of *Classes Préparatoires* in the Lycée Charlemagne and examiner for the *École de St-Cyr*.

The second example is the *Nouvelles Annales de mathématiques*, presented as the « Journal of the candidates to the Polytechnic and Normal schools » (Verdier, 2009). It was created in 1842 by Orly Terquem et Camille Gerono, who were both teachers at the Collège Sainte-Barbe, and it was edited until 1927.

The journals are a good vehicle to move ideas between teachers of *Classes Préparatoires*, and to propagate new methods among secondary schools teachers. In this manner, they constituted the condition for change, which is the training of teachers. An example concerning descriptive geometry is a paper of Amédée Mannheim published in the *Annales de mathématiques* of 1882. He was professor at EP and wrote a *Cours de géométrie descriptive de l'École polytechnique* in 1881, where he proposed to make the drawing without the ground line (Mannheim, 1881). He presented his new views to teachers of Preparatory schools in a paper entitled « First elements of descriptive geometry » of the *Annales*, because "they allowed the students to prepare their applications better" (Mannheim, 1882, p. 385). Indeed, Ernest Lebon adopted Mannheim's method in the third edition of his *Traité de géométrie descriptive* intended to the secondary teaching (Lebon, 1901). He was a teacher at the Lycée Charlemagne. The journals created a diving belt between the teaching in EP and the secondary teaching.

What is the place of descriptive geometry in the *Annales de mathématiques*? In 1893, Charles-Ange Laisant edited a textbook entitled *Géométrie (et géométrie descriptive)* which is a « Collection of mathematical problems for *Classes de Mathématiques élémentaires*", they are taken from the *Annales de mathématiques* from 1842 to 1892 and classified by him. He was a former EP student and of the School of Metz. He became professor at the EP in this year 1893. He founded the journal *L'Enseignement Mathématique* in 1899 with Henri Fehr. In the book, he remarked that the problems on descriptive geometry are not numerous and he noted that the problems could not ask for a drawing work in an elementary teaching of descriptive geometry. I will come back later on this remark.

The changes at the turn of the century

At the end of the 19th century many criticisms were addressed to the teaching and the curriculum in *Classes Préparatoires* where « Nothing changed since 1852! ». It follows that many changes were accomplished in the beginning of the 20th century, concerning the curriculum for the entrance to EP in October 1902, the examinations to enter the ECAM in 1903 and to the *Grandes Écoles*, with the creation of the inter-ministerial Commission of the *Grandes Écoles* in 1904. In this same period, an important reform of the curriculum of the secondary schools was put in place in the years 1902–1905.

The criticisms against the teaching in *Classes Préparatoires*

In 1899 Hermann Laurent wrote « Considerations on the teaching of mathematics in the special classes in France » in the new journal *L'Enseignement Mathématique* (Laurent, 1899). He was tutor of calculus and examiner at the EP. In his paper, he criticized the situation in teaching mathematics by explaining that the teaching of mathematics has to be utilitarian, that the teachers' training is begun too theoretical and that the curricula are too heavy. He mentioned the teaching of descriptive geometry specially:

> We should not forget that the purpose of this science, or better this art [is] to indicate on paper the constructions that workers, who have elementary knowledge, will have to make in space. So it is not necessary to introduce the properties of the surfaces of the second order (Laurent, 1899, p. 43, tr. E. B.).

The new programme for entrance to the EP is the object of the comments of Charles-Ange Laisant in a paper published also in *L'Enseignement Mathématique* in 1903: "we also have to react against the invasion of certain theories, which took a place out of proportion with their importance, and to substitute to them more useful notions" (Laisant, 1903, p. 78). Further, he pointed out the perverse effects of examinations, the process by which the teaching is only intended to solve some exercises, always the same exercises.

> The examiners, constrained in a very short time to judge a great number of candidates, have a tendency to come again to some questions and some exercises, with a visible preference. The teachers […] sometimes attribute to them an importance that they have not in the beginning, lengthen their lessons consequently, adapt theories which are far from the letter and the spirit of the curriculum (Laisant, 1903, p. 83, tr. E. B.).

The tension between *professor*s of *Grandes Écoles* and teachers of *Classes Préparatoires* was flagrant in a paper of 1904, « The Reform of the programme for the entrance in *Grandes Écoles* », published in *L'Enseignement Mathématique*. It is a report written by Paul Appell, as president of the inter-ministerial Commission in charge of proposing new programme. He was a « product » of the University, he was a former ENS student and of the Faculty of sciences of Paris, and a *professor* of the Faculty of sciences of Paris from 1885. He wrote that " [The inter-ministerial Committee] strove to give the scientific instrument to the students, which is essential for the applications: all the rigorous and systematic developments concerning the principles are discarded" (Appell, 1904, pp. 485–486). That means that, if in the beginning, the teaching of *Classes Préparatoires* depended of the curriculum of *Grandes Écoles*, little by little, the inverse was true, now, this curriculum depends of the teaching of *Classes Préparatoires*. The *professor*s of University, as those of EP, promoted less theoretical teaching, and they used the journal *L'Enseignement Mathématique* to circulate their ideas.

The changes in curricula of geometry and descriptive geometry in secondary teaching

The University had an important role in the Reform of secondary teaching in the years 1902–1905 (Belhoste, 1990, p. 390). Four of the main actors of this Reform were Gaston Darboux, Paul Appell, Jules Tannery and Émile Borel. Both were former ENS students and then taught at the Faculty of sciences of Paris. In the beginning of the century, the ENS became a leading institution and formed prestigious university professors (Zeldin, 1967).

Charles Méray was an inspirer of the reform of 1902–1905, he was a former student of the ENS and became professor at the Faculty of Dijon. It is interesting to remark that in his *Nouveaux éléments de géométrie* of 1874 and in his « Considerations on mathematical teaching » (1892), he considered that descriptive geometry has to play an important role the teaching of geometry.

Indeed, the teaching of geometry and descriptive geometry became very close with the reform of 1902–1905. On one hand, the importance of drawings was mentioned in the teaching of geometry in all the grades: « the importance of practical exercises, as land survey, working drawings; it is only under the situation to make them numerous that the students will remember descriptive geometry and will like it » (notes, p. 496). Projections and the perspective entered the curriculum of geometry, and so, the teaching of geometry contained many notions of descriptive geometry, like space, projections, movements, translation, rotation. An example is the 8th edition of the *Cours de géométrie à l'usage des élèves de mathématiques élémentaires avec des compléments destinés aux candidats à l'École normale et à l'École polytechnique* (1909). The authors, Charles Vacquant and Antonin Macé de Lépinay, were former ENS students and teachers of *Classes Préparatoires* of the Lycée Saint-Louis and the Lycée Henri IV. Charles Vacquant was the author of a textbook on descriptive geometry (first ed. 1863) and he continued with a textbook on geometry in the beginning of the 20th century. The fact that this last textbook concerned also the candidates to entrance to *Grandes Écoles* was a way of exceeding the official curriculum.

On the other hand, the teaching of descriptive geometry began in the first grade (one year before the *Classes de Mathématiques élémentaires*) and it contained the representations of point, line and plane on two planes of projection, the *rabattement* of a plane on the horizontal plan and the change of the vertical plane. The teaching in the *Classes de Mathématiques élémentaires* contained the *rabattements*, the change of a plane of projection and the rotation around a perpendicular axis of a plane of projection (a remainder of Olivier's method), the projections of a circle, the intersection of a sphere and a plane and the intersection of a cone and a plane.

The *Cours de géométrie descriptive* of Claude Roubaudi (1907) is an interesting example of the spirit of the new curriculum. The author was teacher at the Lycées Saint-Louis and Louis-Le-Grand, his book was intended for students of the First grade and of the *Classes de Mathématiques élémentaires*. The part of the

textbook devoted for the First grade began with the « Notions on projections ». It contained the conical, cylindrical and orthogonal projections and the main properties of these projections. The first chapter of the part for the *Classes de Mathématiques élémentaires* treated changes of planes, rotations and *rabattements* as problems on projections on a plane. Roubaudi called all these various changes « displacement ». The chapter on round bodies began with the projections of a curve and a circle. It was proven that « the tangent to the projection of a curve is the projection of the tangent to this curve at the corresponding point » (Roubaudi, 1907, p.32). In his *Cours*, the applications of projections were the main subject of the problems. That means that, now, the notion of projection is central in the teaching of both geometry and descriptive geometry.

As a conclusion, many notions and results of descriptive geometry went from upper to lower grades, throughout a century: taught in the years 1850 in *Classes Préparatoires*, then after 1867, taught in *Classes de Mathématiques élémentaires*, and they are now taught in the First grade of *Lycées*. We have to remark that, in the beginning of the 20th century, the authors of the textbooks for Secondary school remained as teachers of Preparatory schools for a great part, and very often they taught in the best *Lycées* of Paris. Briot, Vacquant, Roubaudi taught in the same Lycée Saint-Louis.

Conclusion

The teachers of the *Classes préparatoires* played a major role in the changing of the teaching of descriptive geometry, from the years 1850 to the years 1910. It is clear that they occupied a strategic place between the *Lycées* and the *Grandes Écoles*, but the detailed historical study brings an interesting understanding of the process and character of the change itself. Firstly, there is a general movement to modify the subject: to pass from a teaching of notions to a teaching of methods. It is the sense of the success of the methods promoted or introduced by Olivier in 1843: his method of point, line, plane and his method of changes of plane. With Amiot (1853), Briot and Vacquant (1863), these methods became themselves the subject of the teaching and the main purpose of descriptive geometry disappeared. Indeed the proper role of the descriptive method was to represent objects of the space on a plane, and not to know how it would be possible to make it easy.

The interest for the methods is linked to the examinations, for which the teachers have to prepare their pupils. Indeed, the time to make a complete drawing work is long, while the exercises need less time. Consequently, it is better to replace descriptive drawings by geometrical exercises, but the teachers of *Classes Préparatoires* prepared for examinations and the teachers of *Lycées* wanted to prepare their students for the upper grades.

The role of the *Classes Préparatoires* is reinforced by the existence of a strong network of the teachers of these school forms, given a kernel concentrated at

some Parisian *Lycées*, given their foregoing socialization as former ENS students and thanks to journals, which propagated ideas and conceptions. The fact that they were the authors of textbooks is considerable, because these textbooks have an effect, as well on the higher education, that of the *Grandes Écoles*, as on teaching at the level of secondary schools.

The teachers of *Classes Préparatoires* also play a major role in the passage of notions of descriptive geometry into the teaching of geometry, while, in the beginning, they are the only to teach all the geometries. The notion of projection became central in geometry teaching with the reform of 1902–1905, then it was enlarged by the notion of transformation between 1925 and 1962. At each step, the teachings of descriptive geometry and geometry became closer. Finally, in 1962, descriptive geometry became a part of the teaching of geometry in the curriculum of the *Classes de Mathématiques élémentaires* and, in 1966, disappeared from the *Lycées*. It remained a subject of teaching in *Classes Préparatoires* until it disappeared from the programme of the entrance examination to *Grandes Écoles* and schools of engineers in 1970.

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Mathematics education in twentieth century Iceland – Ólafur Daníelsson's impact

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Abstract

Ólafur Daníelsson was a towering figure in mathematics teaching in Iceland during the first three quarters of the 20th century. His position as textbook writer, the mathematics teacher of the first 167 primary teachers, and the teacher of the first 20 cohorts of high-school mathematics-stream graduates, contributed to his superior personal influence. Furthermore, official decisions, such as a limited access to high-school education and later strict entrance examination during 30 years, where his textbooks were a required reading, made his influence last long after his retirement. In parallel with his dedicated work at building up secondary mathematics education, Daníelsson managed to keep up a scientific career, present papers at international congresses and have them published in distinguished mathematics journals.

Introduction

The first Icelandic mathematician to complete a doctoral degree was Ólafur Daníelsson (1877–1957). At the beginning of his career in 1904, the year that Iceland was granted home-rule from Denmark, there were few opportunities for a mathematician in Iceland. By the end of his career in 1941, however, Daníelsson had become an undisputed leader of mathematics education in Iceland, having trained the first primary school teachers and written influential textbooks. Daníelsson's history is the story of a strong impact of a single individual, who established mathematics education policy in the power of his education and authority. He managed also to maintain an admirable scientific profile. We shall investigate the political circumstances and the professional channels that enabled him this.

The purpose of this paper is twofold:

- to analyse the reasons for Daníelson's enormous impact on Icelandic mathematics education lasting until the 1970s, and

- to investigate the professional circumstances of a mathematician at the outskirt of Europe in the early 1900s.

The research method is historical, i.e. a careful analysis of a range of documents. The history is told within the framework of the history of education and schools, and the general history of Iceland by referring to scholars' published works, legislation, regulations, reports and articles. Daníelsson's textbooks were analysed, their forewords as well as their mathematical content. Information about their lifetime was sought in school reports and unprinted reports on the national examination during 1946–1976 (*Skýrsla um landspróf miðskóla*). Biographical information was accessed in biographical lexicons, such as *Kennaratalið* (Kristjánsson, 1958–65). A booklet about Ólafur Daníelsson (Arnlaugsson and Helgason, 1996) is a source referred to extensively in this paper.

Background

Iceland at the turn of the 20th century

Iceland, a remote island in the North-Atlantic, was a tributary to Denmark from the fourteenth century. When the large mountainous country gained home-rule in 1904 it was nearly devoid of roads, bridges, primary schools and other products of the industrial revolution. The population was 78,000. Since 1802, there was only one Latin School, belonging to the Danish educational system. In 1877, it was too small to split into a language-history stream and a mathematics-science stream, so the language-history stream was chosen for political reasons. In the late 1800s, discussions arose whether engineering was relevant for progress in Iceland, but a lack of mathematics stream made it necessary for the students to study one extra year in Copenhagen in preparation for engineering studies. The Latin school turned into the six-year Reykjavík High School (*gymnasium*) in the Danish educational system in 1904. Iceland gained sovereignty in 1918 (Bjarnadóttir, 2006b, pp. 110–120, 131–139).

Ólafur Dan Daníelsson studies and early life

Ólafur Dan Daníelsson, a farmer's son, studied at the Latin School in Reykjavík during 1891–1897. A relative offered to provide him with financial support if he were to study engineering. However, he chose mathematics and went in 1897 to study in Copenhagen where he first had to complete the high school mathematics-science stream. In 1900, Daníelsson published his first scientific paper in the Danish journal *Nyt Tidsskrift for Matematik B*, and in 1901, he earned a gold medal for a mathematical treatise at the University of

Copenhagen (Helgason, 1996). Daníelsson completed a Mag.Scient-degree at the University of Copenhagen in 1904 to be eligible to teach in the Danish high school system. His university professors were geometers Hieronymus Georg Zeuthen (1839–1920) and Julius Petersen (1839–1910), a well-known textbook writer. Upon returning home in 1904, Daníelsson applied for the position of mathematics teacher at the Reykjavík High School, as his former teacher died that same year. However, the position went to the first Icelandic fully-educated engineer who had served in the post of National Engineer, planning the first roads and bridges in the country, but wished then for more comfortable job.
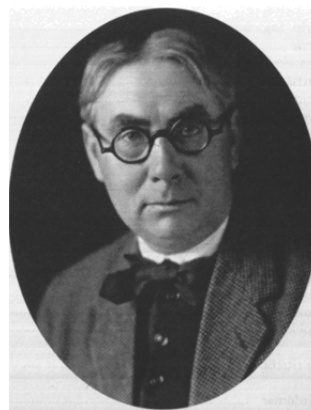
Figure 1. Ólafur Daníelsson

Daníelsson began then to prepare his doctoral thesis, which he defended in 1909: Nogle Bemærkninger om algebraiske Flader der kunne bringes til at svare entydigt til en Plan Punkt for Punkt [Several remarks on algebraic surfaces which could have one-to-one correspondence to a plane]. The thesis was an extension of earlier works by Zeuthen, and other well-known European mathematicians: Rudolph Clebsch, Guido Castelnuovo and Luigi Cremona (Helgason, 1996). Meanwhile, Daníelsson offered private lessons and composed a brief arithmetic textbook, Reikningsbók [Arithmetic], that was very elementary but with a dash of demanding problems, published in 1906 (Arnlaugsson, 1996).

Schooling in Iceland in the early 1900s

School legislation on primary schools in towns and villages and itinerary schools in the country side was enacted in 1907. A number of simple textbooks were published and their quality discussed in teacher journals. A teacher college was established in 1908 and Daníelsson was appointed as its mathematics teacher, serving until 1920. The students were mature people who might have been teaching for a number of years but who had not enjoyed much schooling themselves. The first 167 primary school teachers graduated under Daníelsson's supervision in mathematics (*Skýrsla um Kennaraskólann í Reykjavík*, 1908–1920). He revised his textbook, *Arithmetic*, for their needs and republished it in 1914.

The price of textbooks was an important factor (Bjarnadóttir, 2009b) and became even more important during the Great Depression in the 1930s. Legislation was enacted in 1937 on the establishment of a monopoly, The State Textbook Publishing House, to publish textbooks for the primary school level and distribute for free. By law, the publishing house was to offer a choice of at least two textbooks in each subject, but increasing inflation during the Second

World War and the post-war period limited the choice to a single textbook in most subjects (Bjarnadóttir, 2006a; 2009a).

Around 1930, admission to the Reykjavík High School became restricted to 25 novices a year, but remained open in the newly established Akureyri High School in Northern Iceland. A number of lower secondary schools were established around the country but without any direct connection to higher education. Following the establishment of the Republic of Iceland in 1944, new education legislation was enacted in 1946. It allowed for a national entrance examination to the high schools, to be held in the lower secondary schools. This examination standardized the syllabus for the lower secondary school level during its term in action, 1946–1976 (Bjarnadóttir, 2006b, pp. 179–234).

The restricted admission, a centralized admission examination and The State Textbook Publishing House were to create circumstances of stagnation and monopolistic textbook policy (Bjarnadóttir, 2006a; 2009a).

Daníelsson's contribution to mathematics education

A mathematics stream

In 1877, the authorities were of the opinion that there was insufficient demand for a mathematics-science stream at the Latin School and opted for a language-history stream when Latin Schools in the Danish educational system was divided into two streams. During the First World War, discussion intensified that Iceland had to have its own engineers. Daníelsson, physicist Thorkell Thorkelsson (1876–1961) and philosopher Ágúst H. Bjarnason (1875–1952), took the initiative to promote educational opportunities for students to study mathematical sciences. Bjarnason had taken his Latin-school education in Denmark and his horizon was different from his countrymen's. He wrote several articles in his journal *Iðunn* in 1917–1919. He said in agony early in 1919:

> Now Iceland has acquired sovereignty and has to stand on own feet. We Icelanders are faced with utilizing a large and difficult country and making it submissive to us. We have yet to harness our waterfalls to lead light and warmth over the country, process fertilizers from the air and run all our machines. But do we have the know-how to carry this out? Here, specialists are needed for all kinds of work; but no one knows anything in the fields that we need most, compared with the foreigners' long experience. Yet, we are bursting with arrogance, composing long poems extolling our own greatness, and we even think, some of us, that we are superior to other nations. (Bjarnason, 1919, p. 80; translation: KB)

In October 1919, a mathematics stream was finally established at the Reykjavík High School after extensive lobbyism. As the only person with university

degree in mathematics, Daníelsson was appointed its main mathematics teacher. This meant a greater professional challenge for him, more prestige, less teaching and higher salary than at the Teacher College. He gained time to develop textbooks and to avail himself to research. Thorkelsson was appointed to teach the physics (Bjarnadóttir, 2006b, pp. 163–169).

The authorities had realized that it was more advantageous to prepare Icelanders for engineering studies than to hire foreign engineers who demanded higher salaries and often left after a few years with their experience. The aim of the new mathematics stream was to make the students eligible to attend the Polytechnic College in Copenhagen and study sciences at universities without spending an extra year abroad for preparation.

Textbooks

After his appointment at the Reykjavík High School, Daníelsson began his mission by choosing mathematics textbooks. For the more advanced grades, he chose the Jul.-Petersen's textbook series, written by his professor. He chose *Arithmetik og Algebra til Brug ved Gymnasiet og Realskolen* (Petersen, 1906), *Lærebog i Plangeometri for Gymnasiets sproglige og matematisk-naturvidenskabelige Linier samt Realklassen* (Petersen, 1914), *Lærebog i Trigonometri* (Hansen, 1919), *Lærebog i Differential- og Integralregning* (Hansen, 1921b), *Lærebog i analytisk Plangeometri* (Hansen, 1921a), *Lærebog i Stereometri* (Hansen, 1920) og *Tillæg til Arithmetik og Algebra* (Hansen, 1921c) (*Skýrsla um Menntaskólann í Reykjavík*, 1904–1946). C. Hansen revised professor Petersen's textbooks after his death in 1910. The more elementary textbooks of the Jul.-Petersen's series had already been in use at the Latin School since 1877 for the lower grades.

Daníelsson then began to write his own series and published four textbooks in the 1920s. He rewrote his *Arithmetic* (Daníelsson, 1906; 1914; 1920a) once more, and added *On plane geometry* (Daníelsson, 1920b), *Trigonometry* (1923) and *Algebra* (1927). The last three were the first of their kind in the vernacular. They were used at Reykjavík High School along with the advanced Danish textbooks and in Akureyri High School after it was established in 1930.

Guðmundur Arnlaugsson (1913–1996), was Daníelsson's former student. In his view (Arnlaugsson, 1996, p. 20) the textbooks were a great accomplishment, testifying to enthusiasm, optimism and craftsmanship in writing. He praised the publishers' daring and grandiosity to publish books about such extraordinary topics.

Arithmetic

Daníelsson wrote in the forewords to his first edition of his *Arithmetic*:

> This little booklet is intended … to compensate for two drawbacks which … characterize most … of our arithmetic textbooks; one is that they give no explanations at all, not even of the simplest computation methods, and

> therefore many learn the procedures by heart without understanding their reasons; and more so as many [who teach] … lack sufficient skills to explain the arithmetic down to its roots, without having for that any support from the textbooks … the other … is that their exercises are generally too easy, and each of them is most often aimed at only one computation method. The pupil can therefore guess the method without understanding the problem. (Daníelsson, 1906, p. iii; translation: KB)

This quote describes the situation in Iceland in 1906; no legislation on schools for the general public, no teacher college, and practically no educated teachers. Those who knew more tried to teach those knowing less. The *Arithmetic* was a part of a rise in educational standards during the first decades of the 20th century. It was a tiny textbook, quite elementary, but with some challenging problems and an effort to go beyond cookbook recipes, e.g. with explanations of the Euclidean algorithm for the greatest common divisor. The author dropped the explanation in the 1914 edition. He must have thought that they were premature in an elementary arithmetic textbook. The second edition was by and large written as a continuation of the first edition, adding ratio and proportions in the form of the Rule of Three, percentages and equations. In a new section on geometry, Daníelsson stated for example that the volume and the surface area of a sphere was 11/21 of the volume and surface area of the circumscribing cube, a neat simplification of more complicated formulas that those not acquainted with algebra would have had difficulties in handling.

The third edition in 1920 combined the two earlier editions. It was well suited for beginners at Reykjavík High School which many did not have solid preparation in arithmetic from primary school or home education. Danish textbooks were, however, still used there until Daníelsson's book was adopted in 1927 (*Skýrsla um Menntaskólann í Reykjavík*, 1904–1946). In the lower secondary school in Akureyri, which was working at becoming a high school, it was immediately adopted (*Skýrsla um Gagnfræðaskólann á Akureyri*, 1906–1940).

Algebra

Daníelsson's last textbook was *Kenslubók í algebru* [*A textbook in algebra*], published in 1927. His foreword bears witness to the situation in teaching advanced arithmetic at the time:

> … pupils, who have studied outside the schools, … have come up … to examination … so prepared in algebra that they have perhaps only solved the exercises, but do not know at all the basis of the symbolic language, have sometimes not had any tuition in it. (Daníelsson, 1927, p. 3–4; transl.: KB)

In this textbook Daníelsson carefully laid out the axiomatic foundation of algebra by introducing the commutative, associative and distributive laws of addition and multiplication and used them to prove various properties, such as the relations of subtraction to addition of whole numbers but without exercises.

Many students may have missed its point, and teachers found the book difficult to teach (Arnlaugsson, 1996, p. 19) but it contained an excellent collection of exercises. It was used in the two 6-year high schools. In 1946, it was together with the *Arithmetic* adopted into the syllabus for the national entrance examination of the then four year high schools to remain there into the 1970s. Later books had to adapt to Daníelsson's books, which thus became very influential through the mid-twentieth century and shaped the mathematics education of generations. Eventually they were gradually replaced from 1968 by New-Math textbooks (Bjarnadóttir, 2006b, pp. 254–268).

The *Algebra* contained a number of stories which students were expected to translate into equations. The easiest ones were of the type "think of a number". Other problems concerned the hands of the clock and their movements, and mixing, e.g. different types of wines. The classical mixing problem of the metals in the crown of King Hiero is also found. Problems with water running in pipes into cisterns and out again have their place (Daníelsson, 1927, pp. 93–114). Problems on dividing heritage are given, such as the classical problem of the man who left diamonds to his children such that the first one had one diamond and 1/7 of what was left, the second one was to have 2 diamonds and 1/7 part of what was then left, etc. Finally, all the heirs were allotted equal heritage, and the question concerned the number of heirs and diamonds (Daníelsson, 1951, p. 105). Jens Høyrup (2008) traces this problem back to Fibonacci's *Liber Abaci*, and suggests that a simple version of the problem is originally either a classical, strictly Greek or Hellenistic, or a medieval Byzantine invention, and that sophisticated versions must have been developed before Fibonacci.

Daníelsson thus knew a wealth of ancient problems. He composed his own problems too, concerning daily life of the time, such as about maids that received their salaries in boots and frocks (Daníelsson, 1951, p. 119). Those problems did not appeal to youth around the 1960s, whereas the author of this article has noticed that former students do enjoy refreshing their memory later in life by the ancient problems.

Geometry and Trigonometry

Daníelsson's textbooks *Um flatarmyndir* [*On plane geometry*] of 1920 and *Kenslubók í hornafræði* [*Trigonometry*] of 1923, however, proved to be too ambitious for his students who had little previous acquaintance with geometric concepts. Few teachers had the courage and/or capability to interpret them. Daníelsson said in his forewords:

> … some intellectuals … think that the goal of the geometry teaching is … teaching people to measure cabbage gardens or grass fields. But then a long time and a lot of work would be badly spent … it would be better to have an agronomist measure the piece of land and thus rid many of the future intellectuals of great adversity. No, the purpose … is to train the pupil in precision of his thinking and at the same time his inventiveness (Daníelsson, 1920b, pp. iii–iv; translation: KB)

The textbook *On plane geometry*, intended for novices at the six-year high school, approximately age 14, was quite theoretical. It began by a section on limits to prepare for proving the existence of irrational numbers. Next section contained a list of definitions and the postulate on a line through two points. The author admitted in his foreword that his experience was that students were relieved when that section was over. The first chapter's third section was on parallel lines, followed by exercises. Five were on computing angles, one of them in the hexadecimal system, and all exercises after that through chapter six were on proving on the basis of the definitions and theorems introduced. The following exercises up to the fifteenth and last chapter were alternatively on constructions and proving and computations by recently proved formulas, such as Heron's formula. Eventually, *On plane geometry* was transferred to the upper level of the high school. In 1937, when geometry was required at the lower level, Danish textbooks were translated (*Skýrsla um Menntaskólann í Reykjavík, 1904–1946*).

Petersen's textbook that *On plane geometry* was to replace, contained many more calculation exercises, and its exercises on proofs were fewer by far and printed in italics, so as to warn teachers and students (Petersen, 1943, pp. 75–90). Peterson's textbook was translated into Icelandic in 1943 when it could not be accessed from Denmark, even if it was hard enough. It survived in the Reykjavík School during 1877–1970 with breaks of *On plane geometry* and some New-Math experiments in the 1960s, despite notable criticism. In Petersen's eulogy in 1910 it said:

> First around the turn of the century people began to realize that the advantages of these textbooks [Peterson's series] were more obvious for the teachers than for the pupils ... the great conciseness and the omitted steps in thinking did not quite suit children. These books were excellent when the whole syllabus was to be recalled shortly before examination, but if the students were to acquire new material, one had to demand a wider form for presentation. (Hansen, 2002, p. 51; translation: KB)

In Denmark, Petersen's elementary geometry textbook was intended for the so-called *Mellemskole* [middle school] for age 12–15 (Hansen, 2002, p. 40). A reviewer wrote about the introduction to its 1905 edition:

> ... one reads between the lines the author's disgust against modern efforts, which in this country as in other places deals with making children's first acquaintance to the mathematics as little abstract as possible by letting figures and measurements of figures pave their way to understanding of the geometry's content ...
>
> Working with figures ... aids the beginner in understanding the content of the theorems, which too often has been completely lost during the effort in "training the mind". If the author knew from a daily teaching practice, how often pupils' proofs have not been a chain of reasoning but a sequence of words, he would not have formed his introduction this way ... for the middle school it [the textbook] is not suitable (Trier, 1905; translation: KB).

One might seek some explanation to the lack of success of the *On plane Geometry* to more modern theories on geometric thinking. The theory of Pierre and Dina van Hiele, developed in the late 1950s, suggests that pupils progress through levels of thought in geometry. The van Hiele model provides a framework for understanding geometric thinking (Clements, 2003, pp. 152–154). The theory is based on several assumptions: that learning is a discontinuous process characterized by qualitatively different levels of thinking; that the levels are sequential, invariant, and hierarchical, not dependent of age; that concepts, implicitly understood at one level, become explicitly understood at the next level; and that each level has its own language and way of thinking.

In the van Hiele model, level 1 is the visual level, at which pupils can recognize shapes as whole but cannot form mental images of them. At level 2, the descriptive, analytic level, pupils recognize and characterize shapes by their properties. At level 3, the abstract/relational level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand, even sometimes to provide logical arguments in the geometric domain, whereas at level 4, students can establish theorems within an axiomatic system. The van Hiele levels have proved useful in describing pupil's geometric concept development, even if they may be too broad. Pupils may possess and develop competences and knowledge at several levels simultaneously, although one level of thinking may predominate.

In the 1920s, primary schooling was underdeveloped in Iceland. The prescribed primary syllabus in geometry revolved around the area and volume of the simplest objects. Preparation for entrance to high school would rather be in languages, such as Danish and even Latin, and elementary arithmetic, but the novices had seldom met geometric concepts. They may not have been receptive for tasks suitable for van Hiele levels 3 or 4, having missed training at lower levels. But Daníelsson's ambitions lay in geometry, which he had pursued into a doctoral dissertation, and he seems to have intended to go as far as possible to share his way of thought on that topic with the students and their teachers.

The *Trigonometry* was intended for the mathematics-science stream of the upper level of the high school for only few students. They were therefore more receptive for that difficult topic than younger students were for elementary geometry. It was, however, eventually substituted by textbooks written in Danish which students had by then become accustomed to in other subjects. It seems that Daníelsson had too high ambitions in his own research subject, while his arithmetic and algebra textbooks were to survive him for decades.

Daníelsson's approach to school mathematics was strictly academic. His teaching inspired his students, at least if they showed talents and commitment. One of them said, "What especially influenced us was his enthusiasm and respect for mathematics" (Arnlaugsson, 1996). He explained arithmetic in an intelligible way, supported by proofs if he thought it would be useful, but only allowed space for initiative and creativity in his verbal exercises.

Advanced education

The University of Iceland, established in 1911, only offered studies in theology, medicine and law, in addition to studies of Icelandic history and literature. There were no science subjects, so that the Icelandic Literary Society, established in 1816, and the Iceland's Society of Engineers, established in 1912, and their journals, became important platforms to enhance scientific knowledge in the country. Daníelsson (1913; 1921; 1922) wrote in 1913 an article on geometry and the specific relativity theory in *Skírnir*, the journal of the Icelandic Literary Society, in 1921 in the *Journal of Iceland's Society of Engineers* on the same topic and in 1922 in *Skírnir* on the general theory of relativity.

Daníelsson thus reached far beyond the schools where he taught, in his effort to educate his countrymen. He began training his students in scientific thought while Iceland was still rural and self-study was common. One of his first high school students, Leifur Ásgeirsson (1903–1990), studied by distance learning only, mailing his solutions to his master. Ásgeirsson completed his doctoral degree in mathematics in Göttingen, Germany, in 1933 under the supervision of Richard Courant (Birnir et al., 1998). He published an article in *Mathematische Annalen*, Vol. 113, in 1937: *Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten.* This theorem became attached to Ásgeirsson's name and called Ásgeirsson's Mean Value Theorem. It concerns an ultra-hyperbolic differential equation in a neighborhood of a convex compact set (Hörmander, 2001).

Ásgeirsson left Göttingen in 1933 to become the headmaster of a lower secondary school in the countryside of North-Iceland. When the WWII isolated Iceland from the continent of Europe, Ásgeirsson was appointed as professor at the newly established engineering department of the University of Iceland. Other teachers at the department were also former students of Daníelsson or of his students, for example Arnlaugsson. Sigurður Helgason (1927–), professor emeritus at Massachusetts Institute of Technology, was initially Ásgeirsson's student (Birnir et al., 1998). Both Ásgeirsson and Helgason proposed internationally known theorems, named after them. Daníelsson was thus the mathematical ancestor of a new series of mathematical scientists, as well as a number of excellent mathematics teachers.

Daníelsson's scientific work

Research papers

In the 1920s, Daníelsson increased his efforts into research of algebraic geometry. He attended Scandinavian mathematical congresses and wrote numerous papers in academic journals:

- In the Danish *Matematisk Tidsskrift*: In 1926, 1940, 1945, 1948.
- In *Mathematische Annalen*: Vol. 102 (1930), 109 (1934), 113 (1937), 114 (1937).

Daníelsson attended the Sixth Scandinavian Congress of Mathematicians in Copenhagen in 1925 and the seventh congress in Oslo in 1927, and presented papers in both congresses. His first paper, "En Lösning af Malfattis problem" [A solution of Malfatti's Problem] was published in *Matematisk Tidsskrift* (Daníelsson, 1926). Next, he published a paper in *Matematische Annalen*: "Über korrespondierende Punkte der Steinerschen Fläche vierter Ordnung und die Hauptpunkte derselben" (Daníelsson, 1930), the last one of 44 articles by Einstein, van der Waerden, von Neumann, Landau, Ore, and Kolmogoroff among others. Later, the article "Über orientierbare und nicht orientierbare algebraische Flächen" (Daníelsson, 1937) was number six of a similar number of authors. The same issue contained the dissertation of Leifur Ásgeirsson on differential equations of second order. The journals and mathematicians' congresses were the threads that linked Daníelsson and Ásgeirsson to the mathematical community on the continent, even if they had to earn their livelihood by secondary school teaching.

Smaller works

All his life, Daníelsson was interested in elementary geometry, as by his expression "it is hard to find simpler and neater tasks than artful mathematics problems" (Daníelsson, 1914, foreword). His last paper was published in the *Journal of the Icelandic Society of Engineers* in 1946 and in *Matematisk Tidsskrift* in 1948. The papers contained the following theorem which Daníelsson in his isolation did not know that had been proved by others earlier (Helgason, 1996):
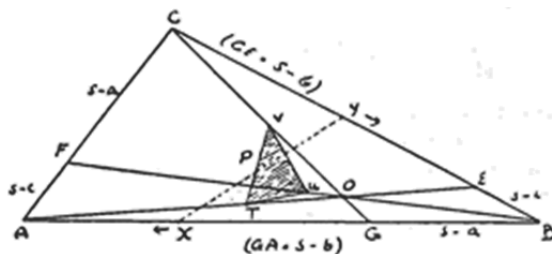
Figure 2

The locus of the "point of bisection", P_1 of a transversal XY, bisecting the perimeter of a given triangle is the perimeter of a given triangle ABC [see Fig. 2], is the perimeter of another triangle TUV, whether the triangles are considered in a Euclidean space or a hyperbolic one. … The vertices of the triangle TUV are the points of bisection of the straight lines drawn from the vertices of the triangle ABC and which bisect its perimeter. (Daníelsson, 1946, p. 70)

Daníelsson chose the points E, F and G, so that AE divides the side a into s–b and s–c, (s is half the perimeter of ABC, and a, b and c the sides opposite the angles A, B and C). BF and CG divide b and c similarly, so by Ceva's theorem AE, BF and CG pass through the same point, O.

Now the point X moves from G to A and point Y moves from C to E in such a way that CY = GX. The transversal XY must then always halve the perimeter of the triangle ABC since both EA and CG do so. As the series of points X and Y are congruent, the locus of the point of bisection P_1 is a straight line from V to T. Similarly, if X moves from A to F and Y from E to B in such a way that AX = EY, the bisecting point of XY moves along the line TU and it continues moving from U to V when X moves from F to C, and Y from B to G.

Daníelsson continued to claim that proving various other rules respecting the triangle TUV was easy in the case of triangle ABC being a Euclidian one, for example that the sides of the triangle TUV are perpendiculars to the bisecting lines of the angles A, B and C (Daníelsson, 1946, pp. 69–71).

Discussion

Daníelsson's extensive influence may be attributed to several factors. His strong personality and firm belief in mathematics as a superb science made him an excellent champion for mathematics education. Moreover, he was the mathematics teacher of the first 167 primary school teachers in Iceland. His former students propagated Daníelsson's vision and interpretation of school mathematics, such as primary school teacher Elías Bjarnason (1927–29) who composed a simplified version for primary school level of Daníelsson's own *Arithmetic* for adolescents (Schiöth, 2008). Bjarnason's textbook was chosen as the sole arithmetic textbook for all children, 10–14 year old, during the 1940s to 1970s by the monopoly State Textbook Publishing House. Bjarnason's textbook may have been considered as the most suitable preparation for secondary level schooling due to its compatibility with Daníelsson's *Arithmetic*. Similarly, the first secondary school mathematics teachers were those who had studied at the mathematics stream at the Reykjavík High School as there was no training of secondary school teachers at the University of Iceland until the 1950s (Bjarnadóttir, 2006b, pp.189–191, 431).

Another factor was that the Reykjavík High School was the sole school of its kind until 1930. When another high school was established in North Iceland, Daníelsson became protector of its mathematics-science stream and the students graduated under his supervision. Admission to Reykjavík High School became restricted in 1929, which created strong competition. Following this, new lower secondary schools were established in the 1930s around the country for the common people who had not had the opportunity to attend school after age fourteen. Daníelsson's *Arithmetic* was adopted in more and more of these

schools to enable their most promising students to transfer to one of the upper grades of the six year Reykjavík High School (Bjarnadóttir, 2013).

In 1946, the two six-year high schools were reduced to four-year upper secondary schools, and a national entrance examination was implemented in a number of lower secondary schools as a precondition for admission to the upper secondary schools. In deference to the hitherto dominant Reykjavík High School, the mathematics syllabus for the entrance examination was taken from the former second grade of this school, which included Daníelsson's *Arithmetic* and *Algebra* textbooks. The syllabus and its exam remained in place from 1946 to 1976, although alternatives to Daníelsson's textbooks were gradually phased in, especially after 1968 when the New Math had been introduced (Bjarnadóttir, 2006b, pp. 179–268).

Daníelsson retired from teaching in 1941 and died in 1957. His influence spanned nearly seven decades, from 1906 when he published his first textbook and 1908 when he began teaching at Iceland's Teacher College, until 1976 when his textbooks were removed from the reading list of the national entrance examination. His legacy as a dedicated mathematician is unquestioned. Mathematics education in Iceland was shaped by his vision.

The conditions in Iceland at this time, such as the restricted access to one of its two high schools, national isolation during the two world wars, the great depression in between, the monopoly of the State Textbook Publishing House, and the national entrance examination with its syllabus defined by a booklist dominated by Daníelsson's textbooks, created circumstances where discussion gradually faded out (Bjarnadóttir, 2006a; 2009a). Many generations of teachers did not know other mathematics textbooks than those by Daníelsson or built on his ideas. As they were so genuine and flawless they were not debated and no discussion took place until long after his death.

One may read from the foreword of the 1906 edition of the *Arithmetic* that Daníelsson was concerned with understanding mathematics as most textbook-writers have been (Bjarnadóttir, 2007). It is not likely that Daníelsson studied the pedagogical theories of Pestalozzi and his followers on primary teaching even if they were favoured in Denmark in the first decades of the twentieth century while he stayed there (Hansen, 2009). He might rather have known the theories by Felix Klein, who was interested in mathematics teaching at the border of high schools and universities (Schubring, 2008), but that was not an issue in 1906 in Iceland which had no university teaching in mathematics. Daníelsson's target group for his *Arithmetic* was different from that of textbook writers in countries with whom Daníelsson was acquainted, in Denmark and Germany. Daníelsson was a pioneer and he had to stick to his own ideas about mathematics learning and teaching.

Daníelsson declared in his first book in 1906 that he wanted to remedy the shortcomings of other textbooks where students learned the procedures by rote without understanding their reason. He tried to provide explanations in the first edition of the book, at least of some topics, but he dropped them in later

editions, probably feeling that they did not reach his audience. His teacher student, Bjarnason (1927–29) adopted the procedures without explanation. This developed into the situation where the State Textbook Publishing House, which was established to ensure equal access for all to textbooks, ended up advocating a single unexplained procedure for each topic. Thus, the diversity in approaches of the early twentieth century eventually faded away.

Conclusions

Ólafur Daníelsson was the right person at the right time when educational authorities finally decided to meet the demands for preparation for technical education. The Great War had disturbed sailing contacts to Copenhagen where the preparation had been sought, so a mathematics stream was established at the Reykjavík High School and entrusted to the hands of Daníelsson. His education and attitude to mathematics was such that he made great demands and offered no compromises. He was the only mathematician in the country for a quarter of a century and had few to consult with. He set himself high goals and achieved them. Higher technical education had to wait until the isolation of World War II, when Ásgeirsson was called back from his rural lower secondary school teaching.

In time, Daníelsson's position at the Reykjavík High School made that influences from his views were felt far outside the school itself. Primary schools and lower secondary schools adapted to the requirements of the school due to the restricted access, both before and during the period of the national examination. It took an international reform movement in the 1960s and 1970s, the New Math, to turn the general attention away from Daníelsson's influence towards different kinds of approach to mathematics teaching in Iceland.

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Teaching traditions in Swedish school algebra – a project description

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Abstract

In this paper we present a project that just has been started. The general purpose of the project is to identify how the algebraic content in Swedish school algebra has been formed and developed during the last fifty years. Curricula and textbooks from elementary school up until upper secondary school from the years 1962, 1969/70, 1980, 1994 and 2011 will be investigated. By means of discourse analysis potential teaching traditions in school algebra will be identified on the basis of mathematical content, degree of difficulty and contextualization. In this paper we will describe the aims, purposes, structure and some very early results based on a pre-study to this project. Moreover, the term "teaching tradition" will be described and discussed on the basis of Almqvist et. al. (2008). We will also take into account the linguist Anward's (1983) terms "actual text" and "produced text" in order to describe how a content is formed and chosen in a pedagogical context.

Background

During recent years Swedish school pupils' results in mathematics at the international tests TIMSS (Trends in International Mathematics and Science Study) and PISA (Program for International Student Assessment) have declined. The last decade's results show that the proportion of low performed pupils in Swedish schools has increased while the proportion of high performed pupils has decreased (National Agency for Education, 2012, p. 108). Moreover, it seems that this is not only a trend in comparison with other countries, but also compared to Swedish school pupils over time. A report from the Swedish National Agency for Education (2010) shows that the proportion of pupils who does not pass mathematics in grade 9 has increased from 5.3% to 7.9% between the years 1998 and 2010. Furthermore, the part of mathematics where Swedish school pupils' results have decreased at most in

comparison with other OECD countries is algebra and geometry (Swedish National Agency for Education, 2012, p. 49).

The negative trends in Swedish school mathematics have among other things led to major efforts on mathematics teaching projects as well as teacher training. In order to make relevant and motivated didactical choices, it is important for a teacher to possess knowledge of how the content of a particular subject has been formed and developed over time. Almqvist et. al. (2008) discuss how meaning-making is formed in educational discourses. They present a pragmatic approach for studies of meaning-making in order to enable discussions on questions regarding how meanings are made in people's actions. In connection with a teacher's choice of content, Almqvist et. al. (2008, p. 14) consider three different levels, each important for how a content is formed and developed:

1. The intrapersonal level, i.e., an individual's meaning-making, for instance how a student learns a specific concept.
2. The interpersonal level, i.e., how meaning-making is formed in the interaction between individuals, for instance in a classroom.
3. The institutional level, i.e., how steering documents (curricula and syllabuses) and teaching materials (for instance textbooks) are related to the formation of a content.

During the last decades the research field of mathematics education has been dominated by studies at the intrapersonal level, such as individuals' concept development (see for instance Sfard, 1991; Sfard & Linchevski, 1994; Tall, 2011) and at the interpersonal level (see for instance Cobb et. al., 1992; Oltenau & Holmqvist, 2012). Meanwhile, studies regarding how mathematics as a school subject has been formed and developed on the basis of curricula, syllabuses, tests or textbooks (i.e. on the institutional level) are not as well represented, especially not in Sweden. Furthermore, studies that grasp over different school levels in order to examine the progression of the subject from early school years up until upper secondary school are also not common. For teachers, such holistic approach is important in order to clarify what their current teaching will lead to during a certain stage of development.

In this paper we will describe a project that just has been started. The project plan, the theoretical framework and some very early results will be presented. The project is concentrated to the institutional level by means of studies of Swedish curricula and textbooks. Furthermore, the project is limited to treat school algebra which is one of the fields where Swedish school pupils' results on TIMSS and PISA have decreased at most compared to other OECD countries (Swedish National Agency for Education, 2012, p. 49).

Aims and purpose with the project

This project is supposed to last for four years and involves two researchers. The general purpose with this project is to contribute with knowledge regarding how the algebraic content in Swedish school mathematics has been formed and developed on the institutional level from elementary school up until upper secondary school. Within the project mathematics curricula and textbooks from the different Swedish school levels between the years 1962 and 2011 will be investigated. By means of discourse analysis (Fairclough, 2003) content patterns in school algebra will be identified regarding the following three dimensions;

- mathematical content,
- degree of difficulty,
- contextualization.

The project will be based on a *diachronic* as well as a *synchronic* perspective. The former perspective refers to the development along a time axis, meanwhile, the latter perspective refers to what actually exists at each school level at one particular moment. The two perspectives are summarized in Figure 1.

Figure 1. The two perspectives of the project.
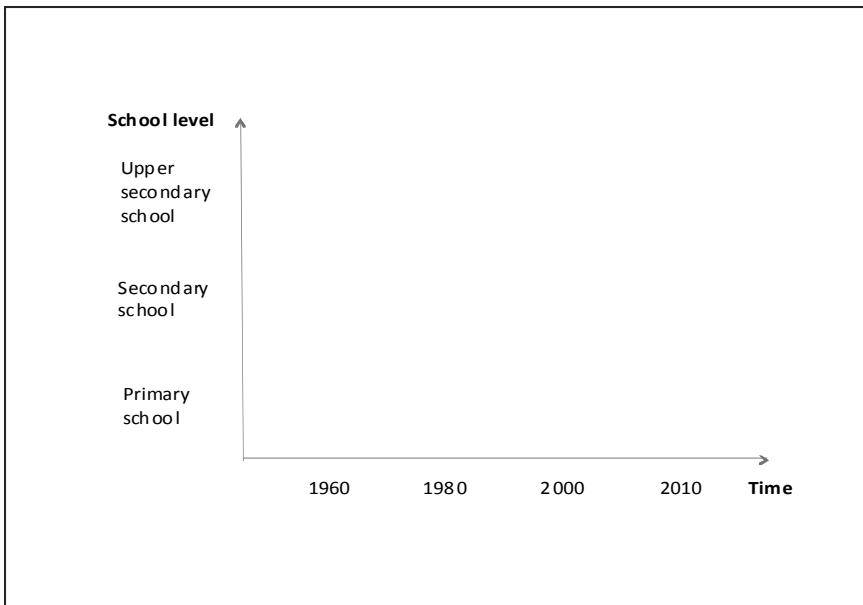
We believe that the usage of a diachronic perspective will enhance the investigation of the algebraic content in a synchronic perspective. The contrasting effect occurring between different time periods may clarify the algebraic content today as well as how the algebraic content has changed during the years.

On the basis of the three dimensions mentioned above (mathematical content, degree of difficulty and contextualization) *teaching traditions* and the

progression between different school levels will be described. (The term teaching traditions will be considered in more detail below.) A first aim is to identify what algebraic content has been included and excluded respectively in the curricula and the textbooks. In order to support the discourse analysis we will use text analytical tools developed within systemic functional linguistics.

A second aim is to analyze how the included contents are contextualized (for instance if the content is connected to everyday contexts or "pure" mathematical contexts) and identify the degree of difficulty. Finally, a third aim is to put the received results of the project in a dialogue practice with teachers from different school levels (which will be described below).

The aims of the project will be concretized by means of the following five research questions:

1. What mathematical content can be identified within Swedish school algebra at the years 1962, 1969/70, 1980, 1994 and 2011?
2. What types of contextualizations and what degree of difficulty can be identified within the algebraic content at the years 1962, 1969/70, 1980, 1994 and 2011?
3. Can different teaching traditions be identified in connection with different school levels and different time eras? If so, what teaching traditions? What differences and similarities may in that case exist between these teaching traditions?
4. What type of progression can be identified between different school levels regarding algebraic content, degree of difficulty and con-textualization?
5. What values can knowledge of teaching traditions, potential mathematical content, degree of difficulty and contextualizations contribute to in connection with a teacher's didactical choices?

Survey of the field

As mentioned above, within the research field of mathematics education there are relatively few studies at the institutional level, based on for instance steering documents, tests and textbooks. The studies at institutional level consider surveys regarding teachers' reactions at curriculum reforms (see for instance Charalambos & Philippou, 2010), how problem solving has changed over time in different syllabuses (see for instance Stanic & Kilpatrick, 1989) and the adoption of etnomathematical perspectives in certain curricula (see for instance Dickenson-Jones, 2008), etcetera.

Studies at the institutional level within the Nordic countries are even more limited. An exception is Hemmi's et. al. (2011, 2013) comparative studies of curricula in school mathematics in Estonia, Finland and Sweden. Hemmi et. al. (2013) have investigated what role the competences proof and argumentation have in these three countries' current mathematics curricula. The results

revealed three different trajectories with specific characteristics, shortcomings and strengths. For instance, the Swedish curricula contained significantly less elements of proof reasoning and argumentation compared to the other two countries. However, the Swedish curricula contained significantly more of the so called "everyday mathematics" compared two the other two countries.

Another Swedish study at the institutional level is Jakobsson-Åhl's (2006) thesis regarding the algebraic content in Swedish textbooks in upper secondary school mathematics during the years 1960–2000. The result revealed that during the time period the algebraic content had changed from being dominated by algebraic manipulations and expressions to becoming more integrated with other school subjects and thus being more anchored with reality as well as everyday activities. Furthermore, Johansson (2006) has considered how textbooks are used and what influence they have in mathematics teaching activities. In the same study, Johansson considered the relation between curricula and the algebraic content in textbooks. The result showed, among other things, that the aims in the curricula with regard to the role of mathematics in society did not reflect the content in the textbooks.

 Among the research studies at the institutional level in Sweden Prytz (2007) has studied geometry instruction at lower secondary school based on a curricular perspective. Prytz (2012) has also used methods from sociology of education to study social structures in mathematics education. Research studies at the institutional level in other school subjects in Sweden are far more represented compared to mathematics (see Liberg et. al., 2012, and *Utbildning och demokrati,* 2008).

Research connected to school algebra at the intrapersonal and interpersonal levels are far more prevalent within the field of mathematics education. One debated topic in the research of algebraic thinking and learning is when algebra should be introduced in schools and what difficulty level algebra should have. Some researchers suggest that individuals' development of algebraic concepts is reflected in the historical development of algebra (see for instance Sfard & Linchevski, 1994; Katz et. al., 2007; Moreno-Armella et. al., 2008). Such an approach implies (among other things) that in an individual's learning process arithmetics and rhetorical algebra always precede abstract algebra (see for instance Cerulli & Mariotti, 2001; Warren, 2003), which would mean that algebra should not be introduced in lower school levels. Furthermore, Mac Gregor (2001), among others, claim that the cognitive development of pupils at lower school levels is insufficient to be able to understand algebra. This has been criticized by among others Blanton & Kaput (2005), Carraher et. al. (2006) and Persson (2010) who all claim that algebra should be introduced at lower school levels. Furthermore, Carraher et. al. (2005) emphasize that a deep understanding of arithmetics requires mathematical generalizations that are algebraic in nature and that algebraic notation makes it easier for young learners to give expressions to such mathematical generalizations.

In a joint international project led by the Swedish professor Roger Säljö a comparative video study has been initiated. The aim of the project is to compare how algebra is introduced in grade 6 and 7 in Sweden, Finland, Norway and the United States. The project plan can be found in (Kilhamn, 2013) and (Kilhamn & Röj-Lindberg, 2013).

Teaching traditions

The content and contextualization of school algebra have been formed and developed over the years. Within the field of didactics such content formation are sometimes referred to as the emergence of *teaching traditions*. Almqvist et. al. (2008) describe teaching traditions as:

> Regular patterns of choices of content which has been developed over time within a specific subject (Almqvist et. al., 2008, p. 14).[1]

According to Almqvist et. al. the content patterns form a certain "education culture" which constitutes what is considered as an adequate teaching and as a relevant content. Almqvist et. al. point out that teaching traditions can provide knowledge with respect to what values a specific education culture holds. This is based on the fact that the choice of content depends on what is considered as important, relevant, correct, etcetera (Almqvist et. al., p. 15).

Teaching traditions are created and maintained by means of what is said and done in a certain culture, including texts of and within the culture. In text analytical studies of, for instance, textbooks the analysis is based on the content that is offered. We may never know exactly how different persons read and understand a text, but by comparing the content of the text with other possible contents we can point at texts that are more reasonable than others.

Almqvist et. al. (2008) refer to the linguist Anward (1983, pp. 100–140) who uses the terms *actual text* and *produced text* in his study regarding how a content is formed and chosen in pedagogical contexts. Anward describes the actual text as the content the teacher accepts and considers as the most relevant. Furthermore, the actual text is represented as a subset of the produced text, which refers to everything that possibly can be said about this content. Apparently, in pedagogical contexts the actual text will be considered as the legitimate content and becomes *included* in the teaching, while the rest of the produced text will be *excluded* from the teaching (see Figure 2).

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[1] The term teaching tradition originates from Raymond Williams (1973) and was introduced in Sweden by Tomas Englund (1986).

Figure 2. The relation between Anward's terms *actual text* and *produced text*.
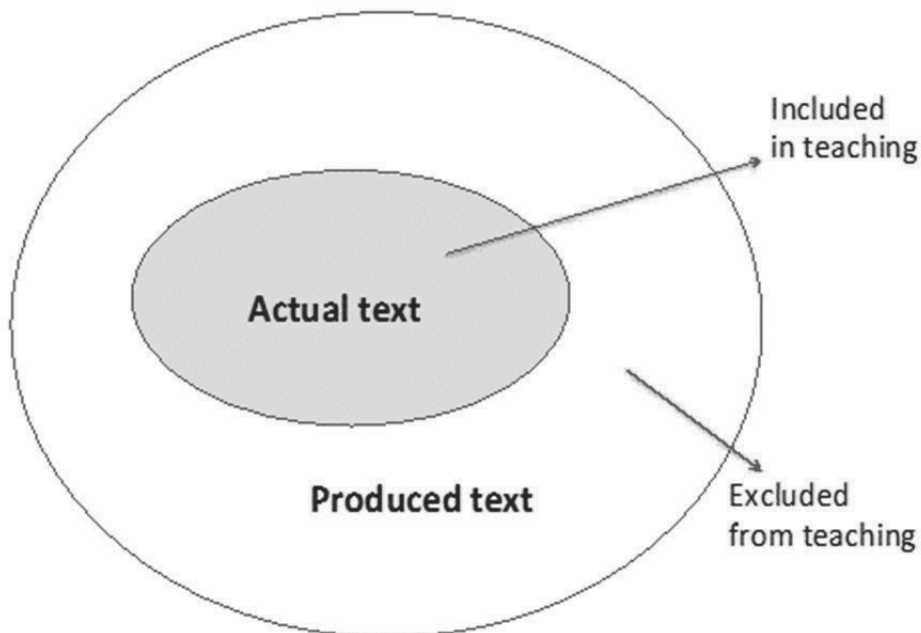
In order to identify teaching traditions in Swedish school algebra we will in this project use Anward's terms actual text and produced text together with Bednarz' et. al. (1996) classification of algebraic content. Bednarz et. al. (1996) describe five different perspectives regarding algebraic content in connection with research as well as teaching:

- **The historical perspective:** Algebra viewed from a historical perspective in order to appreciate and get a better understanding of the complex nature of algebra.
- **The generalization perspective:** Algebra is considered as a generalization of arithmetic and geometry. Geometric patterns and regularities are described by means of algebra.
- **The problem-solving perspective:** Algebra is viewed as a tool for solving problems that cannot be carried out by arithmetic.
- **The modeling perspective:** Algebra is used in order to construct real as well as abstract models.
- **The functional perspective:** Relations between variables is expressed by means of algebra, for instance functions are expressed with algebraic rules and representations. Mathematical analysis is based on this perspective.

In our project we will use Bednarz' et. al. (1996) perspectives of algebraic content together with the following two additional perspectives (constructed within our pre-study):

- **The structural perspective:** The study of common properties of algebraic constructions.
- **The everyday perspective:** The usage of algebra outside the mathematical context (which sometimes is referred to as "everyday mathematics).

The seven perspectives above are based on what algebra actually is and how algebra can be contextualized, but can also be viewed as interest directions connected to algebra and what particularly is emphasized within teaching contexts.

Studies, material and analysis

The general theoretical framework for this project is didactic (rather than historic) and the overall question investigated is "*What?*". The three dimensions mentioned above (algebraic content, contextualization and degree of difficulty) will be used in order to identify and describe teaching traditions in Swedish school algebra at every school level between the years 1962 and 2011.

In the project discourse analysis will be used to investigate the algebraic content in the curricula and in the textbooks. Especially, we want to find out what content is in the foreground and in the background during the given time period. A basic assumption for this analysis is that the content that has been put in the foreground has been considered as important, relevant, correct, etcetera. The usage of material from all school levels and from different time eras enable us to consider aspects from a certain school level or a certain time era that has been excluded. In this project we will use (as we mentioned above) Anward's (1983) terms of "actual text" and "produced text" in order to discuss the actual chosen content within a teaching tradition in contrast to a content within another possible teaching tradition.

Studies and material

The project will consist of two different studies. Within the first study, which is connected to research questions 1–4 above, curricula and textbooks will be studied with focus on the algebraic content. The last five Swedish curricula in mathematics at lower, intermediary, upper and upper secondary school level will be included in the study. These curricula were introduced in Sweden at the years 1962, 1969/70, 1980, 1994 and 2011. The choice of textbooks within the project is based on school level, time and how widespread the usage has been. In order to cover every school level textbooks from grades 2, 5, 8 and first grade at upper secondary school will be included in the study. We will also take

into account that the textbooks belongs/have belonged to the most popular in each grade.

The second study of the project, which is connected to research question 5 above, consists of an intervention study. Two focus groups with active teachers from each school level will be video recorded at two different sessions. During the first session the teachers will discuss curricula and textbooks and how these are used and integrated in their own current teaching. Subsequently the teachers will get a review of the results from the project's first study and exercises in order to discuss curricula and textbooks in a broader perspective. Finally the teachers will return to their focus groups and in the light of the results of the project discuss the same curricula and textbooks that were treated at the first session.

Analysis

The first study consists of three analytical steps. In a first step, algebraic content that has been put in the foreground and in the background respectively will be identified (research question 1). To support the discourse analysis and in order to distinguish between foreground and background in a text we will use textanalytical tools developed within systemic functional linguistics (see for instance Fairclough, 2003). In a second step, we will identify the difficulty level of the algebraic content and how the algebraic content is contextualized (research question 2). The identification of how the algebraic content is contextualized will be based on Bednarz' et. al. (1996) perspectives of algebra together with our own two additional perspectives of algebra (see above). The identification of difficulty level will primarily be based on Hemmi et. al. (2011, 2013) and Jakobsson-Åhl (2006), see above. In the third analytical step the results of the first two steps will be used in order to identify and describe different teaching traditions in school algebra (research question 3) and to identify progression between different school levels (research question 4).

The second study (research question 5), where results from the first study are inserted in a conversation practice, the participating teachers' statements from the first two meetings will be analyzed in relation to the teaching traditions and the progression patterns that were identified in the first study.

Some very early results

In a pre-study to this project the last three preschool curricula in Sweden (1980, 1994 and 2011) have been compared with focus on the mathematical content in general (see Andersson, 2011). The study was based on a quantitative as well as a qualitative text analysis. The former analysis investigated the frequency of some specific mathematical keywords, meanwhile the latter analysis investigated

the structure of the texts, to whom the texts were directed and which mathematical content that were treated in the texts.

The results indicated that specific mathematical keywords such as calculate, measure, solve, etcetera had decreased over the years, while "unspecific" mathematical keywords such as use, describe, interpret, handle, etcetera had increased over the years. A typical example is when the word "calculating" has been changed to the phrase "usage of subtraction methods". Another typical example is when the word "measuring" has been changed to the phrase "descriptions of geometrical forms".

Another result from the study was that the content of "everyday mathematics" had increased over the years and at the same time the "pure" mathematical content had decreased. For instance, the results of the study indicated that the higher level mathematics in preschool (such as quadratic equations) had been replaced by lower level mathematics in preschool (such as numeral system and patterns). These results follow the conclusions of Hemmi's et. al. (2011, 2013) and Jakobsson-Åhl's (2006) studies that was mentioned above.

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Lucienne Félix and Osvaldo Sangiorgi: Interchanges between two Bourbakist militants in the 1960s

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Abstract

The New Math movement involved interchanges among teachers and professors with very different positions and concerns, as was the case of Lucienne Félix, from France, and Osvaldo Sangiorgi, from Brazil. This paper discusses their motivations to the interchange developed in the beginning of the 1960s, taking into account their professional careers and the contexts of their commitment to modernizing the teaching of Mathematics. We also look into Sangiorgi's appropriation of Lucienne Félix's work.

Introduction

One of the most noteworthy features of the so-called "New Math" movement, which emerged and spread through different countries between 1950 and 1960, was the internationalization of debates concerning teaching programs and approaches.

International agencies such as the Organisation for European Economic Co-operation (OECE) or the Organization of American States (OAS) promoted interchanges through major events; for instance, the Royaumont Seminar, in 1959, and the First Inter-American Conference on Mathematics Education (IACME), in 1961. The magnitude of these events has fostered interpretations, from various authors, claiming that the spread of New Math fundamentally resulted from these agencies' and from the European and North American governments initiative. In Latin America, notably, the modernization movement's spread would have been an accomplishment of U.S. agencies, as part of an expansionist strategy towards the continent.[1]

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[1] See Ruiz & Barrantes 1998 and Vasco Uribe 2011 in the list of references.

However, the internationalization of the debates on New Math or "modern mathematics", as it was commonly called in Europe and Latin America, was not promoted only by governments and big agencies. The International Comission for the Study and Improvement of Mathematics Teaching (CIEAEM), which had its first meetings in 1950, was an initiative of teachers and intellectuals engaged in the renewal of the teaching of mathematics. During that period, trade unions, institutions and intellectuals from various countries promoted exchanges, composing what Dias (2008) names a "continental and international network built around the modern mathematics movement" (p. 20).

What motivated these exchange efforts for the renewal of teaching? How did the postwar context favor these initiatives?

This paper intends to contribute to this discussion by studying the meeting that happened in the early 1960s between Osvaldo Sangiorgi, who played a leading role in the modern movement in Brazil, and Lucienne Félix, a professor and author engaged in debates regarding the teaching of mathematics in France and Europe.

On the trail of connected history

Earlier studies on modern mathematics emphasize the ways proposals built in other countries were interpreted in Brazil[2], following the logic of studies in history of education that focus on the study of influences or radiations from the "center" towards the "periphery" (Warde, 2013). Such "one-way" approach has been questioned by proponents of "connected history," which instead values the complex, dynamic and asymmetrical character of relational configurations (Werner & Zimmermann, 2006, p. 38).

The exchange between Lucienne Félix and Osvaldo Sangiorgi in the early 1960s is an example of a connection that cannot be explained through a unidirectional logic. It was not encouraged by the governments or international agencies, and was only possible because both actors, despite being from different places and having different interests, mobilized to meet each other.

As proposed by the "connected-history" perspective, in order to understand this mobilization, the research must take place at the same level as the players and their logics of action, taking into account the social contexts that both enable and trigger these movements (Douki & Minard, 2007).

Following this perspective, this paper examines the exchange between Lucienne Félix and Osvaldo Sangiorgi focusing on their trajectories, keeping in mind the context of their professional practice and involvement in the New Math movement.

An extensive repertoire of sources supports a reconstruction of the meeting and a brief discussion of its effects. Such repertoire includes teaching

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[2] See D'Ambrosio 1987 and Soares 2001 in the list of references.

publishings, articles published in mainstream press newspapers, textbooks, consultations to Lucienne Félix's career files in the *Archives Nationales* of France and to Osvaldo Sangiorgi's Personal Archive[3].

The interaction between Sangiorgi and Félix can be seen as a component of a complex tangle of interactions that took place in the 1950s and 1960s. In addressing this connection, we seek to contribute to the construction of new insights into this tangle.

Lucienne Félix and the spirit of Sèvres

Lucienne Félix had her own path marked by her years at the *École Normale Supérieure de Jeunes Filles*, where she studied during the 1920s and worked as Henri Lebesgue's assistant during the 1930s (Félix, 1986a, 2005).

The *École*, founded in 1881 and located in Sèvres – a commune close to Paris –, was dedicated to the education of female teachers, targeting female secondary school classes. In a time in which women in teaching pursueded different careers than men, having to undergo a specific competitive exam for a position in the public education system, the *École de Jeunes Filles* was not required to follow the same programs of the traditional *École Normale Supérieure* of Paris (*Le cinquantenaire...*, 1932). This relative autonomy and the presence of the mathematicians Émile Picard, Émile Borel and Henri Lebesgue at the *École de Jeunes Filles* enabled an education directed towards the development of mathematical thinking, towards the experience of mathematics as investigation, as a human activity and living science, according to Félix (1957a).

Lucienne Félix's teaching practice in secondary school began in 1923. The laudatory opinions of inspectors – "one of our best mathematicians in the women's lyceums" [4] – granted her a promotion to the prestigious and coveted position as teacher of *Mathématiques Spéciales* classes in the *Lycée de Jeunes Filles* of Versailles, which prepared female students for the competitive exams for the admission to the *Écoles Normales*. However, her teaching practices – inspired by what she called "the spirit of Sèvres" –, with which she sought to incite reflection in her students, put her in conflict with the education designed to fulfill the exam requirements, which were built around standard problems. The education inspectors and principal's criticism, who reproached her for "overloading her classes with digressions,"[5] was mixed up with the persecution of Jews perpetrated by the collaborationist government established during the

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[3] *Arquivo Pessoal Osvaldo Sangiorgi* - APOS was consulted at the Documentation Center of the Research Group of History of Mathematics Teaching in Brazil (GHEMAT).

[4] Original version: "une de nos meilleurs mathématiciennes des lycées de jeunes filles". Notes of an education inspector from the *dossier de carrière* of Lucienne Félix, February, 1926, *Archives Nationales*, France, *côte* F/17/28679.

[5] Original version: "alourdir son cours de digressions". Notes of an education inspector from the *dossier de carrière* de Lucienne Félix, February, 1945, *Archives Nationales*, France, *côte* F/17/28679.

Nazi occupation of France. Deprived of her position in 1940 by the antisemitic laws of the Vichy government, Félix was arrested in August, 1944, in the transit camp of Drancy and reinstated after her release; she was again relieved of her post in *Lycée* in 1945, without any formal justification (Félix, 2005).

From 1946 until her retirement in 1966, she was a teacher at the *Lycée La Fontaine* in Paris. In this female secondary school, she taught the last classes of the first cycle (*quatrième* and *troisième*), the class of *Mathématiques Élémentaires*, at the end of upper secondary school, and also the disciplines of Algebra and Cosmography for the Philosophy class. Between 1950 and 1960, at the invitation of Alphonse Hennequin, Lucienne Félix also worked as an examiner of *Mathématiques Générales* in the University of Paris.

Lucienne Félix, "bourbakist" militant

In her career, Lucienne Félix sought to create contact with different discussion groups on the teaching of mathematics. In the 1920s and again in the 1940s, she attended a group that held its meetings in Sèvres and was guided by Marceline Dionot.[6]

By the end of the 1940s, Félix reports a change-over that came about due to her contact with the work of the French group Bourbaki that, according to her, replaced the study of particular objects and the relationships between them with a study that focused on the structures of these relationships. That approach was seen as an alternative to what Félix called "dogmatic" teaching – based on the repetition of textbooks, on the compartmentalized study of different topics and on the reproduction of examples (Félix, 1986a, 2005). From then on, she started participating in what she called the "bourbakist movement," which had the purpose of "adapting the approach of modern mathematics to the secondary school" (Félix, 2005, p. 84).

But in what sense can one talk about a "bourbakist movement" in the 1950s? Lucienne Félix participated in two major debate forums on math education: the previously mentioned CIEAEM, and the *Association des Professeurs des Mathématiques de l'Enseignement Public* (APMEP).

The first meeting of what would later become the CIEAEM took place on the outskirts of London in August 1950, on the initiative of the mathematician and educator Caleb Gattegno. The invitation was transmitted by Mme. Hatinguais, head of the *Centre International d'Études Pédagogiques* (CIEP) of Sèvres – a French intitution founded in 1945, of which Lucienne Félix was close to and with which she collaborated.

In a period of expansion of educational systems, and in which the influence of the New School reached the secondary school, Gattegno intended, according to Gispert (2010), to combine the "pedagogical modernity" – the pedagogies

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[6] Lucienne Félix was also an active member of the *Association des élèves et anciennes élèves* of Sèvres.

centered on student activity – with the "mathematical modernity." In April, 1952, the CIEAEM meeting was attended by bourbakist Dieudonné and other mathematicians, such as Gustave Choquet and André Lichnérowicz, committed to the renewal of higher education in France.

This same intent of reconciling the rigor adopted by bourbakists in their "Elements of Mathematics" with the active teaching methods was present in the APMEP, which, in April, 1950, created a study committee with the suggestive name "*Axiomatique et redécouverte*" (D'Enfert, 2010). Marceline Dionot was one of the founding members of the committee and promoted its communication with the group from Sèvres (Initiation..., 1953, p. 57). Thus, Lucienne Félix began her career in the Association, being involved even as a member of its policymaking forums (*Bureau* or *Comité*) from 1955 to 1969. One of the activities that Félix was involved with in the APMEP, in the 1950s, was the organization of lectures given by mathematicians, among which were the bourbakists Henri Cartan and Laurent Schwartz.

It was in this environment of debates on teaching practices, of interchange between teachers of the secondary school and mathematicians, that Lucienne Félix published her first books. "*L'Aspect Moderne des Mathématiques*" and "*Exposé Moderne des Mathématiques Élémentaires*", published in 1957 and 1958, addressed teachers and students of the secondary school, as works of "initiation to the modern mathematics". Those were followed by several other works published by her, such as geometry books for students in junior secondary and books for teachers and students of primary school.

In her memories, Lucienne Félix does not allude to the Royaumont Seminar, but she is mentioned in the report as a lecturer and both her books are mentioned as references[7] (OECE, 1961).

Osvaldo Sangiorgi, a "modern" teacher

The training of teachers for the secondary school in universities started in Brazil in 1934, with the creation of the Faculty of Philosophy, Languages and Literature, and Human Sciences[8] (FFLCH) of the University of São Paulo (USP) (Valente, 2005). Osvaldo Sangiorgi earned his degree in Mathematics by the University of São Paulo in 1941 and was part of a small minority of licensed secondary school teachers in the context of the expansion of education and predominance of non-graduate teachers. From 1947 on, he also worked in higher education as an assistant professor of geometry in the Faculty of Philosophy of the Mackenzie Institute, in São Paulo.

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[7] She is also mentioned in the notes on the debate following Willy Servais's conference for having reported a successful experience with junior secondary students (OECE, 1961, p. 79).
[8] Translation from "Faculdade de Filosofia, Ciências e Letras" by John Milton, cf. <http://fflch.usp.br/inicial/en>.

In the 1950s, Sangiorgi was recruited by the emerging publishing house *Companhia Editora Nacional*, which, according to him, "kept an eye out for" good teachers and invited them to write textbooks. He became a prestigious author by publishing the series "Matemática – curso ginasial" (Mathematics – junior high), the book "Matemática e Estatística" (Mathematics and Statistics), for the training of primary school teachers, and the math section of the "Programa de Admissão" (Admission Program), which prepared students for the exam for admission to secondary school (Valente, 2008).

In 1955, Sangiorgi participated as a representative of the Society of Mathematics of São Paulo in the First National Congress of Mathematics Teaching in Secondary Schools, held in Salvador. The Congress, organized by professors from the Faculty of Philosophy of Bahia, was a milestone in the establishment of a national forum for discussing math education in secondary school. Until then, debate and decision on programmes were monopolized by teachers from the traditional *Colégio Pedro II*, in what was then the capital, Rio de Janeiro (Souza, 2008)[9].

In this Congress and in the ones that followed, Sangiorgi had an active role in discussions, proposing changes to the curricula and alluding to "modern" tendencies in teaching and to texts that were being read in Europe, such as the book "*L'Enseignement des Mathématiques*" (Piaget et al., 1955).

Although critical of official programs, Sangiorgi maintained good relationships with government agencies, especially with the Bureau of Education of São Paulo. He used to attend to examination boards and, through the publisher, give lectures and courses to teachers.

In 1960, Sangiorgi was invited by the Brazilian Institute of Education, Science and Culture (IBECC), an organ of the Brazilian section of UNESCO, for the Summer Institute for High School and College Teachers of Mathematics, in the University of Kansas, with a scholarship granted by the Pan American Union and the North American National Science Foundation. The following year, Sangiorgi organized in São Paulo a course that was similar to the one he attended in Kansas and, by its end, he headed the creation of the Study Group for Mathematics Teaching in São Paulo[10] (GEEM), which assembled university professors and primary and secondary school teachers.

In 1961, the new National Education Law[11] granted the states autonomy to develop their own curricula, extinguishing thus the standard curriculum, which had been valid for entire Brazil. In the Fourth Brazilian Congress of Mathematics Teaching, in 1962, the GEEM proposed the "Minimum Subjects for a Modern Program of Mathematics for Upper Primary Schools and Secondary Schools," which was approved by the Congress. The "Minimum Subjects" embodied proposals for curricular reorganization discussed in former

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[9] The current programs in the late 1950s had been approved by the Congregation of the Colégio Pedro II and ratified by the Department of Education and Health in 1951 (cf. Ordenance 1045).
[10] Grupo de Estudos em Ensino de Matemática de São Paulo.
[11] Lei de Diretrizes e Bases da Educação Nacional.

Congresses and novelties such as the study of basic notions of linear functions and set theory in the first years of secondary school.

The proposal was approved with modifications by the Bureau of Education of São Paulo, and adopted by Sangiorgi as basis for the organization of a new series of textbooks entitled "Mathematics – Modern Course,"[12] released in 1963. The sale success of the collection – which received financial support from the Committee of Technical Books and Textbooks (COLTED) – granted it its position as the most known and widespread reference of the New Math movement in Brazil.

The encounter between Sangiorgi and Lucienne Félix

In her first visit to Brazil, in 1962, Lucienne Félix was part of a French pedagogical mission responsible for teaching courses to secondary school teachers in Brasília, Rio de Janeiro and São Paulo.

But why was Lucienne Félix invited to participate in an official mission? She says that, since her dismissal from the preparatory classes, in 1945, she had lost touch with the French Department of Education: "Some of us worked individually and not for any official body, but against them (which was my case)" (Félix, 1986b, p. 80).

The mission was coordinated by the aforementioned CIEP of Sèvres, which was created during the postwar period with the goals of encouraging the training of teachers in the university and promoting international collaboration for innovations in teaching practices, especially in the ongoing experience of the "classes nouvelles" (CIEP, 1971). The exchange with Brazil was established in 1949, when Brazilian teachers participated in internships in the CIEP, and proposed, inspired by the "classes nouvelles", experiences for renewing the teaching in secondary schools in Brazil (Neves, 2010). [13]

The invitation for the mission came from the head of CIEP, so that Lucienne Félix could replace, in the last minute, one of the members of the mission, who was prevented from traveling (Félix, 2005). Her addition to the mission, therefore, can be attributed to the urgency of the replacement and to her good relationship with CIEP, where Félix had even participated as a lecturer for an international internship on Mathematics organized by OECE in 1958 (CIEP, 1971). However, René Haby (2008, p. 36), who also participated in the mission, mentions that Brazilians "were very interested in the novelty that modern mathematics represented," and suggests that Lucienne Félix's coming to Brazil, due to her being known for her work, was requested by teachers and professors.

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[12] Matemática – Curso Moderno.

[13] Contacts established with CIEP also encouraged professor Martha Dantas, from the University of Bahia, to organize the First National Congress of Mathematics Teaching for Secondary School, in 1955 (Garnica, 2008).

Indeed, in a report to UNESCO before the mission, Sangiorgi had already mentioned, amidst GEEM's guidelines of action, the "convenience of using Lucienne Félix's new concepts […] about unity in mathematics teaching, according to our Brazilian patterns" (Sangiorgi, 1962, p. 3).

The mission led to the first contact between Félix and Sangiorgi. He attended a course entitled "Principles and Methods of the New Pedagogy" [14] given by the French visitors and offered to secondary school teachers in São Paulo from August 1st to 25th. Daily newspapers recorded two lectures given by Lucienne Félix during her stay in São Paulo: one organized by GEEM, on August 10th, entitled "Introduction of Modern Mathematics in Secondary Education" (Conferência sobre matemática, 1962); another, promoted by the Mathematical Society of São Paulo, on August 17th, entitled "Bourbaki, his ideas, his work" (Palestra de professora francesa, 1962).

In August, 1965, Félix returned to Brazil at the invitation of GEEM. In Porto Alegre, a southern Brazilian city, she delivered a series of conferences at the School of Philosophy of the Federal University of Rio Grande do Sul (Conferências, 1965). By the end of August, Félix delivered in São Paulo a series of lectures that were attended by hundreds of primary and secondary school teachers (Recursos…, 1965). She also visited schools that were developing new experiments in teaching, such as the *Ginásio Vocacional do Brooklin*. She visited the cities of Salvador and Recife, in the Northeast of Brazil.

In November 1968, then retired, she traveled to Latin America and Brazil for the last time. She gave two lectures for teachers at the invitation of GEEM, entitled "Practice of Modern Mathematics in Class" and "Algebraic and Topological Structures Through Geometric Situations" (Pedagoga…, 1968).

The internationalist activism of Lucienne Félix

What were Félix and Sangiorgi's motivations to seek an interchange that had little to no official support?

Lucienne Félix's books were never translated to Portuguese. Therefore, her trips to Brazil were not aimed at promoting her published work. They were not – as Georges Papy and Zoltan Dienes did – about recruiting Brazilian teachers for internships or research groups either.

In Lucienne Félix's memoirs, her trips overseas were referred to as part of the activity performed as a member of the CIEAEM. Encouraging teacher's research was a shared goal:

> This relationship [with CIEAEM] tacitly entailed the duty of encouraging research in accordance with the spirit of the Committee, within any available means (Félix, 1986b, p. 80).

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[14] Cf. certificate available at GHEMAT Arquives (APOS T 2 243 1).

Brazil was considered by Lucienne Félix, among Latin American countries, "the one closest to France, geographically and intellectually." She mentions, and emphasizes, the "research on modernizing the teaching of mathematics at primary and secondary schools," developed mainly by "Sangiorgi and his team" (Félix, 2005, p. 121). GEEM was, then, one of the targets of her activities:

> One of the tasks of the members [of CIEAEM] was to widen the circle by incorporating the most diverse characters prone to taking part in our very special group, to accepting its spirit and to contributing with their experience and reflection (Félix, 1986b, p. 80).

But Lucienne Félix also indicates that the effort to renew teaching practices was linked to deep democratic aspirations, shared by the founders of the Commission. This idea is confirmed in a testimony by Gattegno:

> Just as Mlle. Félix, I can say that the real reason for the commitment of so many people, including me, was the feeling that no one should be deprived of the joy of discovering mathematics, something that we know is within everyone's reach, because it was within our reach[15].

Not coincidentally, the first meetings of CIEAEM gathered survivors of the war who had devoted themselves to teaching math, even when suffering persecution, and were part of the resistance to Nazism:

> In the occupied countries, teachers had to improvise, to adapt to populations that were mixed, displaced, and refugees from different countries. Above all, there were young people among the persecuted, living in hiding. Professors and teachers of all levels were dedicated to the profession, teaching courses secretly, such as Papy in Belgium and Emma Castelnuovo and her illustrious father in Italy. In German-occupied Poland, Sofia Krygowska created a clandestine university that even provided students with diplomas that would later be recognized (Félix, 1986b, pp. 5–6).

The effort of traveling through Latin America, in precarious conditions, can be seen as part of an internationalist activism committed to the development of education as part of the making of more democratic societies:

> Several times, Brazil, Peru, Chile, Uruguay and Argentina were, in some way, my field of action. It would be very interesting to describe the stage of research regarding the teaching of mathematics in each of these countries which then enjoyed freedom, but whose economic and social development was dangerously unbalanced (Félix, 1986b, pp. 98–99).

But it was also about expanding her "field of action," as she suggests. In the French New Math reform, initiated in 1967, Lucienne Félix was not an

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[15] Gattegno's preface to Félix (1986b), published in French and translated by this author.

authority figure. Abroad, she took that position, giving her opinions on ongoing experiments, and also conducting some of her own. For example, she conducted an experiment in a primary school in São Paulo, exploring the idea of periodicity with small children through games (Félix, 1969).

Lucienne Félix according to Sangiorgi

Lucienne Félix's endorsement of GEEM's actions was celebrated in Sangiorgi's statements to the mainstream press:

> The results obtained by classes whose students were introduced to the so-called Modern Mathematics are such that they are likely to excite the ones responsible for the education of our youth. This impression was confirmed by the renowned French educator and mathematician Lucienne Félix, who, just over a month ago, was among us at GEEM's invitation. (O GEEM..., 1965, p. 10).

Introduced as a "French educator and mathematician", Lucienne Félix was valued both as the author of reference works of "modern mathematics" and as a secondary school teacher.

Her involvement in GEEM's activities granted "modern mathematics" the status that France then enjoyed in Brazil, both in educational circles and among mathematicians.

Furthermore, being in contact with large groups of teachers, Lucienne Félix mentioned her own teaching experience, testifying to the feasibility of implementing "modern mathematics" in the classroom.

Osvaldo Sangiorgi also included elements of Félix's speech in his articles and textbooks for upper primary school. It is interesting to observe how this appropriation was built, since the audiences targeted by each author were different, and the meanings attributed to their productions were also distinct.

The first book by GEEM, for teachers, was published in September, 1962, shortly after Lucienne Félix's stay in São Paulo. In the introduction to the book, Sangiorgi reproduces the importance Lucienne Félix gives to the ideas of set and structure:

> Set and structure are concepts that will enable students – from primary school on and with much less effort than is expended today – to understand, through connections that have not been revealed, the unity that exists in the interpretation of facts, in Mathematics and other fields. The one to expose such relationships to us, last August, was a renowned French mathematician and pedagogue, Lucienne Félix (Sangiorgi, 1965a, p. 3).

In this excerpt, we observe that Sangiorgi, while quoting Félix and endorsing his ideas with her work, develops a defense for New Math that takes into account pressing educational issues in Brazil. On the one hand, by emphasizing

understanding and interpretation, he evokes active methods which were widely broadcast at that time. On the other hand, by promising a learning process with "much less effort," he seeks the approval of those excluded from school due to flunking or failing the dreaded admission exam for secondary school.

Sangiorgi, unlike Lucienne Félix, did not intend to restructure the curriculum of secondary school according to the logic of axiomatic systems. In his books, comments on groups, rings and fields were relegated to the appendices.

Consistent with the promise of a "simplified" form of mathematics and concerned with the circulation of his works, Sangiorgi decided to write them to be intelligible not only to upper primary school students, but also to teachers of the secondary school who, for the most part, had no university education in mathematics.

Thus, the elements borrowed from Lucienne Félix's works were those which, in Sangiorgi's view, could foster an easier understanding of the concepts and tasks to be undertaken by the students.

Lucienne Félix, in her lectures, insisted on the use of gestures, graphs and colors to represent mathematical ideas. A catchphrase of hers was "Pas des phrases" (Recursos..., 1965), on the argument that the use of the fewest words as possible would bolster a better understanding of mathematical relationships by the students, and that non-verbal representations could dodge the pitfalls related to the use of the mother tongue:

> What a beautiful understanding occurs when one can be quiet and express oneself through gestures, without bringing a verbal cloak to stand between the mathematical relationships and the thought! (Félix, 1957b, p. 134).

She presented, in her books, a few suggestions of graphs, warning that it was not about imposing "a typical graph for the child to reproduce mechanically" (Félix, 1965, p. 115). Sangiorgi reproduces some of these graphs, crediting them to Lucienne Félix, in his textbooks. The first example is the one of a "truss" that represents the decomposition of the number 60 into its prime factors (figure 1).

Figure 1. Truss illustrating the decomposition of the number 60 into its prime factors.
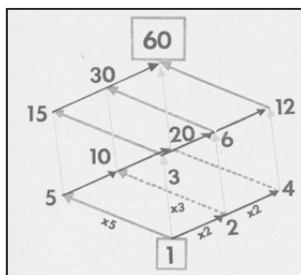
Source: Sangiorgi (1965b, p. 130)

The understanding of the factorization, represented by the "truss" is, then, contrasted with the programmed algorithms traditionally used in school to obtain the least common multiple or the greatest common factor of two numbers.

A second example of a graph reproduced by Sangiorgi is a scheme that represents the classic isosceles triangle theorem (figure 2).

Figure 2. Representation of the isosceles triangle theorem.
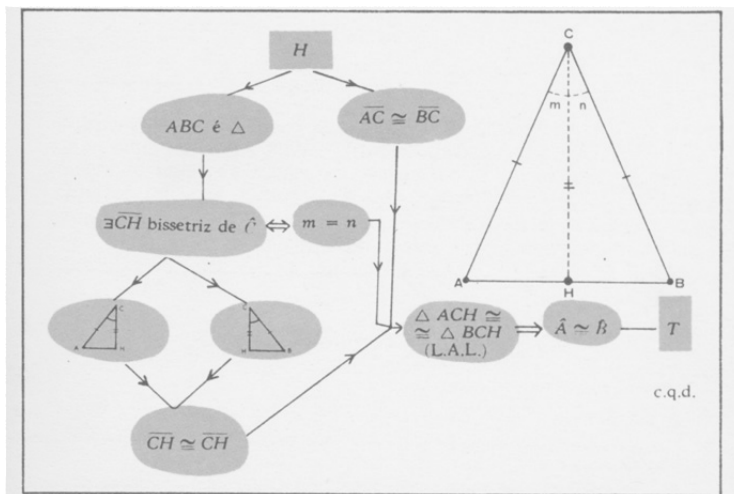
Source: Sangiorgi (1967a, p. 241)

Sangiorgi presents this scheme as one of the possible resources of a "renovated" teaching of deductive geometry, in which students are asked to participate in the construction of the representations:

> The results obtained in classroom experiments with the 'new tools' that aim to put aside, once and for all, the wrong legacy that says that studying geometry is about 'memorizing theorems and more theorems,' are astounding. (Sangiorgi, 1967b, p. 32).

The use of mathematical symbolism was also presented by Lucienne Félix as a necessary tool to express mathematical ideas with consistency and accuracy:

> We use these symbols to summarize the assertions into formulas that are, thus, independent of the subjective nature of language, placing emphasis in their logical and mathematical content (Félix, 1962, p. 1).

To exemplify the eloquence of symbols, Lucienne Félix draws an analogy between the symbol used to represent the relationship of implication (assertion p implies assertion q), and the symbol of "permitted direction of travel" in traffic signs, "a well-known sign to students who live in the city" (Félix, 1957b,

p. 129). But she explains "the green light is only lit after the justification" (Félix, 2005, p. 196), emphasizing the necessary conditions for the students to make conclusions about the existence of relationships of implication or equivalence.

Sangiorgi also recommends the use of symbols, but, instead of being concerned with developing deductive thinking, he expresses the intention of simplifying the reading of mathematical texts by students:

> The illustrious French mathematician and educator, Lucienne Félix, recommends the use of the color green for the symbol of implication, because, in this way, the "road" becomes "open" for deduction (Sangiorgi, 1965b, p. 20).

Lucienne Félix and Sangiorgi, bourbakist militants

Lucienne Félix and Osvaldo Sangiorgi followed different paths.

She had her life and career marked by her condition as a woman and as a Jew. Félix survived the internment camp and the arbitrary actions of her superiors, and built her field of action on the sidelines of official structures.

Sangiorgi followed an upward career as a teacher and author of textbooks. As GEEM's president, he was one of the most prominent leaders of the process of renewal that affected, in different ways, the teaching of mathematics in Brazil.

Félix was concerned with encouraging creative, yet rigorous, mathematical thinking. Sangiorgi wanted to confront the fear and the failures of students towards school mathematics.

Working in different institutional and cultural contexts and with different concerns, Félix and Sangiorgi claimed to be militants of the modernization movement – which she termed "bourbakism". They proposed not only to change the programs and the approach to school mathematics, but to mobilize fellow teachers to engage in the debate and research on teaching.

In building his own version of New Math, Sangiorgi drew on fragments of Lucienne Félix's work and discourse. The GEEM benefited from the prestige built around her figure. In return, Lucienne Félix got in São Paulo wider dialogue than in other cities and countries in Latin America. And she publicized these encounters, in a period in which her audience in France had declined.

Cooperation between Félix and Sangiorgi was built outside the major events and the initiatives of the agencies or governments. It contributed, albeit asymmetrically, to enlarge their space and autonomy of action regarding these same governments that, years later, would attempt to institutionalize New Math. It is an example of an exchange that followed a horizontal path which testifies the multiplicity of paths tracked by international debate on the teaching of mathematics in the 1960s.

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From Lancaster to Pestalozzi – changing views of mathematics education in Latin America during the nineteenth century[1]

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Abstract

We survey some of the main changes in mathematics education in Latin America in the 19th century, starting with the widespread adoption of Lancaster's method of mutual learning , at the beginning of the century, till the general acceptance of Pestalozzi's ideas at the end of the century. We emphasize the role played by normal schools in this modernization of mathematics education in Latin America. We also try to show some common features of this modernization processes and also some particular characteristics of the examples we chose to describe.

Introduction

The independence movements in Spanish speaking America were heirs of the liberal ideas of the Enlightenment (Hobsbawm 1996). Therefore, right after independence, most of the newly empowered governments believed that education would redeem their countries from ignorance and poverty. So, the concern with the preparation of more and better teachers and with the establishment of national school systems was widespread throughout Latin America (Garcia 2002, p. 37). Besides, it was realized that the education systems were important in building the new national states:

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[1] This paper is an expanded version of parts of Chapter 17 in The History of Mathematics Education in Latin America by João Bosco Pitombeira de Carvalho. In Schubring, Gert and Karp, Alexander (eds.) International Handbook of the History of Mathematics Education. (2014) New York. Springer. (335-359).

> The possibility of rupture of the colonial framework required to a great extent that most teachers understand and accept the new educational organization promoted by the State through laws and regulations. (Garcia 2002, 38)

Right after independence, we see attempts at the establishment of public education systems in some of the new nations in Spanish America. In most cases, the legislations that created these systems were very naïve in that they proposed actions completely out of the reach of the new countries, often involved in liberation wars against Spain or foreign interventions of various kinds. Besides, the need of military and civilian personnel to rule the new countries soon made secondary and post-secondary education more urgent. In this paper, we shall deal mostly with the elementary school systems, with a few words about the other levels of formal education.

In Brazil, before 1822, Portugal was able to smash several attempts at independence, some of which also drew inspiration from the Enlightenment (Rodrigues 1975). The country's first constitution, of 1824, stipulated that elementary education was free for all citizens[2] and the first Brazilian law regulating education, of October, 15th, 1827 ordered that a public "school of first letters should be opened in all towns and villages". The law stipulated that boys would study the four elementary operations and practical geometry, the girls the same as the boys in arithmetic but less geometry. Only with the end of slavery in Brazil, in 1888, and the need to substitute slaves by free laborers and artisans, with the consequent need of more education for the working force, does one see effective steps to make elementary schools more inclusive.

Even if we find common threads in the development of the education systems in Latin American countries, each one had its own rhythm and its conception of the role education should play in its society. Also, it is impossible to attempt to cover, in a small number of pages, the situation in all the new nations. Much to our regret, we had to limit ourselves to a few representative examples (Argentina, Brazil, Chile, Colombia, Costa Rica, Mexico, and Venezuela) which show both similarities and individual characteristics.

The beginnings of the education systems in Latin America in the 19th century

What was the situation of education, in particular of mathematics education, in Latin America right after independence from Spain and Portugal? It seems safe to suppose that Prieto´s (2010, pp. 33–34) words about Chile also apply to most of the continent:

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[2] Slaves were not citizens. Also, one needed a minimum legally fixed annual income or own certain amount of property.

> The few schools were run by persons whose intellectual preparation seldom went past reading and writing. Some of them were soldiers made prisoners during the wars of independence, and others came from less reputable occupations.

In Chile, sometimes generous persons established village schools, which survived painfully, with scant help from the municipal governments (Labarca 1939, pp. 71–72). The teachers were priests or laypersons, with very low instruction. Confessional schools in monasteries were more numerous, and they provided a somewhat more systematic instruction. By 1810, the government ordered the establishment of schools for boys and for girls in religious convents and monasteries (Labarca 1939, p. 85) but the fact that this order had to be repeated several times makes clear that it was not obeyed. The priests and nuns who would teach in these schools had to pass an examination to verify their "ability to read, write and count". The exams should include the use of "all kinds of letters" and "examples of the four arithmetic operations" (Labarca 1939, p. 85). In 1830, the Chilean Congress decided, with no avail, that the religious orders that opened an elementary school in the villages where they had monasteries would receive back their confiscated properties. Since the Church did not comply, in 1832 it was given one month to open the schools, otherwise they would be established by the municipalities and the monasteries would be charged for the costs. The schools were opened, but "it would have been better if this had not happened" (Labarca 1939, p. 87) because of their very low quality. In 1841, there were only 56 public elementary schools in the whole country. In Santiago, two years later, there were eight public elementary schools, financed by the municipality, seven convent schools, three schools run by the Church, but not in convents, and 60 private schools, with a total of 2 269 boys and 1 050 girls (Labarca 1939, p. 86).

In Colombia, the education act of 1826 established that in all parishes there should be created at least an elementary school for boys and, where possible, another one for girls. In these schools, the children would be taught the basic facts about religion, moral and civility principles, how to write and read, the first rules of arithmetic, grammar and orthography of the Spanish language, and the political constitutional catechism (Zuluaga 1979, p. 17). The same year another law instituted schools for poor children, in which only Lancaster's method would be allowed.

These examples are sufficient to show the difficulties the new independent nations faced in order to establish a system of public elementary education.

How to establish educational systems with the small number of existing schools, few and poorly trained teachers and scant resources was the challenge facing the new nations. A widely adopted answer to these problems was Lancaster's method of mutual learning, which was used in many Latin American countries to teach children reading and elementary school

mathematics and also in teacher training schools, to prepare teachers who would propagate this method all over the new nations.

Lancaster's method of mutual learning in Latin America

Lancaster's method of mutual learning was adopted in many countries, among them Argentina, Bolivia, Brazil, Chile, Colombia, Costa Rica, Ecuador, El Salvador, Guatemala, Honduras, México, Peru, Uruguay and Venezuela (Munévar 2010, p. 57).

Lancaster´s method (called the monitorial system by him), in which more advanced students taught less advanced ones, made it possible for a small number of adult masters to educate large numbers of students at low cost. From 200 to 1 000 students seated in rows, usually of 10 pupils each. An adult teacher taught the monitors and each monitor taught his row. Besides monitors who taught, there were monitors to take attendance, monitors to examine and promote pupils, monitors to rule writing paper and check slates and books and, over all the monitors, a monitor general. The best pupils became monitors as a reward for their performance. The method emphasized strict discipline and both teachers and students were like cogs in a machine. The main pedagogical characteristic of the system was its inelasticity, its mechanical, repetitious methods, and its lack of a proper psychological basis. In mathematics, the students learned (Lancaster 1810; Reigart 1916):

Lancaster´s method should be called Bell and Lancaster´s method, since its basics ideas – pupils teaching pupils – and strict discipline were first conceived by Andrew Bell (1753–1832), who was for a while the superintendent of the Madras Male Orphan Asylum, in India, an institution for the sons of soldiers. His system was modified by Joseph Lancaster (1778–1838), a Quaker, who was responsible for the popularity of the method in many countries, among them several in Latin America.

- Reading and writing numbers – the integers, common fractions and composite numbers.
- The four arithmetic operations on numbers (addition and multiplication tables).
- Decimal fractions
- The rules of three (simple and compounded).
- Percentages. Interest and discount rates.
- A difference of purpose in the use of Lancaster's method between Europe and the Latin American countries was that

[T]he Lancasterian school was the first step of the school system [In Latin America], and in Europe it consolidated itself as an autonomous and terminal instruction mode, whose purpose was the quick education of industrial workers. (Vidal and Ascolani 2009, 93)

The Lancasterian *normal* schools were in fact training laboratories, for

> They were established in already existing primary schools where student teachers learned by serving as assistants to a master teacher. The Lancasterian method lent itself very well to such a purpose, since it used older students to teach younger ones. (Britton 1994, 27)

Muñoz (1918, p. 95) claims that Argentina was the first country to adopt Lancaster´s method, in 1819, when a Lancasterian school for teachers was opened in Buenos Aires, and the method was officially adopted by the recently created *Universidad de Buenos Aires* (Vidal & Ascolani 2009, p. 93). In 1819, three years before independence from Portugal, we find a Lancasterian school in Rio de Janeiro for the "disadvantaged" (Bastos 1997, p. 125) and one for the military and their number increased considerably in the following years, particularly for the military (Neves 2007, p. 3). The method was officially adopted in 1827, but very soon there were strong objections to its use (Moacyr 1936, pp. 197, 205, 216, 252; Moises 2007, pp. 67–68). By 1837 its use was forbidden by law.

In 1821, Colombia adopted officially Lancaster's method, since the country's poverty required the use of a single method. Between 1821 and 1844, this goal had the highest priority. The task of propagating Lancaster's method by means of the appropriate training of teachers fell to teacher training schools, of which three were opened in 1822, respectively in Bogotá, Caracas and Oaxaca,[3] all of them following Lancaster's method. In 1826 the government ordered that a translation of the *Mutual teaching manual* (*Manual del sistema de enseñanza mutual aplicada a las escuelas primárias de los niños*) be adopted in all primary schools (Zuluaga 2001, pp. 42–43). By 1838 there were in Bogotá approximately 270 primary schools that used Lancaster's method (Zuluaga 2001, p. 62). The progressively increasing criticisms of this method and the discussions about the role of public education slowly eroded the support the method until 1844, when the reform instituted by Ospina became law, and the method was officially forbidden (Zuluaga 2001, p. 73).

In Chile, we see a Lancasterian *normal* school in Santiago in 1821. Shortly thereafter, there were set up Lancasterian schools at Valparaiso and Coquinhos (Muñoz 1918, p. 96). Also in 1822 we witness the establishment of a Lancasterian school in Lima, Peru, restricted to men, but it had a very short life, just one year (Ortiz 2004). Four years later, the Peruvian government established two Lancastrian normal schools in Lima, one for men and another for women, and ordered that the same be done in each province and department (Ortiz 2004, p. 60). Unfortunately, very few of the planned schools were established or survived.

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[3] We remark that at the time Venezuela, like some other present day countries in northern South America and southern Central America were part of Colombia, forming Greater Colombia (Gran Colombia).

The privately instituted *Compañia Lancasteriana* was created in Mexico in 1822 and had a great influence in Mexican education. From 1842 through 1845, the company headed the *Dirección General de Instrucción Primaria*, which was in charge of public primary education in Mexico. By 1870 the Compañia had eleven Lancasterian schools in Mexico City, among them two for adults, and a few in other cities. The company was officially closed only in 1890, ending its great influence upon elementary education in Mexico (Solana, Reys, Martínez 1982).

Lancaster's method was introduced in Costa Rica in the late 1820s and was used in many schools by 1840. Later, it was adopted officially for elementary instruction in the late 1860ties and used until the mid 1880s (Garcia 2002, p. 38; Ruíz y Barrantes 2000, p. 146).

Normal schools and their role in the modernization of education in Latin America

The creation of normal schools was an essential part of the attempts to set up public education systems in the new Latin American nations. Their purpose was to substitute untrained teachers, habituated in practice, by teachers professionally trained in the normal schools. In addition, the role played by normal schools in the modernization of school mathematics in Latin America cannot be overlooked. They were very important in the transition from an instrumental school based on Lancaster´s ideas to one founded upon Pestalozzi´s educational views, as shown by Zuluaga (2001) in the case of Colombia.

There are conflicting claims for the creation of the first of these schools in Latin America. Anyway, we see several public normal schools being established in the first half of the 19th century. Besides the ones mentioned previously (1821 in Santiago, Chile; 1822 in Lima, Peru), we can point out the following: in 1822, the *Escuela Nacional Lancasteriana*, in Mexico City, operated by the *Compañia Lancasteriana*; the *Escuela Normal de Enseñanza Mutua de Oaxaca* (1824), at Oaxaca, which also used Lancaster´s method, followed by the normal schools of Zacatecas (1825) and Guadalajara (1828) and, in 1849, the *Escuela Normal Mixta de San Luis de Potosí*; in Niterói (1835), Brazil; the *Escuela Normal de Preceptores* (1842) in Santiago, Chile.

As a result of the education reforms of the period 1832–1843, which shaped secondary education in Chile, a law of 1841 was passed providing funds to establish a normal school for prospective male teachers, in Santiago, the *Escuela Normal de Preceptores* (Campbell 1959, p. 371) followed in 1854 by a normal school for prospective female teachers, open only occasionally till 1880, when it was "recreated". The teachers formed by these normal schools spread out thinly all over the country, staffing the growing web of elementary schools. The *Instituto Pedagogico* was created in 1889 to prepare teachers for secondary schools.

The first normal school in Argentina was established in 1870, the *Escuela Normal de Paraná*, as part of Sarmiento's political and educational ideas. Influenced by American pedagogical ideas, Sarmiento used them in the *Escuela Normal de Paraná*, a model for normal schools in Argentina. Its first director and teachers were North Americans and its textbooks were translations of American texts. Students from all provinces in Argentina were sent to study at the *Escuela Normal de Paraná*, which was indeed a model for other normal schools created in Argentina (Puiggros 2006). This American influence on the preparation of elementary school teachers is not restricted to Argentina, as shown by Valente (2012) for Brazil.

As the century neared its end, the ideas of Dewey, Francis Wayland Parker and other American educators and psychologists become more and more known in Latin America and applied in the normal schools. In Brazil, the influence of American elementary school mathematics textbooks changed the existing Brazilian textbooks: we witness the appearance of mathematics textbooks specifically conceived for elementary school, not watered down versions of secondary school texts (Valente 2012, p. 65). In the state of São Paulo, in Brazil, right after the republic was proclaimed, in 1889, a state educational reform augmented the curriculum of the city's normal school to four years and established a laboratory school. This was extended to all the state in 1892 (Vidal and Ascolani 2009, p. 122).

The consolidation of public education systems

In Argentina, the school system became more organized starting in the 1950s and 60s, with four types of elementary education: private, municipal, rural and the one provided by the *Sociedad de Beneficencia*, created in 1823 and which provided elementary education for women (Vidal and Ascolani 2009, p. 118). After 1852, partially because of the ideas of Domingo F. Sarmiento, there was an attempt to unify and to make the elementary public school system more inclusive. Argentina, by the end of the 19th century had 40 normal schools spread over the country (Vidal and Ascolani 2009, p. 122), and was able to constitute a lay (non-religious) and modern normal school[4] system which provided the teachers that helped the country reach a high degree of literacy in the early 20th century.

A very promising development happened in the 1840s in Chile, with the establishment of night schools for adults, for both secondary and professional education, supported either by religious orders or by workers´ associations. In 1856, the *Sociedad de Instrucción Primaria* was created by a group of educators and intellectuals who realized that the country could not modernize itself with its

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[4] As in many countries in Latin America, the catholic church had schools, run by nuns, that prepared young girls for teaching in primary schools.

high illiteracy. The purpose of the *Sociedad* was to call attention to the educational problems of Chile and to make primary education more inclusive. To do this, it opened elementary schools for boys and girls and for adults. The *Sociedad* started its activities with four night schools for adults. In its schools the emphasis was on a more practical and useful education.

In the 1880s, education was in a very bad situation in Costa Rica. Illiteracy was high, the economic situation prevented public investments in education and the central government had almost no control of education, which remained in private hands or the Catholic Church.. In elementary education, Lancaster's method was widespread until 1850 and classrooms grouped students of different ages and levels.

One important educational reform was instituted in 1885 and 1886 by Mauro Fernández, minister for public instruction. It comprised the reorganization of elementary education, establishment of public secondary schools, the closing of the Universidad de Santo Tomás in 1888 and the prohibition of religious instruction[5] in all public schools. Fernández was familiar with the ideas of Horace Mann, Pestalozzi, Fröebel, Herbart, Jules Ferry, Andrés Bello, Domingo Faustino Sarmiento[6] and was inspired by Ferry's laws of 1881 and 1882 and was strongly influenced by the reform attempted, without success, by Julián Volio in 1867 (Ruíz y Barrantes 2000, p. 148).

The mathematics programs instituted by the reforms for elementary school comprised arithmetic and geometry. In arithmetic, one studied the operations with the natural numbers, divisibility, the greatest common divisor, fractions and measures of length, volume and time, the "rule of three". Geometry included straight lines and curves, plane geometric figures, space figures, perimeters, areas and volumes. Teaching should be intuitive, without stress on formulas and related to the other curriculum subjects. Elementary school lasted four years, and the different topics of the program were revisited several times, with progressive extension (Ruíz 1994, pp. 39–40).

From 1850 on, the educational system of Brazil begins to take a more organized shape. We have the professional and military schools, *Colégio Pedro II* and similar establishments in the provinces for secondary education, private schools for the children of the rich (often run by Englishmen or Germans, for boys, and by Frenchmen, for girls), the growth in numbers of the normal schools. All these contributed to the improvement of the mathematical level of general education, and to the establishment of a new profession, or, at least, activity: the mathematics teacher (Soares 2007). During this period, Brazil, like several other Latin American countries, followed the French model for secondary and post secondary education, while English and American

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[5] Fernández followed anti-clerical ideas prevailing in France at the time.

[6] Domingo Faustino Sarmiento (1811–1888) was a very influential Argentinean intellectual who was President of his country from 1868 till 1874. He was read not only in Argentina, but in many Spanish speaking Latin American countries.

influences were felt in primary education (Lorenz and Vechia 2005), (Neves 2006, 2007, 2008, 2009), (Gomes 2011) and (Valente 2012).

A very important development in Brazil was the establishment of "grupos escolares", which were created at first in the state of São Paulo, in 1893, and slowly spread all over the country. Until then, Brazilian elementary school classrooms usually had students of widely different ages and knowledge levels. A "grupo escolar" was a school with students grouped in classrooms by knowledge level and who had to progress yearly through a regular curriculum. Each classroom was in charge of a single teacher, and there was a building that housed all the classrooms and the administrative facilities of the school. That is, a "grupo escolar" was an elementary school as we conceive them today, a grade school. They embodied the positivist and republican ideas prevailing at the time in Brazil and were an important means to make the public elementary school system more inclusive and to mold and discipline citizens for the new modern republican society (Vidal 2006). They enforced a strict discipline that stressed punctuality, cleanness, "moral virtues" and the "civic values", which should all be promoted (Souza 2004, p. 127).

A development specific to Brazil, in the second half of the 19th century was the establishment of a very well organized and extensive system of ethnic German schools in the country southernmost states (Kreutz 2000, pp. 163–164). It had extensive pedagogical publications, among them a periodical dedicated to elementary school textbooks (Kreutz 2007). This system was often in conflict with the Catholic church, because most of the teachers in the German schools were Protestants. After the republic, it remained very active, but it eventually clashed with the unifying and nation building drive of the central government, and it was dismantled in the 1930s (Fonseca and Tambara 2012; Kreutz 2000, 2005; Marques 2010; Schubring 2003, 2004).

A changing view of education

In the second half of the 19th century, new pedagogical ideas reached Latin America. Most often, they were first put in practice in the normal schools, which kept increasing in numbers, as the several governments tried to consolidate and expand their respective school systems, and in private schools run by foreigners, particularly Americans. Pestalozzi´s ideas with its emphasis on intuition mainly through the "object lessons" teaching methodology were very influential in changing elementary and middle school education. In elementary school mathematics, the "object lessons" were particularly innovative in geometry, stressing the handling of and experiencing with actual solids, instead of the dry memorization of their elements – vertices, edges, faces – and of their classification.

Intuitive learning was influential in Europe (Schelbauer n.d.; Valdemarin 2000). For example, Felix Klein strongly advocates that the high school course

on deductive geometry be preceded by a course on "object lessons" (Carvalho 2006, pp. 74–75). We can surmise that the ongoing discussions in the political and educational circles in Brazil about the role of intuitive learning in school modernization reported by Schelbauer (n.d.) also happened in other Latin American countries. These discussions refer to several object lessons manuals, among them *Lições de cousas*, by Saffray, published in Portugal in 1908; *Plan d'études et leçons de choses*, by Jules Paroz, 1875; *Exercises et travaux pour les enfants selon la méthode et les procedés de Pestalozzi et de Froebel*, by Fanny Ch. Delon e M. Delon, 1892 and 1913 (Valdemarin 2000).

In Latin America two particularly influential books on object lessons, were Primary object lessons – manual for teachers and parents, by Norman Alisson Calkins, published in 1861 and Lessons on objects, graduated series designed for children between the ages of six and fourteen years: containing also information on common objects, written by Edward Austin Sheldon, published in 1863 (Sheldon 1863). Two editions of Calkins in Argentina, respectively in 1871 and 1872 were used officially in the city of Buenos Aires during the period 1872–1887 (Brafman 2000, p. 183; Gvirtz 2000, pp. 182–183). His book was translated into Portuguese, in 1886 (Auras 2003; Gomes 2011), and was used in Uruguay, where it was translated and published in 1872, and in Chile. It was published in the first Argentinean pedagogical journal, the Anales de la educación común, created in 1858 (Vidal & Ascolani 2009, p. 118).

Concluding remarks

Starting in the early 20th century, most Latin American countries underwent modernization of their education systems. The idea that elementary education should be made inclusive became more and more accepted and there were genuine attempts to broaden the purposes of secondary education, taking into account that not all of its students would proceed to post-secondary academic training. The liberal 19th century ideal of universal education was present in many reforms. The overall presence of French influence diminished as the 19th century ended, and American educational ideas came increasingly to the fore, particularly in elementary and secondary education. Of course, each nation found its own path, according to its educational tradition, history, and culture. Latin American educators became even more aware of the educational trends and discussions in Europe and the United States. Dewey's ideas became more and more influential.

The First World War marks the beginning of a long period of crisis in Latin America, in which many social and political structures suffered great stress. Of course this had considerable impact on education in general. In mathematics education the echoes of the first international reform movement of mathematics education, strongly influenced by Felix Klein, and which led to the creation of IMUK, (Schubring, 1987), reached the continent with considerable

delay, in the twenties and thirties (Carvalho, 2006). The next important turning point in mathematics education in Latin America was the impact of the New Math movement in many of its countries (Barrantes y Ruiz 1998; Carvalho and Dassie 2012).

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The understanding of parallel lines in early nineteenth century textbooks: A comparison between two Norwegian geometry books from 1827 and 1835

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Abstract

Bernt Michael Holmboe (1795–1850), wrote several textbooks on mathematics, and his presentation of geometry was traditional and in conformity with Euclidean ideas. Christopher Hansteen (1784–1873), professor in applied mathematics, wrote a textbook on geometry where he challenged the traditional Euclidean geometry, and he introduced the subject matter in a very "un-Euclidean" way. This paper compares the understanding of parallel lines and Euclid's parallel postulate in the textbooks by Holmboe and Hansteen. It also comments on the understanding and interpretation of parallel lines and the parallel postulate presented in the works of earlier mathematicians from the 5th to the 19th century.

Introduction

The *Elements* (Euclid & Heath 1956) was collected by the Greek mathematician Euclid of Alexandria (approx. 300 BC), and the axiomatic-deductive method that Euclid used has for more than two thousand years been a model for how to prove theorems. A well-known, and probably the most disputed, of the axioms in the Elements is the parallel postulate. The parallel postulate was for a long time accepted as self-evident, but some asserted that it was too complicated to be admitted as an axiom, and it ought to be a theorem. From Antiquity, several attempts have been made to prove it, but all without success. In the early nineteenth century, these attempts led to the discovery of non-Euclidean geometry.

Carl Friedrich Gauss (1777–1855) was probably the first mathematician to doubt the self-evidence the parallel postulate, and to conceive an idea of the possibility of a non-Euclidean geometry. He did, however, write little and

publish nothing on the subject, but his ideas have been deducted from his correspondence and notes (Ewald, 2005, p. 297).

The comprehension of the concepts have changed considerably, and I will in this paper discuss the use of the concept *parallel lines* in two Norwegian textbooks in geometry from 1827 and 1835, the former written by Bernt Michael Holmboe (1795–1850) and the latter by Christopher Hansteen (1784–1873), both professors at the University of Christiania.[1] I am interested in who, and what ideas, influenced Holmboe and Hansteen when they wrote their textbooks.

There was a present debate about the use of Euclidean ideas in textbooks in geometry, and when Hansteen published his textbook in geometry, it was evidently a controversial issue, and an attack on the Euclidean textbooks. The cause for the very bitter controversy between Holmboe and Hansteen in 1835–36 was whether one in geometry textbooks and geometry teaching should be true to Euclid or not, and especially, the handling of parallel lines. Holmboe's books were firmly in the Euclidean tradition that was typical for geometry textbooks in the 18th and 19th century. An additional source to Holmboe's and Hansteen's understanding is the newspaper polemics that occurred in two Norwegian newspapers in 1835–36. (Christiansen 2012a)

These textbooks were written for use in the learned schools of Norway, where the pupils started at the age of 9–10 years, and the duration was normally eight years consisting of four two-year grades. The learned schools gave a classic education, meant to qualify for the university. The University of Christiania, established in 1811, was in function from 1813, and the only use of mathematics in the beginning was for the *examen philologico-philosophicum* – a preparatory exam for other subjects at the university. The lectures in mathematics were on trigonometry, stereometry, basic algebra, and later applied mathematics after Christopher Hansteen's appointment.

Earlier research about the textbooks of Bernt Michael Holmboe and his textbooks may be found in Christiansen (2010, 2012a, 2012b). In these papers, there are general descriptions of geometry textbooks in Norway in the first half of the nineteenth century, and a controversy that occurred. There is also a description of Holmboe's textbooks in arithmetic.

The parallel postulate

Gray (2008, pp. 83–84) says that "The *Elements* is a highly organized, deductive body of knowledge. It is divided into a number of distinct themes, but each theme has a complex theoretical structure". It is the nature of the arguments that makes the *Elements* so convincing and, with some exceptions from the number-theoretic books, they use the axiomatic method. Book One of the

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[1] Christiania was the name of Oslo from approximately 1600 till 1925.

thirteen books in the *Elements* starts with a number of definitions, postulates and common notions. Definition 23 states that

> *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. (Euclid & Heath, 1956, p. 202)

and the fifth postulate, the so-called *parallel postulate*, states

> That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (Euclid & Heath, 1956, p. 202)

$$\alpha + \beta < 2R$$
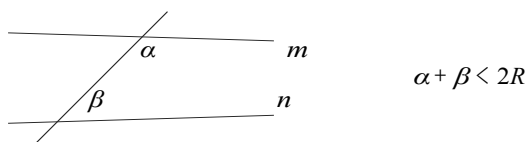
Parallel lines are therefore straight lines that do not meet.

There are several equivalent substitutes for the parallel postulate, and one of them, the so-called *Playfair's axiom*, states that "For every line l and for every point P that does not lie on l, there exists a unique line m through P that is parallel to l" (Greenberg, 2008, pp. 20–21). Assuming that by *lines* is meant *straight lines*, and given that the point and the lines are in the same plane, then it is asserted that a parallel line does exist, and that it is unique. This version was published by John Playfair (1748–1819) in 1795, but it was already mentioned by Proclos in the fifth century. (Euclid & Heath, 1956, Comment by Heath; Greenberg, 2008)

If we remove the parallel postulate and everything depending on it, we get a so-called "*neutral*" geometry, or the "*core*" of the *Elements*. There have been numerous attempts to prove the parallel postulate, but they have failed, mostly for being circular – that is using arguments that are equivalent to the parallel postulate, and many attempts at proving the parallel postulate assume that the core of the *Elements* is valid.

Further understanding of the parallel postulate

Proclos (411–485)

Proclos argued in the fifth century AD for an attempted proof of the parallel postulate. Let two lines, *m* and *n,* be intersected by at third line *k* at *P* and *Q* as shown on the figure, and let the interior angles on the same side add up to two

right angles. Then let a fourth line *l* cross *m* at *P* and enter the space between *m* and *n*. Proclos then argued that the distance between *l* and *m* gradually increases as one moves away from *P*, and therefore *l* must eventually cross *n*. This attempted proof does, however, assume that the distance between *m* and *n also* does not increase indefinitely, but this assumption is wrong since the parallel postulate cannot be taken for granted. The conclusion from Proclos' attempted proof is rather that the parallel postulate is equivalent to the statement that two lines that do not meet, also do not diverge. It is important to know which properties of straight lines that comes from their definitions, and which properties that are deduced as theorems (Gray, 2008, p. 85).
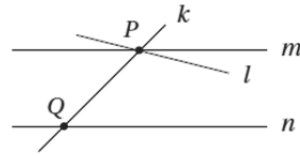
Christopher Clavius (1538–1612)

The influential Jesuit and mathematician Christopher Clavius, who edited and reworked the *Elements* in 1574, tried to argue that parallel lines could be defined as equidistant lines (Greenberg, 2008, p. 213):

> For any line *l* and any point P not on l, the equidistant locus to l through P is the set of all the points on a line through *P* (which is parallel to *l*).

Already in the 10th century, ibn al-Haytham (965–1039) argued for this definition by imagining a rigid segment *P Q* attached to the straight line *l* at *Q*, and perpendicular to *l*. When *Q* is moving along *l*, such that *P Q* is always perpendicular to *l*, then *P* has to move along another line, which is parallel to *l* (Greenberg, 2008; Katz, 2009).
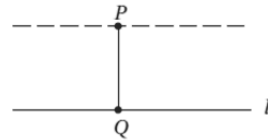
John Wallis (1616–1703)

Wallis was the most influential English mathematician before Newton. He did not to try to prove Euclid's parallel postulate in neutral geometry, instead he published in 1693 a new postulate that he believed to be more plausible:

> Finally – supposing the nature of ratio and of the science of similar figures already known – I take the following as a common notion: to every figure there exists a similar figure of arbitrary magnitude. (Greenberg, 2008, p. 215)

If we restrict our attention to triangles we can formulate Wallis' postulate as:

> Given any triangle $\triangle ABC$ and given any segment *DE*, there exists a triangle $\triangle DEF$ having *DE* as one of the sides such that $\triangle ABC \sim \triangle DEF$. (Greenberg, 2008, p. 216)

and we can intuitively understand the postulate as "you can either magnify or shrink a triangle as much as you like without distortion". Wallis used this to prove Euclid's parallel postulate, but it is equivalent to the parallel postulate, and the proof was flawed.

In a comment to the parallel postulate (Euclid & Heat, 1956, Volume I, pp. 210–211), Heath writes that Wallis proved that if a finite straight line is placed on an infinite straight line, and then moved in the direction of the infinite line, the finite line will always lie on the infinite line. Furthermore, if an angle is moved along an infinite straight line such that one leg of the angle always lies on the infinite line, the angle will remain the same, and if two straight lines are cut by a third, with the sum of the interior angles on the same side less than two right angles, then each of the exterior angles is greater than the opposite interior angle. The latter is of course only true for angles on the same side as the sum of the two interior angles are less than two right angles, but this is not stated in Heath's comment. Heath ends his comment on Wallis' arguments by concluding that

> The whole gist of this proof lies in the assumed postulate as to the existence of similar figures; and, as Saccheri points out, this is equivalent to unconditionally assuming the "hypothesis of the right angle," and consequently Euclid's Postulate 5. (Euclid & Heath, 1956, Volume I, p. 211)

Giovanni Girolamo Saccheri (1667–1733)

The Italian Jesuit Saccheri published in 1733 an attempted proof of the parallel postulate, in a book called *Euclid Freed of Every Flaw*, where he introduced a trichotomy. Unless the parallel postulate is known, the angle sum of a triangle may be one of the following cases, where R is an angle of 90°:

Case L A triangle has an angle sum less than 2R
Case E A triangle has an angle sum equal to 2R
Case G A triangle has an angle sum greater than 2R

and it is assumed that whatever happens to one triangle, happens to them all – there are apparently three geometries compatible with the core of the *Elements*. *Case E* is of course Euclidean geometry, and Saccheri tried to show that this was the only possible case. He did establish a number of interesting propositions (Gray, 2008, p. 87).

By this, Saccheri had discovered the elementary parts of the non-Euclidean geometry, but he either did not recognize it or he was afraid to acknowledge it (Greenberg, 2008).

Johann Heinrich Lambert (1728–1777)

The German/Swiss mathematician Johann Lambert pursued the idea of Saccheri's trichotomy, and he raised issues that will later be found in selections from Gauss, Riemann, von Helmholtz and others. He had a sketch of an argument showing that in *Case L*, the area of a triangle was proportional to the difference between 2R and the angle sum (Greenberg, 2008; Euclid & Heath, 1956, Comment by Heath; Ewald, 2005).

$$S = \alpha + \beta + \gamma$$
$$A \propto 2R - S$$
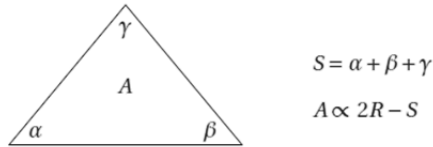
Regarding Euclid's parallel postulate, Lambert wrote that "Not only does it naturally give the impression that it should be proved, but to some extent it makes the reader feel that he is capable of giving a proof, or that he should give it. However, to the extent to which I understand the matter, that is just a first impression" (Greenberg, 2008, p. 223).

Lambert discussed the logical and philosophical status of the parallel postulate, and he criticized attempts to define parallel lines in such a way that it removes difficulties with proving the parallel postulate. He remarked the lack of geometrical precision to draw parallel lines equidistant, and that it is impossible to continue lines indefinitely. Lambert's strategy was to argue that the *truth* of the parallel postulate is not the issue, the question is "whether the Parallel Postulate can be derived by rigorous, logical inferences from the other Euclidean axioms" (Ewald, 2005, p. 157). He did, however, not submit his *Theory of Parallels* for publication – presumably due to his dissatisfaction with not being able to prove the parallel postulate – and it was published nine years after his death (Ewald, 2005).

Adrien-Marie Legendre (1752–1833)

Legendre was writing textbooks after the French revolution, and he made several attempts to prove Euclid's parallel postulate.

He has a suggestion to a proof in the 11th edition of his *Élémens de Géometrie* (Legendre, 1817, p. 22), where he first asserts that if two straight lines *AC* and *BD* are intersected by a third line *AB*, such that $\angle CAB + \angle ABD = 2R$, then *AC* ∥ *BD*.[2] He proves this the following way:

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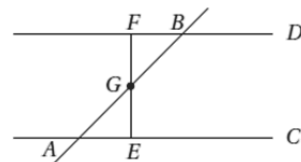
[2] *Si deux droites* AC, BD, *sont avec une troisieme* AB, *deux angles intérieurs* CAB, ABD, *dont la somme soit égale à deux droits, les deux lignes* AC, BD, *seront parallelles.* (Legendre 1817: 22)

Let *G* be the midpoint of the transversal *AB,* and let the straight line *EF,* through *G,* be perpendicular to *AC*. He then concludes that $EF \perp AC \Rightarrow EF \perp BD$. We have that both $\angle GAE + \angle GBD = 2R$ and $\angle GBF + \angle GBD = 2R$, and if we remove $\angle GBD$ from each of these sums, we have that $\angle GAE = \angle GBF$. We also have that $\angle AGE = \angle BGF,$ being vertical angles, and thus, ΔAGE and ΔBGF are equal, having a side and two adjacent angles equal in each triangle. Therefore $\angle BFG = \angle AEG = 1R$ by premise, and Legendre concludes from that that $AC \perp EF$ and $BD \perp EF$, and consequently $AC \parallel BD$.

Legendre then demonstrates in a following theorem that if two straight lines *AI* and *BD* are intersected by a third line *AB*, such that $\angle BAI + \angle ABD < 2R$, then the two lines *AI* and *BD* will meet when prolonged.[3]
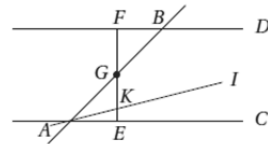
Legendre asserted the hypothesis that for any acute angle and any point in the interior of that angle, there exists a line through that point, and not through the angle vertex, which intersects both sides of the angle. He wrote that it was in conflict with the nature of the straight line not to accept this hypothesis, and it is easily proven dropping a perpendicular from the chosen point to one of the sides of the angle, and then applying the parallel postulate to demonstrate that the perpendicular and the *other side* of the angle will meet (Greenberg, 2008, p. 222).

The textbooks by Holmboe and Hansteen

BERNT MICHAEL HOLMBOE (1795–1850) was born in southern Norway. He worked from 1818 till 1826 as teacher at Christiania Cathedral School, then as a lecturer at the university from 1826 till 1834, and after that as a professor. He was the third person to be appointed professor in mathematics at the new university in Christiania. Holmboe wrote textbooks in arithmetic, geometry, stereometry, trigonometry and higher mathematics. These were the textbooks in mathematics that were predominantly used in the learned schools in Norway between 1825 and 1860, a decade after Holmboe's death. He was probably one of the most influential persons in the development of school mathematics in the first half of the 19th century in Norway. His ways of presenting the subject matter was in many ways

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[3] Si deux lignes droites *AI*, *BD*, sont avec une troisième *AB*, deux angles intérieurs *BAI*, *ABD*, dont la somme soit moindre que deux droits, les deux lignes *AI*, *BD*, prolongées, se rencontreront. (Legendre, 1817, p. 22)

very traditional, and they were challenged by his colleague and former mentor, Christopher Hansteen. As teacher at Christiania Kathedralskole, Holmboe earned a reputation as Niels Henrik Abel's teacher in mathematics. After 1826, Holmboe also held a position as teacher in mathematics at the military academy.

CHRISTOPHER HANSTEEN (1784–1873) was born in Christiania in Norway. He was first a law student in Copenhagen, but became interested in the natural sciences when he met the physicist H. C. Ørsted. He became a teacher in applied mathematics at the University in Christiania in 1814, and he was professor from 1816 till 1861. Hansteen was very productive, and wrote about terrestrial magnetism, northern light, meteorology, astronomy, mechanics, etc. He received further international recognition after an expedition to Siberia in 1828–30 to study the geomagnetism. In 1835, Hansteen wrote a textbook in geometry where he challenged the traditional Euclidean geometry.
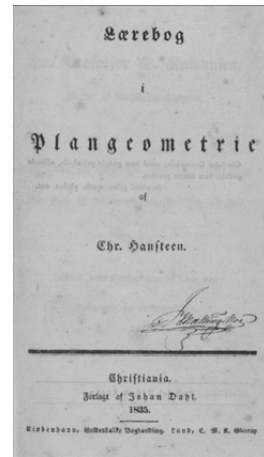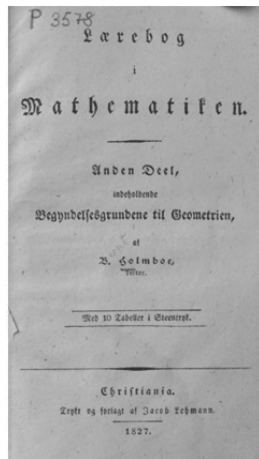
Biographic details about Holmboe and Hansteen, as well as descriptions of the fundamental concepts found in Holmboe (1827, 1833) and in Hansteen (1835) are discussed in Christiansen (2012b).

Holmboe (1827, 1833)

Holmboe's textbook in geometry came in a total of four editions, but only the two first were published in Holmboe's lifetime. There are very few differences

from the first edition to the second, and none concerning the concepts discussed in this paper.

Two of the chapters are called "About two straight lines intersected by a transversal",[4] and "About parallel lines".[5] The first of these chapters gives a thorough description of all pairs of angles this situation produces. This is followed by the consequences of two corresponding angles being equal, and vice versa, the situations which have the consequence that the corresponding angles are equal (Holmboe 1827, pp. 11–16).

The chapter "About parallel lines" has a theorem with a proof which states that when two straight lines are intersected by a transversal, such that an outside angle is equal to its corresponding interior angle, that is $\angle r = \angle p$ on the figure below, then the two straight lines cannot intersect, no matter how far they are prolonged in both directions (Holmboe, 1827, p. 45). The structure of the proof is that if the two lines cross on one side of the transversal, then the two lines and the transversal form a triangle, where $\angle r$ is an outside angle. Holmboe has already demonstrated that an outside angle of a triangle is always greater that any of its interior angles (Holmboe, 1827, p. 34), so therefore $\angle r > \angle p$, which contradicts the condition.

This is followed by Holmboe's definition of *parallel lines*.

> *Two* straight lines in the same plane that do not intersect when prolonged indefinitely to both sides, are *parallel* to each other, or the one is parallel to the other.[6] (Holmboe, 1827, p. 46)

In two following theorems, using the same situation of two straight lines intersected by a transversal, he demonstrates first that if the outside angle is greater than the interior, $\angle r > \angle p$, then the two straight lines are *not* parallel. He next proves that *if* the two lines are parallel, then $\angle r = \angle p$. This last proof is done by assuming that $\angle r \neq \angle p$, and showing that the lines then are *not* parallel (Holmboe, 1827, p. 50).

$$\angle r = \angle p \implies m \parallel n$$
$$\angle r > \angle p \implies m \nparallel n$$
$$m \parallel n \implies \angle r = \angle p$$
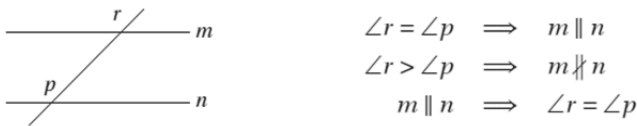
In a following corollary he then states that if two lines are parallel, and inter-sected by a transversal, then the sum of the two interior angles equals 2R. This is a consequence of the previous theorem that proves that $\angle r = \angle p$. This is followed by another corollary stating that if the sum of the two interior angles

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[4] Om to rette Linier, som overskjæres af en tredie
[5] Om parallele Linier
[6] To rette Linier i samme Plan, som til begge Sider forlængede i det Uendelige ikke skjære hinanden, siges at være *parallele* med hinanden, eller den ene at være parallel med den anden.

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is *not* equal to 2R, then the two lines are *not* parallel (Holmboe, 1827, pp. 51–52).

$$m \parallel n \implies \angle o + \angle q = 2R$$
$$\angle o + \angle q \neq 2R \implies m \nparallel n$$
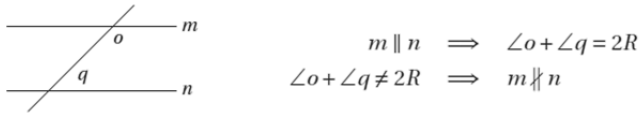
These two corollaries carry many characteristics of corresponding angles in the original text. It is the last one mentioned here that has the same wording as Euclid's parallel postulate, but it is not emphasized in any way.

Holmboe's proof is based on a theorem stating that the part of a plane that is between the two sides of an angle, is always larger than the part of a plane that is between two straight lines and on one side of a transversal, when the transversal cuts the two lines in such a way that an outside angle equals the opposite inside angle (Holmboe, 1827, pp. 48–49). It is an interesting observation that this way of arguing is unlike all other ways of arguing for the parallel postulate in this paper.

Holmboe is in his textbook very true to the ideas of the *Elements* in the way of introducing and presenting the subject matter, but without ever referring to or even mentioning Euclid. The textbook is a collection of definitions, axioms, theorems and proofs, and nothing is used before it is defined or proven.

Hansteen (1835)

In 1835, Christopher Hansteen published a textbook in basic geometry (Hansteen 1835), which in many ways challenged Holmboe's textbooks. Hansteen tried to expand Euclid's definition of straight lines and of parallel lines, and Euclid's parallel postulate (Euclid & Heath, 1956).

Hansteen argued for an understanding of parallel lines where one lets a perpendicular to any kind of line move along this line, in such a way that it always is a perpendicular. Any point on this perpendicular then describes a line, where any point's smallest distance to the original line all over is the same (Hansteen, 1835, IX). Consequently, Hansteen has a definition of parallel lines

> Any line that is being described by a point on the perpendicular to a given line, when it moves along the same with an unaltered angle, is said to be parallel to the directrix. (Hansteen, 1835, p. 59)

Where the characteristics of a line, parallel to another, are
- it always cuts off equal parts of all its perpendiculars
- any perpendicular to one of these lines is also a perpendicular to the other

And a parallel to a straight line has in addition the following characteristics
- the parallel is also a straight line

- as these straight lines never intersect, they form no angle with each other
- if the parallel lines are intersected by a transversal, then the alternate interior angles are equal, the corresponding angles are equal, and the consecutive interior angles equals 2R.
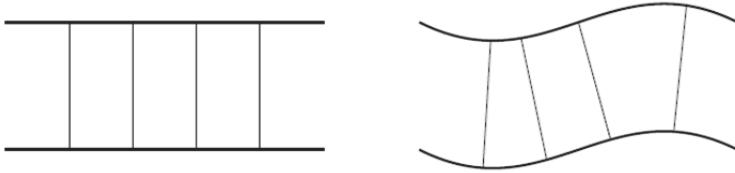
By following these properties of parallel lines, Hansteen transform Euclid's disputed axiom into a corollary, which he proves (Hansteen, 1835, p. 70):

If two straight lines, KB[7] and CD are intersected by a transversal EF in such a way that the sum of the two interior angles $\angle x$ and $\angle y$ is less than 2R, then the two lines must necessarily cross when prolonged in the directions JB[8] and GD. This is demonstrated by showing that $\angle x = \angle m$ $+\angle n$ together with the premise $\angle x + \angle y <$ 2R gives that $\angle m + \angle n + \angle y < 2R$. Since $\angle n + \angle y = 1R$, then must $\angle m < 1R$, and the two straight lines KB and CD must cross.
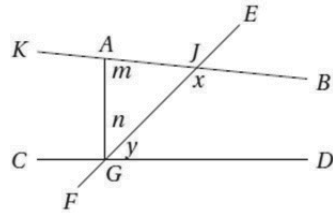
The definition of parallel lines given by Hansteen is exactly the same as the definition given by ibn al-Haytham, see page 4, with the not insignificant difference that Hansteen does not restrict his definition to be valid for straight lines only.

Hansteen does all over let lines and planes be produced by the motion of points and lines, because this method gives the clearest conception of a line's direction in any point. He admitted that one may easily imagine that a point in motion has a certain bearing in any place of its trajectory. Some geometers object to this method since motion involved *time* and *power*, two concepts that are irrelevant to geometry, but belongs in the mechanics. Hansteen states that the motion of an immaterial point requires no power, and that we are only elucidating a motion in our minds (Hansteen, 1835, p. XII). The perpendicular in a point of a curve requires smoothness and differentiability, and one may easily find examples of curves where a parallel according to Hansteen's definition will cross both itself and the given curve.

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[7] KB is misprinted as HB in the original text.
[8] JB is printed as IB in the original text, which was common.

He also claims that if two parallel lines are intersected by a transversal, and the sum of two interior – or exterior – angles equals 2R, means nothing more than that the sum of two adjoining angles equals 2R.

Hansteen's textbook was published in one edition only, but one reason may be that it contained much subject matter outside the school curriculum. He explains that because of a limited production of textbooks in Norway, he has added subject matter beyond the curriculum of the learned schools, but should be of interest for students that want to prepare themselves for a study of the higher mathematics (Hansteen, 1835, p. XVIII). It is also worthwhile to mention as a curiosity that Hansteen in his textbook introduces and describes *Metre* as a new unit of length (Hansteen, 1835, p. 81).

Some concluding remarks

Holmboe is in his textbook very true to Euclid in the way the subject matter is presented, without ever mentioning Euclid's name, and parallel lines are dealt with in a very thorough way. Both Holmboe and Hansteen give proofs of the parallel postulate, but they are both incorrect. The difference between the two authors was rooted in whether one in mathematics education should present the subject matter in a traditional Euclidean way or not. Hansteen lets utilitarian considerations overrule logical deduction and theoretical thinking, the basis of the textbook is real life, with references to artifacts like corkscrews, stovepipes and hourglasses, and he tried to expand Euclid's definition of straight lines and of parallel lines. The publication of Hansteen's textbook in geometry was a controversial issue, and an attack on the Euclidean textbooks. An additional source to Holmboe's and Hansteen's understanding is the newspaper polemics that occurred in two Norwegian newspapers in 1835–36 (Christiansen, 2012a).

Hansteen was intentionally trying to tear down the walls that existed between the classical geometry on one side, and the newer analytical geometry and the infinitesimal geometry on the other. Hansteen makes noteworthy objections to definitions of infinitely long straight lines, by asking with what tool such a prolonging shall be made. He states that it is more proper that a "mechanical artist" derives rules for his practice from the definitions and theorems of the geometry, than that the theoretical geometer shall direct his concepts and definitions towards this practice. The carpenter's planer and the metalworker's file are tools that are suitable for producing homogeneous planes and lines, and the geometer should not neglect to acquire the theoretical principles on which these methods are based. A ruler is described as a tool – made by wood or metal – by which one may produce straight lines in a plane (Christiansen, 2012b).

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Teaching the mathematical sciences in France during the eighteenth century: a few examples from some of the most used textbooks

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Abstract

*In 18th century France the textbooks for teaching the mathematical sciences in higher education – the "*Cours de mathématique*(s)" – were substantially changed in their structure. Around the middle of the century, the French-writing tradition definitely dropped out of the conception of a highly comprehensive mathematics, which, for instance, usually included fortifications, geography, astronomy, optics, perspective, and gnomonic, in favour of a narrower, but more technically approached, one. As we expect, it turns out that these textbooks are very tied to the educational system of the time. In particular, the development of the military schools and the desire to let them match high-standard requirements produced a kind of technical and scientific-oriented textbooks, which contain the specific topics that an officer is not only supposed to know, but also to apply. In the following, I will firstly provide a short overview on the French educational system of the 18th century, with particular attention to the kind of schools where the teaching of mathematics in a broad sense played a central role. Furthermore, I will deal with a selection of some of the most used textbooks, their contents, and their pedagogical approach.*

Introduction

First of all, we need to discuss two preliminary points to dispel any possible misunderstanding.

During the 18th century, the term "*mathématique*(s)" was understood in a much wider sense than we do nowadays. It is rare to find the expression "*sciences mathématiques*" in the textbooks of French author of the 18th century. Nevertheless, I prefer to employ the term "mathematical sciences" rather than "mathematics" to underline its comprehensive meaning and to avoid

misunderstandings. Indeed, at that time, the "pure" subjects, as geometry, arithmetic, or algebra, were not only included in the mathematical sciences but also the "mixed" ones, as for example astronomy, mechanics, optics, fortifications (cf, for instance, Savérien 1753, p. xxviii).

The second preliminary point concerns our choice of the textbooks. We consider the following criteria. Firstly, the textbooks must have been written with a teaching purpose. Secondly, they must provide a complete presentation of the mathematical sciences. The meaning of "complete" depends on each author. Thirdly, we only consider textbooks that are written in French. Fourthly, with regard to time recognition, we use textbooks from the half of the 17th century, when the first textbooks written in French appeared, to the French Revolution, when the educational system underwent major changes. If one looks for textbooks applying these criteria, one will find about sixty titles. Clearly, I cannot provide a satisfactory account of all of them in this paper, so I chose a (very small) selection of some of the most common ones.

Finally, I would like to add a remark on the circumstances that were the reason for this paper. This study takes place in the context of the project "*Traditionen der schriftlichen Mathematikvermittlung im 18. Jahrhundert in Deutschland und Frankreich*", financed by the Deutsche Forschungsgemeinschaft (DFG) at the Bergische Universität in Wuppertal. The final aim of this project is to establish a comparison between the German and French textbooks that were used during the 18th century to teach mathematics in higher education. Eventually, we hope to manage to analyze the emergence of traditions in teaching mathematics in this period, and also to retrace their possible origins in the textbooks written in Latin, especially by the Jesuits. To this purpose, we are moreover working at a comprehensive database, based on the software developed by another DFG project, the "*Personendaten-Repositorium*", at the Berlin-Branderburgische Akademie der Wissenschaften.

The French educational system around the 18ᵗʰ century and the mathematical sciences

With regard to the institutionalized science teaching, we need to consider at first that

> [i]n the context of seventeenth- and eighteenth-century France the term "higher education" is an anachronism. It implies the existence of a carefully articulated system of educational provision, functionally differentiated and age-specific. But at this date no such system pertained anywhere in Europe, let alone in France. (cf Brockliss 1987, p. 2)

Despite this situation, we will try to summarize the main characteristics of the French educational system in this period.[1]

Where the mathematical sciences were taught in France?

Mathematics was taught in a variety of contexts in France during the 18th century. From the most to the least attended establishments, there were: the universities, the colleges, the military schools, the technical schools, and the *maisons particulières*. Moreover, the mathematical sciences were also taught by free teachers.

The universities were in some cases very old institutions dating back to the 12th century. In France at the eve of the French Revolution, there were about 25 universities with 300–400 students each on average. They were financed by private endowments (by the Crown in particular) and by the student fees, so that they were overall quite rich. They had the monopoly of granting degrees in one of the three faculties, namely medicine, law, or theology. Nevertheless, due to the fact that there were some faculty boards to evaluate the students in the propedeutical subjects in order to admit them, mathematics was also a topic dealt with. Indeed, since the colleges had absorbed the teaching of the former faculty of arts, this last one had been reduced to the function to deliver the degrees being necessary for entering one of the three faculties.

In general, the colleges had been more recently created (the first one was founded in Paris during the 15th century) and in a larger number than universities. At the eve of the French Revolution, out of the 348 colleges, only 171 offered a complete teaching, which means the last year of philosophy, and were called the *collèges de plein exercice* (cf Brockliss 1987, p. 22). During the 17th century, they were generally more attended than the universities, but afterwards a period of decline started due to the overall educational provision. The colleges were run both by seculars (cf Brockliss 1987, pages 23 and 481) and by teaching orders, like the Jesuits (up to their expulsion in 1762), the Oratorians, the Benedectines, and to a lesser extent by some others. They were mostly boarding schools and were intended for the students who wanted to continue their studies at a university. The colleges could only bestow a degree if they were affiliated to a university. In colleges, the liberal arts and philosophy were the main teaching subjects. Mathematics was only taught during a part of the last year of philosophy, together with physics.

Of a far more recent creation (second half of the 18th century), the military schools were meant to train the future army officers. They had been founded in small towns, usually where a college previously existed, and will be the basis of the future *académies*. Since they were restructured colleges in most cases, they inherited some of their characteristics. For instance, they were boarding schools run by regular teaching orders, especially by the Benedectines. The

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[1] For the rest of this section, I will refer to Brockliss 1987 and Taton 1964.

mathematical sciences together with equitation and military tactics were taught – and one could find an instruction in these last topics only in these kinds of schools – as well as classics and philosophy.

The technical schools like the École d'Hydrographie (1666), the École Royale des Ponts et Chaussées (1775), the École des Mines (1783) and the *maisons particulières*, founded in the 18th century, were also meant to satisfy the demand for institutional instruction in practical mathematics. Especially the *maisons particulières* were private schools that served to prepare candidates for the entrance exams. Finally, the free teachers were completely independent from the institutional context (and this makes the data recovering even more difficult). Therefore, they are not part of my topic.

To sum up, during the 18th century the mathematical sciences were mainly taught in two kinds of institutions, namely the colleges and the military schools. Nevertheless, only in the military schools the teaching included a wide spectrum of topics and, in addition, only in some schools high standard level was reached. Without any doubt, the most innovative mathematical teaching was delivered at the École du Génie in Mézières. The list below shows the most renowned military schools up to the French Revolution.

Table 1. French military schools around the 18th century

Écoles des Gardes du Pavillon et de la Marine	1689	Brest, Toulon, Rochefort
Écoles d'Artillerie	1720	Auxonne, Besançon, Grenoble, La Fère, Metz, Strasbourg, Valence
École du Genie	1748	Mézières
École Royale Militaire	1751	Paris
École Royale de la Marine	1773	Le Havre
Écoles Militaires	1776	Auxerre, Beaumont-en-Auge, Brienne, Effiat, Pont-à-Mousson, Pontlevoy, Rebais, Sorèze, Thiron, Tournon, Vendôme

Note that the École du Génie was unified to the École d'Artillerie in La Fère in 1756 and then moved again to Bapaume in 1765.

Who were the teachers?

In the military schools and colleges, the teachers who explained the mathematical sciences were officially lecturing either on "philosophy" or on "mathematics". They could belong to a religious order, but in the main they were secular. The list below shows the most famous teachers who also wrote a textbook in the mathematical sciences.

On the other hand, it is much more difficult to evaluate the number of the students and identifying precisely their background or their professional orientation since very few recordings are left.

Table 2. Mathematics teachers in France around the 18th century

Bernard Forest de Bélidor	École d'Artillerie (La Fère)	1720-1738
François Blondel	Université de Paris	
Charles Bossut	École du Génie (Mézières)	mid-18th ct
Charles Camus	École du Génie (Mézières); École d'Artillerie (La Fère)	mid-18th ct; mid-18th ct
Nicolas-Louis de La Caille	Collège Mazarin (Paris)	1739-1762
Bertrand Lamy	Collége de Saumur, Angers ; Seminaire de Grenoble	1669-1675 ; 1665-1687
Jean-François Marie	Collège Mazarin (Paris)	1770s
Jean-Antoine Nollet	Collège Navarre (Paris)	1756-1770
Dominique Rivard	Collège Beauvais (Paris)	1735-1770
Pierre Varignon	Collège Mazarin (Paris)	1690s

Which kind of mathematics was taught?

With regard to the classification of the knowledge, the two important categories of arts and sciences were identified, for instance in the Encyclopedias and in the one by Diderot and d'Alembert. In turn, the arts were divided into the mechanical ones, meaning the manual artisan crafts, and the liberal ones, for instance the languages, the art of war, design. Medicine, law and theology came under the umbrella of the sciences. They were increasingly ordered according to importance or "generality". Philosophy was considered as propedeutical, inasmuch as it provided the conceptual tools that a student would need in his further studies. Even though the contents of a philosophy grade could be very varied, starting from the 17th century, we find that, to a greater or lesser extent, physics and mathematics were also taught.

During the 18th century, the didactic concerns in mathematics (and in the mathematical sciences) in particular underwent some changes. Since mathematics deals with natural bodies in abstract, in the previous century it was considered to be subordinated to physics, which deals with natural bodies *tout court*. Therefore, it was taught after physics but as a matter of fact it was more usually not taught at all. In the 18th century, mathematics was considered as a real science that trains the intellect to argue. Moreover, due to the increasing mathematization of physics (especially of Newtonian physics), mathematics should be taught before physics; otherwise the students were lacking the far more sophisticated required grounding. We find here *in nuce* the two main (sometimes opposed) arguments to justify the usefulness of studying mathematics: because it is a mental training and because it is needed in the technical applications.

In practice, by the 1720s, the average amount of mathematical instruction that a student received went not much beyond than the Euclidean principles. Afterwards, the situation changed substantially. If students had to understand a lecture on mathematical sciences like dynamics, statics, or hydrostatics, they needed at least the basic notions of algebra. We have few evidences to report on the period between the 1720s to the 1760s, but at the end of this period we know (cf Brockliss 1987, p. 385) that even in smaller towns the students were

instructed in conic sections and most of them had also been introduced to calculus.

A common problem in teaching mathematics in this period was the time. Indeed, a teacher had only one year (the philosophy year) available to introduce the students from the basis in arithmetic to calculus, and then to physics. Later on, this will lead to a splitting of the philosophy grades.

By which means mathematics was taught? The case study of three French textbooks from the 18th century

Around the half of the 18th century, some pedagogical concerns inspired the creation of new textbooks. They were needed in order to maximize the students' understanding by replacing the dictation practice during the lectures. To this aim, the students, and not only the teachers, were supposed to have them at hand. This implies that, in the large, the textbooks were decreased in size, from *in folio* or *in quarto* to *in octavo*. For instance, Bélidor's book, which was written around twenty years before the two others and – as we will see – was published in an old-style fashion, is an *in folio* as in the 17th century Claude Françoit Milliet Dechales' textbook (1674) was. We also recall that, for instance, Jean Prestet's textbook (1675) or still the second edition of François Blondel's one (1699) were *in quarto*. On the contrary, La Caille and Camus' textbooks are *in octavo*.

The demand for new textbooks also depends on the fact that the mathematical sciences were gradually becoming a teaching subject that was no more marginal. Indeed, in various kinds of military schools the mathematical sciences were a prominent topic. Therefore, their teaching underwent a rethinking. Moreover, the older textbooks did no more comply with the latest technical requirements, especially after the reorganization of the French Royal Marine (1763) and the resultant demand for well-prepared officers.

Of course, it is also likely that these new textbooks were used for self-teaching outside an institutional context, as it is sometimes written in the prefaces.

In the following, as already mentioned above, I made a choice concerning the textbooks that are considered in detail. Out of more than two hundred works, I have chosen three of them: the textbook by Bélidor (1725), the one by La Caille (1741–1750), and the one by Camus (1749–1751). My criterion has been to consider the textbooks which, as far as we know, were among the most common ones at the time. Moreover, the three scholars were members of the Académie des Sciences and the eulogies that have been read on the occasion of their death (all by Grandjean de Fourcy, permanent secretary of the Académie) are still available in the archives, which makes it easier to recover some biographical information. Under these conditions, Bézout's textbooks, the *Cours*

de mathématiques à l'usage des Gardes du Pavillon et de la Marine (1764–1769) and the *Cours de mathématiques à l'usage du Corps royal de l'Artillerie* (1770–1772), maybe the most widespread textbooks of the time, are notably absent in the following. Indeed, my main concern is to give an overall frame of the mathematical textbooks of the period. Therefore, I decided to omit the analysis of these two textbooks, on the one hand because of reasons of space and on the other hand because, if needed, the very detailed treatment provided by Liliane Alfonsi (cf Alfonsi 2011) can provide important information.

The textbook by Bélidor (1725)

Bernard Forest de Bélidor (1698–1761) had a double career as an army officer and a military engineer. Between 1720 and 1738, he held the chair in mathematics at the artillery school in La Fère. He was also in charge of some administrative duties: in 1758, he held position of inspector of the arsenal in Paris and in 1759, the position of *inspecteur général* of the mines. He was moreover an academic: he entered the Royal Society in 1726 and the Académie des Sciences in 1756.

Bélidor wrote many textbooks for the civil and military engineers. His work *Nouveau cours de mathématiques à l'usage de l'Artillerie et du Génie* was put out as a manuscript version since 1722, and in 1725, it was published in one *in folio* volume. The textbook was originally conceived for the students of La Fère, where Bélidor was a professor, but it was commonly used in all the artillery schools for almost two decades, and also at the École Royale des Ponts et Chaussées. As Grandjean de Fourcy remembers in his eulogy, the fact that the topics treated in Bélidor's textbook were considered as the overall knowledge that an officer needed made him to be acknowledged in a way as the "general professor" of mathematics (cf Fourcy 1761, p. 171).

As Bélidor writes in the preface, he believes that arithmetic, algebra and geometry are the common base of all the mixed mathematics, but he also thinks that the officers and the military engineers should not study mathematics as someone who wants to dedicate his whole life to it. Thus, he made an effort to gather everything that a military engineer has to know in one volume with examples and applications. Bélidor indicates some pedagogical concerns, mainly focused on helping the beginners. On the one hand, he wants an engineer to know the reasons of the results that he uses, since one performs more surely operations when one is aware – for instance – of the nature of the numbers employed. On the other hand, he tries to simplify the exposition by shortening the operations as much as possible.

The textbook consists of sixteen chapters, called "*livres*". The half of them (*Livres* I to VIII) is devoted to pure mathematics, namely arithmetic, geometry, and algebra. After that, we find conic sections and their application to projectiles trajectories, linear trigonometry and levelling, measuring by using the *toisé* unit measure and how to construct frameworks for buildings, the measures

of regular and irregular surfaces and solids, how to divide fields and how to use a sector, how to deal with alloys, the study of moving bodies and bomb throwing, and finally static mechanics, hydrostatic, and hydraulic. Compared to the textbooks that we will take into account in the following, Bélidor's one is written some twenty years before and is an example of a text in which the mathematical sciences are dealt with in an old-style fashion. This means that they are organised in fragmented series of heterogeneous topics, such as, among others, the usage of the old unity of measure of the *toisé*.

Figure 1. Title page of the textbook by Bélidor
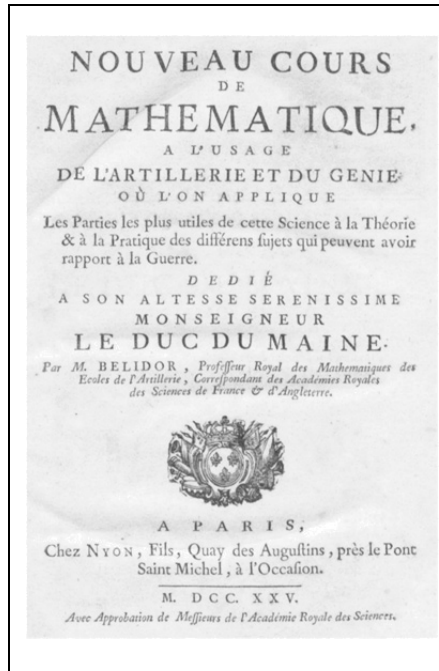
The textbook by La Caille (1741)

Nicolas-Louis de La Caille (1713 – 1762) was an astronomer and a very famous professor. From 1739 to 1762 he taught mathematics at the Collège Mazarin. Moreover, he was a member of many Academies, like the Académie des Sciences (since 1741) or the academies in Saint Petersburg, Bologna, and Göttingen.

After that, he was in charge of the mathematical teaching at the Collège Mazarin and La Caille gave the authorization to print his lectures. The outcome is a complete *in octavo* textbook in the mathematical sciences which consists of four volumes. This textbook should have reasonably replaced the textbook by Jean-Mathurin Mazéas (1758), which was no more published in Paris after

1776.[2] The *Leçons élémentaires de mathématiques* date back to 1741; they were extended by Marie in 1770 and published until 1811. They deal with arithmetic, algebra, geometry, conic sections, and (less than two decades after Bélidor's textbook) with differential and integral calculus. The *Leçons élémentaires d'astronomie* had been firstly published in 1743 and then extended in 1779 by Jerôme Lalande, who was a former student of La Caille. They collect the knowledge that La Caille had learned since he practiced astronomy as a profession. The *Leçons élémentaires d'optique* and the *Leçons élémentaires de mécanique* were respectively published in 1746 and 1750. The former deals with optics, catoptrics, dioptric, and perspective which were quite common subjects at that time. Any description of the instruments and machines is left out because La Caille rather regards them as belonging to experimental physics. The lecture on mechanics, instead, should not only be focused on the machines, but on everything that can be moved and that can move something else, that is on the whole matter. With regard to to his reputation as astronomer, La Caille explains that the textbook about mechanics is originated in the dissatisfaction with the existing books on the topic. This deals with linear motion, shock, and circular motion, trying to reduce their principles to a clear and methodical system.

In his prefaces, La Caille presents a number of pedagogical concerns. First of all, he decided to write the textbook in French, since he believed that this language was more suitable than Latin to explain the mathematical sciences. Moreover, he argued that teaching mathematics in Latin was an ancient, almost abandoned method since this language did not help the students to grasp easily the mathematical contents (cf La Caille 1766, p. 22). It is well-known that there were textbooks in French since the 16th century. Thus, this could attest La Caille's will to distance him from the Jesuit tradition in colleges and universities where some texts might still have been in Latin. Going back to the pedagogical concerns, La Caille says that he tries to "press a bit" the mathematics in the first volume in order to gain a double advantage. Indeed, on the one hand, he manages to deal briefly with pure mathematics, while on the other hand, by not writing all the explications gained the advantage to challenge his public and to make it active in reading. Since the textbook contains such brief accounts, an aloud explication by a teacher is the best way to benefit from it – La Caille says that this is its core ("*âme*"). The textbook should better be considered as a printed exercise book, so that students and teachers could save time by avoiding transcribing the lectures, which were orally dictated. As it will be later the case for Bézout's textbooks, La Caille uses two different character sizes. The smaller one is intended for the most advanced topics. As a general rule, he says that he prefers the indirect methods, as the double false position, even if they are not as elegant as the geometers would have liked to. Indeed, they have

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[2] Mazéas was a philosophy professor at the Collège Mazarin in Paris. His textbook, printed in Paris in 1758, is called *Elémens d'arithmétique, d'algèbre et de géométrie, avec une introduction aux sections coniques.*

a valuable advantage over the direct methods since in most of the cases they are easier to apply in practice and lead to easier calculations.

Figure 2. Title page of the textbook by La Caille
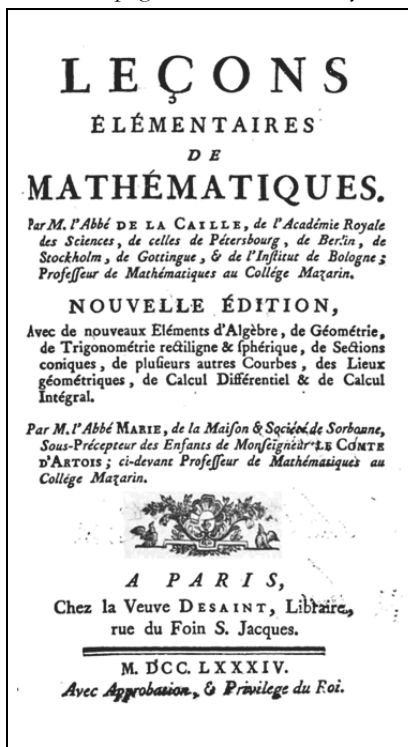
The textbook by Camus (1749-1751)

Charles Camus (1699 – 1768), a former student of Pierre Varignon,[3] covered mainly academic, teaching, and administrative positions. During the mid-18th century, he was a mathematics professor at the military engineering school in Mézières and at the artillery school in La Fère. Moreover, he was the predecessor of Bézout as examiner for the military engineers and for the artillery up to 1768. He was a member of the Académie des Sciences since 1727, and he also took part in the Royal Society, in the Académie Royale d'Architecture, and in the Academy of the Navy.

Camus' *Cours de mathématiques* is based on the lectures delivered at the Académie Royale d'Architecture[4] (as Camus says in his preface, so we suppose

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[3] Varignon was a professor of mathematics at the Collège Mazarin at the end of the 17th century and a member of the Académie des Sciences in Paris, of the Royal Prussian Academy of Sciences in Berlin, and of the Royal Society in London.

[4] This was founded by Louis XIV, who puts its direction into the hands of his *Premier architecte*. It gathered a variable number of members and also delivered some teaching, among which a mathematical one was.

that he also lectured there). It was commissioned for the military engineers by the Minister of war the Comte d'Argenson, who also defined the topics to be included. As a consequence (and as it will later be the case for Bézout), Camus' textbook easily received the governmental imprimatur. Moreover, when the corps of the military engineers and of the artillery were merged in 1756, Camus' textbook started to be also used in the artillery schools. Considering that Camus became the examiner for both schools, it is not surprising that the book achieved success (four complete editions up to 1769). Anyway, the book was also strongly criticized since it had been conceived for the military engineers, but adopted in the artillery schools without any real change. We remark, in particular, that more than two-thirds of the volume on mechanics is devoted to gravity centres, and only a minimal part deals with forces. Camus' textbook is divided in three volumes. The first one, which dates back to 1749, deals with arithmetic. Its main topics are numbers, proportions, alloys, progressions, and combinations. The second volume, published in 1750, is about geometry, namely about lines, surfaces, proportions, solids, plane trigonometry, and some special curves. The last volume from 1751 treats mechanics, and more precisely statics. As already mentioned above, it deals with gravity centres, but also – even if in a far less proportion – with forces and machines. In truth, a fourth volume on hydraulics was also prescribed by d'Argenson, but it had never been written. Moreover, there should have also been another volume where analysis and algebra (intended as calculations with letters or "*calcul littéral*") should have been coupled, but it has been never published. We could interpret the following argument by Camus as an explanation: he considers that the calculations with letters should not be treated together with the numerical ones (that is, arithmetic), since this does not agree with the mandatory topics that engineers must be instructed to (cf Camus 1749, p. iv).

Concerning the pedagogical positions, Camus' textbook is different from the two preceding ones because he does not seem to be concerned with simplifying the task to the beginners. His aim is rather to develop the topics in a very detailed way, because he believed that the existing textbooks were too much superficial. On the polar opposite of Alexis Clairaut, who between 1741 and 1746 wrote some textbooks in which he praised the pedagogical value of letting the reader gradually discover abstract propositions starting from particular problems, Camus' praised the "synthetic" or inductive way and opened his textbook by explaining precisely the meaning of "definition", "theorem", "problem", "corollary", "remark", and "scholium". This choice has been strongly criticized by his contemporaries, who blamed him for having written a book too elaborated for beginners. Even Grandjean de Fourcy cannot avoid to recall in Camus' eulogy that such an inductive method makes the book more difficult and long for beginners to read (cf Fourcy 1768, p. 152).

This textbook, as the one by La Caille, highlights a significant change in topics compared to Bélidor's one. Especially the range of the mathematical

sciences is getting narrower and narrower and, at the same time, more technically specialized, including calculus and a very detailed part on mechanics.

Figure 3. Title page of the textbook by Camus
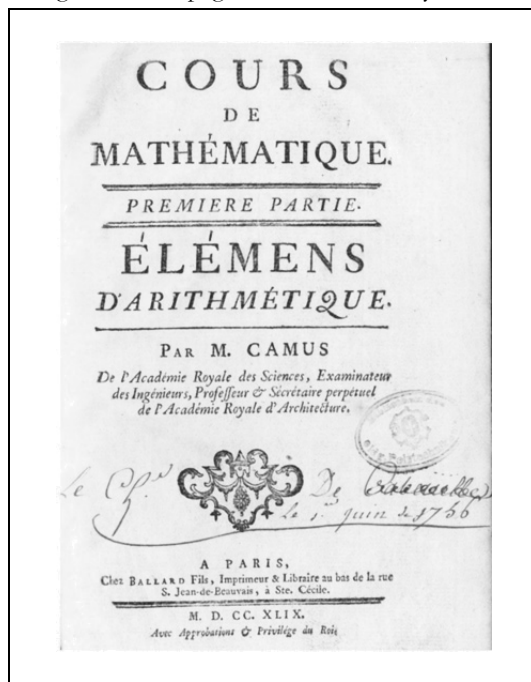
Conclusion

During the 18th century the changes in the structure of French mathematical textbooks went in the direction of a more technical presentation and, at the same time, give a no more highly comprehensive picture of the mathematical sciences. This turn is for sure linked with the changes in the educational system, and especially with the creation of military schools to improve the army standard. Indeed, it is there that at that time the mathematical sciences were taught in-depth, thus becoming a prominent teaching subject.

Among the textbooks that we have taken into account, Bélidor's one recalls more an ancient tradition in which many heterogeneous topics are dealt with. Indeed, its aim is to provide the whole knowledge that a military engineer needs. On the contrary, La Caille and Camus' textbooks, which both were published twenty years later, show some signs of changing in contents. The indicated trend is to limit the number of topics (pure mathematics, astronomy, mechanics, and optics in La Caille's textbook and pure mathematics and mechanics in Camus' one), handling them from a more technical viewpoint. This is also the course that Bézout took in his textbooks.

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Modern mathematics at the 1959 OEEC Seminar at Royaumont

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Abstract

The OEEC Seminar on New Thinking in School Mathematics, held from November 23 to December 5, 1959 at the Cercle Culturel de Royaumont (France) is considered a turning point in the history of the New Math movement, both in Europe and in the United States. In spite of its generally accepted historical importance, the historical circumstances of the Seminar are but little known. Based on a close reading of the published report of the Seminar, some first hand testimonies of participants and critical reviews published in several journals, we attempt to reconstruct the events at the Seminar and to unearth the strategies employed to present a seemingly unanimous view on school mathematics. Our findings point to a much more critical response to Dieudonné's proposals than is usually implied in general accounts.

Introduction

The OEEC Seminar, held from November 23 to December 5, 1959 at the *Cercle Culturel de Royaumont* in Asnières-sur-Oise (France) is considered a major event in the history of the New Math movement, both in Europe and in the United States. At Royaumont Jean Dieudonné launched his famous motto "Euclid must go!", which became a symbol of the radical modernization of school mathematics. In recent histories of mathematics education, Royaumont is considered a turning point, if not a starting point, of the New Math reform. As Kristín Bjarnadóttir (2008, p. 145) observed: "The Royaumont Seminar can be seen as the beginning of a common reform movement to modernize school mathematics in the world." Or in the words of Ole Skovsmose (2009, p. 332): "After the Royaumont seminar, modern mathematics education spread world wide, and dominated a variety of curriculum reforms."

Surprisingly, given the iconic status of the Royaumont Seminar, very little historical work has been done to unearth the processes which have shaped the outcome and the impact of the Seminar. Only recently, Gert Schubring has

begun to search the archives for much needed contextual information (Schubring, 2013). Schubring found that there were significant differences between the intended setup of the Seminar and the actual published report. In particular, he unearthed a preliminary background report on the state of research of mathematics education, which, however, was only superficially discussed during the Seminar and hardly mentioned in the final report. Given the central position acquired by the Seminar in the subsequent reform movement of school mathematics, this neglect seriously hindered the development of institutionalized research in mathematics education.

Our approach is somewhat different: we started from an analysis of the published report of the Seminar (OEEC, 1961), and supplemented this with some contemporary reactions of participants and commentators. Our goal was to understand the ways in which the 'radical' reform programme, formulated by Dieudonné, came to be seen as the main conclusion of the Seminar. Obviously, from reading the published report, it is clear that Dieudonné's position was not universally accepted by all participants, and that a much more balanced approach to the reform of school mathematics was being proposed. Also, given the large involvement of American mathematicians in the Seminar, we hoped to find some points of comparison between both reform movements.

The Royaumont Seminar was organized by the Office for Scientific and Technical Personnel (OSTP), set up in June 1958 by the Organisation for European Economic Co-operation (OEEC) for the purpose of "promoting international action to increase the supply and improve the quality of scientists and engineers in the OEEC countries" (OEEC, 1961). Although the 'Sputnik shock' of 1957 is often referred to as being the crucial factor that started the reform of modern mathematics, the creation of the Office was in fact a response to previous reports on the lack of scientific personnel in Europe. Since the early 1950's it had been a growing concern of both the OEEC and the United States that scientific and technical manpower in Western Europe was severely lagging behind the equivalent manpower in the US and the USSR (Krige, 2006). One of the policy targets of the Office, therefore, was the improvement of scientific education in OEEC countries, and in particular its adaptation to the progress of modern science. The Office organized in June 1959 a conference on physics education, in December 1959 the Royaumont Seminar on school mathematics, and in February 1960 a conference on chemistry education. A similar conference on biology would later take place in September 1962.

The aims of the Royaumont Seminar were practical. They were summarized in four specific goals (OEEC, 1961, p. 12):

1. To clarify and summarize the foremost thinking on mathematics and the mathematics curriculum in the elementary and secondary schools, the recruitment and training of teachers of mathematics and the needed research in mathematical education.

2. To specify (i) the purposes of mathematical education; (ii) the specific changes desirable in the content of instruction; (iii) new goals, new materials and new methods of instruction and (iv) further teacher training necessary for reform in mathematical education.
3. To indicate the specific procedures and means that might be considered in any country seeking to obtain a more adequate supply – both in number and quality– of mathematicians for teaching and research and of mathematically competent persons in science, industry and government.
4. To suggest appropriate follow-up action, both national and international (including further action by OEEC).

The Seminar was not meant to be a research conference. As James R. Gass, Head of Division of the Office, put it: the Seminar "is not intended that it should contribute particularly to the latest developments in professional discussions, but rather that it should produce the 'bilan'." (La Bastide-Van Gemert, 2006, p. 222) The contributions were to be authoritative and wide-ranging, leaving it to the respective governments to take the appropriate actions.

Planning the event

It is beyond doubt that the International Commission on Mathematical Instruction (ICMI) was heavily involved in the organization of the Seminar. In March 1959 an OEEC group of experts convened in Paris, consisting of Marshall H. Stone, Gilbert Walusinski, Gustave Choquet, Howard F. Fehr, Hans Freudenthal, Willy Servais and Albert W. Tucker (La Bastide-Van Gemert, 2006, p. 218). All but the last two were members of the ICMI Commission 1959–1962 with Stone as its chairman and Walusinski as its secretary. Yet, Freudenthal was severely disappointed by the heavy constraints put on the initiative by the Governing Committee of the OEEC. Freudenthal even refused to go to Royaumont – a decision that he would come to regret in his later years (La Bastide-Van Gemert, 2006, p. 215–225).

Each OEEC Member State or participating country was "invited to send three delegates to the Seminar, one an outstanding mathematician, another a mathematics educator or a person in charge of mathematics in the Ministry of Education, and a third an outstanding secondary school teacher of mathematics" (OEEC, 1961, p. 7) There were 30 delegates from 16 European countries, and three more from Canada and the United States. We know very little about the selection process of the delegates or the guest speakers. The two Swiss delegates were designated by the *Office fédéral de l'industrie, des arts, des métiers et du travail* (Pauli, 1979). Also the Dutch delegates were sent by the Ministry of Education, Arts and Science (Leeman, Bunt, & Vredenduin, 1960). But it may have been different in other countries. In Italy, the *Commissione Italiana per*

l'Insegnamento della Matematica, which acted as a national subcommittee of ICMI, stated that it would participate in the Seminar with its own delegate (Unione Matematica Italiana, 1960, p. 91).

On top of the official delegates, 13 guest speakers were invited. Here again the influence of ICMI was clear, but no less of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM), a forum of European mathematicians, teachers, philosophers and psychologists to study the reform of school mathematics. Five guest speakers (as well as one delegate) had been founding members of CIEAEM in 1952 (Bernet & Jaquet, 1998). Gustave Choquet, professor of mathematics at the Institut Henri Poincaré (Paris), was president of CIEAEM. Jean Dieudonné was professor of mathematics at the Institut des Hautes Études Scientifiques (Paris), a prominent leader of the Bourbaki group and a contributor to CIEAEM meetings. Lucienne Félix was a secondary school teacher of mathematics at the Lycée La Fontaine (Paris). The Belgian Willy Servais, who adhered closely to the French group, was invited as secretary of the CIEAEM. There were also three speakers from the United States: Edward G. Begle, who had just secured a major financial subsidy from the National Science Foundation for the work of his School Mathematics Study Group at Yale; Howard F. Fehr of Teachers' College at Columbia University (New York) and Robert E.K. Rourke, a secondary school teacher at Kent School, Connecticut. Begle, Fehr and Rourke all served on the board of the American Commission on Mathematics of the College Entrance Examination Board, which in 1959 published an influential Program for College Preparatory Mathematics. With the delegates Marshall Stone, chairman of ICMI and professor at the University of Chicago, and Albert W. Tucker of Princeton University, also chairman of the American Commission on Mathematics, the American contribution to the Seminar was substantial. Both Stone and Tucker presented a paper. Other invited speakers were Otto Botsch from Heidelberg (Germany [the report systematically uses 'Germany', although all occurrences refer to West-Germany]), the author of a textbook on *Bewegunsgeometri*e, which in German schools already had begun to replace the traditional Euclidean approach, Svend Bundgaard from Aarhus (Denmark) and Luke N.H. Bunt from Utrecht (The Netherlands), both involved in curriculum reform, and Edwin Arthur Maxwell from Queen's College, Cambridge (UK), a successful author of several textbooks. Charles Brunold, general director of secondary education at the French Ministry of National Education, presented a paper on the French educational system and the efforts to fill the gap of qualified teachers. Finally, there was the psychologist William Douglas Wall, director of the National Foundation for Educational Research (UK) and chairman of the International Project for the Evaluation of Educational Attainment. There were sixteen papers presented. Apart from the persons already mentioned, there was a report by Pierre Théron of the French Ministry of Education on the problems involved in implementing the proposed programmes.

In the list of 46 participants, Schubring (2013) counted 16 university mathematicians and an equal number of (probably university trained) secondary school teachers. The rest were government delegates and a few educationalists.

Debates and controversy on Euclid

The participants arrived in Royaumont on Monday morning 23 November, where they were formally welcomed by Robert M. Clark, assistant director of the OSTP, and Charles Brunold, general inspector of secondary education in France. One of the participants later described the general atmosphere of the Seminar (Piene, 1960a, p. 53):

> The venue was a former convent, completely isolated, where participants besides meetings also had their sleeping and dining rooms, so that there was an abundance of leisure during the 14 days the seminar lasted. It was a typical working seminar. One night there was a reception in the OEEC's house in Paris, another a highly entertaining lecture on mathematics education in the Soviet Union (by Prof. Rourke) and one night we were shown a film about fabric structure. Otherwise, the time was spent on short trips and enjoyable and enlightening association with foreign colleagues.
>
> The work of the Seminar was organized along three sections. Professor Marshall Stone (Chicago) was chairman of the seminar and professors Fehr (New York), Dieudonné (Paris) and the Inspecteur général Théron (Paris) were leaders of the three "sections" in which the seminar was divided. Each disposed of a third of the time the seminar lasted. Every day two lectures were held with subsequent discussion, which brought both factual information ("in our country, we do it so and so") and expressions of opinion ("more set theory" – "to hell with Euclid"). Often three groups were created, each discussing a major problem in the lecture. After that, we met again in a plenary meeting with the minutes of each group and continued the discussion.

Section I was concerned with the evolution of mathematics which would call for modifications in the secondary education, section II focused on the teaching of mathematics in secondary schools and discussed the student's ability to learn mathematics, and section III attempted to set out the more technical problems of implementation, e.g. the formation of teachers or the availability of appropriate textbooks. In reality, however, both sections I and II discussed proposals to reform the content of mathematics education, with great emphasis on its mathematical content (and much less on the student's abilities). These discussions lasted for seven days. Three days were spent on section III, and the final day was reserved for reaching an agreement on the final resolutions (Leeman et al., 1960). Schubring (2013) is more specific: section I occupied only one day, whereas the final section lasted for two and a half days.

The actual program of the Seminar is difficult to reconstruct. Marshall Stone gave the opening address, after which the sections started. One participant's report puts five lectures (Dieudonné, Servais, Botsch, Choquet and Félix) in the first section, four lectures (Fehr, Maxwell, Tucker and Bunt) in the second, and five lectures (Théron, Brunold, Rourke, Bundgaard and Begle) in the third, without mentioning Wall's paper (Leeman et al., 1960). This schedule differs from the number and the order of papers in the published Report with only Dieudonné and Tucker in the first section, Choquet, Servais, Félix, Botsch, Maxwell and Bunt in the second, and Théron, Begle and Wall in the third section. The Report further mentions "separate papers" dealing with "research into the teaching of mathematics" (OEEC, 1961, p. 13), and as indicated by Kay Piene in the quotation above, there were also entertaining night lectures, one of them being Rourke's paper on Russian school mathematics.

The official languages at the Seminar were French and English, with simultaneous translations of all addresses and discussions. Written papers were also translated 'on the spot'. In the personal archives of Willy Servais the English translation of his lecture in French is still extant. It bears three dates: Servais dated his manuscript on 25th November, the English translation is dated on 27th November, while the document is given an official code number on 28th December. Alternatively, the paper by Gustave Choquet in the original French version, is dated on 15th December, ten days after the end of the Seminar.

Although there were sixteen papers presented at the Seminar, two of them clearly stood out. One of them was the opening address by Marshall Stone, "Reform in School Mathematics". Stone pointed to the "dislocation" between secondary and university levels of mathematical instruction, as a result of the extraordinary growth of pure mathematics in modern times, and the increasing dependence of scientific thought upon advanced mathematical methods. "No technological society of the kind we are in the process of creating can develop freely and soundly until education has adjusted itself to the vastly increased role played by modern science. […] Thus the teaching of mathematics is coming to be more and more clearly recognized as the true foundation of the technological society which it is the destiny of our times to create." Still, "as a practical matter the initial step toward reforming our teaching of mathematics will probably have to consist of establishing an intrinsically better mathematical curriculum without reference to its ultimate co-ordination with introductory science courses" (OEEC, 1961, pp. 17–18, 21). In the remainder of his speech, Stone amply discussed the need for more efficient teaching methods and new teaching materials, and he concluded with a plea for the creation of institutes for research in mathematics teaching.

Although his talk primarily dwelt on the modernization of teaching methods, Stone also addressed the introduction of modern mathematics into the curriculum. "We cannot put off much longer a fundamental study of the introduction of some modern mathematics of a suitable kind into the secondary

school curriculum." This would mean incorporating "a few subjects or topics of fairly recent origin" and the elimination of "dead, useless, outmoded or unimportant parts of mathematics, however hallowed by tradition" (OEEC, 1961, pp. 16–17). Stone did not specify which new or old parts of mathematics he had in mind.

Several of these points were taken up by the second speaker, Jean Dieudonné. He also started from the observation that secondary school students were not adequately prepared for higher mathematical education. The culprit was the "pure geometry taught more or less according to Euclid," (OEEC, 1961, p. 34) as it was still the habit in French schools. The solution had to be radical. Hence, his provocative thesis "À bas Euclide!" ("Euclid must go!"), without doubt the most often quoted words from the Royaumont Seminar. According to Dieudonné, most of the topics in a Euclidean geometry course have "just as much relevance to what mathematicians (pure and applied) are doing today as magic squares or chess problems!" (p. 36). Instead of the old curriculum, Dieudonné proposed to teach matrices and determinants, elementary calculus, construction of the graph of a function, elementary properties of complex numbers and polar co-ordinates. The problem was how to organize this material into a well-balanced curriculum. Dieudonné conceded that he had no direct experience in teaching to secondary school students, but his outline of a modern curriculum for students from age 14 to 17 was quite concrete, roughly starting from "experimental" mathematics, concentrating on techniques and practical work, to a rigorous, axiomatic treatment of two- and three-dimensional space.

Dieudonné's address created both strong approval and disagreement, which is summarily reproduced in the published report of the Seminar. The report indeed started this section with the subtitle: Sharp Controversy Provoked. "After some discussion, both groups modified their positions on the programme and reached general agreement on a set of proposals which did not remove Euclid entirely from the secondary-school curriculum" (OEEC, 1961, p. 47). Probably some of the controversy also echoed in contributions of other participants. Edwin Maxwell in his lecture on a new syllabus for the calculus, ended with a critical note on geometry: "I feel that the premature introduction of vectors [in analytic geometry] is a possible source of real confusion to the young. The economy of effort which they allow is, of course, very real; but that economy is one for the mature mind, rather than for the beginner" (p. 89).

Some of the vehemence of the debates can be seen through the reports made by participants in the months after the Seminar. The Norwegian delegate Kay Piene (1960a) singled out "til helvende med Euclid!" as one of the lively opinions expressed during the Seminar. Lucien Kieffer (1960), one of the Luxembourg delegates, devoted one third of his report to Dieudonné's lecture. According to Kieffer, Dieudonné also would have cried "Mort au triangle!" ("Death to triangles!", "Nieder mit dem Dreieck!"). But, Kieffer asked, "is the triangle indeed so useless? Man has used triangles for so long, and our

technology cannot do without them – without the triangle the Eiffel tower would crumble!" (The original reference to the Eiffel tower is sometimes ascribed to Emma Castelnuovo, Équipe de Bordeaux, 2009, p. 2.) Kieffer remarked that in particular the official delegates of the French schools had looked quite skeptical upon Dieudonné's proposal. The Swiss delegate, Laurent Pauli, made no mention of Dieudonné's outcry in his original report (Pauli, 1961), but would do so in a later commemorative paper (Pauli, 1979). There he stated that Dieudonné "of the Bourbaki group" had been surrounded by "several of his colleagues." Pauli observed that Dieudonné's actual curriculum proposals did not nearly receive as much attention as did his violent attack on Euclidian geometry. The Dutch delegates wrote an extensive report on the Seminar, in which again Dieudonné (now with "À bas Euclide!") held a prominent place. Yet, they added that the "general opinion had been, if we are not mistaken, that the views of Dieudonné are on the one hand very valuable, but on the other hand should be taken with a grain of salt" (Leeman et al., 1960, p. 220).

The reaction against Dieudonné's anti-Euclidean stance was not based on a desire to keep old geometry in place. Piene (1960b) observed that "the assembly seemed to think that Dieudonné went too far, but found that the usual Euclid-representation can be pruned and modified, and that gymnasium geometry can be reformed in a more algebraic direction and with the use of vectors" (p. 68). All participants agreed that modern mathematical teaching had to concentrate on the acquisition of abstract concepts and mathematical structures, rather than on an encyclopedic knowledge of useless theorems, but many of them considered at least part of Euclid's geometry an ideal stepping stone to attain this higher level. This point was in particular developed by Otto Botsch. In his proposal for reform, Botsch situated himself squarely within the Euclidean tradition, but he distanced himself from the purely deductive way in which it was usually taught.

> Euclid's system has outlasted centuries of development in mathematics. The aims of modern instruction in the schools transcends (sic) the limits of Euclid less than we might suppose. But Euclid is a prefabricated house, and its instruction is static. It is our aim to make instruction dynamic, and this cannot be done by giving our pupils a systematically ordered catalogue of tasks to accomplish, which is essentially what we do in teaching Euclid (OEEC, 1961, p. 77).

His alternative, the *Bewegungsgeometrie*, consisted in a dynamic approach to instructing geometry, already in use in more than half of the secondary schools in Germany. The underlying inspiration of his proposal was Felix Klein's Erlangen Program for geometry based on groups. According to Botsch, the study of geometry should be preceded by the study of physical objects, including paper-folding, drawing, cutting and pasting, and the making of geometrical ornaments. Only then could one begin with the study of simple

symmetrical figures and the properties of axial symmetry. In a later stage, one could move to the study of translations and rotations and to the concept of congruence. Translations in space can subsequently lead to a geometry of vectors and to the study of the properties of groups.

Botsch's proposal appears to have found some support among the participants. Kieffer (1960) considered intuitive geometry, based on the manipulation of triangles, an ideal means to lead the student to deductive reasoning. Furthermore, intuitive geometry was to him an important instrument to improve inductive reasoning. It is highly dangerous, he remarked, to introduce abstract concepts too early to young students who still need a clear, simple language. Another delegate, Walter Saxer from Switzerland, showed himself very critical of Dieudonné's proposal (Saxer, 1965). He remarked that an abstract, axiomatic geometry would only be fruitful for students who would later go on to study mathematics at the university, but for the others, in particular also for the engineering students, this type of mathematics would be inappropriate. He also objected that the axiomatic method left too little room for invention and discovery, still essential ingredients of the mathematical mind. The report observed that "the discussion was influenced by the experience of teachers on the one hand, and the mathematicians' lack of teaching experience on the other. […] What is needed is psychological and experimental research on the formation of concepts and mental growth of pupils, so that instruction in geometry can be organised on valid principles of learning" (OEEC, 1961, p. 80). Still there was a large consensus on the final goal of geometry instruction, viz. that after the early stages of intuitive learning, there should come "the breaking of the bridge with reality – that is, the development of an abstract theory" (p. 80). As Piene (1960b, p. 69) concluded:

> The essential point of the whole discussion was that either the real numbers or a vector program will be able to unite algebra and geometry, and thus provide greater unity and strength to the mathematics curriculum. Experiments in this direction must be encouraged.

The unity of (all) mathematics

A second important theme in the Seminar was the ambition to bring unity in the mathematical curriculum of the secondary school. This was particularly discussed in the treatment of arithmetic and algebra, by respectively Gustave Choquet and Willy Servais. Choquet started from the fact that modern mathematics increasingly tends to do away with the boundaries between arithmetic, algebra, geometry and calculus. It was therefore important that also in primary and secondary schools, arithmetic and algebra would be merged as closely as possible. This could be done by studying the many structures in the set of integers Z, such as order, group and ring. He advocated a structuralist

approach, proposing e.g. to introduce addition and multiplication at the primary level by, respectively, the union of finite disjoint sets and the product of finite sets. With respect to numerical calculations, he objected to the tedious training of long calculations ("the grocer's son can add very well by using his father's adding machine", OEEC, 1961, p. 65), but proposed instead that it would be far more profitable and pleasant for the pupil to do a few calculations in the binary, octad, or duo-decimal systems. "There can be no doubt," he concluded, "that the basic concept of numeration can only be thoroughly understood by the pupil if he has studied several different systems" (p. 67). In his full lecture, he recalled that he had seen 13-year-old children perform these calculations with enthusiasm, rapidly even outperforming their teacher in virtuosity – a remark wisely left out of the official report (Choquet, 1959).

Choquet's address was followed by a proposal for a modern and coherent approach to algebra by the Belgian Willy Servais. Servais also emphasized the unity of mathematics, and hence the integration of all parts of the curriculum:

> In the present state of mathematics, what is needed is not an algebra syllabus alongside an arithmetic and geometry syllabus but a combined syllabus in which mathematics would cease to be split up into watertight sections. Why force a student to solve a problem by arithmetic when algebra gives an immediate solution? Why separate trigonometry from geometry and throw away all the advantages that can be derived from merging it with analytic geometry and algebra? The teaching of algebra and mathematics cannot be modernised simply by bringing in new topics at the last minute and patching them on to the traditional subject matter. The whole edifice must be rebuilt from the foundations and structured in accordance with modern ideas. (Servais, 1959, p. 3)

According to Servais, the teaching of algebra should not be confined to operations with numbers or numerical variables. Modern algebra is the study of operational structures, irrespective of the nature of the objects covered by the operations. Both for psychological and mathematical reasons, sets should be introduced as early as possible. Sets also prove a good foundation for elementary notions of logic which are important not only for the study of mathematics, but which also serve as an important ingredient of intellectual life. Servais' emphasis on logic does not imply that he opted for a strictly deductive instruction. Although he clearly saw mathematics as a deductive science, he favored a more active and exploratory approach to mathematics teaching. Properties of the algebra of sets should be discovered rather than expounded. Definitions should be given progressively to make pupils aware of what they have acquired. He further proposed an early introduction of the notion of function using the Cartesian product of sets, and of relations, in particular equivalence and order relations. This would lead to the concepts of group, vector space and complex numbers, which could be further exemplified both in algebra and in geometry.

Together with the unity of mathematics, some speakers also emphasized the necessity of mathematics for practical applications, even as a way of life. In his talk on applied mathematics, Albert Tucker quoted Warren Weaver, former director of the Rockefeller Foundation, in pointing out the use of mathematical techniques, which were not yet part of the standard curriculum: matrix algebra for linear programming; probability for making decisions on price forecasting or quality control; set theory for treating complex alternatives in business situations. Tucker showed that simple problems of linear programming offered excellent means of utilizing inequalities, intersections, graphic methods and algebraic procedures for solving equations. Luke Bunt likewise pointed to the growing importance of statistics as an auxiliary science for the natural and the social sciences, but also for modern citizenship. He proposed to focus on a limited number of essential concepts and methods, and argued that hypothesis testing and judging the characteristics of a population on the basis of a sample, should be the dominant objective of a course in statistics.

The final section of the Seminar was devoted to problems of implementation of reform such as the preparation of new teaching materials and the reform of teacher training. In particular the paper by Ed Begle captured the attention of most Europeans. Begle was the leader of the School Mathematics Study Group (SMSG) founded in 1958, and consisting of mathematicians, math teachers and other experts. The primary task of SMSG was to "provide a sound basis for a solid college course in calculus and analytical geometry by the end of the 12th grade" (National Science Foundation, 1959, p. 82). Begle had received from the National Science Foundation the enormous sum of $ 1,250,000. Begle's approach was well appreciated, presumably for its open collaboration. "We did not want to leave all the decisions on mathematics to the university mathematicians and all the decisions on pedagogy to the high-school teachers. We wanted all the decisions to be made by the group acting as a whole" (OEEC, 1961, p. 100).

A final discussion paper in this section was presented by William Douglas Wall, director of the National Foundation for Educational Research, London. He indicated some steps to which educational research must look forward. He started to admit that research in mathematics education has often been short-term, scattered and piecemeal, but that in spite of these shortcomings, current knowledge of child development and learning processes could certainly bring about striking improvements to educational practice. However, he continues, results from psycho-pedagogical research are often ignored by educational administrators, teacher-training institutions and the schools themselves. "They prefer the cosy comfort of unverified opinion and rule of thumb to the dangers of objectively verifiable hypotheses" (OEEC, 1961, p. 102). He further pleaded for a co-ordinated, long term and multidisciplinary research effort and explained how such research could be carried out. Wall showed himself critical about the suggestions for change made in the Seminar. "They have a justification in reason and logic perhaps but we have no means of knowing

whether they will be as successful or more so than the old ways, unless into our reform we build from the outset means of objective study and evaluation of results" (p. 103). But, as Schubring (2013) has shown, this road was not taken.

At the end of the Seminar a list of resolutions was adopted unanimously. These can be briefly summarized as:

1. There is a most urgent need for adapting the teaching of school geometry and algebra to the sweeping advances made in modern mathematics.
2. Elementary probability must be recognized as an appropriate part of the mathematics taught in secondary schools.
3. Competent teachers should be attracted and retained in the profession.
4. Teaching of mathematics in secondary schools should be given only by university graduates majoring in mathematics.
5. Each delegation should prepare a small bibliography in order to acquaint mathematics teachers with major publications on the subjects discussed.
6. The OEEC should establish a group of experts to work out a detailed synopsis.
7. The OEEC should encourage the execution of experiments on the proposals made.

The Seminar finally ended on Friday 4 December, one day before the announced end-date. Schubring (2013) found that the participants had been invited to attend the Bourbaki Seminar in Paris the following weekend, but he was unable to ascertain the presence of any of the Royaumont participants. Some participants met each other again in Paris on 7–8 December for the ICMI meeting.

Reception and reaction

The papers of the Seminar were not immediately published, with the exception of Rourke's paper on mathematics instruction in Russia (Rourke, 1960). The few typescript documents in the Servais archives are all marked RESTRICTED. News about the Royaumont Seminar spread first through the participants, who reported to their local communities. Several of these reports were printed. The spirit of Royaumont can also be detected at several conferences, where participants of the Seminar met with other colleagues. At the 1960 Easter meeting of the British *Association for Teaching Aids in Mathematics*, the Royaumont delegate Cyril Hope, "now being described as the 'self-styled Public Relations Officer for the New Mathematics', argued for some 'modern' ideas to be experimented with at school level, and against Euclidean geometry" (Cooper, 1985, p. 164). The ICMI Seminar for Mathematical Instruction, held in Aarhus from May 30 to June 2, 1960 was organized by Svend Bundgaard, with papers by Piene, Choquet and Dieudonné. With the support of the OEEC, Jean J. Van

Hercke, one of the Belgian delegates, organized from August 25 to 31, 1960 an international conference for secondary school teachers from Belgium, Switzerland and Luxembourg in Brussels to discuss new developments in university mathematics since 1930. From September 19 to 24, 1960 an International Symposium on the Coordination of the Instruction of Mathematics and Physics took place in Belgrade, attended by the Royaumont participants Choquet, Fehr, Hope, Maxwell and Stone. This conference was actually preceded by the OEEC meeting of experts in Zagreb and Dubrovnik (respectively from August 21 to September 2, and from September 4 to 17), convened to draw up a Synopsis of a modern curriculum for school mathematics along the lines agreed in Royaumont.

About a year and a half after the Seminar the official report came out as *New Thinking in School Mathematics*, or in the French version *Mathématiques nouvelles*. The book was widely distributed (at first even free on request) and reviewed in many mathematical and educational journals. *New Thinking* did not contain the original papers, but consisted of a collection of excerpts, summaries, comments and conclusions. Only two contributions were reproduced in extenso, the opening address by Marshall Stone and the controversial presentation by Jean Dieudonné. The second part of the book was a Survey of Practices and Trends in School Mathematics, the outcome of a questionnaire submitted to each of the OEEC Member countries and to Canada and the United States. Some reviews of *New Thinking* also make mention of a second OEEC publication *Synopses for Modern Secondary School Mathematics* (or *Un programme moderne de mathématiques pour l'enseignement secondaire*), which came out in October 1961 and which was the result of the follow-up meeting of the Royaumont group of experts in Zagreb/Dubrovnik. A second companion volume edited by OEEC, *School mathematics in OEEC countries: summaries / L'Enseignement des mathématiques dans les pays de l'OECE: monographies*, was a collection of short monographs on mathematical education in the OEEC countries. These monographs had been made available to the participants of the Royaumont Seminar, as the results of the larger survey (which was finally published in *New Thinking*) were not ready in November 1959.

The preparation of the Report was the work of Howard Fehr, assisted by Luke Bunt. It may be assumed that the report was originally written in English. There are several peculiar things to note about the book. The English title was taken from the official title of the Seminar, and was e.g. completely in line with other similar volumes published by the OEEC (or later OECD); *New thinking in school chemistry* (1961) and *New thinking in school biology* (1963). Yet, the French translation differed. Whereas both other volumes bore the title *Pour un nouvel enseignement de la chimie/biologie*, the Royaumont book was called *Mathématiques nouvelles*. This was possibly a reference to the American term 'New Math', but one can only speculate on the reasons why this title was chosen. In Europe the term 'modern mathematics' was much more common than 'new mathematics' and there seems to be no obvious reason why the American developments

would be taken to be the flag of the whole operation. It remains puzzling why nothing in the French title related to teaching or school mathematics.

A second note concerns the selection of papers. There is no doubt that Fehr and Bunt remained faithful to the original texts of the contributors, although most of the papers were heavily truncated and possibly even the order of presentation was adapted to the conclusions of the report. The editors also reported extensively on the arguments presented by the participants in response to the papers. As the papers are not reproduced in their original form but summarized in short paragraphs, as are also the remarks by other participants, it is not always clear whether these paragraphs represent the opinion of one of the speakers, or of the whole group. This editorial policy was quite in line with the original intent of the Seminar, namely to establish a concise formulation of the consensus among experts. Indeed, the report cannot be read as the reflection of the personal preferences of the editors. Bunt's own paper on probability and statistics had been reduced to just over one page, and Fehr's introduction to section II was reproduced anonymously in three pages.

New Thinking came out in May 1961. It would lead too far to make a detailed analysis of all the reviews in the various journals. In general, much attention was given to Dieudonné and his criticism of Euclid. Georges Bouligand (1889–1979), a mathematics professor recently retired from the Sorbonne, devoted much of his review (Bouligand, 1962) to a detailed exposition of Dieudonné's lecture, with only a brief mention of the contributions by Stone, Choquet, Maxwell and Tucker. Bouligand noted in the end with satisfaction that the French contribution had been important. The psychologist John Burville Biggs (1934–) likewise headed his review (1961) with Dieudonné's "Euclid must go", although he assured his readers that the report was not meant to startle. Hans Gollmann (d. 2003) from Graz considered the controversy surrounding Dieudonné's "Euclid must go!" to be caused more by its poignant formulation than by the truly revolutionary content of his proposal (Gollmann, 1962). To Gollmann, the realization of this program was without a doubt a commandment of our times. Abraham Robinson (1918–1974) also endorsed the efforts of the Royaumont group but he urged that they be supplemented by a more careful screening of the topics taken into consideration (Robinson, 1962). Another favorable review appeared in *Mathematics Teaching*, the journal of the *Association for Teaching Aids in Mathematics* (Cooper 1985, p. 164). Some reservations were made, however, in the next issue of *Mathematics Teaching*, where three representatives of respectively the secondary school teachers, teacher training establishments and universities were asked to give their views on the OEEC reports. Joan Blandino of St. Michael's Convent Grammar School (North Finchley) found the real question to be "when and how [teachers] should change their own curricula," (Blandino, Sillitto, & Hilton, 1962, p. 6) for the final responsibility of putting *New Thinking* into practice would lie with the teachers. A.G. Sillitto of Jordanhill College of Education (Glasgow) made a similar point, stating rather dramatically, that for many

teachers in "grossly understaffed schools […] the main duty is, simply survival" (Blandino et al., p. 9). For him also, the initiative had to come from the teachers, who in England –contrary to the situation in France– did not feel at home with the new mathematics. Professor P.J. Hilton, a mathematician of the university of Birmingham, noted that "no one concerned with the teaching or the practice of mathematics in this country can afford to ignore" this publication, and went on to answer the criticism brought forward against the reform proposals that technique was being sacrificed for rigour (Blandino et al., p. 11–12).

A critical review by the English logician Ruben Louis Goodstein (1912–1985) appeared in the *Mathematical Gazette* (Goodstein, 1962). He argued that no reasons at all were given for the specific proposals, and that a differently constituted committee would have made quite different proposals. Goodstein concluded: "Proposals as extreme and eccentric as those under review can I fear only serve to damage the case for reform" (p. 72). Probably the sharpest criticism was given by Alexander Israel Wittenberg (1926–1965), former student of Ferdinand Gonseth at ETH Zürich and currently professor at Laval University in Quebec. He accused the Royaumont group of having imposed only the opinions of some individuals on the OEEC, and by doing so, of having effectively sidelined much needed fundamental research on mathematics education (Wittenberg, 1961). Wittenberg did not accept the argument that school mathematics had to be reformed to prepare students for university studies. If this would be the case for mathematics, why not for every other field of knowledge? Would the secondary school then become nothing more than a propaedeutic training school for the university? Democratic society was better served by giving young people a responsible, humanistic and broad education. Wittenberg also objected to the argument that knowledge of modern mathematics was a basic instrument for everyday life. The fact that quantum mechanics makes use of matrix algebra, is not enough reason to teach the topic to young students. The real problem was, according to Wittenberg, that none of these questions had been addressed in Royaumont. Many of the arguments, he concluded, had been nothing more than "window-dressing" and propaganda to serve a short-sighted and even dangerous *Fachimperialismus*.

Conclusions

Having reconstructed the history of the Royaumont Seminar in some detail, we can now attempt to retrace its position in the New Math reform of the 1960s. The Seminar did not come out of the blue. Most of the invited speakers had known each other for many years, and were selected for very good reasons. The origin of the Seminar must perhaps be placed as far back as the 1952 meeting of CIEAEM in La Rochette par Melun on "mathematical and mental structures", which had brought together Dieudonné, Choquet and Servais in

dialogue with psychologist Jean Piaget and philosopher Ferdinand Gonseth. In several countries the reform of school mathematics was well underway by 1959, with a large number of specialist meetings on a regular basis. Yet, *New Thinking* made no mention of these prior developments. The Seminar at Royaumont seemed to act in complete isolation, not only in space but also in time. From reading the report, one could get no idea about what had been achieved before, or on the variety of opinions and approaches in the field. This resulted in an authoritative, even dogmatic 'bible' of modern mathematics. The endorsement of the Seminar by the OEEC and its continued financial support for the implementation of what had been decided at Royaumont, further contributed to its iconic status.

This is not to say that mathematicians were not aware of the biased message that was promulgated by the Royaumont Seminar. In most reports and reviews it was recognized that the Royaumont view was actually derived from the Bourbaki approach of mathematics. Even if the word Bourbaki did not occur even once in the published report, there was no doubt about its Bourbakist inspiration. In countries where Bourbaki was held in high esteem (such as France or Belgium), this was accepted as an extra argument in favor of the reform. In other countries, where the appreciation for Bourbaki was more conditional, the enthusiasm for the Royaumont reform proposals may have been less unanimous. Further research may throw light on the link between the spread of Bourbakism and the reform of school mathematics in Europe.

From the first reports on the Seminar, it has become clear that Royaumont would almost immediately, long before the publication of *New Thinking*, be identified with the "Euclid must go!" of Dieudonné. From its ubiquitous presence in almost every report of participants, one can conclude that Dieudonné (and the Bourbakists) had indeed succeeded in putting this particular topic in the centre of debates. It did not matter whether the report agreed or disagreed with Dieudonné. "Euclid must go" was a message that stuck. Royaumont thus came to be seen as a synonym for a radical, revolutionary reform of school mathematics, although this was clearly not the general feeling among the participants. Kay Piene (1960b, p. 71) indicated in his report that the ambitious reform had to be carried out by evolution, not revolution. But the reviews of *New Thinking* only enhanced the feeling that Dieudonné's slogan was the true battle cry of the reform. The more moderate proposals or prudent warnings contained in *New Thinking*, if mentioned at all, were easily forgotten. Yet, reducing the Seminar to one slogan can also fire back. Further research should establish whether the figure of Dieudonné and his attack on Euclid did foster or harm the reform movement in Europe.

There is finally the almost unnoticed role of the Americans in the Seminar. The New Math in the United States had its own parallel history, in which the Royaumont Seminar was only a minor episode (Corry, 2007). Yet, during the Seminar, the differences between the French and the American had been very clear. Whereas in France the Bourbaki approach was firmly established in the

universities to the point where university mathematics simply meant "pure mathematics according to Bourbaki", in the United States the problem was first of all to improve the low standards of mathematics teaching in secondary schools (often by non-licensed mathematics teachers) and the poor admission rates of students to mathematical higher education, including applied mathematics and engineering. In France the demand for modernization came from the universities and was aimed at introducing modern 'Bourbaki mathematics' in secondary schools; in the United States the renewal of mathematics education was urged by industry and politics, and aimed at the modernization of teaching methods. As a consequence, the American reform was in terms of 'modern' mathematical content considered more moderate than the French proposals (Kieffer, 1960) and more specifically aimed at enhancing the quality of teachers' education. The introduction of modern mathematics in Europe and the United States ran along parallel but different lines. It remains to be seen whether the Royaumont Seminar had any influence on bringing these lines closer to each other.

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Decimal fractions in school mathematics in Great Britain and North America, 1667–1887

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Abstract

This chapter examines the extent to which decimal fractions were part of school arithmetic curricula in Great Britain and in North America during the period 1667–1887. We assume that whereas the curriculum intended by a textbook author can be identified by studying what that author wrote, a teacher's implemented curriculum is more likely to be identified by studying what students wrote in cyphering books. Analyses of a large number of textbooks and cyphering books revealed that although most textbooks included sections on decimal fractions, before 1792 relatively few students in British or North American schools actually studied decimal fractions. In the nineteenth century, however, higher proportions of North American students than British students studied decimal fractions. The change in the relative emphasis on decimal fractions is explained in terms of the theoretical lens of "lag time".

Introduction

In 1792 the United States of America, influenced by Thomas Jefferson's pioneering efforts with respect to currency reform (Jefferson, 1784a, 1784b), introduced the world's first fully decimalized currency.[1] Soon after this, the United States *almost* followed up with the world's first fully decimalized system of weights and measures (Boyd, 1953, 1961; Clements & Ellerton, 2015; Linklater, 2003). In this chapter, effects on school mathematics of the U.S. government's decision to introduce decimal currency as the official federal currency will be explored. The method used will be to compare the emphases on decimal fractions in school curricula in Great Britain and in the North

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[1] Since 1704 the Russian *rouble* had been equal to 100 *kopecks,* and in Japan—where silver money was basically money by weight—1000 *mommes* was equal to 1 *kan* (Nishikaw, 1987). The new U.S currency was fully decimalized—in the sense that successive subunits of currency increased by a factor of 10.

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America during two different periods – the first, between 1667 and 1791, and the second between 1792 and 1887. The analysis will be assisted by the introduction of a "lag-time" theoretical construct, with data in British and North American school mathematics textbooks and students' cyphering books from the two periods being examined. The lower time bound of 1667 was chosen because that was the year when the oldest of the cyphering books was prepared; and the upper time bound of 1887 was chosen because that was the year when the most recent of the cyphering books was prepared.

In this chapter, a "school" will be defined as any education environment in which at least one "teacher" regularly met with at least two "students," at an agreed place, for the purpose of helping the students to learn facts, concepts, and skills, from at least one of reading, writing, or arithmetic (Ellerton & Clements, 2012). For the period under consideration, our concept of "school" will include within its ambit "academies", "apprenticeship schools", "boarding schools", "common schools", "dame schools", "evening schools", "grammar schools", "local schools", "private schools", "public schools", "subscription schools", and "writing schools" (Cremin, 1970; De Bellaigue, 2007; Monroe, 1917).

Cyphering books and intended and implemented curricula

In order to investigate questions pertaining to the extent to which common fractions (often called "vulgar fractions") and decimal fractions were studied in North American and British schools during the period 1667–1887, cyphering books prepared in Great Britain between 1749 and 1887, and in North America between 1667 and 1861, were examined. These cyphering books are part of the Ellerton-Clements collection which, at the time of writing (September, 2014), comprised 522 cyphering-book units (hereafter termed "CBUs") – 370 prepared in North America, 102 in Great Britain, and 50 in other nations (Ellerton & Clements, 2014).

The term "cyphering book" is used, here, to refer to handwritten manuscripts which focused on mathematical content and had all of the following properties:

1. Either the manuscript was written by a student who, through the act of preparing it, was expected to learn and be able to apply whatever content was under consideration; or, it was prepared by a teacher who wished to use it as a model which could be followed by students preparing their own cyphering books.
2. Usually, all entries in the manuscript appeared in ink – as handwritten notes, or problem solutions, or as illustrations. Headings and sub-headings were presented in decorative, calligraphic style. Occasionally, water-color illustrations were included.

3. The manuscript was dedicated to setting out rules, cases, model examples and exercises associated with a sequence of mathematical topics. Although most cyphering books were specifically concerned with arithmetic, especially commercial arithmetic, some were dedicated to algebra, or geometry, or trigonometry, or to mathematics associated with mensuration, navigation, surveying, fortification, etc.

4. The topics covered were sequenced so that they became progressively more difficult. The content also reflected the expectation that, normally, no child less than 10 years of age would be assigned the task of preparing a cyphering book (Ellerton & Clements, 2014, p. 1).

In fact, there are *more than* 370 North American cyphering books and 102 British cyphering manuscripts in the Ellerton-Clements collection – if the same person prepared more than one of the cyphering books, or persons from the same family prepared obviously-related cyphering books, the set of related books was regarded as one *cyphering book unit*. All of the manuscripts in the collection were purchased separately, and analysis has revealed that they were prepared in many parts of North America and Great Britain.[2] The collection might be thought of as comprising CBUs prepared independently of each other. The same cannot be said of the other two large collections of British cyphering books – one held by the Mathematical Association and the other by John Denniss (see Denniss, 2012 for summaries of those collections). The Ellerton-Clements collection is easily the largest set of North American cyphering books (Ellerton & Clements, 2014).

Although it might seem reasonable to assume that an analysis of the extent to which entries on decimal fractions can be found in the 102 British and 370 North American CBUs in the Ellerton-Clements collection would permit reasonably objective generalizations to be made with respect to the implemented arithmetic curricula in British and North American schools during the period 1667–1887, there are two important caveats:

1. The dates for which the cyphering books were prepared are not distributed evenly across the period 1667–1887, there being much greater numbers of manuscripts for the period 1792–1887 than for the period 1667–1791.

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[2] Considerable details with respect to 212 of the 370 North American cyphering books units—numbers of pages, genders of writers, locations where books were prepared, topics covered, etc.—are provided in Appendix *A* of Ellerton and Clements (2012). The 102 British CBUs were prepared in many parts of Great Britain, including at least 5 from Scotland, 2 from Wales, 13 from Yorkshire, 6 from Manchester, and 5 from London. Most were prepared by boys, but at least 10, and probably about 20, were prepared by females. Most of the British cyphering books gave the impression of having been prepared by children from middle- or upper-class families—with many of them prepared in named boarding schools or academies. The authors have also examined more than 500 other CBUs, located in special collections in North America or the United Kingdom, but data from these books have not been included in this chapter.

2. The oldest of the British manuscripts in the Ellerton-Clements collection carries the date 1749. Also, there is no North American cyphering book in the collection which was prepared after 1861.

In this chapter, the terms "intended curriculum" and "implemented curriculum" will often be used. Since the term "curriculum" was only occasionally used with respect to what was taught in schools before the twentieth century, there is a sense in which our use of it in this chapter is anachronistic. However, we have found the terminology helpful from an analysis perspective, especially when distinguishing between what we have termed "intended" and "implemented" courses of study. Descriptions of such courses of study may or may not have been recorded in handwritten or printed documents – more often than not, they were the result of tradition, living in the minds of persons providing or receiving instruction.[3]

Theoretical base: The concept of lag time for school mathematics

Before we consider the history of the introduction of the concept of a decimal fraction into British and North American schools it will be useful to offer a theoretical base for the discussion which will take place in this chapter. That base is illustrated in Figure 1, in which the distinction is made between three types of mathematics – research mathematics, service mathematics, and mathematics education. Although these three aspects of mathematics are distinguishable, they do have their intersections. Figure 1 emphasizes that forms of mathematics are developed within societies in which "ethnomathe-matical forces" are at work shaping and using, and sometimes modifying, existing forms of mathematics.

The term "lag time" has been theorized and applied in several other areas of scholarship (see, e.g., Bellman & Danskin, 1954; Keraliya & Patel, 2014), and around 1960 some influential persons interested in improving school mathematics were referring to the "lag between the new ideas and their effect of schools" (Organisation for Economic Cooperation and Development, 1961,

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[3] Since about 1980, the International Association for the Evaluation of Educational Achievement (IEA) has distinguished between the intended, implemented, and attained curriculum of education establishments (see, for instance, Robitaille & Garden, 1989; Pelgrum & Plomp, 1993; Westbury, 1980). According to these authors, these different perspectives (or manifestations of curricula) can be characterized as follows:

- The *intended* curriculum refers to the curriculum plans set out in an official document or more simply, planned by an individual textbook author, teacher, or group of teachers, with respect to what they expect will constitute important instructional content and sequence.
- The *implemented* curriculum consists of the content, time allocation, instructional strategies, and so on, that teachers actually realize in their lessons.
- The *attained* curriculum is the knowledge, skills, and attitudes of students that occur as a result of teaching and learning.

p. 11). In the case of implemented mathematics curricula, we will define lag time as the amount of time between when a mathematical development (such as the definition of a decimal fraction) was first made known – probably by a mathematician or a practitioner – and the time when that development was "normally studied" as part of the implemented mathematics curriculum in schools in particular communities.

Figure 1. Different ways of "seeing" problems or situations that might relate to mathematics (from Ellerton & Clements, 2014, p. 321)
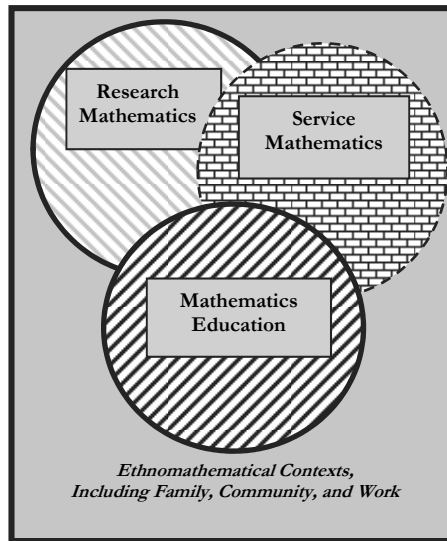
Thus, lag time will vary within and between communities, depending on a teacher's or school's willingness, or lack of willingness, to include a concept or principle or skill in the implemented mathematics curriculum. We recognize that the term "normally studied" in our definition of lag time will likely result in subjectivity, but we hesitate to define the term more precisely. In the case of the present study, which deals with implemented curricula in North America and Great Britain between 1667 and 1887, we shall adopt a pragmatic definition by which we assume that the adoption of a development within a region or community had occurred when that development was formally dealt with in at least 50% of the cyphering books prepared within that region or community.

In Figure 1, the circles representing three different aspects of mathematics are set within "Ethnomathematical Contexts, Including Family, Community, and Work". During the period 1667–1887, ethnomathematical contexts varied enormously, both within and between nations. Advances in mathematics, in the applications of mathematics, and in the interactions of these applications with the needs of evolving communities, stimulated changes to intended, implemented, and attained school mathematics curricula (Ellerton & Clements, 2014; Westbury, 1980).

Sixteenth- seventeenth- and eighteenth-century developments in the concept of a decimal fraction

We now summarize some of the developments in thinking about numbers – their nature, representations, and applications – during the period 1600–1791. We then consider how those developments were accepted and applied by those responsible for developments in "service mathematics", and how the developments in pure and applied mathematics, together with ethnomathematical factors, influenced intended, implemented, and attained mathematics curricula in schools – both in North America and in Great Britain. The main focus will, at first, be on investigating the extent to which decimal arithmetic influenced intended and implemented school mathematics curricula during the period. Later, the extent to which certain profound events in the United States and in Great Britain influenced both the application of decimal concepts and the implemented curricula in schools during the period 1792–1887 will become a focus of attention.

From our perspective, the single most important development in the history of school mathematics was the creation of the Hindu-Arabic numeration system, with its digits 0, 1, 2, 3, 4, 5, 6, 7 8 and 9 and its place-value system of numeration (Devlin, 2011; Franci, 2003). It is believed that this system of representation was introduced in India around 500 CE. The system's method of representing numbers would spread through India and through Arab nations before being introduced into Europe (Høyrup, 2005). From about 1200 CE it would become the cornerstone of the cyphering tradition which emerged as the key to effective mercantile practices in Western European nations and in the Americas, as well as in the East Indies (Devlin, 2011; Ellerton & Clements, 2012; Franci & Rigatelli, 1982; Grendler. 1989; Radford, 2003).

An extension of the original Hindu-Arabic numeration system was called for because there was a myriad of measurement contexts in which continuous quantities – such as time, weight, length, etc. – required more than just whole-number quantities of traditional units for their measurement. At first, this void was covered by common fractions but, later, decimal fractions would be introduced (Adeljaouad, 2006; Dauben & Scriba, 2002; Devlin, 2011).

Although some believe that the concept of a decimalized system of numbers and its application to measurement can be traced back to antiquity, the actual sequence of developments has been difficult to untangle (Dauben, 2002). Here we mention only key developments by François Viète (in France, in 1579), Simon Stevin (in the Netherlands, in 1585), John Napier (in Scotland, 1614, 1619), Henry Briggs (in England, 1617), and John Wilkins (in England, in 1668) (see Tabak, 2004). In 1579, Viète declared that "sexagesimals and sixties are to be used sparingly or never in mathematics, and thousandths and thousands, hundredths and hundreds, tenths and tens, and similar progressions, are to be used frequently or exclusively" (quoted in Boyer, 1991, p. 303); and, in 1585,

Stevin maintained that the universal introduction of decimal coinage, and measures was inevitable.

Stevin's (1585) 36-page *De Thiende* (*The Art of Tenths*), which elaborated on the nature and usefulness of decimal fractions, was first published in Dutch – but it was quickly translated into French and English The spirit of Stevin's idea was captured in an English translation of the subtitle to *De Thiende*: "*Decimal arithmetic: Teaching how to perform all computations whatsoever by whole numbers without fractions, by the four principles of common arithmetic*".

Within 40 years of Viète's and Stevin's groundbreaking work, Napier (1614) and Briggs (1617) used decimal notation in their development of the theory and some applications of logarithms. Napier (1614) wrote directly about decimals when he referred to "numbers distinguished thus by a period in their midst, whatever is written after the period is a fraction, the denominator of which is unity with as many cyphers after it as there are figures after the period" (p. 8). Napier (1619) was the first to use a period, as in 2.0346, to separate the whole number part and the decimal part (Glaisher, 1873). The potential of the ideas of Napier and Briggs was recognized by Johannes Kepler, the German mathematician and astronomer, and by the Flemish mathematician Adrien Ulacq. Soon, the application of decimal fractions, and logarithms, would transform mercantile and military practices related to navigation, surveying, fortification and astronomy (Bruce, 2002; Ellerton & Clements, 2014).

So the development from research mathematics to service mathematics occurred fairly quickly. As early as the 1620s it seemed that the time was ripe for this new aspect of the theory of numbers to become part of the implemented curriculum in schools. Edmund Wingate (1624, 1630), a British mathematician working in France, was among several scholars who advocated the rapid inclusion of decimal-fraction concepts in school mathematics curricula.

Decimal fractions in British and North American school mathematics 1630–1791

In 1630, Wingate (1596–1656) boldly used the "decimal point" – a recently-developed notation – in his *Of Natural and Artificiall Arithmetique* (Glaisher, 1873). He distinguished between "natural or common arithmetique" and "artificial arithmetique". A second edition appeared in 1650 with the revised title *Arithmetique Made Easie*. With artificial arithmetick, Wingate introduced decimal fractions and logarithms, and asked students to use these when making calculations likely to require lengthy multiplications or divisions, or both.

The preface to the eighth edition of Wingate's book, published in 1683, indicated that there had been numerous changes to Wingate's original text, and that the mathematician John Kersey would now be named as the main author

of "Mr. Wingate's Arithmetick." In the preface, Kersey (1683) claimed that Wingate had asked him to revise *Arithmetique Made Easie* so that the four operations, compound operations on money and weights and measures, reduction, practice, the rules of three, simple and compound interest, discount, alligation, fellowship, false position, and arithmetical and geometrical progressions, would be initially developed through whole numbers. Then, later in the book, the same topics should be reconsidered with common fractions; and then, later still, with decimal fractions (Ellerton & Clements, 2012). In Kersey's (1683) revised edition, decimal fractions were not formally dealt with until Chapter 22. Kersey (1683) made clear his own perspective when he wrote that "decimal fractions are being commonly abused, by being applied to all manner of questions about money, weight, &c, when indeed many questions may be resolved with much more facility by vulgar arithmetic" (p. 168).

So, Kersey effectively allowed teachers in British schools to continue to follow the traditional, non-decimal, sequence for elementary *abbaco* arithmetic. Other authors of arithmetics adopted a similar approach. Edward Cocker's popular arithmetic, first published by John Hawkins in 1677, basically avoided decimal fractions, but in 1685 Hawkins published *Cocker's Decimal Arithmetick*. In his preface to this second book, Hawkins acknowledged that, in presenting elementary *abbaco* arithmetic without using decimal fractions in a separate text, he was following the lead of Kersey (Cocker, 1685, p. vi).

The 1685 first edition of *Cocker's Decimal Arithmetick* set out, with great clarity, the calculating power of decimals. Cocker (1685) wrote:

> Here by the way, take notice, that although amongst artificers the two foot rule is generally divided, each foot into 12 inches, &c., yet for him that at any time employ'd in the practice of measuring, it would be most necessary for him to have his two foot rule, each foot divided into 10 equal parts, and each of those parts divided again into 10 other equal parts: so would the whole foot be divided into 100 equal parts, and thereby would it be made fit to take the dimensions of any thing whatsoever, in feet and decimal parts of a foot; and thereby the content of any thing may be found exactly, if not more exactly and near, than if the foot were divided into inches, quarters, and half quarters. (p. 45)

Cocker calculated the "content" of a table top 18 feet 9 inches long by 3 feet 6 inches by multiplying 18.75 by 3.5, and showing the unit "square feet" in his answer. He proceeded to show how much more cumbersome the calculation of the "content" of a table would be if the traditional method were used. 18 feet 9 inches would be converted to 225 inches and 3 feet 6 inches to 42 inches; then, after multiplying 225 by 42, the product 9 450 would be obtained, which after division by 144 would give $65\frac{90}{144}$; then, that mixed fraction would need to be interpreted in relation to the original problem. Cocker (1685) described that method as "tedious" when compared with "the decimal way" (p. 47).

It was one thing, however, to present good reasons for the introduction of decimal arithmetic into schools and another thing for teachers to adopt such a radical innovation. There was a long tradition implicitly defining what school arithmetic should look like, and teachers needed more than mere argument to be persuaded to depart from long-held ideas and practices. Cocker's (1677) traditional arithmetic would prove to be much more popular than his *Decimal Arithmetick,* with revised editions of the non-decimal text being published for the next 150 years. *Cocker's Decimal Arithmetick* would also be reprinted, but it was always much less popular than Cocker's traditional arithmetic textbook.

Many eighteenth-century British authors of school arithmetics included chapters on decimals, with some authors pointing out how a decimalized form of numeration could not only revolutionize school arithmetic but also have major applications. However, almost without exception, the authors of the texts included sections on decimal arithmetic only after the standard sequence of elementary *abbaco* arithmetic (from numeration and the four operations, to the rules of three, to progressions) had been dealt with using traditional whole-number approaches. If an author went beyond the traditional, there would usually be a section on "vulgar arithmetic" before the section on decimal arithmetic. That was the case, as we shall see, with the two most popular arithmetics used in British schools during the period 1750–1850.

British historians (e.g., Denniss, 2012; Michael, 1993; Stedall, 2012) have pointed to the unparalleled popularity, with respect to school mathematics in Great Britain between 1750 and 1850, of Francis Walkingame's *The Tutor's Assistant; being a Compendium of Arithmetic and a Complete Question Book.* A 1785 edition of Walkingame's book devoted its first 97 pages to "Part I: Arithmetic in Whole Numbers", then the next 10 pages to "Part II: Vulgar Fractions", and then the next 42 pages to "Part III: Decimals". The second most popular text was Thomas Dilworth's *Schoolmasters Assistant* – a text which was widely used in North America in the second half of the eighteenth century. Dilworth's book, which was dedicated "to the revered and worthy schoolmasters in Great Britain and Ireland" (Dilworth, 1773, p. iii), devoted its first 110 pages to "Part I: Of Whole Numbers", then the next 12 pages to "Part II: Of Vulgar Arithmetic", and finally the next 46 pages to "Part III: Of Decimal Fractions".

The textbooks suggested that decimal arithmetic should be part of the intended curriculum for teachers. But, the above analysis raises some serious historical questions. Did Walkingame and Dilworth, and numerous other writers whose arithmetics presented whole number arithmetic, vulgar fraction arithmetic, and decimal fraction arithmetic as separate areas of study, include the sections on decimal fractions merely to catch the small number of students who continued to study arithmetic beyond whole numbers and vulgar fractions? What proportion of those teachers who required their students to use texts like those authored by Walkingame and Dilworth ever got around to asking their students to study decimal arithmetic? Such questions relate to the

"implemented curriculum", and answers will be given later in this chapter with respect to school arithmetic in Great Britain and North America.

With three minor exceptions (specifically, Greenwood, 1729; Venema, 1730; and Grew, 1758), before 1776 the only mathematics textbooks available for use in the British colonies in North America were written by persons of European origin who were not living in the colonies at the time of writing or initial publication. Among the most popular arithmetics used in North American schools were those written by Englishmen – Edmund Wingate, Edward Cocker, George Fisher, Thomas Dilworth, James Hodder, and Francis Walkingame (Ellerton & Clements, 2012; Monroe, 1917). Furthermore, the teachers who taught North American students were often from Great Britain. Thus, the intended curricula for school arithmetic in the two nations would be expected to have been similar, and one might therefore also expect the implemented curricula to have been similar. An analysis of data with respect to such expectations will now be presented (see entries in Table 1).

Decimal fractions in implemented arithmetic curricula in schools in Great Britain and North America 1667–1887

Table 1 was constructed with the following four questions in mind:
1. To what extent did (a) British, and (b) North American school students who prepared cyphering books study common fractions before 1792?
2. To what extent did (a) British, and (b) North American school students who prepared cyphering books study common fractions between 1792 and 1887?
3. To what extent did (a) British, and (b) North American school students who prepared cyphering books study decimal fractions before 1792?
4. To what extent did (a) British, and (b) North American school students who prepared cyphering books study decimal fractions between 1792 and 1887?

An examination of entries in Table 1 will reveal that 10 of the 102 British CBUs in the Ellerton-Clements collection were prepared before 1792. So far as the 370 American CBUs were concerned, 35 were prepared before 1792.

For the purpose of analysis, the cyphering books were placed in the following six categories:
1. Books which focused on algebra – with these, there was not much material directly related to common or decimal fractions.
2. Books dedicated to the study of at least one of mensuration, trigonometry, surveying, navigation, or applied geometry – in these, decimal fractions usually played an important role.
3. Books in which neither common nor decimal fractions were mentioned.

4. Books which mentioned common fractions but not decimal fractions.
5. Books which dealt with decimal fractions, but not common fractions.
6. Books which paid attention to both common fractions and decimal fractions.

Table 1 shows the percentages of books in each of these categories. The percentages have been rounded to the nearest whole numbers.

Table 1. Common fractions and decimal fractions in British and North American cyphering book units (CBUs) 1667–1791 and 1792–1887 (% of CBUs in that category)

Category	Presence in CBUs of sections on common fractions or decimal fractions	Great Britain 1667–1791 (Percentages based on 10 CBUs)	Great Britain 1792–1887 (Percentages based on 92 CBUs)	North America 1667–1791 (Percentages based on 35 CBUs)	North America 1792–1887 (Percentages based on 335 CBUs)
A	None (algebra cyphering books)	10% (1 book)	9% (8 books)	3% (1 book)	4% (15 books)
B	Totally integrated presence (mensuration, trigonometry, surveying cyphering books)	20% (2 books)	16% (15 books)	14% (5 books)	7% (25 books)
C	*Neither* common fractions *nor* decimal fractions	30% (3 books)	20% (18 books)	43% (15 books)	33% (109 books)
D	Common fractions *but not* decimal fractions	20% (2 books)	39% (36 books)	29% (10 books)	13% (42 books)
E	Decimal fractions b*ut not* common fractions	0% (No book)	0% (No book)	0% (No book)	4% (12 books)
F	*Both* decimal fractions *and* common fractions	20% (2 books)	16% (15 books)	11% (4 books)	39% (132 books)
Totals		100%	100%	100%	100%

A strong theory should be predictive, and in this case, before the analysis which generated Table 1 was carried out, it was predicted that, before 1792, the percentages of British and North American cyphering books which included sections on decimal fractions should be similar. However, between 1792 and 1887, the percentage of North American cyphering books which included sections on decimal fractions would be expected to be greater than the corresponding percentage for British cyphering books. The reason for such a prediction is that, after 1791, in North America, the national currency was officially fully decimalized, with its dollars and cents, but the official currency in Great Britain remained the non-decimalized sterling, with its pounds, shillings and pence. The United States Mint was established, officially, in 1792.

Discussion

The prediction that, after 1791, implemented arithmetic curricula in North American schools would pay greater attention to decimal fractions than was the case in British schools proved to be correct. If it is assumed that entries in rows *B, E,* and *F* in Table 1 can be associated with decimal fractions being part of implemented curricula, then for the period 1667–1791 the sums of the percentages for those three rows – 40% for Great Britain, and 25% for North America – suggest that in this earlier period *more* attention was paid to decimal fractions in British schools than in North American schools.

But, for the later period, 1792–1887, there was a pronounced change: 32% of British cyphering books, but 50% of North American cyphering books, had entries on decimals. In other words, although the uptake for decimal fractions in North American schools during the latter period was considerable, that was not the case in British schools. It is possible that this statement will need to be modified so that it covers only the period up to 1862, the year when the so-called "Revised Code", incorporating the payment-by-results system, was introduced in Great Britain (Rapple, 1994). That is a matter for further research.

The move towards including decimal fractions in the implemented curriculum in North America is not surprising. By 1792 the work of research mathematicians with respect to the development of decimal concepts which would be relevant to schools had been completed; also, from the point of view of service mathematics, the influential Thomas Jefferson had pushed strongly for forms of decimalized currency and weights and measures which would facilitate everyday calculations within the populace; and from the point of view of mathematics education, there were ethnomathematical factors which made it difficult for schools to avoid teaching decimals in schools. After all, everyone – well, almost everyone – would have thought it desirable that children learn to make federal-money calculations confidently and accurately.

In Great Britain, there was no decimalized currency and therefore less societal pressure on schools to teach decimals. Furthermore, unlike the situation in France, where a metric system of weights and measures was introduced, there was no movement towards a decimalized system of weights and measures.

During the period 1792–1887 it seems that more attention was paid to common fractions in both British and North American schools than had been the case between 1667 and 1791 (this can be seen by examining entries in Rows *D* and *F* in Table 1). The difference between the two nations was that a greater proportion of students in North America than in Great Britain began to study *both* common and decimal fractions.

Another observation from entries in Table 1 is that in North America during the period 1792–1887 more than one-third of those studying arithmetic did not study common fractions (see Rows *C* and *E*). Yet, most North American

students who prepared cyphering books proceeded to the "rule-of-three". Some of the most popular North American school arithmetics in the first half of the nineteenth century (e.g., those by Daniel Adams, Nathan Daboll, Zachariah Jess, Michael Walsh, Charles Davies, Frederick Emerson, and Stephen Pike – see Ellerton and Clements, 2012) delayed the introduction of common or decimal fractions, or the concept of percentage, until after they had applied "whole-number" rules to solve tasks associated with reduction, the rules of three (direct, inverse, and double), simple and compound interest, tare and tret, alligation, fellowship, and false position. Many teachers encouraged their students to keep using convoluted whole-number methods which avoided fractions. Undoubtedly, sometimes this was because the teachers themselves did not understand the new approaches.

Imagine, for a moment, a student being asked to find the simple interest on 546 pounds 18 shillings for five years at 4¼ percent per annum, and that this student lived in a region where dollars and cents, as well as pounds and shillings, were used every day. Imagine, further, that this student had never formally studied percentage, or common or decimal fractions. In fact, around 1825 a youth named Abraham Lincoln, living in Indiana, in the United States, was placed in that situation (see Ellerton & Clements, 2014, pp. 166–167 for details). Somehow, Abraham solved the problem, but his method was rule-based and cumbersome. In the *abbaco* curriculum which Abraham and his teachers followed, the section on simple interest came well before sections on common and decimal fractions, or percentage.

According to Denniss (2012) and Ellerton and Clements (2014), in the nineteenth century British students who prepared cyphering books often copied pages directly from textbooks. From the pattern of entries in Table 1 it can be seen that most British students who prepared cyphering books did not study decimal fractions – yet most of the popular arithmetics (e.g., those by Walkingame and Dilworth) included a section on decimal fractions. In the United States, as the nineteenth century progressed, about one-third of school students who prepared cyphering books studied neither common nor decimal fractions.

If one wishes to compare an "intended" with an "implemented" curriculum, one should consider the question: "Whose intended curriculum are we talking about – the textbook author's, the school's, or the teacher's?" A textbook author might have expected students to study both common and decimal fractions. However, by placing the sections dealing with those topics late in their textbooks authors made it easy for teachers to decide that there was no need, or time, for their students to study fractions, and certainly not decimal fractions. The curriculum implemented by the teacher was not always the same as the intended curriculum of an author. In North America during the nineteenth century, on the other hand, the existence of a national decimalized federal currency placed pressure on teachers to teach their students about decimal fractions. Students wanted to make efficient calculation in dollars and

cents, and parents wanted the same for their children. Therefore, teachers known to be able to teach decimal fractions were likely to be preferred for employment in local schools over teachers who were not able to do so.

It can be concluded, therefore, that if we assume that a textbook defined an intended arithmetic curriculum, then for many students, indeed a majority of the students in Great Britain who prepared cyphering books in the eighteenth and nineteenth centuries, the implemented arithmetic curriculum differed significantly from the intended curriculum. In fact, a *teacher's* intended curriculum was often different from the intended curriculum of the *author* of the arithmetic text which he or she was consulting – if, indeed, the teacher did consult a textbook. Although in Great Britain teachers of arithmetic tended to have arithmetic textbooks (Denniss, 2012), even well into the nineteenth century, that was not the case for teachers of arithmetic in North America (Burton, 1833; Ellerton & Clements, 2012; Wickersham, 1886).

From a historical perspective, the above analysis calls for three closing comments:

1. Those who research the history of the mathematics curriculum in Great Britain in the eighteenth and nineteenth centuries need to take account of both cyphering-book and textbook data.

2. From a lag-time perspective, despite the pioneering efforts of authors such as Edmund Wingate, Edward Cocker, Francis Walkingame, and Thomas Dilworth, most arithmetic teachers in British schools did not require their students to study decimal fractions.

3. With respect to Figure 1, it is intriguing that the genius and hard work of research mathematicians such as François Viète, Simon Stevin, John Napier, Henry Briggs and Edmund Wingate, in introducing decimal and logarithmic concepts and showing how these could be useful to society, and even – in the case of Wingate – making efforts to import the concepts into school mathematics curricula, were not enough to encourage many teachers to include decimal fractions in their implemented mathematics curricula.

With respect to the second of these points, one might ask: Why should teachers in Great Britain have departed from their comfort zones by requiring their students to study decimal fractions? Those who taught arithmetic were familiar with the traditional arithmetic curriculum and, from their perspective, there seemed to be no pressing reason to add to it a study of decimal fractions. Making an effort to get their students to study common fractions might be worth the trouble, but why worry about decimal fractions? Currency calculations, and calculations related to weights and measures, could be performed without using decimals – indeed, such calculations had been done without decimals for centuries. So why change?

With respect to the third point, those who studied navigation, astronomy, mensuration, and surveying quickly began to make use of decimal and logarithmic concepts. The reason was simple – knowledgeable teachers

recognized that decimals, logarithms and trigonometric concepts greatly facilitated the study of navigation, astronomy, mensuration, and surveying. For such teachers, the effort needed to familiarize students with the new material was worth it. But, despite Wingate's best efforts, a similar attitude was not to be found among teachers and students of traditional school arithmetic.

The situation would change in Great Britain after the 1860s, when an externally-set arithmetic curriculum would be enforced in government primary schools, and external examinations would become increasingly important. The compulsory arithmetic examinations would include questions on decimal arithmetic, and a "payments-by-results" system would mean that teachers' salaries would depend on how well their students would answer such questions. That created an altogether different scenario (for details, see Roach, 1971). Despite the fact that in the United States there was no national system of schooling, no formally-prescribed curricula, no payment by results, and, for many students, no access to arithmetic textbooks (Clements & Ellerton, 2012), there was a felt need to teach federal currency. In some, but certainly not in all, cases this was accompanied by a decision to teach decimal fractions formally.

In North America the large-scale introduction of decimal fractions into implemented school curricula had a shorter lag time than in Great Britain. We conjecture that this was because the United States of America introduced decimalized currency in the 1790s, and hence knowledge of how to perform decimal calculations obviously became useful at all levels of society. Nevertheless, entries in Table 1 reveal that many teachers chose to allow their students to avoid learning about decimal fractions. One might conjecture that, in the nineteenth century in both Great Britain and in the United States of America, decimal fractions would rarely be used for calculations related to weights and measures because weights and measures were not decimalized. Cyphering-book evidence suggests that that was true even for problem situations in which decimalization would have been practically and educationally advantageous. From that perspective, it would be interesting to complement the entries in Table 1 with entries derived from cyphering books prepared in France. From the Napoleonic era onwards, a more centralized approach to mathematics education in France was adopted, and it is likely that that encouraged more secondary teachers in France to teach, and more students to learn, and to use, common and decimal fractions than had been the case in the pre-Revolutionary period (Barnard, 2008; De Bellaigue, 2007).

It is impossible to ascertain *attained* curricula merely by examining chapters in textbooks or handwritten entries in cyphering books. Certainly, in the process of studying a cyphering book we often felt that we had some idea of the extent and quality of the learning that was occurring as the student made entries into his or her cyphering book. But it was impossible to determine the extent to which students *reflected* on what they had entered in their cyphering books.

In this chapter we have emphasized not only distinctions between the concepts of intended and implemented curricula, but issues such as *whose* intended curricula we were talking about. Curriculum innovation in school mathematics is usually a complex phenomenon involving far more than merely reacting to ideas which mathematicians or politicians or curriculum developers, or relevant significant others. One of the consequences of the introduction of a decimalized currency in 1792 by the United States was that a greater proportion of students in North America than in Great Britain learned about decimal fractions, and about how these could be applied to solving real-life problems.

This chapter has summarized our large quantitative and qualitative analyses of cyphering-book data related to common and decimal fractions. We analyzed our data from a lag-time perspective and intend to undertake similar analyses in relation to curricular areas other than common and decimal fractions.

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Francesco Severi and mathematics teaching in secondary schools. Science, politics and schools in the first half of the twentieth century

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Abstract

Francesco Severi (1879 –1961) was, as is well known, a top-level mathematician who made very significant contributions in the field of algebraic geometry as well as in various other areas in mathematics. Less well known are his activities in the field of mathematics education. In this paper we intend to illustrate this work, situating it within the framework of the political and institutional history of the first half of the twentieth century. The aspects we will consider are the following: the reasons which led Severi to become concerned with problems pertaining to mathematics teaching and the influence of Federigo Enriques; his brief involvement in the direction of the Italian association of mathematics teachers Associazione Mathesis; his relationship with Fascism and the conflict with Enriques; his vision of mathematics teaching and its reflections in textbooks. Finally we will attempt to show how Severi's approach to education is characterized by a core set of assumptions whose roots lie in the way of conceiving mathematical research that was common to the Italian School of algebraic geometry.

Introduction

The historical period that provides the backdrop for Severi's scientific and academic life comprises the first half of the twentieth century. The institutional context which frames his commitment to education is characterised, in the first two decades, by the Casati Law (1859), in spite of some attempts at reform which were either unsuccessful, or carried out only in part, as was the case with the important reform project proposed by the Royal Commission (1909). The rise of Fascism and the Gentile Reform (1923) nullified any attempt at renovation in the area of science notwithstanding the battle to restore dignity to mathematics carried out by some mathematicians such as Enriques and Guido

Castelnuovo.[1] Severi's activities were especially marked by his relations with Fascism.

Born in Arezzo in 1879, Severi graduated in mathematics in 1900 at the University of Torino under the supervision of Corrado Segre, and in 1902 became assistant lecturer to Enriques at the University of Bologna. The scientific collaboration with Enriques resulted, in 1907, in the award of the Prix Bordin of the Académie des Sciences of Paris for their joint research on hyperelliptic surfaces. In the years between 1906 and 1913 Severi received other important awards and honours, such as the Gold Medal of the Società dei XL, the Guccia Medal, and the *Premio Reale* for Mathematics of the Accademia dei Lincei. In 1905 he had obtained the professorship of projective geometry at the University of Parma, but in 1921 he succeeded in transferring to the chair of algebraic analysis at the University of Rome, an important place for scientific research. Although in earlier times he had been anti-Fascist, in 1929 Severi became a member of the Accademia d'Italia, established by the Fascist regime to take the place of the prestigious Accademia dei Lincei. That marked the beginning of his support for government policies, and in 1932 he enrolled in the Fascist Party. In 1938, when the racial laws caused the removal from teaching of all Jewish mathematicians, he assumed the professorship of higher geometry, which had been held by Enriques. The following year he created, in Rome, the Istituto Nazionale di Alta Matematica (INDAM), of which he was president until his death in 1961, turning it into an important centre for research, see (Roghi 2005).

Why Severi became concerned with mathematics education: relationships with Enriques and political agenda

Two factors are of prime importance for fully understanding the reasons which led Severi to become concerned with problems pertaining to mathematics teaching: his relationship, first of collaboration and then of conflict, with Enriques, and his singular political agenda.

The influence of Enriques

Severi's collaboration with Enriques began right after he earned his degree, intensified during the period in which Severi was Enriques's assistant in Bologna, and reached its peak during their joint work on hyperelliptic surfaces. To be sure, the influence of Enriques is one of the principal factors underlying

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[1] See Giacardi & Scoth 2014 and the texts of the programmes on the website http://www.mathesistorino.it/?page_id=564.

Severi's interests in mathematical epistemology and teaching. To confirm this, we need only look at the writings and events of the period from 1902 to 1920.[2] In 1903 he collaborated with Enriques and Alberto Conti, the director of *Il Bollettino di Matematica* – a journal addressed mainly to the mathematics teachers in the lower level of secondary schools – to write the report on extensions and limits of the teaching of mathematics in lower and upper levels of secondary schools, which is based on Enriques's pedagogical tenets (formative role of mathematics, reduction of programmes, importance of intuition, usefulness of connecting the teaching of mathematics with that of physics).

In 1906 Severi published his *Complementi di geometria proiettiva* (1906) as a complement to Enriques's *Lezioni di geometria proiettiva* (1903). The two textbooks were born in symbiosis, and give evidence that Severi accepted the epistemological and didactic vision of his mentor. In the 1914 paper entitled "Razionalismo e spiritualismo" Severi sided with Enriques against the idealism of Benedetto Croce, proclaiming the cognitive and aesthetic value of science and illustrating the harmful consequences of the "movement against science" (Severi 1914, p. 187) in society and education. These and other writings demonstrate an acceptance of many of Enriques's methodological assumptions:

- Knowledge proceeds by successive approximations.[3]
- Geometry is seen as a part of physics.[4]
- Mathematical concepts have a historical and psychological genesis.
- Analogies and inductions play an important role in discovery.
- The experimental and intuitive approach is preferable in mathematics teaching.

The direction of the Associazione Mathesis and first divergences from Enriques

Severi's burning ambition to occupy top-level positions within the mathematics and academic communities inevitably led to his first clashes with Enriques on the academic plane. He himself said, "My will is tenacious to the point of obstinacy" (Severi 1953, p. 69). When Severi became president in 1909 of the Associazione Mathesis, an association of mathematics teachers of secondary schools, he attempted to insert himself into the work of the Italian subcommission of the *Commission Internationale de l'Enseignement mathématique* (later ICMI), whose three delegates – at the time, Castelnuovo, Enriques and

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[2] The most important of the papers are: Enriques, Severi, Conti 1903, Severi 1906, Severi 1910, Severi 1914, Severi 1919.
[3] Severi 1914, p. 189: "Every truth is a step along the way to a more profound truth." This and all other translations of quotations from the Italian are by the authors.
[4] Severi 1910, pp. 45-46: "Geometry knows well that of which it speaks: the physical world. It differs from physics only in method: predominantly experimental for the one, deductive for the other. And even the method loses its deductive character when discovery is concerned. At the frontiers of science … one goes forward by dint of fortunate inductions and thought experiments. And there is no lack of cases in which one resorts to genuine physical experiments."

Vailati – were nominated directly by the ICMI Central Committee. The Associazione Mathesis was not officially part of the delegation. To reach his objective, and in particular to carry out an inquiry on the teaching of mathematics in the various kinds of schools in Italy, Severi sought the support of Vito Volterra:

> And since we firmly believe that in a matter as delicate as the one involving methods of teaching, not only useful but necessary and paramount is the counsel of those who are able to treasure everyday experience carried out especially in middle schools, so we intend to conduct the inquiry on our own and report on the outcome, together with the proposals, in a separate report, which will be presented at the Cambridge congress.[5]

He even suggested that Vailati should be encouraged to resign: "Poor Vailati, afflicted as he is by his long illness, might do well to step down … and then much could be put to rights by having a replacement elected by the Mathesis".[6] Severi's attempts to impose himself were not successful because Enriques and Castelnuovo believed that it was important that the subcommission, while collaborating with the Mathesis, maintain its "freedom to act" and not be obliged to conform to the directives of the Association. This first setback was followed by another. During his term as president, Severi sent repeated requests (in January 1909, February and April 1910[7]) to the different ministers for education at the time asking them to consider the proposals put forward by the Mathesis during its national congresses in Florence (16–23 October 1908) and in Padua (20–23 September 1909). These proposals concerned the reform of the Teacher Training Schools (*Scuole di Magistero*), the abolition of the choice between Greek and mathematics beginning in the second year of *liceo*, which had been introduced by the Orlando Decree of 1904, and the reinstatement of the written exam in mathematics for all categories of schools. Despite his efforts, Severi was able to obtain from the Minister only a few general promises, and in all likelihood this drove him to look for different ways to achieve his ends and impose his will on the mathematical and academic communities. Thus on 6 November 1910 he announced his resignation and that of the entire Mathesis executive committee:

> We intend to communicate our decision to the largest daily newspapers, so that public opinion will pause, at least for a moment, to consider whether the slight regard in which cultural Societies, such as ours, are held by executive power, constitutes the most suitable means for stimulating that disinterested attachment to Education, which, despite everything, teachers still show themselves to hold… If with the resignations we are able to achieve the aim of interesting

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[5] F. Severi to V. Volterra, Padova, 13 April 1909, in P. Nastasi 2004, pp. 177-178.

[6] F. Severi to V. Volterra, Padova, 20 April 1909 in P. Nastasi 2004, p. 180.

[7] See "Il Consiglio Direttivo dal Ministro della P.I.", *Bollettino della Mathesis* 1909, pp.1-2; "I voti del Congresso di Padova presentati al Ministro della P.I.", *Bollettino della Mathesis* 1910, pp. 1-4; "Il Ministro Credaro e la Mathesis", ibid., pp. 29-30.

public opinion in the questions of didactics that the Mathesis has defended, we hold ourselves amply compensated for the effort expended for the Society during the past two years.[8]

Severi's mandate was too short to leave a noticeable mark, but in any case he deserves the credit for having put his finger on the two main weaknesses of the Mathesis. On one hand, he called for the reform of the *Bollettino della Mathesis*, the official journal of the Association, which was supposed to be transformed from a simple administrative tool into a journal with articles about science and education. On the other hand, he hoped for a strengthening of the Association's congresses, which were to offer rich programs and, above all, fight absenteeism.[9] His wishes would be carried out by the presidents who succeeded him, first Castelnuovo and then Enriques, both of whom, like Severi himself, were components of the Italian School of algebraic geometry.

In 1914 Croce, in his article "Se parlassero di matematica?", sharply attacked Severi for having invaded territory that did not belong to him – that of philosophy – in the paper "Razionalismo e spiritualismo":

> I have a fervent request of Prof. Severi, who is a cultivated man, and that is not to get involved in discussing concepts that belong to a field he is not in and not to enter into something for which I am not certain he has an aptitude …, but for which he is certainly not prepared. (Croce 1914, p. 80)

Croce's attack contributed to Severi's growing distance from Enriques and in the years that followed the scientific and cultural rivalry with Enriques became gradually more evident. In 1921 Severi brought to light an error of Enriques, leading to a heated polemic that would last over twenty years. That same year, supported by Tullio Levi Civita, Severi had the better of Enriques for the transfer to Rome to the chair of algebraic analysis left vacant by Alberto Tonelli. Enriques would assume the chair in higher geometry in 1923, thanks only to Castelnuovo's renunciation of it.[10] This rivalry, as has been said, led to a genuine "chase" on scientific, academic, educational, editorial and cultural planes, as it will be shown by what follows, see (Faracovi, 2004).

Relationship to Fascism and the conflict with Enriques

Severi's political career was singular: he was a Socialist during the period he was in Padua; as rector in Rome, he resigned after the murder of Giacomo Matteotti; he was a signer of Croce's Manifesto of the Anti-Fascist Intellectuals; he was a supporter of those who opposed the fascistization of the University of

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[8] "Dimissioni del CD", *Bollettino della Mathesis* 1910, p. 90.
[9] "Programma del prossimo Congresso sociale", *Bollettino della Mathesis* 1910, pp. 51-52.
[10] See "Il trasferimento di Enriques a Roma", in T. Nastasi 2011, pp. 256-302.

Rome. However, quick to understand the mechanisms of political power and exploit them to his own advantage, following his nomination as a member of the Accademia d'Italia in the spring of 1929,[11] he supported Fascism without reserve. In 1929–1931 he had no qualms about collaborating on the draft of a new form of oath of loyalty to the Fascist party,[12] and, later, about using the racial laws to assume absolute control over Italian mathematics. Thus he began to be involved in the process of the fascistization of culture, contributing to widen the breach between Italian mathematicians and the international mathematics community that was one of the reasons for the ensuing weakening of mathematics research in Italy.[13] When he later became conscious of this process of weakening, he attempted to revitalize Italian research by creating in 1939 the Istituto Nazionale di Alta matematica (INDAM, the National Institute for Higher Mathematics). On this aspect of Severi's personality, Francesco Tricomi wrote:

> Severi … wanted to be (and to a certain extent, was) the 'godfather' of Italian mathematics during the Fascist period. We in any case have the consolation of knowing that – while, as a rule, totalitarian regimes put the worst elements in positions of control, only because they are violent or subservient or both – in the case of Severi, the man was, from a scientific point of view, irreproachable (Tricomi 1967, p. 55).

The "Severi case" has been amply studied by historians,[14] so here I will only mention Severi's overt opposition to Enriques because of its reflections on his activities in education. The most important facts were the following: Severi refused to collaborate with the *Enciclopedia italiana* on the mathematics section, of which Enriques was director, writing: "with a man such as Enriques, … I can no longer have anything in common, much less a relationship akin to subordination".[15] He opposed the request that university chairs be established for history of science, presented by Enriques to the Accademia dei Lincei in 1938. That same year Italy's shameful racial laws were put into effect, which, among other things, excluded people of Jewish extraction from teaching in universities (Israel 2010) and Severi unhesitatingly exploited them in order to rise to a position of absolute predominance in Italian mathematics. He immediately transferred to the chair of higher geometry held by Enriques, and in February 1939 he also assumed the direction of the University School for the History of the Sciences created by Enriques in 1924, leading at last to its

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[11] Enriques's name was included on the early lists of candidates of scientific disciplines but was stricken at the last moment; see Capristo 2001.

[12] F. Severi to G. Gentile, Barcelona, 15 February 1929, in Guerraggio & Nastasi 1993, pp. 211-213.

[13] See Guerraggio & Nastasi 1993; "Conclusions" in Brigaglia & Ciliberto 1995, pp. 197-204; Israel 2010, chap. 6.

[14] See footnote 13, Faracovi 2004 and Roghi 2005.

[15] F. Severi to G. Gentile, Arezzo, 24 May 1928, in Guerraggio & Nastasi 1993, pp. 209-210.

closure. As president of the Vallecchi publishing house in Florence, he took advantage of the circular issued by Minister of Education Giuseppe Bottai in August 1938, which ordered school principals to eliminate from use all textbooks written by Jewish authors, to replace the successful geometry textbooks for secondary schools by Enriques and Amaldi with his own textbooks, published by Vallecchi.

Severi's opinion of Fascist school policy

Severi's attitude towards the Gentile Reform was in many respects similar to that of Enriques: he was convinced of the superiority of the *ginnasio-liceo*,[16] because of its frank formative aims, he was in favour of combining mathematics and physics teaching but held that too few hours were dedicated to mathematics, and that the number of hours assigned to teachers (22) was too heavy (Severi 1927–1928, p. 116). Moreover, he conceived of knowledge as a personal conquest and opposed encyclopaedism. There were, however, points where their opinions differed: Severi, in fact, tended to share the nationalistic and autarchic vision of scientific research and only later became aware of the harm that scientific isolation could lead to. Further, Enriques's dialogue with Gentile was on the philosophical plane; the fact is that he did not want to renounce his idea of the fusion of scientific knowledge and humanistic idealism that was the basis of the cultural program to which he had dedicated his whole life (Giacardi 2012, § 3.3). In contrast, Severi's relationship with Gentile assumed a political overtone and he adapted himself to Fascist directives concerning education, as can also be seen in his *Curriculum vitae*, where he states that he "had also contributed with his writings to the most elementary fields of mathematics, to renovate teaching methods" in middle schools, "adapting them to the new lines of knowledge and new pedagogical needs determined by Fascism" (Severi 1938).

Furthermore, when in 1939 the Grand Council of Fascism approved the twenty-nine declarations contained in the *Carta della Scuola* (School Charter) presented by Bottai with the aim of a further fascistization of Italian schools, Severi declared that he agreed "to every single part of it" (Severi 1939, p. 63). He shared the idea of assigning educational value to manual work, and he approved the principle affirmed by Bottai according to which "the humanistic school, be it classical or scientific, as a preparation for the university studies, must be pruned back" (Severi 1939, p. 65). In fact, classical or scientific studies must be directed to those who in future will be the ruling class of the nation; while the other young people will be given the "chance to follow their preferred vocational path" (Severi 1939, p. 65). This, according to Bottai, was an essential condition for the effectiveness of the university and "the prosperity of the

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[16] See "Riunione straordinaria promossa dal consiglio direttivo, Roma 11 febbraio 1923", *Periodico di Matematiche* 1923, pp. 156-157.

University must be measured rather with the decrease and not with the increase in the school population" (Severi 1939, p. 65). For example, Severi disapproved of the "combined degrees" (*lauree miste*) in physical and mathematical sciences established in 1921, aimed at qualifying young people to teach these disciplines in secondary schools, because, being easier to award with respect to the degrees in mathematics and in physics, they had attracted "undesirable elements" and "the damage had had repercussions for secondary schools, through the deficient preparation of the teachers, which [had] then … been deleterious to the preparation of the students" (Severi 1941, p. 199).

Mathematics teaching: methodological assumptions

In spite of their differences, the cornerstones of Severi's methodological and pedagogical vision were nevertheless very close to those of Enriques, although the epistemological considerations upon which they were founded were not as broad and amply illustrated.

Severi dealt with problems concerning the teaching of mathematics in various articles in addition to textbooks.[17] In particular a synthesis of Severi's vision of mathematics teaching appears in the entry "Didattica della matematica" that he wrote for the *Enciclopedia delle Enciclopedie* (Severi 1931), which includes an historical excursus about the teaching of geometry in Italy that goes from the use of the textbooks by Legendre and Bertrand at the beginning of the nineteenth century up to the Gentile Reform.

First of all Severi believed that secondary schools must have an essential formative aim and a "frank humanistic basis", but humanism must not be "disjoined from scientific thought" because "true humanism is integral by nature". Thus it is necessary to transmit to the student a unitary vision of culture, and strictly scientific teachings must be "maintained in the same plane", as historic, literary and philosophic ones (Severi 1940a, p. 70).

To these ends mathematics can play an important role because it trains the faculties of intuition and abstraction and develops an aptitude for "observing, abstracting, and deducing" (Severi 1940a, pp. 72–73). Mathematics teaching should have an intuitive character in lower middle schools and a rational character in upper middle schools, it must proceed by successive approximations from the concrete to the abstract, and allow time for the ideas to "filter slowly through the minds, if it is desired that they leave traces that are useful and lasting" (Severi 1931, p. 365).

In teaching, priority must be given to intuition because it develops in a way that is natural and direct, as a "synthesis of sensations, observations and experiences", almost without any wilful effort at attention on the student's

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[17] The principal articles are the following Enriques, Severi, Conti 1903, Severi 1911, 1919, 1927, 1931, 1939, 1940, 1940a, 1951, 1951a.

part", and because only intuition provides the raw material for the logical machine. In his words: "It is necessary to take middle school teaching of mathematics back to its practical and intuitive origins; and this not only for practical reasons […], but above all precisely for the educational goals of secondary studies" (Severi 1931, p. 368).

At the same time he criticizes the pseudo-rigour and incoherence of certain textbooks. He mentions, for example, the introduction of the concept of direction for distinguishing straight line from curves, which "implies that the concept of direction is held to be more intuitive, where instead it descends from the notion of straight line and of tangent at every point of a curve!" (Severi 1927–1928, p. 114). Another aspect often stressed by Severi is the importance of using the utmost parsimony in formulating programs, reducing them for each discipline to things which are truly essential and which have unquestionable educational value. In particular, Severi suggests abandoning the cyclical method by which subjects already treated in an intuitive way in middle schools are repeated and developed in a rational manner in secondary schools, and "bringing teaching closer to the current state of science" (Severi 1940a, pp. 72–73).

In order to give new impetus to teaching by means of continuous and fruitful contact with the real world, it would be useful for teachers to link mathematics teaching to that of physics. From a pedagogical point of view, it is important that they stimulate "the youthful desire for conquest", involve the students in the process of constructing knowledge and exhort them to acquire mathematical truths for themselves, because, "allowing them to find everything nice and ready, does them no good" (Severi 1927, p. V). The role played by teachers in guiding the students in learning is in fact central according to Severi:

> Having discovered the main path [to learning], it is necessary to travel it anew, and to clear away the difficulties that are too serious for non-experts, so that the student can travel them along with us, following us, without excessive effort, in the process of constructing knowledge (Severi 1927, p. V).

Finally, Severi, like Enriques, believed that the history of science can play a significant educational role in facilitating students' comprehension of certain mathematical concepts. For example in introducing real numbers in secondary schools it is preferable to follow the historical path and present them as ratios of magnitudes as Euclid did; later the teacher can gradually arrive to their definition as Dedekind cuts (Severi 1931, p. 365; Severi 1927, p. VI).

Severi himself used history in his lessons at university as well as in the courses of specialisation and advises: "don't forget the masters, because an ingenious idea is worth more in creative power than all of its consequences." (Severi 1955, p. 38).

Severi's vision of teaching of mathematics reflected in textbooks

How this vision of teaching translated into practice emerges above all from the textbooks for lower and upper secondary schools, which constitute Severi's most important and lasting legacy regarding secondary teaching.

Significantly, beginning in 1926 he directed the book series entitled *Collezione di testi di matematica per le scuole medie* for the Vallecchi publishing house in Florence. The series included his own textbooks for geometry, arithmetic, algebra (with trigonometry, financial mathematics and infinitesimal analysis) for the different types of secondary schools (*ginnasio-liceo, scuole tecniche, istituti tecnici, scuole professionali femminili, istituti magistrali,* …).[18] which were often written in collaboration with two teachers, his niece Maria Mascalchi,[19] who taught at the *liceo classico* Massimo d'Azeglio in Turin, and Umberto Bini, teacher at the liceo scientifico Cavour in Rome.

The distinguishing features of these books are: the use of an intuitive approach, which does not exclude due attention to rational aspects, suitably adapted according to school level and type of school; some use of history of mathematics; questions to facilitate learning; a great number of exercises; clarity and precision. Moreover, Severi was a fervent supporter of the need for brevity of treatment, stripping it of anything that is not essential to the comprehension of the structure of a mathematical theory, and of making room for more modern topics (Severi 1934, I, p. V).

The best known of Severi's textbooks is entitled *Elementi di geometria* (2 vols, 1926, 1927, adapted for the various types of schools. This text is distinguished by its particular approach to the principal topics of geometry (congruence, equivalence, parallel theory, theory of proportions), as well as for the methodological framework dictated by the concern that "the intuitive underpinnings of each notion introduced does not escape the students" (Severi 1939, pp. 9–10), and that the programs be slimmed down by eliminating superfluous subjects. About this Severi claims:

> The experience of the decade that has passed since the Gentile Reform has shown the necessity of thinning out and simplifying in order to lighten the load on students, without harming the formative function of mathematics teaching, and in particular of geometry. I have been a tenacious advocate of these

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[18] The list of Severi's textbooks – all published by Vallecchi – can be found in the website http://www.mathesistorino.it/?page_id=886.

[19] Maria Mascalchi (Lucca 1902 – Torino 1976), recalled by her students (among whom Primo Levi) as a Fascist of no great charm, graduated in mathematics from the University of Torino in 1919 and in 1928, after having taught at the Istituto Tecnico in Venice, was appointed to the professorship of mathematics and physics in the Liceo Classico D'Azeglio in Torino. Here she saw to the adoption of the textbooks by Severi, and after the issue of the Fascist School Charter in 1939, she directed the Laboratory for wood and metal working (Archivio del Liceo Massimo d'Azeglio, Torino, Fascicolo insegnanti, 123/1).

reductions, and once translated into act, I held it my duty to adapt to them (Severi 1934, I, pp. V–VI).

It was Gentile who wrote the preface of the 1926 edition:

> I am pleased to see that books such as these by Prof. Severi are beginning to be published for the study of mathematics in secondary schools. […] And to me these books seem to correspond wonderfully to our desire that these subjects, which always run the risk of ending up in one of two opposite extremes, either stiffening into abstruse abstraction, or falling into intolerable triviality, also be presented in the most suitable form for beginners: the heuristic form of the concept arrived at by means of intuitions that are concrete, evident and attractive (Severi 1926, p. V).

Without going into details about all the topics treated, we will mention only Severi's handling of congruence, parallels theory and real numbers.

With regard to congruence, Severi turns to Euclid's approach, that is, the use of movement, but he frames it in a complete logical structure. As a primitive he assumes the notion of congruent line segments, and defines movement as the one-to-one correspondence that transforms each segment into an equal segment, adopting, however, from the very beginning, "the language of physical movement". In fact he states that "the concept of congruence can never be detached from that of movement, because the two concepts are indissolubly linked in the mind" (Severi 1926, p. XI). This approach is linked to Severi's firm belief that geometry is a "chapter of physics" and its teaching must be brought closer to that of physics. For this reason Severi criticizes those authors, such as Enriques and Amaldi, who adopted Hilbert's approach to the congruence theory, which is irreproachable from a logical point of view, but

> besides the serious didactic drawback of forcing the assumption as a postulate of one of the cases of the congruence of triangles … it offers others of no less seriousness. The student cannot in fact understand why one has gone to such lengths of reasoning to prove the congruence of certain figures, which he would be able to verify immediately through superposition …, it leads further to an artificial and harmful hashing of the concept (Severi 1926, p. XI, XII).

At the same time he also criticises those textbooks where movement is introduced, but not placed within a complete logical framework.[20]

Instead, for the theory of parallels Severi distanced himself from Euclid, whose definition of parallels "presupposes an integral concept of the plane". Since the student can only ever utilise a part of the plane it is necessary that the geometry that he is taught be "realisable in a drawing" (Severi 1931, p. 367) and he thus defines as parallel two equidistant straight lines, postulating that "in a

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[20] See, for example, the textbook *Elementi di geometria* (Venezia 1878) by Aureliano Faifofer (1843–1909), which Severi nevertheless considered "the first good Italian treatise of elementary geometry" (Severi 1926, p. XI).

plane, the locus of the points located by a part of a given straight line and having from this a given distance, is still a straight line" (Severi 1926, pp. 111–112).

With regard to the real numbers, Severi states "In our schools, for decades and decades, ever since Faifofer transported Dedekind's theory into elementary teaching, the real numbers have become the thing most abstract and indigestible" (Severi 1927–1928, p. 113). For this reason, he introduces the real numbers in the upper level of secondary schools, in the way that he understood them to have emerged historically, that is, as relations among homogenous magnitudes and thus starts by considering the approximate values of the ratio of two incommensurable magnitudes, gradually arriving at the definition by means of Dedekind cuts (Severi 1927, Chap. IV).

To complete his text, at the end of each chapter Severi introduces numerous problems (almost 500), the most complex of which are accompanied by hints towards the solution. For the best students he inserts various complements: continuous fractions (Severi 1927, p. 23); conic sections (ibid., pp. 202–203); area of the ellipse (ibid., p. 204); spherical triangles (ibid., pp. 218–219); the Pappus-Guldinus theorem (ibid., p. 239); the graphic representation of functions (ibid., chap. XV), and more. For teachers he adds appendices to clarify the logical layout of the treatment (Severi 1926, pp. 173–184, Severi 1927, pp. 263–271).

The same didactical tenets, adapted for youngsters from 11 to 14 years old, characterise the textbook co-authored with Maria Mascalchi, *Nozioni di Aritmetica per le scuole secondarie e di avviamento professionale* (1935, 8th rpt. 1938). Here again the teaching of the discipline is accompanied by empirical observations, and that didactical requirement is compensated by the rigour of exposition and sobriety of language. The rules are given after suitable explanations and examples and are sometimes accompanied by "observations" that aim at either clarifying critical points or highlighting possible errors. As the authors say:

> The rules are actually almost always accounted for by examples and an embryo of reasoning, … the formulation has been limited and retouched to permit the greatest possible brevity and clarity; the fundamental concepts are introduced without exclusion of methods, in order to reconcile with a minimum of effort the necessities of teaching and respect for logic (Prefazione, 1938, p. VII).

For example, the concept of fractions and the related properties are made to descend from both the division of the magnitudes into equal parts and from the consideration of fractions as "operating symbols of a potential multiplication and division" (1938, pp. 57–58).

Algorithms are illustrated step by step by numerical examples, and particular attention is paid to approximations and to those arguments (interest, discounts and so forth) that are especially useful for beginning a trade, the book being

aimed at those who will not pursue further study. Before addressing this kind of problems, the authors dedicate a section to the solution of first-degree equations, also explaining the conditions for solvability (1938, pp. 108–110).

Some parts of the text are devoted to illustrating methods of rapid calculation (1938, pp. 25–27, 77–78) and mental calculation, while others present problems that arouse curiosity (1938, pp. 144–146), partly drawn from the book *Giochi di aritmetica e problemi interessanti* (1924) by Giuseppe Peano, and from *Matematica dilettevole e curiosa* (1st ed. 1913, 3rd ed. 1929) by Italo Ghersi.

The textbook presents a rich selection of problems (about 600) of various kinds: some require the simple application of the rules; others are drawn from real life experiences; others are connected to simple notions of physics; still others require "reasoning" on the basis of notions presented.

The treatment is enriched by short digressions (the monetary system, daylight saving time, longitude and latitude, systems of numeration other than base 10) and by a few historical notes on the origin of the decimal-based metric system and the calendar. Numerical tables of the primes from 2 to 3 000, of squares and square roots conclude the volume.

The texbook was updated after the introduction of the *scuola media unica* by the minister Bottai. In the revised 1941 edition (F. Severi, M. Mascalchi, *Nozioni di Aritmetica pratica con cenni storici per il 1° e il 2° anno della scuola media*), the graphic aspect is more refined, and questions are often introduced to verify the student's comprehension and solicit an active learning. Historic notes (concerning numbers, fractions, calendar, and so forth) are introduced at the end of each chapter. In contrast to the textbooks by Mascalchi for the third, fourth and fifth classes of elementary school, which presented drawings, problems and observations that were clearly Fascist propaganda,[21] here the only references to Fascism appear in three exercises (pp. 8, 34, 60)[22] that introduce the "Balilla", the Fascist youth organization, and seem to be inserted opportunistically, inasmuch as the phase of fascistization of the schools was in full swing.

Conclusions

Our examination of Severi's commitment to questions regarding mathematics teaching allows us to discern a core set of didactical assumptions shared by the members of the Italian School of algebraic geometry, which consist in a common way of conceiving mathematical research, and constitute an ulterior indicator of the appropriateness of the term "School" in speaking of the Italian geometers: attributing an educational value to mathematics, in hopes of attaining a scientific *humanitas*; preferring to use the faculty of intuition and the

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[21] On the topic of Fascist propaganda in elementary school books, see Luciano 2013-2014.
[22] There were also three references to Fascism in the 1938 edition: pp. 10, 29, 56.

heuristic procedures in teaching; aiming at rigour in substance, rather than formal rigour; establishing connections between mathematics and other sciences; and giving importance to the history of mathematics in teaching and in research.

These tenets show that when we use the word School speaking about the group of Italian researchers in algebraic geometry, we are referring not only to a group of researchers trained by the same *maestri*, from whom they draw topics of investigation, methodologies, approaches to research and a particular scientific style, and a place where talents are developed and contacts made, but also an environment in which a common way of viewing and conveying mathematical knowledge, directed their activities in education, in spite of the fact that their motivations and even the strategies they employed sometimes followed different channels.

For Corrado Segre, the father of the Italian School of algebraic geometry it was above all the intimate connection that he saw between teaching and research that led him to become concerned with questions regarding mathematics education. Instead, Castelnuovo's motivation was mainly social, see (Giacardi 2010). What led Enriques to become interested in problems of education were his strong philosophical, historical and interdisciplinary interests, and especially the studies on the foundations of geometry. He adopted a wide range of strategies and worked on different fronts: institutional, publishing (journals, book series, textbooks), and cultural. Further, he addressed his activities to different categories – secondary school teachers, researchers, philosophers, scientists, people of culture – inviting their cooperation.

As we have seen, Severi's itinerary was of yet a different nature: his interest in problems concerning the secondary teaching of mathematics was inspired both by his relationship, first of collaboration and then of rivalry, with Enriques, and by political reasons. After his unsuccessful attempt to insert himself into ICMI, and the sparse results as president of the Mathesis Association, in the course of about a decade Severi marshalled his ideas into line with the school policies of the Fascist regime, while holding to the pedagogical tenets of the Italian School of algebraic geometry. The route he favoured for improving mathematics teaching was the publishing of textbooks, a choice which reflected his political attitude towards Fascism, but which, as we have tried to show, was certainly a mirror of his conviction in the high and formative role of mathematics.[23]

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[23] After the fall of the Fascist regime, Severi was accused by the commission charged with the purge of university personnel of having carried out activities in defence of Fascism and of having collaborated with the Republican Fascist government. After a first deliberation (23 December 1944) that resulted in Severi's dismissal from service, he presented an appeal in the form of a lengthy, detailed document in his defence. After various vicissitudes and following testimony in his favour, the commission arrived at the following conclusion: "Severi did not receive from Fascism anything more than what he merited; he did however consent that his famous name, his

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moral rectitude, his high scientific merits, and his past as an anti-Fascist intellectual be used, not only for the good of Italy, but also for the political aims of the regime. [This is] A much more serious fault, in that this adhesion, in reference to a personage such as Severi, effectively constituted, for the regime, a noteworthy reinforcement". Ultimately, since the activity carried out by Severi was of a predominantly scientific nature, it was not deemed sufficiently serious to lead to a declaration of his being "unworthy of serving the State", and the commission sentenced him to a lesser penalty, that is, censure (Archivio Centrale dello Stato, Roma: *MPI, Professori epurati*, B 31 *Francesco Severi*, N. 372, Seduta del 9 maggio 1945, Sezione I).

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Problems in old Russian textbooks: How they were selected

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Abstract

This article is devoted to the study of problem sets in Russian textbooks and problem books. While the subject matter of problems in such texts often remains unchanged over decades, if not centuries, the structure of the entire set of problems devoted to any given subject matter sometimes changes radically – problems come to be selected more systematically, with a view to helping students recognize and overcome difficulties that may arise. This article attempts to trace such changes and to connect them with changes taking place in education and, more broadly, in the life of the country.

Introduction

This article is devoted to the study of problem sets in Russian textbooks and problem books, and forms one part of a study that is being conducted by the author (another part of this study is represented in Karp, 2013). Paradoxically, although studies of textbooks are quite numerous (such as, for example, those by Schubring (1987, 1999), or Howson's (1995) book, which deals with relatively recently published textbooks, or Polyakova's (2002) study of Russian mathematics education as a whole, which devotes considerable attention to textbooks), these studies usually say very little about problems for students to solve on their own. One of the reasons for this is obvious: until a certain point, textbooks simply did not contain such problems. Nonetheless, even later textbooks, which do contain such problems, have clearly not been sufficiently studied from this perspective – the author knows of no studies that systematically focus on this aspect of Russian educational literature.

Meanwhile, studying problem sets helps one to understand better how educators have envisioned the goal of education: what problems they wanted to teach students to solve, and how they wished to attain this goal. Naturally, the problem sets found in textbooks cannot be equated with the work that students

actually did in class – if only because teachers did not necessarily follow the textbooks in every detail. At the same time, the types of problems offered in a textbook, and the form and sequence in which they were presented, helps us to form a picture at the very least of how the author of the textbook envisioned the lesson and of the methodological thinking by which the author was guided.

Background

In this article, we will analyze several textbooks and problem books published between the mid-nineteenth and the mid-twentieth centuries. Naturally, not all Russian textbooks, and even fewer problem books, will be examined. The discussion will, however, concern probably the most popular editions, and for the later Soviet period, the only editions used in schools.

The oldest text discussed below is Fyodor Busse's textbook from 1845. This is by no means the first Russian textbook, of course. Kolyagin and Savvina (2013), in an article entitled "Textbooks and Curricula: How It All Began," for some reason begin their account from the beginning of the nineteenth century (noting, indeed, that textbooks were already in existence prior to this date, but that it was at this point that government regulation of mathematics education truly began). In fact, the history of Russian textbooks begins long before this date, at the very least from Euler's textbooks, and the role of the government in the regulation of the life of gymnasia (state-run) and military academies was always fundamental (Karp and Vogeli, 2010). To be sure, up to a certain time, such educational institutions were very few in number, and the general atmosphere in gymnasia was in a certain sense relatively free (students could flunk mathematics and yet successfully continue their education, see Karp, 2007). Regulations became more rigid gradually, particularly during the reign of Czar Nicholas I (1825–1853). The latter addressed educational institutions directly in a rescript from 1826: "I see with sadness that they lack the proper and necessary uniformity on which the upbringing of children as well as their education must be founded" (*Perepiska*, 1828, p. 1). In addition, the number of educational institutions continued to grow.

Yet in 1821, the minister of education had to sign personally letters to the head of the St. Petersburg school district with a proposal to procure a book on mathematics for the educational institutions overseen by the latter. The proposal was enacted and seven books in all were purchased for the entire school district, which included, for example, the Archangelsk gymnasium, which was quite far from St. Petersburg (*O knige*, 1821). By 1845, both the need for books was greater and the content of education was defined more rigidly than before. The title page of Busse's textbook states: "Manual Composed for Gymnasia by Order of the Ministry of Education."

But textbooks were composed not only for gymnasia: the system of secondary education (to use today's terminology) included not only gymnasia,

but also so-called *real schools* (Realschulen) and military educational institutions – the cadet corps. Even more importantly, educators began writing textbooks not only on orders from the Ministry of Education, but on their own initiative (in pursuit of commercial aims, among others), although these textbooks did have to be officially vetted for use in state-run schools.

The death of Nicholas I marked the start of a period of so-called Great Reforms (which included the emancipation of the peasants from serf dependence). The textbooks of Avgust Davidov, a professor at Moscow University, were written during this period, and, as all of his work, they undoubtedly reflect to no small extent the renewal of civic life that was taking place around him.

The textbooks of the following generation, written by Andrey Kiselev, appeared by contrast during the subsequent period of stagnation, during the reign of Nicholas I's grandson; their author has always been viewed as an embodiment of tradition and pedagogical experience – he did, in fact, spend a quarter of a century working at a *real school* and in a cadet corps (Karp, 2012). His textbooks remained in use in schools for almost one hundred years (the last generation of students who used his textbooks graduated in 1976), and during the Soviet period they were used under fundamentally new conditions: first of all, as the only textbooks in their subjects in the whole country; and second of all, in the context of universal education, with a number of students orders of magnitude greater than any that had existed before the Revolution.

The textbooks named above, along with other books that will be discussed below, undoubtedly reflected not only changing curricular or exam-based requirements, but also, even if in a far more complex fashion, general developments that were taking place in Russia and the country's changing reality.

Some methodological considerations

There is a vast literature devoted to the question of how problems should be classified (for example, in Russian, Stolyar, 1974, Sarantsev, 2002). Problem sets, which in their own right constitute unified texts, have received far less attention (see Karp, 2002). In actual fact, collections of problems sometimes fail to cohere into any unified texts, consisting simply of lists of problems known to the author, presented without any design. Usually, however, the author of a textbook (or problem book) must willy-nilly make certain conscious decisions – what to include, what not to include in the list of problems offered to the students. There exist problem books whose authors have aimed to construct an orderly and sequential system of problems, in which each problem has its pedagogical role (Polya's and Szegö's (1998) *Problems and Theorems in Analysis* remains a classic example of such a problem book). The Russian mathematicians Glazman and Lyubich (1969) compared their problem book to

music lessons, "each of which is devoted to a specific aspect of musical preparation and which together form the foundation for the performance skills of the future musician" (p. 7).

In analyzing a system of problems, attention must naturally be paid to their subject matter and to certain quantitative characteristics – the overall number of problems, their thematic distribution, their formal distribution (for example, in contemporary problem sets, the number of multiple choice tasks, short answer tasks, and essay questions), and so on. This kind of analysis, however, is limited. To draw a parallel with yet another field, we could compare the analysis of our texts to the analysis of poetic texts: it is certainly useful to determine the meter of a poem and its subject matter, but it would be naïve to think that the analysis will be complete once we have done so.

It is important to determine the difficulty of the problems, understood both in terms of the number of steps required to obtain an expected solution, and as a certain psychological characteristic. What is most important for us, however, is the structure of the text that we wish to analyze: this concept we take to include the relative positions and sequencing of the problems, their organization into different groups, and the existence of connections between different problems (to be sure, any relatively comprehensive analysis usually takes up a great deal of space and therefore cannot be carried out in full within the confines of this paper).

Analyzing the structure of problem sets is difficult if only because in order to carry out such an analysis, one must take into account the specific characteristics of the audience to which the problems are addressed. In the past, the Russian psychologist Kalmykova (1981) objected to drawing a rigid contrast between productive reasoning and the reasoning which simply reproduces a previously learned strategy, pointing out that even if students have already solved problems very similar to the problem which they are working on now, the solution of the new problem need not necessarily constitute an act of memory – the solvers must still find some way to re-code the present problem. The roles of memory and, conversely, of creativity can also differ for different solvers. From the point of view of the professional mathematician, the solutions of the equations $x^2–4x+3=0$, $x^2–4x=–3$ and $\frac{1}{3}x^2–\frac{4}{3}x+1=0$ are indistinguishable, but for the student who is just beginning to study quadratic equations this is not the case. Identical problems are problems that are solved in ways that are identical from the point of view of the solver. Consequently, in analyzing sets of problems, one must be very attentive to their differences, attempting to establish the extent to which these differences are deliberate and what motivated them.

In conclusion, we should say that, just as with the analysis of a literary text, it appears reasonable to indicate the general ideas and perspectives in terms of which a text will be analyzed, without however attempting to define its "coding" in advance and to impose this coding on all encountered instances.

The discussion below will often address individual problems, and not only the types and groups to which they may be said to belong.

Problems in algebra

We will first look at problems in algebra – a subject less bound by tradition than geometry.

Davidov's textbooks

Davidov's 1866 textbook is divided into parts, which are divided into chapters, which are composed of sections. Problems for students to solve on their own are included in each chapter. Thus, for example, chapter VI of the first Part, "Division of Monomials and Polynomials" concludes with 48 problems, in 33 of which the students are asked "to divide." The very first of them is as follows: "Divide $a^2-2ab+b^2$ by $a-b$."

One can observe a certain rise in the level of technical difficulty as one moves toward the end of the list of problems. Thus, problem 31 reads as follows: "Divide $(ax+by)^2+(ay-bx)^2+cx^2+cy^2$ by x^2+y^2." Problems that involve division with a remainder appear indiscriminately in the general list, without being singled out in any way.

In problems 34–37, students are asked to divide a product or to represent a given expression in the form of a product, one of whose factors is given, i.e. the division is made more difficult by introducing some additional task, or the division assignment is formulated in a less direct manner. In several other exercises, the task is formulated as follows: "Show that such-and-such an expression is divisible by such-and-such an expression." Finally, problems 42–48 are word problems, which effectively ask students to write down the answer as the quotient of two polynomials. The last problem in the set is an example of such an assignment:

> 48. A blend of loose tea is made up of three kinds of tea: a pounds of the first kind, one pound of which costs x rubles; b pounds of the second kind, one pound of which costs m rubles more than the first kind; and c pounds of the third kind, one pound of which is n rubles cheaper than the second kind. How much does one pound of the blend cost?

Davidov's textbook remained in use in schools for over 60 years; therefore, it makes sense to deviate from chronology, and, rather than moving on to the next textbook that was published, to examine other editions of this same textbook. Jumping immediately to 1914, to the twenty-first "reviewed and revised edition," we see that at the beginning of the list of problems, eight new problems have appeared, with problems 1–2 containing three assignments each

– here, students are asked to divide a monomial by a monomial. Problems 3–5 ask students to divide a polynomial by a monomial, while problems 6–8 offer a kind of reformulation of this assignment: for example, to represent a polynomial in the form of a product, one of whose factors is given. Then follow problems 1–37 from the first edition in their entirety (now numbered 9–45). Then several new problems appear, including quite easy factorization problems – numbers 46–50 (for example, #46: factor $4a^2b^4-9a^4b^2$). These are followed by two more easy problems, in which students are asked to show that a given polynomial is divisible by $x–1$ and so forth. And finally, the problem set concludes with problems 38–48 from the old edition.

The post-Revolutionary edition of 1922 includes the same problem set in this chapter as the 1914 edition, but it also contains a new chapter on "Factorization," which however does not contain any problems.

Kiselev's pre-revolutionary textbooks

It should be noted that gradually various problem books began to appear to supplement the textbooks. The author of yet another long-lived textbook, Andrey Kiselev, even wrote in the introduction to his algebra textbook that

> certain teachers assume that the lower grades of secondary educational institutions do not need any algebra textbook, and that a good collection of algebra problems will suffice (1888, p. 1).

Disagreeing with this point of view, Kiselev went ahead and wrote a textbook; without hesitating to derogate Davidov's textbook right in his introduction, he clearly felt, however, that as far as problems were concerned, everything was already in order, and therefore he did not include problems for students to solve on their own in his pre-Revolutionary textbooks.

Shaposhnikov's and Val'tsev's books

Probably the most successful of the problem books (at least in terms of its length of service in schools) turned out to be Shaposhnikov's and Val'tsev's text (the first edition of which came out in 1888). The same section on "The Division of Polynomials" in this problem book is made up of five subsections: *Dividing Monomials (problems 411–460), Dividing a Polynomial by a Monomial (461–480), Dividing Polynomials (481–526), Division Using Formulas (527–624), Multiplying* and *Dividing Polynomials with Letter Coefficients (625–654)*. Each subsection begins with a brief explanation and instructions for solving the problems, but after this the text contains practically no words. The problems are evidently sequenced in order of ascending difficulty; for example, students are first asked to divide polynomials by a number, then by a monomial with one first-degree variable, then by a monomial with one variable and a coefficient, and so on. Many

problems are given with doublets – identical problems – as the authors themselves wrote: "every minute shade of the general diversity is represented in two congeneric forms" (p. IX). Some problems are marked with an asterisk, which denotes the presence of some special features; one problem singled out in this way, for example, is # 493, in which division of polynomials leaves a remainder (no other special subsection is created for such problems). The number of problems is very great and the problems presuppose highly developed computational skills on the part of the students. Let us provide one example (# 520):

$$(6x^8 + 10\frac{1}{2}x^4y^4 + 36x^2y^6 + 16y^{10} - 50xy^8 - 8x^5y^4) \div (4\frac{1}{2}xy^2 - 4y^4 + 3x^3).$$

Shaposhnikov's and Val'tsev's problem book went through numerous editions, but without many changes. In the twenty-fifth, post-Revolutionary edition, the problems in the section we have been examining here are completely identical to the ones discussed above. Later, however, when the problem book was approved as the official (i.e. only) problem book used in Soviet secondary schools, certain revisions were introduced. For example, the subsection on "The Division of Monomials" now began with eight new problems (each of which contained two assignments) involving the division of numbers. Some sections were shortened, for example, "Division of Polynomials," whose 46 problems were reduced to 28. Furthermore, some of the most technically difficult assignments, such as No. 520 cited above, were omitted. And in general, many of the remaining assignments were also substantially reworked. Thus, for example, the old # 497 – divide *3a⁴–8a³+7a²–2a* by *3a²+2a* – which had appeared in the first third of the list, was replaced by # 369: divide *3a⁴–8a³+7a²–2a* by *(3a²+2a) – (a²–2a+1)* – which now became the last assignment in its subsection. The subsection on *Division Using Formulas* was divided into two parts – *Short-Cut Multiplication* and *Short-Cut Division* – while the last subsection, devoted to the division of polynomials with letters (somewhat strangely, since such division had been carried out also in preceding subsections) was eliminated. Two word problems were added to the section on "Short-Cut multiplication," one of which was the following: how will the area of a square change if the length of one of its sides is increased by 1, while the length of an adjacent side is reduced by 1?

In the 1950 edition, the section on *The Division of Polynomials* changed only slightly: in particular, the numerical problems with which the section on *The Division of Monomial* began were again removed.

Kiselev's books again

It should be noted that, recognizing the need for a problem book, Kiselev himself finally published such a text in 1928. It differs from the problem book discussed above, first, by the number of problems that it contains – they are far fewer. The section on *The Division of Monomials* contains nine problems only; six

problems are devoted to dividing a polynomial by a monomial; 13 problems deal with dividing a polynomial by a polynomial; and then begins the section *Factorization.* From a technical point of view, Kiselev's assignments are simpler than Shaposhnov's and Val'tsev's. Probably the most technically difficult assignment is the following: # 279: $(3ax^5-15a^2x^4+6a^3x^3) \div (x^2-5ax+2a^2)$.

Kiselev's problem book, however, contains assignments of a completely different type – it has more "words" than the problem book examined above (although still not very many). For example, students are asked to explain why in certain cases division is impossible. Students are asked not only to divide, but also to find a quotient and a remainder; and there is even an assignment in which students are asked to establish that the remainder, after a given polynomial is divided by the polynomial $x-1$, is equal to the value of this polynomial when $x=1$.

Post-Revolutionary editions of Kiselev's textbook did contain problems, but even fewer than his problem book; instead, a revised version of Shaposhnikov's and Val'tsev's problem book was used in tandem with Kiselev's textbook. Thus, for example, only three problems (seven assignments) in Kiselev's textbook were devoted to the division of monomials.

Larichev's books

The last text that we will mention here is the problem book by Pavel Larichev, whose first part came out in 1948, and which replaced Shaposhnikov and Val'tsev. Here, three subsections are devoted specifically to the division of polynomials: *Division of Monomials, Dividing a Polynomial by a Monomial*, and *Division of Polynomials*, containing, respectively, problems 219–240, 241–259, and 260–270. The next subsection, *Exercises for Review*, contains many division problems, but often with additional requirements. For example, in # 234 students are asked to solve the equation $20x^3 \div 4x^2-14+x=4$, and in # 257 they are asked to find the numerical value of an expression given the value of a variable in it, after first simplifying the expression (by means of division). In the first three subsections and in general, the tasks usually contain more "mini-assignments" than those in Shaposhnikov and Val'tsev. Thus, usually students are asked not only to divide one polynomial by another, but to divide and then to check their results. Another conspicuous difference in this problem book is that the tasks appear in groups; thus, # 270 contains ten tasks – in the odd ones, students are asked to divide polynomials by $x+1$, and in the even ones by $x-1$. Such assignments had been found in other books discussed above, but in those cases they had simply appeared next to each other, while here they are united into one set. In general, all of the problems in the subsection on *Division of Polynomials* contain from four to ten tasks that are identical or similar.

Larichev's problem book went through numerous editions and revisions. In the seventh edition, the whole topic *Division of Polynomials* was moved from the chapter on *Monomials and Polynomials* (which was now renamed *Operations on*

Whole Algebraic Expressions) to the chapter on *Factorization*, thus shifting to a more subordinate capacity, which led to certain abridgements, in particular, dividing polynomials by a monomial now became part of the subsection on *Division of Polynomials* (rather than a separate subsection).

Problems on quadratic equations

Partly in order to reiterate what has already been said, and partly in order to confirm the character of the changes that took place, below we briefly analyze problems pertaining to another topic – quadratic equations – which have usually been found in the second parts of the problem books and textbooks discussed above. Without going into detail, we present the results of the analysis of the relevant sections in the form of a table.

Table 1. Problems on quadratic equations in different textbooks and problem books.

Books	Brief description	Notes on further editions
Davidov	Three chapters on this topic: 1. 69 problems on solving quadratic equations: the first 7 are equations in standard form; 9-63 are reducible to quadratic equations; from # 26 on, equations with unknowns in the denominator appear; from # 54 on, equations with letter coefficients; 64-69 deal with forming equations with given roots and investigating equations. 2. 30 word problems reducible to quadratic equations. 3. 69 equations reducible to quadratic equations, most of them irrational, but also a few biquadratic equations and so forth, which are not singled out in any way.	Subsequent editions contain many (33) new word problems, most of them with a geometric content. There are no other changes.
Shaposhnikov and Val'tsev	The number of problems is significantly greater than in Davidov's textbook. Problems are grouped into subsections, for example, problems in the section on *Solving Numerical Equations* are subdivided into the following groups: a) solving incomplete equations (without a constant term), b) solving incomplete equations (without a second term), c) solving complete equations, etc. Then follow equations with letter coefficients, again grouped into sections. In each section, the level of difficulty gradually increases. There are many problems for further investigation and word problems. Biquadratic and other equations are separated from irrational equations.	The pre-Revolutionary editions are practically identical. In later Soviet editions, the solution of numerical quadratic equations is separated from the solution of equations with letter coefficients, whose number increases somewhat. Graphic solutions are added. Certain types of equations are more clearly distinguished.
Kiselev's Problem Book	The sections are approximately the same as in Shaposhnikov and Val'tsev, the problems are arranged in ascending order of difficulty, but there are many fewer problems. Irrational equations are separated from biquadratic equations.	
Larichev	An even more detailed subdivision into groups; a large number of "doublets" – pairs of identical problems. Oral problems are arranged in a separate group. The investigation of quadratic equations with parameters is separated from the main part – the solution of such equations – more systematically than in Shaposhnikov and Val'tsev.	Subsequent editions do not contain very significant changes.

Discussion of problems in algebra

The changes that occurred may not seem very significant – students still solved something pretty similar to what they used to solve in the past – but in reality they are radical. Among these changes one can distinguish those which were motivated by changes in curricula and other external requirements, and those which were motivated by methodological considerations. To the first type belongs, for example, the division of the topic on *Quadratic Equations* into two parts in later editions of Shaposhnikov's and Val'tsev's problem book, so that equations with number and letter coefficients were covered with a break between them; in this case, evidently, official topic planning had been changed somewhat. The significant increase in word problems with a geometric content in later editions of Davidov's textbook was an obvious result of exam-based demands. Many similar examples could be given.

More significant, however, are changes of the second type – methodologically motivated changes. There is an obvious leap from Davidov to Shaposhnikov and Val'tsev and Kiselev. This leap may be observed, first of all, in the fact that problem sets begin to be structured in a far more detailed and explicitly indicated fashion. The authors define ideas that are new to the students much more deeply and precisely, and subdivide problem sets accordingly. They conceive of the problem set as a kind of unified text, in which the position of each problem is significant (one can trace, by looking at various editions of the same book, cases in which problems were moved from one position to another – such moves were by no means random – and the relocated problem was supplemented by new problems, resulting in some new group of problems, addressing some new idea). In general, the basic principle of problem set construction – an increasing level of difficulty – can be found in Davidov's book, too, but in texts that appeared a little over twenty years later, this principle is adhered to more strictly, and to some degree cyclically – for example, first all the steps and cases are covered using numerical equations, and then once more using equations with letter coefficients. Problems begin to be divided on the basis of the use that can be made of them – for example, as difficult problems, or as oral problems.

Larichev's problem book represents yet another step. In the first place, a significant role is played in it by an idea that had appeared earlier: the idea of using identical problems – problems for reinforcement, as it were, whose solutions are identical from the point of view of the students. Previously, problems that appeared next to each other had been very similar, but still differed somewhat, and there were very few problems that were completely identical. Now, in certain problems (not all), students were offered assignments that were identical, in large numbers. Another aspect of Larichev's problem book is that the problems in a subsection, at least in outward appearance, are somewhat more varied than in Shaposhnikov's and Val'tsev's and even Kiselev's texts. And in Larichev without a doubt "formula problems"

predominate (apart from subsections devoted to word problems, of course); but subsections devoted to a single assignment (like "divide"), common to all problems, are very rarely found in Larichev.

If we ask what caused these changes, we should recall the substantial increase in the number of students that occurred during these years, and consequently also in the number of teachers. One can wonder to what extent teachers in the 1860s themselves understood how to select problems that would allow them ultimately to teach students to solve problems from Davidov's text. Even if we assume that Davidov simply paid no attention to the sequencing of problems, limiting himself to offering lists of problems that students were supposed to learn how to solve, and expecting that teachers would figure out on their own how this was to be done, such a stance must be taken note of. Such an assumption, however, is unlikely accurate: otherwise, why would Davidov have added the problems on the division of monomials mentioned above? He clearly did not foster great hopes that teachers themselves would come to such assignments on their own, and yet he expected that his problems would be used in order to learn, and not just be perceived as an end in themselves. But in any case, the number of teachers during the 1880s grew, while their independence shrank.

It is interesting, although not entirely clear, to what degree the revolution in methodology and the recognition of the need to present problem sets in a more detailed and structured fashion were connected with developments taking place abroad. During the pre-Revolutionary years, the writers of Russian problem books did not hesitate to refer to foreign texts, but they usually did so only when discussing the theoretical construction of a course.

Larichev's latest books reflected an orientation toward universal education (even if during the years when his book was written, the fail rate in mathematics could reach up to 20%). As might be supposed, the repetition and even plain rote memorization of solutions was seen, under such circumstances, to be the only way to teach poorly performing students to meet minimal standards. What is significant, however, is that by continuing to particularize problem sets in accordance with methodological principles and to explore new possibilities for problem design (which educators began discussing later, emphasizing the importance of problems in which the same material was represented in different forms or formulated verbally in different ways), the problems in his problem book retained a high level of quality and diversity.

Problems in geometry

When we analyze sets of problems in geometry, we can observe historical processes similar to those that took place in algebra, but time-honored tradition in this case played a much greater role; in particular, many typical problems (above all, compass and straightedge constructions) automatically predominated

for a long time, and it was also somewhat more difficult to structure and subdivide the material in terms of "minute shades of the general diversity" (to use Shaposhnikov's and Val'tsev's expression).

Neither the first edition of Busse's textbook (1845), nor the eighth (1888), contained any problems for students to solve on their own. Only various examples were offered, i.e. in essence, theoretical material. Davidov's 1864 textbook contained 256 problems for students to solve on their own (covering the entire course, including plane geometry and solid geometry). In later editions, 337 so-called computation problems were added. Subsequently, the problem sets did not change: the publication of Zikhman's book, titled "Complete Solutions and Detailed Explanations in All Possible Ways (1–8) of All 256+337 Problems without Exception in A. Davidov's *Elementary Geometry*" as late as 1909 is indicative. In fact, even the first 256 problems contain computation exercises, but compass and straightedge construction problems unquestionably predominate.

Although one probably cannot say that the problems are positioned chaotically, it is difficult to see any clear structure behind them. Thus, for example, the problems in the chapter on areas begin with problem # 161, in which students are asked to find the locus of the vertices of all triangles that are equal in area and have a common base; then follow ## 152–165, in which students are asked to find the areas of different polygons; and in # 166 students are asked to construct a triangle given a side, the opposite angle, and the area. The computation problems in this chapter are structured better, but they are rather monotonous – almost all of them require students to find the area of some polygon, and the only differences between them consist in: (1) what this polygon is, (2) whether the formula can be applied directly or requires some other, additional steps, and (3) what the numerical givens are.

Kiselev's geometry textbook, which came out in 1892, begins with an introduction in which readers are informed that

> The book is supplied with a considerable number of exercises, comprised in part of certain theorems that did not make it into the text, but are of interest, but mainly of compass and straightedge problems and computation problems (p. VII).

In reality, however, the book contains very few problems. The section on areas contains eight problems with proofs, six computation problems, and nineteen compass and straightedge construction problems (all of the problem sets are divided into three such parts). The very first problems of each of these subsections are rather difficult. For example, the first problem in the chapter on areas, # 275, is as follows: "Draw two lines through the vertex of a triangle to divide it into three parts, the ratio of whose areas is m:n:p." There are certain connections between problems, for example, in the next problem, # 276, students are asked to draw a line through a point on the side of a triangle to

divide the triangle into two triangles with equal areas, for which the steps carried out in # 275 are useful. These connections, however, are not at all obvious.

Kiselev's textbook remained in schools for a very long time, but the problems did not change very much. In 1927, in keeping with the spirit of the times, many applied problems were added, which were then successfully removed when the spirit of the times changed. The formulations of problems were corrected, some of them – very few – were deleted, but on the whole little was changed.

It should be borne in mind, however, that along with Kiselev's textbook, Nikolay Rybkin's (1903) problem book was used as a mandatory text in Soviet schools. The first edition of Rybkin's plane geometry came out in 1903, and in it we see the appearance of the same planned and systematic structurization as in contemporaneous books on algebra. The section on *Areas of Polygons* contains 129 problems, which are divided into subsections, in each of which one can constantly observe a methodical transition from one idea or technique to another, from one aspect of study to some similar aspect, from easy or even oral problems to more difficult ones, and so on. Furthermore, problems are sequenced not according to the increasing difficulty computations, but according to the increasing difficulty of the geometric material (the author himself emphasizes in his introduction that this was in fact his goal). Furthermore, although the title of the problem book does emphasize the fact that it contains computation problems, in reality it also includes problems that require proofs, which are however not segregated in any way from the rest of the problems, which indeed seems quite natural in this problem book, since it contains problems with very varied formulations.

Rybkin's textbook remained in use in schools for many decades, but revisions to it were minimal.

Conclusion

Inevitably, when we compare old problem books with modern ones, we should probably refrain from engaging in simplistic lamentations about the fact that today's students would be unable to solve the problems in old texts – life has changed, and demands have changed, and today's students perhaps are able to do something that their peers were unable to do one hundred years ago. Analyzing the structure of the problems offered in these books, on the other hand, seems to us to be far more fruitful. Unfortunately, in many cases in today's Russian textbooks and problem books, one finds that "same-type" problems, which serve a purpose when relied on judiciously, are far too numerous to be useful – students are given many problems, but almost all of them are identical; while in the books analyzed above we observed the gradual appearance of preparatory problems, now not infrequently these are the only

kinds of problems offered, and the number of substantive problems is very small.

It would do well to ponder the future fate of the methodological thinking that evolved in Russia by the late nineteenth century (Karp and Vogeli, 2011). The ability to teach problem solving is above all the ability to construct problem sets. Primitivization, promoted under the banner of exam preparation, or in the spirit of solving only that which will be useful in real life, or simply due to a disbelief that children can be taught something, can destroy such thinking. It is also useful to ask why and how this thinking evolved and what threats it faces. As in any professional activity, so in the teaching of mathematics, one cannot restrict oneself to voicing broad objectives: what is needed is a detailed professional accounting of what has been achieved and what is now being done. In this respect, the analysis of old problem books turns out to be important and timely.

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Warren Colburn and the inductions of reason

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Abstract

The mathematician and mathematics educator Warren Colburn (1793–1833) is usually credited with introducing the ideas of the Swiss pedagogue and educational reformer Johann H. Pestalozzi (1746–1827) into American school mathematics. It is not clear that Colburn was as strongly influenced by Pestalozzi as has often been claimed, but Colburn did pioneer the teaching of arithmetic to children younger than 10 or 12 years, encouraged a focus on beginning with the child's experience, and helped begin the shift from a written approach to an oral approach in teaching mathematics. The so-called cyphering-book tradition had long been dominant in American schools, and Colburn was prominent in initiating the decades-long movement away from that tradition. His arithmetic and algebra textbooks, which advocated an inductive rather than a deductive approach to instruction, were enormously popular and generally seen as extremely influential. Although Colburn's innovative ideas regarding the teaching of mathematics were frequently misunderstood, ignored, or opposed by teachers, especially teachers of older children, those ideas continued to influence U.S. school arithmetic until the 1920s. During that decade, Edward L. Thorndike (1874–1949) began the movement back to memorization and attention to the deductive learning of rules. His behavioral psychology came to dominate arithmetic instruction in the United States, and its influence remains strong. Nonetheless, Colburn can be seen as the originator of many teaching practices being advocated today. His emphasis on giving serious attention to the young learner as one who can find his or her own way in mathematics, arriving at generalizations through induction from appropriate practical experience, was almost two centuries ahead of its time. American mathematics educators owe an immense debt to Warren Colburn that they would do well to acknowledge, understand, and appreciate.

Introduction

The most influential American mathematics educator of the early nineteenth century, Warren Colburn, taught school mathematics for only a few months as a Harvard undergraduate and then for only two and a half years after graduating

at the age of 27 in 1820. In 1823, he left education to become a manufacturing executive, but between 1821 and 1825, he managed to write and publish three path-breaking mathematics textbooks, the first of which – *An Arithmetic on the Plan of Pestalozzi, With Some Improvements* – sold more than two million copies during its first 35 years in print (Edson, 1856, p. 13; Johnson, 1904, p. 312) and more than three and a half million during its first 70 years (Cajori, 1890, p. 106). Thomas Sherwin, long-time principal of the English High School at Boston, termed the book, which was known by a variety of titles (see Table 1), "the ne plus ultra of primary arithmetics" (Edson, 1856, p. 17). For David Eugene Smith (1916), the book was

> the first great external influence, one based on a mixture of child psychology and common sense, that caused any change in the sluggish course of American arithmetic, and it is one of the few influences that have been exerted on the subject which are really significant. (p. 113)

Table 1. Textbooks by Warren Colburn

Date	Title	Description
1821	*An Arithmetic on the Plan of Pestalozzi, With Some Improvements* [1]	First U.S. textbook with mental arithmetic, oral drill and practice, and no rules without reasons; designed for children as young as 5 or 6 years; first part was text; second part had explanations and problems
1822	*First Lessons in Arithmetic, on the Plan of Pestalozzi, With Some Improvements*	Revision of preceding text in which all sections, not just those involving fundamental operations, were preceded by simple illustrative problems; number of pages increased from 108 to 172
1826	*Colburn's First Lessons: Intellectual Arithmetic Upon the Inductive Method of Instruction*	Retitled version of *First Lessons* from 1822
1847	*Warren Colburn's First Lessons: Intellectual Arithmetic Upon the Inductive Method of Instruction—New Edition, Revised and Improved*	Only part of the original preface included; directions added for eight preliminary lessons in which pupils count objects; no reference to accompanying plates
1863	*Warren Colburn's First Lessons—New Edition: Intellectual Arithmetic Upon the Inductive Method of Instruction*	Original preface restored (except for paragraph on Pestalozzi); introduction to written arithmetic by Colburn's son; introduction by George Emerson; and 11 pages added on written arithmetic
1884	*Warren Colburn's First Lessons: Intellectual Arithmetic Upon the Inductive Method of Instruction*	Thoroughly revised and enlarged edition with written arithmetic and pictures added, notation introduced earlier, and counting numbers taken up individually and separately
Sequel		
1822	*Arithmetic; Being a Sequel to the First Lessons in Arithmetic*	Written to follow (but not assume) the *First Lessons* text; for children 8 to 10 years
1828	*Arithmetic Upon the Inductive Method of Instruction: Being a Sequel to the Intellectual Arithmetic*	Title change of the 1822 *Sequel* and with minor editorial changes but not a revision
Algebra		
1825	*An Introduction to Algebra Upon the Inductive Method of Instruction*	Written to make transition from arithmetic to algebra as easy as possible

[1] This title comes from the microfilm of a Harvard University copy and also from Ellerton and Clements (2012, p. 119), but the 1821 edition has also been cited as *First Lessons in Intellectual Arithmetic* (Kegley, 1947, p. 22; Monroe, 1912a, p. 423; Richeson, 1935, p. 76; Smith, 1916, p. 113), or as having the same title as the 1822 edition (Monroe, 1917, p. 63).

American arithmetic before 1821

Arithmetic was a rudimentary subject in the grammar schools of Colonial America, treated seriously only in schools of trade and commerce (Monroe, 1917, p. 17). The emphasis was on written arithmetic, so it was not ordinarily taught until pupils were 11 or 12 and could read and write. Textbooks were not common; instead, pupils worked on problems given by the teacher, who then checked their solution against a key (Kimmel, 1919, pp. 197–198). They typically wrote out their solutions in so-called cyphering books (see Ellerton & Clements, 2012, for a thorough treatment of the cyphering-book tradition in North America). In an account published almost 50 years after he had left his writing school in Boston at age 15 or so, William Fowle (1858) reported that printed arithmetic books were not used in those schools until after he had left. He said,

> The custom was for the master to write a problem or two in the manuscript of the pupil every other day. No boy was allowed to cypher till he was 11 years old, and writing and cyphering were never performed on the same day. . . . All the sums [the master] set for his pupils were copied exactly from his old manuscript [from his own school days]. Any boy could copy the work from the manuscript of any further advanced than himself, and the writer never heard any explanation of any principle of arithmetic while he was at school. Indeed, the pupils believed that the master could not do the sums he set for them. (p. 336)

In a recollection of his school days written for Henry Barnard's *American Journal of Education* some 70 years later, Joseph Buckingham (1863), who had been attending school only a few days a year, said that in 1790 or 1791 (roughly at age 10), he was deemed old enough to learn to cypher and therefore was permitted to go to school more regularly. He recounted his experience with arithmetic:

> I told the master I wanted to learn to cipher. He set me a *sum* in simple addition—*five columns* of figures, and *six figures* in each column. All the instruction he gave me was—add the figures in the first column, carry one for every ten, and set the overplus down under the column. I supposed he meant by the *first* column the left hand column; but what he meant by carrying one for every ten was as much a mystery as Samson's riddle was to the Philistines. I worried my brains an hour or two, and showed the master the figures I had made. You may judge what the amount was, when the columns were added from left to right. The master frowned and repeated his former instruction— [add] up the column *on the right,* carry one for every ten, and set down the remainder. Two or three afternoons (I did not go to school in the morning) were spent in this way, when I begged to be excused from learning to cipher, and the old gentlemen with whom I lived thought it was time wasted. (p. 130)

Monroe (1917; see also Monroe, 1912b) characterizes the cyphering-book method of arithmetic teaching as going from the abstract to the concrete,

emphasizing memorization, not assisting the pupil who might need help, proceeding deductively from rule to problem, making no provision for drill, relying on written work only, avoiding class or group instruction, and yielding tangible results in the cyphering books produced. According to Monroe (1917), Warren Colburn (Figure 1) changed all that:

> Arithmetic was given a place of increased importance as a school subject; the content of the texts was changed almost abruptly; the aim of arithmetical instruction was modified to include mental training as an important factor; and much of the instruction in arithmetic became oral. Warren Colburn exerted a greater influence upon the development of arithmetic in the United States than any other person. (p. 53)

Figure 1. Warren Colburn

Artist: unknown

Colburn's method

Colburn always maintained that he learned from his pupils what to put in his textbooks, and that does seem to have been the case. He tried to find out what would interest young children and determined that, by handling arithmetic orally instead of in written form and by beginning with small numbers, he could introduce arithmetic to children as young as 5 or 6 years. In the original preface to the book that went through multiple editions and became known as *First Lessons* (Table 1), he wrote,

> The names of a few of the first numbers are usually learned very early; and children frequently learn to count as far as a hundred before they learn their letters.
> As soon as children have the idea of more and less, and the names of a few of the first numbers, they are able to make small calculations. And this we see

them do every day about their playthings, and about the little affairs which they are called upon to attend to. The idea of more and less implies addition; hence they will often perform these operations without any previous instruction. If, for example, one child has three apples, and another five, they will readily tell how many they both have; and how many one has more than the other. (Colburn, 1823, p. iii)

Colburn thought that rather than being presented with a procedural rule for operating on numbers, but with no reason for doing so, and then being presented with a set of abstract numbers too large for reasoning about, pupils should begin by reasoning about a small number of concrete objects in understandable situations (Michalowicz & Howard, 2003; Monroe, 1913a). Here are the first exercises Colburn (1823) presents in *First Lessons:*

1. How many thumbs have you on your right hand? how many on your left? how many on both together?
2. How many hands have you?
3. If you have two nuts in one hand and one in the other, how many have you in both?
4. How many fingers have you on one hand?
5. If you count the thumb with the fingers, how many will it make? (p. 2)

Colburn wanted to resolve the problem of learners not understanding the results they were getting or what those results might be good for. In an address to the American Institute of Instruction in 1830, he characterized the learner being taught by the instructional methods commonly used during the latter part of the seventeenth century as follows:

> When [the learner] had got through [the problem] and obtained the result, he understood neither what it was nor the use of it. Neither did he know that it was the proper result, but was obliged to rely wholly on the book, or more frequently on the teacher. As he began in the dark, so he continued; and the results of his calculation seemed to be obtained by some magical operation rather than by the inductions of reason. (Monroe & Colburn, 1912, p. 466)

That last phrase – "the inductions of reason" – captures much of the essence of Colburn's method, which for *First Lessons* proceeded according to the following three principles:

1. Proceed from the practical example to the abstract number.
2. Use small examples in order to make clear the reasoning in them.
3. Make the pupil discover himself the rule. (Keller, 1923, p. 163)

Colburn offered these principles as guidelines, however, and not as rules. He seems to have been averse to setting out any theory of teaching. Addressing the question of the best mode of teaching his system, he said that, as usually posed, "it does not admit of an answer" (Monroe & Colburn, 1912, p. 466).

> The method must be suited to the teacher; and the teacher again, to be successful, must adapt his method to the scholar. . . . The best method for any

particular instructor is that by which he can teach the best. It is that which is suited to his particular mode of thinking, to his manners, to his temper and disposition; and generally, also, it will be modified by the character of his school. (p. 467)

Part of Colburn's approach to arithmetic was to introduce a new topic by asking what he called a "practical question," which was then followed by the same problem in abstract form. For example, in *First Lessons,* Colburn (1823) introduced a section on multiples of fractions with the following problems:

1. If a breakfast for 1 man cost 1 third of a dollar, what would a breakfast for two men cost?
2. How much is 2 times 1 third?
3. If it takes you 1 third of an hour to travel 1 mile, how long will it take you to travel 3 miles?
4. How much is 3 times 1 third?
5. If 1 man can eat 1 third of a pound of meat at a meal, how much can 5 men eat?
6. How much is 7 times 1 third?
7. If 1 man can eat 2 thirds of a pound of meat for dinner, how many thirds of a pound would 3 men eat?
8. How much is 2 times 2 thirds? (p. 78)

Colburn published the first edition of the *Sequel* in 1822, the same year he produced the revision of *First Lessons.* He intended the *Sequel* to be studied by the pupil after he or she had completed the *First Lessons,* but that was not a prerequisite, and pupils could begin the *Sequel* as soon as they could read. Colburn distinguished between what he termed *principles* (processes such as multiplying integers and dividing fractions) and *subjects* (applications of arithmetic such as compound interest and mensuration). The book had two parts: The first part had graded lists of problems accompanied by the occasional note with a definition or interpretation; the second part developed the principles behind each list of problems. After finishing the problems in each list, the student was expected to read the development of the corresponding principle, explain the reasoning behind it, and only then commit to memory the accompanying rule, which should have been formulated inductively while the problems were being solved (Monroe, 1913b).

Monroe (1913b) gives an example of how Colburn, in the second part of the *Sequel,* is able to take the learner's point of view. "The way he guides the learner in the development of the principles adds a touch of genius to the whole work" (p. 299):

A boy wishes to divide $\frac{3}{4}$ of an orange equally between two other boys; how much must he give them apiece?

If he had 3 oranges to divide, he might give them 1 apiece, and then divide the other into two equal parts, and give one part to each, and each would have $1\frac{1}{2}$

orange. Or he might cut them all into two equal parts each, which would make six parts, and give 3 parts to each, that is, $\frac{3}{2} = 1\frac{1}{2}$, as before. But according to the question, he has $\frac{3}{4}$ or 3 pieces, consequently he may give 1 piece to each, and then cut the other into two equal parts, and give 1 part to each, then each will have $\frac{1}{4}$ and $\frac{1}{2}$ of $\frac{1}{4}$. But if a thing be cut into four equal parts and then each part into two equal parts, the whole will be cut into 8 equal parts or eighths; consequently, $\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$. Each will have $\frac{1}{4}$ and $\frac{1}{8}$ of an orange. Or he may cut each of the three parts into two equal parts, and give $\frac{1}{2}$ of each part to each boy, then each will have 3 parts, that is $\frac{3}{8}$. Therefore $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$. Ans. $\frac{3}{8}$. (Colburn, 1833, pp. 166–167)

Monroe observes that two more problems are explained briefly in a similar manner, and then Colburn (1833) makes the following observation:

> In the three last examples the division is performed by multiplying the denominator. In general, if the denominator of a fraction be multiplied by 2, the unit will be divided into twice as many parts, consequently the parts will be only one half as large as before, and if the same number of the small parts be taken, as was taken of the large, the value of the fraction will be one half as much. If the denominator be multiplied by three, each part will be divided into three parts, and the same number of the parts being taken, the fraction will be one third of the value of the first. Finally, if the denominator be multiplied by any number, the parts will be so many times smaller. Therefore, *to divide a fraction, if the numerator cannot be divided exactly by the divisor, multiply the denominator by the divisor.* (pp. 167–168)

As Monroe notes, Colburn's approach was to begin with a crude approach to solving the problem, one that learners might easily devise for themselves using their intuition. Only after the learners had found their cumbersome way to a solution did Colburn offer a shorter way and rule. He let learners find their own way to a solution, never telling them directly how to do any example.

According to Colburn's biographer Theodore Edson (1856), the *Sequel*

> is certainly a work of great ingenuity, which shows a great mastery of the principles of education, and which [Colburn] himself considered a book of more merit and importance than the First Lessons. Of the Sequel, indeed, it may be said, not only that its true value has not, in general, been sufficiently estimated, but, that its actual influence on the use, the understanding, and popularity of the First Lessons has been appreciated only by particular observers. (p. 16)

Edson went on to point out that at the time of his death, Colburn was in the process of revising the *Sequel* so that its distinctive characteristics would be better understood and appreciated. But as Monroe (1917) observed:

> The Sequel has no such interesting history as the First Lessons. The original form was not revised. While it enjoyed a fair degree of popularity, it was small compared with that of the First Lessons. Editions were printed in 1841, 1849, and as late as 1860. (p. 64)

Apparently, part of the problem was that the *Sequel* had competition from other textbooks, whereas the *First Lessons* did not (Kegley, 1947).

Colburn's *Algebra* was published in 1825 and remained in print until at least 1842 but was never revised. According to Heller (1940), Colburn "combined the French theme of algebra for generalization with the English theme of algebra for the solution of numerical problems" (p. 49) in an innovative book that was elementary enough to be used by young learners yet substantial enough to provide a solid foundation for further study. Thomas Sherwin, the Boston high school principal cited above, said that in the Algebra book, Colburn had "accomplished much, by rendering the study interesting, and by gradually leading the student to a knowledge of pure algebraical symbols and processes. Mr. Colburn did much to place algebra within the reach of the mass of learners" (Edson, 1856, p. 17). An original demonstration of the Binomial Theorem based on inductive reasoning is one of the features of the book. "Instead of the deductive rigor of the French or the bald statement of rule of the English, [Colburn devised] an inductive presentation inspired by the Pestalozzian doctrines" (Heller, 1940, p. 49).

Pestalozzi's influence

In his preface to the 1822, 1823, 1826, and up to 1845 editions of *First Lessons* (but not the 1821 edition), Colburn acknowledged his debt to Johann Pestalozzi:

> In forming and arranging the several combinations the author has received considerable assistance from the system of Pestalozzi. He has not, however, had an opportunity of seeing Pestalozzi's own work on the subject, but only a brief outline of it by another. The plates also are from Pestalozzi. In selecting and arranging the examples to illustrate these combinations, and in the manner of solving questions generally, he has received no assistance from Pestalozzi. (Colburn, 1823, p. ix)

By *combinations,* Colburn meant number facts, such as "Nine and two are how many?," "Seven less four are how many?," and "Five times seven are how many?" By *examples,* he meant so-called practical problems accompanying the combinations, such as the following:

> Three boys, Peter, John, and Oliver, gave some money to a beggar. Peter gave seven cents; John, four cents; and Oliver, three cents; how many did they all give him?

A man bought thirty apples at the rate of 3 for a cent; how many cents did they come to?

And by *plates,* Colburn meant tables that were sometimes included with the text, such as that in Figure 2.

Figure 2. A Pestalozzian number chart
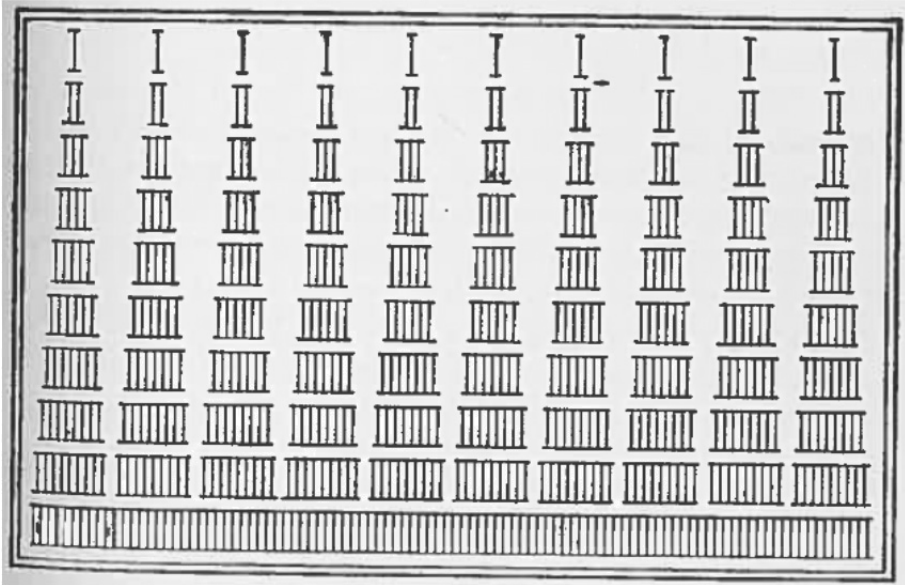
Adapted from Public Education in the United States: A Study and Interpretation of American Educational History, by Ellwood P. Cubberley, 1919, p. 303.

The brief outline to which Colburn referred has not been determined with certainty, but available evidence (Keller, 1923, pp. 166–169) suggests that it came principally from a book by Joseph Neef (1808), a Swiss associate of Pestalozzi's brought to the United States by the philanthropist William Maclure to introduce Pestalozzian teaching methods into American schools. Much of the wording in Colburn's preface echoed comments in Neef's book, the plates that Colburn gave at the end of his book were clearly modeled after Neef's description, and the corresponding exercises were those suggested by Neef. Neef's commentary on the practice of cyphering expressed sentiments with which Colburn would certainly agree: "It is generally believed, that cyphering and calculating are identical things. But this general belief is a palpable mistake. Calculating and cyphering differ from each other as widely as spoken and written numbers" (p. 15).

Edson (1856) offered the following appraisal of Colburn's debt to Pestalozzi:

[Colburn's] "First Lessons" was, unquestionably, the result of his own teaching. He made the book because he needed it, and because such a book was needed

in the community. He had read Pestalozzi, probably, while in college. That which suited his taste, that which he deemed practicable and important, he imbibed and made his own. He has been sometimes represented as owing his fame to Pestalozzi. That in reading the account and writings of the Swiss philosopher, he derived aid and confidence in his own investigations of the general principles of education, is true. But, his indebtedness to Pestalozzi is believed to have been misunderstood and overrated. (p. 12)

For some reason, although Colburn retained the paragraph acknowledging his debt to Pestalozzi in the preface to *First Lessons,* "on the Plan of Pestalozzi" was dropped from the retitled book in 1826 (Table 1). Instead, "Colburn's," "Intellectual," and "Inductive Method of Instruction" were added. Whoever made the change apparently wanted to give Colburn credit for the method being employed and to indicate that the method advocated mental arithmetic (*intellectual*) and proceeded from the specific to the general (*inductive*). When, in 1830, Colburn addressed the American Institute of Instruction on the teaching of arithmetic, he made no mention of Pestalozzi, instead referring to "the old and new systems of teaching arithmetic" (Monroe & Colburn, 1912, p. 464).

"*Oral instruction* and the *inductive method* are the features of Colburn's books which have received general recognition, and which were most effective in changing school practices" (Monroe, 1913b, p. 301). They are also the principal features that Colburn's approach to arithmetic has in common with that of Pestalozzi. Both Pestalozzi and Colburn believed that children need not wait until they knew how to write in order to learn arithmetic. Instead, they could begin at age 5 or 6, learning it orally by starting with small numbers of objects from their daily life. They should not be expected to memorize meaningless rules, nor should they be exposed to number symbols unless and until they had mastered mental arithmetic. They should proceed to develop their own generalizations about numbers by induction from concrete instances. Pestalozzi appears to have stressed the role of counting in developing number concepts somewhat more than Colburn did, whereas Colburn did more to emphasize the decimal system and operations with numbers. Pestalozzi used displays such as that shown in Figure 2 to help children develop intuitive ideas of numbers and their relations. In contrast, especially since he was producing textbooks, Colburn gave more attention to reorganizing the content and elaborating it pedagogically. In the *Sequel,* for example, Colburn (1833) followed the addition of whole numbers with multiplication rather than subtraction, and he introduced the multiplication of fractions before addition or subtraction. He also eliminated some arithmetic topics, such as the rule of three, which he considered more confusing than helpful, and powers and roots, which he saw as belonging to algebra. Colburn's influence on the teaching and learning of mathematics, though owing a debt to Pestalozzi, went well beyond oral instruction and the inductive method.

Colburn's contribution

Regardless of how influential Pestalozzi's ideas were on Colburn's work, *First Lessons* blazed a trail in U.S. education:

> The publication of this book marked our first adoption of Pestalozzian ideas in teaching, and was the only phase of Pestalozzianism to be widely adopted before 1860. . . . The book must be ranked with Webster's *Speller* as one of the greatest American textbooks. Mental arithmetic, by 1850, had become one of the most important subjects of the school, and everywhere Colburn's book was in use. The sale of the book was enormous, and its influence great. Like all successful textbooks, it set a new standard and had many imitators. (Cubberley, 1919, p. 304)

During the decades after Colburn's death, the mental arithmetic he proposed became incorporated into the broader idea of mental discipline. As Smith (1916) noted, movements such as that launched by Colburn tend to get carried

> to an extreme and to turn some of the good into evil. Pestalozzi had suggested that a child should think in his number work; Colburn had sold an enormous number of books with the word "intellectual" in the title; and so the extremists proceeded to act upon the principle that if it did the child good to think a little, it would do him much more good to think much more. Accordingly the idea of mental discipline came to the fore. . . . For the next half century [after 1827], the idea of mental discipline dominated the teaching of arithmetic, and arithmetic dominated the curriculum. (p. 113)

Toward the end of the nineteenth century, the influence of mental discipline theory on the thinking of U.S. educators began to decline. Already at the turn of the century, psychologists such as William James and E. L. Thorndike were doing experiments to challenge the idea that training in mental operations would transfer to thinking and learning in general. Some years later, Thorndike (1924) conducted a study to show that school subjects such as mathematics had little value in training the mind. The controversy about mental discipline – especially among mathematics educators – continued on, however, for more than a few decades into the twentieth century (Stanic, 1986), and the theory remained alive long after mathematics education in the United States had adopted the behavioristic emphasis on the deductive "rule-example-practice" approach.

   Although mathematics as a finished product is commonly seen as providing the epitome of deductive reasoning, mathematics in the making relies heavily on plausibility and induction from specific examples. Warren Colburn was the first American mathematician and mathematics educator to recognize that the learning of school mathematics, too, should make use of inductive reasoning. Current efforts by U.S. mathematics educators to include more applied problems in the curriculum, encourage students to explore those problems,

compare their different solution strategies, and discuss their findings have their roots in Colburn's method of teaching arithmetic that he developed almost two centuries ago.

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Abraham Gotthelf Kästner and his "Mathematische Anfangsgründe"

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Abstract

In the 18th century, the term "mathematics" was understood in a much wider sense than we do nowadays. The aim of this paper is to investigate which disciplines were under the umbrella of mathematics in the 18th century. For this purpose, the mathematical "Anfangsgründe" are very fruitful sources. The study of these textbooks might be a contribution to learn more about studying and teaching mathematics at German universities in the 18th century. The work in hand deals with the mathematical disciplines which you can find in the ten-volume textbook "Mathematische Anfangsgründe" by Abraham Gotthelf Kästner (1719–1800). His textbook was very popular and often used during his lifetime.

Introduction

We know little about learning and teaching mathematics at German universities in the 18th century. There are different reasons for that, for instance the heterogeneous educational system with different regulations. Another reason might be that the focus of the researchers on the history of mathematics was based on mathematical research. In the 18th century, research was part of the scientific academies, while the universities were responsible for teaching. Usually the university professors did not produce new knowledge and research results.

With the help of academic journals we are able to reconstruct the situation at academic societies in the 18th century. In case of the universities the "Anfangsgründe" are very fruitful sources. The "Anfangsgründe" are scientific, introductory textbooks, which were very popular and often used in 18th century Germany. Above all they were created to assist teaching mathematics at German universities, and also for the use by students at any level. By means of

these textbooks we can find out which contents should be taught at German universities during the 18th century.

There are a handful of popular "Anfangsgründe"-authors; one of them is Abraham Gotthelf Kästner (1719–1800), professor of mathematics and physics at the University of Göttingen. In 1758 he started his work *Mathematische Anfangsgründe* which ended up to be a ten volume creation. His textbooks were leading in the second half of the 18th century and superseded the *Anfangs=Gründe aller mathematischen Wissenschaften* by Christian Wolff (1679–1754). Wolff is known as the founder of the so-called "Anfangsgründe"-tradition. His four-volume *Anfangs=Gründe* were firstly published in 1710 and were without any competition until the second half of the 18th century. Within the "Anfangsgründe"-tradition it seems that Kästner is overshadowed by Wolff. Therefore it is interesting to have a look at Kästner and his *Anfangsgründe*.

In the first step, I will present the educational system of 18th century Germany in order to embed the "Anfangsgründe". In the second step, I will point out some characteristics of this specific type of literature. Thirdly, I will give a short biography of Kästner and will take a look at his *Mathematische Anfangsgründe* with the focus on the classification of mathematics. At the end, I will give a brief conclusion and close with some further research questions.

The educational system in 18th century Germany

It is not possible to give a uniform description of the educational system in 18th century Germany.[1] It was heterogeneous because of the confessional and territorial splitting of Germany (cf. Schindling, 1994, p. 3). Nevertheless, there are some works relating to single universities (Kühn, 1987; Müller, 1904).

There were four faculties at German universities: The three "higher" faculties law, medicine, and theology, and the "lower" arts faculty. Mathematics was not an independent academic discipline but part of the arts faculty. In the 19th century mathematics became an independent academic discipline. The lower position of the subject within the arts faculty is also shown in the appointment of the teachers. They often had the position only until they received a better paid option in one of the higher faculties (cf. Turner, 1975, p. 499).

In order to qualify for one of the three higher faculties, every student had to pass a propaedeutic study within the arts faculty. To do so, they also had to attend lectures on mathematics. One reason for the propaedeutic study might be to guarantee a unified knowledge level. There was no compulsory education in Germany at this time. You could enroll at the university without having

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[1] For further information see (Schubring, 2005, chapter II. 2.6). This work contains a lot about European universities and its teachers in the 18th century.

attended a public school before. Hence mathematics was taught on a lower level than today.

Another interesting difference from today is the language. The German language did not come into the universities until the 18th century. Previously Latin was used. The use of the German language also fits into the visions of the Enlightenment, which accelerated the separation of medieval and outdated learning contents and the dogmatic teaching method, the broad dissemination of knowledge and the establishment of German as a scientific language. With the use of the German language one could reach people outside of the university milieu (cf. Kühn, 1987, p. 17).

Mathematical "Anfangsgründe"

The term "Anfangsgründe" is not related to a specific subject. In 18th century Germany you can find "Anfangsgründe" for a lot of disciplines. We are interested in those of mathematics, especially in those which were popular, often used, and comprehensive. The following statements are the results of the study of the mathematical "Anfangsgründe", but they are also applicable for "Anfangsgründe" of other subjects (cf. Kröger, forthcoming, chapter 1).

The "Anfangsgründe" are scientific, introductory textbooks. You can translate this term with "elements" or "basics". Above all, these textbooks were created to assist teaching at German universities, and also for the use by students at any level.

The authors described their textbooks as concise, clear, and exhaustive, so that you could learn the basics of mathematics in a very short time.

The "Anfangsgründe" were the first German-language textbooks which were used at German universities. In the 18th century, the philosophers of the Enlightenment had the aim to disseminate knowledge which comes along with the establishment of German as scientific language instead of Latin. So you could reach a broader audience, and also those people who did not have any Latin knowledge or were not allowed to study, for instance women. In particular the "Anfangsgründe" could also be used for autodidactic studies. The introduction of the German language at universities provided the basis for German-language textbooks. The rapid increase of these textbooks and the fact that some of them were very popular and often used, shows that there was a need and demand for them (cf. Kühn, 1987, p. 66). There already existed some German mathematical textbooks, but they were mainly created for future merchants and thus only for a small addressed audience. They focused on a small part of mathematics, mostly arithmetic. The lack of comprehensive mathematical textbooks could be corrected by the "Anfangsgründe". By means of these textbooks, a uniform mathematical teaching was possible.

Another fact is the promotion of mathematics during the 18th century in order to establish it as an independent discipline (cf. Sommerhoff-Benner,

2002, p. 304). An indication for the independence might be textbooks like the comprehensive "Anfangsgründe".

In some "Anfangsgründe" you can find not only pure but also applied mathematics. The authors treated a lot of themes which are now considered as physics (such as mechanics or statics) or even no mathematics at all (like fortification or ballistics). They did not confine the content to one topic but tried to present all mathematical disciplines in such a way that also people without any previous mathematical knowledge could understand the contents. Therefore it was important that beginners could learn mathematics from its elements. For that reason the authors started with rudimentary explanations (for instance the definition of a number and the four basic arithmetic operations). Then we find theorems and their proofs, problems and their solutions, images, examples closely related to life, historical remarks, and references to further literature. The "Anfangsgründe" should be the basis for further, independent and self-motivated studies on mathematics.

During the 18th century, the "Anfangsgründe" were very popular, but after this period they were out of use. A possible explanation for that might be the changes within the educational system and within the sciences. Very important in this context is the Prussian educational reform in Germany in 1810. The reformers established different kinds of schools, worked on curricula, and required new adapted textbooks. In some "Anfangsgründe" you can find disciplines which belonged to mathematics in the 18th century, but became independent in the 19th century – for instance physics. In addition to that there was a vast increase of new knowledge, mainly in analysis (differential and integral calculus) which should be included in the textbooks. These developments can be mentioned as reasons why the "Anfangsgründe" were out-dated in the 19th century.

There are a handful of authors who wrote popular "Anfangsgründe". Christian Wolff (1679–1754) is known as the founder of the so-called "Anfangsgründe"-tradition. His four-volume *Anfangs=Gründe aller mathematischen Wissenschaften* were firstly published in 1710 and were reprinted until 1800, a long time after his death. They were without any competition for almost 50 years. Then new mathematical textbooks appeared which were written by the next generation of mathematicians, namely Abraham Gotthelf Kästner (1719–1800), Wenceslaus Johann Gustav Karsten (1732–1787), Johann Andreas von Segner (1704–1777), and Heinrich Wilhelm Clemm (1725–1775). All of them were professors of mathematics. The textbooks were planned to help as lecture notes and were adapted for the lectures. The "Anfangsgründe" do not represent mathematical research, but knowledge which should be taught at universities.

I decided to take a closer look at Abraham Gotthelf Kästner, because his *Mathematische Anfangsgründe* were leading in the second half of the 18th century (cf. Kühn, 1987, p. 72). They were so popular that Johann Andreas Christian Michelsen (1749–1797), teacher of mathematics at a grammar school in Berlin,

labeled Kästner as teacher of mathematics of whole Germany (cf. Müller, 1904, p. 58). However, within the textbook-tradition it seems that Kästner is over-shadowed by Wolff.

Abraham Gotthelf Kästner – Short biography

Abraham Gotthelf Kästner is known as philosopher and mathematician. He was born on the 27th of September 1719 in Leipzig, and died on the 20th of June 1800 in Göttingen. He never attended a public school, but was privately educated by his father Abraham Kästner and his uncle Gottfried Rudolph Pommer. The latter awakened Kästner's interest in mathematics and provided him some mathematical books from his private library. Because of his early and widespread knowledge Kästner was known as a child prodigy.

Figure 1. Abraham Gotthelf Kästner

Artist: Johann Heinrich Tischbein the Elder (1722–1789)

At the age of 10, Kästner attended lectures on law which his father gave at the University of Leipzig. In 1731, on Kästner's 12th birthday, he enrolled at the University of Leipzig to become a lawyer, like his father wanted. However, he was more interested in other topics, especially in mathematics, physics, and philosophy.

In 1736, Kästner published the mathematical treatise *De theoria radicum in aequationibus*. With this he qualified to work as a private lecturer at the University of Leipzig in 1739. In 1746, he became an extraordinary professor of

mathematics in Leipzig. Ten years later, he went to the University of Göttingen as a full professor of mathematics and physics, where he remained until his death.

Figure 2. Titlepage of Kästner's *Anfangsgründe der Arithmetik…*

Kästner also corresponded with well-known scholars, for instance Leonhard Euler, Pierre-Louis de Maupertuis, Georg Christoph Lichtenberg, and Johann Heinrich Lambert. Kästner sent his treatise to Euler, who was very impressed with it. So Kästner became known among the scientists very early. In 1747, Kästner started translating various books from Dutch, English, and French into German, and also the journal of the Royal Swedish Academy of Sciences. Beyond that he wrote a lot of reviews and articles for popular scientific journals, for instance for the *Allgemeine Deutsche Bibliothek* and the *Göttingische Anzeigen von gelehrten Sachen*. Because of this work, Kästner came in contact with

a lot of literature of numerous scientific branches (cf. Baasner, 1991; Kästner, 1768a).

Kästner was a member of numerous scientific academies and societies, for instance the societies in Berlin, London, and St. Petersburg. In figure 2 you can see the title page of the first volume of Kästner's *Anfangsgründe*. It was common to list the author's membership in academies as a guarantee for the textbook's excellence (cf. Baasner, 1991, p. 10). The memberships can be regarded as an indicator for Kästner's popularity, his interests in scientific research, and his reputation as a mathematician.

Kästner's *Mathematische Anfangsgründe*

In 1758, Kästner began publishing his series *Mathematische Anfangsgründe*. In the beginning they were 6 volumes. Not the complete work, but some volumes were reprinted. Kästner also extended the content and added new volumes in the course of years, so that there were 10 volumes of this series until 1801. Kästner's textbooks seemed to be a role model for other textbook-authors, because some mathematical textbooks from the 1770s and 1780s based on his *Anfangsgründe* (cf. Müller, 1904, p. 58).

On the title page of the first volume of Kästner's *Anfangsgründe* (figure 2) you can find the information that this one is "der mathematischen Anfangsgründe ersten Theils erste Abtheilung" (the first subdivision of the first part of the complete *Anfangsgründe*). Kästner separated his series into four parts ("Theile") with various subdivisions ("Abtheilungen") (table 1).

Kästner was not only interested in the reprint of his textbooks but also in the up-to-dateness of the contents. This is conspicuous on the fact that the third part of the *Anfangsgründe* was split in two subdivisions since the third edition 1780/1781, because the contents increased so that one volume would not be sufficient (cf. Kästner, 1792a, p. vi).

By means of the separation of Kästner's *Anfangsgründe* into parts and subdivisions you can see that Kästner differentiated between elementary and higher mathematics as well as pure and applied mathematics.

Part 1 is dedicated to elementary pure mathematics, namely arithmetic and geometry including trigonometry, and its applications. In addition you can find the perspective as a mathematical discipline in the first subdivision. Kästner said that this one belongs to applied mathematics, but he took this theme within the first part of the series, because it could be learned without much previous knowledge, as a foretaste of the applied mathematics (cf. Kästner, 1800, p. *8^r).

The knowledge of elementary pure mathematics, which you can find in part 1, is an essential condition for learning applied mathematics, which is presented in part 2 of the *Anfangsgründe*. The first subdivision consists of mechanical and

optical sciences, the second subdivision of astronomical and architectural sciences. The latter ones are not mentioned on the title page.

In part 3 of this series you can find the higher pure mathematics in form of algebra and analysis, which Kästner explains in different subdivisions. This knowledge is important for higher applied mathematics in form of higher mechanics and hydrodynamics, which is presented in part 4 of the *Anfangsgründe*.

Table 1. Overview of Kästner's *Mathematische Anfangsgründe*

Part / subdivision	Title	Editions
1 / 1	*Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie, und Perspectiv* [Elements of arithmetic, geometry, plane and spherical trigonometry, and perspective]	1758, ²1763, ³1774, ⁴1786, ⁵1792, ⁶1800
1 / 2	*Fortsetzung der Rechenkunst in Anwendungen auf mancherley Geschäffte* [Continuation of arithmetic in applications on several commercial operations]	1786, ²1801 (published by Bernhard Thibaut)
1 / 3	*Geometrische Abhandlungen. Erste Sammlung. Anwendungen der Geometrie und ebenen Trigonometrie* [Geometric treatises. First collection. Applications of geometry and plane trigonometry]	1790
1 / 4	*Geometrische Abhandlungen. Zweyte Sammlung. Anwendungen der Geometrie und ebenen Trigonometrie* [Geometric treatises. Second collection. Applications of geometry and plane trigonometry]	1791
2 / 1	*Anfangsgründe der angewandten Mathematik. Mechanische und optische Wissenschaften* [Elements of applied mathematics. Mechanical and optical sciences]	1759, ²1765, ³1780, ⁴1792
2 / 2	*Anfangsgründe der angewandten Mathematik. Astronomie, Geographie, Chronologie und Gnomonik* [Elements of applied mathematics. Astronomy, geography, chronology, and gnomonic]	1759, ²1765, ³1781, ⁴1792
3 / 1	*Anfangsgründe der Analysis endlicher Größen* [Elements of analysis of finite quantities]	1760, ²1767, ³1794
3 / 2	*Anfangsgründe der Analysis des Unendlichen* [Elements of analysis of infinite quantities]	1761, ²1770, ³1799
4 / 1	*Anfangsgründe der höhern Mechanik* [Elements of higher mechanics]	1766, ²1793
4 / 2	*Anfangsgründe der Hydrodynamik* [Elements of hydrodynamics]	1769, ²1797

Kästner's classification of mathematics

By reference to the structure of Kästner's *Anfangsgründe* you can see the broad field of disciplines which belonged to mathematics in the 18th century. My leading question is: How did Kästner classify the mathematical disciplines?

The classification or hierarchy of mathematics is already visible in the structure of Kästner's textbook, namely by means of the separation into different parts and subdivisions. Beyond that you can also find very detailed

remarks on the classification in the *Anfangsgründe* (Kästner, 1800, pp. 1–23), which I will present in the following passages.

First of all, Kästner divided the mathematical sciences into pure and applied mathematics ("mathesis pura vel abstracta" and "mathesis applicata"). Pure mathematics deals only with magnitudes. Further features are regarded within applied mathematics. For instance, the length of an object belongs to pure mathematics, but the distance of two objects to applied mathematics.

Pure mathematics is separated into elementary and higher mathematics. Arithmetic and geometry belong to the elementary pure mathematics. Higher mathematics consists of algebra and analysis.

The application of pure mathematics on objects in nature is called applied mathematics. Kästner wrote that you can find the direct application of arithmetic within the context of budgeting, trade, and commercial calculations, the application of geometry within the context of field measuring. There are a lot of other sciences with are only complete because of pure mathematics. Kästner divided the applied mathematics into three main parts: Mechanical, optical, and astronomical sciences. Besides, Kästner presented the perspective and the architectural sciences fortification, civil architecture, and artillery as applied mathematics. Concerning the three latter ones Kästner wrote in his *Commentarius* that it would be possible to create a new, fourth part of applied mathematics, namely the architectural sciences. Otherwise these themes would belong to the mechanical sciences (cf. Kästner, 1768b, p. 42).

Although Kästner gave a detailed classification of mathematical sciences it might irritate that he did not determine a concrete number of applied mathematical sciences. In his textbook Kästner described them as "mehr als zwölf Wissenschaften, deren jede ihre Grundsätze hat"[2] (Kästner, 1792b, p. *2r). In his *Commentarius* he wrote that the applied mathematics consists of "dreyzehn oder vierzehn Wissenschaften, die sich allenfalls in drey oder vier Hauptabtheilungen bringen liessen"[3] (Kästner, 1768b, p. 42). This is an indication that the system of the applied mathematics was not totally fixed. Instead it seems that it was open for new sciences. The aerometry serves as an example. Kästner emphasized in his paper *Ueber die Verbindung von Mathematik und Naturlehre* that it is thanks to Wolff that the aerometry became an mathematical discipline because Wolff collected all the material and presented it in an mathematical way (cf. Kästner, 1772, p. 88f.).

In summary, Kästner gave a detailed classification of the mathematical disciplines which he all presented in his *Anfangsgründe*. The disciplines which belong to applied mathematics can be divided into four main parts which are presented in table 2.

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[2] More than twelve sciences with their own principles. Translated by Desirée Kröger.
[3] 13 or 14 sciences which can be classified in three or four main parts. Translated by Desirée Kröger.

Table 2. Overview of Kästner's classification of applied mathematics

Mechanical sciences	Optical sciences	Astronomical sciences	Architectural sciences
Statics (deals with the equilibrium of solids)	Optics (behavior and properties of light)	Astronomy	Fortification (military constructions and buildings)
Mechanics (movement of solids)	Catoptrics (reflection of light by mirrors)	Chronology (arrangement of events in order of occurrence)	Civil architecture
Hydrostatics (properties of liquids in balance)	Dioptric (refraction of light by lenses)	Geography (the land, the features, and the phenomena of the Earth)	Artillery (protection during war)
Hydraulics (properties of liquids in motion)		Gnomonic (construction of sundials)	
Aerometry (properties of the air)			

The comparison with other textbooks shows that Kästner's *Anfangsgründe* contain all the disciplines which were under the umbrella of mathematics in the 18th century. Wolff treated the same mathematical disciplines as Kästner, but he did not explain the hierarchy of mathematics. Kästner's classification was also a role model for other textbook-authors, for example Karsten, who wrote the eight-volume *Lehrbegrif der gesamten Mathematik* (1767–1777). Karsten referred explicitly to Kästner's classification of applied mathematics and borrowed it (cf. Karsten, 1767, p. *8r). In contrast to Kästner, Karsten tried to add new themes to the applied mathematics, namely pneumatics. Karsten justified the independent treatment of pneumatics from aerometry with the fact that it would be also common to distinguish between hydrostatics and hydraulics (cf. Karsten, 1771, p. b3v).

Conclusion

In summary, Kästner's *Anfangsgründe* can be seen as an introductory textbook for beginners of mathematics which should prepare and lay the foundation for further studies on mathematics. That is the reason why you can find simple explanations at the beginning of each chapter. Because Kästner's *Anfangsgründe* were often used and very popular in the second half of the 18th century, they can be regarded as an indication of the mathematical knowledge in Germany during that time.

On the basis of Kästner's *Anfangsgründe* we could see which disciplines belonged to mathematics in the 18th century. There was already the distinction between pure and applied mathematics, but it is different from today. There were a lot of disciplines which now belong to physics, which became an independent discipline in the 19th century. One of Kästner's merits is the detailed classification of the mathematical disciplines, which was representative

for the 18th century. Kästner also separated between elementary and higher mathematics in the same way we do today.

An interesting fact is that Kästner did not determine a certain number of disciplines which belong to applied mathematics. An explanation might be that other disciplines could become part of it, like aerometry or pneumatics.

There was no combination of teaching and research at universities in the 18th century.[4] While you could find teaching at universities, research was the business of academic societies (cf. Grau, 1988, p. 16). Nevertheless there are some hints that research was not disregarded at universities and in mathematical textbooks. Kästner used mathematical monographs for some parts of his *Anfangsgründe*, gave a lot of remarks on mathematical research and referred to further scientific literature (cf. Kröger, forthcoming, chapter 3.3.9). So it is interesting to analyze if and how mathematical research was treated at universities and in textbooks during this period. One thesis of my research is that the boundaries already blurred during the 18th century. This aspect you can also see at the establishment of the scientific society of Göttingen in 1751 which was associated to the local university.

Kästner was named as the teacher of mathematics for the Germans by Michelsen (cf. Müller, 1904, p. 58). Müller wrote that a lot of mathematical textbooks from the 1770s and 1780s based on Kästner's *Anfangsgründe* (cf. Müller, 1904, p. 58). So it is important to study the textbooks in order to find out why Kästner's *Anfangsgründe* were so popular and what was actually new in them.

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[4] For further information see (Schubring, 2002. This work contains an overview regarding the relation between teaching and research in various European countries in the 18th century.

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Change and stability: Dutch mathematics education, 1600–1900

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Abstract

From each of three periods a Dutch mathematics curriculum is described and analyzed, with emphasis on the intended and the implemented curriculum in each case. Comparison shows that there are factors common to each case and that some factors and some actors are essential for the design and implementation of a successful mathematics curriculum. Other factors contribute to successful implementation.

Introduction

The design of new mathematics curricula frequently results in a heated debate on the merits of specific subjects and on which topics to include in the content. The arguments which are used often originate in personal, subjective preferences; decisions rarely are the result of an objective consideration of important factors such as aims, intended student population, means and time available. Hardly ever the actors involved look back to consider which factors made some mathematics curricula succeed or fail in the past. However, as is remarked in (Krüger & van Maanen, 2014), such research has the advantages that one is able to study both the process and the outcome; an unbiased approach is more likely. Matos (2012) points out that insight in past curricula and their problems provides a better understanding of present day situations.

These observations led to the question which factors and actors were important in the design and implementation of mathematics curricula in the past (Krüger, 2012). We formulated the following research question.

Which are the factors and actors that influence to a high degree the content of mathematics curricula?

Three sub questions structure the research more specifically:

- Which factors and whose ideals are influential on the content of the formal curriculum?

- Which factors and which actors influence the translation of the formal curriculum into the implementation?
- Which factors and which actors are important for (successful) implementation of a curriculum?

The research concerns mathematics curricula, situated at the level between primary and university education, which in our age is called secondary education. Three curricula with an emphasis on mathematics, from the 17th, 18th and 19th century, were selected as cases for analysis and comparison.

Method and theory

For each of the three curricular cases, data from manuscripts, archives, contemporary textbooks and other publications have been collected. To structure the data, concepts from curriculum design theory are used, see for example (Goodlad, Klein & Tye, 1979; Van den Akker, 2003; Westbury, 1980).

Especially the distinction in domains or stages has been useful:

- the *intended* curriculum, the ideal of the person(s) who initiated the curriculum and the documents which are considered to constitute the formal curriculum;
- the *implemented* curriculum, which involves the interpretation and implementation of the formal curriculum, by teachers and through teaching materials;
- the *attained* curriculum, the experiences of students and the success (or lack of success) of the curriculum in relation to the students.

Figure 1 Spider web model of curriculum components
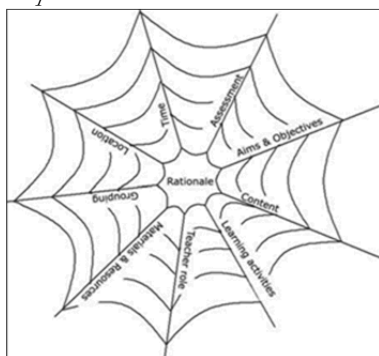
In the spider web model (Van den Akker, 2003) several curriculum components, such as aims, content, learning activities and location, influence each other. This model is taken as starting point for the analysis of historical cases. A description of each case, including analysis of the data, is followed by a comparison of the results to establish which conditions were influential in each of the cases.

The implementation at the start of each curriculum and during about 30 or 40 years after the start has been researched. To determine if a historical curriculum was a success three criteria were used.

1. Students: the schooling attracts students during a longer period of time and students have advantage of this schooling later on.
2. The position of the curriculum in relation to comparable curricula.
3. Appreciation: the appreciation in society, i.e. through favourable reports, mentioning in letters or attempts to imitation.

Comparison with recent developments in mathematics curricula should result in criteria which are important for present-day curriculum design for mathematics. In this paper the intended and implemented curriculum of each case is discussed.

1. The *Dutch Engineering School* (Duytsche Mathematique) was established in 1600 and affiliated to Leiden University. Its aim was to provide an efficient and effective course to train military engineers for the Dutch army. The Engineering School offered lectures in Dutch (Duytsche) instead of Latin language. It flourished at least until 1666, but was closed down in 1681.

2. The three *Foundations of Renswoude* (Fundaties van Renswoude) were founded in 1756. In Delft, The Hague and Utrecht a Foundation was affiliated to an orphanage, to provide daily care and schooling to a group of boys, selected from the orphanage. The aim was to provide talented orphans with an education and training for technical professions, so they could help to improve the country. The focus in the research is on the *Foundation of Renswoude in Utrecht* (Fundatie van Renswoude Utrecht), which still exists, albeit without school attached to it.

3. The *HBS* (Hogere Burgerschool), was established by the Dutch government in 1863, as part of the first Dutch legislation for secondary education. This was a new type of school in the Netherlands; a secondary school for children of citizens who would not enter university, but who would take up higher technical or administrative positions in society. The HBS-certificate did not provide entrance to university exams, but an increasing number of graduates went on to study at a university. The HBS was abolished in 1963 when a new structure for secondary education was introduced.

For the Dutch Engineering School, the main historical data were collected from manuscripts and archives in the university library of Leiden and of Groningen and from the Regional Archive in Leiden. For the Foundation of Renswoude, the Archive in Utrecht contains a wealth of information. For the HBS the National Archive in the Hague, some regional archives and the digital archives of the Parliament provided the majority of the historical data.

The ideal and the formal curriculum

All three curricula were initiated by a person with influence, based on position and/or wealth. Information about the ideals of these people is usually not explicit, but the reasons to initiate a new curriculum which emphasized mathematics, may be reconstructed to some extent in all three cases.

Three actors, three ideals

The *Dutch Engineering School* was initiated by *prince Maurice of Orange*, a commander of the army of the Dutch rebels. In the war with Spain, the two commanders Maurice and his cousin William Louis, had become very successful in recapturing and defending the fortified towns, around which the war efforts were concentrated. They used modern techniques for which army engineers, well trained in mathematics, with a good understanding of the new methods, were indispensable. Mathematics was seen as a necessary tool for army engineers, military and civil architecture, and also for high quality surveying (Van den Heuvel, 1991; Muller & Zandvliet, 1987). There were no institutes to provide the necessary type of schooling. See also (Krüger, 2010).

The *Foundations of Renswoude* were initiated by a wealthy widow, *Maria Duyst van Voorhout, Baroness of Renswoude,* who bequeathed her capital to three orphanages under condition that they selected talented boys from the orphanage to teach them

> "Mathematics, Drawing,[…], practices in building dykes to protect our Country against floods or similar Liberal Arts…." (HUA 771, inv. 1).

There had been several floods; the dykes along the coast and rivers were in a bad state. Since the beginning of the century economic conditions deteriorated, resulting in widespread poverty and a large number of orphans and abandoned children. Mathematics was not only seen as a useful tool, but also as offering a superior way of reasoning, which could be transferred to other domains of knowledge and to daily life (Alberts, 1994). There was a lack of mathematical trained professionals in all relevant areas, but hardly any institutes which offered schooling of the right quality, also those schools were fairly expensive (Krüger, 2012).

During the first half of the 19th century there were several attempts to introduce legislation for secondary education, but they all failed, until 1863. The prime minister, also minister of Interior Affairs, *Mr J.R. Thorbecke*, succeeded in guiding his proposal for legislation of secondary education, including the *HBS*, through Parliament, with very few changes. The HBS was inspired by the Prussian 'Realschule', but differed in some aspects. Thorbecke saw the HBS as a form of general education, preparation for higher positions in industry and commerce. The HBS also prepared students for a study at the recently established Polytechnic School (at present the University of Delft); mathematics was considered necessary as a support for physical science and as a preparation for the Polytechnic School. This was according to the idea that natural laws were expressed in mathematical form (Bos, 1997, pp. 174, 175; Explanatory Memorandum, 6-6-1862).

Translation into formal documents

The first known formal curriculum in the Netherlands is the *Instruction* (Instructie), written by Simon Stevin (9 January 1600) for the Dutch Engineering School (Molhuysen, 1913; NL-Ldn-RAL). It is rather detailed, with specification of content, the order of the subjects in the course, teaching methods, learning activities and the transition to the position of army engineer. For the Foundation of Renswoude in Utrecht, there are several documents which form part of the formal curriculum. The *General Regulations* (17 May 1756) and the first *Instruction for the Mathematician* (July 1761) are the most important for the mathematics curriculum (HUA 771, inv. 5, inv. 8). These documents contain mainly articles on organisation, accountability and supervision and little on the content of the curriculum, contrary to the formal document for the Dutch Engineering School.

For the HBS there are more documents: the law on secondary education, May 1863; the *Explanatory Memorandum,* June 1862 and the *Regulation* and *study programmes* for the final examinations, 1870 (Hubrecht, 1880; Hubrecht, 1882; SGD18611862_0000568). The text of the law was formulated by Mr Thorbecke, with help of his advisor, prof. P.L. Rijke, and the main civil servant for Education, dr. D.J. Steyn Parvé, who was appointed as Inspector of secondary education in 1864. The Regulations for the final examinations and the study programmes for the final examination were proposed and made obligatory by the successors of Thorbecke, Mr J. Heemskerk Azn and mr. C. Fock .

The content of the mathematics curricula and other curricular components

The description of the content in the formal curriculum is quite different in each of the three cases.

The specification of the curriculum for mathematics is rather detailed in the case of the Dutch Engineering School (Molhuysen, 1913). Probable reasons were the pressing need for trained engineers in combination with the lack of examples of similar courses; this Engineering School was indeed an innovation in Dutch education (Krüger, 2010; Muller & Zandvliet, 1987). Moreover, Stevin had very specific ideas about the basic knowledge and skills necessary for a military engineer and how they should be taught. As the aim of the curriculum was to train engineers as fast as possible, in the basic course he cut out topics which he considered superfluous for engineers (*Instruction*, 1600). Examples are algebra, which at the time was not considered necessary for surveyors and conic sections.

In the 18th century the regents of the Foundation of Renswoude in Utrecht gave no more than an indication of the content, they left the choice of subjects and topics to the mathematics teacher. However, they took great care to find an

excellent teacher, Laurens Praalder (Krüger, 2012; Krüger & van Maanen, 2014).

In 1862, in his *Explanatory Memorandum*, Thorbecke advised on the content of mathematics for the HBS, the only subject for which he thought advice was necessary. However, within a few years, study programmes for the final examinations of all subjects were prescribed, one of the most elaborate programmes concerned mathematics (Table 1).

Table 1. The mathematics topics mentioned in four formal curricula; V: the subject is mentioned, but not specified (1761)

	1600	1761	1862 HBS, guidelines	1870 HBS, study programme
Mathematics		V		
Arithmetic	Four operations whole numbers Four operations fractions Four operations decimal numbers Rule of three in all three numbers		Continuation of primary school	Different methods for solutions Correct number of decimals in approximations Logarithms
Algebra	Excluded		Quadratic equations Arithmetic series Geometric series Newton's binomial theorem	Arithmetic series Geometric series Newton's binomial theorem Arithmetic series of higher order Indeterminate equations Exponential equations
Geometry	Measuring a circle, part of circle, area Subdividing figures Checking calculations		Plane Solid	Plane Solid Volumes of polyhedrons Volumes of cylinder, cone and sphere Geometric characteristics of the spherical triangle Skill in mathematical reasoning Awareness of the relation between various topics
Surveying trigonometry	Calculating area with decimal numbers Fieldwork Mapping Working from maps	V	Trigonometry	Solution of simple trigonometric equations Application of plane trigonometry to simple problems in applied mathematics
Applications	Measurement on paper of dykes Calculation of volume			
Fortification Architecture	Definitions Mapping of towns Drawing perimeter of fortifications Working from drawings	V		
Descriptive geometry			Up to curved surfaces	Up to curved surfaces
New geometry			Not mentioned	Harmonic intersection, transversals and centres of similitude (optional)

Various components of the formal curricula

In each of the three cases it was clear for all concerned, which types of students were intended. The Dutch Engineering School was meant for all men who were able and willing to take courses in engineering in Dutch language in the town of Leiden. The Foundation of Renswoude was meant for talented and reasonably well behaved boys from the affiliated orphanage. The HBS was meant as a general education for the sons of well-to-do citizens, who were to take up a leading position in industry or commerce, but who would not study at university. These schools were established in several towns all over the country, so the HBS catered for more students than did the Dutch Engineering School and the Foundation of Renswoude. Also the students entering the HBS would have finished primary education, including Dutch and French language and arithmetic and they would on average be younger.

The duration of the course was specified in the HBS curriculum only (Table 2). In the 17th and 18th century the type and order of learning activities were specified: first theory, followed by practice, alternating with theory (the Engineering School) or theory combined with practice (the Foundation). Other components which are mentioned in formal documents of all three curricula are the teacher's role, the location in which learning would take place, financial means and some aspects of transition (to the next stage after finishing the course). In the 18th and 19th century, other subjects than mathematical sciences were taught as well; this meant that a smaller proportion of instruction time was available for mathematical subjects. The duration of the course, the position of practice, other subjects taught in the same period and the proportion of teaching time, available for mathematics are shown in Table 2.

Table 2. Some curricular components compared[1]

	1600	1761	1870
Duration	Not specified	Not specified	5 years
Practice or fieldwork	Combined with theory	Some, mostly after the first two years	None mentioned
Other subjects	none	reading, writing, drawing, some geography and French	17 subjects, i.e. science, languages, economics
Proportion of time for mathematics	100%	65%	20%

Mathematics was an important part of all three curricula, but its relative share diminished, as the requirements for a successful career in technology increased. These were both professional and social. In the 18th century, one had to be able to speak and write Dutch properly, but one also had to know some French, German and/or English in order to get and maintain a good position as a professional, such as in water management or as an instrument maker. Moreover, social skills, such as decent table manners, being able to read a

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[1] 1600: Krüger (2010), 1761: Gaemers (2004), 1870: Bartels (1963)

newspaper and discuss the news items were an advantage, so the orphans in the Foundation were taught these and other social skills as well (Krüger, 2012). In the 19th century the number of subjects had grown even more, resulting in an increase in the duration of the theoretical curriculum.

Another important development was in learning materials: the availability of printed texts. In the early 17th century they were expensive, rather scarce and written for self-instruction of adults; in the second half of the 19th century there was an abundance of textbooks, written specifically for use in secondary schools and relatively cheap. The character of the learning materials changed and as secondary education became more important books also became affordable for more students (Smid, 2008).

The implementation of the intended curriculum

Obviously, information about the implemented curriculum is only second hand, observations are not possible. An important source for the Dutch Engineering School is a manuscript, *Mathematical Works* (*Mathematische Wercken*), lecture notes by Frans van Schooten sr., professor at the Engineering School from 1615 – 1645. The manuscript dates from around 1622 and is one of the Leiden manuscripts described by (van Maanen, 1987). Information about the implementation in the Foundation of Renswoude comes from the Archive of the town of Utrecht (HUA 771) and some manuscripts which are students' notes. Information about the implementation in the HBS is available through reports of national inspectors (National Archive), yearly reports to the Parliament (digital archive of the States General), information about specific schools in regional archives, contemporary publications and some biographies.

In all three cases, great care was taken to find teachers of high quality, with good content knowledge.

The Dutch Engineering School

The *Instruction* by Simon Stevin was the most detailed on content and learning activities of the three formal curricula. The interpretation by professor Frans van Schooten[2], as represented in *Mathematische Wercken,* was rather faithful to the *Instruction*. However, there were some additions. Some extra constructions, such as a spiral and a geometric rose, were added, wine gauging was a topic and the manuscript shows a large variety of different techniques in geometric calculations, following on the treatment of commonly used techniques. In the extensive treatment of surveying, trigonometric tables were used (not

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[2] The sole successor of Ludolf van Ceulen and Simon van Merwen, who taught together from 1600 – 1610. Frans van Schooten was surveyor and assistant to Van Ceulen and started teaching in an unofficial capacity after Van Ceulens demise; he was officially appointed in 1615.

mentioned by Stevin). Throughout the manuscript decimal notation is used in calculations, which is unusual for that time, as is evident from manuscripts and books from the 17th century. Van Schooten also paid much attention to performing calculations in sufficient decimals, in order to minimalize the error in the result (van Schooten, 1627). Most of these adaptations were in line with the stated aim of the course, but it is clear that there was also opportunity to gain knowledge for other professions than military engineer. In fact a lot of students went into another profession, in which they needed practical mathematics (Krüger, 2010). The content of the course was determined by the teacher, based on the formal curriculum, the aims of the course, the prior knowledge of the students, and the requirements of the follow-up, the students' future profession. Assessment became a component on request of the students (Molhuysen, 1913). A certificate of the Dutch Engineering School was seen as advantageous for a future career as a surveyor. See also Krüger (2010).

Surveying and fortification were applied mathematics; fieldwork formed the practice of surveying and fortification. The interpretation of Van Schooten contained somewhat more theory than was absolutely necessary, thus also catering for the more talented and more eager students.

The Foundation of Renswoude in Utrecht

In 1761 the regents of the *Foundation of Renswoude* mentioned very few subjects in their *Instruction*. Unlike Stevin, they left the content to the professional, the mathematics teacher. It is quite possible that none of them had much detailed knowledge of the required mathematics, but even more important is that by 1750 there were examples of this type of education, especially in the towns. The inclusion by the regents of military and civil architecture in the formal curriculum of the Foundation reminds one of the military and civil architecture, taught at Leiden university at the time. Praalder, the mathematics teacher, interpreted this very global *Instruction* based on his own long experience as a mathematics teacher, surveyor and examiner of marine officers. The regents were happy with the results, as is apparent from the minutes of their meetings.

The curriculum consisted of three phases:
- phase 1: the basic theory, mainly mathematics, but also drawing, French language, etc;
- phase 2: apprenticeship started, theory in more specialised topics continued;
- phase 3: the student became a professional. In this phase the student continued practice, at a higher level and studied theory more independently.

The influence of transition on the content was thus more noticeable than in the Duytsche Mathematique, from phase 2 onwards the amount of mathematics and the topics depended on the specialisation chosen. Remarkable in the implementation of this curriculum was the high quality of the learning

environment and of the learning materials provided: well-furnished study rooms, a rich supply of books and instruments, supervision on homework.

As in the Dutch Engineering School in 1620, the implemented curriculum contained not only theoretical concepts and exercises, but also applications and some practical work. It prepared for a range of professional activities, in which surveying often was a requirement. If we compare the curricula of the Engineering School around 1622 with that of the Foundation around 1780, arithmetic and geometry were an important part in both. In 1780 algebra was taught to all students as a necessary mathematical topic, up to and including second degree equations and geometric series. Fortification (architecture) was by now one of the possible specialisations after the general course, chosen by only a limited number of students. The principles of surveying and trigonometry, with the addition of the use of logarithms, were taught to many, but not all students. The content of arithmetic was similar to the content in the Dutch Engineering School, geometry still was based on Euclid. See also Krüger (2012) for more details.

The HBS

In 1862 Thorbecke advised a fairly general list of topics for the mathematics curriculum of the *HBS*. The schools which were subsidized by the national government were obliged to teach all 18 subjects mentioned in the law, the schools subsidized by the councils were free to choose the subjects they offered. All schools were free to implement their own curriculum, regarding specific content and time allotted to a subject. The final exam was national and involved 16 of the subjects mentioned in the law; the examiners were organized in regional exam committees, each of which decided themselves on the content of the exams, both written and oral. The number of towns with a HBS grew fairly rapidly: in 1864, one year after the legislation, there were nine such schools; in 1880 the number was 36. The first final exams were in 1866, very soon there were complaints about the variation between the regions in the examinations. The national inspectors too, urged for more guidelines. So Regulations for the final exams were introduced, with a study programme for each subject. Written exams would be the same for all candidates, the oral exams remained regional. The study programme of 1870 for mathematics shows a far more detailed description, and also an increase in topics to be examined, compared to the list of topics by Thorbecke (Table 1).

However, the interpretation by teachers in schools varied somewhat, and from textbooks and resolutions of meetings it is clear that at least in some schools the mathematics curriculum contained more topics than were mentioned in the study programme for exams. The reasons given for this local expansion of the curriculum included the demands of the admission exams of some colleges, the advantage for development of mathematical reasoning and the value for development of concepts.

Thus the teacher still had a large influence on the content, but by now the final examinations also influenced the content directly. So did the text books, of which there was an abundant choice, as many teachers published textbooks. Aspects of transition, especially the demands of entrance exams and the programme of the Polytechnic School, also were a noticeable influence.

Table 3 presents an overview for mathematics of the topics mentioned in the study programme. The topics which were not mentioned, but were taught as required knowledge for the final examinations, are in italics, i.e. *quadratic equations*. The topics which would not be in the final examinations, but were taught by some teachers, are in bold, i.e. **complex numbers**.

Table 3 The implementation of mathematics for HBS in some schools

Italics: not mentioned in study programme, prerequisite for the topics mentioned
Bold: not mentioned in the study programme, taught by some teachers

Arithmetic	*Repeat of operations with whole numbers, fractions and decimals, metric system, lowest common denominator, greatest common divisor; Proportionality, with applications; Extraction of roots*
	Correct number of decimals in approximations, Logarithms
Algebra	*Operations with whole and broken terms, Roots, Exponents: whole, broken and negative, Linear and quadratic equations*
	Arithmetic series, Geometric series, Newton's binomial theorem, also for powers higher than two, Arithmetic series of higher order, Indeterminate equations, Exponential equations
	Higher order equations, Convergence of series, Continued fractions, Combinations and permutations, Complex numbers
Geometry	*Plane, including circles, area*
	Solid, Volumes of polyhedrons, Volumes of cylinder, cone and sphere, Geometric characteristics of the spherical triangle
Trigonometry	*Trigonometric ratio's*
	Solution of simple trigonometric equations, Application of plane trigonometry to simple problems in applied mathematics
Descriptive geometry	Up to curved surfaces
New geometry (optional)	Harmonic intersection, transversals and centres of similitude

The optional subject, new geometry, was taught by some, but not all teachers. So because of the autonomy of schools and of the teachers within the schools there still was variation in what was taught, but the subjects of the final examination was the same for all students.

The HBS offered general education, with a strong emphasis on preparation for the Polytechnic School and other technological studies. Many mathematics teachers in those early years were engineers, so they were able to refer from

their own experience to the professional use of mathematics. The growing number of teachers, who had studied mathematics at university level, did not have this professional experience outside the school environment (Bartels, 1963).

Stability and changes in the content

In all three implemented curricula arithmetic and geometry were taught to new students. In the 19th century *arithmetic* seemed to contain a new topic: the correct number of decimals in approximations and logarithms. However, already Frans van Schooten paid attention to the importance of performing calculations with the correct number of digits in relation to the units used, though this was not mentioned as a separate topic.

Geometry remained the classical Euclidean plane and solid geometry. In the curriculum of the HBS descriptive geometry also was important, as a preparation for technical studies. An optional part of the study programme was 'new geometry', which was meant to offer new, effective methods in synthetic geometry. The topics were harmonic intersections, transversals and centres of similitude; authors of textbooks for the HBS referred to Chasles and sometimes German authors (Versluys, 1868, 1897). Maybe this subject was intended to bring some new element into the mathematics curriculum of the HBS and also to serve as a preparation for analytic or for projective geometry, part of the curriculum in technical studies. 'New geometry' was taught in at least some schools, as is evident in the publication of textbooks from different authors, of which new editions appeared until the end of the 19th century. Analytic geometry did not become part of the HBS curriculum until after the second world war. This is unlike the situation in France, where analytic geometry formed a part of the preparation for entrance into the Polytechnic School (Barbin, 2012). It was from 1878 onwards, part of the mathematics curriculum of the gymnasium, which prepared students for university education (Smid, 1997).

Algebra did not belong to the intended curriculum of the Dutch Engineering School and did not occur in *Mathematische Wercken*. It is possible that Frans van Schooten sr. taught algebra to some students, as an extension of the core curriculum or as a private course. A manuscript by Van Schooten on algebra, with cossic notation, is in the university library of Groningen (UBG, Hs 443). His son and successor, Frans van Schooten jr., taught algebra in 1659 and logarithms in 1655 (Dopper, 2014). In the Foundation of Renswoude as well as in the HBS, algebra was an important part of the curriculum and the content of this subject in the school curriculum expanded considerably from the mid of the 18th to the end of the 19th century (Table 1, Table 3).

Trigonometry was taught in all three curricula, but the treatment of the topic and its role changed. In the 17th and 18th century trigonometry was a skill used in surveying. In the lecture notes by Frans van Schooten, f55^r contains a

drawing which shows a definition of sinus, tangens and secans and how to calculate them. Following that there are many different examples in contexts in which trigonometric tables are used. It suggests strongly that the students were shown what is meant by the three trigonometric proportions and learnt the use of them and the properties through contexts and practice. In the students' notes by Jan Mentz, a student of the Foundation of Renswoude, the theory of trigonometry was treated first, with proofs of several properties, followed by some practical exercises usually with a context from surveying (*Trigonometrie*). This is somewhat similar to the book which was used by many students (Morgenster & Knoop, *Werkdadige Meetkonst*). The geometrical rules which occurred in the notes by Jan Mentz were similar to those in *Mathematische Wercken*. In the 19th century there was more emphasis on trigonometric equations, the subject appeared theoretical, hardly any authentic applications were shown.

From the early 17th to the start of the 20th century theoretical treatment of mathematics gained in importance in these curricula; in the 19th century applications received little attention, apart from mechanics. All three curricula aimed to prepare students for technical professions in which the use of mathematics was important; in all three at least some of the theory preceded practice. However, the amount of theoretical knowledge and time spent on theory expanded from the 17th to the 19th century. The curriculum of mathematics became more general and the number of topics increased. This was in line with the development in mathematics as a discipline. The increasing emphasis on general education probably was an influential factor as well (Schubring, 2012).

Discussion and conclusion

In all three cases, the curriculum was a success, due also to the content of the mathematics programme. In each century there also were examples of less successful curricula, with similar aims as the curricula described in this paper. Further research should establish likely causes of this lack of success, i.e. the initial lack of qualified teachers.

One of the obvious differences between the three curricula is the position of mathematics relative to other subjects; from only mathematical subjects in the 17th century, to one out of sixteen subjects in the 19th century. This is to some extent a consequence of the slightly different aims of the schooling: in the 17th century military engineers and a few other mathematical practitioners, in the 18th century a broad range of highly qualified mathematical practitioners, but also chirurgeons and in the 19th century general education as a preparation for higher positions in industry and commerce. It was not yet 'mathematics for all' (Schubring, 2012), but a lot more students in more schools than previously received mathematics education in the 19th century. Also in the early 17th

century thorough knowledge of mathematics was a sufficient basis for a mathematical practitioner (Rogers, 2012), in the 18th century it was an advantage if an engineer could read at least German and French publications in his field (Krüger, 2012).

Another difference is the changing role of *examinations*. The centralized final examinations, which were introduced in the 19th century for the HBS, became a major and to some extent stifling influence on the mathematics curriculum. In combination with increasing demands of other subjects, there was less time to teach other topics than those for this examination. One notices towards the 20th century a gradual decrease of the influence of teachers on the content and a less flexible curriculum.

There are however many similarities between these cases with regards to importance of some curricular components.

In each case the ideals of an *initiator* were the source of the design of a new curriculum; also there were people in the right position willing to act on those ideals. The formal curricula differed in the specification of the content of mathematics. However, in all cases the authors paid attention to the *aims* of the initiator and to the matter of *transition* of the student to the next stage in his career, after finishing the curriculum.

In the implementation the *teachers* had a central role; they selected topics and decided on the order and the ways in which to teach those, selected learning materials and learning activities, made adjustments according to the needs of the students, took into account demands of transition to the next stage and were responsible for most of the assessments. It is significant that in the 17th and 18th century the first teachers were selected with great care; they had good knowledge of the relevant content and a good reputation as a teacher. In the 19th century the emphasis was on theoretical knowledge, possession of a university degree or equivalent in mathematics; quite a few of the first mathematics teachers in the HBS had an engineering degree. Thus in all three examples the teachers were highly educated, autonomous in the writing of their teaching texts and in the 18th and 19th century fairly independent in the choice of learning materials and textbooks. This is somewhat similar to the situation in the 19th century in Prussia and different from France at the time (Smid, 2008). Typically quite a few of the mathematics teachers of the HBS wrote and published textbooks, based on their own ideas on content and didactics of mathematics instruction. Thorbecke and most members of Parliament assumed that the teaching itself would not pose serious problems; pedagogical requirements were mentioned in the legislation, but they were not assessed. However, teachers who were highly qualified, but who lacked pedagogical qualities, sometimes posed problems, as is apparent in reports from school directors and from other sources.

Supervision was not an issue in the Dutch Engineering School, but if a professor did not teach well he would lose his students and the course might be discontinued. In the Foundation of Renswoude there seems to have existed a

system of accountability and supervision, which worked well (Gaemers, 2004). In the case of the HBS the school principal kept an eye on teaching practices, as did the national inspectors. These inspectors had a strong role in supervision, with the aim to improve the curriculum where possible and to advise the minister. They also played a role in alignment of the curricula between schools, as part of improvement of the teaching. It was a system mainly based on guidance and advice, far removed from the monitoring and assessment described by Karp (2012) for the Soviet Union between 1930 and 1950.

There are some other factors which contributed to a successful implementation in each case, such as teaching location and finance. A suitable *location* was considered important in all three cases. In 1600 the university governors provided teaching rooms and areas where fieldwork could be practised; in the 18th century a spacious house with good quality rooms for lessons was built, both for teaching to groups and teaching to individuals. It also provided space for students to study outside teaching hours, with learning materials available and with some supervision. In the 19th century Thorbecke refused to provide subsidy for a school unless a building of decent quality and with sufficient rooms was available; the law on education required that the classrooms were of sufficient size and would not cause health problems.

Finance, the amount and the allocation, were important in each case, but most obvious in the Foundation of Renswoude, as apart from educational means, there had to be provisions for housing and care of the boys and some support for students to establish themselves in their profession after they left the Foundation. (Bartels, 1963). It is highly probable that to a large extent the success of the HBS was due to the willingness of the national government to finance a number of model schools, to pay relatively high salaries to qualified teachers in those schools and to provide some subsidy for the establishment of similar schools by town councils. The financial situation of the university was one of the arguments to close the Dutch Engineering School in 1681. Lack of finance, due to external causes, nearly caused the end of the Foundation of Renswoude in 1810.

So in spite of the differences between these curricula, there were important similarities in the characteristics of some curriculum components. Examples such as a formal curriculum in line with the ideals and aims, teachers with the right content knowledge, the autonomy of teachers combined with the character of the supervision, the attention for transition to the next stage in the student's career, the importance of a good teaching location and the role of finance still are of value for present day mathematics curricula.

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Emma Castelnuovo's commitment to creating a new generation of mathematics teachers

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Abstract

In the year of Emma Castelnuovo's hundredth birthday, this short communication was planned to give an account of her work and, above all, of her involvement in the training of young mathematics teachers. The text is based on interviews given by Emma Castelnuovo, and on "oral history", that is on the accounts of the people that worked alongside her.

Introduction

Emma Castelnuovo celebrated her 100th birthday on December 12th 2013. Her international experience began in the last 40s, soon after the circulation of her first works in the field of intuitive geometry. During that time, she got acquainted with Paul Libois and the École Décroly in Brussels. In the 50s she became one of the founding members of the CIEAEM, where she played an important role.

Emma realized that the methods of the École Décroly could help opening the vision of Italian teachers and young students. To this end she promoted, through grants, study trips to Brussels.

It was also from the École and from Libois that she took the idea of using mathematical exhibitions as an important educational means, not only for the pupils who prepared them but also for the young teachers who collaborated.

In the 60s Emma's works became an important reference for teachers who were studying new forms of teaching according to the ideas of modern mathematics.

By means of the interviews given by Emma, her papers, and the accounts of the people who worked with her, we will try to give an idea of her work, underlining her role in the training of teachers (see Menghini, 2013).

The first period

Emma Castelnuovo was born on the 12th of December 1913. [1] She was the fifth child of the mathematician Guido Castelnuovo and of Elbina Enriques, the sister of another well-known mathematician: Federigo Enriques.

Among her professors at the University of Rome, where she graduated in 1936, we find – beside her father and her uncle – Gaetano Scorza and Tullio Levi-Civita.

That is, Emma grew up in a very significant familiar and mathematical milieu. The aforementioned mathematicians, well known at international level, were also interested in the teaching of mathematics, in its pedagogy.

After graduating Emma was involved, for two years, in setting up the library of the new Department of Mathematics of the "University City" in Rome. She did this work with her study-mate Lina Mancini Proia, who collaborated with Emma in many of her activities.

Owing to the racial laws of 1938, Emma was forced to leave the Department of Mathematics. She was not even allowed to teach in Italian public schools; she therefore began her teaching activity in the Jewish School in Rome, attended by Jewish students who had been expelled from public schools. Emma's former students remember her skill in finding simple ways of explaining advanced mathematical concepts (Limentani, 1993).

With the enthusiasm that followed the end of fascism and of World War II, Emma – together with the university professor Tullio Viola and Liliana Ragusa Gilli, a younger teacher of mathematics – decided to organize a series of conferences under the name of Istituto Romano di Cultura Matematica. It was a winning idea: in this difficult period of Italian history, about a hundred teachers of mathematics attended the talks, which were held by mathematicians, physicists, philosophers, educators (Castelnuovo, 2007). This was the first of Emma's activities in the training of teachers (and of herself).

Among those conferences we might recall that of Colonel Carleton W. Washburne, an educator and a student of John Dewey, who presided the *Allied subcommittee of Education* responsible for 'de-fascistizing' the Italian school programs.

But what we remember above all is a conference held by Emma in 1946, concerning a "new method of teaching geometry in middle school" (Castelnuovo, 1946). Emma had also started to teach in the Scuola Media Tasso in Rome. It was an Italian middle school (pupils aged 11–14), where she worked until her retirement, and this conference describes how, after reading the *Elements* of Clairaut, she suddenly changed her teaching style.

In her conference, Emma explains how she decided to substitute a descriptive method with a constructive and active method: We can start from

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[1] Emma Castelnuovo died on April 13th, 2014.

concrete experiments, like the measurement of the area or of the perimeter of a rectangle and create new concepts by means of applications and examples.

Inspired by Clairaut, Emma believes we must not present the definition of parallel or perpendicular lines, but we should simply use these concepts. It is through their use that they become familiar and clear to the pupil. The pupil should participate in a creative work, and taste the "flavor" of discovery.

Two years later, in 1948, Emma published the book *Geometria intuitiva* (Castelnuovo, 1948). What is typical of Emma's teaching, not only in the field of geometry, is the use of simple concrete materials, of limit situations, of counterexamples (for instance a folding meter helps to show how to transform a quadrilateral into a different one, and to analyze limit situations).

Emma's book, very far from the teaching programs proposed at the time (Menghini, 2010), was an opening to international success.

The international experience

Emma sent a copy of her book on intuitive geometry to Francois Goblot, the editor of the French pedagogical review *Cahiers Pédagogiques*. He advised her to participate in a teacher conference in Sévres, "Les classes nouvelles". It was 1949. In the session which concerned the teaching of mathematics an inspector suggested that she could speak about her experience. Her talk was *not* a success: she was accused of teaching "les mathématiques avec les mains sales" ("mathematics with dirty hands") by many French teachers, but she received compliments from a group of Belgian teachers who worked at the École Decroly, with Paul Libois. Emma knew that Libois was a young geometer who had studied in Rome with Enriques and with her father. Now she learned of his activity in the field of mathematics education: he was a professor at the Université Libre, in Bruxelles, and cooperated with the École Decroly. He had married Laure Fontaine, who in 1953 became the director of the École. The École Decroly became an important reference for Emma and for many Italian teachers (see next section).

In 1951, after a holiday in France, Emma decided to visit Jean Piaget in Geneva. She obtained an appointment after informing an assistant of Piaget that she had taught children aged 11–14 and that she wanted to speak about their concept of angle.

It is not surprising to find Emma among the founding members of the CIEAEM (Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques), which started its work as a study group in 1950, but was officially founded in 1952. It was Caleb Gattegno who invited Emma to join the CIEAEM after reading her book about intuitive geometry (Gattegno, 1953; Félix, 1986; Furinghetti, Menghini et al., 2008).

The CIEAEM was interested in the learning process of a child, looking at this process from many points of view. Its philosophy was linked to Gattegno's

opinion that mathematics teaching has to take into account the child's mind, mathematical theories, social, scientific and technological theories. The study of this complexity must be based on practice.

The vision of the CIEAEM always matched the vision held by Emma. The initial activities (Rencontres) were meetings of about 30 people, mostly teachers, and at least one class of pupils. They would start from a subject and the material was prepared in advance, but there was not a pre-fixed lesson. The teachers worked with the pupils, discussed, and let the pupils discuss. There are no proceedings of the first meetings (Castelnuovo, 1981).

Emma's first Rencontre was in 1954 and concerned modern mathematics. This topic was not yet a matter of division within the CIEAEM, on the contrary, all participants considered the concept of structure as a fundamental one. The meeting gave rise to the first publication of the CIEAEM (Piaget et al. 1955). Emma didn't contribute to the book, but she wrote the preface to the Italian edition, together with Luigi Campedelli, a professor at the university of Florence (Campedelli & Castelnuovo, 1960).

Figure 1. Variation of the angles of a triangle

Figure 2. Variation of the area of rectangles
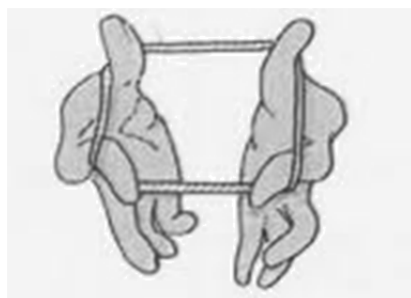
The Rencontre of 1957, organized by Pedro Puig Adam, concerned the use of concrete materials in mathematics teaching. On this occasion Emma, working with a middle school class annexed to the Italian Lycée in Madrid, showed the variation of the angles of a triangle using a rubber band and two nails (Figure 1), and the variation of the area of rectangles with the same perimeter by using her "logo" (Figure 2): a knotted string held with four fingers (Castelnuovo, 1957). There was also in this case a publication (Gattegno et al., 1958), and Emma contributed with the paper "L'Object et l'action dans l'enseignement de la géométrie intuitive", in which she discusses and expands the experience of the Rencontre.

In 1959 at the famous congress of Royaumont organized by the OECE. (Organisation Européenne de Coopération Economique, now OECD) (Furinghetti, Menghini et al. 2008; OEEC, 1961) Italy was represented by Emma and Campedelli (Menghini, 2007).

As it is well known, the conference was held under the influence of the French mathematical school. After Dieudonné's talk, and his famous cry against Euclid and against the triangle, Emma openly observed that the table at which the speakers were sitting was held by a structure made of triangles, which are rigid and non-deformable (see Figure 3). This episode remained in the memory of many people attending the conference, and even turned into a "legend" about Emma's declaring that without triangles the French would not have the Tour Eiffel (Équipe de Bordeaux, 2009).

Figure 3. The triangle is non-deformable
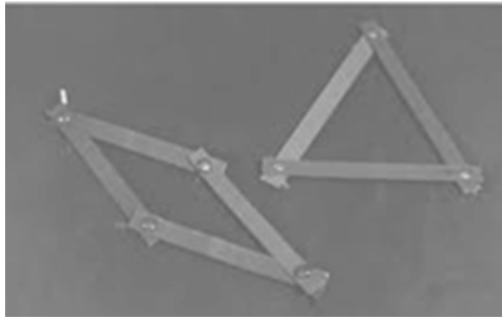
Gattegno quit the CIEAEM in 1960, because he didn't like the CIEAEM's adoption of a privileged program that followed the French point of view. Piaget, too, quit in 1960. In the meantime, many Italians entered, following Emma.

The CIEAEM of the 60s was still a gathering of few people. Among the most active members we remember Sofia Krygowska from Poland, Georges Papy and Willy Servais from Belgium, Tamas Varga from Hungary, and later Claude Gaulin from Canada. The small number of participants helped to create a good working atmosphere. Papy, who was president of the CIEAEM from 1963 to 1970, directed the meetings, and even if not everybody shared his didactical opinions, his unquestioned culture was a stimulus to study and reflect upon teaching questions (Mancini Proia, 2003).

Papy left the commission in 1970. The participants were increasing and it was difficult to maintain the structure of a working group. The Rencontres were becoming conferences, always attended by a large number of teachers, of all levels. In the 70s and 80s, under the presidency of Krygowska, Gaulin, Emma, and Michele Pellerey, the subjects of the CIEAEM referred to inter-disciplinarity and trans-disciplinarity. "Mathematics for all" became a standard theme (Figure 4).

Also in this evolution, the CIEAEM was always in harmony with Emma's way of working, and many of her younger collaborators joined it. Emma chaired the CIEAEM for two years, from 1979 to 1981. Under her presidency the two Rencontres of Oaxtepec in Mexico and of Pallanza in Italy were organized. Concluding her mandate in Pallanza, she said:

> Never present mathematics as something finished, something that you know and your pupils ignore. Stimulate their interests on issues they may feel, they can live; let theories arise from concrete situations, from reality […]. To do this we must study, read, think, rebuild (translated by the author).

And she concluded:

> By means of mathematics you are training men. And this is beautiful. (Castelnuovo, 1981, p. 355).

Figure 4. The CIEAEM 28, Louvain-la-Neuve, 1976: Stefan Turnau, Anna Sofia Krygowska, Emma Castelnuovo, Claude Gaulin, Willy Servais, Guy Brousseau.

Emma was a member at large also in the larger international organization ICMI (the International Commission on Mathematical Instruction), from 1975 to 1978 (Furinghetti & Giacardi, 2008). The ICMI has a long history that started in 1908 (Schubring, 2008). In 1967 Hans Freudenthal became its president. In that period the ICMI started to collaborate with UNESCO, the quadrennial ICME (*International Congress on Mathematics Education*) started, and also the journal *Educational Studies in Mathematics* (*ESM*) began its publication, edited by Freudenthal. Like many Italians, Emma was strongly involved in that period: she participated in the UNESCO Colloque in Bucarest in 1968, in the first ICME in 1969, and she published some papers in *ESM*.

Freudenthal greatly appreciated Emma's work. According to Pellerey, Emma inspired his Realistic Math Education, which was born in the Netherlands in this same period, and involved primary school and high school.

After being invited by the IREM (Institut de recherche sur l'enseignement des mathématiques) of Niamey and by UNESCO, Emma went to Niger four times, from 1977 to 1982, to teach in classes corresponding to our middle school (Figure 5). It was a very important experience. The letters written by the Nigerian students testify their happiness to "build" mathematics and to feel like the Italian students (Lanciano, 2013; Berté, 2013).

Figure 5. Emma Castelnuovo in Niger

The international fame of Emma grew particularly in Hispanic countries (Mexico, Argentina, Spain, Dominican Republic). Her book on intuitive geometry was translated in 1963 in order to be used in middle schools and for teacher training in Spanish-speaking countries; in 1964 Emma resided in Argentina to hold talks with teachers; in 1978 she represented the CIEAEM, with Dieudonné and Servais, at the *Conférence Interaméricaine d'Amérique du Sud* in Caracas. At a Congress in Cuba in 1997 she even met Fidel Castro, who had gone to the hotel of the foreign participants to check their expenses. In Spain in 1991 the still very active *Sociedad Madrileña de Profesores de Matemáticas (SMPM) "Emma Castelnuovo"*, (http://www.smpm.es) was founded.

Bruxelles and the École Decroly: a place for training teachers

In 1950 Emma started regularly visiting the École Decroly and the Université Libreof Brussels, where Libois taught geometry. In 1962 she made the journey with her collaborators, Liliana Ragusa Gilli, Lina Mancini Proia, and Ugo Pampallona. This was the "quartet" of mathematics teachers that led Roman (and partly Italian) mathematics education in the 1960s and 1970s. It was also joined by Michele Pellerey – a that time a mathematics teacher, and later rector of the Salesian University, who played an important role in CIEAEM – and from time to time by other university professors. The members of this group were, like Libois, former students of Guido Castelnuovo and Enriques, and had inherited from them how to give attention to intuition, to the applications of mathematics, and to the renewal of mathematics teaching (see Mancini Proia, 2003).

> At the École I was struck by the methodology: the teachers said only a few words, they proposed exercises to the pupils and these worked alone, trying to answer the questions. This was a more efficient method than the colloquial method that I used, and I tried to copy it (Mancini Proia, 2003, p.19; translated by the author).

In 1965 Emma established the grant *Premio Guido Castelnuovo*, thanks to a legacy of her father (and, according to Emma's friends, also thanks to the income from her books). The grant offered, each year, a study trip to Bruxelles for about a dozen of mathematics teachers. Until 1974 many young teachers could enjoy such a wonderful training opportunity.

Figure 6. Pupils measuring the shadows at the École Decroly

During each journey, four days were devoted to the École Decroly, taking part in the Journées pédagogiques, which the École organized in March (from 1956)

for all those who wanted to know its methods. During these days the visitors – who were teachers of all scholar levels – took it in turns to observe lessons at different scholar levels. Discussions with the teachers followed.

Figure 7. Emma at the École Decroly with the pupils measuring the shadows

The pedagogical methods of the École were (and are) characteristic: the teacher oriented the students, encouraging them to build upon their understanding, mathematics was connected to reality; there wasn't a program, but various centers of interest that connected all the subjects. For instance, a certain month the theme was painting, and in mathematics they would study projections, perspective, the composition of colors, … Beside the teacher's lessons, there were also discussions and work groups about particular topics.

One day was also devoted to a visit to the École Berkendael where George and Frédérique Lenger Papy held their lessons, George taught in the high school and Frédérique to students in compulsory education. George Papy's were frontal and formal lessons, generally dealing with set theory and linear algebra. Madame Papy's lessons were also frontal lessons centered on graphs and sets, but she adopted a system based on arrows, Venn diagrams, colors that George himself had developed in order to give evidence to the mathematical structures (Vanpaemel, De Bock, Dirk & Verschaffel, 2012). When describing some of the lessons of George Papy, Emma underlined that he made every effort to avoid any reference to intuition (Castelnuovo, 1965). Lina Mancini often recalled this episode. Papy had written on the blackboard: $a + b = b + a$. "Ah, cela veut dire que on peut mélanger" ("this means that we can intermix"), said a pupil, accompanying the sentence with a hand gesture corresponding to

the exchange of two objects. "Non!", Papy answered, "Cela veut dire $a + b = b + a$" ("this means $a + b = b + a$").

A further day was devoted to a visit to the Université Libre, where Paul Libois' students displayed their mathematical exhibits. By using wood, iron, and plastic, they constructed and presented models of quadrics, of projective geometries and others; these manually constructed models could even address sophisticated topics. There was always also a dinner at Libois' home. Alongside the Italians, there were some teachers of the École, like Simone Trompler and Francis Michel, and some young collaborators of Libois, like Xavier Hubaut and Francis Buekenhout.

Figure 8. Paul Libois with Lina Mancini Proia

Italian teachers owe a great deal to Libois (Figure 8), for instance such interesting suggestions as the use of the shadows produced by the sun or by a lamp to introduce geometrical transformations (Figures 7 and 8).
From Libois, Emma also borrowed the idea of mathematical exhibitions as an important educational approach. Pupils learn both while preparing the exhibit and while explaining it to others.

In 1971 and 1974 Emma organized two exhibitions in Rome (Figures 9 and 10). In 1974 she also went with pupils of the third class and her collaborators to the École Decroly (Figure 11) and to Lausanne, to present the exhibition (Barra, 1974; Castelnuovo, 1972; Castelnuovo & Barra, 1976).
Exhibitions were organized over the years also as a means to train teachers, and not only in Italy (for instance, in many congresses of the CIEAEM and the ICME). They provided training not only for teachers visiting the exhibition, but above all for the young teachers, who worked with the pupils to organize the exhibit. Emma was so convinced of the didactic value of exhibitions that she also organized one exhibition in Niger, at the end of her stay, by using only simple materials.

Figures 9. A pupil illustrates the routes of the airplanes
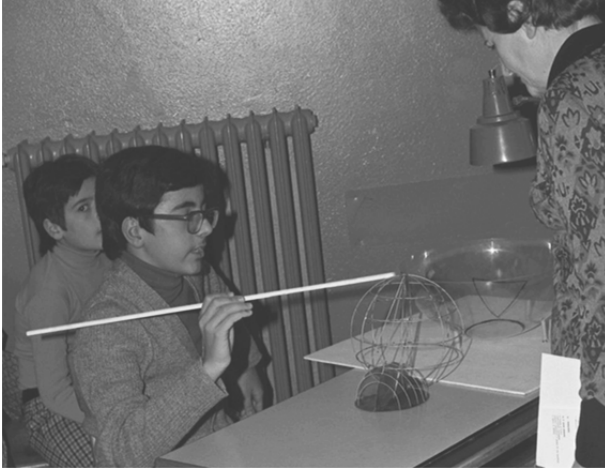
Figure 10. Hans Freudenthal visits the exhibition, accompanied by Hans-Georg Steiner
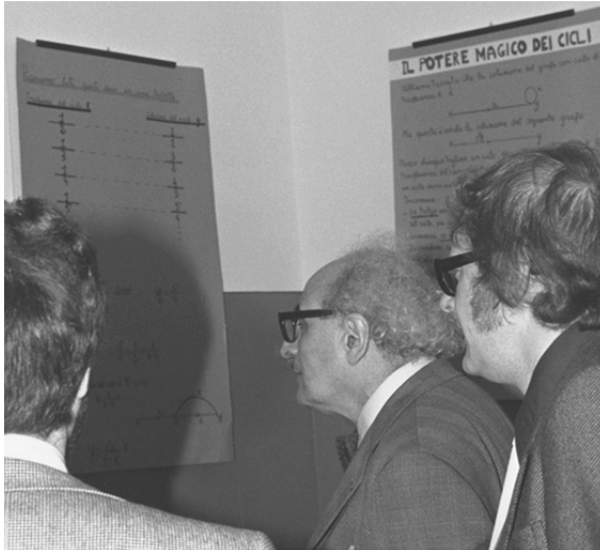
Teacher training

In Emma's opinion, transmitting new ideas and looking in greater depth at educational questions through reading and studying were important in the training of teachers, but of much greater importance was the comparison of different experiences. It was not *her* method that had to be transmitted, but, for a long time, the main method to be used and shared was the one of the École Decroly, and the Premio Castelnuovo was devoted to this.

At the beginning of the 1960s, Lucio Lombardo Radice, professor at the University of Rome, had the idea to send a (female) student to listen to Emma's lessons. This was the beginning of the tradition of preparing final theses in Mathematics Education in order to obtain the degree in Mathematics. The students attended the lessons of a teacher (not only Emma's) for a year, they identified the pupils' difficulties, and prepared their dissertation on traditional or new topics. This also meant the recognition of Mathematics Education as a university subject. The Premio Castelnuovo started in those years to support these kinds of activities, as an annual grant for the students.

Figure 11. 1974: Pupils leave for Brussels with the materials for the exhibition.

Another aid to Mathematics Education as a university subject came from the CNR (Consiglio Nazionale delle Ricerche, the National Research Council), which established post lauream grants in the field of Mathematics Education. This recognition of the role of a researcher in Math Education was also due to the influence and work of Emma.

Through the CNR, the leading group of Roman mathematics education organized two important training courses for young teachers. The first course was held by members of the School Mathematics Project (SMP), Geoffrey Howson, Peter Bowie, and the teacher M.me Joan Blandino, in Pallanza in 1973: it was a 15-day course with about 30 participants. The English professors gave their lessons in the morning, and in the afternoon the younger teachers, organized into groups, worked under the supervision of the older teachers to prepare teaching materials. As it is well known, at the base of the work of the SMP, there is the conviction that problems are the best starting point, a point of view shared also by Emma.

A second course was organized in Rome, where Tamas Varga presented the Hungarian project of the National Pedagogical Institute (OPI) for elementary school. This resulted in an analogous project (RICME, Ricerche per l'Innovazione del Curricolo Matematico nelle Elementari, Researches for Innovating the Mathematics Curriculum in Elementary School) in Italy led by Michele Pellerey (Pellerey, 1976).

Emma didn't like to train teachers by explaining her methodology in a talk. She preferred to describe how she worked in the classroom, or better, to show it directly, or – even better – to let the trainee teacher take part in her experiments Her papers and books had the sole aim of leading the teachers to reflect upon their own experience, to look objectively both and at their way of teaching and the way to help the pupil in overcoming difficulties.

But of course there are general pedagogic ideas in her teaching. Emma's book *La Didattica della Matematica* (1963), translated into many languages, gives an account of her main didactical ideas and of the stimuli received from various fields.

Emma always taught in middle school, to pupils aged 11–14. As in her teaching of intuitive geometry, Emma applied to all her teaching the idea that the learning process goes from the concrete to the abstract. Pupils have to know facts before the theories that explain them; the approach to mathematics has to be "experimental" and "active". The use of simple instruments and direct experience help to discover some fundamental properties.

Figure 12. Cylinder made by threads
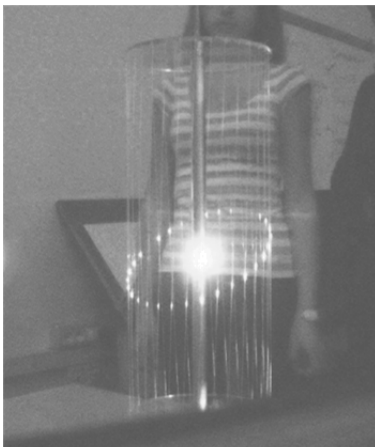
Emma is a follower of the active school of Maria Montessori and Ovide Decroly, but what she appreciates even more, is Piaget's conception of material, the use of the object, the action of the pupil in using it. Piaget's experiments, seen as educational experiences, allow the development of certain laws which are necessary for the acquisition of a concept, and guarantee more freedom in the construction of mathematics. At a later stage the pupil may be able to grasp the more abstract symbols of mathematics and draw from them some less evident propositions. Such a method requires a personal effort from the pupil, who has to embark in creative work.

Emma was very severe at the beginning of her math lesson: the pupils' desks had to be in order, only the necessary equipment for the math lesson was to be on them, and chewing gum was not allowed. She exacted total attention. But when pupils started to discuss a mathematical problem together, they had total freedom and Emma didn't stop them until she saw that the discussion had

ended. There was a laboratory devoted to mathematical films, such as those of Nicolet (Castelnuovo, 1953), and to particular experiments. For instance, a cone made by parallel threads, or a similarly made cube, were cut by a plane of light created by a black slide with a cut in the middle (Figure 12). Enthusiasm arose when, among the sections of the cube, the hexagon appeared.

There are some central points in the Emma's work: movement, limit cases, infinitesimal reasoning, simple models, and the presentation of cases in which a certain property doesn't hold. Teaching and evaluation are completely intertwined.

Emma's legacy

In 1977 Emma was called by the Ministry of Education to take part in the elaboration of a new syllabus for middle school. In the period from 1940 to 1962 many changes had occurred, in particular middle school was gradually extended to *all* pupils (see Vita, 1986). But, notwithstanding general references to intuition, the teaching of mathematics remained linked to a rational approach. According to the 1977 commission, teaching should start, instead, from concrete facts and proceed via problems. The new general subject included, besides mathematics, elements of the natural sciences, physics, and chemistry (see Bernardi, 2012). The teaching throughout the syllabus had to be carried out by the same teacher. Emma did not agree with this – even though in her teaching we can easily find examples taken from the natural sciences – because she believed that the mathematician and the naturalist have their own particular and differing ways of teaching. Apart from this, the new syllabi completely mirrored the ideas held by Emma.

One may note the weight given to the "intuitive capacities of the pupils", since the teacher has to "lead the intuitions and the conjectures to new forms of organized reasoning". It may also be noted that the syllabus has not a yearly division, but it is structured into themes, among which we find "mathematics of certain and probable events" and "geometric transformations". None of these themes appeared in the previous programs. Emma retired in 1979, so she never taught the sciences and never taught "by law" what she had always taught.

The structure and the content of these programs influenced all the following programs in Italy, as well as in other countries; however, we cannot say that the majority of teachers have adopted them, and also nowadays there are still difficulties in breaking with traditional teaching (that is the teaching before 1979).

As we have seen, Emma contributed to the creation of the "researcher in Mathematics Education" at university level, but her work had always something to do with practice. Since the late 1970s, Mathematics Education research shifted towards theory (Furinghetti, Matos & Menghini, 2013), particularly at international level. As a consequence, her work was often ignored. More recent

Italian research deals with the concept of Mathematical Laboratory, and thus it refers to Emma's work.

The OECD-PISA (Organisation for Economic Co-operation and Development: Programme for International Student Assessment) tests are also based on a kind of mathematics that forces teachers to adopt Emma's prominent use of problems, examples, interdisciplinarity – even though such tests are centered on the evaluation of the individual student, while in Emma's activity teaching and evaluation were interwoven.

But the major heritage consists in the work of the teachers who have learnt from her in her classroom, in her books, in the frequent meetings at her home – where everybody could go and discuss with her. This is found in Italy (Castelnuovo, 2008), in Spain (Casalderrey & Ramellini, 2004) and possibly in many other countries.

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Fortification teaching in seventeenth century French Jesuit colleges

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Abstract

The modern theory of fortification was a reply to the progress of artillery in the beginning of the 16th century. Invented in Italy, it was brought by Italian engineers to the North of Europe, along with the new versions of Greek classics on mathematics. This was a matter of lines and angles constrained by the power of cannons and the reach of weapons, in such a way that the authors came to use geometrical demonstrations to prove that their constructions offered the best resistance to the attacks of the enemies. From demonstration to teaching, there was a natural evolution due to an increasing need of training the young Nobility towards the honour of bearing arms for the (French) King. Naturally, after a period of Protestant/professional publications, the Jesuits introduced fortification in their colleges, supplying the demands of both their audience and the authorities of the country. They developed a specific method based on the simultaneous use of both text and image, resulting in a training of both brain and hand.

Introduction

The modern theory of fortification was born in Italy in the beginning of the 16th century. Among its godfathers were famous artists or scholars as Leonardo and Michelangelo, but also lesser-known military architects as Francesco de Marchi or Girolamo Cataneo. All of them had faced the challenge of improving the resistance of Italian cities walls at the times of the new method of besieging, which resulted from the mastering of gunpowder and cannon casting.

First of all, their problem was to build the fortresses according to particular shapes in such a way that cannonballs fired in straight line shouldn't reach directly the weak points of fortified walls; but they had also to make sure the defenders would have any spot of the fortress within the reach of their weapons. Moreover, the quantity of earth behind the walls had to be well measured to make the bastions solid enough, but cheap enough to be built even in lack of money. How could they deal with all of these constraints?

The answer can be quite difficult, but one thing is sure: geometry was essential and we'll explain why. The first modern fortifiers were reluctant to write specifically on geometry, but they knew it well, and their works were deeply established in practical methods of field geometry as well as Euclid's *Elements*. As war moved from Italy to the North of Europe, the Italian engineers spread the new method of fortifying throughout the continent, bringing with them the recently translated corpus of Greek mathematics that was published in Venice and other Italian cities. In fact practical geometry had been maintained lively everywhere but the theoretical corpus at hand tended to miss the most difficult aspect, namely: the proofs. Does that mean that before the Renaissance people had only to verify that things worked well? It depends on the people: if surveyors and measurers might have needed but few theoretical frames, all the circle squarers wouldn't have been able to find their way through this difficult problem. We will now explain why defence was a matter of geometry and why it turned to use more and more practical and theoretical geometric skills.

What is fortification?

As many of 17th century authors reminded their readers, fortification allows people to avoid battles (in a limited range). When the attackers became able to shoot precisely in straight line, the defenders had to rethink the shape of the walled enclosures of their cities. But it was just the final improvement of a very old evolution; let's follow Matthias Dögen[1] in his attempt to explain this process.

History of defence

Matthias Dögen was a Dutch engineer from German origin who worked a major part of his life for the United Provinces; as a writer he has a special style, quoting a huge quantity of ancient authors, from Cesar to Vitruvius, Pliny and Polybius in his introductory discourse. Nevertheless, his explanations are completely clear, enlightened by his plate A (Figure 1):

According to *Figura I* (top right), the first invention of men to protect themselves can be found in Jericho's very thick walls. But any defender had to be discovered to fire at the enemy and thus became an easy target. So, builders invented niches and slots (*Figura II*) to cover the archers inside the fortress. An important drawback is that this produced a dead angle, a space for enemy bombers to work securely.

Figura VI shows the principle of *flanking*: as it is impossible to shoot an attacker when he reaches certain spots against the walls, you just have to build

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[1] Dögen, 1648, p. 12, plate A.

towers growing out of the walls, in order to shoot him from behind. Various shapes have been imagined till the well-known round tower of medieval castles (*Figura IX*).

But when the armies discovered (or received) the metallic cannonball used with metallic guns, they became able to break the wall from the distance, so that defenders had to reply in the same way: the big city walls became eventually dangerous (they easily collapsed under cannon fire) for the population inside. The towers were lowered and extended to have them capable of receiving artillery; these *bastions* were given a pentagonal shape (*Figura X*) determined by the trajectory of bullets.

Figure 1. The tales of defence in (Dögen, 1648)
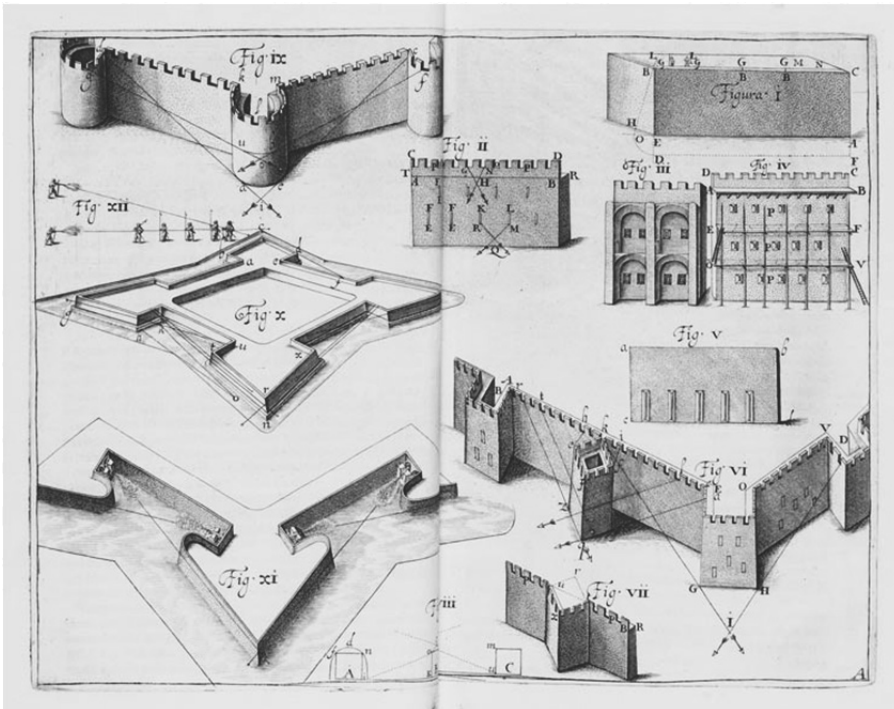
Artist: F. Matham (or H. Hondius ?)

Now, we can understand the constraints on the shapes military architects had to propose to the city councils:

The angle of the bastion (*flanked angle*) could not be too sharp, or else the quantity of earth behind the walls would have been too small to resist a battery of cannons. On the other hand, if this angle had been too open, the length of the wall between bastions (*curtain walls*) would have been too big for the reach of usual riffles of the times (muskets and harquebuses).

The length of the *line of defence* (leading from the angle of the bastion to the birth of the newt bastion on the curtain wall) must be less than the reach of the defending weapons, for the flanking to be effective.

In order to model theses constraints efficiently, military architects had no choice but using the geometrical language they of course knew perfectly.

Fortification as an active geometry

The Italian first authors of the 16th century had written but few lines about the geometrical fundamentals of their new domain of expertise but it changed with the shift to French engineers at the end of the same century. We can mention Claude Flamand, Jacques Perret and Jean Errard, as the principal transmitters of the Italian method, the latter being the only "well-remembered". Is that by chance that the three of them were Protestant and that each of their works were dedicated to the French King Henry IV?

Flamand worked for the Duke of Württemberg in Montbéliard (now in France), and he was deeply influenced by (Italian inspired) German writings on military architecture, including Albrecht Dürer's *Etliche Underricht* (Dürer, 1527) and Daniel Specklin's *Architectura von Vestungen* (Speckle, 1589). He published his major book on fortification in 1597, and in the same time an extensive treatise on mathematics (respectively Flamand, 1597a and Flamand, 1597b). We know but a few about Perret, except that he came from Savoy and probably worked for France. The frontispiece of his book (Perret, 1601) as well as Flamand's ones is surrounded by quotations of the Bible, which is typically Protestant at that time.[2] We don't find this activist behaviour with Jean Errard, who is completely silent on these matters. But as Errard would be celebrated as "the father of fortification *a la française*", forgetting his reformatted roots may have been a winning strategy. But before Errard, nobody had gone so far in the rigour for explanation and persuasion.

The title of Errard's book, *La Fortification reduicte en art et demonstree* (Errard, 1600 and Errard, 1619–22) mentions two new aspects of the pedagogy of this science: 1° "reduction into art", that is careful and well-organised description of the vocabulary, methods, and algorithms of construction, and 2° "demonstration", which can be understood both as "showing through pictures" and "mathematical proof". In order to show the reader how fortification is a part of geometry, we follow Jean Errard's fortification of the enneagon. In fact, Errard tells his reader how he answered (on behalf of the King of France) and gave advice to Venetian ambassadors about this method of construction, as the Venetian Republic planned to erect a new fortress against the Turks and the Austrians[3]. Let's read Errard's own words[4]:

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[2] But the Counter-Reformation will not leave that privilege to the "heretics".
[3] Errard (1600), p. 68.
[4] We translate from French into English, as we do for all the other texts in French.

> In order to construct & draw the fortification of the Enneagon, which is a nine-angled and nine-sided figure, we must divide 360 degrees by 9, and the result will be the angle at the centre, that is 40 degrees, which taken from 180 will leave 140 degrees for the two angles of, namely sixty degrees (Errard, 1619–1622, p 25v)

Of course you remarked that the angles at the base couldn't be of 60 degrees! It's amazing that it wasn't corrected for this third French edition that had been carefully perused, as Alexis Errard (the editor was Jean Errard's nephew) claims. Several other mistakes were omitted, though we must admit that it was not in the mainly corrected parts of this work. Now that the fundamental triangle is built, we just have to follow the fortification program (easier if we follow the picture simultaneously, see figure 2):

Figure 2. Fortifying the enneagon

> Let be described on side AB, the isosceles triangle ABC. To have the line of the face of the bastion, let be made angle ACD of 45 degrees, which are the three quarters of the angle at the base[5]. Then let be made line AE equal to BD and drawn line BE. Then let angle BAD be divided into two equal parts by line AF, and be DH taken equal to EF and drawn the curtain wall FH.
>
> From point F let be drawn a perpendicular on AD like FG, which will be the line of the flank. Then will be made BI equal to AG: in this manner will be described the two half-bastions AGF & BIH (Errard, 1619–1622, p. 25v)
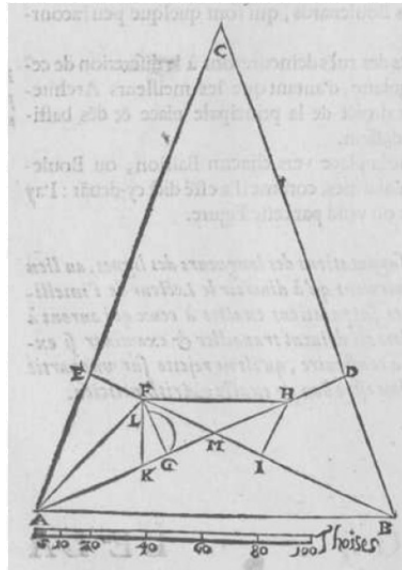
Errard deduces the whole fortified enneagon by reproducing this basic construction eight times, as you can see on figure 3. The actual fortress which has long been considered the greatest fortified polygon can still be seen in Italy: Palmanova (or Palma Nova, as Errard mentioned it) is located 20 km South of Udine (try Google Earth: 45°54'13.78"N 13°18'31.66"E)

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[5] Same false value as before. Anyway the proportion is of no use in the construction.

Figure 3. The completed enneagon
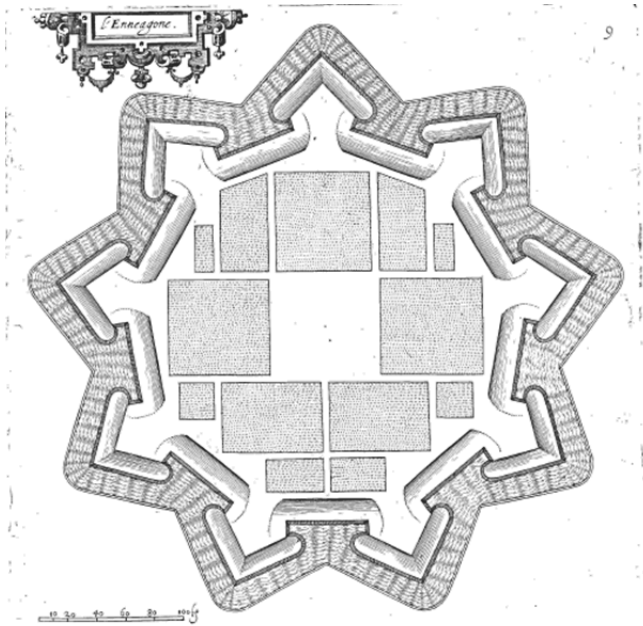
Artist: J. Errard himself, in Errard, 1619–1622.

The second part of Jean Errard's text is devoted to the "proof" of its construction, especially the calculations of lengths and angles measures, using the classical propositions of Euclid's *Elements*. But what was the need of it if the readers were to be engineers or military architects? The fact is that many surviving copies of Errard's treatise can be traced back to prestigious families or milieus all over Europe, and this new place for mathematics in the learning of fortification might have been well accepted in the first half of the 17th century. This is confirmed by the study of a variety of books written in the Netherlands after Simon Stevin and Samuel Marolois. The treatise of the latter (Marolois, 1615) is preceded by an extensive treatise on pure and applied geometry (Marolois, 1614) often bound with the latter and a treatise on perspective and architecture by the same author. Errard and Marolois shared similar purposes: educating young Noblemen to the Profession of Arms through a complete and logical explanation of the Art. Both French and Dutch schools of fortification were thus deeply characterised by the validating role of mathematics; could Jesuits have been insensible to it?

The Jesuit appropriation

As it has been showed by Jesús Luis Paradinas Fuentes, introducing significant mathematics in Jesuit colleges at the beginning (from 1551) was not the

purpose of the Company (Paradinas Fuentes, 2012). Despite the efforts of several Spanish Jesuits and Clavius himself within the Roman College, mathematics was often reduced to the strict fundamentals of geometry or even nothing at all (Paradinas Fuentes, 2012, p. 170) as the principal aim of the teachers was to lead students to theology. Clavius had even proposed a three level curriculum, from a kind of common core to expert studies but we don't find much space for mathematics in the early version of the *Ratio Studiorum* (1586) as well as in the official one, published in 1599 (Farrell, 1970):

Rules for the Professor of Mathematics

1. He should spend about three quarters of an hour of class time in explaining the elements of Euclid to the students of physics. After two months, when his students are somewhat familiar with the subject, he should add some geography or astronomy or similar matter which the students enjoy hearing about […]

2. Every month, or at least every second month, he should have one of the students solve some celebrated mathematical problem in the presence of a large gathering of students of philosophy and theology. Afterwards, if he wishes, the solution may be discussed.

3. Once a month, generally on a Saturday, the class period should be given over to a review of the subject matter completed that month

No mention of arithmetic! Moreover, the only official use of geometry is for geography and astronomy (so it must imply studying the geometry of the sphere as in medieval times). Though, Clavius had suggested a strong increase of mathematical studies in the colleges in 1581, not restricted to first year students: the first six books of Euclid, practical arithmetic and geometry with the use of instruments, cosmography and computus for all students; for second year students, books 11 and 12 of Euclid, trigonometry, geography, algebra… These propositions went far beyond what was acceptable for the professors of philosophy and they generated a general outcry (Paradinas Fuentes, 2012, p 174 *sq*): they were eventually abandoned in the syllabus.

Thirty years later, the reality is slightly different, as we can ascertain if we consider several remaining manuscripts of the Jesuit college courses. For instance, a *Tractatus geometricus* written by Franz Haffner from Solothurn (Haffner, 1627) gives a precise and carefully illustrated account of the course dictated by Vincent Leotaud in Tournon College; it is written in Latin and organised according to Euclid's standards: definitions, propositions, and theorems with their demonstrations. However it includes several new corollaries and even a special part named: *Digression on the practice of what has been tackled before*. It meets a number of previous centennial discourses on practical geometry that were usually given in the books as a transition between theory and practice: you have to pick the flower of your efforts! Geometry has to be useful for human everyday life, not only in a Platonic world. So Leotaud taught his students how to measure distant and inaccessible lengths or heights, using a

kind of Jacob staff in the shape of the Holy Cross… But in the following sections about surfaces and solids, he also teaches the transformation of figures and several problems on hydraulics. He even gives a part (in French) about mathematical recreations, the very year when Leurechon's *Mathematical Recreations* were first published. If you add a part on cosmography, you find a short encyclopaedia on what every priest should know, but of course no fortification[6] as in Clavius propositions.

Nevertheless, the change in the curriculum happened probably due to a meeting between supply and demand: several professors as fathers Bourdin, Fournier, Della Faille etc. were highly skilled in pure or applied mathematics (Dainville, 1954) and as the population of colleges increased constantly from 1600 (Dainville, 1957), we can suggest that more and more future officers had to be taught not only in Philosophy and Theology. This is confirmed, among others, by this extract of the preface to the first posthumous edition of Bourdin's *Architecture militaire* (Bourdin 1655, p. 5)

> [The author] has thought that his time would be well employed if he'd spend it composing this piece, because he clearly saw that it enclosed in a very small number of pages the most beautiful lessons which are to be found in big volumes by other authors, and proposing them by a very easy method, it would be very useful to infinitely many Noblemen, who today are eager to have the honour of bearing the arms to serve the King and for the glory of France […]

Moreover, the success of the colleges among a wide range of the French population could not displease the Order in the context of political struggle (Dainville, 1957 & Romano, 1999). No matter whether the courses were compulsory or not, with a great number of attenders or only for the few, practical mathematics and mixed mathematics found a place in the daily life of certain students, as we can infer from the extended quantity of pages about mathematics we can read in a manuscript of Pierre Bourdin's course in the Parisian *College de Clermont* (Bourdin, 1636), which is the first example we found so far in public libraries.

In the beginning was Pierre Bourdin

The course has been jotted down by a student named Paul Le Mercier, about whom we hardly know anything else but his talent in drawing and writing[7]. Le Mercier took notes (probably the integral dictated lessons) of courses, in the *Collège de Clermont* by Bourdin from 1636 to 1639 (general mathematics, theoretical and practical geometry, military architecture, optics…) as well as a

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[6] Another course can be found in Solothurn public library, written in Dole in 1626-27 by another young catholic nobleman named Samuel Zeltner. The contents are similar, including the encyclopedic aspect.

[7] As he mentions himself as "Paulus Le Mercier Aquilensis", he might be linked to Jacques Le Mercier, a famous French architect who spent a part of his life around 1610 in Italy.

course on gnomonic by Roberval in the *Collège de Gervais* in 1637. The quality of the contents suggests that Le Mercier was a brilliant student.

The major part of this colossal work (more than 1 700 pages in two volumes) is devoted to an extensive practice of geometry, leading to the art of fortification both in Latin and in French. The summary is impressive: the mathematics course consists of approximately 1 000 pages composed in Latin and/or in French. The first Latin part corresponds to the first year (1636–1637), it deals with general and theoretical/practical geometry (3–91); mechanics (92–107); "practical military geometry" including the use of contemporary instruments as Jacob's staff, Danfries's graphometer and trigometer, proportional compass, etc.; surveying, map drawing; military arithmetic (234–261) and architecture (262–), which is a very extensive part, including ordinary and military perspective (to the end of the volume).

The second volume starts with 700 pages on theology, which leave place to the definitions of rational geometry (705–744) and trigonometry (745–762); then a bilingual military architecture (787–854); arithmetic (855–886, including a use of the Pythagorean table in French); optics and vision (887–936). The last part, printed in French, consists of early versions of Bourdin's textbooks on fortification and geometry (1639 and 1641). Generally speaking, the texts are centred on the practice of constructions and the use of instruments, the point is not proofs and properties of the figures: we could judge that the discourse is mainly descriptive and focused on visual evidence rather than purely mathematical proof, as if the author needed first his students' consent to convince them.

The course was planned over the three years, and we can see a major evolution between 1636 and 1639: in 1636, all the geometrical drawings were hand-made, except some plates on fortification at the end of the volume (about the shape of fortresses), whereas in 1637 some of them were already printed (and the back of the leaves was used for hand-written text), and after 1638 it is the case of the majority of them.

We can make some assumptions about this: the teacher could have been led to have his plates printed (probably near the Clermont College, for there were many booksellers/printers in the rue Saint Jacques in Paris), because of the lack of time for the students to make meticulous drawings; or he could have earned some money by selling the manufactured plates? Or else, he could have wanted to try excerpts of his future book in his own classroom? Anyway, the change is obvious. We can trace the evolution of the whole course including fortification, as at least three libraries hold different copies of it[8], one of them completely printed dated from around 1640 and the other two partly printed, partly hand-written. The fact is that the images in general have a fundamental role in Bourdin's teaching, not only as illustrations for the text but also as means of

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[8] Namely, the library of the *Prytanée militaire* in La Flèche (France), the *Max Planck Institute* in Berlin and the *Newberry Library* in Chicago.

learning by themselves. First, as a kind of table showing a variety of new interdependent concepts at the same time, unlike the concepts of basic geometry that are presented independently. Moreover in the case of fortification, the primal concepts receive diverse new names, according to their functions. Figure 4 shows an example of map bearing the names of the important lines and angles of a fortified shape, as if it were places names on a geographical map.

Following Errard's style (which was consistent with the military habits), Bourdin gives his students the usual *ichnography* (plan of the fortresses), then the *orthography* (profile of the ramparts), plus a kind of bird's eye view called 'military perspective', which appears to be mentioned here for the first time, though it was used by the Italian authors a century before. With every shape mentioned in the text, figures (usually in both versions) are given for the reader to follow the ideas line after line. Bourdin may have been aware of many book of his time, as he mentions four different contemporary theories of fortification: French, Dutch, Italian and composite. The texts and illustrations must have been inspired by different sources that were at hand in Clermont College library (*Catalogue*, ca 1760, better than the printed version of 1764), among which we notice especially Jean Errard's *Fortification* second and third editions, his disciple Jean Fabre's *Pratiques* (Fabre, 1629) and the 1628 edition of Marolois's complete works, including geometry, perspective and fortification.

Figure 4: *Nomenclator militaris* in (Bourdin, 1636)
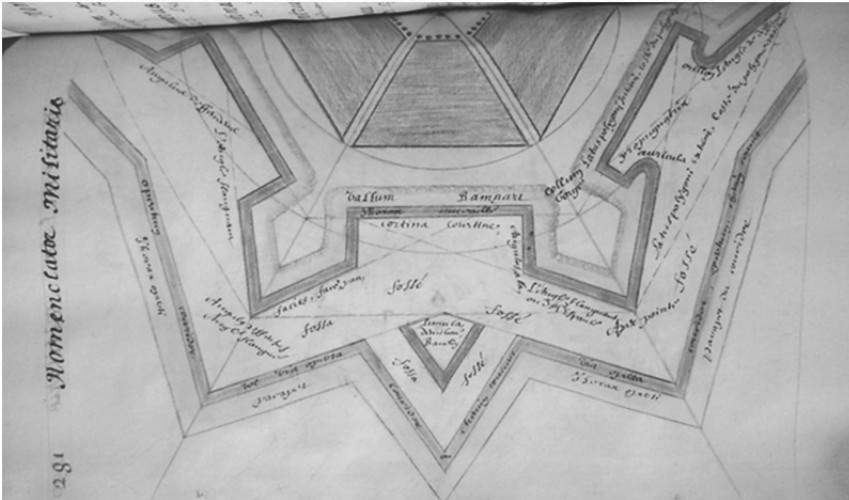
Drawing by Paul Le Mercier.

Bourdin's Military architecture was so popular that it was eventually published posthumously (Bourdin, 1655), before the complete course on mathematics (Bourdin, 1661). Bourdin's foreword is very clear:

Here is the Art of fortifying a variety of places, regular as well as irregular, processed universally & particularly; briefly & clearly; practically & scientifically.

Universally since it presents everything that has been invented & practised by the most capable engineers in Italy, in France, in Holland, in Germany, & in other countries where weapons flourish

Particularly since it digests all these inventions & pairing them so properly he makes them as to be only one with several faces.

Briefly since it contains in a few pages everything remarkable in all the methods which has been used by Jean Errard French engineer of Henry the Great; Lorini, Fiammeli and other Italian authors; Marolois, Fritach, Goldman, Dögen […]

In short: scientifically, since it gives the reasons of its practices & it demonstrates by Trigonometry the accuracy of its processes.

In direct line from Jean Errard, but with the new mathematical trend: trigonometry, Bourdin gives his readers a completely explained description of the methods of fortifying at their time. Mathematics and use of the instruments converge to make the student able to understand both the reasons of such constructions and the techniques of doing it by hand on the paper, then on the field. The role of figures is fundamental, as it will be in the edition of the complete course whose title mentions that it "contains in one hundred figures a general idea of all parts of this science" (Bourdin, 1661). Of course it is usual that a book on geometry relies on illustrations, but the importance of them is slightly different when the matter is practical geometry: in practical geometry you have to act on figures, first by constructing them, then measuring them, and the shapes are more useful than symbolic, as in fortification. That is one of the reasons why the light on the concepts is given by the figures more than by the texts. The focus on illustrations as part of the explanation will be developed and improved after Bourdin's death by two other Jesuit authors.

Bourdin's famous followers: Fournier and Bitainvieu

In 1636, Georges Fournier was already known as a professor on hydrography and navigation and he had replaced Bourdin as a mathematics teacher in the College of La Flèche (where he died in 1652), when Bourdin had come to Paris in 1636. What was Bourdin's influence on Fournier or Fournier's on Bourdin? Nothing is left except several indications that Fournier may have participated in the edition of Bourdin's *Architecture militaire* before 1655, especially when we compare the plate of this edition with Fournier's own book ones. Since Fournier was not a specialist of fortification, we can even think that he had been deeply inspired by his elder. Fournier's methods are not very different from Bourdin's ones, but the aims of the book and, in a way, the "questioning presentation" are worth quoting. In its preface, Fournier justifies first the book,

the author, war and legislation, as we can see through main titles (Fournier, 1648, p. 2 *sqq*):

> 1. Bearing arms is the noblest occupation of civil life;
>
> 2. The aim of the arms is to protect the weak against violence;
>
> 3. Only kings can declare war;
>
> 4. Who gave Sovereigns this power? People rights and Nations consent [it is strange not to read anything about God here].

When he arrives to the content of the book (title 7), Fournier reminds the reader of all that is needed to a young Nobleman to go into the Profession of Arms. He describes the complete summary of the *Art militaire* and consequently all the mathematics you have to study, that's clever! In the heart of the treatise, when listing the maxims of fortification, Fournier devotes a chapter to the explanation of each of them: he is a real teacher in the sense of the one who is preoccupied with the clarity of his message. Nevertheless, mathematics is not as central as it was in Bourdin's treatise, because Fournier's priority is not the proof of the methods: in 1648, things begin to be well-established and teachers have a lesser need to convince their students.

The follower of these two authors will be the most successful as far as editions are concerned: he had the strange name of Silvere de Bitainvieu, which is an anagram of *Jean du Breuil* (his real name) *Jesuite* (if you remember that there was no real typographical difference between *i* and *j*, as well as between *u* and *v*). Du Breuil had already published a famous book on perspective in 1642, and he brought fortification to the good society, as he mentions in his foreword (Bitainvieu, 1665):

> Please allow me to give my advice on this subject, despite the fact that you are Noblemen, & even Princes: don't settle for reading this book & understand it, as I believe you will do very easily, considering the efficiency of your mind, but push yourself to take Ruler & Compass to work by your own hand and make on the paper what you see in the book, not by running from a figure to another, without any continuity and reflexion; but starting with the first one, possessing it well, and practising it even better before going to the second one, and from the second one to the third one. For by going this way from one to the other, you will surely progress and improve your skills day after day.

It is amazing to realize that we might be facing an analogous alternative today, trying to make appealing a science that needs time and pain, trying sometimes to slow down our busy students. Can we compare our students' state of mind to the eagerness of the 17th century young noblemen? We know that the mathematics course was not very popular among young college students, but many adults took private lessons about mathematics and fortification (Dainville, 1957); the republication of our three authors' books show this

popularity as does the quotation of Fournier by Menno van Coehoorn, who was to become the well-known Dutch fortifier, the 'Dutch Vauban'. These books were to serve as first handbooks for professional training of officers and military architects, before the publications of best-sellers like Manesson Mallet's *Travaux de Mars*.

Conclusion

It's amazing to realise that three major Jesuit authors on fortification published their works without their signature. Was it an unfavourable period or were Jesuit authorities against the publication of such books? In fact Fournier's *Hydrographie* (1643) was signed by the author, but in its *Perspective* (1642) Du Breuil was only mentioned as "un Parisien religieux de la Compagnie de Jesus". Later writers as Claude François Milliet de Chales in 1677 among many others wouldn't have this problem of publishing anonymously anymore. Yet, 19th century priests who wrote on mathematics didn't write their names but their initials on the frontispieces of their books, but this was not the case back in the 17th century; we haven't found any convincing explanation so far.

The first three authors we mentioned were real pioneers in teaching military architecture. Coming after a series of Protestant authors, they published along with professional writers, officers or real architects, but they developed their own pedagogy based on the mastering of the instruments, the progressivity of the learning and the juxtaposition of texts and figures, making visible sense on purpose. Fortification was the ideal domain for experimenting this way of learning since the practice came before the proof, and gestures before brain, images before words (despite figures were drawn on the right pages and texts on the left ones; but the eyes go first on the right pages, don't they?). Moreover, the aesthetic of the works catches the eye. In this way, teaching military perspective was essential and the former students could take their place both speaking in Parisian salons and fighting on the field.

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Errard, Jean (1600). *La fortification reduicte en art et demonstrée, par J. Errard de Bar le Duc, Ingenieur du Tres Chrestien Roy de France & de Navarre. s.l.* [Paris]: *s.n.*

Errard, Jean & Errard, Alexis (1620). *La fortification demonstrée et reduicte en art, par feu J. Errard de Bar le Duc, Ingenieur du Tres Chrestien Roy de France & de Navarre.* Paris: *s.n.* [Pierre Delon].

Fabre, Jean (1629). *Les practiques du Sieur Fabre sur l'ordre et reigle de fortifier, garder, attaquer et defender les places.* Paris: Samuel Thiboust.

Farrell, Allan P. (S. J.) (1970). *The Jesuit Ratio Studiorum of 1599 translated into English, with and introduction and explanatory notes.* Washington D. C.: Conference of Major Superiors of Jesuits.

Flamand, Claude (1597a). *La Guide des fortifications, et conduitte militaire...* Montbéliard: Jacques Foillet.

Flamand, Claude (1597b). *Les mathematicques & Geometrie, departies en six livres contenant ce qu'est le plus necessaire, pour l'utilité de la vie humaine.* Montbéliard: Jacques Foillet.

Fournier, Georges (1648). *Traité des fortifications, ou architecture militaire, tirée des places les plus estimées de ce temps pour leurs Fortifications.* Paris: Jean Henault.

Haffner, Franz (1627). *Tractatus geometricus scripsit Franciscus Haffner Salodorensisi Anno 1627, dictabat R. P. Vincentis Leautaud Mathematicarum Professor.* Ms, Solothurn public library.

Manesson Mallet, Allain (1671). *Les Travaux de Mars ou la Fortification nouvelle tant reguliere qu'irreguliere, divisée en trois parties.* Paris: self-published, Jean Henault (printer).

Marolois, Samuel (1615). *Fortification ou Architecture militaire tant offensive que defensive…* Hagae Comitis [The Hague]: Hendrick Hondius.

Milliet de Chales, Claude François (1677). *L'Art de fortifier, de défendre & d'attaquer les places.* Paris: Estienne Michallet.

Paradinas Fuentes, Jesús Luis (2012). The teaching of Mathematics in the Jesuit Ratio Studiorum. In K. Bjarnadóttir, F. Furinghetti, J. M. Matos, & G. Schubring (Eds.), *"Dig where you stand" 2, Proceedings of the second International Conference on the History of Mathematics Education.* Lisbon, UIED, 165–180.

Perret, Jacques (1601). *Des fortifications et artifices, architecture et perspective... s.l.* [Paris]: *s.n.*

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American mathematical journals and the transmission of French textbooks to the USA

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Abstract

In the nineteenth century, American journals became an important vehicle for the dissemination of the scientific knowledge, including the French mathematics exposed in famous textbooks in particular written for École Polytechnique teaching and admission by the leading mathematics scholars of the period 1785–1825. Specialized United States mathematical journals frequently and accurately referred to passages of these works published in France. This article discusses the forms of this circulation (articles, questions, and courses), the origin of the borrowings (textbooks, treatises) and the evolution of the contents during the first half of the century while translations of French textbooks were published in America. It also examines the identity of the journals contributors who introduced theorems, exercises or proofs initially written by French professors or textbook authors. Considering the thematic network of contributors in term of common uses and references, this work shows that the diffusion of French mathematics in American papers relied on specific educational needs.

Introduction

In the United States, science and mathematics practice, education and diffusion experienced deep transformations in the first half of the 19th century[1]. The creation of specialized journals, learned societies and autonomous mathematical chairs in colleges witnessed the beginnings of the mutations the country was experiencing in the acceleration of the scientific knowledge diffusion and in the structuration of education. Textbooks, that used to be imported or reprinted from English works in the colonies, started to be written by American authors and participated to the growth of the education publishing market (Karpinski, 1940). American mathematical publications of the very early nineteenth century

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[1] For an overview on American mathematics between 1800 and 1875, before the "emergence of the American mathematical research community", see chapter one of Parshall & Rowe (1994).

were still influenced by English methods in arithmetic (Michalowicz & Howard, 2003), algebra (Pycior, 1989, pp. 126–128) or geometry (Ackerberg-Hastings, 2002, p. 69). At the beginning of the 1820s, *curricula* of a few colleges were reformed, introducing French more analytic textbooks. In the two following decades, more than twenty French textbooks translations were published[2]. French mathematics education, this is to say mathematical work done within the French institutional framework as defined in (Crosland, 1992, p. 12) and designed for the use of instruction by or for teachers and students, was not only imported in the United States through textbooks translations. It was also displayed and diffused in articles or questions references in American mathematical journals.

This study intersects two fields on the history of mathematics in the United States. The first deals with the consideration of the scientific journal as a specific way to transmit and diffuse knowledge in the nineteenth century. Works as (Hogan, 1985), (Timmons, 2003) and (Kent, 2008) especially showed how the constitution of a publication community and the introduction of education and research-oriented contents built up attempts to initiate the professionalization of mathematics. The other concerns foreign – notably French – influences mathematics education in America was exposed to, right after the War of Independence. A very large study was made on the subject in (Cajori, 1890) in which a bibliography of nineteenth century colleges' *curricula* and textbooks was compiled. Lao G. Simons (Simons, 1931) gave a list of French textbooks translations American scholars produced between 1818 and 1850. More recent works draw very general overviews (Parshall & Rowe, 1994), or focused on specific topics, specific textbooks or very short time periods as (Pycior, 1989) did for algebra textbooks.

Examining the two approaches, this article describes the transmission to America of French mathematics education through American mathematical journals[3]. It relies on a systematic analysis of contents that were diffused and their original French vehicles (were textbooks only concerned?). It also questions uses the American mathematical journals' contributors made of the references in their writings. The paper leans also on a prosopographical study of these contributors in order to evaluate the correlation between needs in term of mathematical education in America and circulations of French contents through journals.

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[2] The corpus of the American translations of French mathematical textbooks is described and analyzed in (Preveraud, 2014, Chapter 4).

[3] The case of general-interest journals that essentially reviewed translations of French mathematics textbooks is mentioned in (Preveraud, 2014, pp. 130-134).

Corpus and methodology

The sources of this study were four American mathematical journals published in the period 1818–1878[4], starting after the first ever French textbook translation[5] and ending before the publication in 1878 of the *American Journal of Mathematics* considered as the first research-oriented journal in that country (Parshall & Rowe, 1994, p. 49). It was a period of specialization and professionalization of mathematics in the United States. *The Mathematical Diary, The Mathematical Miscellany, The Cambridge Miscellany of mathematics, physics and astronomy* and *The Mathematical Monthly* were all mixed-level mathematics journals, introducing problems, articles sometimes but not always research-oriented, or offering courses notes. These journals had a short time life, but their successive publication shows significant mathematical editorial activity during the antebellum period.

Table 1. Four mathematical journals in the United States (1825–1861)

Title of the publication	Dates	Place of publication	Editor	Editorial contents	Issues
The Mathematical Diary	1825–1832	New York, NY	Robert Adrain (1825–1826), James Ryan (1826–1832)	Questions	13 (2 vol.)
The Mathematical Miscellany	1836–1839	Flushing, NY	Charles Gill	Questions & articles	8
The Cambridge Miscellany of Mathematics, Physics and Astronomy	1842–1843	Cambridge, Ma	Benjamin Peirce, Joseph Lovering	Questions & articles	4
The Mathematical Monthly	1858–1861	New York, NY	John D. Runkle	Questions, notes & articles	36 (3 vol.)

In every issue, each of the four publications proposed several questions to be answered by readers, whose solutions were published in the following issues. In *The Mathematical Diary* and *The Mathematical Miscellany,* questions and solutions comprised almost the entirety of the scientific contents. This model of journal where mathematics were exposed through problems rather than research articles found a breeding ground in America as it did previously in England with *The Ladies' Diary* (Albree & Brown, 2009). The role of problems/solutions in these first American journals was predominant in the edification of a mathematical community because readers, issue by issue, were encouraged to answer, criticize, and improve others' mathematical contents in their submitted communications. Regarding education, the importance of problems/solutions was brought out by editors. "It is well known to mathematicians, that nothing contributes more to the development of mathematical Genius, than the efforts

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[4] The first American mathematical journals, *The Correspondent* and *The Analyst,* showed no significant reference to French mathematical works. Between 1818 and 1878, only one publication could not be analyzed: *The Mathematical Companion* (1828–1831) could not be located.
[5] Sylvestre-François Lacroix's *Elements of Arithmetic* by Harvard Professor John Farrar in 1818.

made by the student to discover the solutions of new and interesting questions" remarked Adrain (Adrain, 1825, p. iii).

Progressively, editors added a new type of contents, specially intended for students: transcripts of courses in *The Cambridge Miscellany* or courses notes in *The Mathematical Monthlty*. In their journal, Gill, Peirce and Lovering, as well as Runkle made explicit their desire to target a young readership by creating a "Junior department" at the beginning of each issue. This dedicated section contained questions, articles and courses adapted for a student' audience. Education concerns were made clear by editors in chief. A common point of the four journals was the large range of readership they intended to catch as *Mathematical Monthly*'s editor in chief explained: "[the journal] should embrace students in one extreme and professed mathematicians in the other" (Runkle, 1858–1859, pp. i-ii).

In these four publications, references in problem solving, articles and courses that were related to French mathematics contents were exhaustively looked for in each issue. The attention was focused on quotations of famous French textbooks written in the period 1785–1825, when many French mathematicians were asked to write for the/their teaching at new *École polytechnique* (opening in 1794) and within the frame of secondary education institutions named as *lycées* (created in 1802)[6]. During that period, fewer than ten authors shared almost the total print-run. Gaspard Monge's *Géométrie descriptive* for both *École polytechnique* and *École normale*, or Augustin Louis Cauchy and his *Cours d'analyse* for *École polytechnique* were rather designed for higher education and almost entirely used in that framework. Sylvestre-François Lacroix and his series for *École centrale des quatre-nations* as well as Étienne Bézout reprints of his series for *les gardes du pavillon et de la marine* were used in secondary schools and some of their texts read by *École polytechnique* admission candidates. Prints of Adrien-Marie Legendre's *Éléments de géométrie* and Pierre Louis Marie Bourdon's *Éléments d'algèbre* were also widely used within the *lycées*.

It is also important to highlight the case of recapitulative treatises, as for example *Mécanique céleste* by Pierre Simon de Laplace, *Mécanique analytique* by Joseph Louis Lagrange or *Traité des propriétés projectives des figures* by Jean-Victor Poncelet. Those books did have an ambiguous position towards education: they were usually not meant to be taught, and rather designed to compile and expose the whole knowledge about a specific mathematical subject. Nevertheless, since Laplace, Lagrange and Poncelet were employed for teaching in French higher education institutions respectively at *École normale de l'an III, École polytechnique* and *École d'artillerie et du génie de Metz*, they used their own treatises with their students. Even though treatises did not pertain entirely to education matter[7], they did participate to the circulation of French mathematics education between France and the United States.

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[6] See (Dhombres, 1985).
[7] The status of *Mécanique céleste* was studied in (Hahn, 2005).

The systematic study of French textbooks and treatises references in American journals was conducted under the following methodology. An initial investigation enabled to identify occurrences of the authors' names that have just been listed. In most cases, when a quotation of a patronymic was found, it was immediately followed by a title of a book. Less frequently, contributors to American journals used French publications without quoting the author but only the title of their work. A second inquiry was then pursued in order to seek keywords occurrences as "géométrie", "algèbre", "calcul", "traité", "éléments", etc. Some other French works were found but were discarded from the study because they were not directly addressed to education: articles in Gergonne's *Annales de mathématiques pures et appliquées* or Liouville's *Journal de mathématiques pures et appliquées,* memoirs of Académie des sciences and research-oriented books as Legendre's *Théorie des nombres* or Laplace's *Théorie analytique des probabilities.*

Textbook references supporting problem solving

Famous French textbooks were quoted by contributors to American journals. The most commonly cited authors were Lacroix and Legendre who wrote *Traité élémentaire de calcul différentiel et de calcul intégral* and *Éléments de géométrie.* These two publications were quoted during the whole period, a reason why this article will focus on their use in the four journals. The exhaustive corpus of French textbooks quoted and used in American mathematical journals is gathered in the following table.

Table 2. French mathematical textbooks quoted or used in American mathematical journals (1818–1878)

Title	Author	First edition	Number of references
Éléments de géométrie	Legendre	1794	7
Traité élémentaire de calcul différentiel et de calcul intégral	Lacroix	1802	7
Cours d'analyse	Cauchy	1821	2
Éléments de géométrie	Lacroix	1799	1
Compléments aux éléments d'algèbre	Lacroix	1800	1
Éléments d'algèbre	Bourdon	1817	1
Cours de mathématiques à l'usage des écoles militaires	Allaize, Billy, Puissant, Boudrot	1813	1

Éléments de géométrie by Adrien Marie Legendre, first published in 1794, was widely used in schools of France during the whole 19th century. Its presentation of Euclidian geometry, using algebraic symbolism and a new arrangement of properties, was perceived by American scholars as a good compromise for their teaching between classicism and modernity (Preveraud, 2013, pp. 46–47). The textbook was even twice translated in the 1820s (Schubring, 2007, pp. 46–50), and twice again in the 1840s, not including the reprints. In American journals of mathematics, it was quoted mainly for

highlighting the solutions to geometrical questions. Because contributors needed to write succinctly and have their solution edited, they often referred to a property statement as ellipses length relations between axes and chords (Runkle, 1860, p. 269), or used formula as the one that gives the volume of a tetrahedron (Ryan, 1827–1832, p. 76) and specifically referred to Legendre if the reader wanted to find complete statements and proofs. For example, in a question published in the *Mathematical Miscellany*, one asked to determine the number of diagonals of a polyhedron. In one of the answers, New-York teacher Marcus Catlin used the relation s = E – v + 2 between the number of faces (v), edges (E) and angles (s) of a polyhedron and quoted its exact location in the second textbook edition of Legendre: "See Livre VII, Prop. 25, Leg. Geom, 2nd Ed. Paris" (Gill, 1836–1839, p. 160).

Another French bestseller frequently quoted in American journals was Lacroix's *Traité élémentaire de calcul différentiel et de calcul intégral*. Unlike *Legendre's Geometry*, this book had not been translated in the United States[8]. In American journals, the main use of *Traité élémentaire de calcul différentiel et de calcul intégral* was to help contributors in elaborated mechanical questions where analytical methods gave complicated differential equations to integrate. In *The Mathematical Miscellany*, for example, Charles Avery, professor at Hamilton College needed to solve the following equation: " $(d^2\varphi)/(dt^2) + g\varphi + F(t) = 0$ " (Gill, 1836–1839, p. 230). Instead of explicitly exposing a long reasoning, he gave credit to Lacroix's solving methods of this kind of differential equations ("La Croix, page 407") and furnished almost directly the integrated solution.

The quotation of those two French textbooks helped contributors in their solutions, but also enabled them to avoid consuming too much space in the journal. Nothing is more relevant at a time when journal existence was precarious and relied, in particular, on the cost of paper.

Despite the common use for these textbooks, others were used in a very different way. This was the case of Augustin Louis Cauchy's *Cours d'analyse*, published in 1821 for his teaching at *École polytechnique*. This high-level mathematics textbook was not reported in American journals for its complex and elaborated results and presentation of analysis, nor its help in problem solving. What interested American contributors were the elementary[9] notes Cauchy wrote at the end of his treatise. Contributors translated some of them in the junior sections of *The Mathematical Miscellany* or *The Mathematical Monthly,* with pedagogical goals.

For example, Editor Charles Gill gave a translation for young students of a note on the theory of positive and negative quantities. It was, said Gill, to "assist the students in mastering the first principles of the use of symbols in

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[8] West Point Professor Charles Davies widely used Lacroix's book for the writing of *Elements of Differential and Integral Calculus* (1836) (See Preveraud, 2014, pp. 245-247).
[9] Referring to the foundations of mathematics as defined in (Gérini & Otero, 1993, p. 51).

algebra", a science where "great care should be taken to obtain a correct and precise idea of the symbols used in the science, and the operations performed upon them" (Gill, 1836–1839, p. 204). In this note, Cauchy made a distinction between the number, associated with the idea of magnitude measurement, and the quantity that emerges when "one considers every magnitude of a set specie as serving for increasing or decreasing of another set magnitude of the same species" (Cauchy, 1821, p. 403). For Cauchy, the quantity could only be endowed with a sign placed in front of a number. In other words, if *A* points out a number, +*A* indicates a positive quantity and −*A* a negative one. Cauchy proposed a definition of a negative quantity, similar to what English textbooks used to give: a negative quantity refers to a diminution and has to be subtracted.

This pedagogical approach established a breaking with the analytical methods of Lacroix or Bourdon's algebras where negative quantities were never defined but late introduced as absurd solutions of first degree problems (Lacroix, 1815, pp. 80–91). In the United States, both approaches were used in textbooks publishing (Pycior, 1989). But after the first translations of Lacroix (1818) and Bourdon (1830, 1831), there was a shift in the way negative quantities were introduced in translations of French algebras. For example, Charles Davies's translation of Bourdon's *Éléments d'algèbre* (1835) proposed a mixed solution: an early definition of the negative quantity followed by the interpretation of negative solutions to first degree problems. The analytical, radical and original French approach was progressively given up or softened by a so-called rigor of definitions more appropriate to methods of American and British algebras previously in use. Thus, Lacroix and Bourdon's algebras were only quoted once in American journals. In that context, 1836 translation of Cauchy's note by Gill in his journal, aimed at transmitting to beginners in algebra a frame about the theory of negative quantities, participated in the change of American mathematical algebra textbooks. Gill's note was indubitably well adapted to the mid-1830s needs for algebra instruction.

Treatise references diffused new mathematical contents

Regarding to French mathematics education transmitted to the United States through American mathematical journals, this article discusses now the case of treatises whose references are gathered in the following table.

Table 3. French mathematical treatises quoted or used in American mathematical journals (1818–1878)

Title	Author	First edition	Number of references
Mécanique céleste	Laplace	1799–1825	13
Mécanique analytique	Lagrange	1788	10
Traité des fonctions elliptiques	Legendre	1826	9
Traité des fonctions projectives des figures	Poncelet	1822	4
Traité de géométrie supérieure	Chasles	1852	2
Géométrie de position	Carnot	1803	1

The two most cited books dealt with mathematical mechanics. In *Mécanique céleste* and *Mécanique analytique*, Laplace and Lagrange wrote physics treatises where rules of nature were displayed under the domination of mathematics. Lagrange's work wasn't given any English version. Laplace wasn't translated in America until 1829 with *Mécanique celeste by Marquis de Laplace* written by Salem mathematician Nathaniel Bowditch, and very partial English translations produced in England and Ireland weakly circulated within the United States during the first part of the century (Preveraud, 2014, pp. 412–413). As Lagrange and Laplace's books both gave differential equations governing movements of bodies and the way to solve it, they were widely-used by American contributors facing applied mechanical problems to solve, essentially in *The Mathematical Diary*, before the publication of Bowditch's translation. An example of use was given in Rutgers Professor Theodore Strong's solution. He had to "investigate the nature of the curve described by a body projected obliquely along a given inclined plan, the resisting arising from friction being taken into consideration" (Adrain, 1825–1826, p. 34). Strong quoted Laplace's expression of the friction for an inclined plan situation and adapted the formulae with his own notations. Speaking of celestial mechanics, journals diffusion answered the lack of sustainable books in vernacular language on the subject.

In *Traité des fonctions elliptiques* (1825), Adrien-Marie Legendre synthesized the computation of integrals whose general form was $\int \frac{P(x)}{\sqrt{R(x)}} dx$ with P a rational function and R a polynomial of degree less than three. The French mathematician brought those integrals down to three species easily computable with tables (Legendre, 1825, pp. 14–17). The quotation of his treatise in American journals dealt as well with analytic problem solving. About ten American contributors to mathematical journals quoted the reduction to one of the three species and used the tables of Legendre when integrating a complex equation in geometry, mechanics or pure analysis problems. In *The Mathematical Miscellany,* Marcus Catlin faced the integration of:

$$\frac{n^2 c^2}{1 + n^2 \sin^2\theta} \frac{d\theta}{\sqrt{1 - e^2 \sin^2\theta}} + R^2 d\theta \sqrt{1 - e^2 \sin^2\theta} - \frac{c^2 d\theta}{\sqrt{1 - e^2 \sin^2\theta}}$$

He noticed that the above equation "involves the three kinds of elliptic functions treated of by Legendre in his *Fonctions elliptiques*, see p. 19 of that work" (Gill, 1836–1839, p. 240). Quotations of Legendre's treatise were numerous between 1827 and 1837, right after its publication in France. This quick transatlantic circulation of his work was supported by journals and not by any translations or articles.

Other treatises were quoted at length and not meant only to serve as problem-solvers. Some contributors gave complete excerpts of new pure geometry treatises, as Jean-Victor Poncelet's *Traité des fonctions projectives des figures* (1822) or Michel Chasles's *Traité de géométrie supérieure* (1852). The revival of new pure synthetic geometry occurred in France in the first half of century within a group of mathematicians willing to cut loose from the complexity of algebraic

or analytical methods for problem solving (Kline, 1990, pp. 840–852). Their new methods relied on theory of projections (Poncelet, 1822), theory of transversals (Carnot, 1803) and theory of poles and polar lines (Poncelet, 1822). In *Traité de géometrie supérieure* (1852) Michel Chasles produced a compilation of all the methods in pure geometry. In America, first echoes were produced in *The Cambridge Miscellany* (Peirce & Lovering, 1842, p. 97). But most of the references were located in *The Mathematical Monthly* in the form of articles where theorems and proofs were entirely transcribed and translated, or introduced as courses notes for students.

For example, numbers nine and ten of the journal contained two contributions signed by Yale Professor Henry A. Newton and one of his students, Arthur W. Wright. Newton exposed different "geometrical construction of certain curves by points" (Runkle, 1860–1861, pp 235–244). He defined the polar to a point O with respect to an angle P as "that straight line which, together with the line drawn from the point to the vertex of the angle, divides the angle harmonically" (Runkle, 1860–1861, p. 235), exactly as Michel Chasles did in *Traité de géométrie supérieure* (Chasles, 1852, p. 251). Newton gave a method of construction of polar PO': O' is obtained by the intersection of quadrilateral ABCD diagonals, with D and B belonging to one side of the angle P, A and C to the other.

Figure 1. From Henry A. Newton's article in *The Mathematical Monthly* (Runkle, 1860–1861, p. 235)
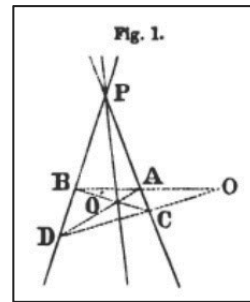
The truth of the construction was provided by the following theorem borrowed to Chasles but not proven by Newton: "in every quadrilateral, the two diagonals and the lines drawn from their intersection to the intersections of the opposites sides form a harmonic beam" (Chasles, 1852, p. 251). It was Wright, a few pages later, who gave the proof to *The Mathematical Monthly* reader. He did so in a general article about the methods of projections (Runkle, 1860–1861, pp. 293–305). He worked on a quadrilateral ABDC and imagined then the figure to be projected in such a manner that points P and O passed to infinity, lines AB and CD became parallels and quadrilateral ABDC was turned into a parallelogram[10] (Runkle, 1860–1861, pp. 297–298). In the projected figure, Wright easily assumed the harmonic division which was also true for the original figure. This proof that relied on the conservation of the harmonic ratio by projection was found in Poncelet's *Traité des proprieties projectives des figures* (Poncelet, 1822, p. 82).

The Cambridge Miscellany and *The Mathematical Monthly* were the first vectors of new pure geometry contents in the United States, years before Francis H. Smith

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[10] The names of the points have been changed to fit Newton's notations.

or William Chauvenet produced appendices as introductions to modern geometry in their respective geometry textbooks *Elements of geometry* (1867) and *A Treatise on Elementary Geometry* (1869) (Preveraud, 2014, pp. 289–295).

Diffusion, translations and educational needs

The contents of French textbooks and treatises references revealed a group of transmitters who were mostly involved in education as shown in Table 4. For each contributor, his activity and residence at time of publication are indicated. Most of these prosopographical details were found in journals themselves: if not, *Appletons' Cyclopedia of American Biography* (Wilson & Fiske, 1887) was used.

Table 4. Contributors who quoted French textbooks and treatises in American mathematical journals (1818–1878).

Journal	Contributor	Activity	Residence
Mathematical Diary	Nathaniel Bowditch	Actuary	Boston life Insurance Company, MA
	Theodore Strong	Professor	Rutgers College, NJ
	Henry J. Anderson	Professor	Columbia College, NY
	Robert Adrain	Professor	Rutgers, NJ/University of Pennsylvania, PA
	Samuel Ward	Student	Columbia College, NY
	Eugene Nulty	Teacher	?
	Thomas J. Megear	Artist	-
	James Macully	Teacher	Secondary school, VA
	Analyticus	?	?
	L'inconnu	?	?
Mathematical Miscellany	Benjamin Peirce	Professor	Harvard College, MA
	Charles Gill	Professor	Saint Paul College, NY
	Theodore Strong	Professor	Rutgers College, NJ
	William Lehnart	Principal	York Academy, PA
	Marcus Catlin	Professor	Hamilton College, NY
	Charles Avery	Professor	Hamilton College, NY
	John B. Henck	Student	Harvard College, MA
	George Perkins	Teacher	Clinton Liberal Institute, NY
Cambridge Miscellany	Charles Gill	Professor	Saint Paul College, NY
	Theodore Strong	Professor	Rutgers;, NJ
	William Brown	Student	Clinton Liberal Institute, NY
Mathematical Monthly	Matthew Collins	Professor	Trinity College, IR
	George W. Hill	Student	Rutgers College, NJ
	John B. Henck	Civil engineer	?
	Pike Powers	Principal	Staunton Academy, VA
	David W. Hoyt	Teacher	?
	John M. Richardson	Teacher	Secondary school, GA
	Hugh Godfray	Professor	Cambridge College, GB
	Thomas Hill	President	Antioch College, OH
	Henry A. Newton	Professor	Yale College, CT
	M.C. Stevens	Professor	Harverford College, PA
	Arthur W. Wright	Student	Yale College, CT
	Thomas Sherwin	Principal	English High School, MA

Most of these men were college professors, as Strong from Rutgers, Newton from Yale, Anderson from Columbia or Peirce from Harvard. Secondary schools teachers (Perkins, Lenhart) and students (Wright, Ward) were also

found. Prosopographical studies on part or totality of the contributors of a journal as in (Preveraud, 2011) for *The Mathematical Diary*, in (Hogan, 1985) for *The Mathematical Miscellany* or in (Kent, 2008) for *The Cambridge Miscellany* indicated that the percentage of men involved in education is higher in the group of contributors quoting French textbooks and references than in the general population of contributors (Preveraud, 2014, pp. 389–392). The diffusion of French mathematical education in American journals came within a group of mathematicians strongly connected to higher or secondary education.

The previous analysis of French references contents has highlighted the mathematical tools the contributors used, the given and proven results they published, and the problems they were interested in. But it missed the relationships between authors and the circulation of references in terms of education needs. Were textbook references used both by teachers and students? Did celestial mathematics (Laplace and Lagrange) treatises' users also quote pure analysis textbooks? Did pure geometry treatises circulate independently of more elementary textbooks? In order to locate the connections between references and education needs of the contributors, social networks analysis provided specific tools (UCINET software and its drawing extension NETDRAW). The analysis relied also on the methodology of studies who gave knowledge to intellectual and social circulation of mathematical ideas through quantitative methods as in (Goldstein, 1999).

By counting the number of common references for every contributor, a thematic network of contributors was built. Each vertex of the network represents one of the contributors that quoted French textbooks or French treatises in one of the American journals[11]. Connections between individuals were obtained by computing common references of contributors: the more reference in common two contributors had, the more the link between them was type drawn bold. For example, Strong and Bowditch gave four common references of French publications, whereas Gill and Collins only one. Isolated men (as James Macully) gave reference(s) that no one else quoted.

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[11] The color of the vertex refers to the activity of the contributor, its shape to the journal he published (MD for *The Mathematical Diary*, MMI for *The Mathematical Miscellany*, CM for *The Cambridge Miscellany*, MMO for *The Mathematical Monthly*). In case the contributor quoted French references in several journals, he was attributed the publication in which he wrote the more.

Figure 2. Thematic network of contributors quoting French textbooks or treatises in American mathematical journals (1818–1878).
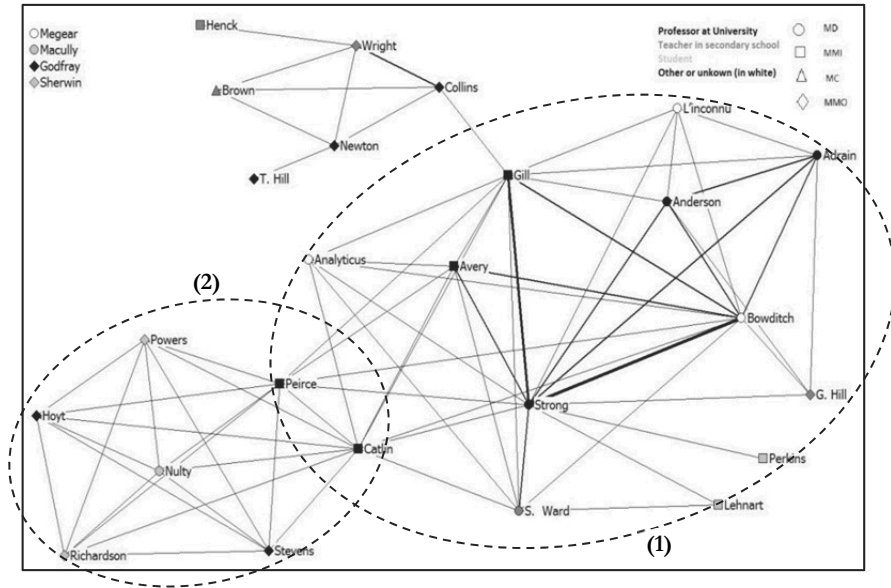
One can point out a large sub-network (1) where Lacroix's *Traité élémentaire de calcul différentiel et de calcul intégral* was widely quoted. Meanwhile, those contributors all referred to one or both of the two treatises about mathematics mechanics: Laplace's *Mécanique céleste* and Lagrange's *Mécanique analytique*. In other words, transmission in American journals of the high-level French mathematics contents as those included in Laplace and Lagrange's works, came within a group of men who had to use and quote also the more elementary textbook of Lacroix in order to solve differential equations displaying movements of bodies. A large part of this group of men's contributions occurred between 1825 and 1839, in the two first journals of the studied period. Uses and needs of Lacroix's book disappeared in the 1840s with the death of some of contributors but mainly because mathematical physics problems to be solved with analysis seemed to interest fewer and fewer mathematicians in the studied journals[12]. For the analytical mathematics and applied mathematics to mechanics, the choice in terms of education readings was strongly correlated to high level and research interests.

Also, this thematic sub-network fits quit well the personal and professional network: Anderson and Adrain taught at Columbia; Catlin and Avery worked at Hamilton College; Adrain and Gill, both editors in chief of *The Mathematical Diary* and *The Mathematical Miscellany* drove exchanges between this group of contributors; Bowditch, Strong and Adrain were members of the same learned

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[12] Mechanics and physics problems occupied 60% of all the questions in *The Mathematical Diary*, 19% in *The Mathematical Miscellany*, 0% in *The Cambridge Miscellany* and 14% in *The Mathematical Monthly* (Preveraud, 2014, pp. 363-367).

societies and published their works in the *Memoirs of the Academy of Arts and Sciences;* Bowditch was the tutor of Peirce, etc. This crossed approach indicates a close superposition of the social network and the thematic network, which tends to prove that diffusion of pure and applied analysis texts occurred in a community of mathematicians who knew each other and exchanged within the framework of their professional activities.

On the contrary, the introduction and circulation of Legendre's *Éléments de géométrie* came in a group of individuals (2) who were professionally, geographically and temporarily not or weakly connected. Plus, when they quoted Legendre, most of them only quoted him and did not refer to any other French publication. For example, none of them quoted modern geometry textbooks[13]. The lack of personal connections between these contributors has to be correlated with the good diffusion of Legendre's textbook in American teaching and publishing through its early translations (1818 and 1828). Those translations were mostly faithful to the original. Later ones (1834, 1844) transformed more deeply the French book contents and structure (Preveraud, 2014, Chapter 5 and Schubring, 2007, p. 49). They were widely used in colleges and high schools during the whole 19th century and became, as well as later French editions of Legendre's textbook, a source for other American geometry textbooks writing (Preveraud, 2014, Chapter 5).

Legendre's *Éléments de géométrie* was only quoted by teachers, whom four taught in secondary schools geographically isolated from each other: Hoyt worked in Massachusetts, Powers in Virginia, Richardson in Georgia, and Nulty in Pennsylvania. Men who quoted Legendre were not familiar with each other and worked at different moments of the time period 1818–1878. For example, Peirce directed *The Cambridge Miscellany* but Stevens wrote in Runkle's journal. The reputation and the large print-out of the text and of its translations that continued to be published for higher and secondary education even after 1850 enabled Legendre's *Éléments de géometrie* to be spread largely to individuals for their use in class without the support of a dense network like Lacroix's *Traité élémentaire de calcul différentiel et de calcul intégral* needed.

Concluding remarks

First textbooks and treatises references in American mathematical journals enabled early contributors to shorten their proofs in problem solving writings. Later authors used French education contents rather to bring out to American readers new mathematical contents produced in France. Thus, the form of quotations changed. At first only furtive references (author, title and concerned

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[13] Modern geometry treatises were quoted by mainly *Mathematical Monthly* writers: Collins, Newton, Wright, Henck, Brown and Hill.

pages), they took the shape of shorts translations: theorems and proofs were produced with pedagogical goals.

Transmission of French education contents through journals was strongly correlated to uses of the existing textbooks corpus in American education. Early (1818, 1828) and well diffused translations of Legendre's *Éléments de géométrie* facilitated its dissemination to isolated secondary schools teachers who widely used it in their communications. On the opposite, the absence of Lacroix and Bourdon's books quotations in journals must be associated with the decline of algebra textbooks publications only relying on pure French analytical methods. In case of a translation's lack (as for Lacroix's *Traité élémentaire de calcul différentiel et de calcul intégral*, Laplace's *Mécanique céleste* before 1829 or pure geometry works), journals were the only way to diffuse foreign new contents as shown in (Bret, 2012, p. 960), and transmission concerned a small group of teachers at colleges and their students, all professionally and personally related.

Thus, American mathematical journals did not produce, transmit and diffuse scientific contents independently of education context. Partly because some of their contributors were involved in teaching mathematics, they responded to the evolution of mathematical science in France but within the yardstick of scholars' needs and uses, approaches and tendencies of American textbooks publishing.

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Swedish mathematics curricula, 1850–2014. An overview

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Abstract

This overview study provides a detailed analysis of four aspects of the Swedish mathematics curricula from the period 1850–2014: general structure, time for mathematics, topics and finally knowledge and progression.

Introduction

This paper is about a study of Swedish national curriculum documents from the period 1850–2014 that concerned mathematics in year 1–9. The study is a part of the on-going research project *The development of School mathematics and reforms of the Swedish school system in the 20th century – a comparative and historical study of changes of contents, methods and institutional conditions.*

The paper's focus is on more structural aspects of the curriculum documents. I describe the length (number of pages and words) of the mathematics curricula, amounts of time for mathematics, different topics and how knowledge in mathematics and progression was expressed.

The relevance of knowing about these structural aspects is that it adds to a broader picture regarding attempts to reform school mathematics, for instance the introduction of New Math in Sweden in the late 1960s. What the people that led the introduction of New Math in Sweden, but also on an international level, wanted and longed for is one thing. What New Math meant for teachers is another question.

Answers to the latter question can be given if we consider more closely the official documents that were used to implement the reform(s); documents the teachers were obliged to consider. Moreover, taking the teacher perspective can also answer questions about why a reform was a success or a failure.

Short background about school types

The three main school types in the period 1850–2014 are considered: *Folkskolan*, *Läroverket* and *Grundskolan*. The facts about these schools in this section are from Larsson & Westberg (2011, pp 103–142).

 Folkskolan was formed in 1842 and comprised six years (1–6). However, the first official national curriculum document regarding mathematics appeared first in 1878. *Folkskolan* was intended for the lower classes and the vast majority of Swedish children and youths. By 1960, *Folkskolan* comprised at least seven years for those students who did not change to secondary education, but students could attend *Folkskolan* for up to nine years.

 Läroverket provided lower and upper secondary education. This school type stemmed from the mediaeval cathedral schools. In the beginning of the 20th century, *Läroverket* was divided in two parts: *Realskolan* (4–9) and Gymnasiet (9–12).[1] Throughout its existence, *Läroverket* recruited students mainly from the upper middle class and above. Only a small portion of children and youths enrolled. During the 19th century, this share never exceeded 4.5 percent. The greatest share was noted in 1965: ca. 12 percent attended Gymnasiet.

 In 1962, *Grundskolan* (1–9) replaced *Folkskolan* and the lower part of *Läroverket*. Due to the size of the reform, it was implemented in stages over a ten year period, beginning in 1962. The upper part of *Läroverket* – Gymnasiet – was replaced by Gymnasieskolan (10–12).

 One of the purposes of the *Grundskolan* reform was to avoid a situation where educational programmes were dominated by children and youths from certain social classes. For that reason, *Grundskolan* did not contain specialized programmes, like for instance theoretical or practical programmes. However, in years 7–9 the students could choose between different mathematics courses (basic and advanced).

Sources

The main material for this study is Swedish national curriculum documents from the period 1850–2014. The study is limited to the years 1–9 of the school system, which today corresponds to the ages 7–15.

 The source material comprises documents issued by the parliament or the government with the intention to regulate practice in all schools within *Folkskolan*, *Läroverket* or *Grundskolan* In this respect the curriculum documents were national.[2]

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[1] Students who went to Gymnasiet left Realskolan after year 8.

[2] All documents were issued by the secular state. This does not mean the church were not involved in the school system, indeed it were, but it did not run own schools or issue parallel curriculum documents. There are some differences in legal status of the investigated documents, where the most important difference concerns the documents regarding *Folkskolan*. Before 1919,

The study is further restricted to texts that described the contents of courses and the teaching methods. These texts are denoted course plans.

Furthermore, texts issued by the central school authorities with the intention of explaining the course plans have also been studied. This type of texts is denoted commentary material.

For the sake of brevity, the term curriculum is used to denote course plans and commentary material.

Throughout the period 1850–2014, the national curriculum documents comprised general parts and specific parts for each subject. The general part I denote the general curriculum. The specific parts about mathematics I denote the mathematics curriculum.

In 1955, both *Realskolan* and *Folkskolan* were given new curricula. Since *Grundskolan* was introduced over a ten year period, beginning in 1962, they were applied in schools waiting to be reformed. In this study, I have not included these curricula. Nor have I included the curriculum documents that were used in the test schools preceding the *Grundskolan* reform.

The reason for this choice is that I consider the first curriculum of *Grundskolan* as a final product, where the curricula of the 1950s were temporaries or pilot projects leading up to this product. The main concern of this paper is to give an overview of the period 1850–2014 and then it is the final products that are interesting, not the processes that preceded them.

The general structure of the mathematics curricula

Before the 1950s, the mathematics course plans of *Folkskolan* and *Realskolan* were relatively short. On average, the contents of all years were described in not more than two pages. In the mathematics course plans of *Folkskolan*, another two or three pages was spent on recommendations regarding teaching methods. The course plans of *Realskolan* did not contain such recommendations. Neither did the curriculum contain general parts on teaching methods.

Actually, the mathematics curricula of *Folkskolan* comprised only course plans, no commentary materials. This was also the case for *Realskolan* until 1935 when commentary material on mathematics instruction was issued.

The *Folkskolan* mathematics curriculum of 1919 (1–7) covered contents and teaching methods in approximately 1800 words. Contents were described with about 700 words (2 pages) and teaching methods with about 1 100 words (2.5 pages). The 1919 curriculum was in effect until 1955.

In the mathematics curriculum of *Realskolan* of 1933, in effect until 1955, the contents of the mathematics courses (year 5–9) were described in

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the *Folkskolan* curriculum documents contained recommendations rather than directives. Local school authorities were supposed to care for the latter.

approximately 240 words. The new commentary material, covering teaching methods, comprised approximately 4 200 words.

In contrast, the mathematics curriculum of 1962, covering year 1–9, was longer, in total approximately 10 500 words. It described the mathematical content, planning and teaching methods and eventually how, in what order and to what extent each mathematical topic ought to be treated. However, this section covered not only more years, but also two separate course tracks (basic and advanced) in year 7–9.[3] The mathematics curriculum of 1962 contained just a course plan, no commentary material.

This type of lengthy curriculum document was not exclusively for mathematics, but for all subjects and other facets of everyday life in the schools. The lengthier curriculum documents can be seen as a reflection of the central school authorities to centralize the power over the school system (c.f. Oftedal Telhaug, Mediås & Aasen 2006, pp. 255–256).

In 1969, *Grundskolan* was given its second curriculum. It included two sections on mathematics: a course plan (about 3 000 words) and commentary material (about 7 000 words). Moreover, the New Math was formally introduced to Sweden through the 1969 curriculum. The general structure of the mathematics curriculum was also similar to that of 1962: they contained descriptions of the mathematical content, planning and teaching methods and in what order the content ought to be treated.

This general structure of the mathematics curriculum was kept in the third curriculum of *Grundskolan* that took effect in 1980, but the course plan was considerably shorter: about 1 900 words. But the commentary material, a 60–page booklet, was considerably longer: about 14 300 words.

The status of the commentary material of 1980 was somewhat ambiguous if we consider its explicit intentions in the foreword as well as in the introduction. On one hand, it should clarify and concretize the content of the course plans. Thus, the directives of the course plan were further explained. On the other hand, it was explicitly established that the material did not contain any regulations or prescriptions; it should rather be used as a support and a basis for discussions when the teachers planned their teaching.

In contrast, in the commentary material of 1969 it had been established that it contained additional instructions, comments and examples concerning the course plan.

The new intentions of the 1980 commentary material can be seen as a reflection of a protracted process of decentralization. Since the middle of the 1970s, leading politicians declared the need to decentralize the Swedish school system, giving more power to local authorities, schools and teachers (Oftedal Telhaug, Mediås & Aasen 2006).

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[3] This division in basic and advanced mathematics courses in year 7-9 remained until the 1994 curriculum. The purpose of the advanced course was to prepare the students for later studies that contained more mathematics.

A second change that can be tied to this ambition is that the course plans of 1980 did not contain year-by-year descriptions. Instead, the content of three periods was described (1–3, 4–6 and 7–9).

The fourth curriculum of *Grundskolan* appeared in 1994. This time, the content received even shorter descriptions. For each subject there were descriptions of what the students ought to know and master in year 5 and 9 (the last year of *Grundskolan*). Moreover, teaching methods were not mentioned at all. Instead, the course plan for each subject contained lengthy sections on motives and purposes for the subjects. Such sections were not included in the previous course plans.

The mathematics curriculum of 1994 was divided into two works: a short course plan (about 1 100 words) and a commentary material (approximately 14 000 words). Both had the same structure: lengthy sections on motives and purposes of school mathematics, descriptions of what the students should master in year 5 and 9 and very little on teaching methods.

The function of the 1994 commentary material in relation to the course plan was more clearly described than in 1980. The commentary material was not binding, but should provide support for the teachers.

The 1994 mathematics curriculum also contained things that had never appeared in previous mathematics curricula. One novelty was sections on assessment and how it was supposed to be used in order to support the students as well as the teachers' own work.

A second novelty was a lengthy appendix in the commentary material (about 5 200 of the 14 000 words) regarding all previous mathematics curricula.

A third novelty in the 1994 commentary material was a lengthy list (11 pages) of references to other school regulations, central school board reports, articles in teacher journals and scientific works. In the 1980 commentary material there was a similar but shorter list (less than 1 page) and mainly references to works from the central school board or teacher journals.

The curricula of 1994 can also be seen as a reflection of the on-going decentralization process. In comparison to previous curricula it said very little about teaching methods and when certain topics should be taught and in what order; these issues were left to the teachers to decide. These changes are linked to a central administrative principle behind the 1994 curriculum: the state should govern schools and teachers through goals and assessments, not through regulations about how to plan and execute the teaching (Oftedal Telhaug, Mediås & Aasen 2006).

The fifth curriculum of *Grundskolan*, the most recent, was introduced in 2011. This time the mathematics curriculum had two parts: a course plan (about 3 000 words), which was together with the general parts and the other subjects, and the second part as a separate commentary material (a booklet, about 13 300 words). In total it comprised about 16 300 words, an all-time high.

The 2011 mathematics curriculum, as well as the whole curriculum, was based on the same administrative principle as the 1994 curriculum. This meant

that very little was said about teaching methods; the focus was on purposes, content and assessment, even though there was some minor changes. The purposes of mathematics education contained fewer paragraphs, while assessment received considerably more paragraphs. The content was given a more extensive treatment as well. The contents of three stages were described, (1–3, 4–6 and 7–9). However, to what extent and in what order different topics should be treated within these stages were not described. Two more significant changes were the cancellations of the sections on the history of mathematics education and the list of references.

The intention of the 2011 commentary material was also somewhat different. The explicit purpose was to give a broader and deeper understanding of selections and standpoints behind the course plan.

A common feature of all mathematics curricula from 1980 and onwards, is that the number of words has been constantly about 50 percent higher than in two mathematics curricula of the 1960s. Thus, the national mathematics curriculum has remained a possible way to gain national influence over mathematics instruction, despite the ambitions of decentralization.

Time for mathematics in the curricula

In the curricula before 1980, time for teaching each subject was prescribed by means of a system with so-called lesson hours per week. For each year of study, the curriculum prescribed how many lesson hours in mathematics the students were supposed to have per week. In the 1980 curriculum, a slightly new system was applied. It prescribed the number of lesson hours over three three-year periods.

From the 1994 curriculum, the concept lesson hour system was abandoned; instead the total number of 60 minute hours for all nine years (total time) was given. There were no directives about how these hours should be distributed over the nine years.

In order to make comparisons, the total times in the pre-1994 curricula have been calculated, see Table 1 below. This was done by multiplying the length of the years of study (number of weeks), the number of lesson hours per week and the length of the lesson hours (number of minutes). The products were divided by 60 in order to express time in 60 minute hours. Total time was obtained by adding the numbers of 60 minute hours intended for mathematics teaching per year of study.

It is noticeable that the unit lesson hour has never been set to 60 minutes in the curricula. The stipulated length has varied between 40 and 55 minutes during the investigated period. Even in the same curriculum it could vary, but then only by five minutes, depending on when during the day a lesson was given. This variation in the same curriculum is the reason for the max and min numbers in Table 1 below.

The numbers for total time before 1994 is a bit imprecise since the number of lesson hours could fluctuate from year to year depending on whether schools days with mathematics lessons were holidays or not. My calculations do not take into account holidays that occurred on regular schooldays. Therefore, the difference between the 1980 and 1994 curricula is a bit smaller than indicated in Table 1.

Table 1. Time for mathematics[4]

	Folkskolan				Secondary schools, year 4-9						Grundskolan				
	1878	1889	1900	1919	1856	1859	1878	1905	1928	1933	1962	1969	1980	1994	2011
Year	Number of lesson hour per week														
1	3,5	2,5	4	3							4	4			
2	3,5	3	4	4							4	4			
3	6	4	5	4							5	5	13		
4	6	4	5	5	5	8	4	4			5	5			
5	5	4	5	5	5	5	5	5	5	4	5	5			
6	5	4	5	5	6	7	5	5	5,5	5	5	5	15		
7				5	6	8	5	5	4	4	4	4			
8					6	7	5	4	5	3	4	4	12		
9					6	7	7	5	5	4	4	4			
sum	29	22	28	31	34	42	31	28	25	20	40	40	40		
Number of study weeks per year															
	34/30	32	32	37,5	36	36	36	38	38	38	39	39	40		
Length of lesson hour (minutes)															
max	50	50	50	45	55	55	55	45	45	45	45	45	40		
min	50	45	45	45	50	50	50	45	45	40	40	40	40		
Total time (h, 1h = 60 min)															
max	748	573	747	872	1122	1386	1023	798	698	570	1170	1170	1067	900	900
min	748	516	672	872	1020	1260	930	798	698	507	1040	1040	1067	900	900
Average time (h/year)															
max	125	96	124	125	187	231	171	133	140	114	130	130	119	100	100
min	125	86	112	125	170	210	155	133	140	101	116	116	119	100	100

Topics in the mathematics curricula

Before the 1960s, there was little variation in the topics of *Folkskolan*. Reckoning ("räkning") and geometry ("geometri") were main topics. Actually, until the 1919 curriculum, reckoning and geometry were separate subjects. In 1919 the single subject reckoning and geometry was formed. The term *mathematics* ("matematik") was never used in *Folkskolan* curricula, neither was the term *arithmetic* ("aritmetik"). In all curricula, reckoning was a topic from year one and onwards; geometry was introduced in year five.

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[4] Regarding the secondary schools of 1856, 1859 and 1878, the numbers presented in Table 1 concern course programmes with most mathematics lessons. These curricula contained a classical and a realistic course programme for year 4-9. The numbers in Table 1 concerns the realistic programmes. The curricula of 1905, 1928 and 1933 contained only a realistic programme in year 4-9. Also in the 1962 curricula there were course programmes with different amounts of mathematics lessons in the last year. The numbers concerns the programmes with the largest number of lessons. The number of study weeks per year in *Folkskolan* in 1878 is given as 34/30. 34 concerns the two first years, while 30 concerns the last four years

In all curricula of *Realskolan*, on the other hand, the term *mathematics* was used for one single subject. Arithmetic, geometry and algebra were the main topics. In which years the topics were supposed to be taught varied somewhat, but mainly, arithmetic was supposed to be taught in year 4–6, algebra in year 7–9 and geometry in year 6–9. It might be that the boundaries between arithmetic and algebra changed over time, but there might also have been an overlap between the topics, since in the curricula of 1859 and 1878 arithmetic was supposed to be taught until year 9 (1859) and later on year 8 (1878).

In the first mathematics curricula of *Grundskolan*, introduced in 1962, all these topics were kept, arithmetic in year 1–9, geometry in year 4–9 and algebra in year 7–9. Until then the mathematics courses comprised these three topics.

For each new mathematics curricula from 1969 to 1994 we can observe changes in how the mathematics courses were divided into topics. Some topics were all new, for instance computing machines, statistics and probability and functions. Some can be considered subtopics of the old topics arithmetic and algebra, e.g. natural numbers, decimal numbers and equations.

Table 2.

1969	*1980*
Natural numbers (1-7)	Problem solving (overarching topic) (1-9)
Measurement (1-9)	Elementary arithmetic (1-9)
Geometry (1-9)	Real numbers (1-9)
Decimal numbers (4-8)	Percentage (4-9)
Rational numbers (4-9)	Measurements and units (1-9)
Negative numbers (4-8)	Geometry (1-9)
Counting machines (7-9)	Algebra and functions (1-9)
Statistics and probability (2-9)	Descriptive statistics and probability (1-9)
Functions (6-9)	Computer (7-9)
Real numbers (7-9)	
Equations (1-9)	
Mathematical models (9)	

The mathematics curriculum of 1994 contained fewer topics. Arithmetic replaced an array of subtopics and the only topics that disappeared were problem solving and computer. Still, the ability to solve problems was mentioned in the 1994 mathematics curriculum, but the concept was not further developed and cannot be considered a specific topic.

However, problem solving appeared again as a topic in 2011. Apart from that and some changes of names, the mathematics curriculum of 2011 brought little changes when it comes to topics.

Table 3.

1994	*2011*
Arithmetic (1-9)	Number sense and the use of numbers (1-9)
Geometry (1-9)	Algebra (1-9)
Statistics (1-9)	Geometry (1-9)
Algebra (1-9)	Probability and statistics (1-9)
Probability (6-9)	Relations and changes (1-9)
Functions (6-9)	Problem solving (1-9)

Of course, this brief overview regarding topics does not say much about the content of each topic and how it has changed over time. Still, thanks to the long time span, the overview shows a clear trend in how the mathematics curricula have changed, especially since the late 1960s.

The trend is that topics or subtopics that before were treated in later years, became parts of the mathematics courses from year one, e.g. geometry, equations (algebra) and eventually also functions. In addition to that, statistics and probability has become a part of the curricula from year one. Consequently, the prescribed school subject mathematics has become more versatile, especially in the early years.

Given that the times for mathematics teaching have been almost on the same level since the 1950s, see the forgoing section, something needs to have been removed from the mathematics curricula. The curriculum documents do not contain any numbers about the relative size of each topic with respect to teaching time, but there are some clues to at least major changes.

When it comes to year 1–9, arithmetic seems to have lost ground in the sense that less time should be spent on complex computations. Since the 1960s, the curriculum documents contain directives about how less time should be spent on such computations, especially when calculators became available.

Geometry is the second topic to have lost some ground, especially in year 7–9. However, this changed already in connection to the 1962 curriculum. Much less time was supposed to be spent on proofs and other theoretical aspects.

Knowledge and progression in mathematics

The ideas of knowledge and progression can be conceived as essential in a context of teaching and learning. We wish that the students' knowledge develops due to teaching, and they should make progress over time. These ideas clearly permeate today's Swedish curriculum, but has it always been like that? The basic answer for the period 1850–2014 is yes, which is about to be apparent. However, the ways to express knowledge and progression has changed, but not in a distinct direction.

This section is restricted to arithmetic and the general parts of the mathematics curricula that concerned all topics. The reason for this is that arithmetic is the only topic that has been a major topic in all three school forms over the whole period. Thus we may follow changes between each curriculum.

In the analysis, I have considered only those sections of the mathematics curricula that focused on descriptions of what the students were supposed to learn, i.e. the content of teaching. The reason for this is that the same content was treated in for instance sections on teaching methods.

Two aspects of how the curricula have expressed knowledge and progression are considered: mathematical concepts and abilities linked to mathematical concepts. The notion mathematical concepts refers to

mathematical objects like for instance numbers in arithmetic and lines in geometry, but also operations on and relations between objects. The notion of abilities concerns what the students were supposed to do with the mathematical concepts, e.g. reckoning, understand them, use them for communication or apply them in some context.

The analysis was guided by the following questions: a) what terms were used to express mathematical concepts? b) what terms were used to express abilities? c) how were concepts and abilities structured?

All terms have been entered into an Excel sheet for each curriculum document. In order to use excel sorting functions and to eliminate doublets, the endings of some terms have been altered.

The period 1850-1950 – abilities and mathematical concepts

During this period the number of unique terms to describe the content of the arithmetic courses was relatively stable. Not many unique new terms were added. This applies for mathematical concepts as well as abilities.

In the *Folkskolan* curricula the term *four rules of reckoning* was used and of course terms for each rule. In Swedish the terms were "addition", "subtraktion", "multiplikation" and "division". Non-Latin terms were in parallel use, e.g. "fråndragning" (take away). Rule of three, was mentioned only in 1878.

Apart from the operations, two types of numbers were described: whole numbers and fractions. The latter were divided in general fractions and decimal fractions. In 1919 the term *percent* appears for the first time in *Folkskolan*. Numbers and numerals were further described by terms like *multiplier*, *divisor*, *number ranges* and *four digit numbers*.

These terms expressed progression when they were used to describe which exercises the students should work with. The curricula successively allowed more complicated exercises by expanding the range of numbers, 0–10, 0–100 and allowing factors, divisors and multipliers with more digits. A second way to express progression was to allow more than one type of arithmetical operation in the solution of an exercise. Terms for abilities did not express progression.

Regarding the terms for abilities the most common term was *reckoning*. There were two subcategories: mental reckoning and written reckoning.

The term *apply* indicated that reckoning should be used with certain types of numbers, but also outside school in various contexts.

Apart from terms directly connected to reckoning, there were a few terms that referred to cognitive abilities (*to apprehend numbers*) and communicative abilities (*to denote and to name numbers*).

In the 1919 mathematics curricula, two new terms appeared that would remain in later curricula: *skills* and *insight*.

In comparison, the descriptions of the content of the arithmetic courses in the curricula of the lower part of *Läroverket* were different. No terms for

operations were used; the courses were about arithmetic and four rules of reckoning. In the years parallel to *Folkskolan* (mainly year 4–6), the terms *whole numbers fractions* and *decimal fractions* were used.

Rule of three was included in year 6 in every curriculum except for 1905 and 1928. The term *percent* appeared for the first time in 1905, then in year 6.

For the years 7–9, the curricula said very little about arithmetic beyond the term *arithmetic*. Before 1905, the terms *roots* and *greatest common divisor* were also used, but then only in 1856. After 1905 the term *square root* was used, and in 1905 and 1933 the term *irrational numbers* was used as well.

Another difference was fewer terms for abilities: just *reckoning*, *mental reckoning* and *apply*. This concerned all years of *Realskolan*. A third difference is that the *Realskolan* curricula expressed progression in arithmetic only by new types of numbers: whole numbers, fractions, roots and irrational numbers.

The period 1962-2014 – mathematical concepts

A majority of the terms used to describe the mathematical concepts in arithmetic courses of *Folkskolan* and *Realskolan* remained, or were replaced by terms quite similar in meaning, in all mathematics curricula of *Grundskolan*. An example of terms with similar meaning is the term *general fraction*; it was replaced by the terms *fraction concept* and *fraction form*.

Decimal fraction disappeared, however, and instead *decimal numbers* was inserted. *Rule of three* disappeared as well.

A few terms that were new in the 1960s remained in all ensuing curricula. This was *natural*, *rational* and *real numbers*. These terms denote general categories for numbers and they did not exclude concepts in previous curricula.

Taken together, the terms about mathematical concept mentioned so far, or terms similar to them in meaning, can be considered a core of terms in the arithmetic courses during the period 1850–2014. The only exception is terms that described which exercises the students should work with, which remained only in the first curricula of *Grundskolan* of 1962.

Actually, the 1962 curriculum contained an even greater set of this type of terms, e.g. *tens*, *hundreds*, *one digit factor*, *whole number divisor*, *fraction divisor*, *transition from decimal form to fractions*. However, some new terms referred just to properties of numbers rather than aspects of exercises and calculations, e.g. *odd*, *even*, *prime* and *divisibility*. Surprisingly, these terms disappeared after the 1962 curriculum.

Thus, the curriculum of 1962 pursued the way of the previous curricula of *Folkskolan* in how to express progression. A difference from before is that this was done also in year 7–9. Remember that the curricula of *Realskolan* did not express progression except for new types of numbers: whole numbers, fractions and irrational numbers.

By the 1969 curriculum, almost all terms that were used to describe different types of exercises disappeared, which included several terms with origin in the *Folkskolan* curricula. This coincided with a new way of expressing progression.

The focus shifted from calculation exercises the students ought to handle to concepts the students needed to understand in order to calculate.

The description of the content was also structured in a new way. As already mentioned, this curriculum contained more explicit topics. The relevant topics for arithmetic were natural numbers, decimal numbers, rational numbers, negative numbers and real numbers.

Under each topic so called "moments" were listed, mainly mathematical concepts; the course plan suggested in which grades each moment should be taught. *Addition*, *subtraction*, *multiplication* and *division* were general "moments" under each topic. Specific "moments" under natural numbers were for instance *greater/smaller than*, *equal to*, *number system with other bases than ten*, *commutative, associative and distributive laws* and *sets*; under decimal numbers we find *exponents*; under real numbers we find *number line* and *order*.

The way to express progression changed again in the curriculum of 1980, back to the order of 1962. Consequently, there were more terms to describe the exercises. New terms this time were *multi digit*, *equal denominator*, *two decimals*, and *multiplication of two negative numbers*. In comparison, this set of terms was much smaller than in 1962. One reason for this is that the 1980 curricula just prescribed guidance regarding the last years of three periods (1–3, 4–6 and 7–9), whereas in 1962, the content of each 9 year was prescribed. But even so, the 1980 mathematics curriculum was in general also less detailed.

With the 1994 curriculum, the character of the mathematics curriculum changed once more. This time, the course plan was briefer. It prescribed what the students were supposed to master in year five and nine. The terms used to describe the content barely transcended the core terms mentioned above. *Proportionality* and *number pattern* were the only new terms.

Progression in arithmetic was expressed quite briefly as well, the main difference between year five and nine concerned rational numbers. In year five, the students should *handle* so-called *simple fractions* and *numbers in decimal form*. In year nine, the students were supposed to *handle rational numbers in general*. They should also *handle percent* and *proportionality*.

In the 2011 curriculum, the description of the mathematical concepts was a bit longer than in 1994, but few new terms were added. Progression in arithmetic was also expressed in a quite simple way, at least with respect to terms regarding mathematical concepts. The mathematical curriculum described the central content of the three stages: year 1–3, year 4–6 and year 7–9. The concepts and calculations that should be treated in year 1–3 mainly concerned natural numbers. In year 4–6: rational numbers. In year 7–9: real numbers.

In the 2011 curriculum an innovation was the so-called knowledge requirements. They specified what the students should know in year 3, 6 and 9. In year 6 and 9, these requirements were linked to a grading system. For year 3, the requirements also contained conditions regarding the exercises, but since the same was not done for year 6 and 9, no progression was expressed.

The period 1962-2014 – abilities

In the case of terms regarding abilities we can discern a core from 1850 to 2014. Reckoning was one of the core abilities. There were two types or modes of reckoning: mental and written. Aside from reckoning there were terms for use or apply. Until the curricula of 1962, the Swedish term was "tillämpa". After that, the most used Swedish term was "använda". The related term *skill* ("färdighet") was also used in the mathematics curricula until 2011.

Apart from this core, the descriptions of the content in the mathematics curricula during the period 1962–2014 also brought new terms.

Estimate reckoning, a third type of reckoning, was a term that was introduced in the 1962 curriculum and it remained in all subsequent curricula.

The 1962 curricula also added types of terms that refer to what I denote as generic non-specific abilities. They are generic in the sense that they were used in several subjects, not only mathematics. They are non-specific in the sense that the terms do not indicate a certain action. For instance the term to understand can be used in several contexts (generality); moreover the term does not reveal what you can do when you understand something (non-specific).

Table 4. Terms for generic non-specific abilities, 1962–2011

1962	*1969*	*1980*	*1994*	*2011*
capability		capability	capability	–
familiarity	familiarity		familiarity	
insight	insight		insight	
	understand	understand	understand	
		master		

In relation to the terms regarding reckoning, the terms in Table 4 constitute an important complement, especially the terms *insight* and *understand*. Knowledge in reckoning was more than being able to reckon. The use of these kinds of terms, e.g. *insight*, we also find in the *Folkskolan* curriculum of 1919, so the use of this kind of terms was not something unheard of in 1962.

The disappearance of this kind of terms in the 2011 curriculum is probably related to a much richer use of more specific terms, see Table 5 and 6.

Beside the terms for the generic non-specific abilities, the curricula of the period 1962–2014 also brought specific terms concerning abilities. They are specific in the sense that they indicate an action. However, they are still generic since they can be used and make sense in other contexts. I discern two types of terms: generic cognitive and generic communicative. The terms of both these categories signify something that requires cognition in some way. Yet, the latter category of terms also contains a link to interpersonal communication. Thus, the latter is a subcategory of the former.

Table 5. Terms for generic cognitive abilities, 1962–2011

1962	1969	1980	1994	2011
–	–	analyze	create models	analyze
		compare	discover	evaluate
		consider	draw conclusions	follow reasoning
		draw conclusions	evaluate	plausibility assessments
		cvaluate	examine	select
		take a stand	generalize	to reason

Table 6. Terms for generic communicative abilities, 1962–2011

1962	1969	1980	1994	2011	
express	–	discuss	argue	account for	discuss
present		speak	explain	argue	express
write			formulate problems	describe	formulate problems
				designate	specify

Many of these terms appeared in the parts of the mathematics curricula that applied for all topics, but it added to the idea that arithmetic is more than reckoning. Thus, the subject mathematics became not only more versatile in the sense of more topics, also the descriptions of what it meant to know something in each topic became more versatile, especially from 1980 and onwards.

In that perspective, the 1980 mathematics curriculum stands out, since it contained a structure regarding the generic abilities. This structure appeared in connection to a certain topic: problem solving. According to the curriculum, efficient problem solving involved three steps: 1) to understand the problem and have a method to solve it; 2) to master the required numerical calculations; 3) to analyze, evaluate and draw conclusions from the result.

As a topic, problem solving had not only a prominent position early on in the course plan, but also a superior status: it should be a part of all other topics. Thus, the three steps not only ordered generic abilities (understand, analyze, evaluate and draw conclusions); the generic abilities were also linked to reckoning and other topics. We should also note that a great part of the abilities in the 1980 curriculum was linked to problem solving.

In the 1994 curriculum, problem solving disappeared as a superior topic and it became an item among others. Its superior status returned in 2011, but it was mentioned quite a way into the commentary material. Moreover, the abilities involved were not arranged in an equally clear manner as in the 1980 curriculum. For example, the problem solving process was not described in steps. Moreover, the 2011 curriculum contained much more terms regarding abilities that were not related to problem solving. Thus, problem solving did not structure abilities and mathematical content as in the 1980 curriculum.

Closing remarks

The general argument for a study like this is that it provides information about a relatively long period. We discern continuities and discontinuities, things that have been stable over time and things that been shifting.

However, we must remember the sources at hand (just curriculum documents) and the quantitative approach (counting words, counting lesson hours and creating lists of terms). Obviously, this study provides little understanding of how the meanings of the terms may have changed over time.

Another aspect not being treated in this study is the purpose of the mathematics courses and how changes in purpose were linked to changes in terms regarding the content. A closely related issue is the social origin of the students as well as the teachers. A central circumstance in that respect was the existence of a parallel school system (*Folkskolan* and *Läroverket*) and the abandonment of it in the 1960s, i.e. the introduction of *Grundskolan*. It would be interesting to compare changes in the curriculum documents, with changes in recruitment of students as well as teachers.

Nonetheless, this study constitutes a basis for further more qualitative studies of for instance textbooks and teacher journals, but also for more sociological questions. This overview gives a good idea about what concepts in mathematics education to focus on. Especially since curriculum documents, most likely, were something teachers and textbook authors did not ignore.

The more specific relevance of this study pertains to the fact that no previous scientific works on the history of Swedish mathematics education contains an overview of the national curriculum documents.[5] There is, however, an overview in the commentary material of 1994. The difference, content wise, is that the overview presented here contains more details concerning general structure, time, topics, knowledge and progression; whereas the 1994 overview is more general and contains parts about teaching methods and intentions.

The relevance of having a narrower scope and a higher level of details is that it adds new perspectives on attempts to reform school mathematics. Not at least how reforms posed new demands on teachers. Take for instance the major school reforms in the 1960s and the introduction of New Math.

New Math was not just about concepts and terminology related to set theory. With the same reform, the teachers got even more to handle: new topics and a new way of expressing progression. These features of the introduction of the New Math in Sweden have not been observed in previous research (c.f. Kilborn 1977, Prytz (2012).

The study presented in this paper also shows that the aftermath of the New Math in the 1980s was not just a question of going back to basics, even though

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[5] See for instance the two more comprehensive works on the history of Swedish mathematics education: Prytz (2007) and Lundin (2008).

we can observe such tendencies. From 1980 there is a clear increase in terms regarding abilities, but also clear variations in these terms between the curricula.

My study also indicates that it was only in the 1980 curriculum that terms regarding abilities were related to the mathematical content in a more clear and structured way. This was due to the superior position of problem solving in the mathematics curricula, especially in the course plan. This conclusion might be a result of the fact that the analysis is focused on descriptions of the content in the course plans. In the commentary material of 1994 and 2011, terms regarding abilities were linked to mathematical content. Even so, there is a difference in clarity and in how things were put in the foreground or not.

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Academics, textbooks and reform of mathematics education in secondary French schools (1890–1905)

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Abstract

At the turn of the 20th century, the French secondary education of mathematics faced two issues. On the one hand it had to find its right place in the secondary education dominated by the classical literary studies. On the other hand it had to take into account some of the latest developments of mathematical research. This paper proposes a study of a set of textbooks which is representative of this will of renewing mathematics education. This study has been conducted with the reforms of the early 1890s as a background, reforms which caused new textbooks to be published. These textbooks expressed both the will of reviewing the mathematics principles and the will to incorporate elements of the most recent theories in the mathematics curricula. They were written by important members of the French mathematical community of which some campaigned to reform the educational system. To conclude, this paper will propose indications on the influence of these books on the following curricula of the mathematics in secondary education.

Introduction

At the end of the 19th century, the French secondary education for boys[1] was divided between what was called two orders of education: a classical education and a modern education. The first one, the most prestigious, was based on the study of the Greco-Latin humanities. It only gave a restricted part to scientific education. The latter one was more developed in the modern education but was marked by a utilitarian aspect. The study of the more theoretical part of mathematics was kept for the elementary *classe de mathématiques* which belonged to the classical education.

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[1] The first law organizing a secondary education for girls was passed in 1880. This education had neither the same duration nor the same curricula and was not given in the same places.

The promoters of scientific education campaigned for an increase of scientific studies. Yet at the end of the 1880s, the defenders of literary studies prevailed. The major part of scientific studies was rejected in the last year of classical education. Thereafter, a new Minister of Public Instruction, Léon Bourgeois (1851–1925), achieved in 1890 and 1891 two successive reforms which had important consequences on the two types of education, especially for scientific instruction.

At the same time there were attempts by mathematicians to adapt the mathematics secondary education to the new epistemological context of this period. The reconstruction of the analysis on arithmetical bases (cf. Boniface, 2003 and Dugac, 2002), the discovery of functions without derivative, the invention of non Euclidean geometries allowed developments in mathematics. This had also caused new questionings about the founding principles of mathematics.

It is in this dual background (institutional an epistemological) that this article analyses a set of textbooks published in the 1890s under the direction of the French mathematician Gaston Darboux (1842–1917). Written after Bourgeois' reforms it is representative of this will to renew mathematics secondary instruction.

The question of scientific education in 1890-1891

As in many European countries, the French educational system was built up during the 19th century (cf. Prost, 1968 and Belhoste, 1995). The secondary instruction was based on the learning of Latin and Greek and was meant for the social elite[2]. This education took place in the *lycées* created by Napoléon in 1802, a network of about a hundred state-run schools in the whole country in 1890[3]. The best ones, the most prestigious, were located in Paris.

The secondary education led to diplomas called *baccalauréats*. The *baccalauréat ès lettres* was "the crowning achievement" of literary studies. The best way for scientific studies was the *baccalauréat ès sciences*. It allowed access to *mathématiques speciales*, an advanced mathematics course which trained for the competitive examinations for admission to prestigious schools as *École Polytechnique* and *École Normale Supérieure*. The education which led to these two *baccalauréats* was called classical education. The teachers of the highest grades of classical education mostly came from the *École Normale Supérieure* and had passed the *agrégation*, a very selective competitive examination. This school became one of the most prestigious French schools throughout the 19th century. It trained some of the most distinguished French scholars, especially in sciences after 1865.

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[2] There were about 160 000 pupils in the secondary schools at the end of the 19th century, some five per cent of the concerned year-classes. The statistics of this paragraph come from *Enquête sur l'enseignement secondaire (III)*.

[3] There was also a network of private schools which concerned about 50% of the pupils.

In addition to this classical education, *L'enseignement secondaire spécial* was created to meet the needs of the modern society born from the industrialization process of the nation (cf. Day, 1972–1973 and Delesalle, 1979). Modern education came into being in the 1830s and was formalized in 1865 by the Minister of Public Instruction Victor Duruy (1811–1894). For the latter this education had to become the secondary education for the people.

This modern education was based on the study of French, foreign languages and sciences and had a utilitarian aim. It was to train the technicians and the executives the nation needed. Pupils of *enseignement spécial* studied in the same *lycées* as pupils of classical education. But they did not have the same teachers and their diploma was not the prestigious *baccalauréat*. For training the teachers of *enseignement séecial* Victor Duruy created another *École Normale* and an *agrégation de l'enseignement special*. This modern education had an unquestionable success among the lower-middle class[4] but it was considered as second-class teaching.

After the fall of Napoléon III the Republicans came into power at the end of the1870s. Education was a priority after the traumatic defeat of 1870 against Germany. In 1880 they increased the part of sciences in secondary classical education. They also reformed *enseignement spécial*: the length of the studies was increased from four to six years, the final diploma became a *baccalauréat* and the curricula became less utilitarian. These reforms gave rise to reactions from defenders of classical education. A problem of competition between modern education and classical education was evoked and some spoke about a threat against the moral unity of the nation (cf. Marion, 1889). The increase of scientific studies in classical education was questioned. In 1885 and in 1890 the number of hours for scientific subjects decreased. Scientific education was at its lowest level in classical education at this period. However, the question of the "duality" of secondary education still remained (cf. Hulin, 2005).

In 1890 Bourgeois was appointed *Ministre de l'Instruction Publique*. Supporter of a scientific humanism (cf. Hulin, 2011), he began to reform the *baccalauréat*. But his main reform was in 1891 the reform of *enseignement spécial*. He changed its name in *enseignement moderne* and created three diplomas for this education: "Literature-Philosophy", "Literature-Sciences" and "Literature-Mathematics" (Figure 1). The latter one was the same for the two orders of education. The opponents of the reform expressed their fears of a decrease of the level of classical education or of a flight towards modern education of many pupils interested in scientific studies.

Ultimately Bourgeois suppressed the *agrégation* of modern education. Teachers of the two orders of secondary education became the same. Thus Bourgeois could say that he constituted, without Greek or Latin, a system of classical instruction. However, we note that the length of studies was still only of six years for modern education, one year less than for classical education.

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[4] 45% of the pupils of secondary public education in 1879 were in the modern education (these figures do not take into account pupils of elementary mathematics).

Figure 1: The secondary educational system after the reforms of 1890, 1891[5]
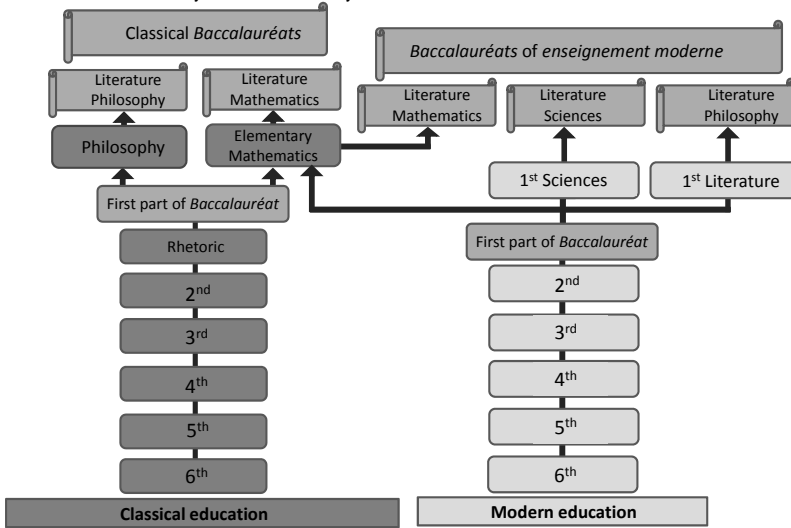
Figure 2: Effects of Bourgeois' reforms on the *baccalauréats* (the vertical axis gives the number of successful candidates at the examination of July[6])
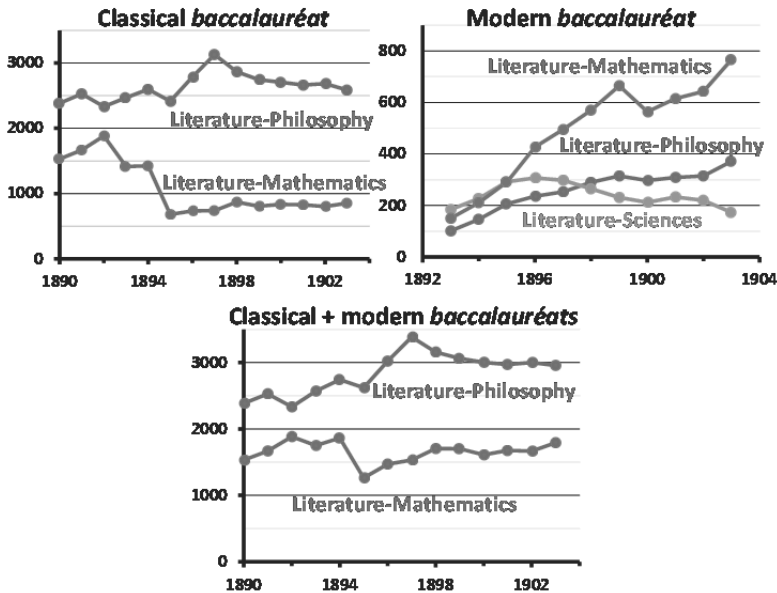
The results of Bourgeois' reform on the *baccalauréat* were those which had been expected (Figure 2). After a few year of transition period, there was a decrease

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of the number of scientific classical *baccalauréats*. On the other hand, we can see in particular the increasing importance of the "Literature-Mathematics" *baccalauréat* when we study the results of the new modern education diplomas. And when we combine these data, it appears that the number of scientific diplomas remained relatively stable over this period. In a decade there was a transfer from *baccalauréat ès sciences* to modern *baccalauréat* of about fifty per cent of the number of successful candidates. All were pupils of elementary *classe de mathématiques*, a grade of classical education.

In a way we can say that modern education modernized the elementary *classe de mathématiques* of classical education. This must have certainly played its part in the reform of 1902 which unified secondary education. The scientific education and especially mathematics education appears as an important factor of this unification[7].

New teachings for the new mathematics of the 19th century

The reviewing of the different curricula was one of the consequences of these reforms. This coincided with a will of renewal of mathematics teaching. This will was expressed for years by some members of the mathematical community directly involved in the educational system.

For example, in 1874, Charles Méray (1835–1911) published a textbook intended to secondary education *Nouveaux Éléments de Géométrie.* This mathematician was professor of differential and integral calculus at the Faculty of Science in Dijon. In his book, he stated that he "changed the bases of the reasoning by replacing the usual axioms by other facts[8]" (Méray, 1874, pp. XII). So, following works of the mathematician Jules Hoüel (1823–1886), Méray's book proposed to merge plane geometry and solid geometry in a geometry based on the movements of translation and rotation (cf. Bkouche, 1996).

In 1880, Justin Bourget (1822–1887) wrote an algebra textbook which wanted to present algebra "in a new light which complies with the progress this science achieved by the study of the quantities called imaginary" (Bourget, 1880, pp. I). Bourget was the *Recteur de l'Académie d'Aix* and thus a prominent person in the French educational system. His book introduced negative numbers with what he called "complex magnitudes with two opposite senses". Thus, for example, +2 was introduced as 2_0 and -3 as 3_π. Bourget called 2 or 3 the absolute values, or modulus. 0 and π were called the arguments.

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[7] The president of the commission designated in 1898 to prepare a reform of secondary education quoted in particular the success of former pupils of *enseignement moderne* at the *École Polytechnique* in order to refuse the removal of this secondary teaching (cf. Ribot, 1900).
[8] Translation by the author. It is the same for all translations into English in this article.

These two examples show one of the main problems with the mathematics curriculum in secondary education during this period: the reviewing of the mathematical principles. This problem is connected to the questionings of the mathematicians of the 19th century on the founding principles of mathematics. The search of rigor in analysis is one of the main causes of these questionings. In particular, the works of Bernhard Bolzano (1781–1848) and Karl Weierstrass (1815–1897) led to what was called by Felix Klein (1849–1925) in 1895 "the arithmetizing of mathematics". We also know, in the 19th century, the importance of the necessity to teach for the mathematicians in this search for more solid founding principles (cf. Belhoste, 1998 and Dugac, 2003)[9]. All this had led in particular to the construction of irrational numbers, whether based on the sequences or on the idea of a cut in the system of rational numbers. The construction of integers, fractions, negative numbers without using the concept of magnitude are problems which were subsequently treated. These theories sometimes caused reactions of rejection (cf. Rafy, 1903 and Goldstein, 2011) but the problem of their introduction in the secondary curriculum of arithmetic and algebra arose at the end of the 1880s.

Another serious question was to introduce (or not) the concept of derivative function in secondary education. For some it was a challenge to introduce the bases of analysis in the secondary schools' curricula although it appears that this teaching was sometimes provided (cf. Bioche, 1914).

In geometry we find the same issues. For instance the definitions of a straight line, of a surface are definitions which are continuously reworked in the 19th century's textbooks. The question of introducing some elements of recent theories such as notion of transformation or projective geometry also arose.

The mathematics curricula of classical education were finally not very much modified in 1890. The algebra curriculum pointed out how to introduce negative numbers. They had to be introduced by studying the position of a point on a straight line and with the help of formula of the uniform motion. Before, they were usually introduced by solving first degree equations or by calculating values of polynomials. This introduction was considered as an important change (cf. Tannery, 1892).

It was the main change in algebra curriculum. In geometry concepts as: translation and rotation[10], power of a point, radical axis and radical center took place in the new curriculum. But these changes did not affect the essential part of the geometry curriculum for which "the reference remains Euclid reviewed and updated by Lacroix" (Belhoste 1990, pp. 380).

New textbooks were published following this change of curricula. Two of them have to be mentioned: *Premières Leçons d'Algèbre Élémentaire*, written by Henri Padé (1863–1953) in 1892 and *Traité d'Arithmétique* by Eugène Humbert

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[9] Richard Dedekind (1831-1916) and Jules Tannery (1848-1910) linked both their definition of irrational numbers to their mathematics teaching (cf. Boniface, 2002)

[10] According to Bkouche (cf. Bkouche, 1996), that introduction of the concept of transformation is to place in the context of "Erlangen program" and not in the one of Meray's geometry.

(1858–1936) in 1893. As the books of Méray and Bourget, they belonged to that line of textbooks which proposed to renew the contents of mathematics teaching. The first one was a small book which dealt with the new way to introduce negative numbers in the curriculum. Padé proposed a purely arithmetical definition of negative numbers[11]. The second one was an ambitious treatise of arithmetic. It also introduced negative numbers, which was an innovation in arithmetic. But Humbert went further by introducing the concept of congruence. He demonstrated Fermat's theorem and Wilson's theorem. He also proposed an introduction of an irrational number as a measurement of an incommensurable length by using what we call Cauchy sequences.

However, one of the most remarkable examples of this kind of textbooks is probably the "comprehensive course for the elementary *classe de mathématique* published under the direction of Gaston Darboux". The authors of this set of five books and their publishing manager were important members of French mathematical community at the end of the 19th century. Their books proposed a new approach and new contents for the elementary mathematics curriculum.

The course of Darboux[12]

Which authors for which books?

The books where published during the period 1894–1901. The first one was the arithmetic textbook written in 1894 by Jules Tannery *Leçons d'Arithmétique Théorique et Pratique.* The second one was a textbook of cosmography which will not be discussed here. The third one is the algebra textbook written by Carlo Bourlet (1866–1913) which was published in 1896 *Leçons d'Algèbre Élémentaire.* Bourlet also wrote the next one published in 1898 *Leçons de Trigonométrie Rectiligne.* The same year the textbook of plane geometry appeared: *Leçons de Géométrie Élémentaire (Géométrie Plane)* and in 1901 the last one was published, the textbook of solid geometry. The two last textbooks were written by Jacques Hadamard (1865–1963). All were published by Armand Colin who was (and still is) an important French publisher of educational books.

Gaston Darboux has not written any books. Yet his role was essential from a mathematical point of view as we will see later on this paper. At the end of the 19th century he was one of the most important French mathematicians. He taught at the *École Normale Supérieure* after brilliant studies in this school. In 1880 he succeeded Chasles in the geometry chair at the *Sorbonne.* In 1884 he was elected to the *Académie des Sciences.* He published over sixty texts in mathematics about all topics. Most of these texts dealt with geometry but he also published

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[11] André-Jean Glière (cf. Glière, 2007) conducted a detailed study of Padé's book in his thesis on the history of the negative numbers.

[12] I shall so appoint this set of textbooks during this article.

about analysis, mechanics and astronomy. Darboux also had an important institutional position in French educational system. In 1888 he was elected to the *Conseil Supérieur de l'Instruction Publique* (High Council for Public Instruction). In 1893 he was appointed dean of the *Faculté des Sciences de Paris*. He was a member of many committees and wrote reports in several occasions for the *Ministère de l'Instruction Publique*. Darboux was also member of the editorial board of the *Revue de l'Enseignement Secondaire et de l'Enseignement Supérieur*, a journal destined to promote secondary and higher education. Darboux was therefore an important person of the educational system reforms.

Jules Tannery, the first author, was like Darboux (and it is the same for the two other authors) a former brilliant student of the *École Normale Supérieure*. He taught at this school from 1881 until his death and he became director of scientific studies in 1884. His mathematics works are however less important than those of Darboux. At the end of the 1880s he also began to play an institutional role and he was elected to the *Conseil Supérieur de l'Instruction Publique*. But he had already published articles in support of institutional reforms in the *Revue Internationale de l'Enseignement*[13], especially to reform the *baccalauréat ès sciences* (cf. Tannery, 1886*a*).

In the 1890s Carlo Bourlet and Jacques Hadamard were two young and promising mathematicians. They published many mathematical works. The first one taught at the *lycée* Henri IV, then at the *lycée* Saint Louis in *mathématiques spéciales*. The second one taught at the *Faculté des Siences de Paris*.

Darboux: an involved publication director

What was the role of Darboux in this editorial project? He is quoted by each of the authors in their prefaces. Tannery wrote:

> On the advice of M. Darboux I tried to take what we could of the old definition: a magnitude is all which is likely of increase or decrease. […] I have to send my best thanks to M. Darboux. He helped me of his advice for the set and also for the details. (Tannery, 1894, pp. VII–VIII)

Bourlet mentioned in algebra textbook:

> On the advice of M. Darboux (advice which was very pleasant for me to follow) I resolutely abandoned the method called elementary to study the variations of a function. I adopted that of the derivative function. (Bourlet, 1896, pp. VI)

And Hadamard wrote, in his first textbook:

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[13] This journal was published by the *Société de l'Enseignement Supérieur* which promoted higher education. Darboux was also member of this society.

> M. Darboux […] made the task peculiarly easy for me by the invaluable advice which he kept giving me […] At many times I took advantage of important indications from M. Darboux for the writing of this textbook. (Hadamard, 1898, pp. VIII)

The preface of the textbook of solid geometry suggests two specific points that seem to have been written under the guidance of Darboux: the exposition of the theory of parallel lines and planes before the theory of perpendicular lines and planes and the definition of the equality of two dihedral angles.

All this indicates that Darboux had a key role in this project. Did he guide some parts of these books as prefaces suggest? Or did he only support choices of the authors? The particulars provided by the authors show that their thanks to Darboux were not only an act of courtesy for a prominent figure of mathematics whose role was simply to ensure the success of these books. Darboux's interventions as a member of the *Conseil Supérieur de l'Instruction Publique* clearly indicate his interest in secondary education (cf. Darboux, 1898). The involvement of Darboux in these texts seems indisputable and this course deserves to be called the "course of Darboux".

Tannery: a specific contribution to the "course of Darboux"

Tannery's mathematics works were less important that those of Darboux and Tannery reached a less prominent position in the educational system. However he held a significant position in this editorial project.

In many ways Tannery was in an intermediate position between Darboux and the authors Bourlet and Hadamard. Tannery was only a few years younger than Darboux. But, if Darboux and Tannery belonged to the same generation, Darboux was already teaching at the *École Normale Supérieure* when Tannery began to prepare his thesis in this school. Bourlet and Hadamard were about twenty years younger. They were former pupils of Darboux and Tannery. However, the latter seems to have held a preferential role in their training (cf. Maz'â and Shaposhnikova, 2005).

His position of professor and of director of scientific studies at the *École Normale Supérieure* probably explains he was in the center of a network of contributors. The footnotes of his book show comments or proposals from Tannery's former pupils or other people in the mathematical community[14]. Most of them were teachers in secondary education. They suggested him new demonstrations of properties or new approaches of some problems. Here is an example which also shows one of the defining features of his book, the will of the author to present in one book the various concepts of the time about arithmetic. Tannery's arithmetic is based on cardinal numbers of which he inferred ordinal numbers. However, he specified that some authors adopted an

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[14] The language employed suggests that Tannery received these indications during personal meetings or by mail.

opposite point of view (he quoted Kronecker and Helmholtz).In the following chapter about the addition of integers he quoted a teacher from Lille (a town in northern France), M. Lamaire. He wrote that it was Lamaire who pointed out Helmholtz's demonstration of associativity and commutativity of sum of integers from the ordinal point of view to him. These demonstrations are given in a long footnote.

As he wrote the book about arithmetic all the following textbooks relied upon it. It is quoted more than thirty times in the textbooks of algebra and plane geometry. Here are two examples. In algebra, when defining the limit of a function at a number, Bourlet quotes the arithmetic textbook to clarify that his definition was a generalization of the definition gave in the latter book. In plane geometry Hadamard proposes a demonstration of the intercept theorem using a theorem from the arithmetic textbook on proportional magnitudes.

Finally we should keep in mind that teachers are the intended readers of the textbooks. And, among the teachers of elementary mathematics, a significant proportion of them were former pupils of Tannery[15]. So we can say that Tannery and his arithmetic textbook occupy a central place in this course which is a little bit the "course of Darboux and Tannery".

New contents for mathematics teaching in the 'course of Darboux'

Different studies could be conducted about the 'course of Darboux': exercises and methods proposed by the authors, organization of the texts for a stand-alone use by pupils[16], etc. This paper is focused on the problems of mathematics principles and of introduction of new mathematics theories in secondary education. The new contents proposed by the books of this course about these two topics were an attempt to adapt education to the modernity of mathematics of the time (cf. Bkouche, 2013). These new contents concerned both parts of the mathematics curriculum and additions to this curriculum.

To review the principles of mathematics

The problem of principles goes through all the textbooks but the most emblematic one is probably the arithmetic textbook. Tannery in this book offered an arithmetic built only on the integers without using the concept of magnitude (cf. Renaud, 2013).

He wrote in *Introduction à la Théorie des Fonctions d'une Variable,* a previous textbook intended for students in higher education:

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[15] In 1903 at least 35% of mathematics teachers are former pupils of Tannery in French public *lycées*. This proportion is slightly lower during the 1890s.
[16] See the use of textbooks that was recommended by Tannery (cf. Tannery, 1895).

> One can completely constitute the analysis on the idea of integer and with the notions related to the sum of integers. One should not use any other postulate or any other experience data. (Tannery, 1886*b*, pp. VIII)

Then he explained his book started with construction of irrational numbers due to lack of space. It is in his arithmetic textbook for pupils of secondary education that Tannery proposed his conceptions of the arithmetic principles. From integers regarded as cardinal numbers he defined a fraction like "a set of two integers which play different roles". Then he defined an irrational number as a cut in the system of rational numbers[17]. Finally he defined the "directly measurable magnitude" as a set which was in a one-to-one correspondence with the set of rational or irrational numbers. So Tannery transformed the arithmetic of magnitudes into arithmetic of numbers.

Concerning the algebra textbook, this will of reviewing principles was expressed through the introduction of negative numbers. Bourlet introduced them by using what he called "segments carried by a straight line" (namely vectors) and algebraic measures. In France he is the first author of textbooks to devote so long an introduction to negative numbers and he invented on this occasion the expression "algebraic measure" (cf. Glière, 2007).

In geometry Hadamard also dealt with the problem of definitions. He devoted two annexes to this problem. In annex B, he briefly evoked the non-Euclidean geometries of Boylai and Lobatchevski. And he went on saying that the parallel postulate is a definition. Then he wrote:

> Yet there are terms which were not defined and cannot be. Because we can define a notion only by means of previous notions. What is impossible for the first notions which have been introduced.
> But as these notions are clear by themselves and have from then on a number of obvious properties, the role of the definition […] is then performed by the properties in question that we admit without demonstration.
> It is the way in which we proceeded for the straight line. It did not receive a definition we gave what we could call an indirect definition by admitting the fundamental properties (Hadamard, 1898, pp. 282–283).

About the non-Euclidean geometries we need to remember that they were still rejected a few years before by some of the most important mathematicians. In 1869, the mathematician Joseph Bertrand (1822–1900) made a fool of Lobatechevsky's essay in his proceedings to the *Académie des Sciences* (cf. Gispert, 1990).

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[17] Tannery only dealt with positive irrational numbers.

In annex D Hadamard proposed a definition of the notion of area[18]. He wrote that his definition presents the advantage to prove what was considered before as a postulate:

> I. Two equal polygons have the same area for all their positions in the space.
> II. The polygon P", sum of the adjacent polygons P and P', has for area the sum of the areas of P and P'.
> (Hadamard, 1898, pp. 289)

Hadamard's reference to non-Euclidean geometries shows too the will of the authors to include some of the last developments of mathematical research. We are going to analyze some other examples of this will, the most emblematic ones.

Texts which show the most recent theories

We have already mentioned the example of the irrational numbers in the arithmetic textbook. Tannery took advantage of it to give an insight of the set theory.

He considered what he called "determined sets of numbers". A set of numbers was determined if it was possible to know if a number belonged or not to this set. He defined the concept of supremum and infimum and demonstrated they existed for every set of rational or irrational numbers[19] bounded from above (this set was of course bounded from below because Tannery only dealt with positive numbers).

Tannery was probably the first French mathematician who had previously proposed elements of the set theory in his textbook *Introduction à la Théorie des Fonctions d'une Variable*. This book had been published in 1886. According to the author it was written after lectures he gave in 1883. So we can suppose Tannery taught these elements at the *École Normale Supérieure*. The mathematician Camille Jordan (1838–1922) had given the first recognized lessons including the set theory at the *École Polytechnique* at the end of the 1880s (cf. Gispert, 1995). This shows the boldness of this book intended for the secondary education despite insertion of these notions in chapters announced outside the curriculum.

And, in his last chapter, Tannery proposed bases of what he called "higher arithmetic". This chapter gives elements of the congruence theory as Humbert's textbook. But Tannery also demonstrated the law of quadratic reciprocity and he used Gauss's subscript theory to solve congruence of first degree[20]. None of the notions he treated in this chapter were in secondary

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[18] Hadamard defined the area of a triangle as the product of an altitude by the corresponding base multiplied by a determinate number k. Then he justified the choice of ½ for k. Hadamard defined similarly the notion of volume with a tetrahedron in his book of solid geometry.

[19] Let us remark that Tannery did not use Dedekind's expression: "real numbers".

[20] Let p is a prime number, g a primitive root of p and a an integer. Tannery called "subscript of a in the base g" every integer   verifying: $g^{\alpha} \equiv a \,(\mathrm{mod}.\, p)$. He called "congruences of first

curriculum. They were neither in the curriculum of special mathematics nor in the curriculum of the Bachelor degree of Mathematics.

To conclude with Tannery's *Leçons d'Arithmétique* here is what the mathematician James Pierpont (1866–1938) wrote about this book in the *Bulletin of the American Mathematical Society* in 1899:

> It is thus a pioneer, perhaps even the inaugurator, of a revolution in secondary instruction in mathematics and as such will receive praise or censure according as the person in question is thoroughly awake to the crying necessity of reform in secondary mathematical instruction, or is not. (Pierpont, 1899, pp. 455)

Another very interesting example is the definition of a derivative function in Bourlet's textbook. The part of the curricula which was involved was variations of a trinomial magnitude and variations of the quotient of two trinomial magnitudes. Bourlet used derivative to study variations of these functions. We have already seen that some thought it was a challenge to introduce it in secondary school curriculum. Bourlet wrote this about the introduction of the derivative function:

> At first sight this can appear as an audacity. But, if we refer to the difficulties presented by the rigorous and complete exposure of any of the elementary methods, we shall be forced to admit that derivative function, without being more difficult to conceive, is a much more regular and easier method. (Bourlet, 1896, pp. VI–VII)

We could say it was a moderate audacity or that it was an audacity for classical education. Indeed, in 1891, this concept had already been introduced in modern education in the curriculum of '1st Sciences':

> Representation of a function by a curve [21] – Notion of the derivative –The derivative is the slope of the tangent to the curve.
> Variation of the following functions:

$$y = ax^2 + bx + c, \quad y = \frac{ax+b}{a'x+b'}, \quad y = \frac{ax^2+bx+c}{a'x^2+b'x+c'}$$

> (Bulletin administratif, 1891, t. 49, pp. 655)

Bourlet wrote that this introduction was a positive experiment. Can we say that modern education had been used as a bench test for the curriculum of mathematics? Or that the modern education curricula were not subjected to the same heaviness than those of classical education?

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degree" the equations: $ax \equiv b \pmod{p}$. In an annex Tannery gave a table of primitive roots and subscripts of prime numbers. This table was extracted of Gauss' *Disquisitiones artithmeticae*.

[21] We note by the way that the word "function" appeared in the modern curriculum but not in the classical one.

How did Bourlet present the derivative function? He gave the definition in epsilon-delta of the limit of a function at a number. Then he defined a continuous function, a derivative function before applying it to study the variations of a function. All this may appear now classical but it was radically new at that time and at such a level of the educational system. Let us keep in mind that in 1875 the mathematicians Houël and Darboux disagreed about the differentiability at 0 of the function $x^2 \sin(1/x)$. The first one, who sought the limit of the derivative, claimed that it had not derivative at 0. The second one calculated its derivative at 0 by using today's definition. He was one of the first in France who defined independently from Weierstrass the notions of limit and of continuity with inequalities (cf. Gispert, 1990). Fifteen years later it seems that epistemological problems had been overcome and that elements of analysis could be introduced in secondary teaching, at least in modern education at first.

We find again this will to renew mathematics education in the textbooks of geometry. Here is for instance the list of the topics outside curriculum we find in the plane geometry textbook: signs of the segments, transversal theory, anharmonic ratio, harmonic range, poles and polars in the circle, inversion, problem of the tangent circles, properties of the inscribed quadrilateral, Peaucellier's reverser. They are found in a chapter entitled "Additions to the third book". But the most innovating concept introduced in geometry is probably the concept of the group of transformations. Hadamard defined it in annex A of the plane geometry textbook. In annex B devoted to Euclid's postulate he clarified the axioms of the Euclidean group of displacements. And then, in annex H of the solid geometry textbook, he applied the theory of groups to prove the existence of regular polyhedral. We can say that we find nearly the same boldness in the annexes of the geometry textbooks as in the chapters off curriculum of the arithmetic textbook. The English journal of mathematics, the *Mathematical Gazette*, just deplored the lack of the imaginaries in these books in order for them to be a "complete exposition of elementary modern geometry" (*Mathematical Gazette*, 1901–1904, pp. 121).

What was the influence of the course of Darboux?

In 1902 the following reform unified secondary education and gave a more important role to scientific education. Its place in a humanistic formation was recognized (cf. d'Enfert and Gispert, 2010). In the following diagram (Figure 3) we can see the former orders of secondary education. But we can see a lot of innovations. Let us note two main ones. The Section C offered a real scientific training in second cycle for pupils of former classical education[22]. And the former modern education had the same length as the classical one.

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[22] Darboux was considered as the initiator of the Section C (cf. Falcucci, 1939).

Figure 3: The unified secondary education, 1902.
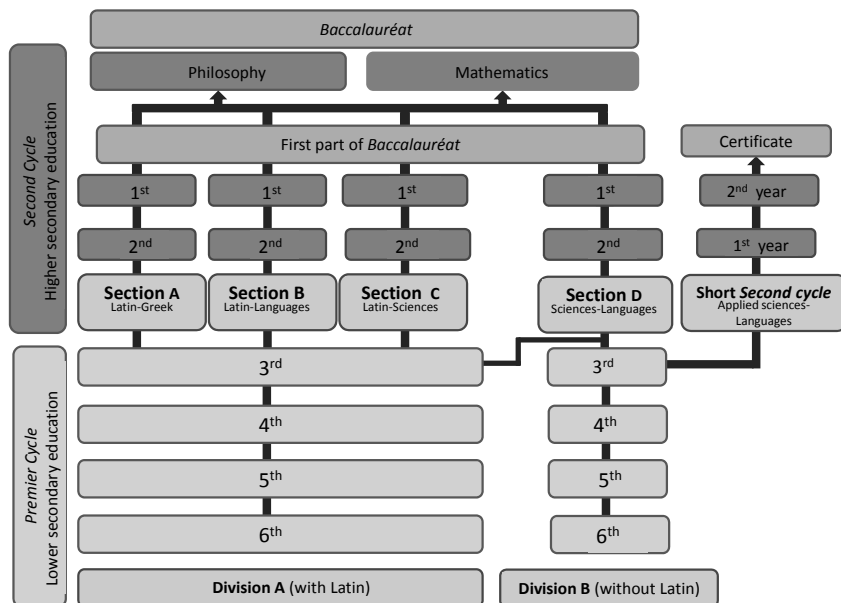
This reform provided the occasion of new curricula. Darboux was the President of the Commission in charge producing the new scientific curricula and Tannery was a member of the subcommittee for the mathematics curriculum. What has remained of the will of the authors to make the mathematics curriculum evolve?

Almost nothing had been retained of the arithmetic textbook in the curriculum of secondary education. Yet the construction of the irrational numbers appeared in the curriculum of *mathématiques spéciales* in 1904.

In 1902 the major change in the mathematics curriculum was the introduction of elements of analysis in secondary education's algebra curriculum. The concept of derivative function was taught from the second grade in two sections of the new educational system. These elements were related to physics teaching. The bases of integration were also introduced in *classe de mathématiques*.

In plane geometry the notion of inversion appeared in 1902 in *classe de mathématiques*. In 1905 poles and polars in the circle, properties of the inscribed quadrilateral, Peaucellier's reverser were introduced in the curriculum.

Some of the textbooks of Darboux's course were modified with the new curricula. Those of arithmetic and geometry knew numerous editions: at least ten editions for the first one, eight for the second one. They became reference books in educational publishing.

Furthermore this movement of renewal of the mathematics secondary education carried on. It carried on and evolved because mathematics changed.

The mathematician Emile Borel (1871–1956) wrote in a textbook intended to pupils of the lower secondary education (fourth and third grades) about the definition of the polynomial function of the first degree: "One of the main purposes of the Mathematics is the study of functions" (Borel, 1905, pp. 152).

During the 1850s mathematics was for Bertrand the science of the magnitudes (cf. Bertrand, 1849). In the 1880s it was for Tannery the science of numbers. The concept of function cautiously introduced in the mathematics curriculum of modern education in 1891 was one of the main purposes ten years later. The changes of mathematics curricula in secondary education are a reflection of the mathematical ideas of this period.

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Epistemology, methodology, and the building of meaning in a new community of mathematics educators in England, 1950–1980

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Abstract

This paper describes the paradigm shift in attitudes to the teaching and learning of mathematics amongst a group of English mathematics teachers from the publication of the Mathematical Association's 'Report on Mathematics in Secondary Schools' (1959), to the Association of Teachers of Mathematics' 'Notes on Mathematics for Children' (1977). The epistemological claims of the new paradigm are demonstrated, and the philosophical basis is examined. Finally, some links are made with the development of research in mathematics education.

Introduction

In earlier papers (Rogers, 2001; 2011; 2012) I explored social and political aspects of the development of the mathematics curriculum in the UK and their associated pedagogical practices. Here I recall some events of the recent past at a time of great change in England, which involved the education system in general. The full complexity of this change can only be indicated, and my choice of events and documents here is necessarily limited. I have been active in some of these enterprises, and my views are influenced by my involvement. Since the historian's enterprise involves interpretation which has an ideological background in a set of beliefs contemporaneous to the writer, and which determine the questions and the ways in which they are asked, I appeal to those colleagues who have shared some of these experiences to compare their interpretations with my own.

This is an attempt to examine more closely some of the events and contexts surrounding the development of a particular style of practice in mathematics teaching in England from the early 1950s to the early 1980s. This period saw

expansion in the Teacher Training Colleges, and the rhetoric of 'training' into 'education' with the inception of a Bachelor of Education (B. Ed.) degree to raise the academic status of teachers,[1] and to achieve the political ideal of an all-graduate teaching profession.

The 1944 Education Act established examinations at age 11 to select pupils for Grammar Schools, where they were expected to progress to university or professional occupations, and for Secondary Technical and Modern schools, to provide the majority of the work force. Another complication was the raising of the school leaving age from 14 to 16 years between 1947, and 1972 with the problem of how to manage and what to teach these new older pupils. In 1951 the examination for school leavers was changed to the General Certificate of Education for older school leavers with a wider range of subjects but there was little change in the mathematics curriculum. The Labour government began changes to Comprehensive Schools in 1965 thus bringing the grammar and secondary modern pupils together to be taught (ideally) in mixed classes. In concert with this change, the Schools Council[2], the Nuffield Foundation[3], and Local Education Authorities began to encourage 'grass roots' curriculum development. The social changes exposed the difficulties and challenged many preconceptions about content and teaching methods in mathematics, and expectations of pupils' achievement (Swan, 1950). At the same time the 'Modern Mathematics' movement was a significant force for curriculum change. A considerable amount of material appeared in England from the United States, and the principal influence from Europe was the *Royaumont Seminar* which encouraged the modernisation of the mathematics curriculum for both economic and pedagogic reasons (OEEC, 1961a; 1961b). In this paper I will consider some of the circumstances surrounding a significant paradigm shift in pedagogical practice, which concerns the foundation and early days of the Association of Teachers of Mathematics.

The dominant paradigm

The dominant paradigm for educational practice, was the belief that some were born to lead, and others to follow (Rogers, 2001; Fyfe, 1947). Consequently, for mathematics, very different syllabuses, expectations, and pedagogical practices were to be found in the state grammar and secondary modern schools. These traditions were reinforced by reports published by the Mathematical Association (M.A.)[4] and the Government Inspectors (HMI, 1958).

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[1] The B. Ed. degree was established in 1964 and the first B.Ed. teachers graduated in 1968.
[2] The Schools Council for Curriculum and Examinations (1964-1984) was a quasi-autonomous organisation funded by government, and set up to encourage and support curriculum development.
[3] The foundation is an independent Charity supporting Educational Research and Development.
[4] For a detailed account of the history of the MA, see (Price, 1994).

In 1956 the M.A. finally produced their report, *The Teaching of Mathematics in Primary Schools*, on which work had started 17 years earlier. A number of committees had met to study the problem, so that by the time the final committee was appointed in 1950 a new doctrine had emerged, which was then embodied in the 1956 report. Earlier differences of opinion are clear from the Preamble to the Report where the members of the Primary School Sub-Committee declare:

> In fact, thinking and doing go so much hand-in-hand that it is almost as if the children were thinking in terms of action. It has also been suggested … that the fundamental patterns of mathematical thinking appear to be fundamental to all thinking. Hence we have found it convenient to analyse our ideas about Mathematics in Primary Schools in a three-fold way:
> 1. the interaction between a child's mind and the concrete situations of his environment – i.e. the substance of his experience
> 2. the growth of mind and mental powers as they are exhibited in the child's growth towards mathematical thinking
> 3. the mathematical ideas themselves and their development and inter-relation. (M.A., 1956, p. 1)

The content of the final report shows that the committee had incorporated a number of Piagetian principles. The Introduction and first chapter set out the place of mathematics in the Primary School where we find the generally recognised reasons for teaching mathematics; utility and scientific value, personal enjoyment, and the child's right to our heritage, but also the claim that the patterns of mathematical thinking are fundamental patterns of all thinking. In this report we find a comprehensive development of ideas about number and space; about organisation of the classroom and different groupings of pupils; about different kinds of understandings and suitable kinds of assessments and records of achievement, the different experiences by which pupils have progressed and the emphasis on understanding mental processes: "*The true test of mastery of a mathematical principle or process is to be found in the ability to apply it to a new situation, and not in a repetition of an already standardised situation committed to memory*." (M.A., 1956, p. 87). Of particular note is a section on "Material aids to teaching" (ibid, pp. 93–106) describing various apparatus, including graphs, diagrams, and displays with suggestions about how they may be employed to make a growing mathematical idea clearer through experiment, and to foster memorising and skills (Gattegno 1963a; Servais, 1970). Finally, there is an annotated section on research on teaching and the nature of mathematical thinking. The willingness and ability to promote a study of research is another innovative idea not usually found in a manual intended for teachers at this time.

A report on Mathematics in Secondary Modern Schools was produced in 1959, which was intended to be read in conjunction with the Primary report.

However here the tone is quite different; in the statement of terms of reference we find:

> The mathematical ideas and teaching methods here discussed are those which we consider suitable for the great bulk of secondary school population between 11 and 15 years of age, who have shown no early signs of readiness for Mathematics as an abstract study, or at least have not achieved the attainments in arithmetic traditionally associated with the ablest of their age group ... but readiness for many mathematical techniques comes with an intellectual maturity only reached by most modern school pupils some time after entry, if at all. (M.A 1959, pp. 1–2)

Clearly, expectations for these pupils are limited, 'intellectual maturity' is the principal criterion, and the attitude is a vocational or manual rather than a professional future for the pupils. At this time many Secondary Modern Schools were still operating, and the writers of this report had not learnt anything from the message in the Primary Report. *A Second Report on the Teaching of Arithmetic in Schools* was published in 1964 that discussed the teaching of arithmetic as part of a general mathematics course, from the end of the primary school to age 16. This is intended:

> ... for teachers of boys and girls of good ability such as are found in grammar schools and in the grammar streams of comprehensive schools. There are many other pupils who are still not ready at 11-plus to begin such a course as is here envisaged and appropriate guidance for their teaching may be found in ... *Mathematics in secondary modern schools*. (M.A., 1964, p. 1)[5]

By this time, the modern mathematics movement was already under way, and so arithmetic:

> ...should be taught in such a way that it leads on to the rest of mathematics, not only algebra and trigonometry, coordinate geometry and calculus, but also the new topics now being introduced into school syllabuses... (ibid, 3)

How arithmetic should be taught so that it leads to subjects not normally taught to these pupils is not explained. Apart from statistics (which was not a new topic), hardly any mention is made of any 'new' mathematics. Towards the end of the report, we find some suggestions for classroom procedure:

> The teacher has ..... two lines of approach. The first is traditional and familiar: it is teacher-centred and relies on exposition and examples worked on the blackboard, but the pupil learns largely by listening. In the second the emphasis is on the pupil's activity: the teacher's task is to contrive the appropriate stimulus to learning, sometimes by posing a challenge or problems and guiding

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[5] Many comprehensive schools adopted the practice of 'streaming' pupils into 'able' and 'less able' pupils, hence the reference to "grammar streams" in this quotation.

> the pupil to discover a solution, at other times by suggesting and activity ....
> leading to the discovery of a generalisation. (M.A., 1964, p. 84)

Apart from this brief hint at 'guided discovery' the suggestions only concern classroom organisation, the choice of routine examples, marking of pupils' work, and so on, and no discussion of the pedagogical challenges produced by the comprehensive educational system. Clearly, the problems of teaching pupils in mixed classes were being felt by members of the M. A. much earlier (Swan, 1950; Nunn 1951) but even after the changes, it was difficult to break out of the long-established pattern. Some teachers in comprehensive schools avoided the problems by sorting pupils into different 'ability groups' using standardised arithmetic tests.

Changes in the primary and secondary schools

Before 1952, the Mathematical Association was the principal authority whereby the beliefs and practices about teaching in the secondary schools were handed on. The members were largely university mathematics graduates who had little or no knowledge of psychology and often no pedagogical training, and their expectations were driven by their training as mathematicians. While they had high expectations for their pupils, these pupils were the privileged few who were mostly found in grammar and private schools. The standards in Primary schools had evolved, and while educationists and official reports demanded more practical applications, classes became smaller and teachers better trained. Despite the influence of Montessori and Froebel on the 'nature of the child' and related practical activities, rote learning of arithmetic was hard to eliminate (Pinner, 1981). By this time, the teachers in the Secondary Modern and Technical schools had attended training colleges where mathematics was limited to basic arithmetic and mensuration, determined by the supposed mental capacities of their pupils. It became clear that these teachers were very concerned with their pupils' problems, and as new entrants to the profession discovered accounts of the work of Piaget, realising that the problems could be seen from different points of view, the epistemological and pedagogical beliefs underlying established practices were seriously challenged. The first response providing a new approach to the problems, and endorsing new practices, was met by the publication of the report by Her Majesty's Inspector Edith Biggs in *Mathematics in Primary Schools* (Biggs 1965) advocating practical mathematics in Primary schools based on Piagetian principles. This was reinforced by the Nuffield Primary Mathematics Project (1964), using a conceptual approach based on research, and issuing a number of Teachers Guides, the underlying mathematics was explained and various activities were suggested, demonstrating structural ideas of modern mathematics to help children (and teachers) to form a conceptual basis for their learning.

The New Mathematics in Europe and the USA

After the Second World War, the European countries realised that their school and university curricula and pedagogical practices had to be revised, and so by 1950 a small group of mathematicians, educationalists, secondary school teachers and psychologists were meeting at the initiative of the mathematician and psychologist Caleb Gattegno, and in 1952 they decided to create the *International Commission for the Study and Improvement of Mathematics Teaching* (CIEAEM) with Gattegno as secretary, see (Bernet & Jacquet, 1998). The first joint publication of the CIEAEM, *L'Enseignement des mathématiques* appeared in 1955. The preface (pp. 5–9), was signed "Le Bureau" but most likely written by Gattegno, clearly showing the united intentions and clearly different interests of the authors[6].

In the 1950s the National Science Foundation was tasked with creating and implementing new curricula for undergraduate and school science and mathematics teaching, realised their need for the development of scientific programmes, and through the ICMI contacted the OEEC, which led to representatives of ICMI and CIEAEM meeting at the Royaumont Seminar in 1959 with the aim of considering modernising mathematics teaching not only in Europe and the United States but globally (Furinghetti, 2014, p. 555). The radical proposals for the development of mathematics curricula, (OEEC 1961a, 1961b), were principally driven by the work of the CIEAEM and brought together by Caleb Gattegno (Vanpaernel, deBock, & Verschaffel, 2012, pp. 486–487), showing that the aim of the 'New Mathematics' was to introduce new content and pedagogical practices to mathematics teaching at all levels.

At this time, European mathematical curricula were quite varied, and early reforms in modernizing their mathematics programmes were by no means uniform (Servais, 1975, pp. 39) since each country had its own mathematical and pedagogical traditions. Servais' conclusion was "What we want is to update teaching of mathematics both in content and in method and to keep it alive as a permanent activity." (ibid: 55). Meanwhile, concerned for the improvement of mathematics in their own schools, in the USA the School Mathematics Study Group (SMSG) was formed[7] publishing some well-designed introductions to various new mathematical topics. Concerned with developments in mathematical pedagogy, a series of translations of Soviet texts were published in the United States (Simon, & Simon; 1963, Kilpatrick, & Wirszup, 1969–72), and much of this material became available in England.

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[6] Among the founders were: Evert W. Beth, Gustave Choquet, Jean Dieudonne, Caleb Gattegno, André Lichnérowicz, and Jean Piaget. The chapters of the book were written independently (possibly at different times) with no attention to any uniformity or editorial policy.

[7] The School Mathematics Study Group (SMSG), founded in 1958, was the largest and best financed of all the National Science Foundation projects of the era, by the combined efforts of the AMS, MAA and NCTM.

ATAM/ATM: building a new community

During the early 1950s, teachers of mathematics became influenced by accounts of Piaget's work and the news of the changes in mathematics teaching emanating from Europe. At this time, Caleb Gattegno was teaching in London where he brought the views of his CIEAEM colleagues, and other European pedagogical developments to the notice of his students and practising teachers. Frustrated with the effect of the social and political changes and seeking some action to alleviate the problems of teaching 'less able' children, in both Primary and Secondary schools, a group of teachers, mathematics education students, and university lecturers took radical action and formed a new professional association for *all* teachers of mathematics. A clear ideological division had developed between the guardians of practice in the M. A. and the ideas represented by Gattegno and his sympathisers. Even as he was becoming involved in the foundation of the CIEAEM and his translation of Piaget (Piaget 1951a), Gattegno's views were clearly expressed in the *Mathematical Gazette*, the journal of the Mathematical Association (Gattegno, 1947; 1949; 1954a,b). The emerging psychological theory meant that many assumptions about teaching mathematics at all levels had to be re-examined. Since it seemed impossible to introduce these ideas into established practice, Gattegno, together with Roland Collins[8] and other sympathisers, broke away from the M. A. and founded the *Association for Teaching Aids in Mathematics* (ATAM).

> There were twelve people – teachers from all over England – at the first meeting of a steering committee held on June 28th, 1952. These founding members had responded to a brief notice in the Mathematical Gazette … inviting the formation of a teachers' cooperative to produce teaching aids. (Tahta & Fletcher. Undated. *The First Ten Years of ATM*. ATM Website)

These teachers were initially concerned with pupils in Primary and Secondary Modern schools, so they emphasised the development of a range of innovative approaches to teaching and the use of apparatus to assist pupils' learning.

> In fact, it is when we are engaged in a dialogue with concrete material that the principal mathematical ideas emerge: every perception or action derived from the concrete duplicates itself in mental imagery; this becomes structured and can then be recalled in its own right. The first objects for mathematical study are the relations between perceptions and actions made virtual in this way. (Servais, 1970, p. 207)

The M. A. had discussed aids to pupils' understanding (M.A. 1947), but even though Gattegno had published his paper on Dynamic Geometry (Gattegno 1954b), little action was taken until the M. A. Primary Report of 1956.

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[8] Roland Collins was a teacher at Doncaster and the author of *Mathematical Pie* a periodical for school pupils.

In November 1955 ATAM published the first issue of their Bulletin *Mathematics Teaching,*

> … in order to provide a fuller and better service for our members and to reach a larger circle of readers …" aiming to "… present articles and news items covering the whole field of mathematics teaching, paying particular attention to the use and development of teaching apparatus and visual aids. We believe that there is need for a periodical devoted to these aspects of mathematics, and it is not our intention to imitate … material to be found in other journals, but to develop along lines of our own. … *The teaching of mathematics requires constant research; and research which aims to advance knowledge of the craft of teaching is just as difficult as research which aims to advance knowledge of mathematical techniques, and perhaps it is even more important. No one can do it better than those who are actively working in the classroom,* and this journal is a means by which practical classroom experience can be passed on to others. (italics mine) (ATM website, members section.)

Teachers were supported with a growing number of pamphlets, explaining new mathematical topics, reviewing books and apparatus, discussing psychological theory and sharing experiences from the classroom. Speakers were invited from Europe and the USA to address the annual Conferences, and accounts of their projects, experiments and experiences were published in the journal. In 1962 the organisation changed its name to the Association of Teachers of Mathematics (ATM) to widen its appeal and show it was concerned with the mathematical education of students *at all levels*, from primary school through to university. The ideas spread rapidly and a new *community of practice* began to develop (Adler, 2000) where newcomers were initiated into the ideas and pedagogical practices that were passed on to others in regional group meetings. The pattern of small discussion groups where teachers talked about their experiences, and then met together at an annual Conference was soon established. An early and continuing feature of the ATM Conferences was the invitation to children from local schools to participate in the workshops, and to become an 'experimental class' where new mathematical topics were introduced.[9] Members of ATM had regular contact with teachers in the former secondary schools and as the new organisation extended its interests, a workshop on *Teaching Mathematics in Modem Schools: The learner-centred approach* clearly indicated the principal direction of development. This was followed by articles in *Mathematics Teaching* and the new association began to assume national importance. "It differed from the Mathematical Association in the emphasis it placed on teaching methods and in that its journal could be read more easily and with greater profit by the teacher in the average school. Moreover, being new it was not weighed down by tradition." (Howson, 1978, p. 186)

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[9] These classes were part of the Research and Development aspect of ATM pedagogy. Vygotski's *Thought and Language* (1962) was one the new publications circulated among members followed by Krutetskii's Psychology of Mathematical Abilities in Schoolchildren (1976).

Gattegno had discovered the short black and white film animations showing the development of mathematical loci by Jean Nicolet, (Nicolet, 1958) (Gattegno, et. al., 1958) and brought then to the attention of ATM members. In a short article he had expressed his view that pupils are able to access

> the whole of school geometry … [when concentrating on their] …perceptual and active experience provided that [the geometry] is presented in an organised form ….. Our idea is to animate any given figure so as to contain a pattern containing an infinite number of figures but to do so in such a way that only the fact that we want to abstract shall be singled out. (Gattegno, 1954b, p. 208)

The concept in question should be seen as the *invariant* of a dynamic pattern, organised so that only one invariant at a time should be apparent to enable pupils to perceive it with their senses, and describe it in their own words, before it becomes formalised later. These and other films became an important feature of the ATM conferences with discussion of mental imagery and how one's powers of visualisation can be used to discover and reinforce important mathematical concepts. The films inspired others to make animations both with pupils in school, and as commercial items. (Gattegno, 1963a, pp. 49–59; 1963b, pp. 102–122; 1980; Tahta, 1981)

In 1962 twenty ATM members came together for a 'writing week'. Group writing became a particular feature of many ATM publications. Individual drafts were critically reviewed by the group, and redrafted, or jointly rewritten, before they were finally accepted by the group, making many difficult personal and intellectual demands on those involved. The product of this first group writing was *Some Lessons in Mathematics: A handbook on the teaching of 'modern' mathematics.* (ATM, 1964) which contained many suggestions for the secondary classroom with some 'modern' ideas and some new approaches to well established curriculum topics. The tone of the book was provocative; posing questions for teachers not only about the content of the curriculum, but also about the relevance of the mathematics, teaching methods, the nature of the learning process, the human element (Gattegno 1970), and motivations for both teachers and pupils. The Introduction states:

> The lessons which follow show a variety of styles and approach, but it seems to us that behind many there is a common strategy. Mathematics does not start from a finished theorem in the textbooks; it starts from situations. Before the first results are achieved there is a period of discovery, creation, error, discarding and accepting. This period is notoriously difficult to discuss, or even to describe in a convincing way, but in this book we have tried to make this period our concern. (ATM 1964, p. 2)

The nature of a *situation pedagogique* (a 'mathematical situation') (Servais, 1975) as developed by Gattegno (Vanpaemel, et. al, 2012) became a focus for debate, and the case for pedagogical reform rested on a desire to produce classroom

environments in which as many pupils as possible worked creatively. It was therefore the teacher's responsibility to understand new knowledge and use it as a basis for a new 'technology for teaching' (Gattegno, 1987). The idea of a period of experimentation before a formal stage was reached, became a hallmark of the ATM approach, and the use of all kinds of apparatus from simple everyday things to specially made 'structural material' were seen as essential to the exploratory period. Trevor Fletcher, in his presidential address hoped that the Association would become more involved in research by searching more deeply into the links between mathematics, psychology and philosophy. He suggested that:

> The difficulties which many pupils have with our subject are emotional, and they cannot be overcome by changing syllabuses or writing textbooks; they may be overcome by understanding the mainsprings of human action and mobilising pupils' entire energies for the task. (Fletcher, 1960, pp. 60–61)

The aims of the Research and Development group were to explore many of the new ideas and assist teachers to achieve understandings of the mathematics they taught, and different methodologies for fulfilling pedagogic action. A significant outcome of these deliberations was the report prepared in 1966 for the Sub-Committee on Mathematical Instruction of the British National Committee for Mathematics entitled: *The Development of Mathematical Activity in Children: the Place of the Problem in this Development* (ATM, 1966). Twenty-eight people contributed to this collection of writings that ranged from factual reports to 'work in progress', on the nature of mathematical activity, by exploring a range of 'problems', their relevance, contexts, aims, place and function in the school curriculum. It was presented to the Mathematical Instruction Section of the ICMI meeting in Moscow, 1966.

> All the organisations concerned with mathematical education at school level were approached and invited to a meeting in London in February, 1965. Papers were subsequently submitted by individuals and groups. The selection, arrangement and editing of these papers was undertaken by W. M. Brookes. No attempt has been made to arrive at an agreed report or to eliminate all disagreements and overlap between the contributions. …. The A.T.M. publishes the full report with the approval of the British National Committee for Mathematics. (ATM, 1966, p. vi)

Another group writing project soon produced *Notes on Mathematics in Primary Schools*, published in 1967. This was largely based on observations of children learning mathematics, and demanded that teachers should themselves engage with the mathematics discussed. The Introduction to this book contains the following important statement:

> Mathematics is the creation of human minds. A new piece of mathematics can be fashioned to do a job in the same way that, say, a new building can be

designed .... The invention of new algebras and new geometries ... have shown that mathematics cannot be an absolute, given *a priori*, or a science built entirely on observations of the real world. Mathematics is made by men and has all the fallibility and uncertainty that this implies. It does not exist outside the human mind, and it takes its qualities from the minds of men who created it. Because mathematics is made by men and exists only in their minds, it must be made or re-made in the mind of each person who learns it. In this sense mathematics can only be learnt by being created. We do not believe that a clear distinction can be drawn between the activities of the mathematician inventing new mathematics and the child learning mathematics which is new to him. The child has different resources and different experiences, but both are involved in creative acts. We want to stress that the mathematics a child knows is, in a real sense, his possession, because by a personal act, he has created it. (ATM 1967, pp. 1–2)

This was not a traditional text-book for teachers. No other book for teachers before this time had made such a statement of philosophy and intention.[10] There are a number of important issues here. The basic epistemological stance respects the child's creative power, a creativity similar in principle to the adult, and the mathematics produced is valued as such. Mathematical *situations* lead to experimentation, so the process of communicating the mathematics to others and refining the ideas is an essential part of the experience which helps to build confidence, autonomy, and relate the new mathematics to that already known. Another significant aspect of the new epistemology is the use of concrete materials as a mediator for mathematical activity (Servais, 1970). For example, while the Cuisenaire Rods can be regarded on one level as a concrete re-presentation of the rational numbers, where visual and tactile experiences enable us to talk about the relationships between the rods with quite simple language ('same', 'larger than', 'half of', etc.), what is more important – and very often unrealised – are their inherent properties which are fundamental to abstract algebraic relations (Gattegno, 1963c, pp. 103–106). Mathematical ideas can emerge when we are engaged with concrete materials, because every perception or action derived from the concrete can duplicate itself in mental imagery that can become structured. Thus the most important objects for mathematical study are the relations between perceptions and actions (Gattegno 1963a: 49–59). From the early 1960s onwards a number of ATM publications debated and refined these ideas.

In 1968 the ATM published *Examinations and Assessment*. Again, there were a number of contributors, and the Foreword explains:

During the period 1965–66 the A.T.M. made a collection of writings about *Problems*, … The present collection arose from a suggestion that the method of working adopted on that occasion might be appropriate for the production of a similar collection on Examinations and Assessment. The intention was to

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[10] Howson (1978, p. 186) states that the ATM publications, *Some Lessons in Mathematics*, and *Notes on Mathematics in Primary Schools*, had a considerable influence on most school text-books published after 1962.

> encourage contributions from all branches of the educational profession – teachers at different levels, administrators, examiners and test constructors, psychologists – and to consider all aspects of evaluative procedures, from large-scale public examinations to an individual's continuous evaluation of his own actions, and to the impact on the pupil and on his teachers of the assessment procedures used. We hope that a similar variety of people will find something interesting and timely in this collection. (ATM, 1968, p. v)

Examinations so often determine teaching and content of mathematics courses. Many issues of language, modes of evaluation, social contexts, stress under pressure, and other affective issues such as how evaluating effects the situation being evaluated, etc., were discussed in this document, foreshadowing many later research investigations.

In 1969 the first ICME meeting was held in Lyon, and members of ATM attended, taking with them a group of children and set up a workshop where French children were invited. To have children at a mathematical conference created quite a stir, and when ICME came to Exeter in England in 1972, the ATM workshop was there again, together with Seymour Papert's remarkable 'Turtles'; both groups reinforcing what children can do, given suitable tools and the awareness of mathematization. (Gattegno 1988)

A well-respected member of ATM, Geoffrey Sillitto,[11] died in 1966, and colleagues contributed to *Mathematical Reflections* (ATM, 1970), which contained an assessment of Sillitto's influence in reforming mathematics in Scottish schools and some of his previously unpublished work. Reflections inspired a number of articles, including contributions from Caleb Gattegno on the human element in mathematics, Lucienne Félix on the concept of function, and Willy Servais on the significance of concrete materials. Later, in 1977 *Notes on Mathematics for Children,* another collective handbook appeared, aimed at teachers of both Primary and Secondary pupils. The central theme of the book is on varieties of transformations in all kinds of contexts, numerical, algebraic, and geometric. This book continued the earlier position, claiming that mathematics begins with bodily actions, perceptions, and speech, and that any pedagogical approach has to acknowledge these facts. Furthermore, our mental powers are fundamentally algebraic by nature and the idea of transformation as a natural capacity of the human mind becomes a unifying algebraic concept. Language and visual perception depend on processes of transformation, and the awareness of transformation can become a technique for dynamically exploring the articulation of mathematical situations. However, while this idea is very powerful, it is not necessarily the only tool for learning mathematics. Another significant aspect of developing this new methodological approach to teaching is the focus on respect for the autonomy of the learner. By now, a clear shift in emphasis had been made; from teaching as an independent activity

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[11] Geoffrey Sillitto was a lecturer at Jordanhill College in Edinburgh and became the convener of the Scottish Mathematics Group that produced the SMG curriculum project (Rogers, 2011).

of the teacher, to acknowledging the nature of a learning process, where learning and teaching were inextricably linked. It had become clear that the original Bourbakist approach to the curriculum had failed:

> … we welcome the impact of modern mathematics on teaching because it points to important characteristics of the nature and role of the subject that passed unnoticed in traditional classrooms. Yet when we look at the tangible outcomes of the recent modern mathematics reforms …. we see that most of the message was misunderstood by those who brought it and garbled by those who had to implement it. …. The recommendation, for example that the elementary teaching of numbers should be based on the explicit teaching of sets now seems a monumental folly. It was proposed because mathematicians knew that the theory of sets had not only added a powerful weapon to their armoury but had succeeded in strengthening and illuminating the foundations of their subject, ….. But instead of having the wisdom to ask themselves what, precisely, might be the implications, if any, for pedagogy they went right ahead and inferred that if the classroom followed the lead given by mathematicians the advantages would be comparable. It is a matter of regret that in several countries, our own not altogether excepted, this example of non-rigorous thinking won a temporary victory. (ATM, 1977, pp. xiv – xv)

Throughout the 1960s and 1970s many curriculum development projects arose, often led by ATM members, some with the object of developing courses to lead to an alternative to the examination, normally taken by school leavers at age 16.[12] In the 1970s a number of ATM members began to obtain posts in teacher training colleges and university departments of education, where they influenced the direction of mathematics education research in the UK and abroad.

Epilogue

Here, I have tried to indicate the most significant aspects of the nature of the changes that occurred in the establishment of new attitudes to teaching and learning in this short period, and how the direction of research was influenced by many ATM members. Not all members of the community agreed with the principles that have been discussed here, nor were they necessarily able to practice them all the time. However, there was sufficient consensus for a large number of teachers to subscribe to the issues identified by ATM and develop them through reflection on their teaching and refining their pedagogical practice. Also, I have not given any account of the important work of Caleb Gattegno,[13] whose enigmatic statement "only awareness is educable" brought

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[12] For a brief summary of the major projects at this time, see (Watson, 1976). For more substantial critiques, see (Cooper, 1994; Howson, 1978; 2010.)

[13] Gattegno published many works on epistemology, psychology and the learning of mathematics, and language. A significant summary of his theory can be found in Gattegno, 1987.

many to focus on the fundamental somatic and social experiences lying at the root of our human functioning. The intuitive beliefs of teachers, based on their sensitive observation of children, the inspiration of Gattegno, and the members of ATM brought together a particular community of practice and encouraged teachers to pursue their ideal, but drew little support from the wider research community, who at the time, were more concerned with quantitative and behaviourist approaches to investigating classrooms.

It is interesting to observe how much of Gattegno's views on the awareness of our somatic nature have in common with the recent work of biologists and cognitive scientists who integrate biology, cognition, and epistemology, asserting that the only world we humans can have is the one we create together through the actions of our coexistence. The Cuisenaire rods referred to above are an example of an iconic re-presentation of algebraic structure that can also be represented symbolically. The crucial idea that the ATM promoted was not merely that these, and other pieces of apparatus are clever teaching devices, but that they can be used as a focus for presenting problems, and encouraging active discussion in the social context of the classroom, so that mathematics was "re-made in the mind of each person who learns it". (ATM, 1967, p. 2)

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Organisations and other links

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The Schools Council Archive can be found at the Institute of Education, London. http://www.ioe.ac.uk/services/1008.html

Analytical geometry in *An elementary treatise on plane and spherical trigonometry, and on the application of algebra to geometry* by S. F. Lacroix

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Abstract

Although most mathematics history books present Lacroix (1765–1843) as a minor mathematician, he was to become recognized both in his own time and in our own for his work as a writer of mathematics textbooks. His importance lies in the way he presents the contents in his textbooks: in an orderly, clear and concise way, from the elementary to the most complex concepts, thus contributing to the dissemination of mathematical knowledge. His works were used during a large part of the nineteenth century, both in schools in France and in a good part of the rest of Europe, particularly in Spain. One example would be his Tratado elemental de trigonometría rectilínea y esférica, y de la aplicación del álgebra á la geometría (An elementary treatise on plane and spherical trigonometry, and on the application of algebra to geometry)*, used as a textbook in Spanish secondary schools and universities for almost a century. In this work, which is part of a dissertation[1] on the study of analytic geometry in Spain during the nineteenth century, we focus on the way Lacroix deals with this part of the mathematics in this book. It shows an analytic geometry typical of the nineteenth century in Spain that preserves some elements from the geometry of Descartes but it also includes elements from the algebraic geometry used nowadays.*

Introduction

Analytical geometry was born in the seventeenth century by the hand of René Descartes and Pierre Fermat. It was to be René Descartes (1596–1650) who would go down in history as the father of analytical geometry thanks to his *La Géométrie*, an appendix to his *Discours de la méthode*, even though the geometry that Descartes describes in his work bears little resemblance to what today we

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[1] This dissertation is currently being developed by Isabel María Sánchez Sierra

consider analytical geometry. Closer to the current conception is what Fermat (1601–1665) developed in his book *Ad locos planos et solidos isagoge* (1679), published posthumously, but written before the appearance of Descartes' *La Géométrie*, and today recognized as having equal merit to Descartes.

In Lacroix's work and generally in all the textbooks of analytical geometry used in Spain in the first half of the nineteenth century, we find an analytical geometry very similar to the methods used by Descartes, and therefore we shall take a brief look at these before moving on to the analysis of Lacroix's work.

The notation used by Descartes in *La Géométrie* is very similar to that used today, but there is an important difference between the two: whereas we consider the parameters and unknown quantities as numbers, Descartes considered them as segments. This gives rise to a difficulty, and whereas an unlimited number of arithmetic operations can be carried out with letters, obtaining new combinations of letters, with segments such combinations are limited to the case in which the "dimension" of the result is one, two, or three, since in the other cases this result cannot be expressed in terms of geometric figures. To overcome this limitation, Descartes resorts to the idea of the unit segment, an arbitrary segment adopted as a unit, and operated with properly, it reduces every combination of segments, whatever its "dimension", to a single segment. Moreover, this unit is implicit, and in fact, neither the unit nor its operations will be explicit. Thus, to operate with segments it is simply necessary to indicate each point with a letter, and the result as the respective combination of these letters according to the rules of algebra. He also shows how to interpret the algebraic operations geometrically and applies all of it to solving geometric problems. To do so he translates the problem to algebraic language using the equivalence between segments and letters described above, then algebraically solves the equation obtained and finally constructs with ruler and compass the algebraic expression obtained as the solution (Smith & Latham. 1952).

As mentioned earlier, we shall find a similar way of addressing analytical geometry in Lacroix's treatise.

The aim of this paper is to analyse how analytic geometry is shown in this text, given the importance of this book in the curricula of secondary schools in Spain during the nineteenth century. This will help us later to find out the influence of Lacroix on Spanish textbooks authors in the time.

Sylvestre François Lacroix: Life and works

Sylvestre François Lacroix (1765, 1843) was born in Paris in the midst of a humble family, but received good academic training despite this. From an early age, he showed a special talent for mathematics, earning himself a recommendation from Monge, who had been his teacher, to enter the École des Gardes de la Marine to occupy a position as mathematics teacher in 1782,

at the age of 17. Lacroix also worked in other schools during the final years of the Ancien Régime, mainly in military schools.

After the Revolution he taught at the École Polytechnique, where he was *instituteur* of analysis from 1799 to 1808, and following that, he was a Professor at the Sorbonne and the Collège de France.

However, Lacroix's importance stems from his work to establish a general education system, and in particular to incorporate mathematics as an integral part of that system. Writing good textbooks was considered of great value in reforming the education system after the Revolution and this is where Lacroix played a major role, since his textbooks had an immense impact on the times by decisively contributing to the dissemination of mathematical contents among students.

Lacroix should be viewed as a writer of mathematics textbooks. His books were used over a period of approximately 50 years, from 1795 to 1845. Most of them were published in successive editions, and used not only in French schools, but in practically all European countries. Thus, in Spain they were used in naval and military academies, in secondary schools, as well as in some universities. He wrote books about almost all branches of mathematics and for almost all educational levels, from secondary to technical schools. As Schubring (1987, p. 42) observed, Lacroix can be considered an author whose work contributed decisively to the establishment of school mathematics in France.

He also had a great influence in Spain, like is showed in Vea works (1995), about the mathematics in Spanish secondary schools during the nineteenth century, and Escribano (2000), who made a descriptive analysis about the most used analytical geometry textbooks in this century.

Lacroix published his first textbook, *Traité de géométrie descriptive,* in 1795. He devoted himself to writing textbooks when he began teaching mathematics in the Ecole Centrale of Paris, at first for his own use and then for a wider audience. In 1797 he published *Traité élémentaire d'arithmétique* and the first volume of his important work *Traité du calcul differential et du calcul integral.* Owing to the urgent need for school textbooks, Lacroix published most of his work over the next four years. His efforts culminated in unique success: in 1803 the Commission charged with choosing the textbooks for the Lyceés selected only Lacroix's books for mathematics, and even in subsequent years his books always figured prominently (Schubring, 1987, p. 42).

The textbook we address here was first published in Paris in 1798 and re-edited many times, both in French and in Spanish. In Spain, this book was included on the lists of textbooks approved by the government for secondary education between 1846 and 1850, but it had already been used as a textbook at secondary schools and universities since the early 19th century (Vea, 1995; Escribano, 2000).

The analytical geometry in the treatise by Lacroix Spanish book

The book we analysed was the eighth edition of Tratado elemental de trigonometría rectilinea y esférica, y de la aplicación del álgebra a la geometría (An elementary treatise on plane and spherical trigonometry, and on the application of algebra to geometry) by S.F. Lacroix, published in Spanish in Madrid in 1846.

This book has 323 pages, 208 of which deal with the application of algebra to geometry and five bookplates with drawings at the back. It is divided into three chapters and an appendix. The first two chapters are devoted to plane and spherical trigonometry, respectively, the third to the application of algebra to geometry, and the Appendix to the fundamental principles of the application of algebra to curved surfaces and double curvatures. Here we focus our study on Chapter III.

Figure 1. Title page of the An elementary treatise on plane and spherical trigonometry, and the application of algebra to geometry by S. F. Lacroix (1846).
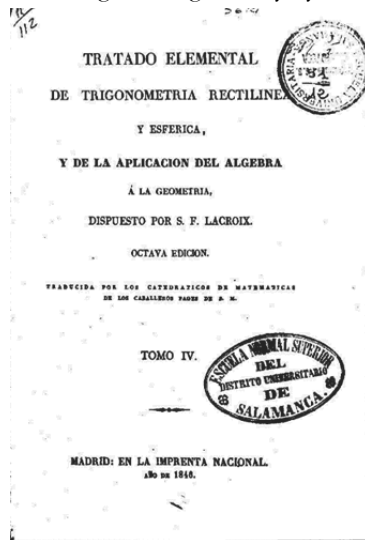
Lacroix classifies applications of algebra to geometry in three types: the first as a method for obtaining geometric results or theorems by expressing the geometric properties through equations and then developing them; the second as a tool for solving geometric problems; and the third to determine a curve.

In the case of problem-solving he considers that algebra can be applied the same as arithmetic, except that in the case of geometry "sometimes lines are sought", and for this one has to know how to translate algebraic operations into geometric operations. That is why he explains how to construct algebraic expressions geometrically. These operations pose two difficulties, the first being

that the expressions must be homogeneous. This is resolved by introducing the concept of the unit segment, the same as Descartes. The second problem is the existence of negative solutions, which would represent negative segments.

In the case of the unit segment, Lacroix explains what it consists of, and provides examples of how to use it in an explicit way, although in the subsequent problems it never appears explicitly. On the other hand, he addresses the topic of negative solutions quite in depth, discussing their interpretation and construction in different cases. Everything he explains is backed up by sample problems.

After demonstrating this way of solving geometric problems using algebra, Lacroix tackles the fundamental idea of Descartes' analysis in which curves are represented by means of equations with two unknown quantities, introducing with this idea a new way of geometrically interpreting an algebraic equation as the locus it represents. Thus, he provides several types of linear equations, the definition of coordinates, the definition of locus and he solves different problems in relation to linear equations and the circle, the distances and the angle between two straight lines. Using linear equations he also solves several geometric problems, some of which were solved in the previous section, as already mentioned.

Below we analyse each of these three types of applications in more detail, and we shall see some of the examples solved.

1. "How the algebra can serve to combine different theorems of geometry[2]"

At the beginning of chapter III he explains what the application of algebra to geometry consists of, and addresses the first of the applications: the theorems of geometry can be simplified by using algebra, and what is more, new theorems can be deduced from the transformations carried out on the equations (p. 99). One of the problems he provides as an example with the solution is the following.

Problem 1: Obtain the expression for the volume of a frustum given its height and the radii of the bases.
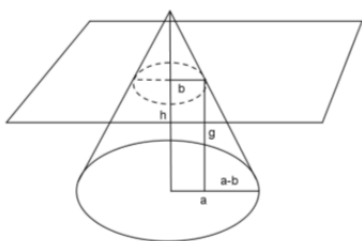
Let a and b be the radii of the bases of a frustum, let g be the height of the frustum and h the height of the whole cone.

[2] Lacroix, 1846, Index.

From the ratio *a-b:a::g:h* he obtains $h = \dfrac{ag}{a - b}$, and from this

$$h - g = \frac{ag}{a - b} - g = \frac{bg}{a - b} \text{ (p. 103).}$$

Since the bases are similar "they will be to each other as the squares of their equivalent lines; such that if we call the lower base *S* and the upper bases" (observe that he identifies the base with its area), we will have $S{:}s{::}a^2{:}b^2$, or $S{:}a^2{::}s{:}b^2$.

Let *m* be the quotient between *S* and a^2 obtaining $S = a^2 m, s = b^2 m$; and the volumes of the whole cone and the removed one will be expressed respectively by

$$\frac{1}{3} hS = \frac{1}{3}\frac{mga^3}{a - b}, \qquad \frac{1}{3}\left(h - g\right)s = \frac{1}{3}\frac{mgb^3}{a - b}$$

(…) subtracting the second expression from the first, will be for the volume of the frustum the next

$$\frac{1}{3}\frac{mg(a^3 - b^3)}{a - b} = \frac{1}{3}mg(a^2 + ab + b^2) = \frac{1}{3}g(ma^2 + mab + mb^2) \text{ (p. 104)}$$

Subsequently he observes that am^2 and mb^2 are the areas of the upper and lower bases of the frustum, and if we take $ab = c^2$, $c = \sqrt{ab}$ it will express the geometric mean between lines *a* and *b* and therefore $mab = mc^2$ will express the area of a circle with radius *c*, from which he concludes:

> (…)[3] the volume of a frustum is equal to a third of its height, multiplied by the sum of the areas of the two bases, and a similar figure built on a radius that is mean proportional between the radii of these bases (p. 104)[4].

He finishes by observing that since $mab = \sqrt{ma^2\ mb^2}$, the area of this figure is the geometric mean between those of the two bases.

He also adds that if three cones of the same height as the frustum proposed are constructed on each of these three bases, the sum of the volumes of these three figures will be equal to that of the frustum of the cone (1846, p. 104).

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[3] The quotes have been translated by Sánchez and González.

[4] (…) que el volumen de un tronco de pirámide ó de un cono es igual al tercio de su altura, multiplicado por la suma de las áreas de sus dos bases, y de una figura semejante construida sobre un radio medio proporcional entre los de estas bases (p. 104).

2. The algebra like a tool for solving geometric problems

After solving the problems about geometry theorems he makes the consideration that in these cases it is not necessary to construct the algebraic solutions geometrically, because of their nature, but he points out that "sometimes lines are sought" (1846, p. 104), introducing in this way the application of algebra to the solving of geometric problems.

In regard to the second type of applications he indicates that the geometric interpretation of an algebraic equation will be given so that a geometric problem can be turned into algebraic language, and once the operations have been carried out the algebraic solution obtained can be used to make the geometric construction.

Let us look at one of the problems he solves to exemplify this.

Problem 2: Inscribe a square into a triangle
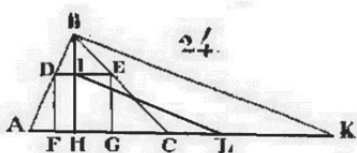
Let the given triangle be *ABC*. We assume square *DEFG* is already inscribed in it. Let the base be *AC=a*, the height, *BH=b*, and the side of the square, *DE=IH=x*. The similar triangles *ABC, DBE* give $\frac{AC}{BH} = \frac{DE}{BI}$, or analytically $\frac{a}{b} = \frac{x}{b - x}$, and

$x = \frac{ab}{a+b}$, a formula that is constructed looking for a fourth proportional to *a+b, a* and *b* (p. 128).

To do so take two points *L* and *K* on the extension of the base so that *HL=a* and *LK=b*. Draw a line KB through this last point, and a parallel line through L, thus forming two similar triangles from which we obtain the proportion, $\frac{b}{a+b} = \frac{IH}{a}$ that is, segment *IH* is the segment sought. To construct the solution draw a line parallel to the base through point *I*, and then two perpendicular lines through points *D* and *E,* respectively, thus obtaining the inscribed square.

For Lacroix it is necessary to explain the general constructions of the different algebraic expressions that provide us with the solutions of an equation, something he calls the "construction of the values of the unknown quantity" since once its expression is obtained the plotting operations corresponding to the ones indicated by the operational algebraic signs should be carried out on the known lines (p. 108).

He explains how to construct quotients, square root expressions, and the solutions to quadratic equations, finishing with the construction of polynomial expressions of degree greater than two. We shall briefly analyse the first two.

He begins by explaining the construction of quotients. He proves that all quotients can be constructed through fourth proportional to the given lines (p. 109).

As regards square roots, he considers two cases. In the first place, the ones in which the root is the sum or the difference of two or more squares, the simplest cases being $\sqrt{a^2 + b^2}$ and $\sqrt{a^2 - b^2}$, which can be constructed, respectively, as the hypotenuse of a right-angled triangle with sides a and b, and as one of the catheti of a right-angled triangle whose hypotenuse would be a and the third side b (p. 111). Secondly, he explains how to construct the expression $\sqrt{ab}$, for which he uses the properties of the chords of a circumference. For example, he uses the fact that a perpendicular line drawn through the diameter is the mean proportional between the two segments it divides it into. He finishes by indicating that all square roots can be constructed with these methods.

In regard to questions concerning the homogeneity of expressions, we have already mentioned that Lacroix, like Descartes, solves this problem using the concept of unit segment and, even though in practice he does not use it in the solving of any problems, he does explain how to transform a heterogeneous expressions into a homogeneous one (p. 113).

As an example, he uses the expression $\sqrt{a + \dfrac{bc}{d^3}}$, which is heterogeneous. Representing the unit with n and multiplying by it properly we have that $\sqrt{a + \dfrac{bcn^3}{d^3}}$ is homogenous (p. 113).

As regards the negative solutions of an equation, Lacroix studies them and gives his geometric interpretation.

> 74. In the application of algebra to geometry the sign – is interpreted in general as it is done with regard to the numbers, inverting in a certain way the wording of the question, or taking the lines that are affected by that sign in the opposite sense to that in which they had been supposed at first (p. 117).[5]

He continues to carry out a broad study of the negative numbers, providing an explanation of why they appear in a problem: the starting formulation is erroneous, or at least in its application to the particular case under consideration, and it must be changed to convert negative solutions to positive ones.

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[5] 74. En la aplicación del álgebra á la geometría el signo – se interpreta en general como se hace respecto á los números, invirtiendo de cierto modo el enunciado de la cuestion, ó tomando las líneas que son afectadas de tal signo en un sentido contrario á aquel en que se las habia supuesto desde luego (p. 117).

We must remember that negative amounts have their origin in those subtractions which cannot be executed in the order in which they are listed, because the amount that must be subtracted is greater than that from which we must subtract. For this reason it is recognised that there is an error in the wording of that question, or at least in its application to the particular case which is being taken under consideration, and removing this error, that is, changing the wording of the question so that it is possible the subtraction which couldn´t be executed before, we should have a positive result (…) (p. 117).[6]

This interpretation of negative numbers is specific from a certain period in France. It appears in the work of other authors, such as Bézout (1730–1783) or Carnot (1753–1823), who directly influence Lacroix, although this idea originally comes from D'Alembert (1717–1783) (Schubring, 2005).

In the dissertation that we are developing we have observed that this way of working with negative solutions is common in the analytic geometry textbooks of the most important Spanish mathematicians, almost until the end of the 19th century (Sánchez, Sierra y González, 2008).

Lacroix clarifies that in some cases, particularly in problems that lead to first degree equations, it is not necessary to make such a change because the very sign of the result indicates the inversion that the formulation is susceptible to (p. 118). These ideas can be verified in the following construction:
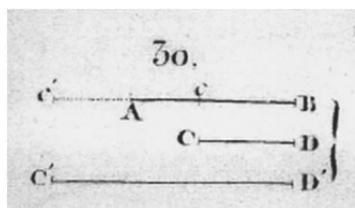
It is also the subtraction which should explain, on the geometric figures, the negative values that the algebra gives to certain lines; to subtract one line from another is enough to take the first over the second, starting from one end of the last one; but on this graphic operation you have to do a few remarks arisen from the way the lines are described. Thus, it should be CD, fig. 30, the line to be subtracted from AB: as the first line is shorter than the second, taking CD from B to c, the difference Ac will be placed on the right of point A; but if you had to subtract C'D' longer than AB, and we always take it on AB, starting from the same end B, the difference of the two proposed straight lines would be recorded as Ac' on AB prolongation, and would be placed on the left of point A, that is, from the opposite side to Ac result of the first operation; therefore, the sign – corresponds to this change of situation (p. 118).[7]

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[6] (…) debemos recordar que las cantidades negativas tienen su orígen de aquellas sustracciones que no pueden efectuarse en el orden en que ellas se hallan indicadas, porque la cantidad que se ha de restar es mayor que la de quien se ha de restar. Por esta circunstancia se reconoce que hay un error en el enunciado de aquella cuestion, ó á lo menos en su aplicación al caso particular que se tiene en consideración, y quitando este error, esto es, modificando el enunciado de la cuestion, de suerte que se haga posible la sustraccion que antes no podia ejecutarse, se deberá tener un resultado positivo(…); (p. 117)

[7] Es tambien por la sustraccion por quien debe explicarse, sobre las figuras geométricas, los valores negativos que el álgebra da á ciertas líneas; para sustraer una línea de otra basta llevar la primera sobre la segunda, partiendo desde uno de los extremos de la última; pero sobre esta operación gráfica hay que hacer algunas observaciones nacidas del modo con que las líneas se

The negative answer tells us that we have taken a larger segment from a smaller one, and thus we do not have to make a change in the formulation, but simply take this consideration into account.

He continues developing this idea and provides several more examples to clarify the concept. He also solves several problems which have negative solutions, as well as constructions of algebraic expressions explained earlier. Below we have a clear example of his way of working with negative numbers.
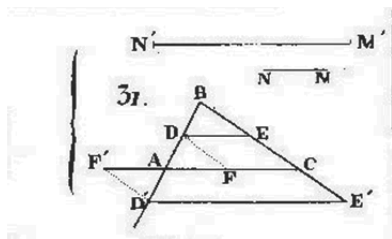
Problem 3: In a given triangle ABC draw a line DE parallel to line AC that is equal to the given line MN.

In this problem, he shows the construction of a quotient and a negative solution.

He assumes the problem is solved and makes $AB=a$, $AC=b$, $MN=c$.

He takes as the unknown quantity the distance $AD=x$, "because the position of a line parallel to a given line is determined by only one of its points," and therefore $BD=a-x$. The similar triangles BAC and BDE give $AB:AC::BD:DE$, or $a:b::a-x:c$.

From the ratio above he obtains the equation $ab - bx = ac$, and from that
$$x = \frac{ab - ac}{b} = \frac{a\left(b - c\right)}{b} .$$

For the construction of the value of x he refers back to the point in which he explained the construction of a quotient, and tells us that to build it, it is sufficient to "subtract from AC=b the straight line CF=c, and draw FD parallel to CB (…)" (pg. 119).

Below he demonstrates, mixing algebraic and geometric methods, that the segment thus constructed is the one sought.

(…) the similar triangles ABC and AFD give the proportion $AC:AB::AF:AD$

$$b : a :: b - c : x = \frac{a\left(b - c\right)}{b} \text{ (p. 119).}$$

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describen. Sea pues CD, fig 30, la línea que debe sustrerse de AB: como la primera es menor que la segunda, llevando en esta la CD desde B hasta *c*, su diferencia A*c* estará colocada á la derecha del punto A; pero si se tuviese que restar C'D' mayor que AB, y esta se llevase siempre sobre AB, partiendo desde el mismo extremo B, la diferencia de las dos rectas propuestas seria anotada en A*c'* sobre la prolongacion de AB, y estaria colocada á la izquierda del punto A, esto es, de un lado opuesto al resultado A*c* de la primera operación; es pues á esta mudanza de situación á quien corresponde el signo - (p. 118).

Finally, he explains the case in which the line given is greater than the base of the triangle:

> If the line MN was longer than AC, it could not exist inside the triangle ABC; it would be necessary to extend the sides AB and BC; but then the point D would become D' on the other side of the point A, and this is precisely what indicates the calculation and construction (p. 120)[8].

Note that he always gives the geometric proof and the algebraic proof and verifies that one does not contradict the other. He also provides the justification of what he had just said:

> In fact, if you have M'N'>AC, it will be c>b; therefore, the amount b-c will be negative; but doing the subtraction of the lines as it has been pointed out in the previous section, F will become F ', and F'D' line, drawn through the point F' in parallel to BBC, will only find the prolongation of the side AB in D' (p. 120).[9]

3. To determine a locus

To end, we shall analyse the third application of algebra to geometry that
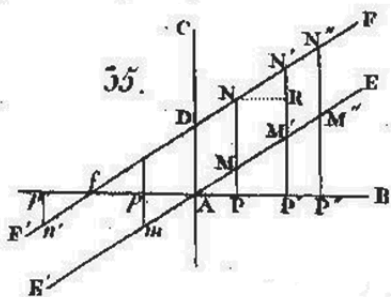
Lacroix gives, based on the concept of locus. He begins this part by explaining this new approach: the curves can be expressed through algebraic equations (p. 132), a fundamental concept of analytical geometry. What is more, Lacroix speaks explicitly of locus and explains what it is, giving several examples, among them the equation of a straight line, which we analyse below.

In this case, he first obtains, using geometrical reasoning, the equation based on the graph, and later he concludes that the locus for the first-degree equation is a straight line.

In the first case he considers the straight line *AE* (fig. 35) and points out that triangles *APM, AP'M', AP''M''*, etc., formed by drawing lines perpendicular to straight line *AB* through each of its points (*PM, P'M'. P''M''...*); are similar triangles, from which the following proportions are obtained:

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[8] Si la linea MN resultase mayor que AC, ella no podría existir en el interior del triángulo ABC; se necesitaria prolongar los lados AB y BC; pero entonces el punto D pasaria á D' del otro lado del punto A, y esto es precisamente lo que indica el cálculo y la construccion.

[9] En efecto, si se tiene M'N'>AC resultará c>b; por lo mismo la cantidad b-c será negativa; pero haciendo la sustraccion de las líneas como se ha indicado en el número anterior, el punto F pasará a F', y la línea F'D', tirada (dibujada) por el punto F' paralelamente á BBC, no podrá encontrar sino la prolongacion del lado AB en D'.

$$\frac{PM}{AP} = \frac{P'M'}{AP'} = \frac{P''M''}{AP''} = ...$$

Let *a* be the constant relationship between the distances *AP, AP', AP''*... and the perpendicular *PM, PM', PM''*... lines, obtaining *PM=axAP, P'M'=axAP', P''M''=axAP''*,..., and he concludes that although all these equations appear to be particular for each point of the straight line *AE*, they can be joined in one, simply by calling *x* the distance from the foot of the perpendicular to point *A*, and *y* to the corresponding perpendicular, so then we have that *y=ax* (p. 134).

That is, Lacroix obtains the explicit equation of a straight line, although he does not call it that.

After explaining what the locus is that corresponds to an equation, he comes back to study it again from the perspective of algebra, thus:

> 87. Of all the equations in two indeterminate, the simplest one is the first-degree equation; it corresponds to the straight line, the simplest of all lines. This equation can be represented by *Cy = Ax + B* (...) (p. 139).[10]

We can say that in this case he gives the general equation of a straight line, even though he does not call it such. Dividing by *C* we obtain $y = ax + b$, giving $\frac{A}{C} = a, \frac{B}{C} = b$, that is, the explicit equation again, and he extracts from it the property that characterizes the straight line, the geometric space corresponding to this equation. He begins by studying equation $\frac{y}{x} = a$:

> If we suppose that *b* is zero, we will have *y=ax*, or *y/x=a*; that is, along the straight line, the ratio of PM to AP will be constant. This property, which is the expression of the similarity of the APM, AP'M'… triangles, on the line AB, belongs to the straight line AE, drawn from the point A which is the origin of coordinates (p. 140).[11]

He then studies the relation $\frac{y}{x}$, or, what is the same, the value of *a,* from which he concludes, based on the right-angle triangle *APM* in the figure, that it "expresses the tangent" of the angle formed by line *AE* and the abscissa axis.

Finally, he again takes up the equation $y = ax + b$, and finds its locus. He deduces that line *DF* parallel to *AE* will be the locus of the equation $y = ax + b$,

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[10] 87. De todas las ecuaciones en dos indeterminadas la más simple es la de primer grado; ella pertenece á la línea recta, la más simple de todas las líneas. Esta ecuacion puede representarse por *Cy =Ax+B* (...).

[11] Supongamos que b sea nula, se tundra y=ax, ó y/x=a; esto es, que en toda la extension de la recta, la razón de PM á AP, fig. 35, será constante. Esta propiedad, que no es otra cosa que la expresion de la semejanza de los triángulos APM, AP'M' & c., sobre la línea AB, no puede pertenecer sino á la línea recta AE, tirada por el punto A del origen de las coordenadas (p. 140).

since *PN=PM+MN=PM+AD, P′N′=P′M′+M′N′=P′M′+A,* etc. He ends by pointing out that coefficient *a* is the same for all straight lines parallel to *AE* (p. 140).

Lacroix solves several problems using systems of coordinates and linear equations. We include the solution of one of them, which, as we shall see, is the generalization of the previously solved problem 2.
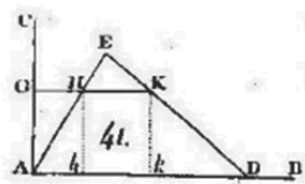
Problem 4: Given two straight lines *AE* and *DE* (fig. 41) by the angles they form with a third line *AB,* and by the part *AD* that they intercept on this third line, find on line *AC,* perpendicular to *AB,* a point *G,* such that drawing a straight line *GK* parallel to *AB,* the part *HK* between *AE* and *DE* will be a given magnitude (*m*).

To solve it on this occasion he uses purely algebraic methods. He establishes the equations of lines *AE* and *ED,* naming *a* and *a′* the tangents that angles *EAD* and *EDA* form, respectively, with line *AB.* He takes this as the axis of abscissas, *A* as the origin of the coordinates and *AD=α.* Line *AE* will have as its equation $y = ax$, since it passes through the point of origin *A.* Since the second one passes through point *D* (0,α) it is deduced that its equation will be $y = -a′(x - a)$ (p. 152).

We are now seeing a way of working that is completely different from the problem 2. Whereas here he uses only algebra, in the previous case he combined algebra with pure geometry.

He obtains the abscissas of points *H* and *K* by taking line *GK* and intersecting it with lines *AE* and *ED.* He considers $y = AG$ and assumes *AG=t,* which by intersecting with lines *AE* and *ED* gives us $t = ax, \quad t = -a′(x - a)$. Finding x in each of these equations we obtain

$$x = \frac{t}{a}, x = \frac{aa′ - t}{a′}$$, which are the *Ah* and *Ak* abscissas, respectively.

The difference between them gives the distance *HK,* which should be equal to the given magnitude *m.*

$$m = \frac{aa′ - t}{a′} - \frac{t}{a}$$

Solving this equation he obtains the length of segment $AG, t = \frac{(a - m)aa′}{a + a′}$ with which he concludes (p. 153).

Using this result, he obtained the solution to the problem of the square solved previously. He supposed that the *HK* segment is equal to the *AG,* "which is equivalent to inscribe a square into a triangle" (p. 153).

In this case, instead of making *HK* equal to *m*, he makes it equal to *t*, "which will give $t = \dfrac{aa' - t}{a'} - \dfrac{t}{a}$, from which it can be deduced that $t = \dfrac{aaa'}{aa' + a + a'}$ " (p. 154).

Conclusions

Our analysis shows that, in Lacroix's work, there are three applications of algebra to geometry, but two different ways of solving geometrical problems with help of algebra. On the one hand, he considers an application of algebra to geometry in which analytical methods are combined with synthetic ones. To solve a problem, instead of constructing the solution solely with ruler and compass, he makes use of algebra to translate the geometric approach to the algebraic approach, solving the corresponding equation in which the letters represent segments and then constructing with ruler and compass the algebraic expression obtained as the solution. This makes it necessary to develop theoretically the construction of algebraic expressions and interpret the negative solutions to an equation. Lacroix's interpretation of these solutions is typical of a period in France (Schubring, 2005), but also in Spain due to the influence of several French authors, including Lacroix, whose work has been used at universities and secondary schools for over half a century.

This method of solving problems, which has much in common with the geometry of Descartes, has both advantages and disadvantages. On one hand, it includes a geometric view of the algebraic solution that is lost in current textbooks but, on the other, being tied to geometry when using algebra complicates the solving of the problem, because after obtaining the algebraic solution it has to be constructed geometrically.

Moreover, Lacroix presents another application of algebra to geometry, similar to current analytical geometry, based on the concept of locus, which uses coordinates to solve problems. Once the geometric approach has been translated to the algebraic approach, it is left out in the finding of the solution. This simplifies the solving of the problem, but a large part of its geometric interpretation is lost in the process.

This analysis shows an analytic geometry typical of the nineteenth century in Spain. It preserves some elements from the geometry of Descartes but it also includes elements from the algebraic geometry appearing nowadays in secondary education mathematics textbooks.

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Maximum and minimum: Approaches to these concepts in Portuguese textbooks

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Abstract

The approach in Portuguese textbooks to the concepts of maximum and minimum has undergone alterations over the years. These changes were influenced by the official curriculum and more recently by the introduction of the graphic calculator in high schools. The concept of the derivative was first mentioned in the national curriculum in 1905 (Aires, 2006; Santiago, 2008). However, not until 1954, with the change of the program, the official curriculum began to make reference to the application of derivatives in the study of the variation of functions. In this study we will begin by examining the first national curriculum, which makes reference to the concepts of maximum and minimum of a function. We will, then, continue analyzing how the concept was shown in Portuguese textbooks, comparing two textbooks and identifying differences and similarities.

Introduction

The analysis of official programs and textbooks is an important part for researchers in Mathematics Education. According to González (2005) "For Mathematics Education researchers, textbooks are a privileged source of information because they can find information about the development of a specific content and the conceptual aspects, activities, problems, exercises sequenced in them" (p. 34).

In this communication we will analyze the textbooks' approach to the maximum and minimum concepts just as an application of the derivative concept.

We will start analyzing the first national curriculum that makes reference to maximum and minimum concepts. Then, we will present the analysis of two

textbooks one from 1955 and another one from 1958. Both of these textbooks were edited in the period of unique book[1] regime, during which the Ministry of Education selected a textbook that must be used in all secondary schools in Portugal. For each one we will explain where and how maximum and minimum concepts appear, and the definitions and examples/applications they show.

Method and theory

According to Schubring (2005) analyzing official programs is a mean to understand the aim of a group in the educational community or the ministry policy; however, by analyzing textbooks we can see the teaching reality because they determine it decisively.

As historical researchers, we use the method defined by Ruiz Berrio (1997), which consists of four phases: heuristic, criticism, hermeneutic and exposition. The heuristic phase consists of searching and collecting documents essential for the research. After the selection of documents we proceed to its criticism analysis, with intern criticism (reliability) and extern criticism (authenticity). In the hermeneutic phase the interpretation of the data is done and, finally, in the exposition phase the description of the results are done.

The official programs references to maximum and minimum

The derivative concept was introduced in Portugal official programs in 1905 (Aires & Santiago, 2014). It made part of the 7th year from *liceu*[2] (the last year before University studies) and was included in the algebra chapter. Let us see the reference in the official program to the derivative:

> Derivative notion; its geometrical interpretation. Derivative of a sum, product, quotient, potency, square. Derivative of circular functions. Revisions. (Decreto-Lei[3] Nº 3 from 03/11/1905)

As we can see, this program did not make any reference to maximum and minimum concepts; it just referred to the derivative notion, its geometrical interpretation and the derivative rules.

Between 1905 and 1950, except between 1936 and 1948, the derivative concept made part of 6th or 7th year of the *liceu* studies (15 and 17 years old

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[1] Translation of "Livro Único"
[2] High school
[3] Law

students) and was included in an algebra chapter or a calculus chapter inside the textbooks.

The applications of the derivative only appeared in national curriculum in 1954, as part of the 6th year of *liceu* in an algebra chapter. Let us see the reference in the official program of the derivative:

> Derivative of a function at a point; derivative function. Derivative of algebraic functions. Application to the study of variation in simple cases. (Decreto-Lei Nº39 807 from 07/09/1954)

We can verify that this program, from 1954, deepens the study about the derivative concept, considering not only the derivative notion and its rules; but also the application to functions variation in simple cases.

As in this period we had the Unique Book regime, in the end of this official program we found out information about the textbook: "*Compêndio de Álgebra, num volume*".

Textbooks approach to the concept of maximum and minimum

According to Aires and Santiago (2011), in 1947 the Unique Book regime started over again and in 1950 was approved as Mathematics textbook "Compêndio de Álgebra" from António Augusto Lopes (Diário de Governo nº 145, II Série, from 24/06/1950). The law foresaw that in 1954 another competition should choose over again a textbook. In this competition António Augusto Lopes' textbook and two other textbooks were selected. This competition had some problems and in 1955 a new competition was opened with another textbook by J. D. da Silva Paulo (as one of the authors of this textbook) competing. This was the person that should choose the textbook and so this was the chosen textbook. However, it was only chosen in 1958 (Almeida, 2014).

In the following we present an analysis of both textbooks: the first one from António Augusto Lopes "*Compêndio de Álgebra para o 6º ano dos liceus*" published in 1955 from Porto Editora in Porto (Fig. 1), and the second one from J. Sebastião e Silva e J. D. da Silva Paulo "*Compêndio de Álgebrapara o 3º ciclo dos liceus*" published in 1958 from Livraria Rodrigues in Lisbon and approved as Unique Book in 22/1/1958 (Fig. 2).

Figure 1: Book cover from 1955   Figure 2: Book cover from 1958
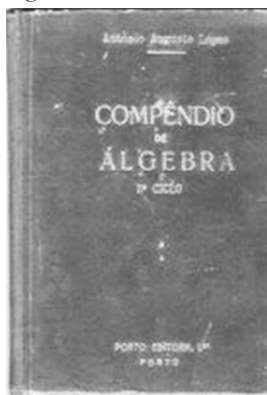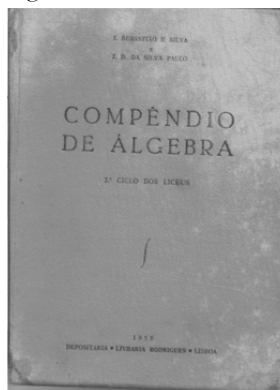
Both of these books have hard covers. The first one is just for the 6th grade of *liceu* and the second one is for 6th and 7th grade of *liceu*.

In the textbook from António Augusto Lopes, we can see that it has seven chapters and the 5th is dedicated to the derivative of functions with one variable and functions variation. This chapter has three parts, the first one entitled "*Definition*", the second one entitled "*Derivative of algebraic functions*" and the last one entitled "*Functions variation*".

It is in the last one that we find the maximum and minimum concepts. This part starts explaining the relation between derivative and direction of variation:

> I.   *If a function is increasing in (a, b), it's derivative is positive or null.*

> II.   *If a function is decreasing in (a, b), it's derivative is negative or null.*

> III.   *If a function is constant in (a, b), it's derivative is null for all values of x in this interval.*

And reciprocally:

> I.   *If the derivative is positive for all values of x in (a, b), the function is increasing in this interval.*

> II.   *If the derivative is negative for all values of x in (a, b), the function is decreasing in this interval.*

> III.   *If the derivative is null for all values in (a, b), the function is constant in this interval.* (p. 169–171)

The first three were followed by the explanation and a graphical representation. In the following we present the explanation from the first one:

Let x_0 and $x_0 + h$ be two values of x, y_0 and $y_0 + k$ the corresponding values of the function. As the function is increasing, k e h has the same sign; $\frac{k}{h}$ is positive: when h tends to zero, the limit of $\frac{k}{h}$ exists, for hypothesis, and can't be neither thing except positive nor null.
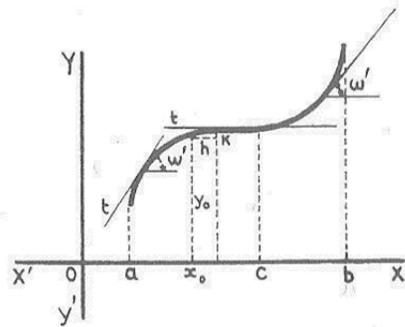
Fig. 52

So the function $y = f(x)$ represented on fig. 52 is increasing in (a, b); the image has an unique tangent in each one of its points: the gradient of the tangent, that is to say, the derivative, is positive (because h and k has the same signal), for all of the values of x in (a, b), except for x=c; the tangent is parallel to X'X. (p. 169)

After that, the author presents ''*Local maximum and minimum (elementary notion)*''. In this part, the author starts establishing a relation between the signal of the derivative and the monotony of the function, then the definitions of the maximum and minimum concepts are presented. The definition of maximum in this book is:

Referring the neighbourhood of the value x = c, we can recognize that the function is increasing on the left side of c, in other words, in an interval (c-h, c) and decreasing on the right side of c, in other words, in an interval (c, c+h). The value of the function for x = c, is bigger than its values for x = c-h and x = c+h:

$$f(c) > f(c - h); f(c) > f(c + h)$$

We can say that the function has a maximum f(c), in x = c.(p. 171)

This definition relates the monotony of the function and the extreme of the function.

After that, the author presents the relation between extremes of the function and the signal of the derivative:

For the values c and c', were the function has maximums or minimums, the derivative is null changing the signal:

a) If it's a maximum, the function moves from increasing to decreasing and, for that, the derivative changes from positive to negative;

b) If it's a minimum, the function moves from decreasing to increasing and, for that, the derivative changes from negative to positive;

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It's indispensable the change of signal; if it doesn't happened, the sense of variation of the function doesn't change (p. 172)

The author notes that the maximum or minimum is the biggest/lowest value in an interval and a maximum could be smaller than a minimum. This note is illustrated with a graphical representation of a function *f(x)* in an interval $[a, b]$ where the maximum *f(c)* is smaller than the minimum *f(c')* (Fig. 3).

Figure 3: Graph representation
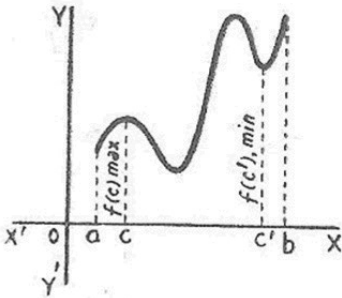
Fig. 56

Then, we find examples of the study of the variation in two different kinds of functions: linear and quadratic. In each example the first step to find the extreme is to calculate the domain, the second step is to differentiate the function, the third step is to make the study of the signal of the derivative and, finally, the conclusion relates the signal of the derivative with the monotony of the function. Those examples are accompanied by the graphical representation of the function, as we can see in the next excerpt:

Study the variation of the function $y = x^2 - 3x + 2$.
1. *Field of existence: (- ∞, + ∞).The function is continuous for all the real values of x.*
2. *Derivative function:* $y' = 2x - 3$. *It's a continuous function in x. Writing it as*

$$y^{\wedge\prime} = 2(x - 3/2)$$

we can conclude: it is negative, null and positive depending on

$$x < \frac{3}{2}; x = \frac{3}{2}; x > \frac{3}{2}; \text{ So that:}$$

1ˢᵗ. For $x = \frac{3}{2}$, *the function admits a minimum* $f\left(\frac{3}{2}\right) = -\frac{1}{4}$, *because, when x changes from* $\frac{3}{2} - h$ *to* $\frac{3}{2} + h$, *the derivative changes from negative to positive;*

2ⁿᵈ. As the derivative is null just for $x = \frac{3}{2}$, *the function is decreasing in the interval* $\left(-\infty; \frac{3}{2}\right)$ *and is increasing in the interval* $\left(\frac{3}{2}; +\infty\right)$.
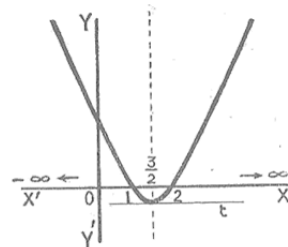
Fig. 59

> 3rd. *The geometrical representation of the function (fig. 59) shows these conclusions and let us know the signal of y when x goes from* $-\infty$ *to* $+\infty$. *Particulary, the function is null when x=1 and x=2.* (p. 174–175)

After the examples the author explains that in this year it is just considered the study of the variation of a function in the simplest cases: linear functions and quadratic functions, but in the 7th year it will be considered functions with a 2nd grade polynomial as the derivative.

At the end of this chapter, as well as in the other chapters, we find eight exercises and the bibliography:

> *V. Herbiet, Compléments d'Algébre, Namur, 1947;*
> *Cours d´Algébre (par une réunion de professeurs), Paris, 1936.*

In this bibliography there are clear French and Belgian influences.

The other textbook by J. Sebastião e Silva and J. D. da Silva Paulo "*Compêndio de Álgebrapara o 3º ciclo dos liceus*" was published in 1958 from Livraria Rodrigues in Lisbon and was approved as a unique book in 22/1/1958.

This textbook has twenty-two chapters: nine for the 6th grade of *liceu* and thirteen for the 7th grade of *liceu*. In 6th grade part, the 7th chapter is dedicated to the derivative. This chapter has four parts, the first one entitled "*Introduction*", the second one entitled "*Derivative concept*", the third one "*Derivative rules*" and the last one entitled "*Derivative Applications*". After that we find exercises and a historical note.

The part called "*Derivative Applications*" starts explaining the direction of variation of a function, establishing the relation between the signal of the derivative and the direction of variation:

> I.   *If the derivative is positive in all point of an interval, the function is increasing in this interval.*
>
> II.   *If the derivative is negative in all point of an interval, the function is decreasing in this interval.*
>
> III.   *If the derivative is null in all point of an interval, the function is constant in this interval.* (p. 225)

This is followed by the explanation with a graphical representation (fig. 4).

Figure 4: Graph representation
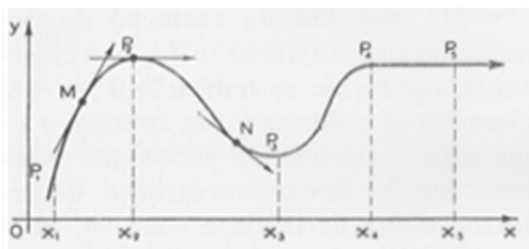
Then begins a part called "*Application of the theorems stated*" that starts explaining how to make the table of monotony with these following two rules and its explanation:

> *Point where the derivative of f(x) change signal, passing from positive to negative, is a point of maximum relative of f(x) (supposing the function continuous in these point).*

> *Point where the derivative of f(x) change signal, passing from negative to positive, is a point of minimum relative of f(x).* (p. 227)

These rules are followed by some examples: quadratic function, cubic function and homographic function.

The author solves some examples in the same way: first he calculates the derivative and its zeros, then he explains the signal of the derivative before and after each zero and builds the table of monotony and finally he establishes the conclusion that the zeros are a maximum or a minimum point. After that the authors suggest the student to draw the graph of the function in an interval.

> *Let be the cubic function:*

$$f(x) = x^3 - 12x + 7$$

> *we have:*

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2)$$

> *The function $f(x)$ is null at the points 2 and -2. And:*

> *for x<-2, x-2<0, x+2<0, so f'(x)>0;*

> *for -2<x<2, x-2>0, x+2<0, so f'(x)<0;*

> *for x>2, x-2>0, x+2>0, so f'(x)>0;*

We have the table:

	$-\infty$		-2		2		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$			23		-9		
		$\nearrow$	máx	$\searrow$	mín	$\nearrow$	

So, at the point -2, the function has a relative maximum, equal to 23, and, at the point 2, a relative minimum, equal to -9.

We recommend the student to draw the graph of this function in the interval $[-5, 4]$, taking as unit 1 cm in the x axis and 1 mm in the y axis. (pp. 228–229)

In the part called "*Concrete Applications*" the authors refers that this theory is applied in various questions (geometry, physics, etc) and two examples were presented:

1. *Among the triangles rectangles that the hypotenuse measures 6 cm, determine those with maximum area.*(p. 230)

2. *It is intended to construct a cylindrical boiler, closed, with volume V, so that total area was minimum. Determinate the radius of the basis, r, and the height, h, of the boiler with that conditions.* (p. 231)

Finally the authors set a point about "*Successive derivatives*" about the second and third derivative, relating it with physics and with inflection points, with exercises followed by their solutions and, after that an historical note.

Conclusion

Comparing both textbooks we can identify some differences and similarities. In both textbooks we can find the relation between the signal of the derivative and the monotony of the function and the definition of maximum and minimum, however, the textbook from 1955 presents the results in a more descriptive way.

Both textbooks show graphs of functions with the tangent line to some points. The textbook from 1955 has three graphics: one for an increasing function, one for a decreasing function and one for a constant function. The textbook from 1958 shows just one graph with a part that increases, another that decreases and another where it is constant. However, the explanation of maximum and minimum definitions is accompanied by a graph only in the textbook from 1955.

The textbook from 1958 is innovative in some aspects: a table of monotony is made and it is the first one that includes concrete applications (optimization problems) and successive derivatives.

The resolution of the exercises also has some differences. In the textbook from 1955 the first step is to calculate the domain, the second step is to find the derivative of the function, then the zeros of the derivative and the signal of the derivative. Finally, the conclusion is that the zero of the derivative is a maximum, justifying that this is because the function is decreasing before the point and increasing after the point. In the textbook from 1958 the author does not calculate the domain in the beginning. He starts calculating the derivative of the function, then the zeros of the derivative and, after that, he explains where the derivative is positive and negative. He then constructs the table of monotony presenting, finally, the maximum and minimum points. Then the author refers that it is advisable for the student to draw the graphic of the function in a given interval.

Another difference is that the textbook from 1955 presents the bibliography at the end of this part whereas the other one does not present any bibliography. The historical note is only presented in the textbook from 1958 as is done in other chapters.

Finally, another difference is that all exercises presented in the textbook from 1955 are arithmetic or geometric, but without a real context. However, the other textbook presents only some exercises in context: one about a cylindrical boiler and another about a rectangular box.

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The emergence of the profession of mathematics teachers – an international analysis of characteristic patterns

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Abstract

When and where emerged the profession of mathematics teachers? Which were the characteristic patterns of professional profiles? Which professional competencies were required to act as a mathematics teacher? Were these patterns the same in different countries or were there revealing differences? A first answer is that the emergence of the profession is intimately tied with the establishment of a respective teacher training so that research has at first to focus on the modes of teacher training. Since there used to be no teacher training before the French Revolution, it is characteristic that a key pattern is constituted by the development when the state assumes responsibility for the education system in the respective country and organizes in particular teacher education. Yet, the state did not assume that responsibility in the various countries at the same time and in particular not necessarily for mathematics as a major teaching subject. The differences in the changes of the state functions are already revealing for deeper cultural-social patterns, which influenced the establishment of public education systems. Mathematics education and their teachers were particularly affected by these differences, given its status between general education and vocational training. Analysing developments in various European and American countries reveals telling differences regarding the forms of teacher training, the competencies required and the competencies acquired, the patterns of teaching profiles within school practice. Particularly important proved to be the position of the mathematics teacher within the staff of his/her secondary school (Gymnasium, college, collège, liceo, liceu, ...): varying here between an isolated specialist, at the margin of the staff and with less rights than, say, the teachers of classical disciplines, and a fully integrated teacher contributing to the school's educational programme and with the same rights for professional promotion. The paper intends to introduce into the research on these questions.

Pre-Modern Times

It was at the beginning of Pre-Modern Times (roughly from 1500 to the French Revolution) that schools – mainly as secondary schools – became systematically organized in Western Europe: first as an impact of the Protestant Reform, and then, as a reaction, within the Catholic Counter-Reform. In both school systems, no teacher formation was established, in general. In the Protestant states and regions, the schools were largely run by the municipalities and it was beyond their capacities to care for teacher education. In a few cases, there were secondary schools run by the respective governments – like the *Fürstenschulen* in Saxony and the *Klosterschulen* in Württemberg –, but the governments saw no need to care for teacher formation. As a matter of fact, there used to be graduates of some university studies who did not find a pertinent position and opted for teaching in a Gymnasium.

On the other hand, in Catholic states and territories, there was no state intervention at all. Jesuit colleges organized secondary schooling in an autonomous manner, yet based on huge grants provided by larger towns, sovereigns or sponsors from nobility. Since all the teachers came from the same order, there was no need for a genuine teacher formation – what the superiors thought to be necessary, would be achieved within the preparation of the novices.[1]

It is highly revealing that the first state initiatives for teacher training concerned primary schooling and not secondary schools. The history of school disciplines uses to focus on secondary schools and thus one disregards often what was going on in primary education. Since it became increasingly disturbing and counter-productive, from the middle of the 18th century at least, that persons contracted as teachers in primary schools were notoriously unqualified – veteran soldiers were often quoted –, governments in Catholic and in Protestant countries began to care. First institutions for teacher training were founded in Austria, in Northern Germany, in Naples. Evidently, this was for preparing to teach the three R's, without special emphasis on mathematics.

The beginning of teacher formation in Modern Times

It is likewise telling that the first concrete institutional measure taken after the French Revolution, and in particular after dissolving all teaching institutions of the *Ancien Régime*, was the establishment of the first school for teacher formation for primary schools at a higher education level in France: the *École normale* of the Year III, hence of 1795 (see Schubring 1982). This foundation corresponded to the conception of the Plan Condorcet of 1792: to establish a

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[1] Christopher Clavius had proposed to establish formation specifically for mathematics teachers, but this was rejected by the Jesuits and did not enter hence into the *Ratio Studiorum* of 1599 (Paradinas 2012, p. 175).

comprehensive and consecutive school system, beginning with primary schools for all. Therefore, the teachers formed at the *École normale* by the revolutionary courses, should thereafter return to their districts and open there the new type of primary schools. And mathematics constituted a key component of this teacher formation, taught by such eminent scientists like Lagrange, Laplace and Monge! Yet, this ambitious program failed: on the one hand, the level of teaching by these scientists was far beyond the reach of the average students, and on the other hand and even more decisive, the opening and running of a net of public primary schools proved not to be viable (Schubring 1982).

Therefore, it was eventually secondary schools, which first became systematically established, as *écoles centrales*, from 1795, with a strong emphasis on teaching mathematics (see Schubring 1984). There is the, at first glance, strange fact that the state did not care for teacher formation for these proper schools. Teacher education was not established, for not creating corporations again (Schubring 1984). In fact, the first institutions of higher education established, the *écoles de santé* and the *écoles de droit* – for forming physicians and jurists – were not entitled to confer titles and diplomas. The best-qualified persons would be chosen in *concours* for concrete positions, be it as physician, as judge or as teacher. A change in this anti-corporatist policy occurred only in the Napoleonic period. When in 1808 the *Université Impériale* was founded, as the administrative unity of all state educational institutions, there were founded various *facultés*, as institutions for professional formation in medicine, law, and theology. As ephemeral propaedeutic institutions, *facultés des lettres* and *des sciences* were created, too: giving some courses complementing the teaching of the last grades in the *lycées*. While they did not serve for teacher formation, there was just one new, but minor institution, which should prepare students for a state examination as teachers: the *École normale supérieure*, functioning from 1810. Having passed that exam on disciplinary studies (without any pedagogical preparation), the *agrégation*, such an *agrégé* was eligible for a position at a *lycée*. Given the marginal status of mathematics teaching after 1815 and the small number of candidates admitted to the *agrégation*, no profession of mathematics teachers emerged in France for a long time. Only in 1910, an association of mathematics teachers, the APMEP, was founded. There is so far, however, no research who were the mathematics teachers at the lycées/collèges during the 19th century so that one does not know which were there their educational careers, their professional profiles and characteristics of their teaching activities (see d'Enfert 2012).

Prussia

Contrary to France, teacher formation constituted an integral element of the profound educational reforms realized in Prussia, from 1810. The measures

taken already in 1810 document a coherent policy for the reform of universities and secondary schools:

- the newly founded Berlin University upgraded the Philosophical Faculty from a propaedeutic one to a professional, having the task to form the teachers for the key disciplines of the Gymnasia;
- the Gymnasia were reformed according to the Prussian neo-humanism: conceiving of human knowledge as an organic unity, the mind had to be developed in its intellectual capacities by acquiring in learning its complementary elements of knowledge; hence school knowledge should be based on three key disciplines: classical languages; history and geography; mathematics and the sciences. The basic curricular document – the (Tralles-)Süvern-Plan of 1810/1816 evidenced this conception;
- mathematics thus having become one of the major teaching subjects in the Gymnasia, and scientific teacher formation being provided at the universities, the state organized that same year the system of state exams for admitting university graduates to teaching positions in the Gymnasia. The exam included that the candidates had to give a sample lesson (Schubring 1991, pp. 38 ff.).
- Later on, in 1826, a probatory year became introduced, to acquire professional pedagogic qualifications, in one of the Gymnasia, under the guidance of an experienced teacher. This general practical peda-gogic formation had been prepared by two "Seminare für gelehrte Schulen" – seminaries for teacher training at secondary schools –, at Berlin and at Stettin, existing already since the early 1800s, where senior teachers tutored teacher candidates in teaching their respective discipline. In 1855, a seminar for training future mathematics teachers was founded at a Gymnasium in Berlin (ibid., pp. 122 ff.).

The school system was expanding; already at the beginning of the reforms, there were about 90 Gymnasia. The original structural plan, the Tralles-Süvern-Plan, had required two mathematics teachers per Gymnasium. In general, it used to be for some time just one, since instead of the second one, planned for the lower grades, functioned a candidate or a generalist teacher. Nevertheless, a sufficient demand for teachers was institutionalized, entailing a steadily increasing demand for university graduates (Schubring 1991, p. 146). Resuming these pivotal data, one can already assert the emergence of a new professional group – in this case for Prussia, then one of the major German states.

One might think that the question about the emergence of the profession of mathematics teachers is thus already resolved for this state. Yet, questions do now only begin if one wants to know more, and at least a bit about the reality of professional life of these teachers. The necessity of such deeper research becomes evident when one has in mind the ominous statement: "mathematicus non est collega", which is often reported as characteristic for 19th century mathematics teachers in Gymnasia. As a matter of fact, this statement was valid

for one German state during the 18th century, for Saxony, where the government had forced the three Gymnasia run by the state to contract for the first time a mathematics teacher. The staff of these schools regarded the newcomers as not welcome intruders – firstly since all of them being teachers of classical languages, they disagreed with this new teaching subject, but secondly since the funds for the salaries had not been augmented. Moreover, the newcomers being mainly practitioners, they were not provided with classical scholarship (Schubring 1991, Morel 2013).

The issue raised thus by this statement is about the position of the mathematics teachers within the social context of the school, and in particular within the staff of his school. Implicitly, most historical studies on mathematics teachers assume the teacher to act as a largely autonomous subject, only governed by the syllabus. However, the mathematics teacher is not alone in his school; he is teaching one among a series of subjects. And it is not him who decides or influences the manner of coexistence of the school disciplines. Rather, the respective school type realizes a conception of knowledge within which each school discipline shares a certain value and consequently a place within a definite hierarchy of disciplines, be it explicitly or implicitly expressed in the prescribed curriculum. Moreover, beyond the intra-school structures of cooperation and of placement, each teacher is confronted with the social view of his discipline as shared by the greater public, and in particular by the parents of the school's students.

The Prussian neo-humanist reform had been aware of the necessary "concertation" and cooperation of the teachers of all the disciplines and therefore stipulated in the teacher exam regulation of 1810 that no teacher should be exclusively qualified in his preferred subject of study (and future teaching discipline), but should prove some competence in the other subjects of "Allgemeinbildung", of general education, – as defined by the three key components of knowledge. As rationale for this stipulation was given that each teacher should be able to contribute, from his side, to the cooperation of all the disciplines towards the general education (Schubring 1991, p. 112). It is important to stress that not only future mathematics teachers had to prove their competencies in languages, but the teachers of classical subjects had to prove some knowledge of mathematics and the sciences, too.

Studying the emergence of the profession of mathematics teachers in Prussia, the aim of the research was therefore – performed in the archives of the former Prussian ministry of education – not only how many graduates were examined each year, from 1810, to be licensed as mathematics teachers, but also what were their qualifications in the other disciplines and – complementary – in how far graduates of philology studies had proved knowledgeable in mathematics and the sciences (Schubring 1991, pp. 126).

The evaluation of all the examinations for the seven provinces of Prussia, from 1810 to 1865/70, yields the remarkable fact that the graduates were in fact not one-sidedly specialized but that they were conferred teaching licenses

("facultas"), in general, for more than one discipline, or that they were at least attested to have proven sufficient general education (Schubring 1991, pp. 126). Purely mathematics specialists, i.e. without any other teaching license were extremely seldom. And during the first two to three decades, one remarks a considerable number of combinations of mathematics with philology. For instance, C. G. J. Jacobi had obtained, in 1824, a license in mathematics and in philology. From the 1840s on, it is instead the combination of mathematics with a license in other teaching subjects (i.e. geography, history or German), which becomes dominant among the possible combinations. The following figure shows the development of the combinations for the case of examinations as "Oberlehrer",[2] made by the examining body in Berlin, for the province of Brandenburg.

Figure 1: Numbers of teacher examinations in Berlin, 1810 to 1860, yielding an Oberlehrer-license, according to combined licenses: white areas: none in math and sciences [M/N]; dotted areas: M/N combined with "other" subjects; shaded areas: combined with philology; black areas: license in M/N without combination (Schubring 1991, p. 143).
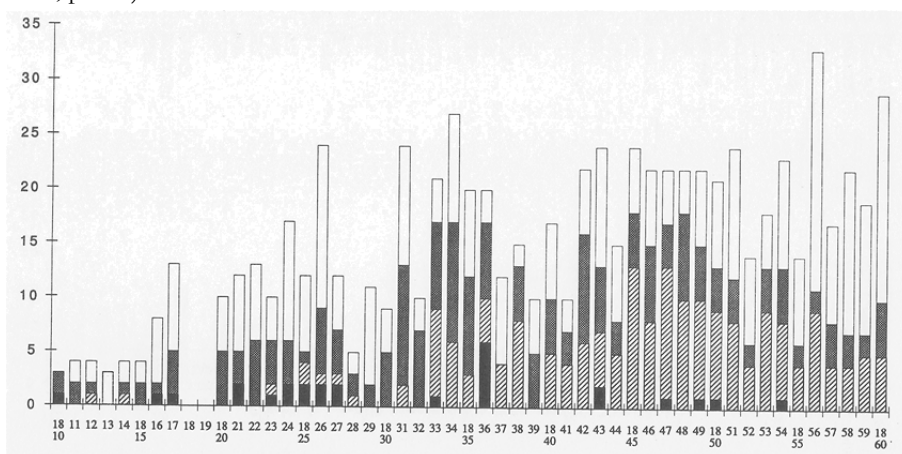
Hence, already from the point of view of the examinations and the teaching licenses, the mathematics teachers were qualified to act as fully integrated members of the staff.

The next step for approaching the historical reality of teaching was to search which subjects the examined teachers had really to teach in their school. Such an assessment was undertaken for three of the seven Prussian provinces, and there for all their Gymnasia. The result was that about the first two decades were a transitional period: Mathematics teaching was given

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[2] "Oberlehrer", as contrasted to the "Unterlehrer" were those who obtained a teaching license either for the upper grades of the Gymnasium or for all grades, whereas the "Unterlehrer" was restricted to the lower grades.

- by generalists (giving almost all subjects in their grade),
- by mathematics and science teachers who at the same time gave a number of other subjects,
- by teachers of history and geography,
- by pure specialists of mathematics and the sciences who had been inherited in the Rhineland from the formerly French *lycées* there,
- and eventually various mixtures of these practices.

The eventually dominant practice became, however, that mathematics in the upper and middle grades was taught by those who had obtained the license in mathematics and the sciences (Schubring 1991, pp. 147). Moreover, the assessment succeeded in the additional result that the conferred licenses for teaching and the real teaching obligations in school practice neatly converged (ibid., p. 157). This confirms clearly that – for Prussia – mathematics teachers constituted a definite and considerable professional group.

There are more criteria to assess the quality of a professional group of teachers. A key issue and indicator for the quality is whether this group participates in professional promotion in the same manner as other professional groups, and here in particular as teachers of classical languages, then usually called philologists. In fact, in the 18th century, in the mentioned case of mathematics teachers in Saxony, they remained always in the lowest position and did not participate in the ascension to higher – and better-paid – positions. When one of these three state schools in Saxony, the *Landesschule Schulpforta*, became Prussian after 1815, due to territorial reduction of Saxony (which had supported Napoleon), it became an issue, in fact, whether the mathematics teacher should continue to obtain no promotion. This issue became immediately settled and confirmed that the mathematics teacher is a colleague of equal rights as all the members of the staff (Schubring 1991, pp. 159). Also later on, when a so-called *Normaletat* (standard budget) became established for each Gymnasium, defining the salaries for all positions, it was clarified that the mathematics teacher would ascend in the hierarchy of the positions like all the others. A new issue in this respect had been whether the mathematics teacher could exert the function of *Klassenlehrer* (the class teacher responsible for all general issues of the class); this function was for a certain time, the better-paid positions were reserved for those teachers exerting this function (ibid., pp. 161–162).

Hanover

The second major state in Northern Germany was the Kingdom Hanover. Following the model of Prussia, the secondary schools had been reformed here, too, from 1830, and the *Abitur* had been introduced. This was complemented in 1831 by regulations for examining university graduates to become teachers. These regulations are characteristic for various other German states, too,

regarding their conception of the profession of mathematics teachers. Contrary to Prussia, where there had been just one type of Gymnasium teachers, independent of the major teaching subject, the exams for Hanover were differentiated according to four distinct types of teachers:

- those who intend to dedicate themselves to the "eigentlichen gelehrten Schulfache" – to the genuine academic teaching profession;
- those who intend to become a "Fachlehrer" – subject teacher – for mathematics and the sciences for all grades of the Gymnasium;
- future teachers of modern languages;
- and "Hilfslehrer" – substitute teachers – who only intend to teach some disciplines in the lower grades of a secondary school.[3]

Figure 2: The 1831 teacher exam regulation for the kingdom of Hanover
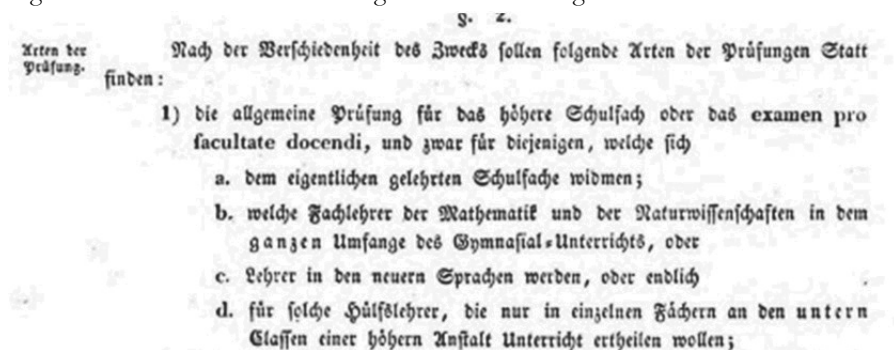
The meaning of the "eigentlichen gelehrten Schulfache" deserves a special comment. Firstly, it implies that the Gymnasium in this state still was understood as a "Gelehrtenschule" like in the centuries before, i.e. as preparing exclusively for university studies – hence preparing to become a *Gelehrter*, a scholar. Secondly, it implied by the adjective "eigentlich" – genuine – that there was a dominant complex of disciplines, namely the classical languages. And thirdly, the "eigentlich" implied an exclusion of others, not-genuine disciplines: hence disciplines taught by teachers not enjoying this status of serving an academic discipline, but teachers only teaching a particular discipline, hence being mere "Fachlehrer". And it is the teacher of mathematics and the sciences who is defined to represent only his "Fach", hence a specialty, but not the venerated world of classical Antiquity – still the ideal of general education in the Kingdom of Hanover.

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[3] Verordnung über die Prüfung der Schulamts-Candidaten, so wie der Lehrer des höhern Schulfachs, und die Errichtung einer wissenschaftlichen Prüfungs-Commission zu Göttingen, St. James's, den 22 April 1831. Gesetzsammlung für das Königreich Hannover, I. Abtheilung, 11. Mai 1831, p. 64.

Saxony

In the Kingdom of Saxony, the state was able only in 1848 to decree a regulation for examining teacher candidates for secondary schools. Its structure corresponded basically to the Hanoverian regulation. Three types of teachers were established:

- the first was again for "genuine" teachers in Gymnasia;
- the second again for "Fachlehrer", restricted to teachers of mathematics and the sciences;
- and thirdly teachers for elementary and technical schools (Morel 2013, p. 64).

The *Fachlehrer* had to prove a certain "Allgemeinbildung" – general education: their oral exam included the other disciplines, i.e. history, geography, pedagogy, German literature, and philosophy – but not classical languages. The genuine teachers were examined, besides the classical languages, only in philosophy and history, but not in mathematics and the sciences (Morel 2013, p. 67). Thus, the mathematics teacher was, like in Hanover, a "colleague", a member of the staff, but with a lesser status since not representing the – classical – ideal of education.

The consequences of such segregated conceptions of education are drastically highlighted by the autobiographical report of Friedrich Paulsen (1846–1908) about the years he passed at the Gymnasium in Altona, the highly traditional *Christianeum*. Altona, near Hamburg, was part of the dukedom Holstein. Paulsen is well known as the historian of German classical education; his seminal work is dedicated to the "gelehrten" schools (Paulsen 1885). As he reports, the Gymnasia of Holstein had been moderately reformed in the 1830s by a professor of philology originating from Saxony. But the reformer had transmitted from there and implemented the conception that modern realistic teaching subjects had to content themselves with being minor subjects and that the focus had to be the classical-humanistic teaching. As a consequence, the practice of the staff in these countries was hence: if a student succeeds in mathematics, it is fine – if not, no problem. Paulsen depicted strikingly how his form, and he himself, too, had internalized this conception of education: as complete disregard of their mathematics teacher – calling him the most sorrowful figure among all his teachers (Schubring 2012, p. 532). Paulsen evidenced that the Gymnasia in the dukedoms Holstein and Schleswig agreed with the traditional humanist conception of the Bavarian and Saxon type and hence sharply contrasted with the Prussian neo-humanism. Mathematics teachers in these states represented hence a quite different professional practice.

Bavaria

Mathematics teachers in Bavaria suffered a unique destiny. During the Napoleonic period, Bavaria had realized important reforms of its system of education. In particular, in 1808, a bifurcated system of secondary schools had been established: on the classical side were the *Lateinschulen* for the lower grades and the *Gymnasial-Institute* for the upper grades; on the realist side were the *Realschulen*, followed by the *Real-Institute*. However, this did not mean a restriction of mathematics to the realist system. On the contrary, mathematics was well represented on both sides, being a major discipline taught in all grades. In the *Real-Institute*, mathematics was even taught between six and eight hours every week. After 1815, however, due to strong political reaction against the earlier commitment to Napoleon, a drastic setback occurred in 1816, when the reform achievements of the Napoleonic period were renounced. On one hand, the realist branch of secondary schools became suppressed completely, while on the other hand, the classical part resumed the earlier dominant position of Latin. Mathematics teachers were dismissed and the teaching of mathematics was reduced to only one hour weekly – now to be taught by the single generalist teacher, usually without specific mathematics training, in his respective grade (Schubring 2012, p. 528).

In 1822, the one weekly hour doubled and mathematics teachers were again admitted to the upper form, the Gymnasium, after passing centralized exams for this instruction. These exams were organized exclusively for contracting mathematics teachers, and occurred in the French form of *concours*. Examiners were mathematics professors from Munich university.[4] Clearly, mathematics teachers were specialists, at the margin of the Gymnasium staffs. Moreover, since none of the sciences was taught in the Bavarian purely classical Gymnasia, mathematics stayed isolated – in the first half of the 19th century – without any connection to applications.

Italy

There has been recent research upon the development of mathematics teaching in Italy from the perspective of the activities of mathematics teachers and in particular of the development of the training of mathematics teachers (Furinghetti & Giacardi 2014; Giacardi & Scoth 2014). It is revealed that Italy presents a country without a genuine professional training of mathematics teachers. During the Napoleonic period, there had been good promises: various parts of Italy being transformed into provinces of the French Empire, in Pisa, a major town in Tuscany, a branch of the *École Normale Supérieure* (ENS) was

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[4] Hauptstaatsarchiv München: Akten des Ministeriums des Innern: No. 19160. Besondere Prüfung für die Lehrstellen der Physik und Mathematik an den Lyceen und Gymnasien des Königreichs im Jahre 1822.

established – thus the characteristic means to form teachers for the humanities and for mathematics in the French *lycées*, by disciplinary studies without any pedagogical component. However, the Pisa *Scuola Normale Superiore* was closed at the beginning of the Restoration and reopened only in 1846. As a consequence of the unification of Italy, it was opened to all Italians in 1862. It is telling for the attitude of leading mathematicians regarding teacher training, that Enrico Betti – director of the SNS from 1865 – "gradually transformed the SNS from an institute for teacher training into an institute for advanced research" (Giacardi & Scoth 2014, p. 215).

In 1875, the government took an initiative to establish a proper institution for teacher training, the *Scuole di Magistero*. Of the 21 Italian universities, in 1875–76, only eight established such institutions; and only three of them (Pavia, Pisa and Rome) offered courses in mathematics. In 1920, these institutions for teacher training were closed; a minor substitute were courses for the "combined degree" (see below).

Netherlands

In the Netherlands, there had been, in 1816, a reform of the Latin schools, the classical secondary schools preparing for university studies, introducing the teaching of "the principles" of mathematics. Yet, the government had not provided for formation and employment of mathematics teachers. The rationale was to have class teachers, focussed on Latin and Greek giving the mathematics lessons, too. Since this proved to be not very effective, all the first decades after this reform, there were two forms of teaching: either by the generalists, or by especially contracted teachers. Since these special teachers used not to be university graduates, but practitioners or primary school teachers, they stood mainly outside the staff and their teaching suffered disciplinary problems. Likewise, their payment used not to be attractive. Even until 1863, the period studied by Smid, there were few university graduates in mathematics entering as mathematics teachers (Smid 1997, p. 174).

In 1828, the government started an initiative to improve the professional formation of future mathematics teachers: special courses should be given at the universities, including pedagogy, methodology, and practical exercises on Latin schools. Yet, missing energy to realize these decisions led to a failure (Smid 2014, pp. 584–585). It was only in 1863 by a law on secondary education and in 1876 by a law on higher education that at least the scientific part of mathematics teacher education became clearly organized and mathematics teachers installed.

Brazil

In Brazil, a system of secondary schools became established from 1837 on, but one has here the case of a country without any measures for teacher education. Practically, this meant to adopt the French model – but in a more radical form: for each vacant position at a *colégio*, there was organized a *concurso* – *concours* or competition – where a jury would assess the qualifications of the candidates; insofar, the Brazilian practice corresponded to the practice in France after the Revolution, from 1795 on. But where should the candidates have acquired relevant qualifications for serving as a mathematics teacher? In France, there had been from 1810 on the *facultés des sciences*, which provided a – for a long time – only very basic program of courses for future teachers; besides that, one had specialized lectures at the ENS for a small selection of students. In Brazil, however, one had adopted just the original French model of special schools as established from 1795 on, until 1810: professional schools for higher education level of training for specific professions. As such schools, *escolas de direito* and *escolas de medicina* were founded – and somewhat later the *escola de minas*. At best, some (applied) mathematics could be studied at the *escola de minas*. Equivalents to the *facultés des lettres* and the *facultés des sciences* were not established. The only other institutions were in fact mathematics was taught systematically was the Military Academy, serving for the training of military and civil engineers. The studies so far undertaken on the mathematics teachers accepted in these *concursos* during the 19th century show that the great majority had pursued some engineering studies (Soares, 2007). It seems thus that each teacher had got some mathematical qualifications (and clearly none pedagogical ones) by some specific career, not permitting therefore to establish some common patterns for a somewhat homogeneous group of mathematics teachers. It would be difficult hence to research the prosopography of mathematics teachers in 19th century Brazil. The situation changed from the 1930s on, when the first universities were founded, including faculties of philosophy with courses for teachers at the secondary schools.

Profiles of professional activities

In present-day work in mathematics education about teacher training one uses to conceive of as essential at least two components: the knowledge of the scientific subject matter and the pedagogical content knowledge (PCK). Regarding PCK, in some countries – like Germany – one differentiates between didactics of mathematics, that is theories about teaching mathematics and introduction into the teaching of the various branches of school mathematics, and formation in educational sciences, including psychology and sociology. As we have seen, formation in the practical side of teaching mathematics and thus in PCK, has always been a neuralgic, or neglected, component of teacher

training for secondary schools – whereas the relation used to be the inverse for teacher training for primary grades: there, the pedagogical component was dominant, and the formation on the scientific component used to be weak. Prussia presented a case with a relatively advanced practice regarding PCK and Italy a case with a weak practice.

Regarding the studies of scientific subject matter at universities by future teachers, no systematic and comparative research has been undertaken on general patterns which disciplines used to be studied and what thus constituted the profiles of professional activity of mathematics teachers. Particularly pertinent seems to be, however, a strong connection between teaching mathematics and teaching physics. We will discuss hence historical evidence for such a professional profile.

In Prussia, we had seen that teachers of mathematics and the sciences had been stipulated in the basic curricular document for the Prussian Gymnasia, the Tralles-Süvern-Plan of 1810/1816 – for one of the three categories of major teaching subjects. One might therefore think – given that "the sciences" constitute a wide spectrum of different disciplines – this did not entail a genuine specialization. As a matter of fact, the files of the teacher exams from 1810 on show that there were separate exams in mathematics and in the sciences and that even within the sciences the candidates received separate licenses for teaching either the exact sciences (physics and/or chemistry) or the 'descriptive' sciences ("beschreibende Naturwissenschaften": geology, botany, zoology). And candidates obtained teacher positions according to these exam licenses and specializations. From 1829 on, there were even instituted special exams for future science teachers (see Schubring 1989, pp. 71 ff.). On the other hand, it proved that future teachers studying mathematics used to study also physics and obtained therefore also a teaching license in physics. Mathematics teachers in Prussia therefore established a professional identity where physics was understood as a science closely related to mathematics, basically conceived of as its application and pertinent to give a meaningful teaching of mathematics.

The case of Bavaria showed the significance of a productive relation between teaching mathematics and teaching physics. Italy presents another case where the professional profile was basically restricted to the mathematics discipline. There, in 1921, a new, additional degree had been instituted, the 'combined degree' (*laurea mista*) in physical and mathematical sciences, aimed at qualifying young people to teach scientific subjects in secondary schools (Giacardi & Scoth 2014, p. 220). Contrary to other countries where mathematics used to be taught by the same teachers as physics, the combined degree in mathematics and physics was unpopular with many Italian mathematicians, even those who had always been in favour of a special degree for teachers. For example, Guido Castelnuovo was critical, predicting that universities offering this special degree would produce 'mathematicians lacking in culture and physicists lacking the skills for experimentation, thus turning out to be mediocre teachers in both disciplines'. This judgment was shared by Vito

Volterra. An exception was Federico Enriques who held that the combined degree had to be maintained and experimented with, 'in the conviction that bringing together mathematics and physics constituted an advantage for scientific and professional purposes' (ibid., p. 220).

Conclusion

We understand thus that the profession of mathematics teachers in the various countries did not only emerge at different times and periods, but also with highly differing conceptions about the professional profile of these teachers: about their disciplinary qualifications, about their pedagogical qualifications and about the profile of teaching practice. Even the career patterns proved to be different, depending upon the status accorded to mathematics teachers. And there are countries – although showing revealing histories of mathematics teaching – where there are not yet studies on their mathematics teachers; for instance, for the USA, the emergence of the formation of mathematics teachers is investigated (Donoghue 2003), but not yet who were the mathematics teachers; and for England, research on mathematics teachers and their formation is missing. More research should be realized on the patterns and profiles of mathematics teachers to achieve prosopographies for characteristic countries and for comparing them.

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"Sickened by set theory?" – New Maths at German primary schools (abstract)

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Introduction of New Math in German primary schools led to a lot of protest in public society, reaching a peak with German magazine "Der Spiegel" headlining "Macht Mengenlehre krank?" ("Sickened by set theory?") in 1974. Although methodical innovations have been implemented permanently from these days on, contents, such as set theory, have been abolished in school mathematics and altogether the reform is said to have failed.

Former set theory taught pupils, though, do have positive memories of their mathematics lessons raising the question for reason of the alleged failure.

Taking a look at the sources from the time one finds a lot of points of criticism against the reform. In spite of their diversity those arguments may be divided into three classes: First of all, there was a wide range of didactical problems. In this place, this means all problems that concerned mainly pupils, including all severities emerging from the new contents and the way they were taught. Then one finds problems seeing teachers as those being mostly affected. Here, all difficulties belonging to class room organisation as well as questions of teacher's education are included. And finally there are problems related to public opinion – making parents those mainly concerned – and political reactions thereto. It is believed that those types of difficulties were the ones that prove crucial to the progress of the reform. It is furthermore assumed that a combination of these problems led to its failure. To find out what was the main cause for the end of New Math, one need to find out, which of these were the most relevant.

A possible source to analyse practically relevant problems are schoolbooks. Indications for all three classes of problems can be found when comparing for example the two editions of "Wir lernen Mathematik I", the schoolbook for first form by authors W. Neunzig and P. Sorger, which "Der Spiegel" particularly refers to. For example, the overall structure of the volume has been modified due to an earlier introduction of numbers in the 2nd edition (probably for political reasons), resulting in a layout less coherent and thus making the book possibly harder to use. Change in presentation of tasks serves as a proof for didactical problems in the first place while propositions for use of additional material such as Logic Blocks (as introduced by Z. P. Dienes) can be seen as a response to organisational problems caused by the teachers' inexperience with a new kind of mathematics lessons. Thus schoolbooks show reactions to criticism and may help to find out, which problems were relevant for the failure of New Math in German primary schools, after all.

The autonomy of secondary school mathematics culture (abstract)

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During the second half of the 19th century the discipline of secondary school mathematics was gradually construed in Portugal with its specific norms and practices (Julia, 1995). Textbooks, expert certified teachers, programmes, special teaching techniques, evaluation procedures, etc. were developed and by the end of the century all these apparatuses were consolidated (Matos, 2013), or, in other terms, the autonomy (Chervel, 1988) of the Portuguese secondary school mathematics was established.

This paper reflects upon the concept of autonomy of school disciplines as proposed by Chervel (1988) by studying representations of the reasons for teaching mathematics and of the appropriate teaching procedures from the 1850s until 1950s in Portuguese *Liceus* and by tracing the ways in which they varied.

Several sources are used: 1) official documents; 2) textbooks prefaces; 3) articles from influent educational actors; and 4) teachers' productions as required by their initial formation.

Five periods will be characterized:

1. Until 1870 scholastic tradition formatted the intended teaching procedures and goals for mathematics teaching were not differentiated from global educational goals.

2. By 1892 secondary schools culture and its disciplines was clearly configured (Nóvoa, Barroso, & Ó, 2003) and intended teaching procedures and goals for mathematics teaching were influenced by the positivist perspectives.

3. Later, until 1931 representations for teaching mathematics and its teaching procedures gradually incorporated the proposals of the New School movement, following both the new republican political tendencies and the emergence of a body of knowledge on "educational science" issued from teacher education institutions.

4. From 1931 until the end of the Second Word War nationalist official perspectives on education limited the scope of goals for teaching mathematics and proposed to centre teaching procedures on drill and practice. Representations on teacher education institutions however continued to include proposals from the New School movement and original teaching perspectives emerge as the use of concrete apparatuses and "laboratory classes".

5. From 1948, and especially by the middle 1960s, goals for teaching mathematics and teaching procedures increased its scope, as modern mathematics emerged.

In conclusion, autonomy of school mathematics is found to be established in the same terms as Julia conceived it (1995) with its specific norms and practices. However, this autonomy did not preclude permeability, as the representations studied were not isolated from global representations stemming from educational, political and social movements.

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Arithmetic textbooks – on the origins of an European tradition (abstract)

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In 1475 the so called "Trientiner Algorithmus" was published, the first text on mathematics printed in German language. It was followed by a few more in the last decades of the 15th century. Then with the textbooks of Johannes Widmann (1489 ff) and Adam Ries (1518 ff) a new sort of text "Rechenbuch" was established, which began a glorious way through Europe in the 16th century.

In my presentation a description of this kind of educational medium will be given (authors, recipients, mathematical content, structure, symbols and language, context of use), the dissemination over all European countries will be retraced and the differences and traditions be shown on a detailed analysis of the didactical design of the introduction to written summations.

Bento de Jesus Caraça: criticism and proposals for the mathematics teaching in Portugal in the 1940s (abstract)

Jaime Carvalho e Silva
Universidade de Coimbra

Bento de Jesus Caraça (1901–1948) was a Portuguese mathematician, an intellectual with a vast number of publications and a political activist opposing the dictatorship in Portugal; because of his political activity after World War II he was expelled from the University in 1946. He was founding member of a mathematics journal "Gazeta de Matemática" still being published today, he was one of the most active members of the newly founded Portuguese Mathematical Society (SPM), where he chaired the Education Committee, and gave a number of talks on the nature of Mathematics, on aspects of the History of Mathematics and the role of culture in the preparation of the citizens.

The Education Committee of SPM produced a text in 1941 with criticism to the official syllabus, namely relating to the reduced content and hours (only 2h a week), to the examinations and its type. And asked to consider new methods including the introduction of cinema in schools and laboratory methods for the teaching of elementary geometry.

Bento Caraça wrote several mathematics textbooks and a very influential popular book to teach mathematics to the "citizen" where he begins with the most elementary notions and goes till the limits of functions and numerical series, with lots of historical discussions and dialogues. He was also the main person behind the "Cosmos Library" a series of original and translated books to disseminate the scientific culture; around 150 books were published.

Bento de Jesus Caraça wrote several texts where he criticized the mathematics teaching in secondary schools, the preparation of teachers and the structure and goals of mathematics teaching. He maintained a dispute with José Sebastião e Silva about the teaching of logarithms.

Recently, a number of manuscripts, kept by his family, were donated to the Mário Soares Foundation that digitized them and made them available on the web. Some of these manuscripts give some more details on the views of Bento de Jesus Caraça about the mathematics teaching in Portugal. The manuscripts help us to understand some concrete proposals he would like to see in action in the secondary mathematics curriculum.

We will give an overview of what we can understand from Bento de Jesus Caraça writings: what he criticized most on the mathematics teaching of the time, and which were his proposals for the improvement of mathematics education.

New geometries for old schools? (abstract)

Klaus Volkert
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Around 1880 there were different new geometries around challenging the teaching of geometry in German grammar schools (Gymnasia): non-Euclidean geometry (a conglomerate of different geometries not in accord with classic Euclidean geometry [like hyperbolic geometry, elliptic geometry but also four-dimensional geometry]), projective geometry and descriptive geometry (à la Monge). A deep necessity was felt to open the teaching of geometry to these new fields in order to "modernize" it. But there was also a lot of opposition against this because many theorems of the new geometries contradicting (seemingly?) common sense had to be accepted. In my talk I will focus on the discussion on points at infinity and on closed straight lines as well on hyperbolic geometry. The importance of concrete intuitive models will also be a point.

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