Universidade Federal de Santa Catarina
Curso de Pós-Graduação em Matemática
Pura e Aplicada

σ - evolution models with low regular time-dependent structural damping

Edson Cilos Vargas Júnior

Orientador: Prof. Dr. Cleverson Roberto da Luz

Florianópolis
Fevereiro de 2019

Universidade Federal de Santa Catarina
Curso de Pós-Graduação em Matemática
Pura e Aplicada

σ - evolution models with low regular time-dependent structural damping

Tese apresentada ao Curso de Pós-Graduação em Matemática Pura e Aplicada, do Centro de Ciências Físicas e Matemáticas, da Universidade Federal de Santa Catarina, para a obtenção do grau de Doutor em Matemática, com Área de Concentração em EDP.

Edson Cilos Vargas Júnior
Florianópolis
Fevereiro de 2019

Ficha de identificação da obra elaborada pelo autor,
através do Programa de Geração Automática da Biblioteca Universitária da UFSC.

Vargas Júnior, Edson Cilos
Sigma - evolution models with low regular time dependent structural damping / Edson Cilos Vargas Júnior ; orientador, Cleverson Roberto da Luz, 2019.
131 p.

Tese (doutorado) - Universidade Federal de Santa Catarina, Centro de Ciências Físicas e Matemáticas, Programa de Pós-Graduação em Matemática Pura e Aplicada, Florianópolis, 2019.

Inclui referências.

1. Matemática Pura e Aplicada. 2. Equações de evolução de segunda ordem. 3. Taxas de decaimento. 4. Método dos multiplicadores. 5. Laplaciano com potência fracionária. I. da Luz, Cleverson Roberto. II. Universidade Federal de Santa Catarina. Programa de Pós-Graduação em Matemática Pura e Aplicada. III. Título.

σ - evolution models with low regular time-dependent structural damping

por
Edson Cilos Vargas Júnior*

Prof. Marcelo Sobottka
Coordenador

Comissão Examinadora

Prof. Dr. Cleverson Roberto da Luz
(Orientador - UFSC)

Prof. Dr. Marcelo Rempel Ebert
(USP)

Prof. Dr. Marcello D'Abbicco
(University of Bari)

Prof. Dr. Ruy Coimbra Charão
(UFSC)

Prof. Dr. Matheus Cheque Bortolan
(UFSC)

Florianópolis, fevereiro de 2019.

*Bolsista da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES

A Deus.
A minha família.
Aos meus amigos.
Ao povo brasileiro.

Agradecimentos

A Deus, causa primeira de todas as coisas, cuja infinita sabedoria e imensurável amor nos guia até ele através dos milênios...

Ao nosso querido irmão mais velho, o Senhor Jesus Cristo, que nos dias mais tristes apaziguou as tempestades do meu barco, que multiplicou os pães da minha mesa e também muitas vezes curou a minha alma que estava enferma. Que me ensina a cada dia a esperar trabalhando e a trabalhar amando...

Aos anjos do Altíssimo, que no anonimato do trabalho Divino me consolaram e me guiaram ao Mestre...

À minha esposa Isabelle, que além de ser o amor da minha vida, soube me entender e me apoiar nos momentos mais críticos e que muitas vezes precisou fazer sacrifícios para me ajudar...

Aos meus pais, Edson e Simone, que me amaram, me aconselharam, me guiaram e me incentivaram desde a minha infância...

Aos meus avós, Elzira, Eugênio (em memória), Inedina e Cilos, que sempre me apoiaram e acreditaram no meu sucesso...

Ao meu irmão Gabriel, que é também um grande amigo e parceiro, sempre valorizou os meus sonhos...

Aos meus familiares, que torceram por mim e que se alegram com as minhas conquistas...

Aos meus amigos, principalmente Giovani, Yuri e Natã, que me apoiaram e sempre têm um tempo para tomar um cafézinho...

Ao meu orientador Cleverson, muito paciente e amigo, diversas vezes me permitiu ampla liberdade para trabalhar, desde a época da graduação...

Ao professor Ruy Charão, que além de um grande amigo, juntamente com meu orientador esteve me instruindo e incentivando...

Ao povo Brasileiro, cujo suor do trabalho financiou este trabalho através da CAPES...

Não me bastaria dizer muito obrigado. Digo que está escrito mais uma página dentre muitas outras no Livro da Eternidade. Essa conquista é vossa também!

Com Mestre Jesus

Com Mestre Jesus, feliz moradia.
Lar abençoado, com paz e harmonia.

Suportando provação, venceremos cada dor.
Unidos de coração, conectados por amor!

Recordemos a cruz: é dura a jornada.
Mas com mestre Jesus, feliz caminhada!

Juntos na oração, vencendo cada dia!
Dividindo o pão, somando alegria!

De alma ávida, busquemos tua luz.
Mais que uma dádiva, és Santo ó Jesus.

Sublime presença, és o sol da nação.
Nos dá esperança, nos dá salvação!

Pelo mestre Jesus, esse simples louvor.
Por ti esses versos, nossa vida, nosso amor!

Resumo

Neste trabalho consideramos modelos de evolução σ sob o efeito de um termo de amortecimento representado pela ação do operador laplaciano com potência fracionária e coeficiente dependendo do tempo dado por $b(t)(-\Delta)^\theta u_t$. O objetivo do trabalho é obter taxas de decaimento do tipo $L^p - L^q$ com $1 \leq p \leq 2 \leq q \leq \infty$ para a solução e sua primeira derivada no tempo, considerando baixa regularidade no coeficiente $b = b(t)$.

Escrita de maneira equivalente, uma importante conjectura afirmava que: "Para $\theta = 0$, quando o coeficiente do amortecimento é efetivo, mesmo sem assumir hipóteses sobre a derivada do coeficiente, é possível obter as mesmas taxas de decaimento para o problema". No presente trabalho fornecemos uma resposta à conjectura, mostrando inclusive que há outras situações na qual a conjectura também permanece válida. Por exemplo, $\theta \neq 0$ ou ainda em casos em que a dissipação é não-efetiva.

Palavras Chave: Equação da onda; Equação de placas; Dissipação friccional; Dissipação viscoelástica; Dissipação fracionária; Taxas de decaimento "sharp"; Dissipação não-efetiva; Dissipação efetiva; Métodos dos multiplicadores; Espaço de Fourier.

Resumo Expandido

Introdução

Neste trabalho consideramos modelos de evolução σ sob o efeito de um termo de amortecimento representado pela ação do operador laplaciano com potência fracionária e coeficiente dependendo do tempo dado por $b(t)(-\Delta)^{\theta}u_t$.

Objetivos

O objetivo do trabalho é obter taxas de decaimento do tipo $L^p - L^q$ com $1 \leq p \leq 2 \leq q \leq \infty$ para a solução e sua primeira derivada no tempo, considerando baixa regularidade no coeficiente $b = b(t)$.

Metodologia

Através de uma revisão bibliográfica, chegamos à seguinte hipótese de pesquisa: "Não é necessário assumir hipóteses sobre a derivada do coeficiente no termo dissipativo, para obter as mesmas taxas de decaimento para modelos de evolução σ com dissipação fracionária cujo coeficiente é bem regular." Nesse sentido, baseado em trabalhos precedentes, criamos um novo método de multiplicadores para obter taxas de decaimento no espaço de Fourier e então provar a hipótese de pesquisa para exemplos importantes de coeficiente $b = b(t)$. Utilizamos resultados conhecidos de "Análise".

Resultados e discussão

Considerando um t_0 adequado, tomamos $b(t)$ "confinada" na curva $g(t) := (1+t)^{\alpha} ln^{\gamma}(1+t)$ para $t \geq t_0$. Além disso, no intervalo $[0, t_0]$

assumimos b positiva e que satisfaça condições adequadas para garantir existência de solução. Nesse contexto, quando comparadas a resultados anteriores que assumem mais regularidade na função b, obtemos as mesmas taxas de decaimento para solução quando $\gamma = 0$ e obtemos taxas melhores quando $\gamma \neq 0$. Para a primeira derivada no tempo da solução, obtemos taxas melhores inclusive quando $\gamma = 0$.

Escrita de maneira equivalente, uma importante conjectura afirmava que: “Para $\theta = 0$, quando o coeficiente do amortecimento é efetivo, mesmo sem assumir hipóteses sobre a derivada do coeficiente, é possível obter as mesmas taxas de decaimento para o problema”. No presente trabalho fornecemos uma resposta à conjectura, mostrando inclusive que há outras situações na qual a conjectura também permanece válida. Por exemplo, $\theta \neq 0$ ou ainda em casos em que a dissipação é não-efetiva.

Considerações finais

Nos casos abordados neste trabalho, mostramos ser verdadeira a conjectura que afirmava que não é necessário assumir hipóteses sobre as oscilações da função b para obter os mesmos resultados já conhecidos. Considerando que assumimos b com baixa regularidade, isto é, poucas hipóteses, o método que desenvolvemos sugere que a conjectura é válida em um contexto mais geral.

O modelo que foi estudado neste trabalho pode representar uma equação da onda ou uma equação de placas com dissipação cujo coeficiente depende do tempo. Mais importante ainda é que o método desenvolvido pode ser estendido e aplicado a outros modelos, como IBq ou equação de placas com inércia rotacional.

Palavras Chave: Equação da onda; Equação de placas; Dissipação friccional; Dissipação viscoelástica; Dissipação fracionária; Taxas de decaimento "sharp"; Dissipação não-efetiva; Dissipação efetiva; Métodos dos multiplicadores; Espaço de Fourier.

Abstract

In this work, we consider $\sigma-$evolution models under effects of a damping term represented by the action of a fractional Laplacian operator and a time-dependent coefficient $b(t)(-\Delta)^{\theta}u_t$. The objective of this work is to obtain $L^p - L^q$ decay rates, with $1 \leq p \leq 2 \leq q \leq \infty$, for the solution and its first derivative in time, considering low regularity in the coefficient $b = b(t)$.

Written in a equivalent manner, an important conjecture was asserting: "For $\theta = 0$, when the coefficient of the damping is effective, without further assumptions on derivatives of the coefficient is still possible to achieve the same known decay rates for the problem". In the present work we provide an answer to the conjecture, showing, in addition, that there are other situations in which the conjecture remains valid, for example $\theta \neq 0$ or even in the case that the damping is non-effective.

Key-Words: Wave equation; Plate equation; Frictional damping; Viscoelastic damping; Fractional damping; Sharp decay rates; Non-effective damping; Effective damping; Multiplier method; Fourier space.

Contents

Introduction

We consider, for $0 \leq \theta \leq \sigma$, the initial value problem for a $\sigma-$evolution equation with fractional damping in $\mathbb{R}^n$:

$$u_{tt}(t,x) + A^{\sigma}u(t,x) + b(t)A^{\theta}u_t(t,x) = 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^n \quad (1)$$

with initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \mathbb{R}^n, \qquad (2)$$

where $A := -\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

The fractional power operator $A^{\delta} : \mathcal{D}(A^{\delta}) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ $(\delta \geq 0)$ with its domain $\mathcal{D}(A^{\delta}) = H^{2\delta}(\mathbb{R}^n)$ is defined by

$$A^{\delta}v(x) := \mathcal{F}^{-1}\big(|\xi|^{2\delta}\mathcal{F}(v)(\xi)\big)(x), \qquad v \in H^{2\delta}(\mathbb{R}^n), \qquad x \in \mathbb{R}^n,$$

where $\mathcal{F}$ denotes the usual Fourier transform in $L^2(\mathbb{R}^n)$ and $|\cdot|$ denotes the usual norm in $\mathbb{R}^n$. The operator A^{δ} is nonnegative and self-adjoint in $L^2(\mathbb{R}^n)$ and the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{2\delta}(\mathbb{R}^n)$. Note that $A^1 = A$ and $A^0 = I$. For $\beta \in \mathbb{N}^n$, say, $\beta = (\beta_1, \ldots, \beta_n)$ we also define $D^{\beta} := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}$, in which $|\beta| := \sum_{j=1}^{n} \beta_j$. The results obtained in this work can be applied to several initial value problems associated to second-order equations, as for example, wave equation, plate equation, among others.

We assume, for a sufficient large $t_0 > 0$, that $b \sim g$ in $[t_0, \infty)$, in other words, there exist $a_1 > 0$ and $a_2 > 0$ such that $a_1 g(t) \leq b(t) \leq$

$a_2 g(t)$ for all $t \geq t_0$, in which $g(t) = (1+t)^{\alpha} ln^{\gamma}(1+t)$. In addition, we consider $b(t) \geq 0$ for $t \in [0, t_0)$ and one of the following hypotheses:

Hypothesis A: For $0 < \theta \leq \sigma$, let $\alpha \in [-1, 1)$ and $\gamma \in \mathbb{R}$ satisfying one of the following:

(i) $\alpha \in (-1, 1)$ and $\sigma(1+\alpha) < 2\theta$;

(ii) $\alpha \in (-1, 1)$, $\gamma \leq 0$ and $\sigma(1+\alpha) = 2\theta$;

(iii) $\alpha = -1$ and $\gamma \geq -1$. In particular $\sigma(1+\alpha) < 2\theta$.

Hypothesis B: For $0 \leq \theta < \sigma$, let $\alpha \in (-1, 1]$ and $\gamma \in \mathbb{R}$ satisfying one of the following:

(i) $\alpha \in (-1, 1)$ and $\sigma(1+\alpha) > 2\theta$;

(ii) $\alpha \in (-1, 1)$, $\gamma > 0$ and $\sigma(1+\alpha) = 2\theta$;

(iii) $\alpha = 1$ and $\gamma \leq 1$. In particular $\sigma(1+\alpha) > 2\theta$.

The cases $(\theta, \alpha) = (0, -1)$ and $(\theta, \alpha) = (\sigma, 1)$ are not included in our hypotheses since its decay rates depends on a_1 and a_2. For suitable constants, our method could be applied achieving sharp decay rates, but in general is not possible, since our method abuse in the use of constants when applying the multipliers. That is, the technique is still valid but must be improved in order to achieve sharp decay rates.

We can consider b a simple function such as $b(t) = 2 + cos(t)$ (case $\alpha = \gamma = 0$) or more complicated functions. To illustrate an interesting *example* for b, we can consider the following (for simplicity we assume $\alpha < 0$ and $\gamma > -1$ if $\mu_1 \neq 0$, but similar examples can be made for the other cases):

$$\begin{aligned} b(t) \quad := \quad & \mu_1 \Gamma(1+\gamma, -\alpha ln(1+t)) + \mu_2 g(t) sin((1+t)^{\eta_2}) \\ & + \mu_3 g(t) cos((1+t)^{\eta_3}) + \mu_4 g(t), \end{aligned} \tag{3}$$

for $t \geq t_0$ big enough, where $\Gamma(s, x) := \int_x^{\infty} y^{s-1} e^{-y} dy$ is the upper incomplete gamma function, η_i, $\mu_j \in \mathbb{R}$ for all $2 \leq i \leq 3$ and $1 \leq j \leq 4$, and at least μ_1 or μ_4 is big enough. The definition for b in $[0, t_0)$ can be anything that make b non-negative. To see why this function can be applied, we apply Lemma A.1.1 with $f(y) = y^{s-1}e^{-y}$ and $\psi(y) = -1$ to obtain: $\Gamma(s,x) \sim x^{s-1}e^{-x}$. Therefore $\Gamma(1+\gamma, -\alpha ln(1+t)) \sim g(t)$ for $t \geq t_0$ big enough. Since μ_1 or μ_4 is big enough, we have $b \sim g$. We can construct several examples with special functions of physical mathematics, like Bessel functions or W-Lambert functions (see Appendix C), just proceeding as made for b in equation (3). That is, *asymptotically* several special functions reduce to the case $b \sim g$. To calculate decay rates for the solution of (1)-(2) for such b, is straightforward by applying Theorem 1.3.1 or Theorem 2.1.1.

The asymptotic profile of (1)-(2) for $\sigma > 0$, $\theta \in (0, \sigma)$, $b(t) = 2\mu(1+t)^{\alpha}$, $\mu > 0$ and $\alpha \in (-1, 1)$, was investigated by D'Abbicco-Ebert in [8]. They proved an anomalous diffusion phenomena for this equation and introduced a classification based on it: the damping is said *effective* when the diffusion phenomenon holds and *non-effective* otherwise. This concept generalized the classification introduced by J. Wirth for $\theta = 0$ in [23] (non-effective case) and [24] (effective case). Furthermore, D'Abbicco-Ebert reported that when $2\theta < \sigma(1+\alpha)$ the damping is effective and non-effective if $2\theta > \sigma(1+\alpha)$. The case $2\theta = \sigma(1+\alpha)$ is treated as a critical case and they do not discuss. In addition, is expected that their work could be extended for a more general class of coefficient b in terms of the following limits:

If $\frac{1}{b} \notin L^1$ and $\lim\limits_{t\to\infty} t^{1-\frac{2\theta}{\sigma}} b(t) = \infty$, the damping is effective;

If $b \notin L^1$ and $\lim\limits_{t\to\infty} t^{1-\frac{2\theta}{\sigma}} b(t) = 0$, the damping is non-effective.

Going back to Hypothesis A and based on the last classification introduced, except for the case $\sigma(1+\alpha) = 2\theta$ and $\gamma = 0$ (critical instance), we have exactly the non-effective damping case. On the other hand, Hypothesis B corresponds to the effective case. Our classification however, is not motivated by whether the asymptotic profile of the solution of the problem has or not an diffusion phenomena, but rely on a new classification in which will be motivated and introduced next. The connection between our new classification and the diffusion phenomena is a open question.

In the case of $b = 1$ and $\sigma = 1$, equation (1)-(2) turns to a wave equation with fractional damping. This equation was approached by

Ikehata-Natsume [13] using the energy method in Fourier space, a technique due to Umeda-Kawashima-Shizuta [22]. However, since the mentioned result was not optimal for $0 \leq \theta < \frac{1}{2}$, the method was improved by Charão-da Luz-Ikehata [2] by using integrable properties of the equation. In this context, the key inequality (together with other techniques) to find the (almost) optimal decay rates for that equation with $0 \leq \theta < \frac{1}{2}$ is given by Lemma 3.2 of [2]:

$$|\xi|^{4\theta}|\hat{u}(t)|^2 \lesssim |\xi|^{4\theta}|\hat{u}_0|^2 + |\hat{u}_1|^2, \qquad \text{for all } |\xi| \leq 1, \tag{4}$$

in which improves the standard inequality given by the energy equation ($\sigma = 1$):

$$|\xi|^{2\sigma}|\hat{u}(t)|^2 \lesssim |\xi|^{2\sigma}|\hat{u}_0|^2 + |\hat{u}_1|^2, \qquad \text{for all } |\xi| \leq 1. \tag{5}$$

The comparison between the powers $2\sigma = 2$ and 4θ lead us to separate in two cases: $0 \leq \theta < \frac{1}{2}$ and $\frac{1}{2} \leq \theta \leq 1 = \sigma$. The first case is when inequality (4) gives a improvement of inequality (5) and second case is when inequality (5) is sufficient to obtain the optimal decay rates.

The method developed by Charão-da Luz-Ikehata [2] was also applied in an abstract second order equation [6] and further in a plate equation with a increasing time-dependent coefficient [7]. In the last case, the decay rates using the energy method hold for a general increasing function but the equivalent inequality (4) was not sufficient to ensure the (almost) optimal decay rate for the particular case

$$b(t) = \mu(1+t)^\alpha, \quad \mu > 0 \text{ and } \alpha \in (0,1] \tag{6}$$

in some cases of θ. However, in the same work, they also considered the particular case (6), obtaining optimal decay rate by using the *diagonalization procedure.* The enhancement for this specific case cast doubts concerning the improvement of the standard inequality given by energy inequality (5) (with $\sigma = 2$ in the case of plate equation) for a time-dependent context. This conclusion lead us to consider the steps in diagonalization procedure to get some relevant information for a better understanding of the problem.

The diagonalization procedure was successfully used in several papers to obtain decay rates for equation (1)-(2). For instance, Wirth in [23] and [24] considered this equation with $\sigma = 1$, $\theta = 0$ and $b(t)$ allowing small oscillations but *close related to* $\mu(1+t)^{\alpha}$ with $\alpha \in [-1, 1)$. A less restrictive oscillations for b, that is, less control on $\frac{d}{dt}b$ was obtained by Hirosawa-Wirth [11] but still not too much general as we would like. For $\sigma = 1$, $\theta \in (0, 1)$ and $b(t) = \mu(1+t)^{\alpha}$, diagonalization procedure was used by Lu-Reissig [18] (decreasing case) and by Reissig [19] (increasing case). More recently, the result was extended by Kainane-Reissig [15] and [16] for $\sigma > 1$, $\theta \in (0, \sigma)$ and b satisfying suitable conditions but *very similar* to $\mu(1+t)^{\alpha} ln^{\gamma}(1+t)$ and requiring a high control on $\frac{d}{dt}b$.

All the mentioned papers not only show the interest in equation (1)-(2) but also reveal a good acceptance of the diagonalization procedure as a suitable method. However, it should notice that this method usually require considerable control on oscilations of b. On the other hand, the method due to Charão-da Luz-Ikehata [2], [6] and [7] in general is not enough to obtain the optimal decay rates in the case of a time-dependent coefficients, moreover only L^2 norms estimates are possible.

In addition, is well known (see [9], [10], [20], [21]) that oscillations in the coefficient can deteriorate or even destroy the decay structure of the equation:

$$u_{tt} - a^2(t)\Delta u = 0. \tag{7}$$

Without control over oscillations of a^2, it is also possible to show results of blow-up of solution of equation (7), (see [4], [5]). Under suitable conditions, defining $A(t) := 1 + \int_0^t a(s)ds$ and $v(t, x) := u(A^{-1}(t), x)$, equation (7) is transformed into:

$$v_{tt} - \Delta v + \tilde{b}(t)v_t = 0, \tag{8}$$

where $\tilde{b}(t) := \frac{a'(A^{-1}(t))}{a^2(A^{-1}(t))}$. Therefore, taking in account the results concerning equation (7) and its relation with (8), it was not clear if equation (1)-(2) admits decay rates allowing substantial oscillations for b and if it has influence in the decay. Indeed, for $\theta = 0$ and $\sigma = 1$ there is a conjecture [25] concerning equation (1)-(2) : "We conjecture that the results of [24] (where only very slow oscillations were treated and decay results of the same structure were obtained) can be extended to general dissipation terms with $tb(t) \to \infty$ without further assumptions on derivatives".

In the present work we provide an answer to the conjecture, showing, in addition, that there are other situations in which the conjecture remains valid, for example, $\theta \neq 0$ or even in the case that the damping is non-effective. Thereby, the objective of this work is to develop a method (inspirit on the works cited above), to obtain sharp decay rates $L^p - L^q$ for the solution of (1)-(2), with $1 \leq p \leq 2 \leq q \leq \infty$, considering only $b(t) \sim (1+t)^{\alpha} ln^{\gamma}(1+t) =: g(t)$ and b non-negative, that is, no control in $\frac{d}{dt}b$ will be assumed. In particular, we will prove that $\frac{d}{dt}b$ has no influence in the decay rates. Going back to the relation between equations (7) and (8), our hypothesis $b \sim g$ does not contradict the results concerning the control over the coefficient of (7). Indeed, $\tilde{b}(t) := \frac{a'(A^{-1}(t))}{a^2(A^{-1}(t))}$ and $\tilde{b} \sim g$ implies in $a'(A^{-1}(t)) \sim a^2(A^{-1}(t))g(t)$, that is, we still have some control in the oscillations of the function a.

Furthermore, it should be noticed that this work can be extended for a more general class of functions g, but for the sake of brevity, we avoid this extension since there is specific calculations required depending on g (for example, see Proposition 2.2.2). In addition, our method can be applied to other equations, for example, plate equation under effects of rotational inertia.

To develop our method, we go back to the origin of the energy method in Fourier space but at the same time considering the knowledge provided by the diagonalization procedure and the method due to Charão-da Luz-Ikehata. For this sake, we consider hyperbolic and elliptic zones *similarly* as considered in the diagonalization procedure, see for example [14], [15] and [16]. In the diagonalization procedure the zones came from WKB analysis, in our case the zones comes together with a energy multiplier, that is, comes from an algebraic understand of the problem (see Proposition 1.1.1, in which has been motivated by the method due to Charão-da Luz-Ikehata [2] and [6]). For each ξ such that $0 < |\xi| \leq R$, we consider $\psi(\xi) := |\xi|^{2\theta} g(t_\xi)$, where t_ξ separates low zone from elliptic and hyperbolic zones (see Sections 1.1 and 2.2 for further details). We introduce a new classification based on the comparison between $|\xi|^\sigma$ and $\psi(\xi)$.

The aim of the Chapter 1 is to investigate, for small frequency, the case $\max\{|\xi|^\sigma, \psi(\xi)\} = |\xi|^\sigma$ which correspond to our assumptions made on θ, σ, α and γ in Hypothesis A. On the other hand, for small frequency, the case $\max\{|\xi|^\sigma, \psi(\xi)\} = \psi(\xi)$ correspond to Hypothesis B and will be treated in Chapter 2.

Throughout this work, we do not discuss the existence of solution to (1)-(2). Therefore, in addition to the conditions aforementioned, we assume suitable condition on b, u_0 and u_1 that ensure existence of solutions u and u_t that make possible the method described in this work.

Useful Functions

Consider $t_0 > 0$ big enough. Throughout this thesis, the following functions will be widely used:

$$g(t) := (1+t)^{\alpha} ln^{\gamma}(1+t),$$

$$\varphi(t) := \begin{cases} (1+t)^{1+\alpha} ln^{\gamma}(1+t), & \text{if } \alpha > -1, \\ ln^{1+\gamma}(1+t), & \text{if } \alpha = -1 \text{ and } \gamma > -1, \\ ln(ln(1+t)), & \text{if } \alpha = -1 \text{ and } \gamma = -1. \end{cases}$$

$$\phi(t) := \begin{cases} (1+t)^{1-\alpha} ln^{-\gamma}(1+t), & \text{if } \alpha < 1, \\ ln^{1-\gamma}(1+t), & \text{if } \alpha = 1 \text{ and } \gamma < 1, \\ ln(ln(1+t)), & \text{if } \alpha = 1 \text{ and } \gamma = 1. \end{cases}$$

In addition, we assume $b \sim g$ in $t \geq t_0$, that is, there exist c_1 and c_2 positive, such that $c_1 g(t) \leq b(t) \leq c_2 g(t)$ for all $t \geq t_0$.

For each $\xi \in B_R := \{\xi \in \mathbb{R}^n \backslash \{0\} : |\xi| \leq R\}$, we define $t_\xi := \varphi^{-1}(N|\xi|^{-2\theta})$. We introduce $\psi : B_R \to [0, \infty)$ defined by

$$\psi(\xi) := g(t_\xi)|\xi|^{2\theta}.$$

Chapter 1

σ - evolution models with low regular time-dependent structural damping I

1.1 Main Estimates in the Fourier Space

In this section, we can assume that the initial data are sufficiently smooth and apply the density argument. Let $u = u(t, x)$ be the corresponding solution of (1)-(2).

We take the Fourier transform in the both sides of (1). Then in the Fourier space one has the reduced equation:

$$\hat{u}_{tt}(t,\xi) + |\xi|^{2\sigma}\hat{u}(t,\xi) + b(t)|\xi|^{2\theta}\hat{u}_t(t,\xi) = 0, \qquad (t,\xi) \in (0,\infty) \times \mathbb{R}^n. \tag{1.1}$$

The corresponding initial data are given by

$$\hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \qquad \xi \in \mathbb{R}^n. \tag{1.2}$$

Throughout this section we shall omit the dependence in ξ inside the functions $\hat{u}(t) = \hat{u}(t,\xi)$, $\hat{u}_t(t) = \hat{u}_t(t,\xi)$ and the density energy $E(t) = E(t,\xi)$, which will be defined further ahead.

When we obtain important estimates in order to prove our results, we apply the multiplier method in Fourier space. Take $Z \subset [0,\infty)\times\mathbb{R}^n$ and $K : Z \to [0,\infty)$. We multiply both sides of (1.1) by $\overline{\hat{u}}_t$ and further by $K(t,\xi)\overline{\hat{u}}$. Then, taking the real part of the resulting identities we have (formally):

$$\frac{1}{2}\frac{d}{dt}\left\{|\hat{u}_t(t)|^2 + |\xi|^{2\sigma}|\hat{u}(t)|^2\right\} + b(t)|\xi|^{2\theta}|\hat{u}_t(t)|^2 = 0 \tag{1.3}$$

and

$$\begin{aligned}&\frac{d}{dt}\left\{K(t,\xi)Re(\hat{u}_t(t)\,\overline{\hat{u}}(t))\right\} + b(t)K(t,\xi)|\xi|^{2\theta}Re(\hat{u}_t(t)\overline{\hat{u}}(t))\\ &\qquad +K(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2\\ &\quad = K(t,\xi)|\hat{u}_t(t)|^2 + \left\{\frac{d}{dt}K(t,\xi)\right\}Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)),\end{aligned} \tag{1.4}$$

for each $(t,\xi) \in Z$ that it makes sense. We define the energy density as:

$$E(t) := \frac{1}{2}\{|\hat{u}_t(t)|^2 + |\xi|^{2\sigma}|\hat{u}(t)|^2\} \qquad \forall\, t \geq 0.$$

By integration the equation (1.3) in $[S,T]$, it follows:

$$E(T) + \int_S^T b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds = E(S). \tag{1.5}$$

Proposition 1.1.1 *Let $\hat{u} = \hat{u}(t,\xi)$ the solution of* (1.1)-(1.2), *$Z \subset [0,\infty)\times\mathbb{R}^n$ and $K : Z \to [0,\infty)$, where $K(t,\cdot)$ is mensurable and $K(\cdot,\xi)$ is a C^1 piecewise function for each $(t,\xi) \in Z$. Suppose that there exist λ_1, λ_2 and $\lambda_3 \geq 0$ such that K and Z satisfies, for all $(t,\xi) \in Z$:*

(0) *If $(s_1,\xi),(s_2,\xi) \in Z$ then $(s,\xi) \in Z$ for all $s_1 \leq s \leq s_2$;*

(1) *$K(t,\xi) \leq \lambda_1 b(t)|\xi|^{2\theta}$;*

(2) *$b(t)K(t,\xi)|\xi|^{2\theta} \leq \lambda_2|\xi|^{2\sigma}$;*

(3) *$\left(\frac{d}{dt}K(t,\xi)\right)^2 \leq \lambda_3 b(t)K(t,\xi)|\xi|^{2\theta+2\sigma}$.*

Then, there exists $C > 0$ such that:

$$\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds \le CE(S) \tag{1.6}$$

for all $(S,\xi),(T,\xi) \in Z$ *such that* $T > S$.

Proof. Fix $(S,\xi),(T,\xi) \in Z$ such that $T > S$. Since inequality (1.6) is trivial if $\xi = 0$, we suppose $\xi \neq 0$. By hypothesis **(0)**, $(s,\xi) \in Z$ for all $s \in [S,T]$.

Observe that conditions **(1)** and **(2)** imply that $K(s,\xi) \le \sqrt{\lambda_1\lambda_2}\,|\xi|^\sigma$ for all $(s,\xi) \in Z$. Thus,

$$|K(s,\xi)Re(\hat{u}_t(s)\,\overline{\hat{u}}(s))| \le \sqrt{\lambda_1\lambda_2}\,|Re(\hat{u}_t(s)\,|\xi|^\sigma\overline{\hat{u}}(s))| \le \sqrt{\lambda_1\lambda_2}\,E(s) \tag{1.7}$$

for all $s \in [S,T]$.

By **(1)**, **(2)** and by density energy equation (1.5):

$$\int_S^T K(s,\xi)|\hat{u}_t(s)|^2 ds \le \lambda_1 \int_S^T b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds \le \lambda_1 E(S) \tag{1.8}$$

and

$$\begin{aligned}
&\left|\int_S^T b(s)K(s,\xi)|\xi|^{2\theta} Re(\hat{u}_t(s)\,\overline{\hat{u}}(s))\,ds\right| \\
&\quad \le \lambda_2 \int_S^T b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds + \frac{1}{4\lambda_2}\int_S^T b(s)K^2(s,\xi)|\xi|^{2\theta}|\hat{u}(s)|^2 ds \\
&\quad \le \lambda_2 E(S) + \frac{1}{4}\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds. \qquad (1.9)
\end{aligned}$$

Using hypothesis **(3)**, we have:

$$\begin{aligned}
&\int_S^T \left\{\frac{d}{dt}K(s,\xi)\right\} Re(\hat{u}_t(s)\,\overline{\hat{u}}(s))ds \\
&= \int_S^T b(s)|\xi|^{2\theta} Re\left(\hat{u}_t(s)\left\{\frac{\frac{d}{dt}K(s,\xi)}{b(s)|\xi|^{2\theta}}\right\}\overline{\hat{u}}(s)\right)ds \\
&\leq \lambda_3 \int_S^T b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds + \frac{1}{4\lambda_3}\int_S^T \frac{\left(\frac{d}{dt}K(s,\xi)\right)^2}{b(s)|\xi|^{2\theta}}|\hat{u}(s)|^2 ds \\
&\leq \lambda_3 E(S) + \frac{1}{4}\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds. \qquad (1.10)
\end{aligned}$$

By integrating equation (1.4) and applying inequalities (1.7)-(1.10) we have:

$$\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds \leq \frac{C}{2}E(S) + \frac{1}{2}\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds,$$

with $C := 2(\lambda_1 + 2\sqrt{\lambda_1\lambda_2} + \lambda_2 + \lambda_3)$. This finish the proof.

□

Definition 1.1.1 *We say that $K : Z \to [0,\infty)$ is a multiplier of energy in $Z \subset [0,\infty) \times \mathbb{R}^n$ if K and Z satisfy the conditions of Proposition 1.1.1.*

If we had a global multiplier of energy K with $Z = [0,\infty) \times \mathbb{R}^n$ and at same time *sharp*, it would be possible to prove the main results of this work using equation (1.5), Proposition 1.1.1, Lemma 1.1.1 and Proposition 1.1.3. Even though global multiplier are possible, they usually does not lead us to sharp decay rates. Therefore, we will separate the problem in zones to find the sharp multiplier of energy in each region. It worth to highlight that these zones are pretty similar to works that use the diagonalization procedure, see [14], [15] or [16] for example.

Let φ as defined in Theorem 1.3.1. We fix $R > 0$ whose choice will be clear in the course of the section. It will satisfy Remark 1.1.3 and, in addition, will satisfy the following restrictions if $\alpha \in (-1,1)$: fulfill inequalities (1.16) and (1.17) if $\gamma < 0$, $R < 1$ if $\gamma = 0$, and realizes inequality (1.18) if $\gamma > 0$. Furthermore, except for the case $\sigma(1+\alpha) = 2\theta$ and $\gamma = 0$, if g is increasing we assume also the inequality (1.19).

For each $\xi \in B_R := \{\xi \in \mathbb{R}^n \backslash \{0\} : |\xi| \leq R\}$, we define $t_\xi := \varphi^{-1}(|\xi|^{-2\theta})$. We consider $\psi : B_R \to [0, \infty)$ defined by

$$\psi(\xi) := |\xi|^{2\theta} g(t_\xi). \tag{1.11}$$

We shall deal with the problem using the following separation zones:

High Zone: $Z^{high} := \{(t, \xi) \in [t_0, \infty) \times \mathbb{R}^n : |\xi| \geq R\}$;

Hyperbolic Zone:

$$Z_{hyp} := \left\{(t, \xi) \in [t_0, \infty) \times B_R : |\xi|^{2\theta} \varphi(t) \geq 1 \text{ and } |\xi|^{\sigma - 2\theta} \geq g(t)\right\};$$

Elliptic Zone:

$$Z_{ell} := \left\{(t, \xi) \in [t_0, \infty) \times B_R : |\xi|^{2\theta} \varphi(t) \geq 1 \text{ and } |\xi|^{\sigma - 2\theta} \leq g(t)\right\};$$

Low Zone: $Z_{low} := \left\{(t, \xi) \in [t_0, \infty) \times B_R : |\xi|^{2\theta} \varphi(t) \leq 1\right\}$.

Remark 1.1.1 *The number t_0 is chosen such that $\varphi(t_0) > 1$, $g(t_0) \leq 1$ if g is non-increasing, $g(t_0) \geq 1$ if g is increasing, $b \sim g$ for $t \geq t_0$, $|g'(t)| \leq \left(\frac{|\alpha|+|\gamma|}{1+t}\right) g(t)$ for $t \geq t_0$ and such that φ and g are monotone (without change of monotonicity) for $t \geq t_0$. Furthermore, throughout this chapter we will assume t_0 big enough to ensure the application of the results of the Appendix.*

The next proposition provide us the multiplier of energy in each zone, with exception to low zone. Actually, it is possible to find a multiplier of the energy in this zone, but it is not necessary. This is because we have the frequency variable satisfying: $|\xi| \leq \varphi(t)^{-\frac{1}{2\theta}}$. Using boundness for $|\hat{u}|$ and $|\hat{u}_t|$ (given for example by equation (1.5)) and by integration in ξ in this region, a natural decay rates appear due to the radius $\varphi(t)^{-\frac{1}{2\theta}}$. This process is made in details in Proposition 1.2.1.

Furthermore, we disconsider Z_{ell} for $\alpha = -1$ because it is empty if t_0 is big enough.

Proposition 1.1.2 *We have the following multipliers of energy:*

Hyperbolic Zone: *In* Z_{hyp}, $K(t,\xi) := g(t)|\xi|^{2\theta}$;

Elliptic Zone: *In* Z_{ell} *and* $\alpha \neq -1$, $K(t,\xi) := \dfrac{1}{g(t)}|\xi|^{2\sigma-2\theta}$;

High Zone: *In* Z^{high}, $K(t,\xi) := \min\left\{\frac{1}{g(t)}, g(t)\right\}|\xi|^{\min\{2\sigma-2\theta,2\theta\}}$.

Proof. To each zone, we must to verify conditions of Proposition 1.1.1. We notice that condition **(0)** is satisfied by the fact that g and φ are monotone. Also, we have $g \sim b$ and $|g'(t)| \leq \frac{c_0}{(1+t)}g(t)$ in $[t_0,\infty)$. For $\alpha \neq -1$, we have $\varphi(t) = (1+t)g(t)$ and therefore:

$$\frac{1}{(1+t)} \leq g(t)|\xi|^{2\theta}, \tag{1.12}$$

for all $(t,\xi) \in Z_{ell} \cup Z_{hyp}$. For $\alpha = -1$ (and $\theta \neq 0$), we have:

$$|\xi|^{\sigma}(1+t) = \left[|\xi|^{2\theta}(1+t)^{\frac{2\theta}{\sigma}}\right]^{\frac{\sigma}{2\theta}} \gtrsim \left[|\xi|^{2\theta}\varphi(t)\right]^{\frac{\sigma}{2\theta}} \gtrsim 1, \tag{1.13}$$

for all $(t,\xi) \in Z_{hyp}$. In Z^{high} similar inequality is also true:

$$|\xi|^{\sigma}(1+t) \geq R^{\sigma}. \tag{1.14}$$

To verify condition **(3)** of Proposition 1.1.1, we use inequalities (1.12), (1.13) and (1.14).

First consider the hyperbolic zone:

(1) $K(t,\xi) = g(t)|\xi|^{2\theta} \lesssim b(t)|\xi|^{2\theta}$;

(2) $b(t)K(t,\xi)|\xi|^{2\theta} = b(t)g(t)|\xi|^{4\theta} \lesssim g(t)^2|\xi|^{4\theta} \lesssim |\xi|^{2\sigma}$.

In hyperbolic zone for $\alpha \neq -1$,

$$
\begin{aligned}
\textbf{(3)}\ \left(\frac{d}{dt}K(t,\xi)\right)^2 &= g'(t)^2|\xi|^{4\theta} \\
&\lesssim \frac{1}{(1+t)^2}g(t)^2|\xi|^{4\theta} \\
&\lesssim g(t)^2|\xi|^{4\theta}g(t)^2|\xi|^{4\theta} \\
&\lesssim g(t)K(t,\xi)|\xi|^{2\theta+2\sigma} \\
&\lesssim b(t)K(t,\xi)|\xi|^{2\theta+2\sigma}.
\end{aligned}
$$

In hyperbolic zone for $\alpha = -1$,

$$
\begin{aligned}
\textbf{(3)}\ \left(\frac{d}{dt}K(t,\xi)\right)^2 &= g'(t)^2|\xi|^{4\theta} \\
&\lesssim \frac{1}{(1+t)^2}g(t)^2|\xi|^{4\theta} \\
&\lesssim g(t)K(t,\xi)|\xi|^{2\theta+2\sigma} \\
&\lesssim b(t)K(t,\xi)|\xi|^{2\theta+2\sigma}.
\end{aligned}
$$

Now, let us consider the elliptic zone with $\alpha \neq -1$:

(1) $K(t,\xi) = \dfrac{1}{g(t)}|\xi|^{2\sigma-2\theta} \leq g(t)|\xi|^{2\theta} \lesssim b(t)|\xi|^{2\theta}$;

(2) $b(t)K(t,\xi)|\xi|^{2\theta} = \dfrac{b(t)}{g(t)}|\xi|^{2\sigma} \lesssim |\xi|^{2\sigma}$;

(3)
$$
\begin{aligned}
\left(\frac{d}{dt}K(t,\xi)\right)^2 &= \frac{g'(t)^2}{g(t)^4}|\xi|^{4\sigma-4\theta} \lesssim \frac{|\xi|^{4\sigma}}{(1+t)^2g(t)^2|\xi|^{4\theta}} \\
&\lesssim |\xi|^{4\sigma} \lesssim b(t)K(t,\xi)|\xi|^{2\theta+2\sigma}.
\end{aligned}
$$

Finally, consider Z^{high}. In this region, the right side of the inequalities can depend on R:

(1) $K(t,\xi) \lesssim g(t)|\xi|^{2\theta} \lesssim b(t)|\xi|^{2\theta}$;

(2) $b(t)K(t,\xi)|\xi|^{2\theta} \lesssim \dfrac{b(t)}{g(t)}|\xi|^{2\sigma} \lesssim |\xi|^{2\sigma}$;

(3) In this region, we have:

$$\left(\frac{d}{dt}g(t)\right)^2 \lesssim \frac{g(t)^2}{(1+t)^2} \lesssim g(t)^2|\xi|^{2\sigma};$$

$$\left(\frac{d}{dt}\frac{1}{g(t)}\right)^2 \lesssim \frac{g'(t)^2}{g(t)^4} \lesssim \frac{1}{g(t)^2(1+t)^2} \lesssim \frac{1}{g(t)^2}|\xi|^{2\sigma}.$$

Therefore,

$$\begin{aligned}\left(\frac{d}{dt}K(t,\xi)\right)^2 &\lesssim \min\left\{\tfrac{1}{g(t)^2}, g(t)^2\right\}|\xi|^{2\sigma}|\xi|^{2\min\{2\sigma-2\theta,2\theta\}}\\ &\lesssim b(t)\ \min\left\{\tfrac{1}{g(t)}, g(t)\right\}|\xi|^{\min\{2\sigma-2\theta,2\theta\}}|\xi|^{\min\{4\sigma-2\theta,2\theta+2\sigma\}}\\ &\lesssim b(t)K(t,\xi)|\xi|^{2\theta+2\sigma}.\end{aligned}$$

□

Note that in Z^{high} we have:

$$|\xi|^{\min\{2\theta,2\sigma-2\theta\}} = \begin{cases}|\xi|^{2\theta} & \text{if } \sigma \geq 2\theta\\ |\xi|^{2\sigma-2\theta} & \text{if } \sigma < 2\theta,\end{cases}$$

$$\min\left\{\frac{1}{g(t)}, g(t)\right\} = \begin{cases}\frac{1}{g(t)} & \text{if } g \text{ is increasing}\\ g(t) & \text{if } g \text{ is non-increasing.}\end{cases}$$

This allow us to calculate the multiplier in Z^{high} in each case. Moreover, this multiplier can be improved but we avoid this procedure. Even though the multiplier is not sharp, the decay rates obtained in the high zone are better than the decay rates of the another regions. The explanation for this behaviour is because our equation has not regularity loss property. For a more general class of equations this point must be considered and improved.

The following lemma plays a fundamental role in order to prove Proposition 1.1.3. This result is a suitable adaptation of some ideas of [17].

Lemma 1.1.1 *Let $E : [S_0,\infty) \to [0,\infty)$ differentiable and non-increasing, $f : [S_0,T_0) \to [0,\infty)$ continuous, where $S_0 \in \mathbb{R}$ and $S_0 < T_0 \in \mathbb{R}\cup\{\infty\}$. Suppose that exists $C > 0$ such that*

$$\int_S^{T_0} f(s)\, E(s)\, ds \le C E(S), \qquad \forall\, S \in [S_0,T_0),$$

then, for every $0 < \epsilon < 1$ holds: $E(t) \le \frac{E(S_0)}{1-\epsilon} e^{-\frac{\epsilon}{C}\int_{S_0}^t f(s)ds}$, for all $t \in [S_0,T_0)$.

Proof. Define $\rho(t) := \frac{1}{C}\int_t^{T_0} f(s)E(s)ds$ for $t \in [S_0,T_0)$. For $0 < \epsilon < 1$, consider the "Lyapunov" function $\mathcal{L}(t) := (1-\epsilon)E(t) + \epsilon\rho(t)$, for $t \in [S_0,T_0)$. Therefore:

$$\mathcal{L}'(t) = (1-\epsilon)E'(t) + \epsilon\rho'(t) \le -\frac{\epsilon}{C} f(t)E(t) \le -\frac{\epsilon}{C} f(t)\mathcal{L}(t),$$

for $t \in [S_0,T_0)$.

That is, $\mathcal{L}(t) \le \mathcal{L}(S_0)e^{-\frac{\epsilon}{C}\int_{S_0}^t f(s)ds}$ for $t \in [S_0,T_0)$. Since $(1-\epsilon)E(t) \le \mathcal{L}(t) \le E(t)$, the result follows.

□

Proposition 1.1.3 *For a fixed $\xi \in \mathbb{R}^n$ and given zone Z, we define $S_0(\xi) = \inf\{s \in [t_0,\infty) : (s,\xi) \in Z\}$, $T_0(\xi) = \sup\{s \in [t_0,\infty) : (s,\xi) \in Z\}$ and $\nu := \min\{2\sigma - 2\theta, 2\theta\}$. Then, there exists $C > 0$ independent of ξ such that:*

$E(t) \lesssim e^{-\frac{1}{C}|\xi|^\nu \int_{t_0}^t \min\left\{\frac{1}{g(s)}, g(s)\right\}ds} E(0)$ for all $(t,\xi) \in Z^{high}$.

If Z_{hyp} has no zero measure, $E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{S_0(\xi)}^t g(s)ds} E(S_0(\xi))$ for all $S_0(\xi) \le t < T_0(\xi)$.

If Z_{ell} has no zero measure, $E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{S_0(\xi)}^t \frac{1}{g(s)}ds} E(S_0(\xi))$ for all $S_0(\xi) \le t < T_0(\xi)$.

Proof. Let $K : Z \to [0,\infty)$ a multiplier of energy in $Z \subset [0,\infty)\times\mathbb{R}^n$. By Proposition 1.1.1, we know that:

$$\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds \lesssim E(S), \quad \forall\,(S,\xi),(T,\xi) \in Z \text{ with } T > S.$$

Further, by equation (1.5) and by property **(1)** of multiplier of energy:

$$\int_S^T K(s,\xi)|\hat{u}_t(s)|^2 ds \lesssim \int_S^T b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds \leq E(S),$$

for all $(S,\xi),(T,\xi) \in Z$ with $T > S$.

Therefore $\int_S^T K(s,\xi)E(s)ds \lesssim E(S)$, for all $(S,\xi),(T,\xi) \in Z$ with $T > S$. Fixing $\xi \in \mathbb{R}^n$, such that $\{t \in [t_0,\infty) : (t,\xi) \in Z\}$ has non-zero measure, we apply Lemma 1.1.1 and conclude:

$$E(t) \lesssim e^{-\frac{1}{C}\int_{S_0(\xi)}^t K(s,\xi)ds} E(S_0(\xi)), \tag{1.15}$$

for all $S_0(\xi) \leq t < T_0(\xi)$. Using Proposition 1.1.2 we know that the corresponding multiplier of energy in each zone, applying in inequality (1.15) we conclude the result.

□

Remark 1.1.2 *In the last proposition, when $T_0(\xi) < \infty$ the inequalities also hold for $t = T_0(\xi)$. Furthermore, the estimates for $E(t) = E(t,\xi)$ are uniform in ξ, that is, the constants in the right side of inequality does not depends on ξ.*

A careful analysis of Proposition 1.1.3 makes us observe another interesting detail: in the first inequality the integral begins in t_0 while in the remaining inequalities the integral begins in $S_0(\xi)$. Furthermore, the energy in the right side of first inequality is valued in zero, while in the other cases is valued in $S_0(\xi)$. In this context, the ξ *independence* (in the range of integration of t and in the time variable of E) of the first inequality make the pointwise estimates in Fourier space for Z^{high} ready to be integrated and conclude the estimates (see Proposition 1.2.3). From now, our idea is improve the estimates in Z_{hyp} and Z_{ell}, in such a way that it has the desired independence on ξ in time variable. The Proposition 1.1.4 will be fundamental for this upgrade.

We have defined t_ξ as the unique solution of $|\xi|^{2\theta}\varphi(t_\xi) = 1$ and $\psi(\xi) = |\xi|^{2\theta}g(t_\xi)$, and now we want to investigate $\max\{|\xi|^\sigma, \psi(\xi)\}$ in B_R. In this chapter, we consider the case $\max\{|\xi|^\sigma, \psi(\xi)\} = |\xi|^\sigma$ for small frequencies. For the remaining case, it is necessary an improvement of the estimates in elliptic zone using a substantially different method, this is the reason why we treat it in Chapter 2. Furthermore, this justify the new classification introduced based in that maximum. In this sense, the restrictions assumed in Proposition 1.1.4 comes from this classification.

Proposition 1.1.4 *Let $\alpha > -1$, γ, θ and σ satisfying Hypothesis A. Thus, there exists small $R > 0$ such that $\psi(\xi) < |\xi|^\sigma$ for all ξ in B_R, except for the case $\gamma = 0$ and $\sigma(1+\alpha) = 2\theta$ such that holds $\psi(\xi) = |\xi|^\sigma$ for all $\xi \in B_R$.*

Proof. For $\xi \neq 0$, let t_ξ be the unique solution of the equation $(1 + t_\xi)^{1+\alpha} ln^\gamma(1 + t_\xi) = |\xi|^{-2\theta}$. Applying Lemma C.1.4 with $\tau = 1 + t_\xi$, $\mu = 1 + \alpha$, $\beta = \gamma$ and $\lambda = |\xi|^{-2\theta}$, we have for $\alpha \neq -1$:

$$(1+t_\xi) = \begin{cases} \left(\frac{1+\alpha}{|\gamma|}\right)^{\frac{\gamma}{1+\alpha}} |\xi|^{-\frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|}|\xi|^{\frac{2\theta}{|\gamma|}}\right)\right]^{-\frac{\gamma}{1+\alpha}} & \text{if } \gamma < 0, \\ |\xi|^{-\frac{2\theta}{1+\alpha}} \text{ if } \gamma = 0, \\ \left(\frac{1+\alpha}{\gamma}\right)^{\frac{\gamma}{1+\alpha}} |\xi|^{-\frac{2\theta}{1+\alpha}} \left[W_0\left(\frac{1+\alpha}{\gamma}|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{-\frac{\gamma}{1+\alpha}} & \text{if } \gamma > 0, \end{cases}$$

where W_0, W_{-1} are the two real-valued branches of W-Lambert's function (see Appendix C for further details concerning this special function). To carefully apply Lemma C.1.4, we need to consider the following condition:

$$R < \left(\frac{(1+\alpha)e}{|\gamma|}\right)^{-\frac{|\gamma|}{2\theta}}, \quad \text{if } \gamma < 0. \tag{1.16}$$

Case $\gamma < 0$: In this case $\sigma \leq \frac{2\theta}{1+\alpha}$. Since $\psi(\xi) = \frac{|\xi|^{2\theta}\varphi(t_\xi)}{1+t_\xi} = \frac{1}{1+t_\xi}$, we have:

$$\psi(\xi) = \left(\frac{1+\alpha}{|\gamma|}\right)^{\frac{|\gamma|}{1+\alpha}} |\xi|^{\frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|}|\xi|^{\frac{2\theta}{|\gamma|}}\right)\right]^{-\frac{|\gamma|}{1+\alpha}},$$

for $\xi \in B_R$, R small enough. By Corollary C.1.1, we have the limit:

$$\lim_{r\to 0^+} r^{\sigma - \frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|} r^{\frac{2\theta}{|\gamma|}}\right)\right]^{\frac{|\gamma|}{1+\alpha}} = \infty,$$

therefore, there exists $R(\gamma,\alpha,\sigma,\theta) > 0$ such that:

$$|\xi|^{\sigma-\frac{2\theta}{1+\alpha}}\left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|}|\xi|^{\frac{2\theta}{|\gamma|}}\right)\right]^{\frac{|\gamma|}{1+\alpha}} > \left(\frac{1+\alpha}{|\gamma|}\right)^{\frac{|\gamma|}{1+\alpha}}, \tag{1.17}$$

for all $0 < |\xi| \ \leq R$. That is, $\psi(\xi) < |\xi|^\sigma$ for all $\xi \in B_R$.

Case $\gamma = 0$: $\psi(\xi) = \frac{1}{1+t_\xi} = |\xi|^{\frac{2\theta}{1+\alpha}}$.

If $\sigma(1+\alpha) = 2\theta$, trivially $\psi(\xi) = |\xi|^\sigma$ for $\xi \in B_R$. In the case $\sigma(1+\alpha) < 2\theta$, we have for $R < 1$, $\psi(\xi) < |\xi|^\sigma$ for all $0 < |\xi| \leq R$.

Case $\gamma > 0$: In this case, necessarily $\sigma < \frac{2\theta}{1+\alpha}$. Since $\psi(\xi) = \frac{1}{1+t_\xi}$, we have:

$$\psi(\xi) = \left(\frac{\gamma}{1+\alpha}\right)^{\frac{\gamma}{1+\alpha}}|\xi|^{\frac{2\theta}{1+\alpha}}\left[W_0\left(\frac{1+\alpha}{\gamma}|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}},$$

furthermore, by Corollary C.1.2 the following limit holds:

$$\lim_{r\to 0^+}\frac{r^{\sigma-\frac{2\theta}{1+\alpha}}}{\left[W_0\left(\frac{1+\alpha}{\gamma}r^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}}} = +\infty.$$

In this case, there exists $R(\gamma,\alpha,\sigma,\theta) > 0$ such that:

$$\frac{|\xi|^{\sigma-\frac{2\theta}{1+\alpha}}}{\left[W_0\left(\frac{1+\alpha}{\gamma}|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}}} > \left(\frac{\gamma}{1+\alpha}\right)^{\frac{\gamma}{1+\alpha}}, \tag{1.18}$$

for $0 < |\xi| \ \leq R$. Then, $\psi(\xi) < |\xi|^\sigma$ for all $\xi \in B_R$.

□

The last proposition is necessary to treat the estimates in elliptic zone, in special to deal with the separation line between elliptic zone and hyperbolic zone. As mentioned before of Proposition 1.1.2, the elliptic zone is empty if $\alpha = -1$ and therefore, even though the same proposition holds in this case, the result of Proposition 1.1.4 is not necessary.

Remark 1.1.3 *We know that φ is increasing (therefore φ^{-1} is also increasing). Thus, if $R < \varphi(t_0)^{-\frac{1}{2\theta}}$, we have $t_\xi = \varphi^{-1}(|\xi|^{-2\theta}) > \varphi^{-1}(\varphi(t_0)) = t_0$, for all $\xi \in B_R$. Furthermore, taking in account the definition of the zones, given $\xi \in B_R$, $t \geq t_\xi$ if, and only if $(t,\xi) \in Z_{ell} \cup Z_{hyp}$ and $t_0 \leq t \leq t_\xi$ if, and only if $(t,\xi) \in Z_{low}$. This remark will be widely used in demonstrating the next proposition.*

When $|\xi|^{\sigma-2\theta}$ is in the domain of g^{-1}, we define $t_1(\xi) := g^{-1}\left(|\xi|^{\sigma-2\theta}\right)$. The number $t_1(\xi)$ is precisely the point where occur a change between hyperbolic behaviour and elliptic behaviour. But sometimes $t_1(\xi)$ simply does not exists, in which means that elliptic zone or hyperbolic zone has zero measure. Since we are interested in applying Proposition 1.1.3, we must to care about the lower $S_0(\xi)$ and up bounds $T_0(\xi)$ limits of the proposition. Therefore, throughout the demonstration below, the existence of $t_1(\xi)$ is discussed only when $t_1(\xi)$ plays role to calculate $S_0(\xi)$ or $T_0(\xi)$.

Proposition 1.1.5 *There exists $C > 0$ such that the following estimates hold:*

If Z_{ell} has no zero measure, $E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_0}^{t}\frac{1}{g(s)}ds}E(0)$, for all $(t,\xi) \in Z_{ell}$.

If Z_{hyp} has no zero measure, $E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{t_0}^{t}g(s)ds}E(0)$, for all $(t,\xi) \in Z_{hyp}$.

Proof. By Proposition 1.1.4, we know that $\psi(\xi) \leq |\xi|^\sigma$ for all $\xi \in B_R$. Initially we consider the case $\psi(\xi) < |\xi|^\sigma$ for all $\xi \in B_R$ and the case $\alpha = -1$.

Case g increasing:

In this case, we have $0 \leq \alpha < 1$ (with $\gamma > 0$ if $\alpha = 0$) thus $\sigma < 2\theta$. Since g can be seen as a bijection between $[t_0,\infty)$ and $[g(t_0),\infty)$, $t_1(\xi)$ is well defined if $|\xi|^{\sigma-2\theta} \geq g(t_0)$ for all $\xi \in B_R$, that is, if:

$$R \leq g(t_0)^{\frac{1}{\sigma-2\theta}}. \tag{1.19}$$

Since $\psi(\xi) < |\xi|^\sigma$ for all $\xi \in B_R$, it follows directly from the definition of ψ that $t_\xi < t_1(\xi)$ for all $\xi \in B_R$. Therefore, for each fixed $\xi \in B_R$, $[t_0, \infty) = [t_0, t_\xi] \cup [t_\xi, t_1(\xi)] \cup [t_1(\xi), \infty)$, where $(s, \xi) \in Z_{low}$ for $s \in [t_0, t_\xi]$, $(s, \xi) \in Z_{hyp}$ for $s \in [t_\xi, t_1(\xi)]$ and $(s, \xi) \in Z_{ell}$ when $s \geq t_1(\xi)$. That is, $S_0(\xi) = t_\xi$ and $T_0(\xi) = t_1(\xi)$ in hyperbolic zone, $S_0(\xi) = t_1(\xi)$ and $T_0(\xi) = \infty$ in elliptic zone.

By Lemma A.1.2, $|\xi|^{2\theta} \int_{t_0}^{t_\xi} g(s)ds \lesssim |\xi|^{2\theta}(1+t_\xi)g(t_\xi) = |\xi|^{2\theta}\varphi(t_\xi) = 1$. Applying Proposition 1.1.3 in the hyperbolic zone we have (for any $C \geq C_2$):

$$E(t) \lesssim e^{-\frac{1}{C_2}|\xi|^{2\theta} \int_{t_\xi}^t g(s)ds} E(t_\xi) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta} \int_{t_0}^t g(s)ds} E(0), \qquad (1.20)$$

for all $(t, \xi) \in Z_{hyp}$. On the other hand, choosing $C_1 \geq C_2 \frac{(1+\alpha)}{(1-\alpha)}$ (with C_1 and C_2 greater than or equal to the constant appearing in Proposition 1.1.3) and using Lemma A.1.2, we have:

$$\begin{aligned} -\frac{1}{C_1}|\xi|^{2\sigma-2\theta} \int_{t_1(\xi)}^t \frac{1}{g(s)} ds - \frac{1}{C_2}|\xi|^{2\theta} \int_{t_0}^{t_1(\xi)} g(s)ds \\ \leq -\frac{1}{C_3}|\xi|^{2\sigma-2\theta} \frac{(1+t)}{g(t)} + C_4, \end{aligned}$$

for all $t \geq t_1(\xi)$. Thus, using inequality (1.20) in $t = t_1(\xi)$ by applying Proposition 1.1.3 in Z_{ell}:

$$\begin{aligned} E(t) &\lesssim e^{-\frac{1}{C_1}|\xi|^{2\sigma-2\theta} \int_{t_1(\xi)}^t \frac{1}{g(s)} ds} E(t_1(\xi)) \\ &\lesssim e^{-\frac{1}{C_1}|\xi|^{2\sigma-2\theta} \int_{t_1(\xi)}^t \frac{1}{g(s)} ds} e^{-\frac{1}{C_2}|\xi|^{2\theta} \int_{t_0}^{t_1(\xi)} g(s)ds} E(0) \\ &\lesssim e^{-\frac{1}{C_3}|\xi|^{2\sigma-2\theta} \frac{(1+t)}{g(t)}} E(0) \\ &\lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta} \int_{t_0}^t \frac{1}{g(s)} ds} E(0), \end{aligned}$$

for all $(t, \xi) \in Z_{ell}$.

Case g decreasing:

For $\alpha \neq -1$, by definition of ψ and due the fact $\psi(\xi) < |\xi|^\sigma$, follows $t_1(\xi) < t_\xi$ for all $\xi \in B_R$. Furthermore, this implies $Z_{ell} = \varnothing$. Applying

Proposition 1.1.3 in hyperbolic zone:

$$E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{t_\xi}^{t} g(s)ds} E(t_\xi),$$

for all $(t,\xi) \in Z_{hyp}$. By Lemma A.1.2, we have that $|\xi|^{2\theta}\int_{t_0}^{t_\xi} g(s)ds \lesssim |\xi|^{2\theta}(1+t_\xi)g(t_\xi) \lesssim 1$, and therefore:

$$E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{t_0}^{t} g(s)ds} E(0),$$

for all $(t,\xi) \in Z_{hyp}$.

The case $\alpha = -1$ is similar, applying Corollary A.1.1 instead of Lemma A.1.2.

Case $g = 1$:

By Proposition 1.1.4, we have $\sigma < 2\theta$. In this case $Z_{ell} = \varnothing$. Since $|\xi|^{2\theta}(1+t_\xi) = 1$, applying Proposition 1.1.3 in Z_{hyp} we have:

$$E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{t_\xi}^{t} g(s)ds} E(t_\xi) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{t_0}^{t} g(s)ds} E(0)$$

for all $(t,\xi) \in Z_{hyp}$.

Let us consider the case $\psi(\xi) = |\xi|^\sigma$ for all $\xi \in B_R$, that is, by Proposition 1.1.4 is the case $\sigma(1+\alpha) = 2\theta$, $\gamma = 0$ and $\alpha \in (-1,1)$. Furthermore, $|\xi|^\sigma = \psi(\xi)$ implies $t_\xi = t_1(\xi)$. For g decreasing, the proof is the same as in the case $\psi(\xi) < |\xi|^\sigma$. When $g = 1$, the proof is again as in the case $\psi(\xi) < |\xi|^\sigma$, the only difference is that $Z_{ell} = Z_{hyp}$.

Finally, for the case g increasing, Z_{hyp} has zero measure. Indeed, if $t \geq t_\xi$ then $g(t) \geq g(t_\xi) = g(t_1(\xi)) = |\xi|^{\sigma-2\theta}$. Thus $[t_0,\infty) = [t_0,t_\xi] \cup [t_\xi,\infty)$, where $(s,\xi) \in Z_{low}$ if $s \in [t_0,t_\xi]$ and $(s,\xi) \in Z_{ell}$ if $s \in [t_\xi,\infty)$. Using the Lemma A.1.2, we have:

$$\begin{aligned}
-\frac{1}{C_1}|\xi|^{2\sigma-2\theta}\int_{t_\xi}^{t}\frac{1}{g(s)}ds &\leq -\frac{1}{C_2}|\xi|^{2\sigma-2\theta}\frac{(1+t)}{g(t)} + \frac{1}{C_2}|\xi|^{2\sigma-2\theta}\frac{(1+t_\xi)}{g(t_\xi)} \\
&\lesssim -\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_0}^{t}\frac{1}{g(s)}ds + \frac{1}{C_2}, \qquad (1.21)
\end{aligned}$$

because

$$|\xi|^{2\sigma-2\theta}\frac{(1+t_\xi)}{g(t_\xi)} = \psi(\xi)^2\frac{(1+t_\xi)}{|\xi|^{2\theta}g(t_\xi)} = |\xi|^{2\theta}g(t_\xi)(1+t_\xi) = |\xi|^{2\theta}\varphi(t_\xi) = 1.$$

By applying Proposition 1.1.3 and inequality (1.21), we have for all $(t,\xi) \in Z_{ell}$:

$$E(t) \lesssim e^{-\frac{1}{C1}|\xi|^{2\sigma-2\theta}\int_{t_\xi}^t \frac{1}{g(s)}ds}E(t_\xi) \lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_0}^t \frac{1}{g(s)}ds}E(0).$$

□

1.2 Integration in each zone

In this section we will apply the pointwise estimates in Fourier space of the previous section, fix the time variable and integrate ξ in $\mathbb{R}^n$. This procedure is make by considering the zone separation introduced in the beginning of Section 1.1 and a proof divided in several propositions to deal with each zone. During the step of integration in ξ, we often use results of Appendices A and B.

To proof the results, we shall consider in this section $\hat{q}$ conjugate of $q \in [2,\infty]$, that is $\hat{q} \in [1,2]$ and $\frac{1}{\hat{q}} + \frac{1}{q} = 1$. Furthermore, $u_0, u_1 \in L^p(\mathbb{R}^n)$ and s conjugate of $p \in [1,2]$, that is, $s \in [2,\infty]$ and therefore $s \geq \hat{q}$. We define $r := \infty$ if $p+\hat{q} = p\hat{q}$ and $r := \frac{p}{p+\hat{q}-p\hat{q}} \geq 1$ if $p+\hat{q} \neq p\hat{q}$. That is, r is conjugated of $\frac{s}{\hat{q}}$, since $\frac{p+\hat{q}-p\hat{q}}{p} + \frac{\hat{q}}{s} = 1$. In addition, we take $\mu, \beta \in \mathbb{N}^n$ and $\varphi = \varphi(t)$ given by (1.27). In this section several times will appear the condition $|\beta| - \sigma + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$, which is equivalent to condition $|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right) > \sigma$ that rises in Theorem 1.3.1.

Proposition 1.2.1 *Consider the conditions above. Let* $u(t,x)$ *the solution of* (1)-(2)*, then the following estimates hold for* $t \geq t_0$*:*

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||_{L^p}^{\hat{q}} \\ &+ \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||_{L^p}^{\hat{q}}.\end{aligned}$$

Let $|\beta| - \sigma + \frac{n}{p\hat{q}}(p+q-p\hat{q}) > 0$ *if* $u_1 \neq 0$ *and any* $\beta \in \mathbb{N}^n$ *if* $u_1 = 0$*, then:*

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||_{L^p}^{\hat{q}} \\ &+ \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||_{L^p}^{\hat{q}}.\end{aligned}$$

Proof. Let $\tau := \varphi(t)^{-\frac{1}{2\theta}}$. Since $|\hat{u}_t(t)|^{\hat{q}} \lesssim |\xi|^{\hat{q}\sigma}|\hat{u}_0|^{\hat{q}} + |\hat{u}_1|^{\hat{q}}$, we have:

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \int_{Z_{low}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma}|\hat{u}_0|^{\hat{q}}d\xi + \int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_1|^{\hat{q}}d\xi \\ &\lesssim \left\| |\cdot|^{\hat{q}|\mu|+\hat{q}\sigma} \right\|_{L^r(B_\tau)} ||\hat{u}_0||_{L^s}^{\hat{q}} + \left\| |\cdot|^{\hat{q}|\mu|} \right\|_{L^r(B_\tau)} ||\hat{u}_1||_{L^s}^{\hat{q}}.\end{aligned}$$

Using Hausdorff-Young inequality (see [1]) and Lemma B.1.1 with $k = \hat{q}|\mu| + \hat{q}\sigma$ for the first term on the right side of above inequality, $k = \hat{q}|\mu|$ for the second term, we have

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||_{L^p}^{\hat{q}} \\ &+ \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||_{L^p}^{\hat{q}},\end{aligned}$$

for $t \geq t_0$.

Let $|\beta| - \sigma + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$ if $u_1 \neq 0$ and any $\beta \in \mathbb{N}^n$ if $u_1 = 0$.

Thus,

$$\int_{Z_{low}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \int_{Z_{low}} |\xi|^{\hat{q}|\beta|} |\hat{u}_0|^{\hat{q}} d\xi + \int_{Z_{low}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} |\hat{u}_1|^{\hat{q}} d\xi$$
$$\lesssim \left\| |\cdot|^{\hat{q}|\beta|} \right\|_{L^r(B_\tau)} ||\hat{u}_0||^{\hat{q}}_{L^s} + \left\| |\cdot|^{\hat{q}|\beta|-\hat{q}\sigma} \right\|_{L^r(B_\tau)} ||\hat{u}_1||^{\hat{q}}_{L^s}.$$

By Hausdorff-Young inequality and Lemma B.1.3 we have for $t \geq t_0$:

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \quad &\lesssim \quad \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} \\ &\quad + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}.\end{aligned}$$

□

Proposition 1.2.2 *Under the conditions of Proposition 1.2.1, the following estimates hold for $t \geq t_0$:*

$$\begin{aligned}\int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \quad &\lesssim \quad \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} \\ &\quad + \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}; \\ \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \quad &\lesssim \quad \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} \\ &\quad + \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}.\end{aligned}$$

Let $|\beta| - \sigma + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$ if $u_1 \neq 0$ and any $\beta \in \mathbb{N}^n$ if $u_1 = 0$, *then:*

$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||^{\hat{q}}_{L^p} + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p};$$

$$\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||^{\hat{q}}_{L^p} + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p}.$$

Proof. Let $\phi(t) := (1+t)^{1-\alpha}ln^{-\gamma}(1+t)$ and φ as before. Taking in account the Hypothesis A, we have for $\theta \in (0,\sigma)$ and for all $\eta > 0$: $\phi(t)^{-\frac{\eta}{2\sigma-2\theta}} \lesssim \varphi(t)^{-\frac{\eta}{2\theta}}$ for all $t \geq t_0$. The last inequality will be useful for the sake of simplicity.

In Z_{ell} we initially consider $\sigma \neq \theta$ (in the hyperbolic zone this restriction is not necessary). Using Proposition 1.1.5 and Corollary A.1.1:

$$\begin{aligned}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma}e^{-\frac{\hat{q}}{2C}|\xi|^{2\sigma-2\theta}\int_{t_0}^{t}\frac{1}{g(s)}ds}|\hat{u}_0|^{\hat{q}}d\xi \\ &\quad + \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}e^{-\frac{\hat{q}}{2C}|\xi|^{2\sigma-2\theta}\int_{t_0}^{t}\frac{1}{g(s)}ds}|\hat{u}_1|^{\hat{q}}d\xi \\ &\lesssim \left\||\cdot|^{\hat{q}|\mu|+\hat{q}\sigma}e^{-c\phi(t)|\cdot|^{2\sigma-2\theta}}\right\|_{L^r(\mathbb{R}^n)}||\hat{u}_0||^{\hat{q}}_{L^s} \\ &\quad + \left\||\cdot|^{\hat{q}|\mu|}e^{-c\phi(t)|\cdot|^{2\sigma-2\theta}}\right\|_{L^r(\mathbb{R}^n)}||\hat{u}_1||^{\hat{q}}_{L^s} \quad (1.22)\end{aligned}$$

and

$$\begin{aligned}\int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma}e^{-\frac{\hat{q}}{2C}|\xi|^{2\theta}\int_{t_0}^{t}g(s)ds}|\hat{u}_0|^{\hat{q}}d\xi \\ &\quad + \int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|}e^{-\frac{\hat{q}}{2C}|\xi|^{2\theta}\int_{t_0}^{t}g(s)ds}|\hat{u}_1|^{\hat{q}}d\xi \\ &\lesssim \left\||\cdot|^{\hat{q}|\mu|+\hat{q}\sigma}e^{-c\varphi(t)|\cdot|^{2\theta}}\right\|_{L^r(\mathbb{R}^n)}||\hat{u}_0||^{\hat{q}}_{L^s} \\ &\quad + \left\||\cdot|^{\hat{q}|\mu|}e^{-c\varphi(t)|\cdot|^{2\theta}}\right\|_{L^r(\mathbb{R}^n)}||\hat{u}_1||^{\hat{q}}_{L^s}. \quad (1.23)\end{aligned}$$

Using Hausdorff-Young inequality and Lemma B.1.2 with $k_1 = \hat{q}|\mu| + \hat{q}\sigma$ or $k_1 = \hat{q}|\mu|$, $k_2 = 2\sigma - 2\theta$ and $\tau = c\phi(t)$ in inequality (1.22) or $k_2 = 2\theta$ and $\tau = c\varphi(t)$ in inequality (1.23), we have

$$\begin{aligned}
\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi &\lesssim \phi(t)^{-\left(\frac{\hat{q}|\mu| + \hat{q}\sigma + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} \\
&\quad + \phi(t)^{-\left(\frac{\hat{q}|\mu| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ||u_1||^{\hat{q}}_{L^p} \\
&\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu| + \hat{q}\sigma + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} \\
&\quad + \varphi(t)^{-\left(\frac{\hat{q}|\mu| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}
\end{aligned}$$

and

$$\begin{aligned}
\int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu| + \hat{q}\sigma + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} \\
&\quad + \varphi(t)^{-\left(\frac{\hat{q}|\mu| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p},
\end{aligned}$$

for $t \geq t_0$.

Let $|\beta| - \sigma + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) > 0$ if $u_1 \neq 0$ and any $\beta \in \mathbb{N}^n$ if $u_1 = 0$. Thus, using Proposition 1.1.5 and Corollary A.1.1:

$$\begin{aligned}
\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi &\lesssim \int_{Z_{ell}} |\xi|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\xi|^{2\sigma-2\theta} \int_{t_0}^t \frac{1}{g(s)} ds} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad + \int_{Z_{ell}} |\xi|^{\hat{q}|\beta| - \hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\xi|^{2\sigma-2\theta} \int_{t_0}^t \frac{1}{g(s)} ds} |\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim \left\| |\cdot|^{\hat{q}|\beta|} e^{-c\phi(t)|\cdot|^{2\sigma-2\theta}} \right\|_{L^r(\mathbb{R}^n)} ||\hat{u}_0||^{\hat{q}}_{L^s} \qquad (1.24) \\
&\quad + \left\| |\cdot|^{\hat{q}|\beta| - \hat{q}\sigma} e^{-c\phi(t)|\cdot|^{2\sigma-2\theta}} \right\|_{L^r(\mathbb{R}^n)} ||\hat{u}_1||^{\hat{q}}_{L^s}
\end{aligned}$$

and

$$\begin{aligned}\int_{Z_{hyp}}|\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi &\lesssim \int_{Z_{hyp}}|\xi|^{\hat{q}|\beta|}e^{-\frac{\hat{q}}{2C}|\xi|^{2\theta}\int_{t_0}^t g(s)ds}|\hat{u}_0|^{\hat{q}}d\xi \\ &\quad+\int_{Z_{hyp}}|\xi|^{\hat{q}|\beta|-\hat{q}\sigma}e^{-\frac{\hat{q}}{2C}|\xi|^{2\theta}\int_{t_0}^t g(s)ds}|\hat{u}_1|^{\hat{q}}d\xi \\ &\lesssim \left\||\cdot|^{\hat{q}|\beta|}e^{-c\varphi(t)|\cdot|^{2\theta}}\right\|_{L^r(\mathbb{R}^n)}||\hat{u}_0||^{\hat{q}}_{L^s} \qquad (1.25)\\ &\quad+\left\||\cdot|^{\hat{q}|\beta|-\hat{q}\sigma}e^{-c\varphi(t)|\cdot|^{2\theta}}\right\|_{L^r(\mathbb{R}^n)}||\hat{u}_1||^{\hat{q}}_{L^s}.\end{aligned}$$

Using Hausdorff-Young inequality and Lemma B.1.2 with $k_1=\hat{q}|\beta|$ or $k_1=\hat{q}|\beta|-\hat{q}\sigma$, $k_2=2\sigma-2\theta$ and $\tau=c\phi(t)$ in inequality (1.24) or $k_2=2\theta$ and $\tau=c\varphi(t)$ in inequality (1.25), we have:

$$\begin{aligned}\int_{Z_{ell}}|\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi &\lesssim \phi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)}||u_0||^{\hat{q}}_{L^p} \\ &\quad+\phi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)}||u_1||^{\hat{q}}_{L^p} \\ &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||^{\hat{q}}_{L^p} \\ &\quad+\varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p}\end{aligned}$$

and

$$\begin{aligned}\int_{Z_{hyp}}|\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||^{\hat{q}}_{L^p} \\ &\quad+\varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p}\end{aligned}$$

for $t\geq t_0$.

For the case Z_{ell} with $\theta = \sigma$, we directly apply Proposition 1.1.5:

$$
\begin{aligned}
&\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \\
&\lesssim \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} |\hat{u}_0|^{\hat{q}} d\xi + \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|} e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} |\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} \left(\left\| |\cdot|^{\hat{q}|\mu|+\hat{q}\sigma} \right\|_{L^r(B_1)} ||\hat{u}_0||^{\hat{q}}_{L^s} + \left\| |\cdot|^{\hat{q}|\mu|} \right\|_{L^r(B_1)} ||\hat{u}_1||^{\hat{q}}_{L^s} \right) \\
&\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} \left(||u_0||^{\hat{q}}_{L^p} + ||u_1||^{\hat{q}}_{L^p} \right) \\
&\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} + \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p},
\end{aligned}
$$

in which the penult inequality is given by Hausdorff-Young inequality, and the last inequality is provided that $e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds}$ is a exponential-type decay while $\varphi(t)^{-1}$ is algebraic or logarithmic decay.

Let $|\beta| - \sigma + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$ if $u_1 \neq 0$ and any $\beta \in \mathbb{N}^n$ if $u_1 = 0$, using again Proposition 1.1.5 and Hausdorff-Young inequality:

$$
\begin{aligned}
&\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \\
&\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} \int_{Z_{ell}} |\xi|^{\hat{q}|\beta|} |\hat{u}_0|^{\hat{q}} d\xi + e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} \int_{Z_{ell}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} |\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} \left(\left\| |\cdot|^{\hat{q}|\beta|} \right\|_{L^r(B_1)} ||\hat{u}_0||^{\hat{q}}_{L^s} + \left\| |\cdot|^{\hat{q}|\beta|-\hat{q}\sigma} \right\|_{L^r(B_1)} ||\hat{u}_1||^{\hat{q}}_{L^s} \right) \\
&\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^{t}\frac{1}{g(s)}ds} \left(||u_0||^{\hat{q}}_{L^p} + ||u_1||^{\hat{q}}_{L^p} \right) \\
&\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}.
\end{aligned}
$$

□

Proposition 1.2.3 *Consider the conditions of Proposition 1.2.1. Then, there exists $t_0^* \geq t_0$ such that the following estimates hold for $t \geq t_0^*$:*

If $\theta \neq \sigma$,

$$\int_{Z^{high}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}$$

and

$$\int_{Z^{high}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{L^p} + \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{L^p}.$$

If $\theta = \sigma$ and $\omega > \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})$,

$$\int_{Z^{high}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{W^{|\beta|+\omega,p}} + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{W^{|\beta|-\sigma+\omega,p}}$$

and

$$\int_{Z^{high}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||^{\hat{q}}_{W^{|\mu|+\sigma+\omega,p}} + \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||^{\hat{q}}_{W^{|\mu|+\omega,p}}.$$

Proof. In this proof we fix $f(t) := g(t)$ is g is non-increasing and $f(t) := \frac{1}{g(t)}$ if g is increasing. Let $\nu := \min\{2\sigma - 2\theta, 2\theta\}$. Using Corollary A.1.1, we have:

$$
\begin{aligned}
&e^{-\frac{1}{C_1}\int_{t_0}^t f(s)ds} \\
&\quad\lesssim \begin{cases} e^{-c(1+t)^{1-\alpha} ln^{-\gamma}(1+t)}, & \text{if } \ 0<\alpha<1 \ \text{ or } \alpha=0 \ \text{ and } \ \gamma>0, \\ e^{-c(1+t)^{1+\alpha} ln^{\gamma}(1+t)}, & \text{if } \ -1<\alpha<0 \ \text{ or } \alpha=0 \ \text{ and } \ \gamma\le 0, \\ e^{-c\, ln^{1+\gamma}(1+t)}, & \text{if } \ \alpha=-1 \ \text{ and } \gamma>-1, \\ e^{-c\, ln(ln(1+t))}, & \text{if } \ \alpha=-1 \ \text{ and } \gamma=-1. \end{cases}
\end{aligned}
$$

Therefore, given $\eta>0$, for a sufficient big t_0^*, we have for all $t\ge t_0^*$:

$$
e^{-\frac{1}{C_1}\int_{t_0}^t f(s)ds} \lesssim \varphi(t)^{-\eta}. \tag{1.26}
$$

Initially, let $\theta\neq\sigma$ and $t_0^*\ge t_0$ such that $\int_{t_0}^{t_0^*} f(s)ds \ge 1$ (this number exists because $\frac{1}{g}$ and $g\notin L^1(\mathbb{R})$) and such that inequality (1.26) is satisfied. For $t\ge t_0^*$, by Proposition 1.1.3 we have:

$$
\begin{aligned}
&\int_{Z^{high}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \\
&\lesssim \int_{Z^{high}} |\xi|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\xi|^\nu \int_{t_0}^t f(s)ds} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad + \int_{Z^{high}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\xi|^\nu \int_{t_0}^t f(s)ds} |\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim e^{-\frac{\hat{q}R^\nu}{2C}\int_{t_0^*}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\xi|^\nu} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad + e^{-\frac{\hat{q}R^\nu}{2C}\int_{t_0^*}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\xi|^\nu} |\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} \left\| |\cdot|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\cdot|^\nu} \right\|_{L^r(\mathbb{R}^n)} ||\hat{u}_0||_{L^s}^{\hat{q}} \\
&\quad + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} \left\| |\cdot|^{\hat{q}|\beta|-\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\cdot|^\nu} \right\|_{L^r(\mathbb{R}^n)} ||\hat{u}_1||_{L^s}^{\hat{q}}.
\end{aligned}
$$

Let $|\beta|-\sigma+\frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})>0$ if $u_1\neq 0$ and any $\beta\in\mathbb{N}^n$ if $u_1=0$, using Lemma B.1.2 with $\tau=\frac{\hat{q}}{2C}$ (in this case $k_2=\nu\neq 0$) and

Hausdorff-Young inequality:

$$\int_{Z^{high}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||_{L^p}^{\hat{q}} + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||_{L^p}^{\hat{q}}.$$

Suppose now $\theta = \sigma$. In this case, since $\omega\hat{q} > \frac{n}{p}(p+\hat{q}-p\hat{q}) = \frac{n}{r}$, we have $\left\| |\cdot|^{-\omega\hat{q}} \right\|_{L^r(\mathbb{R}^n \setminus B_R)} \lesssim 1$. Using inequality (1.26), Holder inequality and Hausdorff-Young inequality we have:

$$\begin{aligned}
&\int_{Z^{high}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \\
&\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad + e^{-\frac{\hat{q}}{2C}\int_{t_0}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} |\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} \left\| |\cdot|^{-\omega\hat{q}} \right\|_{L^r(\mathbb{R}^n \setminus B_R)} \left\| |\cdot|^{|\beta|+\omega} \hat{u}_0 \right\|_{L^s}^{\hat{q}} \\
&\quad + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} \left\| |\cdot|^{-\omega\hat{q}} \right\|_{L^r(\mathbb{R}^n \setminus B_R)} \left\| |\cdot|^{|\beta|-\sigma+\omega} \hat{u}_1 \right\|_{L^s}^{\hat{q}} \\
&\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||_{W^{|\beta|+\omega,p}}^{\hat{q}} \\
&\quad + \varphi(t)^{-\left(\frac{\hat{q}|\beta|-\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||_{W^{|\beta|-\sigma+\omega,p}}^{\hat{q}}.
\end{aligned}$$

The proof of estimate for $\int_{Z^{high}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi$ is analogous.

□

1.3 Main Theorem

Using the results of the last section, we are able to prove the following theorem:

Theorem 1.3.1 *Let $n \geq 1$ and θ, σ, α, γ satisfying Hypothesis A. Let $1 \leq p \leq 2 \leq q \leq \infty$, $\beta, \mu \in \mathbb{N}^n$ and β satisfying $|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right) > \sigma$ if $u_1 \neq 0$. Consider the function*

$$\varphi(t) = \begin{cases} (1+t)^{1+\alpha} ln^{\gamma}(1+t), & \text{if } \alpha > -1, \\ ln^{1+\gamma}(1+t), & \text{if } \alpha = -1 \text{ and } \gamma > -1, \\ ln(ln(1+t)), & \text{if } \alpha = -1 \text{ and } \gamma = -1. \end{cases} \tag{1.27}$$

Then there exists $t_0^*(\theta, \sigma, \alpha, \gamma, \beta, \mu, t_0) \geq t_0$, such that the solution $u(t,x)$ of (1)-(2) satisfies, for all $t \geq t_0^*$:

(i) *If $0 < \theta < \sigma$ and $u_0, u_1 \in L^p(\mathbb{R}^n)$, then*

$$\begin{aligned} ||D^{\beta} u(t,\cdot)||_{L^q} &\lesssim \varphi(t)^{-\left(\frac{|\beta|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_0||_{L^p} \\ &\quad + \varphi(t)^{-\left(\frac{|\beta|-\sigma+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_1||_{L^p}, \end{aligned}$$

$$\begin{aligned} ||D^{\mu} u_t(t,\cdot)||_{L^q} &\lesssim \varphi(t)^{-\left(\frac{|\mu|+\sigma+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_0||_{L^p} \\ &\quad + \varphi(t)^{-\left(\frac{|\mu|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_1||_{L^p}. \end{aligned}$$

(ii) *If $\theta = \sigma$, $\omega > n\left(\frac{1}{p} - \frac{1}{q}\right)$, $m := \max\{|\mu|, |\beta| - \sigma\} + \omega$ and $[u_0, u_1] \in W^{m+\sigma,p}(\mathbb{R}^n) \times W^{m,p}(\mathbb{R}^n)$, then*

$$\begin{aligned} ||D^{\beta} u(t,\cdot)||_{L^q} &\lesssim \varphi(t)^{-\left(\frac{|\beta|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_0||_{W^{|\beta|+\omega,p}} \\ &\quad + \varphi(t)^{-\left(\frac{|\beta|-\sigma+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_1||_{W^{|\beta|-\sigma+\omega,p}}, \end{aligned}$$

$$\begin{aligned} ||D^{\mu} u_t(t,\cdot)||_{L^q} &\lesssim \varphi(t)^{-\left(\frac{|\mu|+\sigma+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_0||_{W^{|\mu|+\sigma+\omega,p}} \\ &\quad + \varphi(t)^{-\left(\frac{|\mu|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)} ||u_1||_{W^{|\mu|+\omega,p}}. \end{aligned}$$

Proof of Theorem 1.3.1 : Let $\hat{q}$ conjugate of q, $v \in \{u, u_t\}$ and $\eta \in \{\beta, \mu\}$, by Hausdorff-Young inequality, we have:

$$||D^\eta v(t,\cdot)||_{L^q} \lesssim ||\,|\cdot|^\eta \hat{v}(t,\cdot)||_{L^{\hat{q}}} \lesssim \left(\int_{\mathbb{R}^n} |\xi|^{\hat{q}|\eta|} |\hat{v}(t,\xi)|^{\hat{q}} d\xi\right)^{\frac{1}{\hat{q}}}. \quad (1.28)$$

It should be noticed that if $\eta = \beta$ and $u_1 \neq 0$, we have the restriction $|\beta| - \sigma + n\left(\frac{1}{p} - \frac{1}{q}\right) = |\beta| - \sigma + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) > 0$. For each fixed t, we separate the integral in inequality (1.28) in four parts, that is, low zone, elliptic zone, hyperbolic zone and high zone. By applying Propositions 1.2.1, 1.2.2 and 1.2.3 the theorem follows. □

Chapter 2

σ - evolution models with low regular time-dependent structural damping II

2.1 Main Theorem and comparison with previous works

In this chapter we shall discuss the following theorem:

Theorem 2.1.1 *Let* $n \geq 1$ *and* θ, σ, α, γ *satisfying Hypothesis B. Let* $1 \leq p \leq 2 \leq q \leq \infty$, $\beta, \mu \in \mathbb{N}^n$. *Consider*

$$\varphi(t) := (1+t)^{1+\alpha} ln^{\gamma}(1+t) \tag{2.1}$$

and

$$\phi(t) := \begin{cases} (1+t)^{1-\alpha} ln^{-\gamma}(1+t), & if \ \ \alpha \in (-1,1), \\ ln^{1-\gamma}(1+t), & if \ \ \alpha = 1 \ \ and \ \ \gamma < 1, \\ ln(ln(1+t)), & if \ \ \alpha = 1 \ \ and \ \ \gamma = 1, \end{cases} \tag{2.2}$$

defined for $t \geq t_0$ and t_0 big enough. Then, there exist $M = M(a_1) > 0$ and $t_0^* > 0$ both depending on $\theta, \sigma, \alpha, \gamma, \beta, \mu$ and t_0, such that the solution $u(t,x)$ of (1)-(2) *satisfies, for all $t \geq t_0^*$:*

(i) *For $\theta \neq 0$, u_1 and u_0 in L^p:*

If $|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right) > \frac{2\theta}{1+\alpha}$ and $u_1 \neq 0$ or any $\beta \in \mathbb{N}^n$ if $u_1 = 0$,

$$\begin{aligned} ||D^\beta u(t,\cdot)||_{L^q} \ \ &\lesssim \ \ \phi(t)^{-\left(\frac{|\beta|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)}||u_0||_{L^p} \\ &+ \phi(t)^{-\left(\frac{|\beta|-\frac{2\theta}{1+\alpha}+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)} ln^{-\frac{\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)||u_1||_{L^p}. \end{aligned}$$

If $|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{2\theta}{1+\alpha}$ and $\gamma > (1+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right)$,

$$||D^\beta u(t,\cdot)||_{L^q} \lesssim \phi(t)^{-\frac{\theta}{(1+\alpha)(\sigma-\theta)}}||u_0||_{L^p} + ln^{-\frac{\gamma}{1+\alpha}+\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{\phi(t)}{M}\right)||u_1||_{L^p}.$$

If $|\mu| + n\left(\frac{1}{p} - \frac{1}{q}\right) \leq \frac{2\theta}{1+\alpha}$,

$$\begin{aligned} ||D^\mu u_t(t,\cdot)||_{L^q} \ \ &\lesssim \ \ \frac{1}{g(t)}\phi(t)^{-\left(\frac{|\mu|+2\sigma-2\theta+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)}||u_0||_{L^p} \\ &+ \varphi(t)^{-\left(\frac{|\mu|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)}||u_1||_{L^p}. \end{aligned}$$

If $|\mu| + n\left(\frac{1}{p} - \frac{1}{q}\right) > \frac{2\theta}{1+\alpha}$,

$$||D^{\mu}u_t(t,\cdot)||_{L^q} \lesssim \frac{1}{g(t)}\phi(t)^{-\left(\frac{|\mu|+2\sigma-2\theta+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)}||u_0||_{L^p}$$
$$+\frac{1}{g(t)}\phi(t)^{-\left(\frac{|\mu|+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)} ln^{-\frac{\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)||u_1||_{L^p}.$$

(ii) *For* $\theta = 0$, $\omega > n\left(\frac{1}{p} - \frac{1}{q}\right)$, u_0 *and* u_1 *in the required space:*

$$||D^{\beta}u(t,\cdot)||_{L^q} \lesssim \phi(t)^{-\left(\frac{|\beta|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma}\right)}\left(||u_0||_{W^{|\beta|+\omega,p}} + ||u_1||_{W^{|\beta|-\sigma+\omega,p}}\right).$$

If $\alpha \in (-1,1)$*:*

$$||D^{\mu}u_t(t,\cdot)||_{L^q}$$
$$\lesssim \frac{1}{g(t)}\phi(t)^{-\left(\frac{|\mu|+2\sigma+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma}\right)}\left(||u_0||_{W^{|\mu|+\sigma+\omega,p}} + ||u_1||_{W^{|\mu|+\omega,p}}\right).$$

If $\alpha = 1$*:*

$$||D^{\mu}u_t(t,\cdot)||_{L^q}$$
$$\lesssim \frac{1}{g(t)}\phi(t)^{-\left(\frac{|\mu|+2\sigma+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma}\right)}\left(||u_0||_{W^{|\mu|+2\sigma+\omega,p}} + ||u_1||_{W^{|\mu|+\sigma+\omega,p}}\right).$$

Comparison with previous works: The Theorem 2.1.1 generalizes and improves some previous results. In the case $\theta = 0$, we generalized the coefficient b for a more general class in the cases of Hypothesis B and extended the result for $\sigma > 0$, instead of the previously $\sigma = 1$, see Theorem 21 of [24]. Using Corollary A.1.1 we can see that the decay rates achieved in this chapter are the same of [24]. The only disavantage of our results is that for $\alpha = 1$ we required additional regularity in the initial data. The reason for assuming this technical hypothesis, it is because we avoid to separate high zone in elliptic and hyperbolic zones, as we have made in low frequency (see next section). Therefore, for the sake of brevity, we postpone this more complete approach for $\theta = 0$ for a forthcoming work, in which will include a treatment of a more general classes of functions b.

For $\gamma = 0$ and $\theta \in (0, \sigma)$, in the cases that $||D^{\beta}u(t,\cdot)||_{L^q} \to 0$, our decays rates for u match with the decay rates of [8]. In addition, in that paper b is asssumed to be well behaved, that is, $b(t) = \mu(1+t)^{\alpha}$, while in this work b is assumed to be asymptotically equivalent to $g(t) = (1+t)^{\alpha} ln^{\gamma}(1+t)$. In [8] the authors also proved the diffusion phenomenon, but did not discuss decay rates for u_t.

In [15] and [16] is discussed the case $\theta \in (0, \sigma)$ and also decay rates for u_t. Is in comparison with these works that lies the most interesting examples. For simplicity, we shall consider the case where g is increasing (in which corresponds to $b(t)$ increasing in [16]). The comparison for the case where g is decreasing is similar and corresponds to [15].

Hypothesis B corresponds to the effective damping case, which is treated in Theorem 2.1 and Theorem 2.3 of [16]. They considered monoticity and other hypothesis on function b. One of them is the following classification for b with $\eta \in (0, \frac{\sigma}{2\theta}]$:

$$(\mathbf{B6})\; S_{\eta} := \left\{ b(t) : \limsup_{t\to\infty} \frac{1+t}{\Lambda_1(t)^{\eta}} < \infty,\; \limsup_{t\to\infty} \frac{1+t}{\Lambda_1(t)^{\beta}} = \infty \;\; \forall\, \beta < \eta \right\}$$

in which $\Lambda_1(t) := 1 + \int_0^t b(s)ds$. In addition, they consider $\Lambda_2(t) := 1 + \int_0^t \frac{1}{b(s)}ds$. Since $b \sim g$, using Corollary A.1.1 is not difficult to see that $\Lambda_1 \sim \varphi$ and $\Lambda_2 \sim \phi$ for $t \geq t_0^*$ big enough, ϕ and φ given by Theorem 2.1.1. Therefore, we can rewrite condition (B6) replacing Λ_1 by φ and $lim\,sup$ by lim. In addition, we have:

$$\lim_{t\to\infty} \frac{1+t}{\varphi(t)^{\nu}} \begin{cases} < \infty \text{ if } \frac{1}{1+\alpha} < \nu \text{ or } \frac{1}{1+\alpha} = \nu \text{ and } \gamma \geq 0, \\ = \infty \text{ otherwise.} \end{cases}$$

Therefore, if $\gamma \geq 0$, we have $b \in S_{\frac{1}{1+\alpha}}$. However, if $\gamma < 0$ there is no η such that $b \in S_{\eta}$. In this case, the results in [16] does not work for $\gamma < 0$ and therefore our work gives on more contribution, allowing the case $\gamma < 0$.

Using that $\Lambda_1 \sim \varphi$ and $\Lambda_2 \sim \phi$ for $t \geq t_0^*$ big enough and the notation of Theorem 2.1.1, by Theorem 2.1 and Theorem 2.3 of [16], for $|\beta| \geq \max\{2\theta, \sigma\}$ and $|\mu| \geq 0$, it follows for $t \geq t_0^*$:

$$||D^\beta u(t,\cdot)||_{L^q} \lesssim \phi(t)^{-\left(\frac{|\beta|+\frac{n}{p}-\frac{n}{q}}{2\sigma-2\theta}\right)}||u_0||_{W^{m,p}} + \phi(t)^{-\left(\frac{|\beta|-\frac{2\theta}{1+\alpha}+\frac{n}{p}-\frac{n}{q}}{2\sigma-2\theta}\right)}||u_1||_{W^{m,p}} \tag{2.3}$$

and

$$\begin{aligned} ||D^\mu u_t(t,\cdot)||_{L^q} &\lesssim \phi(t)^{-\left(\frac{|\mu|+2\sigma-\frac{2\theta}{1+\alpha}+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)}||u_0||_{W^{m,p}} \\ &+ \max\left\{\varphi(t)^{-\left(\frac{|\mu|+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\theta}\right)}, \phi(t)^{-\left(\frac{|\mu|+2\sigma-\frac{4\theta}{1+\alpha}+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}\right)}\right\}||u_1||_{W^{m,p}}. \end{aligned} \tag{2.4}$$

Initially we emphasize that the rescrition $|\beta| \geq \max\{2\theta, \sigma\}$ is technical and we have assumed $|\beta| + n\left(\frac{1}{p} - \frac{1}{q}\right) > \frac{2\theta}{1+\alpha}$, which is less restrictive since due to Hypothesis B we have $\sigma \geq \frac{2\theta}{1+\alpha}$. It should be notice however, that in [8] they achieved the same resctrition to ensure $||D^\beta u(t,\cdot)||_{L^q} \to 0$, but they only considered the case $\gamma = 0$. In addition, the norms on the right side of inequalities (2.3) and (2.4) are in the Sobolev space $W^{m,p}$ with p beeing the same of Theorem 2.1.1 and suitable $m > 0$. In other words our result shows an interesting property of the equation for $\theta \in (0, \sigma)$: we proved a smooth effect, that is, a regularity gain of the solution for $t \geq t_0^*$. Again, in [8] they achieved the same property only for $\gamma = 0$. Finally, let us compare the decay rates in inequalities (2.3) and (2.4) with our results in Theorem 2.1.1. For simplicity we will consider only the decay rate associated to term u_1 in the right side of the inequalities.

For $\gamma > 0$ our decay rate for $||D^\beta u(t,\cdot)||_{L^q}$ has an improvement in the decay given by the extra term $ln^{-\frac{\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)$. We remember that the case $\gamma < 0$ is not considered in (2.3). To see the improvement in the decay rates for u_t we consider the case $|\mu| + n\left(\frac{1}{p} - \frac{1}{q}\right) > \frac{2\theta}{1+\alpha}$ and

$\alpha \in [0,1)$. In the notation of Theorem 2.1.1 we have:

$$\frac{1}{g(t)} = \phi(t)^{-\frac{1}{2\sigma-2\theta}\left(\frac{(2\sigma-2\theta)\alpha}{1-\alpha}\right)} ln^{-\frac{\gamma}{1-\alpha}}(1+t).$$

Since

$$\begin{aligned} &|\mu| + 2\sigma - 2\theta\left(\frac{2+\alpha}{1+\alpha}\right) + n\left(\frac{1}{p}-\frac{1}{q}\right) + (2\sigma-2\theta)\frac{\alpha}{1-\alpha} \\ &= \left[|\mu| + 2\sigma - \frac{4\theta}{1+\alpha} + n\left(\frac{1}{p}-\frac{1}{q}\right)\right] \\ &\quad + \left[2\sigma(1+\alpha) - 4\theta\right]\frac{\alpha}{(1+\alpha)(1-\alpha)}. \end{aligned}$$

Using the last two equalities, we have:

$$\begin{aligned} &\frac{1}{g(t)}\phi(t)^{-\frac{|\mu|+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}} ln^{-\frac{\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right) = \\ &\phi(t)^{-\frac{|\mu|+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)+(2\sigma-2\theta)\frac{\alpha}{1-\alpha}}{2\sigma-2\theta}} ln^{-\frac{\gamma}{1-\alpha}}(1+t) ln^{-\frac{\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right) = \\ &\phi(t)^{-\frac{|\mu|+2\sigma-\frac{4\theta}{1+\alpha}+n\left(\frac{1}{p}-\frac{1}{q}\right)}{2\sigma-2\theta}} \underbrace{\phi(t)^{-\frac{(\sigma(1+\alpha)-2\theta)\alpha}{(\sigma-\theta)(1+\alpha)(1-\alpha)}} ln^{-\frac{\gamma}{1-\alpha}}(1+t) ln^{-\frac{\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)}_{:= w(t)}. \end{aligned}$$

Since we are considering the case g increasing, that is, $\alpha > 0$ or $\alpha = 0$ with $\gamma > 0$ and we have assumed the Hypothesis B, the function $w(t) \to 0$ when $t \to \infty$. Therefore, for g increasing, $|\mu| + n\left(\frac{1}{p}-\frac{1}{q}\right) > \frac{2\theta}{1+\alpha}$, $\alpha < 1$ and $u_1 \neq 0$, Theorem 2.1.1 improves the inequality (2.4) provided by Theorems 2.1 and 2.3 of [16].

The rest of the chapter is devoted to the proof the Theorem 2.1.1.

2.2 Main Estimates in the Fourier Space

In this section, we can assume that the initial data are sufficiently smooth, say $(u_0, u_1) \in \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$ because of the density argument. Let $u = u(t,x)$ be the corresponding solution of (1)-(2).

We take the Fourier transform in the both sides of (1). Then in the

Fourier space one has the reduced equation:

$$\hat{u}_{tt}(t,\xi) + |\xi|^{2\sigma}\hat{u}(t,\xi) + b(t)|\xi|^{2\theta}\hat{u}_t(t,\xi) = 0, \qquad (t,\xi) \in (0,\infty)\times\mathbb{R}^n. \tag{2.5}$$

The corresponding initial data are given by

$$\hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi) \qquad \xi \in \mathbb{R}^n. \tag{2.6}$$

Throughout this section we shall omit the dependence in ξ inside the functions $\hat{u}(t) = \hat{u}(t,\xi)$, $\hat{u}_t(t) = \hat{u}_t(t,\xi)$ and the density energy $E(t) = E(t,\xi)$, which will be defined further ahead.

When we obtain important estimates in order to prove our results, we apply the multiplier method in Fourier space. Take $Z \subset [0,\infty)\times\mathbb{R}^n$, K and J defined in $Z \to [0,\infty)$. We multiply both sides of (2.5) by $J(t,\xi)\overline{\hat{u}}_t$ and further by $K(t,\xi)\overline{\hat{u}}$. Then, taking the real part of the resulting identities we have (formally):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{J(t,\xi)|\hat{u}_t(t)|^2 + J(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2\right\} + J(t,\xi)b(t)|\xi|^{2\theta}|\hat{u}_t(t)|^2 \\ = \frac{1}{2}\left\{\frac{d}{dt}J(t,\xi)\right\}\left(|\hat{u}_t(t)|^2 + |\xi|^{2\sigma}|\hat{u}(t)|^2\right)\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\frac{d}{dt}\left\{K(t,\xi)Re(\hat{u}_t(t)\,\overline{\hat{u}}(t))\right\} + b(t)K(t,\xi)|\xi|^{2\theta}Re(\hat{u}_t(t)\overline{\hat{u}}(t)) \\ + K(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2 \\ = K(t,\xi)|\hat{u}_t(t)|^2 + \left\{\frac{d}{dt}K(t,\xi)\right\}Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)),\end{aligned} \tag{2.8}$$

for each $(t,\xi) \in Z$ that it makes sense. We define the density energy as:

$$E(t) := \frac{1}{2}\left\{|\hat{u}_t(t)|^2 + |\xi|^{2\sigma}|\hat{u}(t)|^2\right\} \quad \forall\, t \geq 0.$$

By integration equation (2.7) in $[S,T]$, follows:

$$E(T) + \int_S^T b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds = E(S). \tag{2.9}$$

Let φ and ϕ as by Theorem 2.1.1. We fix $R > 0$ and $N > 0$ whose choice will be clear in the course of the section.

We shall deal with the problem using the following separation zones:

High Zone: $Z^{high} := \{(t,\xi) \in [t_0,\infty) \times \mathbb{R}^n : |\xi| \geq R\}$;

Hyperbolic Zone:

$$Z_{hyp} := \left\{(t,\xi) \in [t_0,\infty) \times B_R : |\xi|^{2\theta}\varphi(t) \geq N \text{ and } |\xi|^{\sigma-2\theta} \geq g(t)\right\};$$

Elliptic Zone:

$$Z_{ell} := \left\{(t,\xi) \in [t_0,\infty) \times B_R : |\xi|^{2\theta}\varphi(t) \geq N \text{ and } |\xi|^{\sigma-2\theta} \leq g(t)\right\};$$

Low Zone: $Z_{low} := \left\{(t,\xi) \in [t_0,\infty) \times B_R : |\xi|^{2\theta}\varphi(t) \leq N\right\}$.

Remark 2.2.1 *The number t_0 is chosen such that $\phi(t_0)$, $\varphi(t_0) > 1$, $g(t_0) \leq 1$ if g is non-increasing, $g(t_0) \geq 1$ if g is increasing, $b \sim g$ for $t \geq t_0$, $|g'(t)| \leq \left(\frac{|\alpha|+|\gamma|}{1+t}\right) g(t)$ for $t \geq t_0$ and such that ϕ, φ and g are monotone (without change of monotonicity) for $t \geq t_0$. Furthermore, throughout this work we will assume t_0 big enough to ensure the application of the results of the Appendix.*

In Chapter 1, we choose $N = 1$ in the definition of $Z_{ell}(N)$, $Z_{hyp}(N)$ and $Z_{low}(N)$. However, without loss of generality, following the same steps of Proposition 1.1.2, Proposition 1.1.3 and Lemma 1.1.1, we have the following uniform estimates in ξ:

Proposition 2.2.1 *For a fixed $\xi \in \mathbb{R}^n$ and given zone Z, we define $S_0(\xi) = \inf\{s \in [t_0,\infty) : (s,\xi) \in Z\}$, $T_0(\xi) = \sup\{s \in [t_0,\infty) : (s,\xi) \in Z\}$ and $\nu := \min\{2\sigma - 2\theta, 2\theta\}$. Then, there exists $C > 0$ independent of ξ such that:*

$$E(t) \lesssim e^{-\frac{1}{C}|\xi|^{\nu}\int_{t_0}^{t}\min\left\{\frac{1}{g(s)},g(s)\right\}ds}E(0) \text{ for all } (t,\xi) \in Z^{high}.$$

If Z_{hyp} has no zero measure, $E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\int_{S_0(\xi)}^{t}g(s)ds}E(S_0(\xi))$ for all $S_0(\xi) \leq t < T_0(\xi)$.

If Z_{ell} has no zero measure, $E(t) \lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{S_0(\xi)}^t \frac{1}{g(s)}ds}E(S_0(\xi))$ *for all* $S_0(\xi) \leq t < T_0(\xi)$.

When $T_0(\xi) < \infty$, *the last estimates also hold for* $t = T_0(\xi)$.

Even though in Chapter 1 we assumed Hypothesis A and now we are treating Hypothesis B, it is worth to mention that the results of Proposition 2.2.1 still hold for this case. That is, there is no restriction concerning Hypotheses A or B in this result.

The key point for the meticulous reader is the fact that the estimates for Z_{ell} of Proposition 2.2.1 are not sharp in the case of Hypothesis B. For a while, let us *suppose* that estimates of the Proposition 2.2.1 hold for t_0 instead of $S_0(\xi)$. Using Corollary A.1.1, we would have for $(t,\xi) \in Z_{ell}$:

$$
\begin{aligned}
|\xi|^{2\sigma}|\hat{u}(t)|^2 \lesssim E(t) &\lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_0}^t \frac{1}{g(s)}ds}E(0) \\
&\lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\phi(t)}\big(|\xi|^{2\sigma}|\hat{u}_0|^2 + |u_1|^2\big). \quad (2.10)
\end{aligned}
$$

For each $\xi \in B_R := \{\xi \in \mathbb{R}^n\backslash\{0\} : |\xi| \leq R\}$, we define $t_\xi := \varphi^{-1}(N|\xi|^{-2\theta})$. We introduce $\psi : B_R \to [0,\infty)$ defined by

$$
\psi(\xi) := g(t_\xi)|\xi|^{2\theta}. \quad (2.11)
$$

Taking in account the definition (2.11), in Proposition 2.2.5 we will prove, for $(t,\xi) \in Z_{ell}$:

$$
\psi(\xi)^2|\hat{u}(t)|^2 \lesssim e^{-\frac{2}{C}|\xi|^{2\sigma-2\theta}\phi(t)}\left(\psi(\xi)^2|\hat{u}_0|^2 + |u_1|^2\right). \quad (2.12)
$$

Since Hypothesis B implies $|\xi|^\sigma < \psi(\xi)$ (it will be proved in Proposition 2.2.2), inequality (2.12) show us that inequality (2.10) is not sharp. In fact, repeating the same steps of Chapter 1, our draft version of results was worse than others one already know, for example in [8], [14], [15] or [16]. This compelled us to improve the Energy method, using a very different and new technique.

The following proposition not only inform us that $\max\{|\xi|^\sigma, \psi(\xi)\} = \psi(\xi)$ for all $\xi \in B_R$, but also plays a fundamental role to obtain the desired estimates in Fourier space.

Proposition 2.2.2 *Assume that α, γ, θ and σ satisfy Hypothesis B. Thus, there exists small $R > 0$ such that $|\xi|^\sigma < \psi(\xi)$ for all ξ in B_R. Furthermore, the following estimates hold in B_R:*

$$\psi(\xi) \sim \begin{cases} |\xi|^{\frac{2\theta}{1+\alpha}} \left[ln\left(\frac{1}{|\xi|}\right)\right]^{\frac{\gamma}{1+\alpha}}, & \text{if } \theta \neq 0, \\ 1, & \text{if } \theta = 0. \end{cases}$$

Additionally, for $\alpha = 1$ (and $\theta < \sigma$), we have:

$$|\xi|^{2\sigma} ln(1 + t_\xi) \lesssim \psi(\xi)^2 \quad \text{if } \gamma < 1,$$

$$|\xi|^{2\sigma} ln(ln(1 + t_\xi)) ln(1 + t_\xi) \lesssim \psi(\xi)^2 \quad \text{if } \gamma = 1.$$

Proof. For $\xi \neq 0$, let t_ξ be the unique solution of the equation $(1 + t_\xi)^{1+\alpha} ln^\gamma(1 + t_\xi) = N|\xi|^{-2\theta}$. Applying Lemma C.1.4 with $\tau = 1 + t_\xi$, $\mu = 1 + \alpha$, $\beta = \gamma$ and $\lambda = N|\xi|^{-2\theta}$, we have:

$$(1 + t_\xi) = \begin{cases} \left(\frac{1+\alpha}{|\gamma|}\right)^{\frac{\gamma}{1+\alpha}} N^{\frac{1}{1+\alpha}} |\xi|^{-\frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|} N^{-\frac{1}{|\gamma|}} |\xi|^{\frac{2\theta}{|\gamma|}}\right)\right]^{-\frac{\gamma}{1+\alpha}} & \text{if } \gamma < 0, \\ N^{\frac{1}{1+\alpha}} |\xi|^{-\frac{2\theta}{1+\alpha}} \text{ if } \gamma = 0, \\ \left(\frac{1+\alpha}{\gamma}\right)^{\frac{\gamma}{1+\alpha}} N^{\frac{1}{1+\alpha}} |\xi|^{-\frac{2\theta}{1+\alpha}} \left[W_0\left(\frac{1+\alpha}{\gamma} N^{\frac{1}{\gamma}} |\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{-\frac{\gamma}{1+\alpha}} & \text{if } \gamma > 0, \end{cases}$$

where W_0, W_{-1} are the two real-valued branches of W-Lambert's function (see Appendix C for further details concerning this special function). To carefully apply Lemma C.1.4, we need to consider the following conditions on R and N:

If $\theta \neq 0$,

$$R < N^{\frac{1}{2\theta}} \left(\frac{(1+\alpha)e}{|\gamma|}\right)^{-\frac{|\gamma|}{2\theta}}, \quad \text{if } \gamma < 0.$$

If $\theta = 0$,

$$N > \left(\frac{(1+\alpha)e}{|\gamma|}\right)^{|\gamma|}, \quad \text{if } \gamma < 0.$$

Furthermore, for $\theta = 0$, we have $1 + t_\xi = C(\gamma, \alpha, N)$, and assuming

$$R < \left(\frac{N}{C}\right)^{\frac{1}{\sigma}},$$

in which imply $|\xi|^\sigma \leq R^\sigma < \frac{N}{C} = \frac{N}{1+t_\xi} = \psi(\xi)$, for all $\xi \in B_R$ and trivially $\psi \sim 1$.

In the rest of the proof we consider the case $\theta \neq 0$.

Case $\boldsymbol{\gamma < 0}$: Since $\psi(\xi) = \frac{N}{1+t_\xi}$, we have for $\xi \in B_R$:

$$\psi(\xi) = \left(\frac{1+\alpha}{|\gamma|}\right)^{\frac{|\gamma|}{1+\alpha}} N^{\frac{\alpha}{1+\alpha}} |\xi|^{\frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|} N^{-\frac{1}{|\gamma|}} |\xi|^{\frac{2\theta}{|\gamma|}}\right)\right]^{-\frac{|\gamma|}{1+\alpha}}.$$

By Hypothesis B, it follows that $\sigma > \frac{2\theta}{1+\alpha}$. By applying Lemma C.1.1, we conclude:

$$\lim_{r \to 0^+} r^{\sigma - \frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|} N^{-\frac{1}{|\gamma|}} r^{\frac{2\theta}{|\gamma|}}\right)\right]^{\frac{|\gamma|}{1+\alpha}} = 0.$$

Therefore, there exists $R(\gamma, \alpha, N, \sigma, \theta) > 0$ such that:

$$|\xi|^{\sigma - \frac{2\theta}{1+\alpha}} \left[-W_{-1}\left(-\frac{1+\alpha}{|\gamma|} N^{-\frac{1}{|\gamma|}} |\xi|^{\frac{2\theta}{|\gamma|}}\right)\right]^{\frac{|\gamma|}{1+\alpha}} < \left(\frac{1+\alpha}{|\gamma|}\right)^{\frac{|\gamma|}{1+\alpha}} N^{\frac{\alpha}{1+\alpha}},$$

for all $0 < |\xi| \leq R$. That is, $|\xi|^\sigma < \psi(\xi)$ for all $\xi \in B_R$. By using Lemma C.1.1, it follows that $\psi(\xi) \sim |\xi|^{\frac{2\theta}{1+\alpha}} \left[ln\left(|\xi|^{-\frac{2\theta}{|\gamma|}}\right)\right]^{-\frac{|\gamma|}{1+\alpha}} \sim |\xi|^{\frac{2\theta}{1+\alpha}} \left[ln\left(\frac{1}{|\xi|}\right)\right]^{\frac{\gamma}{1+\alpha}}$, for all $\xi \in B_R$.

Consider $\alpha = 1$. Since $\theta < \sigma$ we have

$$\lim_{r \to 0^+} r^{2\sigma - 2\theta} \left[-W_{-1}\left(\frac{2}{\gamma} N^{\frac{1}{\gamma}} r^{-\frac{2\theta}{\gamma}}\right)\right]^{1-\gamma} = 0. \tag{2.13}$$

Since $\psi(\xi) = \frac{N}{1+t_\xi}$, we have $(1+t_\xi)^{-\frac{2}{\gamma}} = N^{-\frac{2}{\gamma}} \psi(\xi)^{\frac{2}{\gamma}}$. By definition of t_ξ, we have $(1+t_\xi)^2 ln^\gamma(1+t_\xi) = N|\xi|^{-2\theta}$ and therefore $ln(1+t_\xi) = N^{-\frac{1}{\gamma}} |\xi|^{-\frac{2\theta}{\gamma}} \psi(\xi)^{\frac{2}{\gamma}}$. In addition, using limit (2.13), the following estimate holds for all $0 < |\xi| \leq R$:

$$\frac{|\xi|^{2\sigma} ln(1+t_\xi)}{\psi(\xi)^2} = \frac{|\gamma|^{1-\gamma}}{2^{1-\gamma}N}|\xi|^{2\sigma-2\theta}\left[-W_{-1}\left(\frac{2}{\gamma}N^{\frac{1}{\gamma}}|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{1-\gamma} \lesssim 1. \tag{2.14}$$

Case $\gamma = 0$: $\psi(\xi) = \frac{N}{(1+t_\xi)} = N^{\frac{\alpha}{1+\alpha}}|\xi|^{\frac{2\theta}{1+\alpha}}$.

In this case, by Hypothesis B follows $\sigma > \frac{2\theta}{1+\alpha}$. Assuming the following inequality $R < N^{\frac{\alpha}{\sigma(1+\alpha)-2\theta}}$, follows that $|\xi|^\sigma < \psi(\xi)$ for all ξ in B_R. Furthermore, since $\gamma = 0$, we have $\psi(\xi) = N^{\frac{\alpha}{1+\alpha}}|\xi|^{\frac{2\theta}{1+\alpha}} \sim |\xi|^{\frac{2\theta}{1+\alpha}}\left[ln\left(\frac{1}{|\xi|}\right)\right]^{\frac{\gamma}{1+\alpha}}$ for $\xi \in B_R$.

Consider now $\alpha = 1$ ($\sigma > \theta$). Taking in account the limit:

$$\lim_{r\to 0^+} r^{2\sigma-2\theta} ln\left(N^{\frac{1}{2}}r^{-\theta}\right) = 0,$$

the following estimate holds for all $0 < |\xi| \leq R$:

$$|\xi|^{2\sigma} ln(1+t_\xi) = \frac{1}{N}|\xi|^{2\sigma-2\theta} ln\left(N^{\frac{1}{2}}|\xi|^{-\theta}\right)\psi(\xi)^2 \lesssim \psi(\xi)^2.$$

Case $\gamma > 0$: Since $\psi(\xi) = \frac{N}{1+t_\xi}$, we have:

$$\psi(\xi) = \left(\frac{\gamma}{1+\alpha}\right)^{\frac{\gamma}{1+\alpha}} N^{\frac{\alpha}{1+\alpha}}|\xi|^{\frac{2\theta}{1+\alpha}}\left[W_0\left(\frac{1+\alpha}{\gamma}N^{\frac{1}{\gamma}}|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}}.$$

By Hypothesis B, we know that $\sigma \geq \frac{2\theta}{1+\alpha}$. By applying Lemma C.1.2, holds:

$$\lim_{r\to 0^+}\frac{r^{\sigma-\frac{2\theta}{1+\alpha}}}{\left[W_0\left(\frac{1+\alpha}{\gamma}N^{\frac{1}{\gamma}}r^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}}} = 0.$$

Therefore, there exists $R(\gamma,\alpha,N,\sigma,\theta) > 0$ such that:

$$\frac{|\xi|^{\sigma-\frac{2\theta}{1+\alpha}}}{\left[W_0\left(\frac{1+\alpha}{\gamma}N^{\frac{1}{\gamma}}|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}}} < \left(\frac{\gamma}{1+\alpha}\right)^{\frac{\gamma}{1+\alpha}} N^{\frac{\alpha}{1+\alpha}},$$

for all $0 < |\xi| \leq R$. That is, $|\xi|^\sigma < \psi(\xi)$ for all $\xi \in B_R$.

By Lemma C.1.2, W_0 behaves asymptotically like the function ln. Therefore, it holds uniformly in ξ:

$$\psi(\xi) \sim |\xi|^{\frac{2\theta}{1+\alpha}} \left[ln\left(|\xi|^{-\frac{2\theta}{\gamma}}\right)\right]^{\frac{\gamma}{1+\alpha}} \sim |\xi|^{\frac{2\theta}{1+\alpha}} \left[ln\left(\frac{1}{|\xi|}\right)\right]^{\frac{\gamma}{1+\alpha}}$$

for ξ in B_R. Consider $\alpha = 1$, $\sigma > \theta$ and $0 < \gamma < 1$. Taking in account the limit

$$\lim_{r\to 0^+} r^{2\sigma-2\theta} W_0\left(\frac{2}{\gamma} N^{\frac{1}{\gamma}} r^{-\frac{2\theta}{\gamma}}\right)^{1-\gamma} = 0,$$

and similarly to the inequality (2.14), the following estimate holds for all $0 < |\xi| \le R$:

$$\frac{|\xi|^{2\sigma} ln(1+t_\xi)}{\psi(\xi)^2} = \left(\frac{\gamma}{2}\right)^{1-\gamma} \frac{1}{N} |\xi|^{2\sigma-2\theta} W_0\left(\frac{2}{\gamma} N^{\frac{1}{\gamma}} |\xi|^{-\frac{2\theta}{\gamma}}\right)^{1-\gamma} \lesssim 1. \tag{2.15}$$

Consider now $\gamma = 1$ and $\alpha = 1$. In this case, we have:

$$\lim_{r\to 0^+} r^{2\sigma-2\theta} ln(2Nr^{-2\theta}) = 0. \tag{2.16}$$

Using the definition of t_ξ and ψ, we have

$$ln(1+t_\xi) = \frac{\psi(\xi)^2}{N|\xi|^{2\theta}} = \frac{1}{2} W_0(2N|\xi|^{-2\theta}).$$

Applying the property $ln(W_0(x)) = ln(x) - W_0(x)$ for $x > 0$ (see Appendix C), and due the fact that $W_0(x) > 0$ for $x > 0$, we have:

$$\begin{aligned} ln(ln(1+t_\xi)) &= ln(W_0(2N|\xi|^{-2\theta})) - ln(2) \\ &\le ln(2N|\xi|^{-2\theta}) - W_0(2N|\xi|^{-2\theta}) \le ln(2N|\xi|^{-2\theta}). \end{aligned}$$

By the last calculations, by inequality (2.15), and by limit (2.16), we have for $0 < |\xi| \le R$:

$$\frac{|\xi|^{2\sigma} ln(ln(1+t_\xi)) ln(1+t_\xi)}{\psi(\xi)^2} \le \frac{1}{N} |\xi|^{2\sigma-2\theta} ln(2N|\xi|^{-2\theta}) \lesssim 1.$$

□

The following lemma will be fundamental in order to obtain sharp estimates in the elliptic zone and in the high zone. A careful reader will perceive that up to now we have used similar strategies of Chapter 1. It should noticed, however, that from now on all the techniques are completely different and start to be far of the standard energy method.

Lemma 2.2.1 *Let $a_1 > 0$ such that $a_1 g(t) \leq b(t)$ and c_0 such that $|g'(t)| \leq \frac{c_0}{(1+t)} g(t)$ for all $t \in [t_0, \infty)$. Furthermore, consider $M > 0$, $\lambda \geq 0$ and $f : [t_0, \infty) \to [0, \infty)$ differentiable, such that, given $\xi \in \mathbb{R}^n$, satisfy:*

$$\lambda + \frac{c_0}{M} < a_1$$

$$|f'(t)| \leq \lambda |\xi|^{2\theta} g(t) f(t), \quad \forall\, t \in Q(\xi), \tag{2.17}$$

in which $Q(\xi) \subset D(\xi) := \{s \in [t_0, \infty) : |\xi|^{2\theta}(1+s)g(s) \geq M\}$ and satisfy the property: if (s_1, ξ) and $(s_2, \xi) \in Q(\xi)$, then $(s, \xi) \in Q(\xi)$ for all $s \in [s_1, s_2]$. Then, for I_f and J_f defined on $Q(\xi)^3$, hold:

$$\begin{aligned} I_f(s_1, s_2, t) &:= \int_{s_1}^{t} e^{|\xi|^{2\theta} \int_{s_2}^{\eta} b(\tau) d\tau} f(\eta) d\eta \\ &\leq \frac{c_2}{1 - c_2\lambda} \frac{f(t)}{|\xi|^{2\theta} g(t)} e^{|\xi|^{2\theta} \int_{s_2}^{t} b(\tau) d\tau}, \\ J_f(s_1, s_2, t) &:= \int_{s_1}^{t} e^{-|\xi|^{2\theta} \int_{s_2}^{\eta} b(\tau) d\tau} f(\eta) d\eta \\ &\leq \frac{c_2}{1 - c_2\lambda} \frac{f(s_1)}{|\xi|^{2\theta} g(s_1)} e^{-|\xi|^{2\theta} \int_{s_2}^{s_1} b(\tau) d\tau}, \end{aligned}$$

for all s_1, s_2 and $t \in Q(\xi)$, where $c_2 = \frac{1}{a_1 - \frac{c_0}{M}}$.

Proof. Throughout this proof, we consider s_1, s_2 and t in $Q(\xi)$. By hypothesis, all the integrals are well defined. Let $N_0 := \frac{c_0}{M} < a_1$. Therefore, we have for $t \in Q(\xi)$:

$$|g'(t)| \leq \frac{c_0}{(1+t)} g(t) \leq N_0 |\xi|^{2\theta} g(t)^2.$$

Let $f=1$. Thus,

$$\begin{aligned} I_1(s_1,s_2,t) &= \int_{s_1}^{t} e^{|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau} d\eta \\ &= \int_{s_1}^{t} \frac{d}{d\eta}\left(e^{|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau}\right)\frac{1}{|\xi|^{2\theta}b(\eta)}d\eta, \end{aligned}$$

and

$$\begin{aligned} I_1 &\le \int_{s_1}^{t} \frac{d}{d\eta}\left(e^{|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau}\right)\frac{1}{a_1\,|\xi|^{2\theta}g(\eta)}d\eta \\ &\le \frac{1}{a_1|\xi|^{2\theta}g(t)}e^{|\xi|^{2\theta}\int_{s_2}^{t} b(\tau)d\tau} + \frac{1}{a_1}\int_{s_1}^{t} e^{|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau}\frac{|g'(\eta)|}{|\xi|^{2\theta}g(\eta)^2}d\eta \\ &\le \frac{1}{a_1|\xi|^{2\theta}g(t)}e^{|\xi|^{2\theta}\int_{s_2}^{t} b(\tau)d\tau} + \frac{N_0}{a_1}I_1. \end{aligned}$$

Solving for I_1 we conclude that:

$$I_1(s_1,s_2,t) \le \frac{c_2}{|\xi|^{2\theta}g(t)}e^{|\xi|^{2\theta}\int_{s_2}^{t} b(\tau)d\tau}, \tag{2.18}$$

for all s_1, s_2 and t in $Q(\xi)$, with $c_2 := \frac{1}{a_1-N_0}$ is independent of ξ. For the general case, using inequality (2.18), (2.17) and integration by parts:

$$\begin{aligned} I_f &= I_1(s_1,s_2,\eta)f(\eta)\Big|_{\eta=s_1}^{\eta=t} - \int_{s_1}^{t} I_1(s_1,s_2,\eta)f'(\eta)d\eta \\ &\le c_2\frac{f(t)}{|\xi|^{2\theta}g(t)}e^{|\xi|^{2\theta}\int_{s_2}^{t} b(\tau)d\tau} + c_2\lambda I_f. \end{aligned}$$

Solving for I_f we conclude that:

$$I_f(s_1,s_2,t) \le \frac{c_2}{1-c_2\lambda}\,\frac{f(t)}{|\xi|^{2\theta}g(t)}e^{|\xi|^{2\theta}\int_{s_2}^{t} b(\tau)d\tau},$$

for all s_1, s_2 and t in $Q(\xi)$.

On the other hand,

$$\begin{aligned} J_1(s_1,s_2,t) &= \int_{s_1}^{t} e^{-|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau} d\eta \\ &= \int_{s_1}^{t} -\frac{d}{d\eta}\left(e^{-|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau}\right)\frac{1}{|\xi|^{2\theta}b(\eta)}d\eta, \end{aligned}$$

we have:

$$\begin{aligned} J_1 &\le \int_{s_1}^{t} -\frac{d}{d\eta}\left(e^{-|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau}\right)\frac{1}{a_1\,|\xi|^{2\theta}g(\eta)}d\eta \\ &\le \frac{1}{a_1|\xi|^{2\theta}g(s_1)}e^{-|\xi|^{2\theta}\int_{s_2}^{s_1} b(\tau)d\tau} + \frac{1}{a_1}\int_{s_1}^{t} e^{-|\xi|^{2\theta}\int_{s_2}^{\eta} b(\tau)d\tau}\frac{|g'(\eta)|}{|\xi|^{2\theta}g(\eta)^2}d\eta \\ &\le \frac{1}{a_1|\xi|^{2\theta}g(s_1)}e^{-|\xi|^{2\theta}\int_{s_2}^{s_1} b(\tau)d\tau} + \frac{N_0}{a_1}J_1. \end{aligned}$$

Solving for J_1 we conclude that:

$$J_1(s_1,s_2,t) \le \frac{c_2}{|\xi|^{2\theta}g(s_1)}e^{-|\xi|^{2\theta}\int_{s_2}^{s_1} b(\tau)d\tau}, \tag{2.19}$$

for all s_1, s_2 and t in $Q(\xi)$. From now on, we consider the general case for $J_f(s_1,s_2,t) = \int_{s_1}^{t} -f(\eta)\frac{d}{d\eta}J_1(\eta,s_2,t)d\eta$. Using inequality (2.19), (2.17) and integration by parts, we have:

$$\begin{aligned} J_f &= -J_1(\eta,s_2,t)f(\eta)\Big|_{\eta=s_1}^{\eta=t} + \int_{s_1}^{t} J_1(\eta,s_2,t)f'(\eta)d\eta \\ &\le c_2\frac{f(s_1)}{|\xi|^{2\theta}g(s_1)}e^{-|\xi|^{2\theta}\int_{s_2}^{s_1} b(\tau)d\tau} + c_2\lambda J_f. \end{aligned}$$

Solving for J_f the result follows.

□

Before Proposition 1.1.2 we mentioned that in Z_{low} it is not necessary a multiplier, only a boundedness for $|\hat{u}|$ and $|\hat{u}_t|$ given by energy equation (2.9). In the present work however, by assuming Hypothesis B, Proposition 2.2.2 ensures $|\xi|^\sigma < \psi(\xi)$ for all $\xi \in B_R$, with ψ given by (2.11). Therefore, one may wonder if $|\xi|^\sigma|\hat{u}(t)| \lesssim |\xi|^\sigma|\hat{u}_0| + |\hat{u}_1|$ is a sharp estimate. Actually, the next proposition show us that we need to replace $|\xi|^\sigma$ by $\psi(\xi)$ in order to achieve an improved estimate.

Proposition 2.2.3 *Assuming Hypothesis B, the following estimate holds for* $(t,\xi) \in Z_{low}$*:*

$$\psi(\xi)|\hat{u}(t)| \lesssim \psi(\xi)|\hat{u}_0| + |\hat{u}_1|.$$

Proof. Fix $\xi \in B_R$ and let $t_\xi > 0$ as defined before. By integrating equation (2.8) in $[0,t]$, for $t \leq t_\xi$ and $K(t,\xi) := (1+t)\psi(\xi)^2$, we have:

$$\begin{aligned}
&\frac{1}{2}\psi(\xi)^2|\hat{u}(t)|^2 + \int_0^t (1+s)\psi(\xi)^2|\hat{u}_t(s)|^2 ds \\
&\quad = \frac{1}{2}\psi(\xi)^2|\hat{u}_0|^2 - \psi(\xi)^2 Re(\hat{u}_1\,\overline{\hat{u}}_0) + \psi(\xi)^2 Re((1+t)\hat{u}_t(t)\,\overline{\hat{u}}(t)) \\
&\quad + \int_0^t b(s)|\xi|^{2\theta} Re(\hat{u}_t(s)\,(1+s)\psi(\xi)^2\overline{\hat{u}}(s))ds \\
&\quad + \int_0^t (1+s)|\xi|^{2\sigma}\psi(\xi)^2|\hat{u}(s)|^2 ds.
\end{aligned}$$

By Young inequality and by Proposition 2.2.2, we have:

$$\begin{aligned}
\frac{1}{2}\psi(\xi)^2|\hat{u}(t)|^2 \quad \leq \quad & \psi(\xi)^2|\hat{u}_0|^2 + \frac{1}{2}\psi(\xi)^2|\hat{u}_1|^2 + N^2|\hat{u}_t(t)|^2 \\
& + \frac{1}{4}\psi(\xi)^2|\hat{u}(t)|^2 + 2N^2\int_0^t b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds \\
& + \frac{1}{8}\int_0^t b(s)|\xi|^{2\theta}\psi(\xi)^2|\hat{u}(s)|^2 ds \\
& + \int_0^t (1+s)\psi(\xi)^2\psi(\xi)^2|\hat{u}(s)|^2 ds,
\end{aligned}$$

because $[(1+t)\psi(\xi)]^2 \leq [(1+t_\xi)\psi(\xi)]^2 = N^2$. Due the fact that $\frac{1}{2}|\hat{u}_t(t)|^2 + \int_0^t b(s)|\xi|^{2\theta}|\hat{u}_t(s)|^2 ds \leq E(0)$, applying Gronwall Lemma, it follows:

$$\psi(\xi)^2|\hat{u}(t)|^2 \lesssim e^{\frac{1}{2}|\xi|^{2\theta}\int_0^t b(s)ds + 2(1+t)^2\psi(\xi)^2}\left[\psi(\xi)^2\left(|\hat{u}_0|^2 + |\hat{u}_1|^2\right) + E(0)\right] \tag{2.20}$$

for all $(t,\xi) \in Z_{low}$. By using Lemma A.1.2:

$$|\xi|^{2\theta}\int_0^t b(s)ds = |\xi|^{2\theta}\int_0^{t_0} b(s)ds + |\xi|^{2\theta}\int_{t_0}^t b(s)ds \lesssim 1+|\xi|^{2\theta}(1+t)g(t) \lesssim 1 \tag{2.21}$$

for all $(t,\xi) \in Z_{low}$. Since $(1+t)\psi(\xi) \lesssim 1$ in Z_{low}, $|\xi|^{2\sigma}|\hat{u}_0|^2 \le \psi(\xi)^2|\hat{u}_0|^2$ (Proposition 2.2.2) and applying inequality (2.21) in inequality (2.20), the lemma is proved.

□

In the following stages of this work, we shall improve the estimates of energy method in elliptic zone. We start by improving the estimates for $|\xi|^{2\sigma}|\hat{u}(t)|^2$ and then using it to improve the estimates for $|\hat{u}_t(t)|^2$. The idea is to define a suitable $F(t) = F(t,\xi) := c|\xi|^{2\sigma}|\hat{u}(t)| + R_0(t,\xi)$ such that we have some control on $\frac{d}{dt}F(t)$ and satisfying for a suitable K:

$$\int_S^T K(s,\xi)F(s)ds \le CF(S),$$

for (S,ξ) and (T,ξ) in Z_{ell}. One may wonder why we want to find again a multiplier for $|\xi|^{2\sigma}|\hat{u}(t)|^2$. Let us recall the previous estimate, given by standard energy method. Using Proposition 2.2.1 for (t,ξ) in Z_{ell} (in this case $S_0(\xi) = t_\xi$):

$$\begin{aligned} |\xi|^{2\sigma}|\hat{u}(t)|^2 \lesssim E(t) &\lesssim E(t_\xi)e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_\xi}^t \frac{1}{g(s)}ds} \\ &\lesssim (|\xi|^{2\sigma}|\hat{u}(t_\xi)|^2 + |\hat{u}_t(t_\xi)|^2)e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_\xi}^t \frac{1}{g(s)}ds}. \end{aligned}$$

Taking in account the improvement provided by Proposition 2.2.3, since we assumed Hypothesis B, one may wonder if is not possible to improve also the inequality above, replacing $|\xi|^{2\sigma}$ by $\psi(\xi)^2$. Therefore, instead of the standard energy method of Chapter 1 that we simply used $c = \frac{1}{2}$ and $R_0(t,\xi) = \frac{1}{2}|\hat{u}_t(t)|^2$, we have now to find $R_0(t,\xi)$ such that $R_0(t_\xi,\xi) \lesssim |\xi|^{2\sigma}|\hat{u}(t_\xi)|^2 + \frac{|\xi|^{2\sigma}}{\psi(\xi)^2}|\hat{u}_t(t_\xi)|^2$. Assuming such existence, we expect the following:

$$\begin{aligned}\psi(\xi)^2|\hat{u}(t)|^2 &\lesssim \frac{\psi(\xi)^2}{|\xi|^{2\sigma}}F(t)\\ &\lesssim \frac{\psi(\xi)^2}{|\xi|^{2\sigma}}F(t_\xi)e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_\xi}^t \frac{1}{g(s)}ds}\\ &\lesssim (\psi(\xi)^2|\hat{u}(t_\xi)|^2+|\hat{u}_t(t_\xi)|^2)e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\int_{t_\xi}^t \frac{1}{g(s)}ds}. \quad (2.22)\end{aligned}$$

This construction, however, is far from easy and there are several troubles to overcome. In the next two propositions, we provide the desired proof.

In the lemma above, $0 < a_1 \leq a_2$ are such that $a_1 g(t) \leq b(t) \leq a_2 g(t)$ for all $t \geq t_0$. Furthermore, c_0 is such that $|g'(t)| \leq \frac{c_0}{(1+t)}g(t)$ for all $t \geq t_0$.

Proposition 2.2.4 *Let $\hat{u}(t,\xi)$ the solution of (2.5)-(2.6), $N > \max\left\{\frac{4c_0}{a_2}, \frac{3c_0}{a_1}, \frac{4a_2}{a_1^2}\right\}$. Consider $F(t) = F(t,\xi) = \frac{a_4}{2}|\xi|^{2\sigma}|\hat{u}(t)|^2 + K(t,\xi)Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)) + N_1\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}E(t,\xi)$, where $K(t,\xi) := \frac{|\xi|^{2\sigma-2\theta}}{g(t)}$, $N_1 := \frac{N}{2c_0}$ and $a_4 := a_2 + \frac{a_2 c_0}{a_1 N}$. Assuming Hypothesis B, for $a_3 := a_1 - \frac{c_0}{N}$, $a_5 := 1 - \frac{N_1 c_0}{N}$ and $a_8 := N_1 a_1 - \frac{N_1 c_0}{N} - 1$, we have:*

$$\begin{aligned}\frac{d}{dt}F(t) &+ a_5 K(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2 + a_8 K(t,\xi)|\hat{u}_t(t)|^2\\ &\leq a_4|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(\tau)d\tau}\left|Re(\hat{u}_t(t_\xi)\,\overline{\hat{u}}(t_\xi))\right|\\ &\quad +(a_4-a_3)|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(\tau)d\tau}\int_{t_\xi}^t e^{|\xi|^{2\theta}\int_{t_\xi}^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta,\end{aligned}$$

for all $(t,\xi) \in Z_{ell}$, with a_3, a_4, a_5, a_8 positive constants. In addition, there exists $C \geq \frac{1}{a_1} > 0$ such that:

$$\int_S^T K(s,\xi)F(s)ds \leq CF(S),$$

for all $(T,\xi), (S,\xi) \in Z_{ell}$ such that $T > S$.

Proof. Let $(t,\xi), (S,\xi) \in Z_{ell}$ with $t > S$. Using equation (2.8) with

$K_0(t,\xi) := e^{|\xi|^{2\theta}\int_S^t b(\tau)d\tau}$, we have:

$$\frac{d}{dt}\left\{e^{|\xi|^{2\theta}\int_S^t b(\tau)d\tau} Re(\hat{u}_t(t)\,\overline{\hat{u}}(t))\right\} + e^{|\xi|^{2\theta}\int_S^t b(\tau)d\tau}|\xi|^{2\sigma}|\hat{u}(t)|^2$$
$$= e^{|\xi|^{2\theta}\int_S^t b(\tau)d\tau}|\hat{u}_t(t)|^2.$$

By integrating in $[S,t]$, multiplying both sides by $e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}$ and rewriting the equation, it follows:

$$\begin{aligned}
&Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)) - e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau} Re(\hat{u}_t(S)\,\overline{\hat{u}}(S)) \\
&\qquad - e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta \\
&\quad = -e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\xi|^{2\sigma}|\hat{u}(\eta)|^2 d\eta \\
&\quad \leq 0.
\end{aligned} \tag{2.23}$$

For $K(t,\xi) := \frac{|\xi|^{2\sigma-2\theta}}{g(t)}$ and $(t,\xi) \in Z_{ell}$ we have

$$\left|\frac{d}{dt}K(t,\xi)\right| \leq \frac{c_0}{(1+t)}K(t,\xi) \leq \frac{c_0}{N}g(t)|\xi|^{2\theta}K(t,\xi) \leq \frac{c_0}{a_1 N}b(t)|\xi|^{2\theta}K(t,\xi).$$

Defining $Q(t,\xi) := b(t)|\xi|^{2\theta}K(t,\xi) - \frac{d}{dt}K(t,\xi)$, due the fact that $a_1|\xi|^{2\sigma} \leq b(t)|\xi|^{2\theta}K(t,\xi) \leq a_2|\xi|^{2\sigma}$, we have:

$$0 \leq a_3|\xi|^{2\sigma} \leq \left(1 - \frac{c_0}{a_1 N}\right) b(t)|\xi|^{2\theta}K(t,\xi) \leq Q(t,\xi) \leq a_4|\xi|^{2\sigma}. \tag{2.24}$$

Using inequalities (2.24) and (2.23) we have:

$$\begin{aligned}
&\left(b(t)|\xi|^{2\theta}K(t,\xi) - \frac{d}{dt}K(t,\xi)\right) Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)) \\
&\qquad - Q(t,\xi)\, e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau} Re(\hat{u}_t(S)\,\overline{\hat{u}}(S)) \\
&\qquad - Q(t,\xi)\, e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta \\
&\quad \geq a_4|\xi|^{2\sigma} Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)) - a_4|\xi|^{2\sigma} e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau} Re(\hat{u}_t(S)\,\overline{\hat{u}}(S)) \\
&\qquad - a_4|\xi|^{2\sigma} e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta.
\end{aligned}$$

Thus using (2.24) we have

$$
\begin{aligned}
&\left(b(t)|\xi|^{2\theta}K(t,\xi)-\frac{d}{dt}K(t,\xi)\right)Re(\hat{u}_t(t)\,\overline{\hat{u}}(t))\\
&\quad\geq \frac{a_4}{2}|\xi|^{2\sigma}\frac{d}{dt}|\hat{u}(t)|^2-a_4|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\left|Re(\hat{u}_t(S)\,\overline{\hat{u}}(S))\right|\\
&\qquad-(a_4-a_3)|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2d\eta. \qquad (2.25)
\end{aligned}
$$

Applying inequality (2.25) in equation (2.8) with $K(t,\xi):=\frac{1}{g(t)}|\xi|^{2\sigma-2\theta}$, we have:

$$
\begin{aligned}
&\frac{d}{dt}\left\{K(t,\xi)Re(\hat{u}_t(t)\,\overline{\hat{u}}(t))+\frac{a_4}{2}|\xi|^{2\sigma}|\hat{u}(t)|^2\right\}+K(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2\\
&\quad\leq K(t,\xi)|\hat{u}_t(t)|^2+a_4|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\left|Re(\hat{u}_t(S)\,\overline{\hat{u}}(S))\right|\\
&\qquad+(a_4-a_3)|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau}\int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2d\eta. \qquad (2.26)
\end{aligned}
$$

On the other hand, define $J(t,\xi):=N_1\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}=N_1\frac{|\xi|^{2\sigma-4\theta}}{g(t)^2}$. Therefore,

$$
\begin{aligned}
\frac{1}{2}\left|\frac{d}{dt}J(t,\xi)\right| &= N_1\left|\frac{d}{dt}K(t,\xi)\right|\frac{K(t,\xi)}{|\xi|^{2\sigma}}\leq\frac{N_1c_0}{(1+t)}\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}\\
&\leq \frac{N_1c_0}{N}\frac{g(t)|\xi|^{2\theta}}{|\xi|^{2\sigma}}K^2(t,\xi)=\frac{N_1c_0}{N}K(t,\xi) \qquad (2.27)
\end{aligned}
$$

and

$$
J(t,\xi)b(t)|\xi|^{2\theta}=N_1\frac{b(t)}{g(t)}\frac{|\xi|^{2\sigma-2\theta}}{g(t)}\geq N_1a_1K(t,\xi). \qquad (2.28)
$$

Applying inequalities (2.27) and (2.28) in equation (2.7), it follows:

$$
\begin{aligned}
&\frac{d}{dt}\left\{N_1\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}E(t,\xi)\right\}+\left(N_1a_1-\frac{N_1c_0}{N}\right)K(t,\xi)|\hat{u}_t(t)|^2\\
&\quad\leq\frac{N_1c_0}{N}K(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2. \qquad (2.29)
\end{aligned}
$$

By adding inequalities (2.26) and (2.29), we have:

$$\begin{aligned}
&\frac{d}{dt}F(t) + \left(1 - \frac{N_1 c_0}{N}\right) K(t,\xi)|\xi|^{2\sigma}|\hat{u}(t)|^2 \\
&\quad + \left(N_1 a_1 - \frac{N_1 c_0}{N} - 1\right) K(t,\xi)|\hat{u}_t(t)|^2 \\
&\le a_4|\xi|^{2\sigma} e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau} \left|Re(\hat{u}_t(S)\,\overline{\hat{u}}(S))\right| \\
&\quad + (a_4 - a_3)|\xi|^{2\sigma} e^{-|\xi|^{2\theta}\int_S^t b(\tau)d\tau} \int_S^t e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta, \quad (2.30)
\end{aligned}$$

for all $t > S$ such that $(t,\xi),(S,\xi) \in Z_{ell}$. Since $(1+t_\xi)g(t_\xi)|\xi|^{2\theta} = N$ and by Proposition 2.2.2 we have $|\xi|^\sigma < \psi(\xi) = g(t_\xi)|\xi|^{2\theta}$, therefore $(t_\xi,\xi) \in Z_{ell}$ for all $\xi \in B_R$. By setting $S = t_\xi$ in inequality (2.30) we prove the first part of the proposition.

Consider $(T,\xi),(S,\xi) \in Z_{ell}$. Using Lemma 2.2.1, we have:

$$\begin{aligned}
&|\xi|^{2\sigma}\int_S^T e^{-|\xi|^{2\theta}\int_S^s b(\tau)d\tau}\left|Re(\hat{u}_t(S)\,\overline{\hat{u}}(S))\right| ds \\
&\quad \le \frac{1}{a_1 - \frac{c_0}{N}} \frac{|\xi|^{2\sigma}}{|\xi|^{2\theta}g(S)} \left|Re(\hat{u}_t(S)\,\overline{\hat{u}}(S))\right| \\
&\quad \le \frac{1}{2(a_1 - \frac{c_0}{N})}\left(|\xi|^{2\sigma}|\hat{u}(S)|^2 + \frac{|\xi|^{2\sigma-4\theta}}{g(S)^2}|\hat{u}_t(S)|^2\right). \quad (2.31)
\end{aligned}$$

Using integration by parts and Lemma 2.2.1 (in the integral from S to s) again:

$$\begin{aligned}
&|\xi|^{2\sigma}\int_S^T e^{-|\xi|^{2\theta}\int_S^s b(\tau)d\tau}\left(\int_S^s e^{|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta\right) ds \\
&\quad = \int_S^T |\xi|^{2\sigma}\left(\int_s^T e^{-|\xi|^{2\theta}\int_S^\eta b(\tau)d\tau} d\eta\right) e^{|\xi|^{2\theta}\int_S^s b(\tau)d\tau}|\hat{u}_t(s)|^2 ds \\
&\quad \le \frac{1}{a_1 - \frac{c_0}{N}}\int_S^T K(s,\xi)|\hat{u}_t(s)|^2 ds. \quad (2.32)
\end{aligned}$$

Since $a_5 = 1 - \frac{N_1 c_0}{N}$ and $a_6 := N_1 a_1 - \frac{N_1 c_0}{N} - 1 - \frac{a_4 - a_3}{a_1 - \frac{c_0}{N}}$, by integrating inequality (2.30) in $[S,T]$ and applying inequalities (2.31) and (2.32), it follows:

$$F(T)\ + a_5\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2ds + a_6\int_S^T K(s,\xi)|\hat{u}_t(s)|^2ds$$
$$\leq F(S) + \frac{a_4}{2\left(a_1 - \frac{c_0}{N}\right)}\left(|\xi|^{2\sigma}|\hat{u}(S)|^2 + \frac{|\xi|^{2\sigma-4\theta}}{g(S)^2}|\hat{u}_t(S)|^2\right). \quad (2.33)$$

Note that by the conditions on N we obtain $Na_1^2+\frac{c_0^2}{N} > \frac{2}{3}Na_1^2+\frac{1}{3}Na_1^2 > 2a_1c_0 + \frac{4a_2}{3} > 2a_1c_0 + a_2 + \frac{a_2c_0}{a_1N}$, therefore $a_6 > 0$.

Since in the elliptic zone holds $K(t,\xi)^2 = \frac{|\xi|^{4\sigma-4\theta}}{g(t)^2} \leq |\xi|^{2\sigma}$ we have for $(t,\xi) \in Z_{ell}$:

$$F(t,\xi) = \frac{a_4}{2}|\xi|^{2\sigma}|\hat{u}(t)|^2 + K(t,\xi)Re(\hat{u}_t(t)\,\overline{\hat{u}}(t)) + N_1\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}E(t,\xi)$$
$$\leq \frac{1}{2}\left(a_4+1\right)|\xi|^{2\sigma}|\hat{u}(t)|^2 + \frac{N_1}{2}K^2(t,\xi)|\hat{u}(t)|^2 + \frac{N_1+1}{2}\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}|\hat{u}_t(t)|^2$$
$$\leq \frac{1}{2}\left(N_1+a_4+1\right)|\xi|^{2\sigma}|\hat{u}(t)|^2 + \frac{1}{2}\left(N_1+1\right)\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}|\hat{u}_t(t)|^2. \quad (2.34)$$

Since $a_7 := \min\left\{\frac{a_4}{2} - \frac{1}{N_1}; \frac{N_1}{4}\right\} > 0$ follows:

$$\begin{aligned} F(t,\xi) &\geq \left(\frac{a_4}{2} - \frac{1}{N_1}\right)|\xi|^{2\sigma}|\hat{u}(t)|^2 + \frac{N_1}{4}\frac{K^2(t,\xi)}{|\xi|^{2\sigma}}|\hat{u}_t(t)|^2 \\ &\geq a_7\left(|\xi|^{2\sigma}|\hat{u}(t)|^2 + \frac{|\xi|^{2\sigma-4\theta}}{g(t)^2}|\hat{u}_t(t)|^2\right), \end{aligned} \quad (2.35)$$

for all $(t,\xi) \in Z_{ell}$. On the other hand, using inequalities (2.33), (2.34) and (2.35), we have:

$$\begin{aligned}
&\int_S^T K(s,\xi)F(s)ds \\
&\le \frac{N_1+a_4+1}{2}\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds + \frac{N_1+1}{2}\int_S^T K(s,\xi)|\hat{u}_t(s)|^2 ds \\
&\le C_1\left(a_5\int_S^T K(s,\xi)|\xi|^{2\sigma}|\hat{u}(s)|^2 ds + a_6\int_S^T K(s,\xi)|\hat{u}_t(s)|^2 ds\right) \\
&\le C_1\left(F(S) + \frac{a_4}{2\left(a_1-\frac{c_0}{N}\right)}\left(|\xi|^{2\sigma}|\hat{u}(S)|^2 + \frac{|\xi|^{2\sigma-4\theta}}{g(S)^2}|\hat{u}_t(S)|^2\right)\right) \\
&\le C_1\left(1+\frac{a_4}{2a_7(a_1-\frac{c_0}{N})}\right)F(S),
\end{aligned}$$

for all $(T,\xi),(S,\xi) \in Z_{ell}$ such that $T > S$, where $C_1 := \max\left\{\frac{N_1+a_4+1}{2a_5}, \frac{N_1+1}{2a_6}\right\}$. Finally, for $C := \max\left\{C_1\left(1+\frac{a_4}{2a_7(a_1-\frac{c_0}{N})}\right), \frac{1}{a_1}\right\}$ and the proposition is proved.

□

Remark 2.2.2 *Since φ is increasing, φ^{-1} is also increasing. Thus, if $N > \varphi(t_0)$ and $R \le 1$, $t_\xi = \varphi^{-1}(N|\xi|^{-2\theta}) > \varphi^{-1}(\varphi(t_0)) = t_0$, for all $\xi \in B_R$.*

Proposition 2.2.5 *Let $\hat{u} = \hat{u}(t,\xi)$ the solution of (2.5)-(2.6) and assume Hypothesis B. Therefore, there exists $C > 0$ such that the following estimates hold:*

$$\begin{aligned}
\psi(\xi)|\hat{u}(t)| &\lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\phi(t)}\left(\psi(\xi)|\hat{u}_0| + |\hat{u}_1|\right), \\
\frac{|\xi|^{\sigma-2\theta}}{g(t)}|\hat{u}_t(t)| &\lesssim e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\phi(t)}\left(|\xi|^{\sigma}|\hat{u}_0| + \frac{|\xi|^{\sigma}}{\psi(\xi)}|\hat{u}_1|\right), \\
|\hat{u}_t(t)| &\lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}\left(|\xi|^{\sigma}|\hat{u}_0| + |\hat{u}_1|\right) \\
&\quad + \frac{|\xi|^{2\sigma-2\theta}}{\psi(\xi)g(t)}e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\phi(t)}\left(\psi(\xi)|\hat{u}_0| + |\hat{u}_1|\right),
\end{aligned}$$

for all $(t,\xi) \in Z_{ell}$.

Proof. Since Hypothesis B holds, by Proposition 2.2.2 we have $|\xi|^\sigma < \psi(\xi)$ for all $\xi \in B_R$. Therefore, fixing $\xi \in B_R$, $T_0(\xi) := \sup\{t \geq t_\xi : (t,\xi) \in Z_{ell}\}$ is such that $T_0(\xi) = \infty$ if g is increasing or $g = 1$ and $T_0(\xi) = t_1(\xi)$ if g is decreasing, in which $t_1(\xi)$ is such that $g(t_1(\xi))|\xi|^{2\theta} = |\xi|^\sigma$. Strictly speaking, the existence of $t_1(\xi)$ for g decreasing is proved in the following way: by Hypothesis B, we have $2\theta < \sigma(1+\alpha) \leq \sigma$, and therefore, assuming:

$$R < g(t_0)^{\frac{1}{\sigma-2\theta}},$$

follows that $t_1(\xi) := g^{-1}(|\xi|^{\sigma-2\theta})$ is well defined.

Define the following function for t such that $(t,\xi) \in Z_{ell}$:

$$\mathcal{L}(t) := \frac{1}{2\hat{C}}F(t) + \frac{1}{2\hat{C}^2}\int_t^{T_0(\xi)} K(s,\xi)F(s)ds,$$

with $K(t,\xi) = \frac{|\xi|^{2\sigma-2\theta}}{g(t)}$, $\hat{C}$ anf F given by Proposition 2.2.4 ($\hat{C} = C$) and the convergence of the improper integral (when $T_0(\xi) = \infty$) is ensured by Proposition 2.2.4. By Proposition 2.2.4, we have $\frac{1}{2\hat{C}}F(t) \leq \mathcal{L}(t) \leq \frac{1}{\hat{C}}F(t)$, for all $(t,\xi) \in Z_{ell}$. Using again Proposition 2.2.4, we have:

$$\frac{d}{dt}\mathcal{L}(t) = \frac{1}{2\hat{C}}\frac{d}{dt}F(t) - \frac{1}{2\hat{C}^2}K(t,\xi)F(t) \leq R(t,\xi) - \frac{1}{2\hat{C}}K(t,\xi)\mathcal{L}(t), \tag{2.36}$$

where

$$\begin{aligned} R(t,\xi) := & -\frac{a_8}{2\hat{C}}K(t,\xi)|\hat{u}_t(t)|^2 + \frac{a_4}{2\hat{C}}|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(\tau)d\tau}\left|Re(\hat{u}_t(t_\xi)\,\overline{\hat{u}}(t_\xi))\right| \\ & + \frac{(a_4-a_3)}{2\hat{C}}|\xi|^{2\sigma}e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(\tau)d\tau}\int_{t_\xi}^t e^{|\xi|^{2\theta}\int_{t_\xi}^\eta b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta. \end{aligned}$$

Rewriting inequality (2.36) and multiplying both sides by the term $e^{\frac{1}{2\hat{C}}\int_{t_\xi}^t K(s,\xi)ds}$, we have:

$$\frac{d}{dt}\left(\mathcal{L}(t)e^{\frac{1}{2\hat{C}}\int_{t_\xi}^t K(s,\xi)ds}\right) \leq e^{\frac{1}{2\hat{C}}\int_{t_\xi}^t K(s,\xi)ds}R(t,\xi),$$

and by integration in $[t_\xi, t]$ and multiplying both sides by $e^{-\frac{1}{2\hat{C}}\int_{t_\xi}^t K(s,\xi)ds}$ we have:

$$\begin{aligned}\mathcal{L}(t) \quad \leq \quad & \mathcal{L}(t_\xi)e^{-\frac{1}{2\hat{C}}\int_{t_\xi}^t K(s,\xi)ds} \\ & + e^{-\frac{1}{2\hat{C}}\int_{t_\xi}^t K(s,\xi)ds}\int_{t_\xi}^t e^{\frac{1}{2\hat{C}}\int_{t_\xi}^\eta K(s,\xi)ds}R(\eta,\xi)d\eta. \end{aligned} \tag{2.37}$$

Let $D(\xi) := \{s \in [t_0, \infty) : |\xi|^{2\theta}(1+s)g(s) \geq N\}$ and $Q(\xi) := \{s \in [t_0, \infty) : (s, \xi) \in Z_{ell}\} = [t_\xi, T_0(\xi))$, in which the last equality is possible by Remark 2.2.2 and the interval include $T_0(\xi)$ when $T_0(\xi) = t_1(\xi)$. In this case, $Q(\xi) \subset D(\xi)$ and $f(\eta) := e^{\frac{1}{2\hat{C}}\int_{t_\xi}^\eta K(s,\xi)ds}$ is well defined in $Q(\xi)$. By Proposition 2.2.4 holds $\hat{C} \geq \frac{1}{a_1}$, and by the definition of elliptic zone, we have $|f'(\eta)| \leq \lambda|\xi|^{2\theta}g(\eta)f(\eta)$, with $\lambda = \frac{a_1}{2}$, for all $\eta \in Q(\xi)$. Furthermore, assuming N as in Proposition 2.2.4, we have $N > \frac{2c_0}{a_1}$ and therefore we can apply the second inequality of Lemma 2.2.1 with $M = N$, $s_1 = s_2 = t_\xi$:

$$\begin{aligned} & \int_{t_\xi}^t |\xi|^{2\sigma}e^{\frac{1}{2\hat{C}}\int_{t_\xi}^\eta K(s,\xi)ds}e^{-|\xi|^{2\theta}\int_{t_\xi}^\eta b(\tau)d\tau}\left|Re(\hat{u}_t(t_\xi)\,\overline{\hat{u}}(t_\xi))\right| d\eta \\ & \quad \lesssim |\xi|^{2\sigma}\left|Re\left(\frac{1}{\psi(\xi)}\hat{u}_t(t_\xi)\ \overline{\hat{u}}(t_\xi)\right)\right| \\ & \quad \lesssim |\xi|^{2\sigma}|\hat{u}(t_\xi)|^2 + \frac{|\xi|^{2\sigma}}{\psi(\xi)^2}|\hat{u}_t(t_\xi)|^2. \end{aligned} \tag{2.38}$$

On the other hand, by Proposition 2.2.4, $\frac{a_4-a_3}{a_1-\frac{c_0}{N}} = a_8 - a_6 < a_8$. By integration by parts and by applying second inequality of Lemma 2.2.1

with $Q(\xi)$ as before, $f=1$, $\lambda=0$, $M=N$, $s_1=s$ and $s_2=t_\xi$:

$$
\begin{aligned}
&\frac{(a_4-a_3)}{2\hat{C}}|\xi|^{2\sigma}\int_{t_\xi}^{t} e^{-|\xi|^{2\theta}\int_{t_\xi}^{s} b(\tau)d\tau}\left(\int_{t_\xi}^{s} e^{|\xi|^{2\theta}\int_{t_\xi}^{\eta} b(\tau)d\tau}|\hat{u}_t(\eta)|^2 d\eta\right) ds\\
&\quad -\frac{a_8}{2\hat{C}}\int_{t_\xi}^{t} K(\eta,\xi)|\hat{u}_t(\eta)|^2 d\eta\\
&\quad = \frac{(a_4-a_3)}{2\hat{C}}|\xi|^{2\sigma}\int_{t_\xi}^{t}\left(\int_{s}^{t} e^{-|\xi|^{2\theta}\int_{t_\xi}^{\eta} b(\tau)d\tau} d\eta\right) e^{|\xi|^{2\theta}\int_{t_\xi}^{s} b(\tau)d\tau}|\hat{u}_t(s)|^2 ds\\
&\quad -\frac{a_8}{2\hat{C}}\int_{t_\xi}^{t} K(\eta,\xi)|\hat{u}_t(\eta)|^2 d\eta \qquad (2.39)\\
&\quad \le \frac{(a_4-a_3)}{2(a_1-\frac{c_0}{N})\hat{C}}\int_{t_\xi}^{t} K(s,\xi)|\hat{u}_t(s)|^2 ds - \frac{a_8}{2\hat{C}}\int_{t_\xi}^{t} K(\eta,\xi)|\hat{u}_t(\eta)|^2 d\eta \le 0.
\end{aligned}
$$

By inequality (2.34) follows:

$$
\mathcal{L}(t_\xi) \lesssim F(t_\xi) \lesssim |\xi|^{2\sigma}|\hat{u}(t_\xi)|^2 + \frac{|\xi|^{2\sigma}}{\psi(\xi)^2}|\hat{u}_t(t_\xi)|^2.
$$

Furthermore, using inequality (2.35), we have

$$
|\xi|^{2\sigma}|\hat{u}(t)|^2 + \frac{|\xi|^{2\sigma-4\theta}}{g(t)^2}|\hat{u}_t(t)|^2 \lesssim F(t) \lesssim \mathcal{L}(t).
$$

Therefore, applying inequalities (2.38) and (2.39) in inequality (2.37), we have:

$$
\begin{aligned}
&|\xi|^{2\sigma}|\hat{u}(t)|^2 + \frac{|\xi|^{2\sigma-4\theta}}{g(t)^2}|\hat{u}_t(t)|^2\\
&\quad \lesssim \mathcal{L}(t_\xi)e^{-\frac{1}{2\hat{C}}\int_{t_\xi}^{t} K(s,\xi)ds} + e^{-\frac{1}{2\hat{C}}\int_{t_\xi}^{t} K(s,\xi)ds}\int_{t_\xi}^{t} e^{\frac{1}{2\hat{C}}\int_{t_\xi}^{\eta} K(s,\xi)ds} R(\eta,\xi)d\eta\\
&\quad \lesssim \left(|\xi|^{2\sigma}|\hat{u}(t_\xi)|^2 + \frac{|\xi|^{2\sigma}}{\psi(\xi)^2}|\hat{u}_t(t_\xi)|^2\right) e^{-\frac{|\xi|^{2\sigma-2\theta}}{2\hat{C}}\int_{t_\xi}^{t}\frac{1}{g(s)}ds}, \qquad (2.40)
\end{aligned}
$$

for all $(t,\xi)\in Z_{ell}$. Note that we have already reached the goal of inequality (2.22). From now, we want to apply the glue procedure to remove the dependence of t_ξ in the right side of inequality (2.40). We will use several times the Remark 2.2.2, that is, $t_\xi > t_0$ for all $\xi \in B_R$.

Consider $-1 < \alpha < 1$, by applying Lemma A.1.2:

$$|\xi|^{2\sigma-2\theta}\int_{t_0}^{t_\xi}\frac{1}{g(s)}ds \lesssim |\xi|^{2\sigma-2\theta}\frac{(1+t_\xi)}{g(t_\xi)} \lesssim \frac{|\xi|^{2\sigma}}{\psi(\xi)^2} \lesssim 1, \text{ for all } \xi \in B_R. \tag{2.41}$$

Consider now $\alpha = 1$ and $\gamma < 1$. In this case, by direct integration and using Proposition 2.2.2,

$$\begin{aligned} |\xi|^{2\sigma-2\theta}\int_{t_0}^{t_\xi}\frac{1}{(1+s)ln^\gamma(1+s)}ds &\lesssim |\xi|^{2\sigma-2\theta}ln^{1-\gamma}(1+t_\xi) \quad (2.42)\\ &\lesssim ln(1+t_\xi)\frac{|\xi|^{2\sigma}}{\psi(\xi)^2} \lesssim 1, \end{aligned}$$

for $\xi \in B_R$. Using again Proposition 2.2.2, for $\alpha = 1$ and $\gamma = 1$ we have:

$$\begin{aligned} |\xi|^{2\sigma-2\theta}\int_{t_0}^{t_\xi}\frac{1}{(1+s)ln(1+s)}ds &\lesssim |\xi|^{2\sigma-2\theta}ln(ln(1+t_\xi)) \quad (2.43)\\ &\lesssim ln(ln(1+t_\xi))ln(1+t_\xi)\frac{|\xi|^{2\sigma}}{\psi(\xi)^2} \lesssim 1, \end{aligned}$$

for $\xi \in B_R$.

Using inequalities (2.40), (2.41), (2.42) and (2.43), and applying Proposition 2.2.3:

$$\begin{aligned} \psi(\xi)|\hat{u}(t)| &\lesssim (\psi(\xi)|\hat{u}(t_\xi)| + |\hat{u}_t(t_\xi)|)\, e^{-\frac{|\xi|^{2\sigma-2\theta}}{4\hat{C}}\int_{t_\xi}^{t}\frac{1}{g(s)}ds}\\ &\lesssim (\psi(\xi)|\hat{u}_0| + |\hat{u}_1|)\, e^{-\frac{|\xi|^{2\sigma-2\theta}}{4\hat{C}}\int_{t_0}^{t}\frac{1}{g(s)}ds} \end{aligned} \tag{2.44}$$

and similarly,

$$\begin{aligned} \frac{|\xi|^{\sigma-2\theta}}{g(t)}|\hat{u}_t(t)| &\lesssim \frac{|\xi|^\sigma}{\psi(\xi)}\,(\psi(\xi)|\hat{u}(t_\xi)| + |\hat{u}_t(t_\xi)|)\, e^{-\frac{|\xi|^{2\sigma-2\theta}}{4\hat{C}}\int_{t_\xi}^{t}\frac{1}{g(s)}ds}\\ &\lesssim \left(|\xi|^\sigma|\hat{u}_0| + \frac{|\xi|^\sigma}{\psi(\xi)}|\hat{u}_1|\right) e^{-\frac{|\xi|^{2\sigma-2\theta}}{4\hat{C}}\int_{t_0}^{t}\frac{1}{g(s)}ds}, \end{aligned}$$

for all $(t,\xi) \in Z_{ell}$. Applying Corollary A.1.1 we conclude that the two first estimates of the proposition hold. Finally, let $w := \hat{u}_t$ and rewriting equation (2.5),

$$w_t(t,\xi)+b(t)|\xi|^{2\theta}w(t,\xi)=-|\xi|^{2\sigma}\hat{u}(t,\xi), \qquad t\in[t_\xi,T_0(\xi)],$$

and solving this first order ODE in w, we find:

$$\begin{aligned}\hat{u}_t(t,\xi) &= \hat{u}_t(t_\xi,\xi)e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(s)ds} \\ &\quad -e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(s)ds}\int_{t_\xi}^t e^{|\xi|^{2\theta}\int_{t_\xi}^\eta b(s)ds}|\xi|^{2\sigma}\hat{u}(\eta,\xi)d\eta.\end{aligned} \tag{2.45}$$

By using inequality (2.44), applying the first inequality of Lemma 2.2.1 with $M=N$, $Q(\xi)\subset D(\xi)$ both as defined before inequality (2.38), with $f(\eta):=e^{-\frac{|\xi|^{2\sigma-2\theta}}{4\hat{C}}\int_{t_0}^\eta \frac{1}{g(s)}ds}$, $\lambda:=\frac{a_1}{4}$ and by Corollary A.1.1, we have for a suitable $C>0$:

$$\begin{aligned}&\left|e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(s)ds}\int_{t_\xi}^t e^{|\xi|^{2\theta}\int_{t_\xi}^\eta b(s)ds}|\xi|^{2\sigma}\hat{u}(\eta,\xi)d\eta\right| \\ &\qquad\lesssim (\psi(\xi)|\hat{u}_0|+|\hat{u}_1|)\frac{|\xi|^{2\sigma-2\theta}}{\psi(\xi)g(t)}e^{-\frac{|\xi|^{2\sigma-2\theta}}{4\hat{C}}\int_{t_0}^t\frac{1}{g(s)}ds} \\ &\qquad\lesssim (\psi(\xi)|\hat{u}_0|+|\hat{u}_1|)\frac{|\xi|^{2\sigma-2\theta}}{\psi(\xi)g(t)}e^{-\frac{1}{C}|\xi|^{2\sigma-2\theta}\phi(t)},\end{aligned} \tag{2.46}$$

for all $(t,\xi)\in Z_{ell}$. On the other hand, since $|\xi|^{2\theta}\int_{t_0}^{t_\xi}b(s)ds\lesssim 1$, by energy equation (2.9) and by Corollary A.1.1, for a suitable $C>0$, we have for $(t,\xi)\in Z_{ell}$:

$$\left|\hat{u}_t(t_\xi,\xi)e^{-|\xi|^{2\theta}\int_{t_\xi}^t b(s)ds}\right|\lesssim(|\xi|^\sigma|\hat{u}_0|+|\hat{u}_1|)\,e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}. \tag{2.47}$$

Applying inequalities (2.46) and (2.47) in inequality (2.45) the result is proved for $\hat{u}_t$.

□

In the next proposition, we use the standard energy method, that is, the Proposition 2.2.1 with Proposition 2.2.4 in order to obtain decay rates for the solution in hyperbolic zone. It should be noticed that an additional glue step is necessary to obtain the desired estimates.

Furthermore, if g is non-decreasing and since we are using Hypothesis B, by using Proposition 2.2.2 we have: given $\xi \in B_R$ and $t \geq t_\xi$, implies that $\varphi(t)|\xi|^{2\theta} \geq N$ and $g(t) \geq g(t_\xi) = \frac{\psi(\xi)}{|\xi|^{2\theta}} > |\xi|^{\sigma-2\theta}$, that is $(t,\xi) \in Z_{ell}$. Therefore, in this case we have $Z_{hyp} = \varnothing$. That is, to obtain the desired estimates in Z_{hyp} is sufficient consider the case g decreasing.

Proposition 2.2.6 *For g decreasing, the following estimates hold in Z_{hyp}:*

$$\psi(\xi)|\hat{u}(t)| \lesssim e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}\left(\psi(\xi)|\hat{u}_0| + |\hat{u}_1|\right),$$

$$|\hat{u}_t(t)| \lesssim \frac{|\xi|^\sigma}{\psi(\xi)} e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}\left(\psi(\xi)|\hat{u}_0| + |\hat{u}_1|\right).$$

Proof. Since g is decreasing, we have $-1 < \alpha \leq 0$ (with $\gamma < 0$ if $\alpha = 0$) and therefore by Hypothesis B we have $\sigma > 2\theta$. Since g can be seen as a bijection between $[t_0, \infty)$ and $(0, g(t_0)]$, $t_1(\xi) = g^{-1}(|\xi|^{\sigma-2\theta})$ is well defined if $|\xi|^{\sigma-2\theta} \leq g(t_0)$ for all $\xi \in B_R$, that is, if $R \leq g(t_0)^{\frac{1}{\sigma-2\theta}}$.

Since $|\xi|^\sigma < \psi(\xi)$ for all $\xi \in B_R$, it follows directly from the definition of ψ that $t_\xi < t_1(\xi)$ for all $\xi \in B_R$. Therefore, for each fixed $\xi \in B_R$, $[t_0,\infty) = [t_0, t_\xi] \cup [t_\xi, t_1(\xi)] \cup [t_1(\xi), \infty)$, in which $(s,\xi) \in Z_{low}$ for $s \in [t_0, t_\xi]$, $(s,\xi) \in Z_{ell}$ for $s \in [t_\xi, t_1(\xi)]$ and $(s,\xi) \in Z_{hyp}$ when $s \geq t_1(\xi)$. That is, in the notation of Proposition 2.2.1, $S_0(\xi) = t_1(\xi)$ and $T_0(\xi) = \infty$ in hyperbolic zone. Since $(t_1(\xi), \xi) \in Z_{ell}$, by applying Proposition 2.2.5 in $t = t_1(\xi)$, we have for $2C_2 \geq C$:

$$|\xi|^\sigma |\hat{u}(t_1(\xi))| \lesssim \frac{|\xi|^\sigma}{\psi(\xi)} e^{-\frac{1}{2C_2}|\xi|^{2\sigma-2\theta}\phi(t_1(\xi))}\left(\psi(\xi)|\hat{u}_0| + |\hat{u}_1|\right),$$
$$|\hat{u}_t(t_1(\xi))| \lesssim \frac{|\xi|^\sigma}{\psi(\xi)} e^{-\frac{1}{2C_2}|\xi|^{2\sigma-2\theta}\phi(t_1(\xi))}\left(\psi(\xi)|\hat{u}_0| + |\hat{u}_1|\right),$$

therefore

$$E(t_1(\xi)) \lesssim e^{-\frac{1}{C_2}|\xi|^{2\sigma-2\theta}\phi(t_1(\xi))} \frac{|\xi|^{2\sigma}}{\psi(\xi)^2}\left(\psi(\xi)^2|\hat{u}_0|^2 + |\hat{u}_1|^2\right). \tag{2.48}$$

By using Proposition 2.2.1 we have for $C_1 > \max\left\{\frac{2C_2}{3(1+\alpha)}, C\right\}$:

$$E(t) \lesssim e^{-\frac{1}{C_1}|\xi|^{2\theta}\int_{t_1(\xi)}^t g(s)ds} E(t_1(\xi)), \tag{2.49}$$

for all $(t,\xi)\in Z_{hyp}$.

Since $C_1 \geq \frac{2C_2}{3(1+\alpha)}$ and $|\xi|^{2\theta}\varphi(t_1(\xi)) = |\xi|^{2\sigma-2\theta}\phi(t_1(\xi))$, by using Lemma A.1.2 we have for a suitable $C>0$:

$$\begin{aligned}
&-\frac{1}{C_1}|\xi|^{2\theta}\int_{t_1(\xi)}^{t} g(s)ds - \frac{1}{C_2}|\xi|^{2\sigma-2\theta}\phi(t_1(\xi))\\
&\quad \leq -\frac{1}{C}|\xi|^{2\theta}\varphi(t) + \left(\frac{2}{3(1+\alpha)C_1} - \frac{1}{C_2}\right)|\xi|^{2\sigma-2\theta}\phi(t_1(\xi))\\
&\quad \leq -\frac{1}{C}|\xi|^{2\theta}\varphi(t) \qquad (2.50)
\end{aligned}$$

for all $t\geq t_1(\xi)$. Finally, using inequality (2.48) in inequality (2.49) and considering (2.50), the proposition is proved.

□

Differently of Chapter 1, in the Z^{high} is important to improve the estimates of Proposition 2.2.1 for $\alpha=1$. Actually, we prove an general improved estimate for g increasing.

Proposition 2.2.7 *Suppose that g is increasing. Let $\nu := \min\{2\sigma - 2\theta, 2\theta\}$ and C big enough. Therefore, there exists $t_0^*\geq t_0$ such that the following estimate holds for $|\xi|\geq R$ and $t\geq t_0^*$:*

$$\begin{aligned}
&|\xi|^{\sigma}|\hat{u}(t,\xi)| \lesssim e^{-\frac{1}{C}|\xi|^{\nu}} e^{-\frac{R^{\nu}}{C}\int_{t_0}^{t}\frac{1}{g(s)}ds}\left(|\xi|^{\sigma}|\hat{u}_0| + |\hat{u}_1|\right),\\
&|\hat{u}_t(t,\xi)| \lesssim \Big(e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}\\
&\qquad + \frac{1}{|\xi|^{2\theta}g(t)}e^{-\frac{R^{\nu}}{C}\int_{t_0}^{t}\frac{1}{g(s)}ds}\Big) e^{-\frac{1}{C}|\xi|^{\nu}}|\xi|^{\sigma}\left(|\xi|^{\sigma}|\hat{u}_0| + |\hat{u}_1|\right).
\end{aligned}$$

Proof. Using equation (2.5) and solving for $\hat{u}_t$ in $[t_0,t]$ we find:

$$\begin{aligned}
\hat{u}_t(t,\xi) &= \hat{u}_t(t_0,\xi)e^{-|\xi|^{2\theta}\int_{t_0}^{t} b(s)ds}\\
&\quad - e^{-|\xi|^{2\theta}\int_{t_0}^{t} b(s)ds}\int_{t_0}^{t} e^{|\xi|^{2\theta}\int_{t_0}^{\eta} b(s)ds}|\xi|^{2\sigma}\hat{u}(\eta,\xi)d\eta. \qquad (2.51)
\end{aligned}$$

Let $t_1^* > t_0$ such that $\int_{t_0}^{t_1^*}\frac{1}{g(s)}ds \geq 1$, whose existence is ensured due the fact that $\frac{1}{g}$ is not in L^1. By the same reason, and due our hypothesis on g also holds $\lim_{t\to\infty} g(t)=\infty$, we can assume:

$$\varphi(t_1^*) > \max\left\{\frac{c_0}{R^{2\theta}a_1 - \frac{R^\nu}{C}}, R^{\nu-2\theta}\right\} \text{ and } g(t_1^*) \geq 1. \tag{2.52}$$

In addition, holds

$$e^{-\frac{1}{C}|\xi|^\nu \int_{t_0}^t \frac{1}{g(s)}ds} = e^{-\frac{1}{C}|\xi|^\nu \int_{t_0}^{t_1^*} \frac{1}{g(s)}ds} e^{-\frac{1}{C}|\xi|^\nu \int_{t_1^*}^t \frac{1}{g(s)}ds} \leq e^{-\frac{1}{C}|\xi|^\nu},$$

for all $t \geq t_1^*$. By Proposition 2.2.1, we have for $t \geq t_1^*$, $|\xi| \geq R$ and C big enough:

$$\begin{aligned} |\xi|^\sigma |\hat{u}(t,\xi)| &\lesssim e^{-\frac{2}{C}|\xi|^\nu \int_{t_0}^t \frac{1}{g(s)}ds} \left(|\xi|^\sigma|\hat{u}_0| + |\hat{u}_1|\right) \\ &\lesssim e^{-\frac{1}{C}|\xi|^\nu} e^{-\frac{R^\nu}{C}\int_{t_0}^t \frac{1}{g(s)}ds}\left(|\xi|^\sigma|\hat{u}_0| + |\hat{u}_1|\right), \end{aligned}$$

and therefore

$$\begin{aligned} &\int_{t_1^*}^t e^{|\xi|^{2\theta}\int_{t_0}^\eta b(s)ds} |\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta \\ &= e^{|\xi|^{2\theta}\int_{t_0}^{t_1^*} b(s)ds} \int_{t_1^*}^t e^{|\xi|^{2\theta}\int_{t_1^*}^\eta b(s)ds} |\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta \\ &\lesssim e^{|\xi|^{2\theta}\int_{t_0}^{t_1^*} b(s)ds}\left(\int_{t_1^*}^t e^{|\xi|^{2\theta}\int_{t_1^*}^\eta b(s)ds} f(\eta)d\eta\right) e^{-\frac{1}{C}|\xi|^\nu}|\xi|^\sigma\left(|\xi|^\sigma|\hat{u}_0| + |\hat{u}_1|\right) \end{aligned} \tag{2.53}$$

for all $t \geq t_1^*$, where $f(\eta) := e^{-\frac{R^\nu}{C}\int_{t_0}^\eta \frac{1}{g(s)}ds}$. Let $M := R^{2\theta}\varphi(t_1^*)$ and $\lambda := \frac{1}{CR^{2\theta-\nu}} < a_1$ for C big enough. For each $\xi \in \mathbb{R}^n \setminus B_R$, we have $Q(\xi) := [t_1^*, \infty) \subset \{s \in [t_0,\infty) : |\xi|^{2\theta}(1+s)g(s) \geq M\}$. Since $g(\eta) \geq g(t_1^*) \geq 1$, we have $|f'(\eta)| = \left|-\frac{R^\nu}{Cg(\eta)}f(\eta)\right| \leq \lambda|\xi|^{2\theta}g(\eta)f(\eta)$ for all $\eta \in Q(\xi)$. Since $\lambda < a_1$, by condition (2.52) we have $\lambda + \frac{c_0}{M} < a_1$ and by Lemma 2.2.1 follows:

$$\int_{t_1^*}^t e^{|\xi|^{2\theta}\int_{t_1^*}^\eta b(s)ds} f(\eta)d\eta \lesssim \frac{1}{|\xi|^{2\theta}g(t)} e^{-\frac{R^\nu}{C}\int_{t_0}^t \frac{1}{g(s)}ds} e^{|\xi|^{2\theta}\int_{t_1^*}^t b(s)ds}, \tag{2.54}$$

for all $t \geq t_1^*$. By applying inequality (2.54) in inequality (2.53), we have:

$$\int_{t_1^*}^{t} e^{|\xi|^{2\theta}\int_{t_0}^{\eta} b(s)ds}|\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta \tag{2.55}$$
$$\lesssim e^{|\xi|^{2\theta}\int_{t_0}^{t} b(s)ds}|\xi|^{\sigma-2\theta}\frac{1}{g(t)}e^{-\frac{R^\nu}{C}\int_{t_0}^{t}\frac{1}{g(s)}ds}e^{-\frac{1}{C}|\xi|^\nu}\left(|\xi|^\sigma|\hat{u}_0|+|\hat{u}_1|\right).$$

Furthermore,

$$\begin{aligned}\int_{t_0}^{t_1^*} e^{|\xi|^{2\theta}\int_{t_0}^{\eta} b(s)ds}|\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta &\leq e^{|\xi|^{2\theta}\int_{t_0}^{t_1^*} b(s)ds}\int_{t_0}^{t_1^*}|\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta\\ &\lesssim |\xi|^\sigma e^{|\xi|^{2\theta}\int_{t_0}^{t_1^*} b(s)ds}\left(|\xi|^\sigma|\hat{u}_0|+|\hat{u}_1|\right).\end{aligned} \tag{2.56}$$

By using inequalities (2.55) and (2.56), we have for $t \geq t_1^*$:

$$\begin{aligned}&e^{-|\xi|^{2\theta}\int_{t_0}^{t} b(s)ds}\int_{t_0}^{t} e^{|\xi|^{2\theta}\int_{t_0}^{\eta} b(s)ds}|\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta\\
&= e^{-|\xi|^{2\theta}\int_{t_0}^{t} b(s)ds}\Bigg(\int_{t_0}^{t_1^*} e^{|\xi|^{2\theta}\int_{t_0}^{\eta} b(s)ds}|\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta\\
&\quad+\int_{t_1^*}^{t} e^{|\xi|^{2\theta}\int_{t_0}^{\eta} b(s)ds}|\xi|^{2\sigma}|\hat{u}(\eta,\xi)|d\eta\Bigg)\\
&\lesssim |\xi|^\sigma e^{-|\xi|^{2\theta}\int_{t_1^*}^{t} b(s)ds}\left(|\xi|^\sigma|\hat{u}_0|+|\hat{u}_1|\right)\\
&\quad+|\xi|^{\sigma-2\theta}\frac{1}{g(t)}e^{-\frac{R^\nu}{C}\int_{t_0}^{t}\frac{1}{g(s)}ds}e^{-\frac{1}{C}|\xi|^\nu}\left(|\xi|^\sigma|\hat{u}_0|+|\hat{u}_1|\right).\end{aligned} \tag{2.57}$$

Using that $\varphi(t_1^*) > R^{\nu-2\theta}$ in which imply that $\varphi(t)|\xi|^{2\theta} \geq |\xi|^\nu$, and by applying Corollary A.1.1 (with t_1^* instead of t_0) in inequality (2.57), there exists $t_0^* \geq t_1^* > t_0$ such that:

$$e^{-|\xi|^{2\theta}\int_{t_1^*}^{t} b(s)ds} \lesssim e^{-\frac{2}{C}|\xi|^{2\theta}\varphi(t)} \lesssim e^{-\frac{1}{C}|\xi|^\nu}e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}. \tag{2.58}$$

In particular, holds $e^{-|\xi|^{2\theta}\int_{t_0}^{t} b(s)ds} \lesssim e^{-\frac{1}{C}|\xi|^\nu}e^{-\frac{1}{C}|\xi|^{2\theta}\varphi(t)}$ for $t \geq t_0^*$. Using that $|\hat{u}_t(t_0,\xi)| \lesssim (|\xi|^\sigma|\hat{u}_0|+|\hat{u}_1|)$, using $1 \lesssim |\xi|^\sigma$, using inequality (2.57) and inequality (2.58) in equation (2.51) the result follows.

□

2.3 Proof of Results

In this section we prove the main result of the chapter. We have to apply the pointwise estimates in Fourier space of the previous section, fix the time variable and integrate ξ in $\mathbb{R}^n$. This procedure is make by considering the zone separation introduced in the beginning of Section 2.1 and a proof divided in several propositions to deal with each zone. During the step of integration in ξ, we often use results of Appendices A and B.

To proof the results, we shall consider in this section $\hat{q}$ conjugate of $q \in [2, \infty]$, that is $\hat{q} \in [1, 2]$ and $\frac{1}{\hat{q}} + \frac{1}{q} = 1$. Furthermore, $u_0, u_1 \in L^p(\mathbb{R}^n)$ and s conjugate of $p \in [1, 2]$, that is, $s \in [2, \infty]$ and therefore $s \geq \hat{q}$. We define $r := \infty$ if $p + \hat{q} = p\hat{q}$ and $r := \frac{p}{p+\hat{q}-p\hat{q}} \geq 1$ if $p + \hat{q} \neq p\hat{q}$. That is, r is conjugated of $\frac{s}{\hat{q}}$, since $\frac{p+\hat{q}-p\hat{q}}{p} + \frac{\hat{q}}{s} = 1$. In addition, we take $\mu, \beta \in \mathbb{N}^n$, $\varphi = \varphi(t)$ given by (2.1) and $\phi = \phi(t)$ given by (2.2) in Theorem 2.1.1. In this section several times will appear the expression $\frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q})$, which is equivalent to $n\left(\frac{1}{p} - \frac{1}{q}\right)$ that rises in Theorem 2.1.1.

Proposition 2.3.1 *Consider the conditions above, $\theta \neq 0$ and assume the Hypothesis B. Then, there exists $t_0^* \geq t_0$ such that the following estimates hold for $t \geq t_0^*$:*

$$\int_{Z_{low}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu| + \hat{q}\sigma + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||_{L^p}^{\hat{q}} + \varphi(t)^{-\left(\frac{\hat{q}|\mu| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||_{L^p}^{\hat{q}}.$$

If $u_1 \neq 0$ and $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) > 0$ or $u_1 = 0$:

$$\int_{Z_{low}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||_{L^p}^{\hat{q}} + \varphi(t)^{-\left(\frac{\hat{q}|\beta| - \frac{\hat{q}2\theta}{1+\alpha} + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) ||u_1||_{L^p}^{\hat{q}}.$$

If $u_1 \neq 0$, $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) = 0$ *and* $\gamma > \frac{(1+\alpha)(p+\hat{q}-p\hat{q})}{p\hat{q}}$*:*

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \quad &\lesssim \quad \varphi(t)^{-\frac{\hat{q}}{1+\alpha}}||u_0||_{L^p}^{\hat{q}} \\ &+ ln^{-\frac{\hat{q}\gamma}{1+\alpha}+\frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{\varphi(t)}{N}\right)||u_1||_{L^p}^{\hat{q}}.\end{aligned}$$

Proof. Let $\tau := N^{\frac{1}{2\theta}}\varphi(t)^{-\frac{1}{2\theta}}$. Since $|\hat{u}_t(t)|^{\hat{q}} \lesssim |\xi|^{\hat{q}\sigma}|\hat{u}_0|^{\hat{q}} + |\hat{u}_1|^{\hat{q}}$, we have:

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi \quad &\lesssim \quad \int_{Z_{low}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma}|\hat{u}_0|^{\hat{q}}d\xi \\ &\quad + \int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_1|^{\hat{q}}d\xi \\ &\lesssim \quad \left\||\cdot|^{\hat{q}|\mu|+\hat{q}\sigma}\right\|_{L^r(B_\tau)} ||\hat{u}_0||_{L^s}^{\hat{q}} \\ &\quad + \left\||\cdot|^{\hat{q}|\mu|}\right\|_{L^r(B_\tau)} ||\hat{u}_1||_{L^s}^{\hat{q}}.\end{aligned}$$

Using Hausdorff-Young inequality (see [1]) and Lemma B.1.3 with $k_1 = \hat{q}|\mu| + \hat{q}\sigma$ or $k_1 = \hat{q}|\mu|$ and $k_2 = 0$ in both cases:

$$\begin{aligned}\int_{Z_{low}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi \quad &\lesssim \quad \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||_{L^p}^{\hat{q}} \\ &+ \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||_{L^p}^{\hat{q}}.\end{aligned}$$

for $t \geq t_0^* \geq \varphi^{-1}(N)$.

By Proposition 2.2.2, we have $\psi(\xi) \sim |\xi|^{\frac{2\theta}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)$. Applying Proposition 2.2.3, we have:

$$\begin{aligned}&\int_{Z_{low}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \qquad (2.59) \\ &\lesssim \int_{Z_{low}} |\xi|^{\hat{q}|\beta|}|\hat{u}_0|^{\hat{q}}d\xi + \int_{Z_{low}} |\xi|^{\hat{q}|\beta|-\frac{\hat{q}2\theta}{1+\alpha}} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi \\ &\lesssim \left\||\cdot|^{\hat{q}|\beta|}\right\|_{L^r(B_\tau)} ||\hat{u}_0||_{L^s}^{\hat{q}} + \left\||\cdot|^{\hat{q}|\beta|-\frac{\hat{q}2\theta}{1+\alpha}} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_\tau)} ||\hat{u}_1||_{L^s}^{\hat{q}}.\end{aligned}$$

Using Hausdorff-Young inequality and Lemma B.1.3 with $k_1 = \hat{q}|\beta|$ and $k_2 = 0$ for the first term on the right side of inequality (2.59), $k_1 = \hat{q}|\beta| - \hat{q}\frac{2\theta}{1+\alpha}$ and $k_2 = -\frac{\hat{q}\gamma}{1+\alpha}$ for the second one, we have:

i) If $u_1 \neq 0$ and $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) > 0$ or $u_1 = 0$, for $t \geq t_0^*$:

$$\int_{Z_{low}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||_{L^p}^{\hat{q}} + \varphi(t)^{-\left(\frac{\hat{q}|\beta| - \frac{\hat{q}2\theta}{1+\alpha} + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) ||u_1||_{L^p}^{\hat{q}}.$$

ii) If $u_1 \neq 0$, $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) = 0$ and $\gamma > \frac{(1+\alpha)(p+\hat{q}-p\hat{q})}{p\hat{q}}$, we have, for $t \geq t_0^*$:

$$\int_{Z_{low}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \varphi(t)^{-\frac{\hat{q}}{1+\alpha}} ||u_0||_{L^p}^{\hat{q}} + ln^{-\frac{\hat{q}\gamma}{1+\alpha} + \frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{\varphi(t)}{N}\right) ||u_1||_{L^p}^{\hat{q}}.$$

In which t_0^* is chosen to ensure conditions on τ of Lemma B.1.3: for $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$, $\tau = N^{\frac{1}{2\theta}}\varphi(t)^{-\frac{1}{2\theta}} \in \left(0, e^{-\frac{2|k_2|}{k_1 + \frac{n}{p}(p+\hat{q}-p\hat{q})}}\right) \Leftrightarrow t \geq \varphi^{-1}\left(N e^{\frac{4\theta|\gamma|}{|\beta|(1+\alpha) - 2\theta + \frac{n}{p\hat{q}}(1+\alpha)(p+\hat{q}-p\hat{q})}}\right)$. On the other hand, if $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) = 0$, $\tau = N^{\frac{1}{2\theta}}\varphi(t)^{-\frac{1}{2\theta}} \in (0,1) \Leftrightarrow t \geq \varphi^{-1}(N)$.

□

Proposition 2.3.2 *Under the conditions of Proposition 2.3.1, there exists $t_0^* \geq t_0$ such that the following inequalities hold for $t \geq t_0^*$:*

If $\theta \neq 0$ and $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) > 0$ and $u_1 \neq 0$ or any $\beta \in \mathbb{N}^n$ if $u_1 = 0$,

$$\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|} |\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \phi(t)^{-\left(\frac{\hat{q}|\beta| + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ||u_0||_{L^p}^{\hat{q}} + \phi(t)^{-\left(\frac{\hat{q}|\beta| - \frac{2\hat{q}\theta}{1+\alpha} + \frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right) ||u_1||_{L^p}^{\hat{q}}. \quad (2.60)$$

If $\theta \neq 0$ *and* $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) = 0$ *and* $\gamma > \frac{(1+\alpha)(p+\hat{q}-p\hat{q})}{p\hat{q}}$,

$$\begin{aligned}\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi &\lesssim \phi(t)^{-\frac{\hat{q}\theta}{(1+\alpha)(\sigma-\theta)}}||u_0||^{\hat{q}}_{L^p} \\ &+ ln^{-\frac{\hat{q}\gamma}{1+\alpha}+\frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{\phi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}. \quad (2.61)\end{aligned}$$

If $\theta \neq 0$ *and* $|\mu| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) \leq 0$,

$$\begin{aligned}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)}||u_0||^{\hat{q}}_{L^p} \\ &+ \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p}.\end{aligned}$$

If $\theta \neq 0$ *and* $|\mu| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$,

$$\begin{aligned}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)}||u_0||^{\hat{q}}_{L^p} \\ &+ \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}\sigma-2\hat{q}\theta\left(\frac{2+\alpha}{1+\alpha}\right)\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)+} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}.\end{aligned}$$

If $\theta = 0$,

$$\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \lesssim \phi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma}\right)}\left(||u_0||^{\hat{q}}_{L^p}+||u_1||^{\hat{q}}_{L^p}\right),$$

$$\begin{aligned}&\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi \\ &\quad\lesssim \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma}\right)}\left(||u_0||^{\hat{q}}_{L^p}+||u_1||^{\hat{q}}_{L^p}\right).\end{aligned}$$

Proof. Initially consider $\theta \neq 0$. By Proposition 2.2.2, we have $\psi(\xi) \sim |\xi|^{\frac{2\theta}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)$ and by applying Proposition 2.2.5, we have:

$$\begin{aligned}
&\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \\
&\quad\lesssim \int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)}|\hat{u}_0|^{\hat{q}}d\xi \\
&\qquad+ \int_{Z_{ell}} |\xi|^{\hat{q}|\beta|-\frac{2\hat{q}\theta}{1+\alpha}}e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)}ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi \\
&\quad\lesssim \left\||\cdot|^{\hat{q}|\beta|}e^{-\frac{\hat{q}}{c}|\cdot|^{2\sigma-2\theta}\phi(t)}\right\|_{L^r(B_R)}||\hat{u}_0||^{\hat{q}}_{L^s} \\
&\qquad+ \left\||\cdot|^{\hat{q}|\beta|-\frac{2\hat{q}\theta}{1+\alpha}}e^{-\frac{\hat{q}}{c}|\cdot|^{2\sigma-2\theta}\phi(t)}ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_R)}||\hat{u}_1||^{\hat{q}}_{L^s}.
\end{aligned}$$

Using Hausdorff-Young inequality and Lemma B.1.5 (case I) with $k_1 = \hat{q}|\beta|$ and $k_2 = 0$ for the first term on the right side of above inequality, whereas we take $k_1 = \hat{q}|\beta| - \hat{q}\frac{2\theta}{1+\alpha}$ and $k_2 = -\frac{\hat{q}\gamma}{1+\alpha}$ for the second one, inequalities (2.60) and (2.61) follows.

Let us consider estimates for $\hat{u}_t$ with $\theta \neq 0$. Using Propositions 2.2.2 and 2.2.5, we have $\psi(\xi) \sim |\xi|^{\frac{2\theta}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)$ and:

$$\begin{aligned}
&\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi \\
&\lesssim \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma}e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}|\hat{u}_0|^{\hat{q}}d\xi + \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}|\hat{u}_1|^{\hat{q}}d\xi \\
&\quad+\frac{1}{g(t)^{\hat{q}}}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)}e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)}|\hat{u}_0|^{\hat{q}}d\xi \qquad (2.62)\\
&\quad+\frac{1}{g(t)^{\hat{q}}}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)-\frac{2\hat{q}\theta}{1+\alpha}}e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)}ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi.
\end{aligned}$$

To estimate the two first integrals, we consider $Z_{ell} \subset Q_1(t) := \{\xi \in B_R : |\xi|^{2\theta}\varphi(t) \geq M\} \cup Q_2(t) := \{\xi \in B_R : |\xi|^{2\theta}\varphi(t) \leq M\}$, where in Q_1 we use Lemma B.1.4 as mentioned and in Q_2 we proceed as in Proposition 2.3.1 and M is provided by Lemma B.1.4 with the following parameters: $k_1 = \hat{q}|\mu| + \hat{q}\sigma$ for the first integral and $k_1 = \hat{q}|\mu|$ for the second, with $k_2 = 0$, $\omega = 2\theta$, $\tau = \varphi(t)$ in both cases. Furthermore, by applying Hausdorff-Young inequality, we have:

$$\begin{aligned}
&\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad \lesssim \int_{Q_1(t)} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)} |\hat{u}_0|^{\hat{q}} d\xi \\
&\qquad + \int_{Q_2(t)} |\xi|^{\hat{q}|\mu|+\hat{q}\sigma} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad \lesssim \left\| |\cdot|^{\hat{q}|\mu|+\hat{q}\sigma} e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)} \right\|_{L^r(Q_1(t))} ||\hat{u}_0||_{L^s}^{\hat{q}} \\
&\qquad + \left\| |\cdot|^{\hat{q}|\mu|+\hat{q}\sigma} e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)} \right\|_{L^r(Q_2(t))} ||\hat{u}_0||_{L^s}^{\hat{q}} \\
&\quad \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_0||_{L^p}^{\hat{q}},
\end{aligned} \tag{2.63}$$

and similarly,

$$\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)} |\hat{u}_1|^{\hat{q}} d\xi \quad \lesssim \quad \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ||u_1||_{L^p}^{\hat{q}} \tag{2.64}$$

for $t \geq t_0^* \geq \varphi^{-1}(M)$ and M given by Lemma B.1.4.

For the third integral, we apply Lemma B.1.5 with $k_1 = \hat{q}|\mu| + 2\hat{q}(\sigma-\theta)$ and $k_2 = 0$, therefore for $t \geq t_0^*$:

$$\begin{aligned}
&\frac{1}{g(t)^{\hat{q}}} \int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)} e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)} |\hat{u}_0|^{\hat{q}} d\xi \\
&\quad \lesssim \frac{1}{g(t)^{\hat{q}}} \left\| |\cdot|^{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)} e^{-\frac{\hat{q}}{c}|\cdot|^{2\sigma-2\theta}\phi(t)} \right\|_{L^r(B_R)} ||\hat{u}_0||_{L^s}^{\hat{q}} \\
&\quad \lesssim \frac{1}{g(t)^{\hat{q}}} \phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ||u_0||_{L^p}^{\hat{q}}.
\end{aligned} \tag{2.65}$$

To estimate the last integral of inequality (2.62) we separate in six cases. Remember that this estimative is only necessary when $u_1 \neq 0$. Let $k_1 = \hat{q}|\mu| + 2\hat{q}\sigma - 2\hat{q}\theta\left(\frac{2+\alpha}{1+\alpha}\right)$, $k_2 = -\frac{\hat{q}\gamma}{1+\alpha}$ and $\nu = \nu(t)$ as in Lemma B.1.10. By Lemma B.1.5 (consider $B(t)$, case I and case II as in the mentioned lemma) and by Hausdorff-Young inequality, we have:

$$I := \frac{1}{g(t)^{\hat{q}}}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)-\frac{2\hat{q}\theta}{1+\alpha}} e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi$$
$$\lesssim \begin{cases} \frac{1}{g(t)^{\hat{q}}}\left\| |\cdot|^{k_1} e^{-\frac{\hat{q}}{c}|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_R)} ||\hat{u}_1||^{\hat{q}}_{L^s} & \text{if case I holds,}\\ \frac{1}{g(t)^{\hat{q}}}\left\| |\cdot|^{k_1} e^{-\frac{\hat{q}}{c}|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B(t))} ||\hat{u}_1||^{\hat{q}}_{L^s} & \text{if case II holds.}\end{cases}$$

Therefore,

$$I \leq \begin{cases} \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}\sigma-2\hat{q}\theta\left(\frac{2+\alpha}{1+\alpha}\right)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}>0,\\ \frac{1}{g(t)^{\hat{q}}} ln^{-\frac{\hat{q}\gamma}{1+\alpha}+\frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{\phi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2<-\frac{1}{r},\\ \frac{1}{g(t)^{\hat{q}}}\varphi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}\sigma-2\hat{q}\theta\left(\frac{2+\alpha}{1+\alpha}\right)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right)||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}<0,\\ \frac{1}{g(t)^{\hat{q}}} ln^{\frac{\hat{q}\gamma}{1+\alpha}}\left(ln\left(\frac{\varphi(t)}{N}\right)\right)||u_1||^{\hat{q}}_{L^p}, \qquad \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2=-\frac{1}{r},\\ \frac{1}{g(t)^{\hat{q}}} ln^{-\frac{\hat{q}\gamma}{1+\alpha}+\frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{M^{\frac{1}{2\sigma-2\theta}}\varphi(t)^{\frac{1}{2\theta}}}{N^{\frac{1}{2\theta}}\phi(t)^{\frac{1}{2\sigma-2\theta}}}\right)||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}=0 \text{ and } -\frac{1}{r}<k_2\leq 0,\\ \frac{1}{g(t)^{\hat{q}}} ln^{-\frac{\hat{q}\gamma}{1+\alpha}+\frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{\varphi(t)}{N}\right)||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}=0 \text{ and } -\frac{1}{r}\leq 0<k_2,\end{cases}$$

$$= \begin{cases} \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}\sigma-2\hat{q}\theta\left(\frac{2+\alpha}{1+\alpha}\right)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}>0,\\ \left(\frac{1}{g(t)^2}\nu(t)\right)^{\frac{\hat{q}}{2}}||u_1||^{\hat{q}}_{L^p} \qquad \text{if } k_1+\frac{n}{r}=0,\\ \left(\frac{1}{g(t)^2}\varphi(t)^{-\left(\frac{|\mu|+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)+\frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})}{\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right)\right)^{\frac{\hat{q}}{2}}||u_1||^{\hat{q}}_{L^p}, \\ \qquad\qquad \text{if } k_1+\frac{n}{r}<0.\end{cases} \tag{2.66}$$

Applying Lemma B.1.6 and Lemma B.1.10 with $\omega = |\mu| + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})$ in inequality (2.66), since $\omega = |\mu| + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) = 2\theta\left(\frac{2+\alpha}{1+\alpha}\right) - 2\sigma$ if $k_1 + \frac{n}{r} = 0$. Therefore:

$$\frac{1}{g(t)^{\hat{q}}}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+2\hat{q}(\sigma-\theta)-\frac{2\hat{q}\theta}{1+\alpha}} e^{-\frac{\hat{q}}{c}|\xi|^{2\sigma-2\theta}\phi(t)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi$$
$$\lesssim \begin{cases} \frac{1}{g(t)^{\hat{q}}}\phi(t)^{-\left(\frac{\hat{q}|\mu|+2\hat{q}\sigma-2\hat{q}\theta\left(\frac{2+\alpha}{1+\alpha}\right)+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma-2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}, & \text{if } k_1+\frac{n}{r}>0, \\ \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p}, & \text{if } k_1+\frac{n}{r}\le 0. \end{cases} \tag{2.67}$$

Using inequalities (2.62), (2.63), (2.64), (2.65) and (2.67), we have for $\theta \neq 0$ and $t \geq t_0^*$:

$$\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi$$
$$\lesssim \max\left\{\varphi(t)^{-\left(\frac{|\mu|+\sigma+\frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})}{\theta}\right)},\ \frac{1}{g(t)^2}\phi(t)^{-\left(\frac{|\mu|+2\sigma-2\theta+\frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})}{\sigma-\theta}\right)}\right\}^{\frac{\hat{q}}{2}}||u_0||^{\hat{q}}_{L^p}$$
$$+\max\left\{\varphi(t)^{-\left(\frac{|\mu|+\frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})}{\theta}\right)},\ \frac{1}{g(t)^2}\phi(t)^{-\left(\frac{|\mu|+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)+\frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})}{\sigma-\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)\right\}^{\frac{\hat{q}}{2}}||u_1||^{\hat{q}}_{L^p}.$$

We calculate the maximum in the last inequality using Lemma B.1.7 with $\omega = |\mu| + \sigma + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})$ for the term associated to u_0 and Lemma B.1.8 with $\omega = |\mu| + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q})$ for the term associated to u_1. Therefore, the result is proved for $\theta \neq 0$.

Finally, let $\theta = 0$. By Proposition 2.2.2 $\psi \sim 1$, using Proposition 2.2.5, using Lemma B.1.5 with $k_1 = \hat{q}|\beta|$ and $k_2 = 0$, and applying

Hausdorff-Young inequality:

$$\begin{aligned}\int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi &\lesssim \int_{Z_{ell}} |\xi|^{\hat{q}|\beta|}e^{-\frac{\hat{q}}{C}|\xi|^{2\sigma}\phi(t)}(|\hat{u}_0|^{\hat{q}}+|\hat{u}_1|^{\hat{q}})d\xi \\ &\lesssim \left\| |\cdot|^{\hat{q}|\beta|}e^{-\frac{\hat{q}}{C}|\cdot|^{2\sigma}\phi(t)}\right\|_{L^r(B_R)}\left(||\hat{u}_0||^{\hat{q}}_{L^s}+||\hat{u}_1||^{\hat{q}}_{L^s}\right) \\ &\lesssim \phi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\sigma}\right)}\left(||u_0||^{\hat{q}}_{L^p}+||u_1||^{\hat{q}}_{L^p}\right)\end{aligned}$$

and

$$\begin{aligned}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim e^{-\frac{\hat{q}}{C}\varphi(t)}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|}\left(|\hat{u}_0|^{\hat{q}}+|\hat{u}_1|^{\hat{q}}\right)d\xi \\ &\quad+\frac{1}{g(t)^{\hat{q}}}\int_{Z_{ell}} |\xi|^{\hat{q}|\mu|+2\hat{q}\sigma}e^{-\frac{\hat{q}}{C}|\cdot|^{2\sigma}\phi(t)}\left(|\hat{u}_0|^{\hat{q}}+|\hat{u}_1|^{\hat{q}}\right)d\xi \\ &\lesssim e^{-\frac{\hat{q}}{C}\varphi(t)}\left\| |\cdot|^{\hat{q}|\mu|}\right\|_{L^r(B_R)}\left(||\hat{u}_0||^{\hat{q}}_{L^s}+||\hat{u}_1||^{\hat{q}}_{L^s}\right) \qquad (2.68) \\ &\quad+\frac{1}{g(t)^{\hat{q}}}\left\| |\cdot|^{\hat{q}|\mu|+2\hat{q}\sigma}e^{-\frac{\hat{q}}{C}|\cdot|^{2\sigma}\phi(t)}\right\|_{L^r(B_R)}\left(||\hat{u}_0||^{\hat{q}}_{L^s}+||\hat{u}_1||^{\hat{q}}_{L^s}\right).\end{aligned}$$

The first member of the right side of inequality (2.68) has a exponential decay, in which is always less than a algebric-logarithmic decay. To estimate the second member, we use Lemma B.1.5 with $k_1 = \hat{q}|\mu|+2\hat{q}\sigma$ and $k_2 = 0$, and by Hausdorff-Young inequality the result follows.

□

Proposition 2.3.3 *Under the conditions of Proposition 2.3.1, there exists $t_0^* \geq t_0$ such that the following inequalities hold for $t \geq t_0^*$:*

If $\theta \neq 0$,

$$\begin{aligned}\int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi &\lesssim \varphi(t)^{-\left(\frac{\hat{q}|\mu|+\hat{q}\sigma+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||^{\hat{q}}_{L^p} \\ &\quad+\varphi(t)^{-\left(\frac{\hat{q}|\mu|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_1||^{\hat{q}}_{L^p}.\end{aligned}$$

If $\theta \neq 0$, $u_1 \neq 0$ *and* $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0$ *or* $u_1 = 0$*:*

$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \lesssim \varphi(t)^{-\left(\frac{\hat{q}|\beta|+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)}||u_0||^{\hat{q}}_{L^p}$$
$$+\varphi(t)^{-\left(\frac{\hat{q}|\beta|-\frac{2\hat{q}\theta}{1+\alpha}+\frac{n}{p}(p+\hat{q}-p\hat{q})}{2\theta}\right)} ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}.$$

If $\theta \neq 0$, $u_1 \neq 0$, $|\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p+\hat{q}-p\hat{q}) = 0$ *and* $\gamma > \frac{(1+\alpha)(p+\hat{q}-p\hat{q})}{p\hat{q}}$*:*

$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \lesssim \varphi(t)^{-\frac{\hat{q}}{1+\alpha}}||u_0||^{\hat{q}}_{L^p} + ln^{-\frac{\hat{q}\gamma}{1+\alpha}+\frac{p+\hat{q}-p\hat{q}}{p}}\left(\frac{\varphi(t)}{M}\right)||u_1||^{\hat{q}}_{L^p}.$$

If $\theta = 0$,

$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}}d\xi \lesssim e^{-\frac{\hat{q}}{C}\varphi(t)}||u_0||^{\hat{q}}_{L^p} + e^{-\frac{\hat{q}}{C}\varphi(t)}||u_1||^{\hat{q}}_{L^p},$$
$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}}d\xi \lesssim e^{-\frac{\hat{q}}{C}\varphi(t)}||u_0||^{\hat{q}}_{L^p} + e^{-\frac{\hat{q}}{C}\varphi(t)}||u_1||^{\hat{q}}_{L^p}.$$

Proof. For g non-decreasing $Z_{hyp} = \varnothing$ and the result is trivial. So in this proof we assume g decreasing. By Proposition 2.2.6 and by Proposition 2.2.2, we have:

$$|\xi|^{\hat{q}|\beta|}|\hat{u}(t)|^{\hat{q}} \lesssim e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}\left(|\xi|^{\hat{q}|\beta|}|\hat{u}_0|^{\hat{q}} + \frac{|\xi|^{\hat{q}|\beta|}}{\psi(\xi)^{\hat{q}}}|\hat{u}_1|^{\hat{q}}\right) \tag{2.69}$$

and

$$\begin{aligned} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t)|^{\hat{q}} &\lesssim \frac{|\xi|^{\hat{q}\sigma}}{\psi(\xi)^{\hat{q}}}e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}\left(|\xi|^{\hat{q}|\mu|}\psi(\xi)^{\hat{q}}|\hat{u}_0|^{\hat{q}} + |\xi|^{\hat{q}|\mu|}|\hat{u}_1|^{\hat{q}}\right) \\ &\lesssim e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}\left(|\xi|^{\hat{q}\sigma+\hat{q}|\mu|}|\hat{u}_0|^{\hat{q}} + |\xi|^{\hat{q}|\mu|}|\hat{u}_1|^{\hat{q}}\right), \end{aligned} \tag{2.70}$$

for all $(t,\xi) \in Z_{hyp}$. The estimates for $\hat{u}_t$ and $\theta \neq 0$ is achieved by using the same calculations as made in inequalities (2.63) and (2.64) since that calculations only require $|\xi| \leq R$ and $|\xi|^{2\theta}\varphi(t) \geq N$, in which is also true for hyperbolic zone. In the case of $\hat{u}$ for $\theta \neq 0$, using inequality (2.69) and Proposition 2.2.2, we have:

$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t)|^{\hat{q}}d\xi$$
$$\lesssim \int_{Z_{hyp}} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}|\xi|^{\hat{q}|\beta|}|\hat{u}_0|^{\hat{q}}d\xi + \int_{Z_{hyp}} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}\frac{|\xi|^{\hat{q}|\beta|}}{\psi(\xi)^{\hat{q}}}|\hat{u}_1|^{\hat{q}}d\xi$$
$$\lesssim \int_{Z_{hyp}} e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}|\xi|^{\hat{q}|\beta|}|\hat{u}_0|^{\hat{q}}d\xi$$
$$+\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|-\frac{2\hat{q}\theta}{1+\alpha}}e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi. \tag{2.71}$$

To estimate the first term of inequality (2.71) we just use the same argument as is in inequalities (2.63) and (2.64), since it only requires the restriction $|\xi| \leq R$ and $|\xi|^{2\theta}\varphi(t) \geq N$, in which is also true for hyperbolic zone. For the second term we fix $k_1 = \hat{q}|\beta| - \frac{2\hat{q}\theta}{1+\alpha}$ and $k_2 = -\frac{\hat{q}\gamma}{1+\alpha}$, therefore:

$$\int_{Z_{hyp}} |\xi|^{\hat{q}|\beta|-\frac{2\hat{q}\theta}{1+\alpha}}e^{-\frac{\hat{q}}{C}|\xi|^{2\theta}\varphi(t)}ln^{-\frac{\hat{q}\gamma}{1+\alpha}}\left(\frac{1}{|\xi|}\right)|\hat{u}_1|^{\hat{q}}d\xi$$
$$\lesssim \left\| |\cdot|^{k_1}e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)}ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_R)} ||\hat{u}_1||^{\hat{q}}_{L^s}, \tag{2.72}$$

and for $M \geq N$ big enough we have:

$$\left\| |\cdot|^{k_1}e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)}ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_R)}$$
$$\lesssim \left\| |\cdot|^{k_1}e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)}ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(\xi\in B_R:\ |\xi|^{2\theta}\varphi(t)\leq M)}$$
$$+\left\| |\cdot|^{k_1}e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)}ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(\xi\in B_R:\ |\xi|^{2\theta}\varphi(t)\geq M)}.$$

To estimate the first term of the last inequality, we proceed as in inequality (2.59). For the second we apply Lemma B.1.4 and using that

$k_2 \leq k_2 + \frac{1}{r}$, follows:

$$\left\| |\cdot|^{k_1} e^{-\frac{\hat{q}}{C}|\cdot|^{2\theta}\varphi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_R)}$$
$$\lesssim \begin{cases} \varphi(t)^{-\frac{k_1+\frac{n}{r}}{2\theta}} ln^{k_2}\left(\frac{\varphi(t)}{M}\right) & \text{if } k_1 + \frac{n}{r} > 0, \\ ln^{k_2+\frac{1}{r}}\left(\frac{\varphi(t)}{M}\right) & \text{if } k_1 + \frac{n}{r} = 0, \text{ and } k_2 < -\frac{1}{r}, \end{cases}$$

applying last inequality in (2.72), further applying in inequalities (2.71) and applying the Hausdorff-Young inequality the result follows.

For $\theta = 0$, by Proposition 2.2.2 we have $\psi(\xi) \sim 1$ and therefore the result is straightforward for both cases: by using inequality (2.70) or inequality (2.69), the exponential part does not depend on ξ and the right side can be estimated using Hausdorff-Young inequality.

□

In the next proposition we obtain decays rates for Z^{high}. However, it should be noticed that we do not cared about the sharp decay rates in this region, our idea is only to ensure that the decay rates will be determined by the Z_{ell}.

Proposition 2.3.4 *Consider the conditions of Proposition 2.3.1. Then, for every $\eta > 0$, there exists $t_0^* \geq t_0$ such that the following estimates hold for $t \geq t_0^*$:*

If $\theta \neq 0$,

$$\int_{Z^{high}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \phi(t)^{-\eta}||u_0||^{\hat{q}}_{L^p} + \phi(t)^{-\eta}||u_1||^{\hat{q}}_{L^p},$$
$$\int_{Z^{high}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \phi(t)^{-\eta}||u_0||^{\hat{q}}_{L^p} + \phi(t)^{-\eta}||u_1||^{\hat{q}}_{L^p}.$$

If $\theta = 0$ and $\omega > \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q})$,

$$\int_{Z^{high}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \phi(t)^{-\eta}||u_0||^{\hat{q}}_{W^{|\beta|+\omega,p}} + \phi(t)^{-\eta}||u_1||^{\hat{q}}_{W^{|\beta|-\sigma+\omega,p}},$$
$$\int_{Z^{high}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \phi(t)^{-\eta}||u_0||^{\hat{q}}_{W^{|\mu|+\sigma+\omega,p}} + \phi(t)^{-\eta}||u_1||^{\hat{q}}_{W^{|\mu|+\omega,p}}.$$

If $\alpha = 1$ and $\theta \neq 0$, we also have:

$$\int_{Z^{high}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \frac{\phi(t)^{-\eta}}{g(t)^{\hat{q}}} ||u_0||^{\hat{q}}_{L^p} + \frac{\phi(t)^{-\eta}}{g(t)^{\hat{q}}} ||u_1||^{\hat{q}}_{L^p}.$$

If $\alpha = 1$, $\theta = 0$ and $\omega > \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q})$, also holds:

$$\int_{Z^{high}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \lesssim \frac{\phi(t)^{-\eta}}{g(t)^{\hat{q}}} ||u_0||^{\hat{q}}_{W^{|\mu|+2\sigma+\omega,p}} + \frac{\phi(t)^{-\eta}}{g(t)^{\hat{q}}} ||u_1||^{\hat{q}}_{W^{|\mu|+\sigma+\omega,p}}.$$

Proof. In this proof we fix $f(t) := g(t)$ is g is non-increasing and $f(t) := \frac{1}{g(t)}$ if g is increasing. Let $\nu := \min\{2\sigma - 2\theta, 2\theta\}$. Using Corollary A.1.1, we have:

$$e^{-\frac{1}{C_1}\int_{t_0}^t f(s)ds} \lesssim \begin{cases} e^{-c(1+t)^{1-\alpha} ln^{-\gamma}(1+t)}, & \text{if } \ 0 < \alpha < 1 \ \text{ or } \alpha = 0 \ \text{ and } \ \gamma > 0, \\ e^{-c(1+t)^{1+\alpha} ln^{\gamma}(1+t)}, & \text{if } \ -1 < \alpha < 0 \ \text{ or } \alpha = 0 \ \text{ and } \ \gamma \leq 0, \\ e^{-c\, ln^{1-\gamma}(1+t)}, & \text{if } \ \alpha = 1 \ \text{ and } \gamma < 1, \\ e^{-c\, ln(ln(1+t))}, & \text{if } \ \alpha = 1 \ \text{ and } \gamma = 1. \end{cases}$$

Therefore, given $\eta > 0$, and sufficient big t_0^*, we have for all $t \geq t_0^*$:

$$e^{-\frac{1}{C_1}\int_{t_0}^t f(s)ds} \lesssim \phi(t)^{-\eta}. \tag{2.73}$$

Initially, let $\theta \neq 0$ and $t_0^* \geq t_0$ such that $\int_{t_0}^{t_0^*} f(s)ds \geq 1$ (this number exists because $\frac{1}{g}$ and $g \notin L^1(\mathbb{R})$) and such that inequality (2.73) is satisfied. For $t \geq t_0^*$, by Proposition 2.2.1 we have:

$$\begin{aligned}
\int_{Z^{high}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}} d\xi &\lesssim \int_{Z^{high}} |\xi|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\xi|^{\nu}\int_{t_0}^t f(s)ds}|\hat{u}_0|^{\hat{q}} d\xi \\
&\quad + \int_{Z^{high}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\xi|^{\nu}\int_{t_0}^t f(s)ds}|\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim e^{-\frac{\hat{q}R^{\nu}}{2C}\int_{t_0^*}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\xi|^{\nu}}|\hat{u}_0|^{\hat{q}} d\xi \\
&\quad + e^{-\frac{\hat{q}R^{\nu}}{2C}\int_{t_0^*}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\xi|^{\nu}}|\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim \phi(t)^{-\eta} \left\| |\cdot|^{\hat{q}|\beta|} e^{-\frac{\hat{q}}{2C}|\cdot|^{\nu}} \right\|_{L^{\hat{p}}(\mathbb{R}^n)} ||\hat{u}_0||_{L^s}^{\hat{q}} \\
&\quad + \phi(t)^{-\eta} \left\| |\cdot|^{\hat{q}|\beta|-\hat{q}\sigma} e^{-\frac{\hat{q}}{2C}|\cdot|^{\nu}} \right\|_{L^{\hat{p}}(\mathbb{R}^n)} ||\hat{u}_1||_{L^s}^{\hat{q}}. \qquad (2.74)
\end{aligned}$$

Using Lemma B.1.4 with $k_1 = \hat{q}|\beta|$ or $k_1 = \hat{q}|\beta| - \hat{q}\sigma$, $k_2 = 0$, $\omega = \nu > 0$, $\tau = \frac{\hat{q}}{2C}$ and Hausdorff-Young inequality:

$$\int_{Z^{high}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}} d\xi \lesssim \phi(t)^{-\eta}||u_0||_{L^p}^{\hat{q}} + \phi(t)^{-\eta}||u_1||_{L^p}^{\hat{q}}. \qquad (2.75)$$

Suppose now $\theta = 0$. In this case, since $\omega\hat{q} > \frac{n}{p}(p + \hat{q} - p\hat{q}) = \frac{n}{r}$, we have $\left\| |\cdot|^{-\omega\hat{q}} \right\|_{L^r(\mathbb{R}^n \setminus B_R)} \lesssim 1$. Using inequality (2.73), Holder inequality and Hausdorff-Young inequality we have:

$$\begin{aligned}
\int_{Z^{high}} |\xi|^{\hat{q}|\beta|}|\hat{u}(t,\xi)|^{\hat{q}} d\xi &\lesssim e^{-\frac{\hat{q}}{2C}\int_{t_0}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|}|\hat{u}_0|^{\hat{q}} d\xi \qquad (2.76) \\
&\quad + e^{-\frac{\hat{q}}{2C}\int_{t_0}^t f(s)ds} \int_{Z^{high}} |\xi|^{\hat{q}|\beta|-\hat{q}\sigma}|\hat{u}_1|^{\hat{q}} d\xi \\
&\lesssim \phi(t)^{-\eta} \left\| |\cdot|^{-\omega\hat{q}} \right\|_{L^r(\mathbb{R}^n \setminus B_R)} \| \, |\cdot|^{|\beta|+\omega} \hat{u}_0 \, \|_{L^s}^{\hat{q}} \\
&\quad + \phi(t)^{-\eta} \left\| |\cdot|^{-\omega\hat{q}} \right\|_{L^r(\mathbb{R}^n \setminus B_R)} \| \, |\cdot|^{|\beta|-\sigma+\omega} \hat{u}_1 \, \|_{L^s}^{\hat{q}} \\
&\lesssim \phi(t)^{-\eta}||u_0||_{W^{|\beta|+\omega,p}}^{\hat{q}} + \phi(t)^{-\eta}||u_1||_{W^{|\beta|-\sigma+\omega,p}}^{\hat{q}}.
\end{aligned}$$

The proof of estimate for $\int_{Z^{high}} |\xi|^{\hat{q}|\mu|}|\hat{u}_t(t,\xi)|^{\hat{q}} d\xi$ is analogous, except for $\alpha = 1$.

For this last case, since $|\xi| \geq R$, we apply Proposition 2.2.7:

$$\begin{aligned}
&\int_{Z^{high}} |\xi|^{\hat{q}|\mu|} |\hat{u}_t(t,\xi)|^{\hat{q}} d\xi \\
&\quad \lesssim e^{-\frac{\hat{q}}{C} R^{2\theta}\varphi(t)} \int_{Z^{high}} |\xi|^{\hat{q}|\mu|+2\sigma\hat{q}} e^{-\frac{\hat{q}}{C}|\xi|^{\nu}} |\hat{u}_0|^{\hat{q}} d\xi \qquad (2.77)\\
&\qquad + e^{-\frac{\hat{q}}{C} R^{2\theta}\varphi(t)} \int_{Z^{high}} |\xi|^{\hat{q}|\mu|+\sigma\hat{q}} e^{-\frac{\hat{q}}{C}|\xi|^{\nu}} |\hat{u}_1|^{\hat{q}} d\xi \\
&\qquad + \frac{1}{g(t)^{\hat{q}}} e^{-\frac{\hat{q}}{C} R^{\nu} \int_{t_0}^{t} \frac{1}{g(s)} ds} \int_{Z^{high}} |\xi|^{\hat{q}|\mu|+\hat{q}(2\sigma-2\theta)} e^{-\frac{\hat{q}}{C}|\xi|^{\nu}} |\hat{u}_0|^{\hat{q}} d\xi \\
&\qquad + \frac{1}{g(t)^{\hat{q}}} e^{-\frac{\hat{q}}{C} R^{\nu} \int_{t_0}^{t} \frac{1}{g(s)} ds} \int_{Z^{high}} |\xi|^{\hat{q}|\mu|+\hat{q}(\sigma-2\theta)} e^{-\frac{\hat{q}}{C}|\xi|^{\nu}} |\hat{u}_1|^{\hat{q}} d\xi.
\end{aligned}$$

Taking in account that $e^{-\frac{\hat{q}}{C} R^{2\theta}\varphi(t)}$ has an exponential behavior, for $t \geq t_0^*$ holds:

$$e^{-\frac{\hat{q}}{C} R^{2\theta}\varphi(t)} \lesssim \frac{\phi(t)^{-\eta}}{g(t)^{\hat{q}}}. \qquad (2.78)$$

For $\theta \neq 0$, using inequality (2.77), inequality (2.78) and proceeding in a similar way as in inequalities (2.74) and (2.75) the result follows.

To conclude the proof for $\theta = 0$, we use inequality (2.77), inequality (2.78) and proceed in a similar way as in inequality (2.76) (taking care with the different regularity) and therefore the result follows.

$\square$

Proof of Theorem 2.1.1 : Let $\hat{q}$ conjugate of q, $v \in \{u, u_t\}$ and $\eta \in \{\beta, \mu\}$, by Hausdorff-Young inequality, we have:

$$||D^{\eta} v(t,\cdot)||_{L^q} \lesssim ||\,|\cdot|^{\eta} \hat{v}(t,\cdot)||_{L^{\hat{q}}} \lesssim \left(\int_{\mathbb{R}^n} |\xi|^{\hat{q}|\eta|} |\hat{v}(t,\xi)|^{\hat{q}} d\xi \right)^{\frac{1}{\hat{q}}}. \qquad (2.79)$$

It should be noticed that if $\eta = \beta$, $v = u$ and $u_1 \neq 0$, we have the restriction $|\beta| - \frac{2\theta}{1+\alpha} + n\left(\frac{1}{p} - \frac{1}{q}\right) = |\beta| - \frac{2\theta}{1+\alpha} + \frac{n}{p\hat{q}}(p + \hat{q} - p\hat{q}) \geq 0$ with the strict inequality or equality depending on the case. For each fixed t, we separate the integral in inequality (2.79) in four parts, that is, low zone (for $\theta \neq 0$, otherwise low zone is empty since t_0 is large), elliptic zone, hyperbolic zone (if g is decreasing) and high zone. By Hypothesis B we have $\phi(t)^{\frac{1}{2\sigma-2\theta}} \lesssim \varphi(t)^{\frac{1}{2\theta}}$ and applying Propositions 2.3.1, 2.3.2, 2.3.3 and 2.3.4 the theorem follows.

Appendix A

Lemma A.1.1 *Let $t_0^* > t_0$, $I = (t_0, \infty)$ or $I = (t_0, t_0^*)$, $f \in C^1(I, \mathbb{R}) \cap C(\bar{I}, (0, \infty))$ and $\psi \in C^1(I, \mathbb{R}) \cap C(\bar{I}, \mathbb{R})$ with $\frac{1}{\psi} \in L^1_{loc}(\bar{I}, \mathbb{R})$ such that there exists $C_0 < 1$ satisfying:*

$$\left|1 - \psi'(t) - \frac{f'(t)}{f(t)}\psi(t)\right| \leq C_0, \quad \forall\, t > t_0.$$

Thus,

$$\begin{aligned} \frac{1}{1+C_0}\left\{f(t)\psi(t) - f(a)\psi(a)\right\} &\leq \int_a^t f(s)ds \\ &\leq \frac{1}{1-C_0}\left\{f(t)\psi(t) - f(a)\psi(a)\right\}, \end{aligned}$$

for all $t \geq a \geq t_0$ such that $t \in \bar{I}$.

Proof. Let $\lambda(t) := e^{\int_{t_0}^t \frac{1}{\psi(s)}ds}$. Thus,

$$J := \int_a^t f(s)ds = \int_a^t \frac{f(s)}{\lambda'(s)}\lambda'(s)ds = \frac{f(s)}{\lambda'(s)}\lambda(s)\bigg|_a^t + \int_a^t \varphi(s)f(s)ds, \tag{A.1}$$

where $\varphi(t) = \left[\frac{\lambda(t)\lambda''(t)}{(\lambda'(t))^2} - \frac{f'(t)\lambda(t)}{f(t)\lambda'(t)}\right] = 1 - \psi'(t) - \frac{f'(t)}{f(t)}\psi(t)$. Therefore,

$$J \leq f(t)\psi(t) - f(a)\psi(a) + C_0 J.$$

Solving for J, we conclude that:

$$\int_a^t f(s)ds \le \frac{1}{1-C_0}\left\{f(t)\psi(t) - f(a)\psi(a)\right\}.$$

On the other hand, by equation (A.1)

$$J \ge f(t)\psi(t) - f(a)\psi(a) - C_0 J.$$

Again, solving J we conclude the result.

□

Lemma A.1.2 *Let $\tilde{g}(t) := (1+t)^\alpha ln^\gamma(1+t)$, with $\alpha, \gamma \in \mathbb{R}$ and defined for $t \ge t_0$ big enough.*

If $\alpha < 1$,

$$\begin{aligned} \frac{2}{3(1-\alpha)}\left\{\frac{(1+t)}{\tilde{g}(t)} - \frac{(1+a)}{\tilde{g}(a)}\right\} \quad &\le \quad \int_a^t \frac{1}{\tilde{g}(s)}ds \\ &\le \quad \frac{2}{(1-\alpha)}\left\{\frac{(1+t)}{\tilde{g}(t)} - \frac{(1+a)}{\tilde{g}(a)}\right\}, \end{aligned}$$

for all $t \ge a \ge t_0 \ge e^{\frac{2|\gamma|}{(1-\alpha)}} - 1$. On other hand, if $\alpha > -1$,

$$\begin{aligned} &\frac{2}{3(1+\alpha)}\left\{(1+t)\tilde{g}(t) - (1+a)\tilde{g}(a)\right\} \\ &\qquad \le \int_a^t \tilde{g}(s)ds \le \frac{2}{(1+\alpha)}\left\{(1+t)\tilde{g}(t) - (1+a)\tilde{g}(a)\right\}, \end{aligned}$$

for all $t \ge a \ge t_0 \ge e^{\frac{2|\gamma|}{(1+\alpha)}} - 1$.

Proof. For the first part, choose $f(t) := \frac{1}{g(t)}$ and $\psi(t) := \frac{(1+t)}{1-\alpha}$, for the second part choose $f(t) := g(t)$ and $\psi(t) := \frac{(1+t)}{1+\alpha}$, and apply Lemma A.1.1, for both cases with $C_0 := \frac{1}{2}$.

□

Corollary A.1.1 *Let* $g(t) = (1+t)^{\alpha} ln^{\gamma}(1+t)$*, defined for* $t \geq t_0$*, with* $\alpha \in (-1, 1)$ *and* $\gamma \in \mathbb{R}$*,* $\alpha = -1$ *and* $\gamma \geq -1$ *or* $\alpha = 1$ *and* $\gamma \leq 1$*. For* $\phi(t)$ *as defined in Theorem 2.1.1, there exist* c_0 *and* c_1 *postive, such that:*

$$c_0\phi(t) \leq \int_{t_0}^{t} \frac{1}{g(s)} ds \leq c_1\phi(t),$$

for all $t \geq t_0^* \geq \phi^{-1}(2\phi(t_0))$*. On the other hand, there exist* $c_0 > 0$ *and* $c_1 > 0$ *such that*

$$c_0\varphi(t) \leq \int_{t_0}^{t} g(s) ds \leq c_1\varphi(t),$$

for all $t \geq t_0^* \geq \varphi^{-1}(2\varphi(t_0))$*, where* φ *is defined in* (1.27)*.*

Proof. For $\alpha \in [-1, 1)$, to estimate the right side of the first inequality we just apply Lemma A.1.2. The left side is obtained also using Lemma A.1.2 and using that, for $t \geq t_0^*$, we have $\frac{1}{2}\frac{(1+t)}{g(t)} - \frac{(1+t_0)}{g(t_0)} = \frac{1}{2}\phi(t) - \phi(t_0) \geq 0$.

For the case $\alpha = 1$, we observe that $\int_{t_0}^{t} \frac{1}{g(s)} ds = \frac{1}{1-\gamma} ln^{1-\gamma}(1+t) - \frac{1}{1-\gamma} ln^{1-\gamma}(1+t_0)$ if $\gamma < 1$, $\int_{t_0}^{t} \frac{1}{g(s)} ds = ln(ln(1+t)) - ln(ln(1+t_0))$ if $\gamma = 1$, the left side is proved using again the restriction $t_0^* \geq \phi^{-1}(2\phi(t_0))$. On the other hand, the estimate of the right side is trivial.

For the right side of the second inequality and $\alpha \in (-1, 1)$ we again apply Lemma A.1.2. For the left side, we use that for $t \geq t_0^*$ holds $\frac{1}{2}(1+t)g(t) - (1+t_0)g(t_0) = \frac{1}{2}\varphi(t) - \varphi(t_0) \geq 0$.

Finally, for the second inequality and $\alpha = -1$ we observe that $\int_{t_0}^{t} g(s) ds = \frac{1}{1+\gamma} ln^{1+\gamma}(1+t) - \frac{1}{1+\gamma} ln^{1+\gamma}(1+t_0)$ if $\gamma > -1$, $\int_{t_0}^{t} g(s) ds = ln(ln(1+t)) - ln(ln(1+t_0))$ if $\gamma = -1$, the left side is proved using again the restriction $t_0^* \geq \varphi^{-1}(2\varphi(t_0))$. The estimate of the right side is trivial.

□

Appendix B

Lemma B.1.1 *For $\tau > 0$, $r \in [1, \infty]$ and $k + \frac{n}{r} > 0$ holds:*

$$\left\| |\cdot|^k \right\|_{L^r(B_\tau)} \lesssim \tau^{k+\frac{n}{r}}.$$

Proof. Initially, consider $r \in [1, \infty)$. Therefore,

$$\begin{aligned} \left\| |\cdot|^k \right\|_{L^r(B_\tau)} &= \left(\int_{B_\tau} |\xi|^{kr} d\xi \right)^{\frac{1}{r}} = \left(\int_0^\tau \left(\int_{\partial S(0,\rho)} \rho^{kr} dS_\rho \right) d\rho \right)^{\frac{1}{r}} \\ &= c_0(n)^{\frac{1}{r}} \left(\int_0^\tau \rho^{kr+n-1} d\rho \right)^{\frac{1}{r}} \leq c(n,r,k)\, \tau^{k+\frac{n}{r}}. \end{aligned}$$

For $r = \infty$ by symmetry is sufficient see that $\sup_{0 \leq x \leq \tau} x^k = \tau^k$.

□

Lemma B.1.2 *For $\tau > 0$, $r \in [1, \infty]$, $k_1 + \frac{n}{r} > 0$ and $k_2 > 0$, holds:*

$$\left\| |\cdot|^{k_1} e^{-|\cdot|^{k_2}\tau} \right\|_{L^r(\mathbb{R}^n)} \lesssim \tau^{-\frac{k_1+\frac{n}{r}}{k_2}}.$$

Proof. Initially, consider $r \in [1,\infty)$. Therefore,

$$\begin{aligned}\left\| |\cdot|^{k_1} e^{-|\cdot|^{k_2}\tau} \right\|_{L^r(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |\xi|^{k_1 r} e^{-r|\xi|^{k_2}\tau} d\xi \right)^{\frac{1}{r}} \\ &= \left(\int_0^\infty \left(\int_{\partial S(0,\rho)} \rho^{k_1 r} e^{-r\rho^{k_2}\tau} dS_\rho \right) d\rho \right)^{\frac{1}{r}} \\ &= c_0(n)^{\frac{1}{r}} \left(\int_0^\infty \rho^{k_1 r+n-1} e^{-r\rho^{k_2}\tau} d\rho \right)^{\frac{1}{r}}. \quad \text{(B.1)}\end{aligned}$$

Making $s = \rho^{k_2}\tau$, we have $\rho = \left(\frac{s}{\tau}\right)^{\frac{1}{k_2}}$, $d\rho = \frac{1}{k_2\tau}\rho^{1-k_2} ds$ and therefore $\rho^{k_1 r+n-1} d\rho = \frac{1}{k_2}\tau^{-\frac{k_1 r+n}{k_2}} s^{\frac{k_1 r+n}{k_2}-1}\, ds$. Applying this change of variable in equation (B.1), we have:

$$\left\| |\cdot|^{k_1} e^{-|\cdot|^{k_2}\tau} \right\|_{L^r(\mathbb{R}^n)} = \left(\frac{c_0(n)}{k_2} \right)^{\frac{1}{r}} \tau^{-\frac{k_1+\frac{n}{r}}{k_2}} \left(\int_0^\infty s^{\frac{k_1 r+n}{k_2}-1} e^{-rs} ds \right)^{\frac{1}{r}}. \quad \text{(B.2)}$$

To estimate the integral in the right side of equation (B.2), we just remember that for $q > -1$,

$$\int_0^1 s^q e^{-rs} ds \le \int_0^1 s^q ds < \infty.$$

Furthermore, $\int_1^\infty s^q e^{-rs} ds < \infty$. Since $\frac{k_1 r+n}{k_2} - 1 > -1$, these remarks with equation (B.2) finishes the proof for $r \in [1,\infty)$.

For $r = \infty$, by symmetry is sufficient consider $v(x) := x^{k_1} e^{-x^{k_2}\tau}$ with $x \ge 0$. Notice that $v(0) = 0$, and $v'(x) = \frac{v(x)}{x}(k_1 - k_2\tau x^{k_2})$ therefore v is increasing for $x \le x_c := \left(\frac{k_1}{k_2\tau}\right)^{\frac{1}{k_2}}$ and is decreasing for $x \ge x_c$. In this circumstances x_c is a global maximum of v. Therefore,

$$\sup_{x\ge 0} v(x) = v(x_c) = c(k_1,k_2)\tau^{-\frac{k_1}{k_2}}.$$

□

Lemma B.1.3 *Consider $r \in [1,\infty]$. Interpreting $\frac{1}{\infty}$ as 0, let $\tau \in (0,\tau_0)$ in which $\tau_0 < e^{-\frac{2|k_2|}{k_1+\frac{n}{r}}}$ if $k_1+\frac{n}{r}>0$, $\tau_0<1$ if $k_1+\frac{n}{r}=0$. Then,*

$$\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_\tau)} \lesssim \begin{cases} \tau^{k_1+\frac{n}{r}} ln^{k_2}\left(\frac{1}{\tau}\right), & \text{if } k_1+\frac{n}{r}>0, \\ ln^{\frac{1}{r}+k_2}\left(\frac{1}{\tau}\right), & \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2<-\frac{1}{r}. \end{cases} \tag{B.3}$$

Furthermore, setting $\hat{B}_{\tau,\tau_0} := \{\xi\in\mathbb{R}^n : \tau\le|\xi|<\tau_0\}$, and $\tau_0 < e^{\frac{2|k_2|}{k_1+\frac{n}{r}}}$ if $k_1+\frac{n}{r}<0$, $\tau_0<1$ if $k_1+\frac{n}{r}=0$ and $k_2>-\frac{1}{r}$, $\tau_0<\frac{1}{e}$ if $k_1+\frac{n}{r}=0$ and $k_2=-\frac{1}{r}$. Therefore,

$$\begin{aligned} &\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} \\ &\lesssim \begin{cases} \tau^{k_1+\frac{n}{r}} ln^{k_2}\left(\frac{1}{\tau}\right), & \text{if } k_1+\frac{n}{r}<0, \\ ln^{\frac{1}{r}+k_2}\left(\frac{1}{\tau}\right), & \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2>-\frac{1}{r}, \\ ln^{\frac{1}{r}}\left(ln\left(\frac{1}{\tau}\right)\right), & \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2=-\frac{1}{r}. \end{cases} \end{aligned} \tag{B.4}$$

If, in addition, $r<\infty$, $k_1+\frac{n}{r}=0$ and $-\frac{1}{r}<k_2\le 0$, we have:

$$\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} \lesssim ln^{\frac{1}{r}+k_2}\left(\frac{\tau_0}{\tau}\right). \tag{B.5}$$

Proof. **Estimates in B_τ for $1\le r<\infty$:**

$$\begin{aligned} \left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_\tau)} &= \left(\int_{|\xi|\le\tau} |\xi|^{k_1 r} ln^{k_2 r}\left(\frac{1}{|\xi|}\right) d\xi\right)^{\frac{1}{r}} \\ &= \left(\int_0^\tau \left(\int_{\partial S(0,\rho)} \rho^{k_1 r} ln^{k_2 r}\left(\frac{1}{\rho}\right) dS_\rho\right) d\rho\right)^{\frac{1}{r}} \\ &= c_{n,r}\left(\int_0^\tau \rho^{k_1 r+n-1} ln^{k_2 r}\left(\frac{1}{\rho}\right) d\rho\right)^{\frac{1}{r}} \\ &= c_{n,r}\left(\int_{\frac{1}{\tau}-1}^\infty (1+s)^{-(k_1 r+n+1)} ln^{k_2 r}(1+s)\, ds\right)^{\frac{1}{r}} \end{aligned} \tag{B.6}$$

in which the last equality is given by the transformation $1+s=\frac{1}{\rho}$.

For the case $k_1 r+n>0$, we set $k_3 := k_1 r+n+1>1$, $k_4 := k_2 r$,

$f(s) := (1+s)^{-k_3} ln^{k_4}(1+s)$ and $\psi(s) := -\frac{1+s}{k_3}$. The idea is to apply Lemma A.1.1:

$$1 - \psi'(s) - \frac{f'(s)}{f(s)}\psi(s) = \frac{k_4}{k_3}\frac{1}{ln(1+s)} + \frac{1}{k_3}.$$

Since $\tau < e^{-\frac{2|k_2|r}{k_1 r+n}} = e^{-\frac{2|k_4|}{k_3-1}}$, we have $ln(1+s) \geq ln\left(\frac{1}{\tau}\right) > \frac{2|k_4|}{k_3-1}$ and taking $C_0 := \frac{1}{2}\left(1+\frac{1}{k_3}\right) < 1$, we conclude for $s \geq \frac{1}{\tau} - 1$:

$$\left|1 - \psi'(s) - \frac{f'(s)}{f(s)}\psi(s)\right| = \left|\frac{k_4}{k_3}\frac{1}{ln(1+s)} + \frac{1}{k_3}\right| \leq C_0.$$

By Lemma A.1.1, we have:

$$\begin{aligned}\int_{\frac{1}{\tau}-1}^{\zeta} &(1+s)^{-(k_1 r+n+1)} ln^{k_2 r}(1+s)\, ds \\ &\lesssim \tau^{k_1 r+n} ln^{k_2 r}\left(\frac{1}{\tau}\right) - (1+\zeta)^{-(k_1 r+n)} ln^{k_2 r}(1+\zeta),\end{aligned}$$

making $\zeta \to \infty$ and replacing in equation (B.6) the result follows.

Now, consider $k_1 r + n = 0$ and $k_2 r < -1$, therefore:

$$\begin{aligned}\int_{\frac{1}{\tau}-1}^{\zeta} &\frac{1}{1+s} ln^{k_2 r}(1+s)\, ds \\ &= \frac{1}{1+k_2 r} ln^{1+k_2 r}(1+\zeta) - \frac{1}{1+k_2 r} ln^{1+k_2 r}\left(\frac{1}{\tau}\right),\end{aligned}$$

since $1 + k_2 r < 0$, making $\zeta \to \infty$ and replacing in equation (B.6) we conclude the proof of inequality (B.3) for $r \in [1, \infty)$.

Estimates in B_τ for $r = \infty$: Let $k_1 > 0$ and $k_2 \in \mathbb{R}$ or $k_1 = 0$ and $k_2 < 0$. Define

$$w(x) := \begin{cases} 0, & \text{if } x = 0, \\ x^{k_1} ln^{k_2}\left(\frac{1}{x}\right), & \text{if } 0 < x \leq \tau. \end{cases}$$

Then, considering our restrictions in τ, is not difficult to see that

w is increasing on $(0,\tau)$ and that w is continuous on $[0,\tau]$. Therefore, $\sup_{0\le x\le\tau} w(x) = w(\tau)$, that is, $\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^\infty(B_\tau)} = \tau^{k_1} ln^{k_2}\left(\frac{1}{\tau}\right)$ and inequality (B.3) is proved.

Estimates in $\hat{B}_{\tau,\tau_0}$ for $1 \le r < \infty$: Similarly to equation (B.6), we have:

$$\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} \\ = c_{n,r}\left(\int_{\frac{1}{\tau_0}-1}^{\frac{1}{\tau}-1}(1+s)^{-(k_1r+n+1)} ln^{k_2r}(1+s)\,ds\right)^{\frac{1}{r}}.$$

Consider the case $k_1+\frac{n}{r}<0$. Setting $\alpha := -(k_1r+n+1) > -1$, $\gamma := k_2r$ and using second inequality of Lemma A.1.2 with $a := \frac{1}{\tau_0}-1 > e^{-\frac{2|k_2|r}{k_1r+n}}-1 = e^{\frac{2|\gamma|}{1+\alpha}}-1:$

$$\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} \\ = c_{n,r}\left(\int_{\frac{1}{\tau_0}-1}^{\frac{1}{\tau}-1}(1+s)^{-(k_1r+n+1)} ln^{k_2r}(1+s)\,ds\right)^{\frac{1}{r}} \\ \lesssim \tau^{k_1+\frac{n}{r}} ln^{k_2}\left(\frac{1}{\tau}\right).$$

Now, consider $k_1+\frac{n}{r}=0$ and $k_2 > -\frac{1}{r}$. Therefore,

$$\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} \\ = c_{n,r}\left(\int_{\frac{1}{\tau_0}-1}^{\frac{1}{\tau}-1}\frac{1}{1+s} ln^{k_2r}(1+s)\,ds\right)^{\frac{1}{r}} \\ = c_{n,r}\left(\frac{1}{1+k_2r} ln^{1+k_2r}\left(\frac{1}{\tau}\right) - \frac{1}{1+k_2r} ln^{1+k_2r}\left(\frac{1}{\tau_0}\right)\right)^{\frac{1}{r}} \\ \lesssim ln^{\frac{1}{r}+k_2}\left(\frac{1}{\tau}\right). \tag{B.7}$$

Consider $k_1 + \frac{n}{r} = 0$ and $k_2 = -\frac{1}{r}$. In this case,

$$\begin{aligned}
\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} &= c_{n,r}\left(\int_{\frac{1}{\tau_0}-1}^{\frac{1}{\tau}-1} \frac{1}{(1+s)ln(1+s)} ds\right)^{\frac{1}{r}} \\
&= c_{n,r}\left(ln\left(ln\left(\frac{1}{\tau}\right)\right) - ln\left(ln\left(\frac{1}{\tau_0}\right)\right)\right)^{\frac{1}{r}} \\
&\lesssim ln^{\frac{1}{r}}\left(ln\left(\frac{1}{\tau}\right)\right).
\end{aligned}$$

To prove inequality (B.5), we use the follow inequality $ln^{\omega}(x) - ln^{\omega}(y) \leq ln^{\omega}\left(\frac{x}{y}\right)$ for all $x \geq y \geq 1$ and $0 < \omega \leq 1$. To check this inequality fix $y \geq 1$ and define $h(x) := (ln(x) - ln(y))^{\omega} - ln^{\omega}(x) + ln^{\omega}(y)$, straightforward calculations shows that $h(y) = 0$ and $h'(x) \geq 0$ for $x \geq y$ and therefore the inequality is proved. Using this inequality with $x = \frac{1}{\tau}$, $y = \frac{1}{\tau_0}$ and $\omega = 1 + k_2 r \leq 1$ and proceeding as first equality used to prove inequality (B.7) we have:

$$\begin{aligned}
\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} &= c_{n,r}\left(\int_{\frac{1}{\tau_0}-1}^{\frac{1}{\tau}-1} \frac{1}{1+s} ln^{k_2 r}(1+s)\, ds\right)^{\frac{1}{r}} \\
&= c_{n,r,k_2}\left(ln^{1+k_2 r}\left(\frac{1}{\tau}\right) - ln^{1+k_2 r}\left(\frac{1}{\tau_0}\right)\right)^{\frac{1}{r}} \\
&\lesssim ln^{\frac{1}{r}+k_2}\left(\frac{\tau_0}{\tau}\right).
\end{aligned}$$

Estimates in $\hat{B}_{\tau,\tau_0}$ for $r = \infty$: For $0 < \tau \leq x < \tau_0$, consider $w(x) := x^{k_1} ln^{k_2}\left(\frac{1}{x}\right)$. Since w is non-increasing on (τ, τ_0), then

$$\sup_{0<\tau\leq x<\tau_0} w(x) = w(\tau).$$

That is, $\left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^\infty\left(\hat{B}_{\tau,\tau_0}\right)} = \tau^{k_1} ln^{k_2}\left(\frac{1}{\tau}\right)$.

$\square$

Lemma B.1.4 *Let $r \in [1,\infty]$, k_1 and k_2 real numbers, $c, \omega > 0$ and $R \leq \left(\frac{1}{e}\right)^{\frac{1}{\omega}}$. Then, there exists $M > 0$ such that, for all $\tau > 1$,*

$$\left\| |\cdot|^{k_1} e^{-c\tau|\cdot|^{\omega}} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q(\tau))} \lesssim \begin{cases} \tau^{-\frac{k_1+\frac{n}{r}}{\omega}} ln^{k_2}\left(\frac{\tau}{M}\right) & \text{if } r \in [1,\infty), \\ \tau^{-\frac{k_1}{\omega}} ln^{k_2}\left(\frac{\tau}{M}\right) & \text{if } \ r = \infty, \end{cases} \tag{B.8}$$

where $Q(\tau) = Q(\tau, M) := \{\xi \in B_R : |\xi|^{\omega}\tau \geq M\}$.

Proof. Initially, consider $r \in [1,\infty)$. We have:

$$\left\| |\cdot|^{k_1} e^{-c\tau|\cdot|^{\omega}} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q(\tau))} \tag{B.9}$$

$$= \left(\int_{Q(t)} |\xi|^{k_1 r} e^{-cr\tau|\xi|^{\omega}} ln^{k_2 r}\left(\frac{1}{|\xi|}\right) d\xi\right)^{\frac{1}{r}}$$

$$= \left(\int_{M^{\frac{1}{\omega}}\tau^{-\frac{1}{\omega}}}^{R} \left(\int_{\partial S(0,\rho)} \rho^{k_1 r} e^{-cr\tau\rho^{\omega}} ln^{k_2 r}\left(\frac{1}{\rho}\right) dS_{\rho}\right) d\rho\right)^{\frac{1}{r}}$$

$$= c_0(n)^{\frac{1}{r}} \left(\int_{M^{\frac{1}{\omega}}\tau^{-\frac{1}{\omega}}}^{R} \rho^{k_1 r + n - 1} e^{-cr\tau\rho^{\omega}} ln^{r k_2}\left(\frac{1}{\rho}\right) d\rho\right)^{\frac{1}{r}}.$$

Let $s = \tau\rho^{\omega}$. Therefore $\rho = \left(\frac{s}{\tau}\right)^{\frac{1}{\omega}}$ and $d\rho = \frac{1}{\tau\omega}\rho^{1-\omega} ds$. After straightforward calculations, we conclude that

$$\rho^{k_1 r + n - 1} d\rho = \frac{1}{\omega}\tau^{-\frac{k_1 r + n}{\omega}} s^{\frac{k_1 r + n}{\omega} - 1} ds.$$

Applying this change of variable in equation (B.9), we have:

$$\left\| |\cdot|^{k_1} e^{-c\tau|\cdot|^{\omega}} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q(\tau))} \tag{B.10}$$

$$= \left(\frac{1}{\omega}\right)^{k_2} \left(\frac{c_0(n)}{\omega}\right)^{\frac{1}{r}} \tau^{-\frac{k_1+\frac{n}{r}}{\omega}} \left(\int_{M}^{\tau R^{\omega}} s^{\frac{k_1 r + n}{\omega} - 1} e^{-crs} ln^{k_2 r}\left(\frac{\tau}{s}\right) ds\right)^{\frac{1}{r}}.$$

In the next step we shall estimate the integral in the right side of equation (B.10). The idea is apply Lemma A.1.1: let $f(s) := s^p e^{-qs} ln^j\left(\frac{\tau}{s}\right)$ with $p := \frac{k_1 r + n}{\omega} - 1$, $q := cr$ and $j := k_2 r$. In this case,

$$\frac{f'(s)}{f(s)} = \frac{p}{s} - q - \frac{j}{sln\left(\frac{\tau}{s}\right)}.$$

For $\psi(s) := -\frac{2}{3q}$, we have:

$$1 - \psi'(s) - \frac{f'(s)}{f(s)}\psi(s) = \frac{1}{3} + \left(\frac{2p}{3qs} - \frac{2j}{3qsln\left(\frac{\tau}{s}\right)}\right). \tag{B.11}$$

Since $R \leq \left(\frac{1}{e}\right)^{\frac{1}{\omega}}$, in the limits of integration of integral (B.10) we have $s \leq \frac{1}{e}\tau$. That is, $0 < \frac{1}{ln\left(\frac{\tau}{s}\right)} \leq 1$. Furthermore, set $M \geq \frac{2(|p|+|j|)}{q} = \frac{2\left(\left|\frac{k_1+\frac{n}{r}}{\omega} - \frac{1}{r}\right| + |k_2|\right)}{c}$. Since in the integral (B.10) holds $s \geq M$, we have:

$$\left|\frac{2p}{3qs} - \frac{2j}{3qsln\left(\frac{\tau}{s}\right)}\right| \leq \frac{2(|p|+|j|)}{3q}\frac{1}{s} \leq \frac{1}{3}.$$

Applying last inequality in equation (B.11) and using Lemma A.1.1 with $I := [M, \tau R^{\omega}]$, we conclude:

$$\begin{aligned}
&\left(\int_M^{\tau R^{\omega}} s^{\frac{k_1 r+n}{\omega}-1} e^{-crs} ln^{k_2 r}\left(\frac{\tau}{s}\right) ds\right)^{\frac{1}{r}} \\
&\leq 3^{\frac{1}{r}} \left(f(M)\psi(M)\right)^{\frac{1}{r}} \\
&= \left(\frac{2}{cr}\right)^{\frac{1}{r}} M^{\frac{k_1+\frac{n}{r}}{\omega}-\frac{1}{r}} e^{-cM} ln^{k_2}\left(\frac{\tau}{M}\right).
\end{aligned} \tag{B.12}$$

Using inequality (B.12) in equation (B.10), we have:

$$\begin{aligned}
&\left\| |\cdot|^{k_1} e^{-c\tau|\cdot|^{\omega}} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q(\tau))} \\
&\leq \left(\frac{c_1(n,\omega,k_2)}{r}\right)^{\frac{1}{r}} M^{\frac{k_1+\frac{n}{r}}{\omega}-\frac{1}{r}} e^{-cM} \tau^{-\frac{k_1+\frac{n}{r}}{\omega}} ln^{k_2}\left(\frac{\tau}{M}\right).
\end{aligned} \tag{B.13}$$

By inequality (B.13) the result is proved for $r \in [1, \infty)$.

For $(k_1, k_2) = (0, 0)$ the inequality (B.8) for $r = \infty$ is trivial. If

$(k_1, k_2) \neq (0,0)$,

$$\lim_{r\to\infty} \frac{2\left(\left|\frac{k_1+\frac{n}{r}}{\omega} - \frac{1}{r}\right| + |k_2|\right)}{c} = \frac{2\left(\left|\frac{k_1}{\omega}\right| + |k_2|\right)}{c},$$

and therefore there exists a uniform lower bound $M^*(k_1, k_2, \omega)$ such that:

$$M \geq M^* \geq \frac{2\left(\left|\frac{k_1+\frac{n}{r}}{\omega} - \frac{1}{r}\right| + |k_2|\right)}{c}$$

for all r big enough. Now, the restriction in M doesn't depend on r and we have the limit:

$$\lim_{r\to\infty} \left(\frac{c_1(n,\omega,k_2)}{r}\right)^{\frac{1}{r}} M^{\frac{k_1+\frac{n}{r}}{\omega}-\frac{1}{r}} e^{-cM} = M^{\frac{k_1}{\omega}} e^{-cM}. \tag{B.14}$$

Since $\tau > 1$, follows $\tau^{-\frac{n}{\omega r}} \leq 1$ for all $r \in [1,\infty)$. Using the last inequality and the limit (B.14) in inequality (B.13), there exist $C(M, k_1, \omega, c)$ and $M \geq M^*(k_1, k_2, \omega)$ such that

$$\left\| |\cdot|^{k_1} e^{-c\tau|\cdot|^{\omega}} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q(\tau))} \leq C\tau^{-\frac{k_1}{\omega}} ln^{k_2}\left(\frac{\tau}{M}\right), \tag{B.15}$$

for all r big enough. Since this estimate is uniform on r, the inequality (B.15) holds for $r = \infty$.

$\square$

Lemma B.1.5 *Let $r \in [1,\infty]$, k_1 and k_2 real numbers, $c > 0$ and $0 \leq \theta < \sigma$. Then, there exist $0 < R \leq \left(\frac{1}{e}\right)^{\frac{1}{2\sigma-2\theta}}$, $M \geq N > 0$ and $t_0^* \geq t_0$ such that, for all $t \geq t_0^*$ hold:*

Case I: *Let $k_1 + \frac{n}{r} > 0$ or $k_1 + \frac{n}{r} = 0$ and $k_2 < -\frac{1}{r}$. Therefore:*

$$\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_R)} \lesssim \begin{cases} \phi(t)^{-\frac{k_1+\frac{n}{r}}{2\sigma-2\theta}} ln^{k_2}\left(\frac{\phi(t)}{M}\right), & \text{if } k_1 + \frac{n}{r} > 0, \\ ln^{k_2+\frac{1}{r}}\left(\frac{\phi(t)}{M}\right), & \text{if } k_1 + \frac{n}{r} = 0 \text{ and } k_2 < -\frac{1}{r}. \end{cases}$$

Case II: *Let $k_1 + \frac{n}{r} < 0$ or $k_1 + \frac{n}{r} = 0$ and $k_2 \geq -\frac{1}{r}$. Therefore, for $\theta \neq 0$:*

$$\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B(t))} \tag{B.16}$$
$$\lesssim \begin{cases} \varphi(t)^{-\frac{k_1+\frac{n}{r}}{2\theta}} ln^{k_2}\left(\frac{\varphi(t)}{N}\right), & \text{if } k_1 + \frac{n}{r} < 0, \\ ln^{\frac{1}{r}+k_2}\left(\frac{\varphi(t)}{N}\right), & \text{if } k_1 + \frac{n}{r} = 0 \text{ and } k_2 > -\frac{1}{r}, \\ ln^{\frac{1}{r}}\left(ln\left(\frac{\varphi(t)}{N}\right)\right), & \text{if } k_1 + \frac{n}{r} = 0 \text{ and } k_2 = -\frac{1}{r}, \end{cases}$$

where $B(t) := \{\xi \in B_R : |\xi|^{2\theta}\varphi(t) \geq N\}$. If, in addition, $r < \infty$, $k_1 + \frac{n}{r} = 0$, $-\frac{1}{r} < k_2 \leq 0$ and assuming Hypothesis B we have:

$$\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B(t))}$$
$$\lesssim ln^{\frac{1}{r}+k_2}\left(\frac{M^{\frac{1}{2\sigma-2\theta}}\phi(t)^{-\frac{1}{2\sigma-2\theta}}\varphi(t)^{\frac{1}{2\theta}}}{N^{\frac{1}{2\theta}}}\right). \tag{B.17}$$

Proof.
Case I: Let $M \geq N$ as in Lemma B.1.4. Therefore in the case I we set:

$$\begin{aligned} B_R &= \{\xi \in B_R : |\xi|^{2\sigma-2\theta}\phi(t) \leq M\} \cup \{\xi \in B_R : |\xi|^{2\sigma-2\theta}\phi(t) \geq M\} \\ &=: Q_1(t) \cup Q_2(t). \end{aligned} \tag{B.18}$$

Estimates in $Q_1(t)$: Let $\tau := \left(\frac{M}{\phi(t)}\right)^{\frac{1}{2\sigma-2\theta}}$, $\tau_0 < e^{-\frac{2|k_2|}{k_1+\frac{n}{r}}}$ if $k_1 + \frac{n}{r} > 0$ and $\tau_0 < 1$ if $k_1 + \frac{n}{r} = 0$. In order to ensure the condition $\tau \in (0, \tau_0)$ we need $t \geq t_0^* > \phi^{-1}\left(\frac{M}{\tau_0^{2\sigma-2\theta}}\right)$.

Assuming this restriction in t_0^*, by applying Lemma B.1.3, that is, using inequality (B.3) , we have:

$$\begin{aligned}
&\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q_1(t))} \\
&\lesssim \left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B_\tau)} \qquad (B.19)\\
&\lesssim \begin{cases} \phi(t)^{-\frac{k_1+\frac{n}{r}}{2\sigma-2\theta}} ln^{k_2}\left(\frac{\phi(t)}{M}\right), & \text{if } k_1+\frac{n}{r}>0,\\ ln^{\frac{1}{r}+k_2}\left(\frac{\phi(t)}{M}\right), & \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2<-\frac{1}{r}. \end{cases}
\end{aligned}$$

Estimates in $Q_2(t)$: We consider $\tau = \phi(t)$. Since t_0 is such that $\phi(t) > 1$ for all $t \geq t_0$ (see Remark 2.2.1), making the restriction $t_0^* \geq t_0$, let $\omega = 2\sigma - 2\theta$ and since $R \leq \left(\frac{1}{e}\right)^{\frac{1}{2\sigma-2\theta}}$, applying Lemma B.1.4 with $Q(\tau) := \left\{\xi \in B_R : |\xi|^{2\sigma-2\theta}\tau \geq M\right\}$, follows:

$$\begin{aligned}
&\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q_2(t))} \\
&= \left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q(\tau))} \\
&\lesssim \phi(t)^{-\frac{k_1+\frac{n}{r}}{2\sigma-2\theta}} ln^{k_2}\left(\frac{\phi(t)}{M}\right). \qquad (B.20)
\end{aligned}$$

In the case $k_1+\frac{n}{r}=0$ and $k_2 < -\frac{1}{r}$ we have $\phi(t)^{-\frac{k_1+\frac{n}{r}}{2\sigma-2\theta}} ln^{k_2}\left(\frac{\phi(t)}{M}\right) = ln^{k_2}\left(\frac{\phi(t)}{M}\right) \leq ln^{\frac{1}{r}+k_2}\left(\frac{\phi(t)}{M}\right)$ for $t \geq t_0^* \geq \phi^{-1}(eM)$. Considering the intersection of the restrictions on t_0^* assumed in the estimates in $Q_1(t)$ and $Q_2(t)$, and by inequalities (B.19) and (B.20) the result follows for the case I.

Case II: Let $\tau_0 := R_0$ with $R_0 < \min\left\{e^{\frac{2|k_2|}{k_1+\frac{n}{r}}}, R\right\}$ if $k_1+\frac{n}{r} < 0$, $R_0 < \min\{1, R\}$ if $k_1 + \frac{n}{r} = 0$ and $k_2 > -\frac{1}{r}$, $R_0 < \min\{\frac{1}{e}, R\}$ if $k_1 + \frac{n}{r} = 0$ and $k_2 = -\frac{1}{r}$. Setting $\tau := \left(\frac{N}{\varphi(t)}\right)^{\frac{1}{2\theta}}$ and applying Lemma

B.1.3 with inequality (B.4), we have for $t \geq t_0^* > \varphi^{-1}\left(\frac{N}{\tau_0^{2\theta}}\right)$:

$$\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(B(t)\cap B_{R_0}\right)}$$

$$\lesssim \left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)} \tag{B.21}$$

$$\lesssim \begin{cases} \varphi(t)^{-\frac{k_1+\frac{n}{r}}{2\theta}} ln^{k_2}\left(\frac{\varphi(t)}{N}\right), & \text{if } k_1+\frac{n}{r}<0, \\ ln^{\frac{1}{r}+k_2}\left(\frac{\varphi(t)}{N}\right), & \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2>-\frac{1}{r}, \\ ln^{\frac{1}{r}}\left(ln\left(\frac{\varphi(t)}{N}\right)\right), & \text{if } k_1+\frac{n}{r}=0 \text{ and } k_2=-\frac{1}{r}. \end{cases}$$

On the other hand,

$$\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B(t)\cap\{\xi\in B_R:|\xi|\geq R_0\})} \lesssim e^{-cR_0^{2\sigma-2\theta}\phi(t)}. \tag{B.22}$$

Using inequalities (B.21) and (B.22) inequality (B.16) is proved.

To prove inequality (B.17) we consider Q_1 and Q_2 as in equality (B.18) and write: $B(t) = (B(t)\cap Q_1(t)) \cup (B(t)\cap Q_2(t))$.

Estimates in $B(t)\cap Q_1(t) = \{\xi \in B_R : |\xi|^{2\theta}\varphi(t) \geq N$ *and* $|\xi|^{2\sigma-2\theta}\phi(t) \leq M\}$ *:* Let $\tau := \left(\frac{N}{\varphi(t)}\right)^{\frac{1}{2\theta}}$ and $\tau_0 := \left(\frac{M}{\phi(t)}\right)^{\frac{1}{2\sigma-2\theta}} < 1$ for $t \geq t_0^* > \varphi^{-1}(M)$. Therefore in the notation of Lemma B.1.3 we have $B(t)\cap Q_1(t) = \hat{B}_{\tau,\tau_0}$ almost everywhere, and by inequality (B.5) follows for $t \geq t_0^*$:

$$\left\| |\cdot|^{k_1} e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B(t)\cap Q_1(t))} \tag{B.23}$$

$$\lesssim \left\| |\cdot|^{k_1} ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r\left(\hat{B}_{\tau,\tau_0}\right)}$$

$$\lesssim ln^{\frac{1}{r}+k_2}\left(\frac{M^{\frac{1}{2\sigma-2\theta}}\phi(t)^{-\frac{1}{2\sigma-2\theta}}\varphi(t)^{\frac{1}{2\theta}}}{N^{\frac{1}{2\theta}}}\right).$$

Estimates in $B(t)\cap Q_2(t)$: Since $B(t)\cap Q_2(t)\subset Q_2(t)$ and $k_1+\frac{n}{r}=0$, proceeding in a similar way of estimates in $Q_2(t)$ in the case I, we have for $t\geq t_0^*\geq t_0$:

$$\begin{aligned}
&\left\| |\cdot|^{k_1}e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)}ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(B(t)\cap Q_2(t))}\\
&\quad\lesssim \left\| |\cdot|^{k_1}e^{-c|\cdot|^{2\sigma-2\theta}\phi(t)}ln^{k_2}\left(\frac{1}{|\cdot|}\right)\right\|_{L^r(Q_2(t))}\\
&\quad\lesssim ln^{k_2}\left(\frac{\phi(t)}{M}\right). \qquad (B.24)
\end{aligned}$$

Finally, since $k_2\leq 0<\frac{1}{r}+k_2$ follows that

$$ln^{k_2}\left(\frac{\phi(t)}{M}\right)\lesssim ln^{\frac{1}{r}+k_2}\left(\frac{M^{\frac{1}{2\sigma-2\theta}}\phi(t)^{-\frac{1}{2\sigma-2\theta}}\varphi(t)^{\frac{1}{2\theta}}}{N^{\frac{1}{2\theta}}}\right),$$

because by Hypothesis B we have $\phi(t)^{\frac{1}{2\sigma-2\theta}}\lesssim\varphi(t)^{\frac{1}{2\theta}}$. Therefore, considering the intersection of the restrictions on t_0^* assumed in the estimates in $B(t)\cap Q_1(t)$ and $B(t)\cap Q_2(t)$, and by inequalities (B.23) and (B.24) inequality (B.17) follows.

□

Lemma B.1.6 *Let $0<\theta<\sigma$ and assume Hypothesis B. For g, φ and t_0 given by Corollary A.1.1 and $\omega\in\mathbb{R}$, the following estimate holds for $t\geq t_0$:*

$$\max\left\{\frac{1}{g(t)^2}\varphi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\theta}\right)}ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right),\varphi(t)^{-\frac{\omega}{\theta}}\right\}\lesssim\varphi(t)^{-\frac{\omega}{\theta}}.$$

Proof. Since $g(t)=(1+t)^{\alpha}ln^{\gamma}(1+t)$ and $\varphi(t)=(1+t)^{1+\alpha}ln^{\gamma}(1+t)$, we have $g(t) = \varphi(t)^{\frac{\alpha}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}(1+t)$. Therefore,

$$\begin{aligned}
&\frac{1}{g(t)^2}\varphi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) \\
&\quad = \varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{-\left(\frac{2\sigma-\frac{4\theta}{1+\alpha}}{\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) ln^{-\frac{2\gamma}{1+\alpha}}(1+t). \quad \text{(B.25)}
\end{aligned}$$

By Hypothesis B, $2\theta < \sigma(1+\alpha)$ with $\gamma \in \mathbb{R}$ or $2\theta = \sigma(1+\alpha)$ with $\gamma > 0$. Thus, for all $t \geq t_0$:

$$\varphi(t)^{-\left(\frac{2\sigma-\frac{4\theta}{1+\alpha}}{\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) ln^{-\frac{2\gamma}{1+\alpha}}(1+t) \lesssim 1.$$

Applying the last inequality in equation (B.25) the result follows.

□

Lemma B.1.7 *Let $0 < \theta < \omega$ and assume Hypothesis B. For g, φ, ϕ and t_0 given by Corollary A.1.1, the following estimate holds for $t \geq t_0$:*

$$\max\left\{\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+\sigma-2\theta}{\sigma-\theta}\right)}, \varphi(t)^{-\frac{\omega}{\theta}}\right\} \lesssim \frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+\sigma-2\theta}{\sigma-\theta}\right)}.$$

Proof. Initially, let $\alpha \in (-1,1)$. Since $g(t) = (1+t)^{\alpha} ln^{\gamma}(1+t)$, $\varphi(t) = (1+t)^{1+\alpha} ln^{\gamma}(1+t)$ and $\phi(t) = (1+t)^{1-\alpha} ln^{-\gamma}(1+t)$, we have $g(t) = \varphi(t)^{\frac{\alpha}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}(1+t)$ and $\phi(t) = \varphi(t)^{\frac{1-\alpha}{1+\alpha}} ln^{-\frac{2\gamma}{1+\alpha}}(1+t)$. Therefore,

$$\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+\sigma-2\theta}{\sigma-\theta}\right)} = \varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{\frac{(\omega-\theta)(\sigma(1+\alpha)-2\theta)}{(1+\alpha)(\sigma-\theta)\theta}} ln^{\frac{2\gamma(\omega-\theta)}{(1+\alpha)(\sigma-\theta)}}(1+t). \quad \text{(B.26)}$$

By Hypothesis B, $\sigma(1+\alpha) > 2\theta$ or $\sigma(1+\alpha) = 2\theta$ with $\gamma > 0$. Since $\omega > \theta$, in both cases holds:

$$\varphi(t)^{\frac{(\omega-\theta)(\sigma(1+\alpha)-2\theta)}{(1+\alpha)(\sigma-\theta)\theta}} ln^{\frac{2\gamma(\omega-\theta)}{(1+\alpha)(\sigma-\theta)}}(1+t) \gtrsim 1,$$

for all $t \geq t_0$. By the last inequality and by equality (B.26) the result follows.

Finally, let $\alpha = 1$. In this case, we have:

$$\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+\sigma-2\theta}{\sigma-\theta}\right)} = \varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{\frac{(\omega-\theta)}{\theta}}ln^{-\gamma}(1+t)\phi(t)^{-\left(\frac{\omega+\sigma-2\theta}{\sigma-\theta}\right)}. \tag{B.27}$$

Remembering the definition of ϕ, we know that for $\alpha = 1$, ϕ has a logarithmic behaviour. Therefore, since $\omega > \theta$ and due the fact that φ has a polynomial-logarithmic behaviour, for $t \geq t_0$ follows:

$$\varphi(t)^{\frac{(\omega-\theta)}{\theta}}ln^{-\gamma}(1+t)\phi(t)^{-\left(\frac{\omega+\sigma-2\theta}{\sigma-\theta}\right)} \gtrsim 1.$$

Applying the inequality above in equation (B.27), the result follows.

□

Lemma B.1.8 *Let $\omega \geq 0$, and assume Hypothesis B. For g, φ, ϕ and t_0 given by Corollary A.1.1, the following estimate holds for $t \geq t_0$:*

$$\max\left\{\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\sigma-\theta}\right)}ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right), \varphi(t)^{-\frac{\omega}{\theta}}\right\}$$
$$\lesssim \begin{cases} \varphi(t)^{-\frac{\omega}{\theta}}, & \text{if } \omega \leq \frac{2\theta}{1+\alpha}; \\ \frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\sigma-\theta}\right)}ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{1}{M}\phi(t)\right), & \text{if } \omega > \frac{2\theta}{1+\alpha}. \end{cases}$$

Proof. Initially, consider $\alpha \in (-1,1)$. In this case, we have $g(t) = \varphi(t)^{\frac{\alpha}{1+\alpha}}ln^{\frac{\gamma}{1+\alpha}}(1+t)$ and $\phi(t) = \varphi(t)^{\frac{1-\alpha}{1+\alpha}}ln^{-\frac{2\gamma}{1+\alpha}}(1+t)$. Therefore,

$$\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\sigma-\theta}\right)}ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right)$$
$$= \varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{-\frac{2}{(1+\alpha)(\sigma-\theta)}\left(\left(\frac{2\theta-\sigma(1+\alpha)}{2\theta}\right)\omega+\sigma-\frac{2\theta}{1+\alpha}\right)}w(t), \tag{B.28}$$

where

$$w(t) = ln^{\frac{2\gamma}{1+\alpha}\left(\frac{\omega-\frac{2\theta}{1+\alpha}}{\sigma-\theta}\right)}(1+t)ln^{\frac{2\gamma}{1+\alpha}}(1+t)ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right).$$

For $\alpha = 1$, we have $g(t) = (1+t)ln^{\gamma}(1+t) = \varphi(t)^{\frac{1}{2}} ln^{\frac{\gamma}{2}}(1+t)$, therefore:

$$\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+2\sigma-3\theta}{\sigma-\theta}\right)} ln^{-\gamma}\left(\frac{\phi(t)}{M}\right) = \varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{\frac{\omega-\theta}{\theta}} w(t), \qquad \text{(B.29)}$$

where

$$w(t) = \phi(t)^{-\left(\frac{\omega+2\sigma-3\theta}{\sigma-\theta}\right)} ln^{-\gamma}(1+t) ln^{-\gamma}\left(\frac{\phi(t)}{M}\right).$$

Remember that, for $\alpha = 1$, $\phi(t) = ln^{1-\gamma}(1+t)$ with $\gamma < 1$ and $\phi(t) = ln(ln(1+t))$ when $\gamma = 1$. Thus, for $\alpha \in (-1, 1]$ w has a logarithmic behaviour.

Define $q := -\frac{2}{(1+\alpha)(\sigma-\theta)}\left(\left(\frac{2\theta-\sigma(1+\alpha)}{2\theta}\right)\omega + \sigma - \frac{2\theta}{1+\alpha}\right)$. Note that $q = \frac{\omega-\theta}{\theta}$ when $\alpha = 1$. Taking in account equations (B.28) and (B.29), we conclude that:

$$\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\sigma-\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right) = \varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{q} w(t).$$

Therefore,

$$\max\left\{\frac{1}{g(t)^2}\phi(t)^{-\left(\frac{\omega+2\sigma-2\theta\left(\frac{2+\alpha}{1+\alpha}\right)}{\sigma-\theta}\right)} ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right), \varphi(t)^{-\frac{\omega}{\theta}}\right\}$$
$$= \max\left\{\varphi(t)^{-\frac{\omega}{\theta}}\varphi(t)^{q} w(t), \varphi(t)^{-\frac{\omega}{\theta}}\right\}. \qquad \text{(B.30)}$$

That is, to prove the result is sufficient analyze $\varphi(t)^q w(t)$. Furthermore, for $\alpha \in (-1, 1)$, we have:

$$\lim_{t\to\infty} \frac{ln(1+t)}{ln\left(\frac{\phi(t)}{M}\right)} = \frac{1}{1-\alpha}$$

and therefore there exist $0 < C_1 < C_2$ such that, for all $t \geq t_0$:

$$C_1 ln\left(\frac{\phi(t)}{M}\right) \leq ln(1+t) \leq C_2 ln\left(\frac{\phi(t)}{M}\right). \qquad \text{(B.31)}$$

Case $\omega > \frac{2\theta}{1+\alpha}$ and $2\theta < \sigma(1+\alpha)$:

In this case, $q > 0$ therefore $\varphi(t)^q w(t) \gtrsim 1$ for $t \geq t_0$. By equation (B.30) the result follows.

Case $\omega > \sigma = \frac{2\theta}{1+\alpha}$, $2\theta = \sigma(1+\alpha)$ and $\gamma > 0$:

By Hypothesis B, in this case $\alpha \in (-1,1)$. By equivalence (B.31) we have $ln\left(\frac{1}{M}\phi(t)\right) \lesssim ln(1+t)$. Therefore for $t \geq t_0$:

$$w(t) = ln^{\frac{2\gamma}{1+\alpha}\left(\frac{\omega - \frac{2\theta}{1+\alpha}}{\sigma-\theta}\right)}(1+t) ln^{\frac{2\gamma}{1+\alpha}}(1+t) ln^{-\frac{2\gamma}{1+\alpha}}\left(\frac{\phi(t)}{M}\right) \gtrsim 1.$$

By equation (B.30) the result follows.

Case $\omega = \frac{2\theta}{1+\alpha}$ and $2\theta < \sigma(1+\alpha)$:

Now $q = 0$ and therefore we must to study the behaviour of w. First, consider the case $\alpha \in (-1,1)$. Since $\omega = \frac{2\theta}{1+\alpha}$ and using equivalence (B.31), follows that $w(t) \lesssim 1$ for $t \geq t_0$. Therefore, $\max\{w(t), 1\} \lesssim 1$ and by equation (B.30) the result follows.

Consider $\alpha = 1$ and $\gamma < 1$. Now, $\phi(t) = ln^{1-\gamma}(1+t)$ and $w(t) = ln^{\gamma-2}(1+t) ln^{-\gamma}\left(\frac{1}{M}\phi(t)\right)$. In addition, for $0 \leq \gamma < 1$ is easy to see that $w(t) < 1$ for all $t \geq t_0$. For $\gamma < 0$ just use that $\frac{1}{M}\phi(t) \lesssim (1+t)$ and therefore $w(t) \lesssim ln^{-2}(1+t) \lesssim 1$. In the case of $\alpha = 1$ and $\gamma = 1$, we have $\phi(t) = ln(ln(1+t))$ and $w(t) = ln^{-2}(ln(1+t)) ln^{-1}(1+t) ln^{-1}\left(\frac{\phi(t)}{M}\right)$. It's very easy to see that $w(t) \lesssim 1$. In all cases, by equation (B.30) the result follows.

Case $\omega < \frac{2\theta}{1+\alpha}$ and $2\theta < \sigma(1+\alpha)$:

In this case, $q < 0$ therefore $\varphi(t)^q w(t) \lesssim 1$ for $t \geq t_0$. By equation (B.30) the result follows.

Case $\omega \le \sigma = \frac{2\theta}{1+\alpha}$, $2\theta = \sigma(1+\alpha)$ and $\gamma > 0$:

In this case $q = 0$ and we have to analyze $\max\{w(t), 1\}$. By Hypothesis B, we have $\alpha \in (-1, 1)$. Furthermore, by equivalence (B.31), follows:

$$w(t) \lesssim ln^{\frac{2\gamma}{1+\alpha}\left(\frac{\omega - \frac{2\theta}{1+\alpha}}{\sigma-\theta}\right)}(1+t).$$

Since $\omega \le \frac{2\theta}{1+\alpha}$, follows that $w(t) \lesssim 1$. Therefore by equation (B.30) the result follows.

□

Lemma B.1.9 *Let us assume Hypothesis B. For g, φ and t_0 given by Corollary A.1.1, for $t \ge t_0$, holds:*

$$\max\left\{\varphi(t)^{-\frac{1}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right), \frac{1}{g(t)}\varphi(t)^{-\left(\frac{\sigma-\theta}{\theta}\right)}\right\} \lesssim \varphi(t)^{-\frac{1}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right).$$

Proof. We have the following equalities:

$$\varphi(t)^{-\frac{1}{1+\alpha}} ln^{\frac{\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) = (1+t)^{-1} ln^{-\frac{\gamma}{1+\alpha}}(1+t) ln^{\frac{\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right) \quad \text{(B.32)}$$

$$\frac{1}{g(t)}\varphi(t)^{-\left(\frac{\sigma-\theta}{\theta}\right)} = (1+t)^{-1}(1+t)^{-\left(\frac{\sigma(1+\alpha)-2\theta}{\theta}\right)} ln^{-\frac{\sigma\gamma}{\theta}}(1+t). \quad \text{(B.33)}$$

If $2\theta < \sigma(1+\alpha)$, for $t \ge t_0$ we have:

$$(1+t)^{-\left(\frac{\sigma(1+\alpha)-2\theta}{\theta}\right)} ln^{-\frac{\sigma\gamma}{\theta}}(1+t) \lesssim ln^{-\frac{\gamma}{1+\alpha}}(1+t) ln^{\frac{\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right).$$

By using the inequality above and equations (B.32) and (B.33), the result follows for $2\theta < \sigma(1+\alpha)$. Finally, let $2\theta = \sigma(1+\alpha)$ and $\gamma > 0$. Since $\frac{\sigma}{\theta} = \frac{2}{1+\alpha}$, rewriting equation (B.33), we have:

$$\frac{1}{g(t)}\varphi(t)^{-\left(\frac{\sigma-\theta}{\theta}\right)} = (1+t)^{-1} ln^{-\frac{2\gamma}{1+\alpha}}(1+t). \quad \text{(B.34)}$$

Since $ln^{-\frac{2\gamma}{1+\alpha}}(1+t) \lesssim ln^{-\frac{\gamma}{1+\alpha}}(1+t) ln^{\frac{\gamma}{1+\alpha}}\left(\frac{\varphi(t)}{N}\right)$ for $\gamma > 0$, the result follows.

□

Lemma B.1.10 *Consider the Hypothesis B, g, φ, ϕ and t_0 given by Corollary A.1.1. Consider $\omega := 2\theta\left(\frac{2+\alpha}{1+\alpha}\right) - 2\sigma$, $p, \hat{q} \in [1,2]$ and the following function, defined for $t \geq t_0$:*

$$\nu(t) := \begin{cases} ln^{-\frac{2\gamma}{1+\alpha}+\frac{2}{p\hat{q}}(p+\hat{q}-p\hat{q})}\left(\frac{\phi(t)}{M}\right) & \text{if } \gamma > \frac{(1+\alpha)}{p\hat{q}}(p+\hat{q}-p\hat{q}) \geq 0, \\ ln^{\frac{2\gamma}{1+\alpha}}\left(ln\left(\frac{\varphi(t)}{N}\right)\right) & \text{if } \gamma = \frac{(1+\alpha)}{p\hat{q}}(p+\hat{q}-p\hat{q}) > 0, \\ ln^{-\frac{2\gamma}{1+\alpha}+\frac{2}{p\hat{q}}(p+\hat{q}-p\hat{q})}\left(\frac{M^{\frac{1}{2\sigma-2\theta}}}{N^{\frac{1}{2\theta}}}\phi(t)^{-\frac{1}{2\sigma-2\theta}}\varphi(t)^{\frac{1}{2\theta}}\right) & \\ & \text{if } 0 \leq \gamma < \frac{(1+\alpha)}{p\hat{q}}(p+\hat{q}-p\hat{q}), \\ ln^{-\frac{2\gamma}{1+\alpha}+\frac{2}{p\hat{q}}(p+\hat{q}-p\hat{q})}\left(\frac{\varphi(t)}{N}\right) & \text{if } \gamma < 0. \end{cases}$$

Then, $\frac{1}{g(t)^2}\nu(t) \lesssim \varphi(t)^{-\frac{\omega}{\theta}}$ for all $t \geq t_0$.

Proof. We have $2\alpha = \frac{2}{\theta}\left(\sigma(1+\alpha) - 2\theta\right) + \frac{\omega}{\theta}(1+\alpha)$, therefore

$$\frac{1}{g(t)^2} = (1+t)^{-\frac{2}{\theta}(\sigma(1+\alpha)-2\theta)}\varphi(t)^{-\frac{\omega}{\theta}} ln^{(\omega-2\theta)\frac{\gamma}{\theta}}(1+t).$$

Then, $\frac{1}{g(t)^2}\nu(t) = (1+t)^{-\frac{2}{\theta}(\sigma(1+\alpha)-2\theta)} ln^{(\omega-2\theta)\frac{\gamma}{\theta}}(1+t)\nu(t)\varphi(t)^{-\frac{\omega}{\theta}}$.

If $2\theta < \sigma(1+\alpha)$, by the limit:

$$\lim_{t\to\infty}(1+t)^{-\frac{2}{\theta}(\sigma(1+\alpha)-2\theta)} ln^{(\omega-2\theta)\frac{\gamma}{\theta}}(1+t)\nu(t) = 0,$$

there exists $c > 0$ such that $(1+t)^{-\frac{2}{\theta}(\sigma(1+\alpha)-2\theta)} ln^{(\omega-2\theta)\frac{\gamma}{\theta}}(1+t)\nu(t) \leq c$ for $t \geq t_0$ and therefore the lemma follows.

Now, considering the case $2\theta = \sigma(1+\alpha)$, by Hypothesis B we have $\gamma > 0$ and therefore:

$$\frac{1}{g(t)^2}\nu(t) = ln^{-\frac{2\gamma}{1+\alpha}}(1+t)\nu(t)\varphi(t)^{-\frac{\omega}{\theta}}. \tag{B.35}$$

In this case, holds the limit:

$$\lim_{t\to\infty} ln^{-\frac{2\gamma}{1+\alpha}}(1+t)\nu(t) = 0. \tag{B.36}$$

In fact, if $\gamma > \frac{(1+\alpha)}{p\hat{q}}(p+\hat{q}-p\hat{q}) \geq 0$, this is true because $\nu(t) = ln^{-\frac{2\gamma}{1+\alpha}+\frac{2}{p\hat{q}}(p+\hat{q}-p\hat{q})}\left(\frac{\phi(t)}{M}\right)$ and $-\frac{2\gamma}{1+\alpha} + \frac{2}{p\hat{q}}(p+\hat{q}-p\hat{q}) < 0$. For $\gamma =$

$\frac{(1+\alpha)}{p\hat{q}}(p+\hat{q}-p\hat{q})$ the limit (B.36) holds because $\nu(t)$ has a $log \circ log$ increasing type, in which always lose for a logarithmic rate. Finally, if $0<\gamma<\frac{(1+\alpha)}{p\hat{q}}(p+\hat{q}-p\hat{q})$, since $\phi(t)^{-\frac{1}{2\sigma-2\theta}}\varphi(t)^{\frac{1}{2\theta}} = ln^{\frac{\gamma}{2\sigma-2\theta}+\frac{\gamma}{2\theta}}(1+t)$, again $\nu(t)$ has a $log \circ log$ increasing type, in which imply that limit (B.36) holds.

Using equation (B.36), in particular $ln^{-\frac{2\gamma}{1+\alpha}}(1+t)\nu(t)$ is bounded, and applying in equation (B.35) the result follows.

□

Appendix C

Consider the following functions: $f_{-1} : (-\infty, -1) \to (-\frac{1}{e}, 0)$ and $f_0 : (-1, \infty) \to (-\frac{1}{e}, \infty)$, both defined by the rule: $x \mapsto xe^x$. These functions are bijective and therefore admits an inverse. Therefore, we can define $W_{-1} := (f_{-1})^{-1}$ and $W_0 := (f_0)^{-1}$, both known as *W-Lambert's function*. Actually, in a more general sense, are the two real-valued branches of W-Lambert's function. The W-Lambert's function plays a fundamental role in this thesis: it is used for explicit calculate t_ξ and $\psi(\xi)$. Since the explicit representation of ψ and t_ξ is not sufficient by itself, in the following are some useful results to prove Propositions 1.1.4 and 2.2.2. The proof of Lemma C.1.1 can be found in [3], and of Lemma C.1.2 in [12].

Lemma C.1.1 *For $u > 0$, we have:*

$$1 + \sqrt{2u} + \frac{2}{3}u < -W_{-1}\left(-e^{-(u+1)}\right) < 1 + \sqrt{2u} + u.$$

Corollary C.1.1 *For $0 < v < \frac{1}{e^2}$, we have:*

$$\frac{1}{3}\, ln\left(\frac{1}{v}\right) < -W_{-1}(-v) < \left(2 + \sqrt{2}\right) ln\left(\frac{1}{v}\right).$$

Proof. Define $u := -(1 + ln(v))$, therefore $e^{-(u+1)} = v$. Notice that $u > 1$ and by applying Lemma C.1.1:

$$\frac{2}{3}(-1 - ln(v)) < 1 + \sqrt{2u} + \frac{2}{3}u < -W_{-1}(-v)$$
$$< \left(2 + \sqrt{2}\right)(-1 - ln(v)) < \left(2 + \sqrt{2}\right) ln\left(\frac{1}{v}\right).$$

Since $\frac{2}{3}(-1 - ln(v)) > \frac{1}{3} ln\left(\frac{1}{v}\right)$ for $0 < v < \frac{1}{e^2}$, we conclude the result.

□

Lemma C.1.2 *For every $u \geq e$, holds:*

$$ln(u) - ln(ln(u)) \leq W_0(u) \leq ln(u) - \frac{1}{2}ln(ln(u)).$$

We immediately conclude that W_0 behaves like the function ln:

Corollary C.1.2 *For every $u \geq e$, holds:*

$$\frac{1}{2}ln(u) \leq W_0(u) \leq ln(u).$$

Lemma C.1.3 *The following properties hold:*

(1) $W_{-1}(xlnx) = ln(x)$, *for* $x \in (0, \frac{1}{e})$;

(2) $W_0(xlnx) = ln(x)$, *for* $x > \frac{1}{e}$;

(3) $W_{-1}(x) = ln(-x) - ln(-W_{-1}(x))$, *for* $x \in \left(-\frac{1}{e}, 0\right)$;

(4) $W_0(x) = ln(x) - ln(W_0(x))$, *for* $x > 0$.

Proof. Let $W \in \{W_{-1}, W_0\}$. Since W is the inverse function of $y \mapsto ye^y$, given x such that $xln(x)$ is in the domain of W, define $y := ln(x)$. Therefore: $W(xln(x)) = W(ye^y) = y = ln(x)$. Taking care with the domain of W, we conclude **(1)** and **(2)**.

On the other hand, let $x \in \left(-\frac{1}{e}, 0\right)$ if $W = W_{-1}$ and $x > 0$ if $W = W_0$. We have $x = W^{-1}(W(x)) = e^{W(x)}W(x)$. That is, $e^{W(x)} = \frac{x}{W(x)}$ and therefore $W(x) = ln(|x|) - ln(|W(x)|)$.

□

Lemma C.1.4 *Let $\mu > 0$ and $\beta \in \mathbb{R}$. Consider $h(t) := t^{\mu} ln^{\beta}(t)$ defined for $t > 1$ and let $\lambda > 0$ such that:*

$$\lambda > \begin{cases} \left(\frac{\mu e}{|\beta|}\right)^{|\beta|} & \text{if } \beta < 0, \\ 1 & \text{if } \beta = 0. \end{cases}$$

Then, $\tau := h^{-1}(\lambda)$ is well defined and

if $\beta < 0$, $\tau = \left(-\frac{\mu}{\beta}\right)^{\frac{\beta}{\mu}} \lambda^{\frac{1}{\mu}} \left[-W_{-1}\left(\frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}\right)\right]^{-\frac{\beta}{\mu}}$,

if $\beta = 0$, $\tau = \lambda^{\frac{1}{\mu}}$,

if $\beta > 0$, $\tau = \left(\frac{\mu}{\beta}\right)^{\frac{\beta}{\mu}} \lambda^{\frac{1}{\mu}} W_0\left(\frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}\right)^{-\frac{\beta}{\mu}}$.

Proof. The proof for $\beta = 0$ is trivial. Now, consider $\beta \neq 0$,

$$\tau^{\mu} ln^{\beta}(\tau) = \lambda \quad \Leftrightarrow \quad \tau^{\frac{\mu}{\beta}} ln\left(\tau^{\frac{\mu}{\beta}}\right) = \frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}. \tag{C.1}$$

For $\beta < 0$, since $h(t) = t^{\mu} ln^{\beta}(t)$ is increasing for $t > \left(\frac{1}{e}\right)^{\frac{\beta}{\mu}} > 1$, we have:

$$\lambda > \left(\frac{\mu e}{|\beta|}\right)^{|\beta|} = h\left(\left(\frac{1}{e}\right)^{\frac{\beta}{\mu}}\right) \quad \Leftrightarrow \quad \tau = h^{-1}(\lambda) > \left(\frac{1}{e}\right)^{\frac{\beta}{\mu}} \quad \Leftrightarrow \quad \tau^{\frac{\mu}{\beta}} < \frac{1}{e}.$$

Using **(1)** and **(3)** of Lemma C.1.3 and equation (C.1), we have:

$$\begin{aligned} \tau &= e^{\frac{\beta}{\mu} W_{-1}\left(\tau^{\frac{\mu}{\beta}} ln\left(\tau^{\frac{\mu}{\beta}}\right)\right)} = e^{\frac{\beta}{\mu} W_{-1}\left(\frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}\right)} \\ &= \left(-\frac{\mu}{\beta}\right)^{\frac{\beta}{\mu}} \lambda^{\frac{1}{\mu}} \left[-W_{-1}\left(\frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}\right)\right]^{-\frac{\beta}{\mu}}. \end{aligned}$$

For $\beta > 0$, $\lambda > 0 = h(1) \Leftrightarrow \tau^{\frac{\mu}{\beta}} > 1 > \frac{1}{e}$. Using properties **(2)** and **(4)** of Lemma C.1.3 and equation (C.1), we have:

$$\tau = e^{\frac{\beta}{\mu} W_0\left(\frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}\right)} = \left(\frac{\mu}{\beta}\right)^{\frac{\beta}{\mu}} \lambda^{\frac{1}{\mu}} W_0\left(\frac{\mu}{\beta}\lambda^{\frac{1}{\beta}}\right)^{-\frac{\beta}{\mu}}.$$

□

Bibliography

[1] BECKNER, W. *Inequalities in Fourier analysis on $\mathbb{R}^n$*. Proc. Natl. Acad. Sci. USA 72 (2) (1975), 638-641.

[2] CHARÃO, R. C.; DA LUZ, C. R.; IKEHATA, R. *Sharp decay rates for wave equations with a fractional damping via new method in the Fourier space.* J. Math. Anal. Appl. 408 (2013), 247–255.

[3] CHATZIGEORGIOU, I. *Bounds on the Lambert function and their application to the outage analysis of user cooperation.* IEEE Communications Letters 17 (2013), no. 8, 1505-1508.

[4] COLOMBINI, F. *Energy estimates at infinity for hyperbolic equations with oscillating coefficients.* J. Differential Equations 231 (2006), 598–610.

[5] COLOMBINI, F. *Loss of derivatives for $t \to \infty$ in strictly hyperbolic Cauchy problems.* Hyperbolic Problems and Regularity Questions: Trends Math., Birkhäuser, Boston, MA, 2006.

[6] DA LUZ, C. R.; IKEHATA, R.; CHARÃO, R. C. *Asymptotic behavior for Abstract Evolution Differential Equations of Second Order.* J. Differential Equations 259 (2015), 5017-5039.

[7] D'ABBICCO, M.; CHARÃO, R. C.; DA LUZ, C. R. *Sharp time decay rates on a hyperbolic plate model under effects of an intermediate damping with a time-dependent coefficient.* Discrete and Continuous Dynamical Systems 36 (2016), 2419-2447.

[8] D'Abbicco, M.; Ebert, M. R. *A classification of structural dissipations for evolution operators.* Math. Meth. Appl. Sci. 39 (2016), 2558–2582.

[9] Hirosawa, F. *On the asymptotic behavior of the energy for the wave equations with time depending coefficients.* Math. Ann. 339 (2007), 819–839.

[10] Hirosawa, F. *Energy estimates for wave equations with time dependent propagation speeds in the Gevrey class.* J. Differential Equations 248 (2010), 2972–2993.

[11] Hirosawa, F.; Wirth, J. *C^m theory of damped wave equations with stabilization.* J. Math. Anal. Appl. 343 (2008), 1022–1035.

[12] Hoorfar, A.; Hassani, M. *Inequalities on the Lambert W Function and Hyperpower Function.* Journal of inequalities in pure and applied mathematics (2008), 9 (2), art. 51.

[13] Ikehata, R.; Natsume, M. *Energy decay estimates for wave equations with a fractional damping.* Differential and Integral Equations 25 (2012), no. 9-10, 939-956.

[14] Kainane, M. *Structural damped $\sigma-$evolution operators.* PhD Thesis, TU Bergakademie Freiberg, Freiberg, 2014.

[15] Kainane, M.; Reissig, M. *Qualitative properties of solution to structurally damped $\sigma-$evolution models with time decreasing coefficient in the dissipation.* Complex Analysis and Dynamical Systems VI. Contemporary Mathematics. vol. 653, Amer. Math. Soc., Providence, RI, 2015, 191–217.

[16] Kainane, M.; Reissig, M. *Qualitative properties of solution to structurally damped $\sigma-$evolution models with time increasing coefficient in the dissipation.* Advances in Differential Equations 20(5-–6) (2015), 433–462.

[17] Komornik, V. *Differential and integral inequalities.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Belgrade 7 (1996), 55-67.

[18] Lu, X.; Reissig, M. *Rates of decay for structural damped models with decreasing in time coefficients.* Int. J. Dynamical Systems and Differential Equations, Vol. 2, Nos. 1/2, (2009), 21-55.

[19] Reissig, M. *Rates of decay for structural damped models with strictly increasing in time coefficients.* Complex Analysis and Dynamical Systems IV, Contemporary Mathematics, vol.554, Amer. Math. Soc. Providence, RI (2011), 187-206.

[20] Reissig, M.; Smith, J. *$L^p - L^q$ estimate for wave equation with bounded time-dependent coefficient.* Hokkaido Math. J. 34/3 (2005), 541–586.

[21] Reissig, M.; Yagdjian, K. *About the influence of oscillations on Strichartz type decay estimates.* Rend. Sem. Mat. Univ. Pol. Torino, 58/3 (2000), 375–388.

[22] Umeda, T.; Kawashima, S.; Shizuta Y. *On the decay of solutions to the linearized equations of electro-magneto-fluid dynamics.* Japan. J. Appl. Math. 1 (1984), 435-457.

[23] Wirth, J. *Wave equations with time-dependent dissipation I: Non-effective dissipation.* J. Differential Equations 222 (2006), 487-514.

[24] Wirth, J. *Wave equations with time-dependent dissipation II: Effective dissipation.* J. Differential Equations 232 (2007), 74–103.

[25] Wirth, J. *On the influence of time-periodic dissipation on energy and dispersive estimates.* Hiroshima Math. J. 38 (3) (2008), 397–410.