FEDERAL UNIVERSITY OF SANTA CATARINA
PHYSICS DEPARTMENT

João Pedro Wojcikiewicz Duarte Silva

COSMOLOGICAL TAXONOMY

Florianópolis

2019

João Pedro Wojcikiewicz Duarte Silva

COSMOLOGICAL TAXONOMY

Dissertação submetido ao Programa de Pós-Graduação em Física para a obtenção do Grau de Mestrado.
Orientador: Dr. Jeferson de Lima Tomazelli

Florianópolis

2019

Ficha de identificação da obra elaborada pelo autor,
através do Programa de Geração Automática da Biblioteca Universitária da UFSC.

Wojcikiewicz Duarte Silva, João Pedro
Cosmological Taxonomy / João Pedro Wojcikiewicz Duarte Silva ; orientador, Jeferson de Lima Tomazelli, 2019.
286 p.

Dissertação (mestrado) - Universidade Federal de Santa Catarina, Centro de Ciências Físicas e Matemáticas, Programa de Pós-Graduação em Física, Florianópolis, 2019.

Inclui referências.

1. Física. 2. Cosmologia. 3. Relatividade Geral. 4. Álgebras de Lie. 5. Teoria Clássica de Campos. I. de Lima Tomazelli, Jeferson. II. Universidade Federal de Santa Catarina. Programa de Pós-Graduação em Física. III. Título.

João Pedro Wojcikiewicz Duarte Silva

COSMOLOGICAL TAXONOMY

Este Dissertação foi julgado aprovado para a obtenção do Título de "Mestrado", e aprovado em sua forma final pelo Programa de Pós-Graduação em Física.

Florianópolis, 15 de Março 2019.

Ivan Helmuth Bechtold
Coordenador

Dr. Jeferson de Lima Tomazelli
Orientador

Banca Examinadora

Dr. Jeferson de Lima Tomazelli
UFSC
Presidente

Dr. Paweł Klimas
UFSC

Dr. Marco Aurelio Cattacin Kneipp
UFSC

Dr. Eliezer Batista
UFSC-MTM

To my parents and siblings for all the encouragement and support.

Acknowledgements

To my parents, Lilian and Eduardo, for all the support and for the cozy little home I could live in throughout my academic life.

To my brothers present in my leisure times to unwind after weeks of intense study. Special thanks to Eduardo Luann for opening my range of possibilities to fully exploit my true potential.

To João Saldanha for being my dearest of friends, for our endless deep conversations on a variety of subjects and for his complete support in my tough days.

To all my friends in the physics department of all areas for the good company and fun times.

To my mentor Dr. Jeferson Tomazelli for his vast knowledge in physics and for his fantastic insights.

To Dr. Jordan B. Peterson and Stefan Molyneux, professor in psychology and philosopher resp., for teaching me the tools and guiding my through a journey of self-knowledge, which helped me discover who I truly am in a time I most desperately needed.

To all the working Brazilian people, for banking this work through the CAPES program.

"You're going to pay a price for every bloody thing you do and everything you don't do. You don't get to choose to not pay a price. You get to choose which poison you're going to take. That's it."

– Jordan B. Peterson

Abstract

The Universe, its origin and fate are the single most relevant subjects that has been challenging human understanding throughout the ages. From an infinitely tall pile of turtles, passing through the Newtonian action at a distance conception, up to the geometrization program launched by Einstein and still unfinished in present times, many were the proposed answers for such deep issues, one more speculative than the other. Fortunately, with the advances of observational techniques and the robust framework of General Relativity, we are able to navigate more safely towards the desired solutions. In the present work we adopt a more conservative approach: we shall review Gravitation theory under the lens of Symmetry Groups presented in the literature and which types of classifications might be conceived from the different degrees of symmetry the Universe manifests through observational data, studying which characteristics we expect a given cosmological model to exhibit. More specifically, we begin by first analyzing spacetimes that admit the maximal number of symmetries; then we flexibilize to those retaining only spatial homogeneity, where the Bianchi classification naturally arises, coming up with the minimum allowed symmetries for the curvature tensor, at the point where we shall be discussing the Petrov classification related to the respective fundamental invariants. After thoroughly studying these fundamental classifications, we will have acquired the indispensible pre-requisites to pursue the more ambitious quest for a theory of Quantum Gravity.

Keywords: Linear Gravitation; Lie Algebras; Einstein Equations; Tetrads; Bianchi Spaces; Petrov Classification; Classic Field Theory

Resumo

O Universo, sua origem e destino são uns dos mais relevantes tópicos que continuam desafiando a compreensão humana ao longo dos tempos. Desde uma pilha infinita de tartarugas, passando pela concepção da ação à distância Newtoniana, até o programa de geometrização iniciado por Einstein e ainda inacabado no presente, muitas foram as respostas propostas para estas questões profundas, uma mais especulativa que a outra. Felizmente, com o avanço de técnicas observacionais avançadas e com o rebusto arcabouço da teoria da Relatividade Geral, estamos aptos a navegar mais seguramente rumo às soluções tão almejadas. Neste trabalho adotamos uma postura mais conservadora: procederemos com uma revisão da teoria da Gravitação sob o prisma dos Grupos de Simetria apresentados na literatura e quais tipos de classificação podem ser concebidos segundo diferentes graus de simetria que o Universo manifesta nos dados observacionais, estudando quais características esperamos que qualquer modelo cosmológico deve exibir. Mais especificamente, começaremos com a análise de espaços-tempo que admitem o número máximo de simetrias; depois flexibilizaremos para aqueles que possuem apenas homogeneidade no setor espacial, onde a classificação de Bianchi emerge naturalmente, seguindo com apenas as simetrias permitidas pelo tensor de curvatura, no momento em que discutiremos a classificação de Petrov relacionado aos respectivos invariantes fundamentais. Depois de estudarmos a fundo

essas classificações fundamentais, teremos adquirido os pré-requisitos indispensáveis para desbravar a ambiciosa busca pela teoria da Gravitação Quântica.
Palavras-chave: Gravitação Linearizada; Álgebras de Lie; Equações de Einstein; Tetradas; Espaços de Bianchi; Classificação de Petrov; Teoria Clássica de Campos

List of Figures

List of Tables

List of Abbreviations

List of Symbols

Contents

Introduction

THE construction of knowledge and our better understanding of the physical phenomena that manifests in Nature are one of the most valuable assets of the human kind. Even back in the day when the acquired knowledge was passed down to generations by the *mythos* – the word of mouth –, explanations of the vastly unknown Nature arose. Still it was not until the Greeks that deeper inquires about Nature and her inner workings emerge. This "uneasiness" led to many questions such as: Why do bodies move? Why do they stay put? What make them move? Why do things "prefer" other things? Where are we living in? What are those lights in the night sky? This pursuit resulted in the very first rudimentary physical models, including models of the Universe, in which the Earth resided in the center and everything else revolved around it.

Roughly two millennia later, Nicolaus Copernicus, upset with the strange aparent orbit of Mars and the increasing demand for the addition of epicycles in order to correct the orbits of "errant stars", or planets, and to explain their retrograde motions in the celestial dome, proposed a model of the Universe (namely, of our Solar system) where the Sun was at the center instead and Isaac Newton formulated the first mathematical model for gravitation via an inverse square action at a distance law, where the first assertion of the *Cosmological principle* appeared. The enormous success of Newtonian mechanics and gravitation consolidated most of the physics for the next centuries, but as technology improved, more it seemed that the Newtonian physics lacked something.

Such discomfort drove the stark reformation of our very way of thinking in the beginning of the 20th century. Challenging both the fundamental premises which preceded all the contemporary theories and our general understanding of them, the birth of disciplines such as Quantum Mechanics and the physics of Relativity marked this new era. Albert Einstein was one of the iconic figures whose concern about the description of electromagnetic phenomena by families of inertial observers led him to the reformulation of Galilean relativity, giving rise to the *Special Theory of Relativity*[11] and later on, going a step further towards gravitational effects, generalized his findings to what we call *General Theory of Relativity*.[12]

To be more precise, Einstein was bothered by the discrepancies that would be found by different inertial observers in the outcome of a given eletromagnetic experimental apparatus, one at rest with it and the other moving with a speed v relative to it, when the traditional Galilean change of reference frame was performed to dinamical physical quantities. In order to address that issue, he postulates two fundamental premises that were very reasonable and were already sustained by experimental data:

1. **Constancy of c:** The speed of light c has the same numerical value in *all* inertial frames of reference;

2. **Principle of relativity:** The laws of Physics must be the same on all inertial frames of reference.

It was then possible to construct a new set of coordinate transformations from the ground up in such a way that any theory build with those two postulates in mind were automatically *covariant*, that is, valid in every inertial frame of reference. However, that comes with a price: the notion of *absolute simultaneity* is now broken, in such a way that it depends on which frame you are in. That prompted a construction of a new 4-dimensional space where now the time is treated as a coordinate, so that the transformation of coordinates is now done by the *Lorentz transformations*. That space was dubbed *spacetime* and each point $P = (ct, x)$ in it represents an *event*. Furthermore, this spacetime structure naturally bears a *causal* structure of events.

Now, since the notion of absolute frames of reference is completely lost, it is necessary to construct *invariant* quantities that are independent of an arbitrary choice of frame of reference. One of such quantities is the so called *interval* Δs^2, which is a quadratic form with a *Lorentz signature* and is characterized by the quadratic differences of the space and time coordinates

$$\Delta s^2 = (c\Delta t)^2 - \Delta \boldsymbol{x}^2 \,.$$

Nevertheless, that still was not enough; the second postulate above was too restrictive and incompatible while dealing with accelerated frames of reference, such as a falling frame on a gravitational field. To address that, Einstein revisited the principle of relativity and appended a few more reasonable others

1. **Constancy of c:** The speed of light c has the same numerical value in *all* frames of reference;

2. **Principle of *General* Relativity:** The laws of Physics must be the same on all frames of reference;

3. **Principle of General Covariance:** The mathematical equations and their numerical constants must be invariant upon a change of reference frame;

4. **Equivalence Principle:** All bodies are equally accelerated in the presence of gravitational forces, following geodesic worldlines, regardless of their nature;

5. **Mach's Principle:** The spacetime structure is influenced by matter.

Just like what happened with the Special Theory of Relativity – as we discussed above –, these postulates drastically changed how we view the gravitational theory, as we will discuss later. Now, spacetime itself was geometrized and gravitational forces were just a consequence of it in the light of the fourth postulate, so there is a natural need to introduce a *Riemannian* metric space to properly describe gravitational phenomena.

This new formulation of gravity trigged a whole new era for Cosmology where Newton's cosmological principle broadens up to accomodate the five postulates of General Relativity and reads as

Cosmological Principle: The laws of Physics must be the same everywhere in the Universe.

Many were the proposed relativistic cosmological models, each and every one still challenged by the ever growing number of more precise data and sophisticated and modern observational techniques, two of such information indispensable for any model: *the Universe is expanding*, as pointed out by the receding motion of galaxies discovered by Hubble and Humason[26] and the discovery and measurement of the *Cosmic Microwave Background*[46] (CMB), both making up the strongest evidences supporting the *big-bang*.

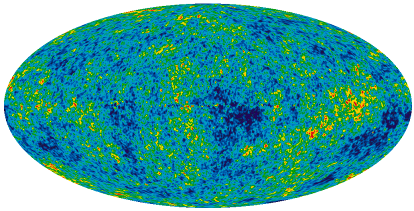


Figure 1: Cosmic Microwave Background (From NASA[3]).

For instance, the *Steady State* model proposed by Bondi and Gold[6] is a theory where the Universe is stationary but still expanding according to Hubble's law. The authors propose what they call the *Perfect Cosmological Principle*, extending the validity of the usual Cosmological Principle to *all epochs*. However, their theory was fundamentally incompatible with the CMB discovery.

The most promising cosmological model to survive the trials of Nature is that of Friedmann, Roberton, Lemaître and Walker indepently found between the 1920s and 1930s, together with the *Inflationary scenario* proposed by Guth[23,34,55] to address several pertinent issues of it, which constitutes the standard model for the Universe nowadays, though it still under scrutiny. Another contesting model that is still revisited is the Brans and Dicke *Scalar Theory of Gravitation*.[7] In this one the authors attempt to fully integrate the Mach's Principle into the theory by the introduction of a scalar field in the place of the inverse of Newton's universal gravitational constant, $\phi \propto G^{-1}$.

In parallel with that, relativistic Field Theory and Quantum Mechanics got traction and, success after success, the Quantum Field Theory was developed, considered nowadays the most fundamental theory where all others emerge at low-energy. Constructed upon the mathematical framework of Symmetry Groups and the *Gauge principle*, it was found that three of the four fundamental interactions of Nature can be accomodated in the so-called *Standard Model*, except for gravity. This along with the apparent convergence of the coupling constants of *all* fields at higher energies,[9] including gravity, points out to a *Theory of Everything*. At this point, the quest for Quantum Gravity begins.

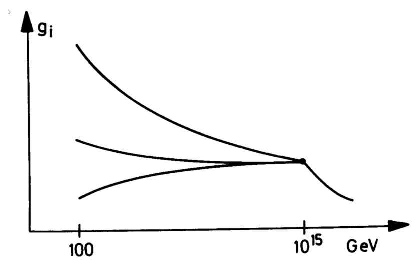


Figure 2: The coupling constants of the fundamental interactions converge at higher energies (Extracted from Boer[9]).

Started by Weyl in the late 1920s, the quantum gravity program quickly became one of the main goals of modern physics but due to the non-linearity of the Einstein equations and to mathematical inconsistences in the quantization program of field theories, at least from what we know thus far, of interacting particles of higher spins, such as the spin 2 *Graviton*, via the gauge principle!

One promising approach to solve this conundrum seems to be the description of gravitational phenomena in a weak-field approximation, where by working directly with Symmetry Groups and the associated spacetime symmetries, it is possible to explore Gravitation in a linear fashion, which, in our view, seems to be the most safe path towards quantization.

In the present work we adopt precisely this approach. By studying the construction of the Classical Theory of Gravitation via Symmetry Groups and by classifying all the possible algebras it comports and their main properties, we prepare the groundwork with the indispensable tools to further pursue Quantum Gravity, resurrecting the most important, and still mostly forgotten works spread over in the literature on this subject.

Although important, this study does not focus on solutions to Einstein equations to specific distributions of matter; our main interest is instead the *geometric aspect* for models described by the three of the most relevant degrees of symmetry in a descending order: from the maximal amount possible to the least crucial symmetries any model ought to have.

In the first chapter we make a comprehensive recapitulation of the necessary tools, where we demonstrate the principal results of Differential Geometry, introducing the notions of Killing vectors and local inertial frames of reference defined by the N-Tuples.

Next, in chapter 2, we assume the spacetime to have the maximal number of symmetries, using the formalism of Killing vectors and deducing its main properties. By demonstrating the subdivision theorem, we show that the standard model of Cosmology follows from a maximally symmetric 3-space, at the point where we discuss it a bit and show some of the main results.

Moving forward to chapter 3, we consider only the homogeneity of the space sector of spacetime. There, in the local N-Tuple frame, we show that the homogeneous 3-spaces defines surfaces of transitivity associated with a Lie group, reducing to one of the nine possible classes of three parameter Lie groups, which makes up the Bianchi classification.

In chapter 4, we consider the inherently symmetries of the curvature tensor, which has to be present in any theory of gravitation. By studying the algebraic properties of this tensor, we can bring it down to its principal axes related with the geometrical invariants, making up the Petrov classification which contains three unique classes plus three special degenerate ones.

Finaly, we present an overall discussion and close this monograph point out some future perspectives of investigations.

CHAPTER 1

Review

WE begin with a quite extensive yet comprehensible review of the main underlying theories that permeates this work. All those addressed subjects are vastly extense, so we will only construct the most important results, but not in a shallow manner, and point the reader to the already established literature if he wants to learn more about them.

1.1 Differential Geometry

We start our description with a simple and ordinary point P in a $(N-1)$−dimensional Euclidian space $\mathbb{R}^{n-1}$. By considering the construction of it in the formalism of *Vector algebra*, endowed with the usual inner product, we can characterize this point as a set of $N-1$ real parameters $\{x^i\}$ and $N-1$ orthonormal basis vectors $\{\hat{\boldsymbol{e}}_i^0\}$, $i = 1, \ldots, N-1$ by the *position vector* $\boldsymbol{r}$ by*

$$\boldsymbol{r} = x^i \hat{\boldsymbol{e}}_i^0 .$$

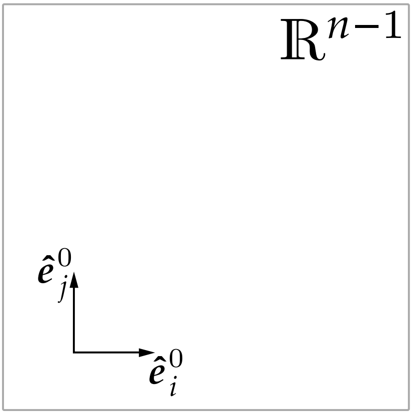


Figure 3: Orthonormal basis vectors on an $(N-1)$-dimensional Euclidian space.

Next we carry out a transformation of coordinates to another system of coordinates by means of a *diffeomorphism*†, characterized by the arbitrary set of basis vectors $\{\boldsymbol{e}'_i\}$ and components

*We are employing Einstein's implicit sum convention throughout the entirety of this whole work. Whenever repeated indices appears, it is undestood that there is a sum spanning all the components of the space.

†A diffeomorphism is a bijetive differentiable map that takes one space into another and it preserves the "smoothness" of the spaces.

$$x'^i = x'^i(x^j)\,, \tag{1.1.1}$$

which is invertible if the Jacobian matrix

$$\mathbb{J} = \left[\frac{\partial x^i}{\partial x'^j}\right] \tag{1.1.2}$$

is not singular. That means we can express a change of coordinates as the matrix relation composed of the differentials dx as

$$d\mathbb{X} = \mathbb{J}d\mathbb{X}' \quad \leftrightarrow \quad d\mathbb{X}' = \mathbb{J}^{-1}d\mathbb{X}\,, \tag{1.1.3}$$

where

$$d\mathbb{X} = \begin{pmatrix} dx^1 \\ \vdots \\ dx^{N-1} \end{pmatrix}, \qquad d\mathbb{X}' = \begin{pmatrix} dx'^1 \\ \vdots \\ dx'^{N-1} \end{pmatrix}$$

if $\det \mathbb{J} \neq 0$.

In that spirit, we can introduce the infinitesimal vector $d\boldsymbol{r} = d\boldsymbol{r}(x^i)$ that can be described by some *local unitary basis* $\{\boldsymbol{e}_j\}$ positively oriented and defined by the tangent curves passing through $P = P(x)$. That is,

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial x^i}dx^i = dx^i\boldsymbol{e}_i\,, \tag{1.1.4}$$

where evidently

$$\boldsymbol{e}_i := \frac{\partial \boldsymbol{r}}{\partial x^i}\,, \quad i = 1, \ldots, N-1\,. \tag{1.1.5}$$

We call dx^i the *contravariant* components $d\boldsymbol{r}$ (denoted by upper indices).

Similarly, we can decompose the vector into the *dual basis* $\{\boldsymbol{e}^i\}$ like

$$d\boldsymbol{r} = dx_i \boldsymbol{e}^i \,, \tag{1.1.6}$$

where here we name dx_i the *covariant* components of the same vector (denoted by lower indices). We will see further ahead on (1.1.29) how both basis vectors are related.

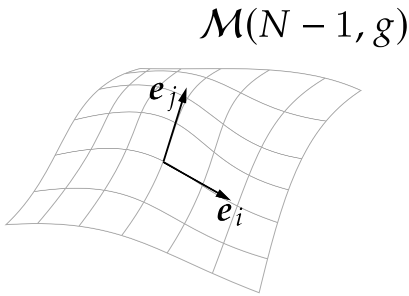


Figure 4: Representation of curvilinear unitary basis vectors on some curved manifold $\mathcal{M}$. Now $\boldsymbol{e}_i$ is a function of the point x on $\mathcal{M}$.

Not only coordinate vectors are subjected to this type of decomposition. In fact, we can apply the same formalism to any vector $\boldsymbol{v}$ and decompose it into the tangent coordinate basis $\boldsymbol{e}_i$. So if v^i are the components of $\boldsymbol{v}$ in the said basis,

$$\boldsymbol{v} = v^i \boldsymbol{e}_i \,, \tag{1.1.7}$$

or, for the dual basis,

$$\boldsymbol{v} = v_i \boldsymbol{e}^i \,. \tag{1.1.8}$$

The set of all tangent vectors passing through P defines the *Tangent space* at P. Moreover, for an arbitrary set of basis vectors*, not necessarily mutually orthogonal, and given that the inner product is well defined, the collection of tangent spaces of all points defines what we call a *Manifold*, denoted by $\mathcal{M}(N-1, g)$, where $N-1$ is the dimension and g the inner product, as we will see next. While this definition can be a bit more flexible regarding the inner product, it is, for all intents and purposes, necessary to restrict that, since its fundamental quality of measuring distances is what we are aiming for in this construction. Such subclass of spaces are called *Riemannian Manifolds*.

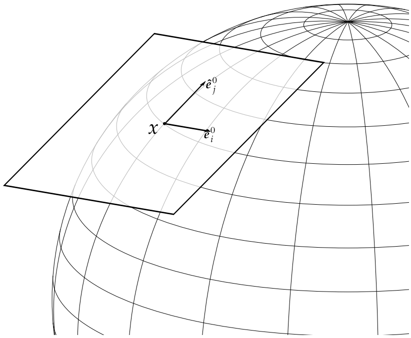


Figure 5: Tangent space

Knowing how vector elements are described on a tangent frame of reference, we are capable of constructing our first *invariant* quadratic form; the line element†. This quantity is nothing more than the length of a vector element.

$$\begin{aligned} ds^2 := |d\boldsymbol{r}^2| &= (dx^i \boldsymbol{e_i}) \cdot (dx^j \boldsymbol{e_j}) \\ &= (\boldsymbol{e_i} \cdot \boldsymbol{e_j}) dx^i dx^j \\ &= g_{ij} dx^j dx^j \,, \end{aligned} \tag{1.1.9}$$

*That is, arbitrary curvilinear vectors.

†This also known to be the *first fundamental form* on the more mathematical literature.

where g_{ij} is the *metric tensor* that represents g discussed above.

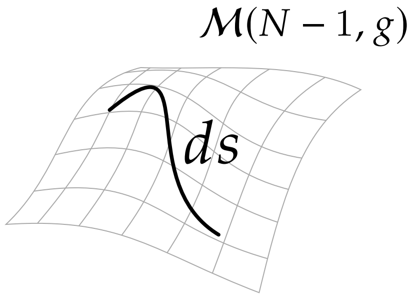


Figure 6: Some arbitrary line element on $\mathcal{M}$.

We define the *signature* of the metric tensor as being the number of positive and negative eigenvalues associated with g_{ij}. Since we are working with real and oriented basis vectors of $\mathbb{R}^{n-1}$, the signature $(+ + \ldots +)$ is all positive. In this case the metric receives the special name of *Riemannian metric*.

The metric admits a matrix representation

$$g_{ij} = (\mathbb{G})_{ij} := \boldsymbol{e_i} \cdot \boldsymbol{e_j} \tag{1.1.10}$$

which can be conceived in any coordinate system and is manifestly symmetric from (1.1.9), where we can define its inverse by

$$g^{ij} = (\mathbb{G}^{-1})_{ij} \tag{1.1.11}$$

so that

$$\mathbb{G}^{-1}\mathbb{G} = \mathbb{1} \iff g^{ij} g_{jk} = \delta^i_k \, ,$$

where δ^i_k is the *Kroenecker delta*.

Now if we bring $\boldsymbol{r}$ into a Cartesian coordinate system and making use of the Jacobian matrix components (1.1.2), we can promptly verify that the metric tensor $\mathbb{G}$ can be written as

$$\begin{aligned} g_{ij} &= \boldsymbol{e_i} \cdot \boldsymbol{e_j} \\ &= \frac{\partial \boldsymbol{r}}{\partial x^i} \cdot \frac{\partial \boldsymbol{r}}{\partial x^j} \\ &= (\mathbb{J}^t)_{ia} (\mathbb{J})_{aj} \end{aligned}$$

$$\therefore\ \mathbb{G} = \mathbb{J}^t \mathbb{J} \tag{1.1.12}$$

which implies

$$\det \mathbb{G} = (\det \mathbb{J})^2 \qquad \Longrightarrow \qquad |\det \mathbb{J}| = \sqrt{g}\,, \tag{1.1.13}$$

where $g := \det \mathbb{G}$.

Here we have to point out that while the above relation is indeed correct *for* and only for Riemannian metrics, the same cannot be said to metrics with no defined signature. For instance, let us consider the $3+1$ (flat) *Minkowski spacetime* $\mathbb{M}^4$ with a *Lorentz signature* $(+---)$, where "+" represents a time coordinate and the "−"ses the spatial sector of the spacetime. By computing the metric determinant, we get $g = -1$, which obviously break the validity of the left-hand side of (1.1.13) if we consider real-valued coordinate functions. What should we do then?

One solution would be to say everything constructed thus far is valid, then to resolve this contradiction we *have* to define the time coordinate as an *imaginary* number, so that when squared gives a negative number. If we opt to do that, the transformation Jacobian would be written in matrix form as

$$\mathbb{J} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (1.1.14)$$

so that $\mathbb{G} = \mathrm{diag}(-1, 1, 1, 1)$ when the transformation $x'_0 = ict$, with $t \in \mathbb{R}$, is performed from a pseudo-Euclidean (complex vector space, with Euclidean metric) to $\mathbb{M}^4$, and everything will work out fine from this point forward. This way of doing geometry was quite used in the genesis of the whole discipline when the mathematical background was not yet fully developed. The first to use the imaginary time componente were Poincaré and Weyl, in order to interpret Lorentz transformations as complex rotations and show the relativistic invariance of Maxwell electrodynamics (see Walter[60])

Another approach, much more modern[40] and unanimously adopted in the present, is to redefine the inner product that yields metrics with any signature, in particular, the Minkowski $(- + ++)$ signature. With that, the metric tensor has the form of

$$\eta \equiv \mathbb{G} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (1.1.15)$$

which implies a line element*

$$ds^2 = -c^2dt^2 + d\boldsymbol{x}^2\,, \qquad (1.1.16)$$

the same used in Special Relativity and, since $g = -1$, we have to adjust (1.1.13) to be always real, that is

*In a Lorentzian metric, the line element is also called a *spacetime interval*, where c is the speed of light.

$$|\det \mathbb{J}| := \sqrt{|g|}\,. \qquad (1.1.17)$$

The above equation is useful when considering the invariant volume element $\sqrt{|g|}dx^4$.

We remark that, as briefly discussed above, this cannot be a Riemannian metric anymore, though the 3−dimensional spatial sector has a signature $(+ + +)$ corresponding to, in fact, a Riemannian metric. Also, due to the square character of the line element, we can equivalently use the *mirrored signature* $(+ - --)$ instead without breaking anything, a convention commonly adopted by physicists and it will be the one adopted in this work.

Points in this space gets the fancy name of *events* and are represented by the tuple (x^0, x^i), with $x^0 = ct$, and vectors that live in it can be decomposed in a similar fashion (1.1.7) as

$$\begin{aligned} \boldsymbol{u} &= u^0 \boldsymbol{e}_0 + u^i \boldsymbol{e}_i \qquad , \quad i = 1, 2, 3 \\ &= u^\alpha \boldsymbol{e}_\alpha \qquad , \quad \alpha = 0, 1, 2, 3\,, \end{aligned} \qquad (1.1.18)$$

where now we make the index distinction of latin letters $(ijk = 1, 2, 3)$ spanning through all spatial coordinates and greek letters $(\mu\nu\alpha = 0, 1, 2, 3)$ through all coordinates, including the "temporal" one. These vectors get the name of 4-vectors in $\mathbb{M}^4$ or N-vectors in a broader space.

Another peculiar property of those N-vectors is that their norm is not always positive. In fact, this very property is suitable for describing the *causal* character of relativity. So, the N-vector $\boldsymbol{u}$ can be classified as

$$u = \begin{cases} \text{timelike} & , ||u|| > 0 \\ \text{spacelike} & , ||u|| < 0 \\ \text{light-like} & , ||u|| = 0 \end{cases} . \tag{1.1.19}$$

At the present, the construction of such inner product falls out of the scope of this section, but we shall return to this in Section 1.2 when we develop the tools necessary to redefine the inner product. Suffice to say that we shall be adopting a more general spacetime $\mathbb{M}^n$, which has indefinite signature and metric $g_{\mu\nu}$, with indices spanning from $\mu = 0, \ldots, N-1$. In this space, eveything established so far is also valid for greek indices.

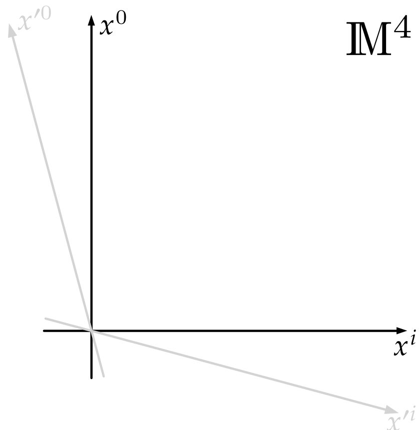


Figure 7: Minkowski diagrams used in special relativity for the metric of same name.

Now back to the main text.

With a defined metric tensor, we can see that the contravariant (1.1.7) and covariant (1.1.8) components of a vector are linked by

$$v_\mu = g_{\mu\nu} v^\nu \tag{1.1.20}$$

or conversely

$$v^\mu = g^{\mu\nu} v_\nu \,. \tag{1.1.21}$$

The two operations above are respectively called *lowering/raising operations* and they play a fundamental role in differential geometry.

Now let us consider a coordinate transformation between two coordinate systems $x^\mu \longrightarrow x'^\mu(x^\nu)$ that has non-singular Jacobian. We can relate both vector basis $\{\boldsymbol{e}_\mu\}$ and $\{\boldsymbol{e'}_\mu\}$ by

$$\boldsymbol{e'}_\mu = \frac{\partial \boldsymbol{r}}{\partial x'^\mu} = \frac{\partial \boldsymbol{r}}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\boldsymbol{e}_\nu \,, \tag{1.1.22}$$

which allow us to establish the transformation rule of the metric tensor (1.1.10)

$$g'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\sigma\rho} \,, \tag{1.1.23}$$

and the components of a vector (1.1.7)

$$\begin{aligned} \boldsymbol{v} = v^\lambda \boldsymbol{e}_\lambda &= \left(\frac{\partial x^\lambda}{\partial x'^\mu} v'^\mu\right) \boldsymbol{e}_\lambda \\ \Longrightarrow \; v^\lambda &= \frac{\partial x^\lambda}{\partial x'^\mu} v'^\mu \,. \end{aligned} \tag{1.1.24}$$

We would get exactly the same result by transforming dx into the new coordinate frame, so by that reason we can say that *contravariant components transform like coordinate differentials*.

Now, by using the transformation laws above, it can be readily shown how the covariant components transform,

$$v_\mu = \frac{\partial x'^\nu}{\partial x^\mu} v'_\nu \,. \tag{1.1.25}$$

While the contravariant components transform as differential of coordinates, we can get this transformation rule if we were to transform a partial derivative of a test function $f = f(x(x'))$, so we say that *covariant compontents transform like partial derivatives of some function of the coordinates*.

Tensors of higher orders can be constructed by doing successive *tensor product* operations on vectors, which expands the vector space dimension as the product of the dimensions of each individual space. By denoting this bilinear operation by $\otimes$ and grabbing two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ from our N-dimensional space, we can define a tensor of *rank* $(2,0)$ as

$$\begin{aligned} \overleftrightarrow{T} &= \boldsymbol{u} \otimes \boldsymbol{v} \\ &= u^\mu v^\nu \boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu \\ &= T^{\mu\nu} \boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu \,, \end{aligned} \tag{1.1.26}$$

or, if we get m vectors from our set, we define a tensor of rank $(m,0)$

$$\begin{aligned} \overleftrightarrow{T} &= \boldsymbol{u}_1 \underbrace{\otimes \ldots \otimes}_{m-1 \text{ times}} \boldsymbol{u}_m \\ &= T^{\alpha_1 \ldots \alpha_m} \boldsymbol{e}_{\alpha_1} \otimes \ldots \otimes \boldsymbol{e}_{\alpha_m} \,. \end{aligned} \tag{1.1.27}$$

The rank notation employed here denotes the amount of contravariant and covariant components, respectively.

With that we can further generalize the transformation law for a tensor with arbitrary rank. If $\overleftrightarrow{T}$ is a tensor of rank (m,n) with components $T^{\alpha_1\dots\alpha_m}{}_{\beta_1\dots\beta_n}$, we can transform its coordinates as

$$\overleftrightarrow{T} = T'^{\alpha_1\dots\alpha_m}{}_{\beta_1\dots\beta_n} \boldsymbol{e}'_{\alpha_1} \otimes \dots \otimes \boldsymbol{e}'_{\alpha_m} \otimes \boldsymbol{e}'^{\beta_1} \otimes \dots \otimes \boldsymbol{e}'^{\beta_n}$$
$$= \left(T'^{\alpha_1\dots\alpha_m}{}_{\beta_1\dots\beta_n} \frac{\partial x^{\rho_1}}{\partial x'^{\alpha_1}} \cdots \frac{\partial x^{\rho_m}}{\partial x'^{\alpha_m}} \frac{\partial x'^{\beta_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial x'^{\beta_n}}{\partial x^{\sigma_n}} \right) \boldsymbol{e}_{\rho_1} \otimes$$
$$\otimes \dots \otimes \boldsymbol{e}_{\rho_n} \otimes \boldsymbol{e}^{\sigma_1} \otimes \dots \otimes \boldsymbol{e}^{\sigma_m}$$

$$\therefore T^{\rho_1\dots\rho_m}{}_{\sigma_1\dots\sigma_n} = T'^{\alpha_1\dots\alpha_m}{}_{\beta_1\dots\beta_n} \frac{\partial x^{\rho_1}}{\partial x'^{\alpha_1}} \cdots \frac{\partial x^{\rho_m}}{\partial x'^{\alpha_m}} \frac{\partial x'^{\beta_1}}{\partial x^{\sigma_1}} \cdots \frac{\partial x'^{\beta_n}}{\partial x^{\sigma_n}} , \tag{1.1.28}$$

where here we made use of the *dual basis* $\{\boldsymbol{e}^\mu\}$ introduced in (1.1.8) and defined by the identity

$$\boldsymbol{e}_\nu \cdot \boldsymbol{e}^\mu = g^{\mu\kappa} \boldsymbol{e}_\nu \cdot \boldsymbol{e}_\kappa = g^{\mu\kappa} g_{\nu\kappa} = \delta^\mu_\nu . \tag{1.1.29}$$

A (m, n) tensor is said to be *contracted* if it is written as a linear combination whose coefficients correspond to a pair of repeated indices, covariant and contravariant, respectively, in its original components, returning a $(m-1, n-1)$ tensor as a result. For example,

$$T^{\alpha_1\dots\alpha_{m-1}}{}_{\beta_1\dots\beta_{n-1}} := T^{\alpha_1\dots\alpha_{m-1}\lambda}{}_{\beta_1\dots\beta_{n-1}\lambda}$$

Furthermore, the scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{u} \cdot \boldsymbol{v} = u^\mu v^\nu (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu) = g_{\mu\nu} u^\mu v^\nu = u^\mu v_\mu$$

is invariant under coordinate transformations. Indeed,

$$u'^\mu v'_\mu = \left(\frac{\partial x'^\mu}{\partial x^\alpha} u^\alpha \right) \left(\frac{\partial x^\beta}{\partial x'^\mu} v_\beta \right) = u^\alpha v_\beta \delta^\beta_\alpha = u^\alpha v_\alpha . \tag{1.1.30}$$

So if we can construct good scalar quantities while developing physical theories, it is guaranteed that those will be exactly the same if seen from *any* frame of reference. The relevance of this will be evident when we start dealing with contracted invariants later on.

Following the logical sequence, we are now apt to construct derivative operations acting on objects in a curvilinear space. For that sake, we take a vector (1.1.7) as a funcion of the x-coordinates and compute a derivative with respect to x^k. This operation takes a vector (rank 1 tensor) and returns a tensor of rank 2,

$$\begin{aligned} \boldsymbol{u}_{,\sigma} \equiv \frac{\partial \boldsymbol{u}}{\partial x^\sigma} &= \frac{\partial}{\partial x^\sigma}(u^\mu \boldsymbol{e}_\mu) = \frac{\partial u^\mu}{\partial x^\sigma}\boldsymbol{e}_\mu + u^\mu \frac{\partial \boldsymbol{e}_\mu}{\partial x^\sigma} \\ &= \left(\frac{\partial u^\mu}{\partial x^\sigma} + \Gamma^\mu_{\sigma\rho} u^\rho\right)\boldsymbol{e}_\mu \,, \end{aligned} \tag{1.1.31}$$

where the *Christoffel symbols of the second kind**are defined by the expansion of $u^\mu \dfrac{\partial \boldsymbol{e}_\mu}{\partial x^\sigma}$ back into the $\{\boldsymbol{e}_\mu\}$ basis and appear as the constants of that expansion. Those symbols are *not* tensors, so they will not transform by a rule such as (1.1.28). That is not surprinsing, for the defining relation explicitly shows their dependence on the coordinate basis.

Those symbols are also the *affine connection* that properly defines the derivative along a curve by means of parallel transportation, as we will see shortly. To be more precise, the construction above defines the affine connection which coincides with the Christoffel symbols of the second kind.

Now, the term inside the brackets are the components of the *covariant derivative* of a vector , which is commonly denoted as†

*One other archaic notation that is eventually used in the literare is $\{{}^{\,\mu}_{\sigma\rho}\}$.

†The covariant and partial derivative shorthand notation $\{\nabla\ \partial\} \rightarrow \{;\ ,\}$ shall be adopted further ahead when things rapidly start to go crazy and polluted notation-wise.

$$u^{\mu}_{;\sigma} \equiv \nabla_{\sigma} u^{\mu} := \frac{\partial u^{\mu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\sigma\rho} u^{\sigma} \,. \tag{1.1.32}$$

Now let us elaborate the Christoffel symbols a wee bit further. First contracting the highlighted relation by a dual basis vector $\boldsymbol{e}^{\lambda}$ we get

$$u^{\mu} \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\sigma}} \cdot \boldsymbol{e}^{\lambda} = \Gamma^{\mu}_{\sigma\rho} u^{\rho} (\boldsymbol{e}_{\mu} \cdot \boldsymbol{e}^{\lambda}) = \Gamma^{\lambda}_{\sigma\mu} u^{\mu}$$

and, since the u^{μ} are independent,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{\partial \boldsymbol{e}_{\beta}}{\partial x^{\alpha}} \cdot \boldsymbol{e}^{\mu} = g^{\mu\lambda} \frac{\partial \boldsymbol{e}_{\beta}}{\partial x^{\alpha}} \cdot \boldsymbol{e}_{\lambda} \,. \tag{1.1.33}$$

Moreover, if the basis vectors satisfy the integrability conditions, then by (1.1.5)

$$\frac{\partial \boldsymbol{e}_{\beta}}{\partial x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial \boldsymbol{r}}{\partial x^{\beta}} \right) = \frac{\partial^2 \boldsymbol{r}}{\partial x^{\alpha} \partial x^{\beta}} = \frac{\partial^2 \boldsymbol{r}}{\partial x^{\beta} \partial x^{\alpha}} = \frac{\partial \boldsymbol{e}_{\alpha}}{\partial x^{\beta}} \,. \tag{1.1.34}$$

This enables us to write (1.1.33) as

$$\begin{aligned} \Gamma^{\mu}_{\alpha\beta} &= g^{\mu\lambda} \frac{\partial \boldsymbol{e}_{\alpha}}{\partial x^{\beta}} \cdot \boldsymbol{e}_{\lambda} \\ &= g^{\mu\lambda} \Gamma_{\alpha\beta\lambda} \,, \end{aligned} \tag{1.1.35}$$

which is the usual form as defined in the literature and it is evident that they are symmetric in $\alpha\beta$. The all-covariant

$$\Gamma_{\alpha\beta\lambda} := \frac{1}{2} \left(\frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} + \frac{\partial g_{\beta\lambda}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} \right) \tag{1.1.36}$$

are called *Christoffel symbols of the first kind**. Now let us see how those objects transfom under a general coordinate transformation.

$$\begin{aligned}\Gamma^{\mu}_{\sigma\rho} &= g^{\mu\lambda}\frac{\partial \boldsymbol{e}_\rho}{\partial x^\sigma}\cdot \boldsymbol{e}_\lambda \\ &= \left(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\lambda}{\partial x'^\delta}g'^{\alpha\delta}\right)\frac{\partial}{\partial x^\sigma}\left(\frac{\partial x'^\gamma}{\partial x^\rho}\boldsymbol{e}'_\gamma\right)\cdot\left(\frac{\partial x'^\varepsilon}{\partial x^\lambda}\boldsymbol{e}'_\varepsilon\right) \\ &= \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\sigma}\frac{\partial x'^\gamma}{\partial x^\rho}\Gamma'^{\alpha}_{\beta\gamma} + \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial^2 x'^\alpha}{\partial x^\sigma \partial x^\rho}\,. \end{aligned} \tag{1.1.37}$$

Alternatively we can use the following identity

$$\partial_\sigma \delta^\mu_\rho \equiv 0 = \frac{\partial^2 x^\mu}{\partial x'^\beta \partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\sigma}\frac{\partial x'^\alpha}{\partial x^\rho} + \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial^2 x'^\alpha}{\partial x^\sigma\partial x^\rho}$$

to bring (1.1.37) in the form

$$\begin{aligned}\Gamma^{\mu}_{\sigma\rho} &= \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\sigma}\frac{\partial x'^\gamma}{\partial x^\rho}\Gamma'^{\alpha}_{\beta\gamma} - \frac{\partial x'^\beta}{\partial x^\sigma}\frac{\partial x'^\gamma}{\partial x^\rho}\frac{\partial^2 x^\mu}{\partial x'^\beta\partial x'^\gamma} \\ &= \frac{\partial x'^\beta}{\partial x^\sigma}\frac{\partial x'^\gamma}{\partial x^\rho}\left(\frac{\partial x^\mu}{\partial x'^\alpha}\Gamma'^{\alpha}_{\beta\gamma} - \frac{\partial^2 x^\mu}{\partial x'^\beta\partial x'^\gamma}\right), \end{aligned} \tag{1.1.38}$$

which clearly does not transform like a tensor. If we were interested in a referential frame in which $\Gamma^i_{jk} = 0$, we would get from the expression above the second order solution next to an arbitrary fixed point P

$$x^\mu - x^\mu_P = x'^\mu - x'^\mu_P + \frac{1}{2}\Gamma'^{\mu}_{\beta\gamma}\big|_P\left(x'^\beta - x'^\beta_P\right)\left(x'^\gamma - x'^\gamma_P\right). \tag{1.1.39}$$

Such frame of reference is baptized *Geodesic frame of reference*. It lies along *geodesic* lines and possesses the quality of being locally

*Similar to the Chirstoffel symbols of second kind, these also have alternative notations in other texts, such as the archaic $\{\alpha\beta,\lambda\}$ but also written as $\Gamma_{\lambda\alpha\beta}$, or with a comma $\Gamma_{\lambda,\alpha\beta}$, $\Gamma_{\alpha\beta,\lambda}$. We shall denote without commas to not mix up with the derivative notation and we put the "lowered index" at last. We judge this form to be the clearest.

flat, so no curvature is measurable in it, which is to be expected as we will see shortly. This result is quite useful; it allows us to simplify our calculations and then, thanks to the tensor character of the theory, will be valid in all frames of reference.

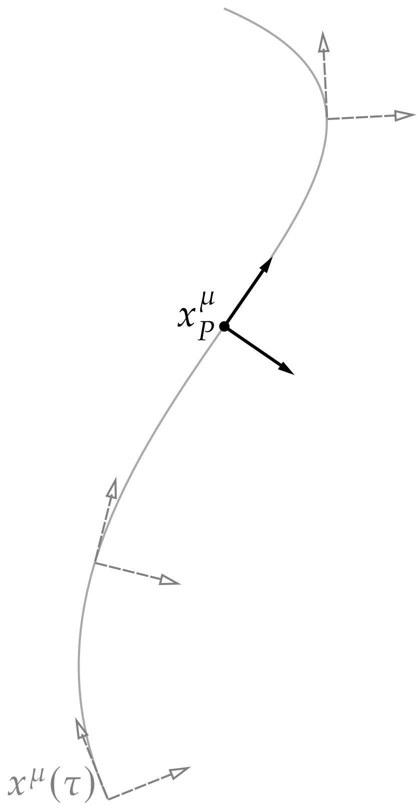


Figure 8: Geodesic frame of reference around x^{μ}_{P}.

We can construct the covariant derivative in tensor form as*

$$\boldsymbol{\nabla} \otimes \boldsymbol{u} = (\nabla_{\alpha} u^{\mu}) \boldsymbol{e}_{\mu} \otimes \boldsymbol{e}^{\alpha} = (g^{\alpha\beta} \nabla_{\alpha} u^{\mu}) \boldsymbol{e}_{\mu} \otimes \boldsymbol{e}_{\beta} \,. \tag{1.1.40}$$

*This is constructed by the tensor product of both $\boldsymbol{\nabla}$ and $\boldsymbol{u}$ expressed in their respective basis.

The covariant derivative is also applicable to general tensors as well, if we *impose* the Leibniz rule

$$
\begin{aligned}
\nabla_\lambda(u^\mu v^\nu) &= (\nabla_\lambda u^\mu)v^\nu + u^\mu(\nabla_\lambda v^\nu) \\
&= \left(\frac{\partial u^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa}u^\kappa\right)v^\nu + u^\mu\left(\frac{\partial v^\nu}{\partial x^\lambda} + \Gamma^\nu_{\lambda\kappa}v^\kappa\right) \\
&= \frac{\partial(u^\mu v^\nu)}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa}u^\kappa v^\nu + \Gamma^\nu_{\lambda\kappa}u^\mu v^\kappa\,,
\end{aligned}
$$

so, if $T^{\mu\nu}(x) = u^\mu(x)v^\nu(x)$,

$$
\nabla_\lambda T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa}T^{\kappa\nu} + \Gamma^\nu_{\lambda\kappa}T^{\mu\kappa}\,. \tag{1.1.41}
$$

Similarly, we can use the fact that scalar functions can be thought of as a scalar product between two vectors, $\phi(x) = \boldsymbol{u}(x)\cdot\boldsymbol{v}(x) = u^\nu(x)v_\nu(x)$, to find the covariant derivative of covariant vectors. This elegant way is only possible because the covariant derivative of scalar fields are reduced to a partial ordinary derivative.

$$
\nabla_\lambda(u^\nu v_\nu) \equiv \frac{\partial(u^\nu v_\nu)}{\partial x^\lambda} = (\nabla_\lambda u^\nu)v_\nu + u^\nu(\nabla_\lambda v_\nu)
$$

$$
\therefore\ \nabla_\lambda v_\nu = \frac{\partial v_\nu}{\partial x^\lambda} - \Gamma^\kappa_{\lambda\nu}v_\kappa\,, \tag{1.1.42}
$$

since u^ν is arbitrary. Covariant derivatives of tensors of higher orders can be construed in a similar fashion by decomposing them into a product of numerous vectors and applying the Leibniz rule along with (1.1.32) and (1.1.42). So if $\overleftrightarrow{T}$ is a tensor of rank (m, n), then we have

$$
\begin{aligned}
\nabla_\lambda T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n} &= \frac{\partial T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}}{\partial x^\lambda} \\
&+ \Gamma^{\alpha_1}_{\lambda\kappa} T^{\kappa\ldots\alpha_m}{}_{\beta_1\ldots\beta_n} + \ldots + \Gamma^{\alpha_m}_{\lambda\kappa} T^{\alpha_1\ldots\kappa}{}_{\beta_1\ldots\beta_n} \\
&- \Gamma^{\kappa}_{\lambda\beta_1} T^{\alpha_1\ldots\alpha_m}{}_{\kappa\ldots\beta_n} - \ldots - \Gamma^{\kappa}_{\lambda\beta_n} T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\kappa} \,.
\end{aligned}
\tag{1.1.43}
$$

With this we can study the metric tensor a bit more and derive one great property it possesses. So, taking the covariant derivative of it and according to (1.1.41) and (1.1.35), we have

$$
\begin{aligned}
\nabla_\lambda g^{\mu\nu} &= \frac{\partial g^{\mu\nu}}{\partial x^\lambda} + \Gamma^{\mu}_{\lambda\kappa} g^{\kappa\nu} + \Gamma^{\nu}_{\lambda\kappa} g^{\mu\kappa} \\
&= \frac{\partial g^{\mu\nu}}{\partial x^\lambda} - \frac{\partial g^{\mu\nu}}{\partial x^\lambda} \\
&= 0 \,.
\end{aligned}
$$

Finally, remembering that $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$,

$$
\begin{aligned}
\nabla_\lambda \delta^\mu_\nu \equiv 0 &= g^{\mu\sigma}(\nabla_\lambda g_{\sigma\nu}) + (\nabla_\lambda g^{\iota\sigma}) g_{\sigma\nu} \\
&= g^{\mu\sigma}(\nabla_\lambda g_{\sigma\nu})
\end{aligned}
$$

so

$$
\nabla_\lambda g^{\mu\nu} = 0 = \nabla_\lambda g_{\mu\nu} \,. \tag{1.1.44}
$$

This is particularly interesting because it enables us to raise and lower tensor indices even inside a covariant derivative.

Now if we contract (1.1.32) by dx^σ, we find the *total variation* or the infinitesimal *parallel transportation* of u^μ (denoted by Du^μ), namely

$$Du^\mu \equiv dx^\sigma \nabla_\sigma u^\mu = du^\mu + \Gamma^\mu_{\sigma\rho} dx^\sigma u^\rho \,, \tag{1.1.45a}$$

$$Du_\mu \equiv dx^\sigma \nabla_\sigma u_\mu = du_\mu - \Gamma^\rho_{\sigma\mu} dx^\sigma u_\rho \,. \tag{1.1.45b}$$

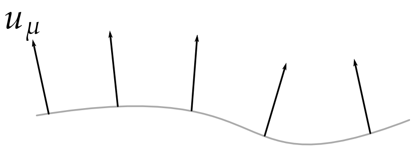


Figure 9: What happens to a vector subjected to parallel transportation.

Considering instead a parametrized curve $x = x(\tau)$, we are able to compute the variation of a vector along it with respect to the parameter τ. Thus, by denoting that as $\frac{D}{d\tau}$ and setting u^μ as the tangent to the curve, $u^\mu = \frac{dx^\mu}{d\tau}$, we have

$$\begin{aligned} \frac{D}{d\tau} u^\mu \equiv u^\sigma \nabla_\sigma u^\mu &= \frac{du^\mu}{d\tau} + \Gamma^\mu_{\sigma\rho} u^\sigma u^\rho \\ &= \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} \,. \end{aligned} \tag{1.1.46}$$

If the total change is such that $\frac{D}{d\tau} u^\mu = 0$, then

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\tau} \frac{dx^\rho}{d\tau} = 0 \,, \tag{1.1.47}$$

which is the *geodesic equation* whose solution, a *geodesic curve*, represents the “straightest” possible curve in a curved space and it also describes the trajetory of test particles free falling in the presence of a gravitational field. In the geodesic frame (1.1.39), we would have

$$\frac{d^2 x^\mu}{d\tau^2} = 0\,,$$

which has a linear solution

$$x^\mu = a^\mu + b^\mu \tau\ . \tag{1.1.48}$$

That means the free fall of particles in the geodesic frame of reference is given by straight lines, exactly as it would as if no curvature was present.

Some discussion follows.

In (1.1.45), the second term accounts for the deviation a vector suffers when subjected to a parallel transport, commonly denoted by δu^μ. Its presence guarantees that when evaluating a derivative at x^μ, correspondind to the increment dx^μ, we are actually evaluating the change of that vector between u^μ and $u^\mu + du^\mu$ as expected.

In fact, on a flat $\mathbb{M}^n$ space, vectors do not change their geometric properties from point to point due to its inherent affine character, but the same cannot be stated about generic manifolds. In general, a vector subjected to a parallel displacement will not "point" to the same "direction" when it goes around a closed loop along geodesic paths.

Enforcing that last bit, we have from (1.1.45b)

$$\frac{\partial u_\mu}{\partial x^\nu} = \Gamma^\kappa_{\mu\nu} u_\kappa\,, \tag{1.1.49}$$

so when we take the roundabout trip along C (see Fig. 10), we shall get a variation Δu_μ. Since we are going around back to the starting point, only the δu_μ variation survives. Thus

$$\Delta u_\mu = \oint_C \delta u_\mu = \oint_C \Gamma^\kappa_{\mu\nu} u_\kappa dx^\nu \,.$$

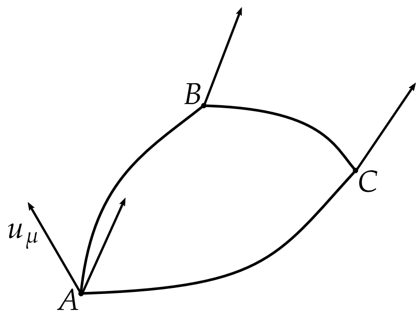


Figure 10: Parallel transport of a vector field u_μ around the circuit $C : ABC$.

Applying Stokes' theorem we get

$$\Delta u_\mu = \int_S dS^{\sigma\rho} \left(\frac{\partial(\Gamma^\kappa_{\mu\sigma} u_\kappa)}{\partial x^\rho} - \frac{\partial(\Gamma^\kappa_{\mu\rho} u_\kappa)}{\partial x^\sigma} \right),$$

where $dS^{\sigma\rho} = \frac{1}{2}\epsilon^{\sigma\rho}{}_{\alpha\beta} dx^\alpha dx^\beta$ is the anti-symmetric surface element.

The values u_i takes inside S are not unique, but they can be approximated, in this infinitesimal regime, by their value on the border without losing its validity. Doing that, results from (1.1.49)

$$\begin{aligned}
\Delta u_\nu &= \left(\frac{\partial(\Gamma^\mu_{\nu\alpha} u_\mu)}{\partial x^\beta} - \frac{\partial(\Gamma^\mu_{\nu\beta} u_\mu)}{\partial x^\alpha}\right)\Delta S^{\alpha\beta} \\
&= \left(\frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} u_\mu - \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} u_\mu - \Gamma^\mu_{\nu\alpha}\frac{\partial u_\mu}{\partial x^\beta} + \Gamma^\mu_{\nu\beta}\frac{\partial u_\mu}{\partial x^\alpha}\right)\Delta S^{\alpha\beta} \\
&= \left(\frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} u_\mu - \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} u_\mu - \Gamma^\mu_{\nu\alpha}\Gamma^\varepsilon_{\mu\beta} u_\varepsilon + \Gamma^\mu_{\nu\beta}\Gamma^\varepsilon_{\mu\alpha} u_\varepsilon\right)\Delta S^{\alpha\beta} \\
&= \left(\frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} - \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} - \Gamma^\mu_{\varepsilon\beta}\Gamma^\varepsilon_{\nu\alpha} + \Gamma^\mu_{\varepsilon\alpha}\Gamma^\varepsilon_{\nu\beta}\right) u_\mu \Delta S^{\alpha\beta} \\
&= R^\mu{}_{\nu\alpha\beta} u_\mu \Delta S^{\alpha\beta}
\end{aligned}$$

where

$$R^\mu{}_{\nu\alpha\beta} = \frac{\partial \Gamma^\mu_{\nu\alpha}}{\partial x^\beta} - \frac{\partial \Gamma^\mu_{\nu\beta}}{\partial x^\alpha} - \Gamma^\mu_{\varepsilon\beta}\Gamma^\varepsilon_{\nu\alpha} + \Gamma^\mu_{\varepsilon\alpha}\Gamma^\varepsilon_{\nu\beta} \tag{1.1.50}$$

is the *Riemann-Christoffel curvature tensor**, the main ingredient needed to work with General Relativity. It is clear from this approach how we can detect and quantify the curvature of Riemannian spaces, but there is still another more direct way of determining that tensor, which is by quantifying the amount of change accounted by two successive covariant derivatives, i.e., a commutator. In general, those derivatives do not commute, as we will verify promptly.

Firstly, let $T_{\beta\nu}$ be the tensor constructed by a covariant derivative of u_ν

$$T_{\beta\nu} = \nabla_\beta u_\nu = \frac{\partial u_\nu}{\partial x^\beta} - \Gamma^\mu_{\beta\nu} u_\mu \,.$$

Then,

*Sometimes the Riemann-Christoffel tensor is defined with the opposite sign.

$$
\begin{aligned}
[\nabla_\alpha, \nabla_\beta] u_\nu &= \nabla_\alpha T_{\beta\nu} \quad - [\alpha \leftrightarrow \beta] \\
&= R^{\mu}{}_{\nu\alpha\beta} u_\mu
\end{aligned}
\tag{1.1.51}
$$

which is precisely the same result as (1.1.50). It is important to note that the Riemann-Christoffel tensor is, in fact, a tensor, even if it merely depends on the Christoffel symbols. That happens because (1.1.51) is an equality, so, since the covariant derivative gives a tensor, the right-hand side of that equation must also be a tensor.

If we contract $\mu\beta$ in (1.1.50)

$$
R_{\nu\alpha} := R^{\lambda}{}_{\nu\alpha\lambda} = \frac{\partial \Gamma^{\lambda}_{\nu\alpha}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\nu\lambda}}{\partial x^{\alpha}} - \Gamma^{\varepsilon}_{\nu\alpha}\Gamma^{\lambda}_{\varepsilon\lambda} + \Gamma^{\varepsilon}_{\nu\lambda}\Gamma^{\lambda}_{\varepsilon\alpha} \tag{1.1.52}
$$

we obtain the so-called *Ricci tensor*, which is symmetric in its indices. In fact, the first, third and last terms follow immediately. To show that the second is also symmetric, we recall (1.1.34),

$$
\begin{aligned}
\Gamma^{\lambda}_{\nu\lambda,\alpha} &= \left(g^{\lambda\kappa} \boldsymbol{e}_{\nu,\lambda} \cdot \boldsymbol{e}_{\kappa} \right)_{,\alpha} \\
&= \boldsymbol{e}_{\lambda,\nu,\alpha} \cdot \boldsymbol{e}^{\lambda} + g^{\lambda\kappa} \boldsymbol{e}_{\lambda,\nu} \cdot \boldsymbol{e}_{\kappa,\alpha} + \Gamma_{\nu\lambda\kappa}\Gamma_{\alpha\varepsilon\sigma}\left(g^{\lambda\sigma} g^{\varepsilon\kappa} + g^{\kappa\sigma} g^{\varepsilon\lambda} \right),
\end{aligned}
$$

which is clearly symmetric as well.

Resuming, by further contracting (1.1.52),

$$
R := g^{\mu\nu} R_{\mu\nu}\,, \tag{1.1.53}
$$

we define the *Ricci scalar*.

The Riemann-Christoffel tensor exhibits many helpful symmetries which shall be derived next. For this purpose we will be doing all our computations in the geodesic frame of reference.

From (1.1.50) it is immediate that

$$R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha} \,. \tag{1.1.54}$$

Now, lowering the first index, we have

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= \Gamma_{\nu\alpha\mu,\beta} - \Gamma_{\nu\beta\mu,\alpha} \\ &= \frac{1}{2}\Big[\left(\cancel{g_{\nu\mu,\alpha}} + g_{\mu\alpha,\nu} - g_{\nu\alpha,\mu} \right)_{,\beta} - \\ &\qquad\qquad - \left(\cancel{g_{\nu\mu,\beta}} + g_{\mu\beta,\nu} - g_{\nu\beta,\mu} \right)_{,\alpha} \Big] \\ &= \frac{1}{2}\left(g_{\mu\alpha,\nu,\beta} - g_{\nu\alpha,\mu,\beta} - g_{\mu\beta,\nu,\alpha} + g_{\nu\beta,\mu,\alpha} \right) \\ &= -\frac{1}{2}\left(g_{\mu\beta,\nu,\alpha} - g_{\nu\beta,\mu,\alpha} - g_{\mu\alpha,\nu,\beta} + g_{\nu\alpha,\mu,\beta} \right) \\ &= -R_{\nu\mu\alpha\beta} \,, \end{aligned} \tag{1.1.55}$$

and, by the symmetry of the metric tensor and the partial derivatives,

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= \frac{1}{2}\left(g_{\mu\alpha,\nu,\beta} - g_{\nu\alpha,\mu,\beta} - g_{\mu\beta,\nu,\alpha} + g_{\nu\beta,\mu,\alpha} \right) \\ &= \frac{1}{2}\left(g_{\mu\alpha,\nu,\beta} - g_{\mu\beta,\nu,\alpha} - g_{\nu\alpha,\mu,\beta} + g_{\nu\beta,\mu,\alpha} \right) \\ &= \frac{1}{2}\left(g_{\alpha\mu,\beta,\nu} - g_{\beta\mu,\alpha,\nu} - g_{\alpha\nu,\beta,\mu} + g_{\beta\nu,\alpha,\mu} \right) \\ &= R_{\alpha\beta\mu\nu} \,. \end{aligned} \tag{1.1.56}$$

There are also two important sum identities, those being the cyclic sum of the covariant indexes and the same for covariant derivatives of the Riemann-Christoffel tensor. Let us compute them:

$$
\begin{aligned}
R^{\mu}{}_{\nu\alpha\beta} + R^{\mu}{}_{\alpha\beta\nu} + R^{\mu}{}_{\beta\nu\alpha} \\
&= \left(\Gamma^{\mu}_{\alpha\beta,\nu} - \Gamma^{\mu}_{\alpha\nu,\beta}\right) \\
&\quad + \left(\Gamma^{\mu}_{\beta\nu,\alpha} - \Gamma^{\mu}_{\nu\beta,\alpha}\right) \\
&\quad + \left(\Gamma^{\mu}_{\nu\alpha,\beta} - \Gamma^{\mu}_{\beta\alpha,\nu}\right) \\
&= 0\,,
\end{aligned}
\tag{1.1.57}
$$

$$
\begin{aligned}
R^{\mu}{}_{\nu\alpha\beta;\lambda} + R^{\mu}{}_{\nu\beta\lambda;\alpha} + R^{\mu}{}_{\nu\lambda\alpha;\beta} \\
&= \left(\Gamma^{\mu}_{\nu\alpha,\beta,\lambda} - \Gamma^{\mu}_{\nu\beta,\alpha,\lambda}\right) \\
&\quad + \left(\Gamma^{\mu}_{\nu\beta,\lambda,\alpha} - \Gamma^{\mu}_{\nu\alpha,\lambda,\beta}\right) \\
&\quad + \left(\Gamma^{\mu}_{\nu\lambda,\alpha,\beta} - \Gamma^{\mu}_{\nu\lambda,\beta,\alpha}\right) \\
&= 0\,.
\end{aligned}
\tag{1.1.58}
$$

Expressions (1.1.57) and (1.1.58) are respectively called *First* and *Second Bianchi Identities* and since they constitute tensor equations, they are valid in all frames. Here it becomes evident just how powerful the geodesic frame is; just imagine the absurd amount of terms we would have if the most generic Riemann- Christoffel were used.

From the second Bianchi identity, we can finally obtain the *Einstein field equations*, which is the core equation of General Relativity and Gravitation. To do that we contract it on the pairs $\mu\lambda$ and $\nu\alpha$,

$$
\begin{aligned}
0 &= R^{\mu\lambda}{}_{\lambda\nu;\mu} + R^{\mu\lambda}{}_{\nu\mu;\lambda} + R^{\mu\lambda}{}_{\mu\lambda;\nu} \\
&= R^{\lambda\mu}{}_{\nu\lambda;\mu} + R^{\lambda}{}_{\nu;\lambda} - R^{\lambda\mu}{}_{\mu\lambda;\nu} \\
&= R^{\mu}{}_{\nu;\mu} + R^{\mu}{}_{\nu;\mu} - R_{;\nu} \\
&= \left(2R^{\mu}{}_{\nu} - \delta^{\mu}_{\nu} R\right)_{;\mu}
\end{aligned}
$$

$$
0 = \left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R\right)_{;\mu} =: G^{\mu}{}_{\nu;\mu}\,, \tag{1.1.59}
$$

where

$$
G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{1.1.60}
$$

is the *Einstein tensor*, so that (1.1.59) has the general solution

$$
G^{\mu\nu} = \Lambda g^{\mu\nu} + \alpha T^{\mu\nu} \tag{1.1.61}
$$

if and only if $T^{\mu\nu}$ is a symmteric tensor such that $T^{\mu\nu}{}_{;\nu} = 0$. In the context of General Relativity, the solution can be interpreted in light of Mach's Principle, so both Λ and $T^{\mu\nu}$ must be associated with the matter content of the universe; in fact, the former is called the *Cosmological constant* whereas the latter, $T^{\mu\nu}$, represents just that and receives the name of *energy-momentum tensor*. Whilst it appears to be loosely connected through Mach's principle, we show in Appendix B that there is a principle of minima shying away from us in the background; so, by formulating this theory via the variational principle, those quantities naturally emerge back to light. There we show that $\alpha = \frac{8\pi G}{3}$, where G is Newton's *Universal Graviatational constant*.

We see how natural it is to construct all those essential quantities to work with geometry from the ground up, starting just with vector definitions and coordinate transformations between two systems of reference. It is remarkable how forthright the covariant derivatives and Christoffel symbols came to be and how clearly the latter depends on the variation of the coordinate basis vectors.

1.2 Local N-Tuples

One other remarkable formalism is the description of geometrical objects, be it vectors, tensors and whatnot, on the *locally inertial frame of reference*, in which the quantities in question are expressed in a flat Minkowskian space $\mathbb{M}^n$, at every point. Effectively, the tensor components will be mapped to the each component of this new basis, removing the coordinate characterization from them and thus virtually transforming them into scalar fields within this frame, while the actual coordinate information is carried by those components.

The components of said basis constitute what we call *N-Tuples*, functions of the point and denoted by $e^{(\alpha)}{}_{\mu}(x)$, with $\mu = 0, \ldots, N-1$ coordinate indices and $(\alpha) = 0, \ldots, N-1$ N-Tuple component indices. In a (3+1) spacetime those objects are also called *tetrads* and *vierbein**.

We start by decomposing the basis vectors (1.1.5) $\boldsymbol{e}(x)$ into a local orthogonormal basis $\hat{\boldsymbol{e}}^{\mathbf{0}}$ with a Lorentzian signature $(+--\ldots-)$,

$$\boldsymbol{e}_{\mu}(x) = e^{(\alpha)}{}_{\mu}(x)\,\hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(\alpha)}\,, \tag{1.2.1}$$

where clearly the N^2 decomposition components $e^{(\alpha)}{}_{\mu}$ are $N \times N$ matrices and depend on the point due to the cartesian basis being constant vectors. It is crucial to understand that even though this basis does not explicitly depend on the coordinates of the point, they *are not the same*

*From german: four legs.

on every point; in fact, we erect a whole new set of $\hat{\boldsymbol{e}}^{\mathbf{0}}$ at every single point of the manifold.

In order to recover the desired signature, we *must* enforce a new inner product definition. Henceforth the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ of $\mathbb{M}^n$, $a \in \mathbb{R}$, the inner product $\boldsymbol{u} \cdot \boldsymbol{v}$ is a symmetric bilinear operation

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= \boldsymbol{v} \cdot \boldsymbol{u} \\ (a\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} &= a(\boldsymbol{u} \cdot \boldsymbol{w}) + \boldsymbol{v} \cdot \boldsymbol{w} \,, \end{aligned} \tag{1.2.2}$$

such that the N-Tuples satisfy*

$$\begin{array}{c} \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(0)} \cdot \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(0)} = 1 \quad ; \quad \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(i)} \cdot \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(j)} = -\delta_{ij} \\ \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(0)} \cdot \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(i)} = 0 \end{array} \,, \tag{1.2.3}$$

where $\hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(0)}$ represents a time direction and can be thought of as a "time versor" $\hat{\boldsymbol{t}}$, whereas the other $\hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(i)}$ are the usual cartesian versors $\hat{\boldsymbol{x}}_i$.

With that, if we employ (1.1.10),

$$\hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(\alpha)} \cdot \hat{\boldsymbol{e}}^{\mathbf{0}}{}_{(\beta)} = \eta_{\alpha\beta} \,, \tag{1.2.4}$$

we readly obtain the local Minkowski space. The norm of this space, called the *Lorentz norm*, is defined as

$$\begin{aligned} ||\boldsymbol{u}|| := \boldsymbol{u} \cdot \boldsymbol{u} &= u^{\mu} u^{\nu} \eta_{\mu\nu} \\ &= (u^0)^2 - (u^1)^2 - (u^2)^2 - \ldots - (u^{n-1})^2 \\ &= (u^0)^2 - \sum_{i=1}^{N-1} (u^i)^2 \,. \end{aligned} \tag{1.2.5}$$

*In the case of any other type of signature, the numbers of positive and negative eigenvalues are enforced analougously.

Now, according to (1.1.10) and (1.1.15), we can relate the actual metric of the spacetime manifold with the local inertial frame metric discussed, namely,

$$\begin{aligned} g_{\mu\nu}(x) &= \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu \\ &= (e^{(\alpha)}{}_\mu(x)\, \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\alpha)}) \cdot (e^{(\beta)}{}_\nu(x)\, \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\beta)}) \\ &= e^{(\alpha)}{}_\mu(x)\, e^{(\beta)}{}_\nu(x)\, (\hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\alpha)} \cdot \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\beta)}) \\ &= \eta_{\alpha\beta}\, e^{(\alpha)}{}_\mu(x)\, e^{(\beta)}{}_\nu(x)\,. \end{aligned} \tag{1.2.6}$$

In a similar fashion to (1.1.29), we can also define a dual N-Tuple basis

$$\begin{aligned} \boldsymbol{e}^\mu \cdot \boldsymbol{e}_\nu \equiv \delta^\mu_\nu &= (e_{(\alpha)}{}^\mu\, \hat{\boldsymbol{e}}^{\boldsymbol{0}(\alpha)}) \cdot (e^{(\beta)}{}_\nu\, \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\beta)}) \\ &= e_{(\alpha)}{}^\mu\, e^{(\beta)}{}_\nu\, (\hat{\boldsymbol{e}}^{\boldsymbol{0}(\alpha)} \cdot \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\beta)}) \\ &= e_{(\alpha)}{}^\mu\, e^{(\beta)}{}_\nu\, \delta^\alpha_\beta \\ &= e_{(\alpha)}{}^\mu\, e^{(\alpha)}{}_\nu\,, \end{aligned} \tag{1.2.7}$$

where we dropped the position dependency to clean things up a bit. From this we can have some fun and obtain some nice properties. Thus, by contracting the expression above with $e_{(\beta)}{}^\nu$

$$e_{(\beta)}{}^\mu = e_{(\alpha)}{}^\mu (e^{(\alpha)}{}_\nu e_{(\beta)}{}^\nu)\,,$$

which is satisfied only if

$$e^{(\alpha)}{}_\nu e_{(\beta)}{}^\nu = \delta^\alpha_\beta\,, \tag{1.2.8}$$

meaning that the N-Tuples are orthogonal, to one another, allowing us to invert (1.2.6),

$$\eta_{\alpha\beta} = g_{\mu\nu} e_{(\alpha)}{}^{\mu} e_{(\beta)}{}^{\nu} ,$$

and perform raising/lowering operations, since we know it is applicable to coordinate indices

$$\eta_{\alpha\beta} = e_{(\alpha)\mu} e_{(\beta)}{}^{\mu} \tag{1.2.9}$$

or

$$\eta^{\alpha\beta} = e^{(\alpha)\mu} e^{(\beta)}{}_{\mu} .$$

Now by contracting (1.2.9) with $\eta^{\beta\gamma}$ and comparing with (1.2.8),

$$e_{(\alpha)}{}^{\mu} (\eta^{\beta\gamma} e_{(\beta)\mu}) \equiv \delta^{\gamma}_{\alpha} = e^{(\gamma)}{}_{\mu} e_{(\alpha)}{}^{\mu} ,$$

yielding

$$\begin{aligned} \eta^{\beta\gamma} e_{(\beta)\mu} &= e^{(\gamma)}{}_{\mu} \\ \eta_{\beta\gamma} e^{(\gamma)}{}_{\mu} &= e_{(\beta)\mu} \end{aligned} , \tag{1.2.10}$$

we see that N-Tuple indices can also be raised/lowered by the local Minkowskian metric $\eta_{\alpha\beta}$.

A similar type of transformation akin to (1.2.6) is also valid to any tensor field. To show that, we pick out one vector (1.1.7) and express it both in the local inertial basis and the decomposed basis (1.2.1),

$$\boldsymbol{v} = v^{(\alpha)} \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\alpha)} = v^{\mu} \boldsymbol{e}_{\mu}$$
$$= v^{\mu} e^{(\alpha)}{}_{\mu} \hat{\boldsymbol{e}}^{\boldsymbol{0}}{}_{(\alpha)}$$

$$\therefore v^{(\alpha)} = v^{\mu} e^{(\alpha)}{}_{\mu} \, . \tag{1.2.11}$$

The same can be done for covariant vectors

$$v_{(\alpha)} = v_{\mu} e_{(\alpha)}{}^{\mu} \, , \tag{1.2.12}$$

and both transformation laws are valid to tensors of any rank due to (1.1.27).

Synthetizing all that was discussed and considering a generic tensor $\overleftrightarrow{T}$ of rank (m, n) to generalize the two expressions above, we have

$$g_{\mu\nu} = \eta_{\alpha\beta} e^{(\alpha)}{}_{\mu} e^{(\beta)}{}_{\nu} \, , \tag{1.2.13a}$$
$$e_{(\alpha)}{}^{\mu} e^{(\alpha)}{}_{\nu} = \delta^{\mu}_{\nu} \, , \tag{1.2.13b}$$
$$e_{(\beta)}{}^{\mu} e^{(\alpha)}{}_{\mu} = \delta^{\alpha}_{\beta} \, , \tag{1.2.13c}$$
$$e_{(\alpha)}{}^{\mu} e_{(\beta)\mu} = \eta_{\alpha\beta} \, , \tag{1.2.13d}$$
$$T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = T^{(\alpha_1) \dots (\alpha_m)}{}_{(\beta_1) \dots (\beta_n)} \cdot \cdot e_{(\alpha_1)}{}^{\mu_1} \dots e_{(\alpha_m)}{}^{\mu_m} e^{(\beta_1)}{}_{\nu_n} \dots e^{(\beta_n)}{}_{\nu_n} \, , \tag{1.2.13e}$$

and their respective inverses. So, by (a) and (e), we can rewrite the spacetime interval as

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= \eta_{\alpha\beta} (e^{(\alpha)}{}_\mu dx^\mu)(e^{(\beta)}{}_\nu dx^\nu) \\
&= \eta_{\alpha\beta} dx^{(\alpha)} dx^{(\beta)} \,, \qquad (1.2.14)
\end{aligned}
$$

where we defined the new forms

$$
dx^{(\alpha)} = e^{(\alpha)}{}_\mu dx^\mu \,.
$$

If φ is a test function, we define the directional derivative along the local (α) direction by projecting the directional derivative into a N-Tuple

$$
\varphi_{,(\alpha)} \equiv \frac{\partial \varphi}{\partial x^{(\alpha)}} = e_{(\alpha)}{}^\mu \frac{\partial \varphi}{\partial x^\mu} \,. \qquad (1.2.15)
$$

It is useful to define the following quantities:

$$
\gamma_{\alpha\beta\gamma} := e_{(\alpha)\mu;\nu} e_{(\beta)}{}^\mu e_{(\gamma)}{}^\nu \,, \qquad (1.2.16a)
$$

$$
C_{\alpha\beta\gamma} := \gamma_{\alpha\beta\gamma} - \gamma_{\alpha\gamma\beta} \,, \qquad (1.2.16b)
$$

the former are called *Ricci rotation coefficients* and the latter will be the *structure constants* of an associated *Lie algebra* later on.

Expression (1.2.16a) is promptly inverted, yielding a practical way to evaluate the covariant derivative of N-Tuples

$$
e_{(\alpha)\mu;\nu} = \gamma_{\alpha\beta\gamma} e^{(\beta)}{}_\mu e^{(\gamma)}{}_\nu \,. \qquad (1.2.17)
$$

Let us work a bit with those definitions to find some symmetries. First, we express the constants only using N-Tuples

$$
\begin{aligned}
C_{\alpha\beta\gamma} &= e_{(\alpha)\mu;\nu} [\underbrace{e_{(\gamma)}{}^{\mu} e_{(\beta)}{}^{\nu}}_{\mu\leftrightarrow\nu} - e_{(\beta)}{}^{\mu} e_{(\gamma)}{}^{\nu}] \\
&= \left[e_{(\alpha)\mu;\nu} - e_{(\alpha)\nu;\mu} \right] e_{(\beta)}{}^{\mu} e_{(\gamma)}{}^{\nu} ,
\end{aligned} \tag{1.2.18}
$$

which are clearly anti-symmetric on $\beta\gamma$.

Next, rearranging (1.2.16a), we obtain

$$
\begin{aligned}
\gamma_{\alpha\beta\gamma} &= e_{(\alpha)\mu;\nu} e_{(\beta)}{}^{\mu} e_{(\gamma)}{}^{\nu} \\
&= -\gamma_{\beta\alpha\gamma} ,
\end{aligned}
$$

which is anti-symmetric on $\alpha\beta$. Summarizing,

$$
\begin{aligned}
C_{\alpha\beta\gamma} &= -C_{\alpha\gamma\beta} \\
\gamma_{\alpha\beta\gamma} &= -\gamma_{\beta\alpha\gamma}
\end{aligned} . \tag{1.2.19}
$$

Finally, we shall invert (1.2.16b) and express $\gamma_{\alpha\beta\gamma}$ in terms of the structure constants. To do that, we compute a cyclic sum in an analogous manner as done to the metric in the Christoffel symbols (1.1.35)

$$
\begin{aligned}
C_{\alpha\beta\gamma} + C_{\beta\gamma\alpha} - C_{\gamma\alpha\beta} = &(\gamma_{\alpha\beta\gamma} - \gamma_{\alpha\gamma\beta}) \\
&+ (\gamma_{\beta\gamma\alpha} - \gamma_{\beta\alpha\gamma}) \\
&- (\gamma_{\gamma\alpha\beta} - \gamma_{\gamma\beta\alpha})
\end{aligned}
$$

$$
\therefore \gamma_{\alpha\beta\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} + C_{\beta\gamma\alpha} - C_{\gamma\alpha\beta}) . \tag{1.2.20}
$$

Our aim is to describe the fundamental geometrical objects in this frame of reference to eliminate the coordinate charactarization in favor of a much simpler scalar quantity, but to do that we still have to establish how the covariant derivatives are projected onto this N-Tuple frame. This is easily done with the help of (1.2.17),

$$\begin{aligned} A_{\mu\nu;\lambda} &= \left(A_{(\alpha)(\beta)} e^{(\alpha)}{}_{\mu} e^{(\beta)}{}_{\nu}\right)_{;\lambda} \\ \therefore A_{\mu\nu;\lambda} e_{(\alpha)}{}^{\mu} e_{(\beta)}{}^{\nu} e_{(\gamma)}{}^{\lambda} &= A_{(\alpha)(\beta);(\gamma)} + \gamma_{\delta\alpha\gamma} A^{(\delta)}{}_{(\beta)} + \gamma_{\delta\beta\gamma} A_{(\alpha)}{}^{(\delta)} \,. \end{aligned} \tag{1.2.21}$$

Back to (1.2.17), we precompute a second covariant derivative that will be needed in a second

$$\begin{aligned} (e_{(\alpha)\mu;\nu})_{;\lambda} &= (\gamma_{\alpha\beta\gamma} e^{(\beta)}{}_{\mu} e^{(\gamma)}{}_{\nu})_{;\lambda} \\ &= \gamma_{\alpha\beta\gamma;\lambda} e^{(\beta)}{}_{\mu} e^{(\gamma)}{}_{\nu} \\ &\quad + (\gamma_{\alpha\beta\gamma} \gamma^{\beta}{}_{\delta\varepsilon} - \gamma_{\alpha\delta\beta} \gamma^{\beta}{}_{\gamma\varepsilon}) e^{(\delta)}{}_{\mu} e^{(\varepsilon)}{}_{\nu} e^{(\gamma)}{}_{\lambda} \,. \end{aligned} \tag{1.2.22}$$

Now the problem boils down to find the curvature tensor $R_{\mu\nu\alpha\beta}$ by computing the commutator (1.1.51) using the above result. So, without further ado

$$\begin{aligned} e_{(\gamma)}{}^{\mu} R_{\mu\nu\alpha\beta} &= e_{(\gamma)\nu;\alpha;\beta} - e_{(\gamma)\nu;\beta;\alpha} \\ \rightarrow R_{(\alpha)(\beta)(\gamma)(\delta)} &= (e_{(\alpha)\lambda;\mu;\nu} - e_{(\alpha)\lambda;\nu;\mu}) e_{(\beta)}{}^{\lambda} e_{(\gamma)}{}^{\mu} e_{(\delta)}{}^{\nu} \\ &= \gamma_{\alpha\beta\gamma;\nu} e_{(\delta)}{}^{\nu} - \gamma_{\alpha\beta\delta;\mu} e_{(\gamma)}{}^{\mu} \\ &\quad + \gamma_{\alpha\beta\lambda} (\gamma^{\lambda}{}_{\delta\gamma} - \gamma^{\lambda}{}_{\gamma\delta}) \\ &\quad + \gamma_{\alpha\lambda\delta} \gamma^{\lambda}{}_{\beta\gamma} - \gamma_{\alpha\lambda\gamma} \gamma^{\lambda}{}_{\beta\delta} \,, \end{aligned}$$

but, since we are on the local tangent frame, the covariant derivatives are reduced to the ordinary directional derivatives (1.2.15). Hence

$$R_{(\alpha)(\beta)(\gamma)(\delta)} = \gamma_{\alpha\beta\gamma,\delta} - \gamma_{\alpha\beta\delta,\gamma} + \gamma_{\alpha\beta\lambda}({\gamma^{\lambda}}_{\delta\gamma} - {\gamma^{\lambda}}_{\gamma\delta}) + \gamma_{\alpha\lambda\delta}{\gamma^{\lambda}}_{\beta\gamma} - \gamma_{\alpha\lambda\gamma}{\gamma^{\lambda}}_{\beta\delta} \quad <++> \; . \tag{1.2.23}$$

We also wish to express the Ricci tensor in terms of the structure constants (1.2.18). For this purpose, we contract the above obtained Riemann tensor on $\alpha\gamma$, employ (1.2.20) and, by using the symmetry of the Ricci tensor, we get

$$\therefore R_{\beta\delta} = -\frac{1}{2}\Big({C^{\alpha}}_{\alpha\beta,\delta} + {C^{\alpha}}_{\alpha\delta,\beta} + {{C_{\beta\delta}}^{\alpha}}_{,\alpha} + {{C_{\delta\beta}}^{\alpha}}_{,\alpha} + {C^{\alpha}}_{\alpha\lambda}{C_{\beta\delta}}^{\lambda} + {C^{\alpha}}_{\alpha\lambda}{C_{\delta\beta}}^{\lambda} - \frac{1}{2}{C_{\delta}}^{\lambda\alpha}C_{\beta\lambda\alpha} + {C^{\lambda\alpha}}_{\delta}C_{\alpha\lambda\beta} + {C^{\lambda\alpha}}_{\delta}C_{\lambda\alpha\beta}\Big) \, . \tag{1.2.24}$$

The utility of this result will become evident in Chapter 3, when we use this formalism to work with Lie algebras, providing a direct way to determine the curvature of the spaces knowing only the structure constants that characterize each algebra.

1.3 Killing Vectors

Usually, the treatment of the highly non-linear Einstein field equations (1.1.59) is very difficult or even analytically impossible for many problems, so it is indispensable that we formulate new ways and tools to help us to gain some insight and make our computation much less distressing and efficient.

One such way is to represent the fundamental quantities in the local N-Tuple frame discussed in the previous section. Another interesting solution is to look up at the *symmetries* of the spacetime carried out by the metric tensor, where the so-called *Killing fields* naturally emerge and are the fundamental objects in this description, responsible for carring out said symmetries. In order to accomplish such task, we shall borrow the already well established (infinitesimal) local approach from field theory.

We highlight that despite of the seemingly ideal aspects enclosed in the motivation above, we have many empirical data that our Universe has a remarkable degree of symmetry regarding its spatial sector, thus justifying the present treatment.

To do that we must construct a covariant toolbox that does not depend on a particular choice of a frame of reference. So, we start by defining the *isometry* of the metric tensor $g_{\mu\nu}$ as a transformation of coordinates that leaves its functional form intact. If we take a diffeomorphism of the metric tensor to another system of coordinates $x \to x'$,

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}(x') \, . \tag{1.3.1}$$

the isometry condition will be given by

$$g'_{\alpha\beta}(x') = g_{\alpha\beta}(x') \, , \qquad \forall x' \, . \tag{1.3.2}$$

Metric tensors (and tensors in general) that satisfy this condition are also called *form invariant*. Since isometries inherently represents some kind of symmetry of the spacetime itself, it must be associated to some symmetry group parametrized by, say, ε. That allows us to exploit the formalism of small pertubations around the point*. Then, by considering a transformation of coordinates to the immediate neighbouring of x,

*A more careful and precise construction can be done if we define the called *Lie Transport*, an operation that drags geometric objects along symmetry directions. These linear approximations and the next results follows naturally in this formalism.

$$x \longrightarrow x' = x + \delta x \,,$$

and setting a pertubation δx to the group transformation along with its parameter,

$$\delta x = \varepsilon \xi(x) \,,$$

where $|\varepsilon| \ll 1$, we can describe the infinitesimal coordinate transformation by

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon \xi^\mu(x) \,, \qquad |\varepsilon| \ll 1 \,. \tag{1.3.3}$$

Now, we plug (1.3.3) into (1.3.1) to obtain

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\delta^\alpha_\mu + \varepsilon \xi^\alpha{}_{,\mu}(x')\right)\left(\delta^\beta_\nu + \varepsilon \xi^\beta{}_{,\nu}(x')\right) g'_{\alpha\beta}(x') \\ &= g'_{\mu\nu}(x) + \varepsilon \xi^\alpha(x) g'_{\mu\nu,\alpha}(x) \\ &\quad + \varepsilon \left(g'_{\mu\alpha}(x) \xi^\alpha{}_{,\nu}(x) + g'_{\alpha\nu}(x) \xi^\alpha{}_{,\mu}(x) \right) \end{aligned}$$

$$\longrightarrow \mathcal{L}_\xi[g_{\mu\nu}] = \xi^\alpha g_{\mu\nu,\alpha} + g_{\alpha\nu} \xi^\alpha{}_{,\mu} + g_{\mu\alpha} \xi^\alpha{}_{,\nu} \,,$$

where we have made several expansions in Taylor series, discarted the terms of order $\mathcal{O}(\varepsilon^2)$ and defined the *Lie derivative* of a tensor T:

$$\mathcal{L}_\xi[T] = \lim_{\varepsilon \to 0} \frac{T(x) - T'(x)}{\varepsilon} \,, \tag{1.3.4}$$

which quantifies how "different" the tensor is in a point immeately next to x. This operation maps the spacetime back into itself $\mathcal{L}_\xi : \mathcal{M} \rightarrow \mathcal{M}$, thus defining an automorphism, and respecting the Leibniz rule.

Indeed, if T is a tensor of rank $(0, m + n)$ whose components are

$$T_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_n} = f A_{\alpha_1 \dots \alpha_m} B_{\beta_1 \dots \beta_n} ,$$

we have

$$
\begin{aligned}
\mathcal{L}_\xi[T_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n}] &= \mathcal{L}_\xi[f A_{\alpha_1\ldots\alpha_m} B_{\beta_1\ldots\beta_n}] \\
&= \xi^\lambda T_{\alpha_1\ldots\alpha_m\beta_1\ldots\beta_n,\lambda} \\
&\quad + T_{\lambda\ldots\alpha_m\beta_1\ldots\beta_n} \xi^\lambda{}_{,\alpha_1} + \ldots \\
&\quad\quad + \ldots + T_{\alpha_1\ldots\lambda\beta_1\ldots\beta_n} \xi^\lambda{}_{,\alpha_m} \\
&\quad + T_{\alpha_1\ldots\alpha_m\lambda\ldots\beta_n} \xi^\lambda{}_{,\beta_1} + \ldots \\
&\quad\quad + \ldots + T_{\alpha_1\ldots\alpha_m\beta_1\ldots\lambda} \xi^\lambda{}_{,\beta_n} \\
&= \xi^\lambda (f A_{\alpha_1\ldots\alpha_m} B_{\beta_1\ldots\beta_n})_{,\lambda} \\
&\quad + f A_{\lambda\ldots\alpha_m} B_{\beta_1\ldots\beta_n} \xi^\lambda{}_{,\alpha_1} + \ldots \\
&\quad\quad + \ldots + f A_{\alpha_1\ldots\lambda} B_{\beta_1\ldots\beta_n} \xi^\lambda{}_{,\alpha_m} \\
&\quad + f A_{\alpha_1\ldots\alpha_m} B_{\lambda\ldots\beta_n} \xi^\lambda{}_{,\beta_1} + \ldots \\
&\quad\quad + \ldots + f A_{\alpha_1\ldots\alpha_m} B_{\beta_1\ldots\lambda} \xi^\lambda{}_{,\beta_n} \\
&= \left(\xi^\lambda f_{,\lambda}\right) A_{\alpha_1\ldots\alpha_m} B_{\beta_1\ldots\beta_n} \\
&\quad + f B_{\beta_1\ldots\beta_n} \big(\xi^\lambda A_{\alpha_1\ldots\alpha_m,\lambda} + \\
&\quad\quad + A_{\lambda\ldots\alpha_m} \xi^\lambda{}_{,\alpha_1} + \ldots \\
&\quad\quad + \ldots + A_{\alpha_1\ldots\lambda} \xi^\lambda{}_{,\alpha_m}\big) \\
&\quad + f A_{\alpha_1\ldots\alpha_m} \big(\xi^\lambda B_{\beta_1\ldots\beta_n,\lambda} + \\
&\quad\quad + B_{\lambda\ldots\beta_n} \xi^\lambda{}_{,\beta_1} + \ldots \\
&\quad\quad + \ldots + B_{\beta_1\ldots\lambda} \xi^\lambda{}_{,\beta_n}\big) \\
&= \mathcal{L}_\xi[f] A_{\alpha_1\ldots\alpha_m} B_{\beta_1\ldots\beta_n} \\
&\quad\quad + f \mathcal{L}_\xi[A_{\alpha_1\ldots\alpha_m}] B_{\beta_1\ldots\beta_n} \, . \\
&\quad\quad + f A_{\alpha_1\ldots\alpha_m} \mathcal{L}_\xi[B_{\beta_1\ldots\beta_n}]
\end{aligned}
\tag{1.3.5}
$$

Immediately, the isometry condition (1.3.2) is codified such that the Lie derivative of the metric tensor vanishes

$$\mathcal{L}_{\xi}[g_{\mu\nu}] = 0\,. \tag{1.3.6}$$

Therefore, the *Killing conditions* are

$$\xi^{\alpha} g_{\mu\nu,\alpha} + g_{\alpha\nu}\xi^{\alpha}{}_{,\mu} + g_{\mu\alpha}\xi^{\alpha}{}_{,\nu} = 0\,, \tag{1.3.7}$$

or, in the covariant form,

$$\therefore\ \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0\,. \tag{1.3.8}$$

Any N-vector field $\xi_{\mu}(x)$ that satisfies (1.3.8) gets the name of *Killing fields* or simply *Killing vectors*, which are the generators of the associated symmetry group that carries the symmetries of the space. To determine all the isometries of the spaces, we just need to find all the Killing fields that satisfy the condition above. Inasmuch as those Killing fields represent the symmetries of the space, we also expect them to be related to conserved quantities by virtue of Noether theorem.

We pick the case of Killing fields associated with the conservation of the total linear momentum

Example:

Linear momentum and associated Killing vector:

$$
\begin{aligned}
\frac{d}{d\tau}\left(\xi^{\mu}P_{\mu}\right) &= \xi_{\mu}\frac{dP^{\mu}}{d\tau} + \frac{d\xi_{\mu}}{d\tau}P^{\mu} \\
&= \xi_{\mu}\left(-\frac{1}{m}\Gamma^{\mu}_{\alpha\beta}P^{\alpha}P^{\beta}\right) + \frac{\partial\xi_{\mu}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\tau}P^{\mu} \\
&= -\xi_{\mu}\frac{1}{m}\Gamma^{\mu}_{\alpha\beta}P^{\alpha}P^{\beta} + \frac{1}{m}\xi_{\mu,\nu}P^{\nu}P^{\mu} \\
&= \frac{1}{m}\left(\xi_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu}\xi_{\alpha}\right)P^{\mu}P^{\nu} \\
&= \frac{1}{m}\underbrace{\xi_{\mu;\nu}}_{\text{anti-sym}}\underbrace{P^{\mu}P^{\nu}}_{\text{sym}} \\
&= 0
\end{aligned}
$$

$$
\therefore\ \xi_{\mu}P^{\mu} = \text{const}\,,
$$

where we made use of the Geodesic equation (1.1.47),
$\frac{dP^{\mu}}{d\tau} + \frac{1}{m}\Gamma^{\mu}_{\alpha\beta}P^{\alpha}P^{\beta} = 0$. ■

In order to relate the Killing vectors to geometry, we will explicitly relate them with the curvature tensor by using (1.1.51) and, employing the first Bianchi identities (1.1.57), simplify the results. More specifically, applying (1.1.51) to ξ_{ρ} on $[\nabla_{\mu}, \nabla_{\nu}]$ and summing the results cyclically yields

$$
\begin{aligned}
\xi_{\mu;\nu;\rho} &= \xi_{\rho;\nu;\mu} - \xi_{\rho;\mu;\nu} \\
&= R^{\lambda}{}_{\rho\nu\mu}\xi_{\lambda}\,.
\end{aligned} \tag{1.3.9}
$$

In other words, *only the Killing vectors and their first covariant derivatives are independent*; further covariant derivatives will always be expressed in terms of ξ_{μ} and $\xi_{\mu;\nu}$ only. Though extremely restritive, we can use this fact for our benefit. For instance, if we know the values

ξ_μ and $\xi_{\mu;\nu}$ takes at some point x_0, we can express *any* Killing vector as a Taylor series around x_0

$$\xi^n{}_\mu(x, x_0) = A_\mu{}^\lambda(x, x_0)\xi^n{}_\lambda(x_0) + B_\mu{}^{\lambda\sigma}(x, x_0)\xi^n{}_{\lambda;\sigma}(x_0)\,, \quad (1.3.10)$$

where n labels one of the N vectors ξ_μ or one of the $\frac{1}{2}N(N-1)$ derivatives $\xi_{\mu;\nu}$ in the spacetime and the coefficients $A_\mu{}^\lambda$ and $B_\mu{}^{\lambda\sigma}$ depend upon x_0 and the metric $g_{\mu\nu}$ in some way but *does not* depend on the Killing vectors $\xi_\lambda(x_0)$ and $\xi_{\lambda;\sigma}(x_0)$, so they are the same for any and *all* Killing vector.

For (1.3.9) to be soluble, it needs to satisfy integrability conditions, which can be found when we take the commutator of covariant derivatives, but now of $\xi_{\rho;\mu}$. Using (1.3.9),

$$\begin{aligned} R^\lambda{}_{\rho\sigma\nu}\xi_{\lambda;\mu} + R^\lambda{}_{\mu\sigma\nu}\overbrace{\xi_{\rho;\lambda}}^{-\xi_{\lambda;\rho}} &= \xi_{\rho;\mu;\sigma;\nu} - \xi_{\rho;\mu;\nu;\sigma} \\ \therefore \left(-R^\lambda{}_{\rho\sigma\nu}\delta^\kappa_\mu + R^\lambda{}_{\mu\sigma\nu}\delta^\kappa_\rho - R^\lambda{}_{\nu\mu\rho}\delta^\kappa_\sigma + R^\lambda{}_{\sigma\mu\rho}\delta^\kappa_\nu\right)\xi_{\lambda;\kappa} & \\ = \left(R^\lambda{}_{\nu\mu\rho;\sigma} - R^\lambda{}_{\sigma\mu\rho;\nu}\right)\xi_\lambda \,. & \end{aligned} \quad (1.3.11)$$

This is yet another restrictive condition, but now it establishes a link between the Killing vectors and their first covariant derivative.

1.4 Useful particular results

In this section we shall enumerate some more handy results that we shall be using further throughout the work but are too specific to deserve a whole section to elaborate them.

1.4.1 Proper time

The *proper time* of a physical entity is defined as the time measured by clocks in the frame of reference of that entity or, equivalently, by the referential where this object is stationary. Thus, there are no spatial variations and the line element reduces to*

$$ds^2 = dt^2 ,$$

which can be inverted as

$$d\tau \equiv dt = \sqrt{ds^2}$$

or, in a path from A to B, considering $\tau_A \equiv 0$ and $\tau_B \equiv \tau$,

$$\tau = \int_A^B \sqrt{ds^2} \, . \qquad (1.4.1)$$

We thus obtain a description that clearly is covariant, since ds^2 is so. For this reason, we distinguish it by using the symbol τ.

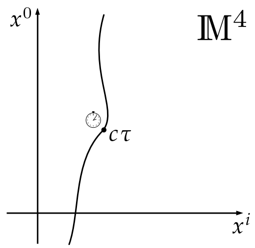


Figure 11: Proper time representation in a Minkowski diagram. It is the time a clock displays in a frame of reference that moves along with the particle.

*From now on we shall adopt $c = 1$.

1.4.2 Synchronous Frame of Reference

This frame of reference is characterized by a family of hypersurfaces in which all the “clocks” carried by the spatial points of every sheet are synchronized among themselves. By virtue of that, the hypersufaces are in free-fall, so they follow geodesic trajectories and have the proper time τ as their time coordinate; we can say that time and spatial coordinates do not mix up, effectively given the following spacetime interval

$$\begin{aligned} ds^2 &= g_{00} d\tau^2 + g_{ij} dx^i dx^j \\ &= d\tau^2 + g_{ij} dx^i dx^j \, , \end{aligned} \tag{1.4.2}$$

where a simple redefinition of coordinates allows us to set $g_{00} \equiv 1$. From that line element, we infer that the metric tensor has the components

$$g_{00} = 1 \qquad ; \qquad g_{0i} = 0 \qquad ; \qquad g_{ij} \equiv -\zeta_{ij} \, , \tag{1.4.3}$$

in which we define the positively defined *spatial metric tensor* ζ of the $N-1$ dimensional spatial sector of the space. By definition, the normal N-vector for the hypersurfaces with τ = const is

$$n_\mu = \frac{\partial \tau}{\partial x^\mu}$$

which gives

$$n_\mu = n^\mu = (1, \mathbf{0}) \, , \tag{1.4.4}$$

already properly normalized; both co- and contravariant representations are identical by (1.4.3).

In this frame of reference the N-velocities

$$u^{\mu} := \frac{\partial x^{\mu}}{\partial s} \tag{1.4.5}$$

point to the time direction, so they are all tangent to x^i = const, coinciding with the normal N-vector

$$u^{\mu} = (1, \mathbf{0}) , \tag{1.4.6}$$

and thus following geodesic curves. Indeed, the geodesic equation (1.1.47) is automatically satisfied, for

$$\begin{aligned} \Gamma^{\mu}{}_{00} &= \frac{1}{2} g^{\mu\sigma} \left(g_{0\sigma,0} + g_{\sigma 0,0} - g_{00,\sigma} \right) \\ &\equiv 0 \; ; \quad \forall \mu , \end{aligned}$$

$$\frac{du^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\alpha\beta} u^{\alpha} u^{\beta} = 0$$

$$\Longrightarrow \frac{du^{\mu}}{d\tau} = 0 ,$$

which is identically satisfied by n_{μ} in (1.4.6).

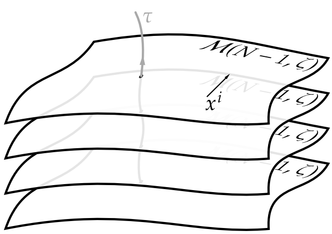


Figure 12: Synchronous frame of reference spatial foliation.

Enumerating all qualities that a synchronous frame of reference must have, we can then *always* construct this frame for any spacetime if we erect it employing the following methodology:

i. Choose a starting everywhere space-like hypersurface Σ_i such that the normals n_μ point to the time direction;

ii. Derive the geodesic curves normal to Σ_i;

iii. Set the time coordinate as the geodesic length s of those curves measured from Σ_i.

It is important to remark that there are infinite ways of defining a syncronous frame of reference thanks to the everywhere spatial subspace described by ζ_{ij}; any coordinate transformation of it will return another syncronous frame.

To write the Einstein equations in this frame of reference, it is useful to split the time and space components up, in particular defining the time derivative of ζ as*:

$$\chi_{ab} := \partial_t \zeta_{ab} = -g_{ab,0} \,, \tag{1.4.7a}$$

$$\chi_{ab} = \chi_{ba} \,, \tag{1.4.7b}$$

$$\chi_a{}^a = \zeta^{ab} \partial_t \zeta_{ab} = \partial_t \ln \zeta \,, \tag{1.4.7c}$$

where operations of lowering/raising indices are carried out by ζ_{ij}. The Christoffel symbols take the form

$$\begin{gathered} \Gamma^0{}_{00} = \Gamma^0{}_{0i} = \Gamma^i{}_{00} = 0 \;\; ; \\ \Gamma^0{}_{ij} = \frac{1}{2}\chi_{ij} \;\; ; \;\; \Gamma^i{}_{0j} = \frac{1}{2}\chi^i{}_j \;\; ; \;\; \Gamma^i{}_{jk} := \Lambda^i{}_{jk} \,. \end{gathered} \tag{1.4.8}$$

*The last one is easily proved by using the identity $\log(detA) = tr(\log A)$.

where the last one are the purely spatial symbols formed by ζ_{ij}. With that, we are able to adress the Ricci tensor (1.1.52),

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} - \Gamma^{\varepsilon}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\varepsilon\nu} + \Gamma^{\varepsilon}{}_{\mu\nu}\Gamma^{\alpha}{}_{\varepsilon\alpha}\,,$$

in this coordinate system. Doing all the calculations, one gets

$$R_{00} = -\frac{1}{2}\partial_t \chi^i{}_i - \frac{1}{4}\chi^i{}_j\chi^j{}_i\,, \tag{1.4.9a}$$

$$R_{0k} = \frac{1}{2}\left(\chi^i{}_{k;i} - \chi^i{}_{i;k}\right), \tag{1.4.9b}$$

$$R_{ij} = P_{ij} + \frac{1}{2}\partial_t \chi_{ij} + \frac{1}{4}\left(\chi_{ij}\chi^l{}_l - 2\chi_{il}\chi^l{}_j\right), \tag{1.4.9c}$$

where P_{ij} is the *totally spatial Ricci tensor* built with ζ_{ij} and $\Lambda^i{}_{jk}$. Finally, the Einstein equations (1.1.61) in mixed components reduce to the following system of differential equations:

$$R_0{}^0 = \frac{1}{2}\partial_t \chi_i{}^i + \frac{1}{4}\chi^i{}_j\chi^j{}_i = 8\pi G\left(T_0{}^0 - \frac{1}{2}T\right), \tag{1.4.10a}$$

$$R_k{}^0 = \frac{1}{2}\left(\chi^i{}_{i;k} - \chi^i{}_{k;i}\right) = 8\pi G T_k{}^0\,, \tag{1.4.10b}$$

$$R_i{}^j = -P_i{}^j - \frac{1}{2}\partial_t \chi_i{}^j + \frac{1}{4}\left(2\chi_{il}\chi^{lj} - \chi_i{}^j\chi_l{}^l\right) = 8\pi G\left(T_i{}^j - \frac{1}{2}\delta_i{}^j T\right). \tag{1.4.10c}$$

One other feature of the synchronous frame of reference is that gravitational fields *cannot* be stationary, because, if it is so,

$$\chi_{ij} = 0\,,$$

causing a contradition to immerse in (1.4.10a) the matter content carried by (or built-in) the energy-momentum tensor, so indeed the metric cannot be stationary. Moreover, from (1.4.10c), in the empty space, we would have

$$P_{ij} = 0\,,$$

corresponding to a null curvature and thus a flat $\mathbb{R}^{n-1}$ Euclidian space.

Real particles also are not at rest in this frame of reference, because in general, the pressure exerted by the matter fields and the cosmological fluid have a spatial component which will not move along the synchronous hypersheets.

CHAPTER 2

Maximally Symmetric Spaces and the Standard Model of Cosmology

SYMMETRIES became a rich soil upon which many physical theories sprout and are considered nowadays indispensible for any *ab initio* theory. Those can be roughly defined by the invariance of something when some sort of transformation is applied to it; for instance, we expect vectors to be the same upon translations in an affine euclidian vector space, or the general principle of relativity to preserve the laws of physics themselves by a general transformation of coordinates. By those two simple cases, we can see how powerful and desirable it is to accommodate symmetries in our theories.

The construction of a spacetime can be (and will be) done from the ground up by explore the desirable symmetries that will take place in the current chapter, where the main ingredients employed will naturally be the Killing fields as they inheritely carry the aforementioned symmetries. In addition to that, we will focus on the properties the spacetime metric has when it admits the maximal number of symmetries, to which they attain the fancy name of *Maximally Symmetric Spaces* (MS Spaces) and have great qualities such as a constant curvature everywhere.

Even though the vast majority of spacetimes does not possess all the available symmetries, the same formalism can be applied to broken down spacetimes, where one part will be MS whereas the other will not. This will be the case for both spatial isotropy and homogeneity, where we will fully recover the famous *Friedmann-Lemaître-Robertson-Walker* (FLRW) metric, that is, the standard cosmological model at the present.

Firstly, we shall expand the theory of Killing vectors seen in Section 1.3, to study the characteristics a MS spacetime has and to construct the necessary framework of said spaces, then we will delve a bit into some particular cases, concluding with a quick review on the FLRW cosmology and some of its main results.

2.1 Maximally Symmetric Spaces

We begin by considering a N dimensional spacetime $\mathcal{M}(N, g)$ endowed with a metric g. As already discussed on Section 1.3, we can express any Killing vector as an expansion around a point x_0 (1.3.10) if we know the values each ξ^n_λ and $\xi^n_{\lambda;\sigma}$ take on such point, where, remembering, n indicates the nth Killing vector from the pool of all $\frac{1}{2}N(N+1)$ vectors obtained by that expansion.

The maximum number of ways to erect (1.3.10) are precisely the number of combinations we can build with ξ^n_λ and $\xi^n_{\lambda;\sigma}$. So, recalling the anti-symmetry of the latter (1.3.8), we have

- ξ^n_λ: N vectors;
- $\xi^m_{\lambda;\nu}$: $\frac{1}{2}N(N-1)$ vectors;

$\therefore$ $\frac{1}{2}N(N+1)$ independent Killing vectors.

We say that a spacetime is *Maximally Symmetric* if it has *all* the $\frac{1}{2}N(N+1)$ independent Killing vectors. This quality attributed to it is not in vain; the linear independency of those vectors are characterized by

$$\sum_n c_n {\xi^n}_\lambda(x) \neq 0 \qquad ; \qquad \sum_n c_n {\xi^n}_{\lambda;\nu}(x) \neq 0 \, ,$$

with constant coefficients c_n, and valid everywhere. If we suppose that we have $M > \frac{1}{2}N(N+1)$ vectors at some point x_0 instead, then the spare *have* to be linearly dependent, so they will obey

$$\sum_n c_n {\xi^n}_\lambda(x_0) = 0 \qquad ; \qquad \sum_n c_n {\xi^n}_{\lambda;\nu}(x_0) = 0$$

at x_0. However, by virtue of (1.3.10), it must be satisfied everywhere, thus being dependent everywhere, so $\frac{1}{2}N(N+1)$ is indeed the maximum number of Killing vectors a N-dimensional spacetime can bear.

We shall break the maximal pool of independent vectors in two subsets: the first corresponding to the N vectors ξ_μ, labelled by (λ), and the second corresponding to the $\frac{1}{2}N(N-1)$ anti-symmetric vectors $\xi_{\mu;\nu}$, labelled by $(\alpha\beta)$. The latter is also anti-symmetric in its label due to the Killing condition (1.3.7).

We say a spacetime is *homogeneous* if there are isometries that take x_0 to x in its neighbourhood so that $x = x_0$. These isometries are generated by *infinitesimal translations*, which are transformations (1.3.3) that take x_0 into a point x in the immediate neighbourhood $x_0 + \varepsilon$. Mathematically speaking, this condition is encoded by translations with respect to the identity,

$$\xi^{(\lambda)}{}_\mu(x_0, x_0) := \delta^\lambda_\mu\,, \tag{2.1.1}$$

that is, when evaluated at x_0, it must return the very point. Remember, here we are grabbing just a subset of N Killing vectors from the maximal amount $\frac{1}{2}N(N+1)$, which is denoted by $(\lambda) = 1, \ldots, N$.

Since x_0 is completely arbitrary, with no particular preference, we can recursively go to infinitesimally neighbouring points by translating as we wish, thus covering the entire N-space, as we would expect from our intuitive notion of homogeneity.

Furthermore, this set of Killing vectors are evidently linearly independent, since

$$\begin{aligned} c_\mu \xi^{(\mu)}{}_\nu(x, x_0)\Big|_{x=x_0} &= c_\mu \delta^\mu_\nu = 0 \\ \Longrightarrow\ c_\nu = 0\,, &\qquad \forall \nu\,. \end{aligned}$$

On the other hand, a space time is named *isotropic* at x_0 if there are isometries that leave that point fixed, in a way that the derivatives take all possible values. By leaving x_0 fixed, we are essentially performing an *infinitesimal rotation* in the neighbourhood of x_0, condition translated mathematically as

$$\begin{aligned}\xi^{(\alpha\beta)}{}_{\mu}(x,x_0) &:= -\xi^{(\beta\alpha)}{}_{\mu}(x,x_0)\,,\\ \xi^{(\alpha\beta)}{}_{\mu}(x_0,x_0) &:= 0\,,\\ \xi^{(\alpha\beta)}{}_{\lambda;\sigma}(x_0,x_0) &:= \left[\frac{\partial}{\partial x^{\sigma}}\xi^{(\alpha\beta)}{}_{\lambda}(x,x_0)\right]_{x=x_0} = \delta^{\alpha}_{\lambda}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\beta}_{\lambda}\,,\end{aligned} \tag{2.1.2}$$

where in the last one the covariant derivative turned into an ordinary one because locally the spacetime is flat.

The first represents the anti-commutativity expected from rotations, the second is a consequence of the first, representing the fixed point condition and the last one is the infinitesimal rotation generator. The subset $(\alpha\beta)$ may be thought as "two translations" along distinct directions,

$$\xi^{(\alpha\beta)}_{\mu} := \left[\xi^{(\alpha)}_{\mu}\right]^{(\beta)}\,,$$

spanning through all the anti-symmetric sector $1,\ldots,\frac{1}{2}N(N-1)$ of (1.3.10). Those are also independent

$$d_{\mu\nu}\xi^{(\mu\nu)}_{\lambda}(x,x_0) = 0\,,$$

for $d_{\mu\nu}$ anti-symmetric due to the anti-symmetry of $(\mu\nu)$. In fact, deriving it covariantly with respect to σ,

$$d_{\mu\nu}\xi^{(\mu\nu)}_{\lambda;\sigma}(x,x_0) = 0\,,$$

and evaluating it on $x = x_0$, gives

$$
\begin{aligned}
0 &= d_{\mu\nu}(\delta^{\mu}_{\lambda}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\lambda}) \\
&= d_{\lambda\sigma} - d_{\sigma\lambda} \\
&= 2d_{\lambda\sigma}
\end{aligned}
$$

$$
\implies d_{\lambda\sigma} = 0\,.
$$

Taking one step further, we can determine a spacetime isotropic *everywhere*. To do this, let us suppose that the same conditions (2.1.2) are also valid for a point immediately close to x_0, that is, $\xi^{(\alpha\beta)}_{\mu}(x, x_0 + dx_0)$. If we are able to find a solution that maintains the same isotropic qualities in an arbitrary neighbouring point, then it shall also attain the homogeneity symmetry and thus be isotropic at every point.

Now, expanding $\xi^{(\alpha\beta)}_{\mu}(x, x_0{+}dx_0)$ around x_0 up to the first order of dx_0,

$$
\xi^{(\alpha\beta)}_{\mu}(x, x_0 + dx_0) = \xi^{(\alpha\beta)}_{\mu}(x, x_0) + \frac{\partial}{\partial x^{\sigma}_0}\xi^{(\alpha\beta)}_{\mu}(x, x_0)dx^{\sigma}_0\,,
$$

we see that $\dfrac{\partial}{\partial x^{\sigma}_0}\xi^{(\alpha\beta)}_{\mu}(x, x_0)$ is a Killing vector by equivalence, considering that dx_0 is arbitrary. So, to compute this derivative, we can use the second property of (2.1.2), considering that $x = x(x_0)$,

$$0 \equiv \frac{\partial}{\partial x_0^\sigma} \xi^{(\alpha\beta)}{}_\mu(x_0, x_0) = \left[\frac{\partial}{\partial x_0^\sigma} \xi^{(\alpha\beta)}{}_\mu(x(x_0), x_0)\right]_{x=x_0} + \left[\frac{\partial}{\partial x_0^\sigma} \xi^{(\alpha\beta)}{}_\mu(x(x_0), x_0)\right]_{x=x_0}$$
$$= \left[\frac{\partial x^\lambda}{\partial x_0^\sigma} \frac{\partial}{\partial x^\lambda} \xi^{(\alpha\beta)}{}_\mu(x(x_0), x_0)\right]_{x=x_0} + \left[\frac{\partial}{\partial x_0^\sigma} \xi^{(\alpha\beta)}{}_\mu(x(x_0), x_0)\right]_{x=x_0} .$$

Since

$$x^\lambda = x_0^\lambda + \xi^\lambda(x(x_0)) ; \qquad |\xi| \ll 1$$

$$\Longrightarrow \frac{\partial x^\lambda}{\partial x_0^\sigma} = \delta_\sigma^\lambda + \frac{\partial \xi^\lambda}{\partial x_0^\sigma}(x(x_0)) ,$$

we have

$$0 = \left[\delta_\sigma^\lambda \frac{\partial}{\partial x^\lambda} \xi^{(\alpha\beta)}{}_\mu(x(x_0), x_0)\right]_{x=x_0} + \left[\frac{\partial}{\partial x_0^\sigma} \xi^{(\alpha\beta)}{}_\mu(x(x_0), x_0)\right]_{x=x_0} + \mathcal{O}\left((\frac{\partial}{\partial x_0}\xi)^2\right)$$

$$\Longrightarrow \frac{\partial}{\partial x_0^\sigma} \xi^{(\alpha\beta)}{}_\mu(x, x_0)\bigg|_{x_0} = -\frac{\partial}{\partial x^\sigma} \xi^{(\alpha\beta)}{}_\mu(x, x_0)\bigg|_{x_0}$$
$$= -\delta_\mu^\alpha \delta_\sigma^\beta + \delta_\sigma^\alpha \delta_\mu^\beta ,$$

where we discarted terms of the second order in $\frac{\partial}{\partial x_0}\xi$. If we set a Killing vector on x_0 as

$$\xi_\mu(x = x_0) = a_\mu$$

and contract $\alpha\sigma$ on the derivative with respect to x_0 above,

$$\frac{\partial}{\partial x_0^\alpha} \xi^{(\alpha\beta)}{}_\mu(x, x_0)\bigg|_{x_0} = -(N-1)\delta_\mu^\beta\,,$$

we can construct Killing vectors

$$\xi_\mu(x) = \frac{a_\beta}{N-1} \frac{\partial}{\partial x_0^\alpha} \xi^{(\alpha\beta)}{}_\mu(x, x_0) \tag{2.1.3}$$

that are connected, both homogeneous and isotropic at every point. Thus, *any spacetime that is isotropic everywhere is also homogeneous*. In this case the full linear independency can be written as

$$\begin{aligned}
0 &= c_\mu \xi^{(\mu)}{}_\lambda(x, x_0) + c_{\alpha\beta}\xi^{(\alpha\beta)}{}_\lambda(x, x_0) \\
\text{(i)}\ {}_{x=x_0} &= c_\mu \underbrace{\xi^{(\mu)}{}_\lambda(x_0, x_0)}_{=\delta_\lambda^\mu} + c_{\alpha\beta} \underbrace{\xi^{(\alpha\beta)}{}_\lambda(x_0, x_0)}_{=0} \\
&\implies c_\mu = 0
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\ {}_{\partial_\rho|_{x=x_0}} &= c_\mu \underbrace{\xi^{(\mu)}{}_{\lambda,\rho}(x, x_0)\Big|_{x_0}}_{=0} + c_{\alpha\beta} \underbrace{\xi^{(\alpha\beta)}{}_{\lambda,\rho}(x, x_0)\Big|_{x_0}}_{=\delta_{\lambda\rho}^{\alpha\beta} - \delta_{\rho\lambda}^{\alpha\beta}} \\
&= c_{\lambda\rho} - c_{\rho\lambda} \\
&\implies c_{\lambda\rho} = 0\,,
\end{aligned}$$

which is automatically satisfied by conditions (2.1.1) and (2.1.2). Then, if a spacetime is homogeneous and isotropic at every point, it is MS.

The converse is also true. To demonstrate this we assume we are in a MS spacetime and represent the linear independency as a $\frac{1}{2}N(N+1) \times \frac{1}{2}N(N+1)$ non-singular matrix $\hat{\xi}$ composed of $\xi_{\mu}^{(n)}(x_0)$ and $\xi_{\mu;\nu}^{(n)}(x_0)$ ordered by $\mu > \nu$, where the last condition ensure we will not overcount vectors. Such matrix has the form

$$\hat{\xi} = \left(\begin{array}{ccc|ccc} {\xi^1}_1 & \cdots & {\xi^N}_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ {\xi^1}_N & \cdots & {\xi^N}_N & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & {\xi^0}_{1;0} & \cdots & {\xi^{\frac{1}{2}N(N-1)}}_{1;0} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & {\xi^0}_{N;N-1} & \cdots & {\xi^{\frac{1}{2}N(N-1)}}_{N;N-1} \end{array}\right),$$

along with

$$\hat{d} = \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ d_0 \\ \vdots \\ d_{\frac{1}{2}N(N-1)} \end{pmatrix} \qquad \hat{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \\ b_{10} \\ \vdots \\ b_{N(N-1)} \end{pmatrix},$$

such that we should be able to solve the system of equations

$$\hat{\xi}\hat{d} = \hat{a}$$

for $\hat{d}$. Breaking into individual parts for ξ_μ and $\xi_{\mu;\nu}$,

$$\begin{cases} \sum\limits_n c_n \xi^n{}_\mu(x_0) & = a_\mu(x_0)\,, \\ \sum\limits_m d_m \xi^m{}_{\mu;\nu}(x_0) & = b_{\mu\nu}(x_0)\,, \end{cases}$$

for arbitrary a_μ and $b_{\mu\nu}$, with $b_{\mu\nu} = -b_{\nu\mu}$ and $n = 1, \dots, N$ and $m = 0, \dots, \frac{1}{2}N(N-1)$. Given $\xi^n_\mu(x_0)$ and $\xi^n_{\mu;\nu}(x_0)$, we feed them back into (1.3.10) and determine $\xi^n_\mu(x)$ everywhere, but since a_μ and $b_{\mu\nu}$ are completely determined in a neighborhood of an arbitrary point x_0 through the previous equations, the whole spacetime has to be both homogeneous and isotropic at every point.

As an example of MS spacetime, we make explicit the calculations for the flat spacetime

Example: Euclidian spacetime

Recalling (1.3.9)

$$\xi_{\mu;\rho;\sigma} = R^\lambda{}_{\sigma\rho\mu}\xi_\lambda\,.$$

In an Euclidian spacetime, the Riemann tensor vanishes and the covariant derivatives reduce to ordinary derivatives,

$$\begin{aligned} &\xi_{\mu,\rho,\sigma} = 0 \\ \Longrightarrow\quad &\xi_\mu(x) = a_\mu + b_{\mu\nu}x^\nu\,, \end{aligned}$$

for a_μ, $b_{\mu\nu}$ constants. From (1.3.8),

$$\begin{aligned} 0 &= \xi_{\mu;\nu} + \xi_{\nu;\mu} \\ &= b_{\mu\sigma}\delta^{\sigma}_{\nu} + b_{\nu\sigma}\delta^{\sigma}_{\mu} \\ &= b_{\mu\nu} + b_{\nu\mu} \\ \implies b_{\mu\nu} &= -b_{\nu\mu} \,. \end{aligned}$$

Chossing the $\frac{1}{2}N(N+1)$ Killing vectors as

$$\begin{aligned} \xi^{(\lambda)}{}_{\mu}(x) &= \delta^{\lambda}_{\mu} && \longrightarrow \text{infinitesimal translations}\,, \\ \xi^{(\alpha\beta)}{}_{\mu}(x) &= \delta^{\alpha}_{\mu}x^{\beta} - \delta^{\beta}_{\mu}x^{\alpha} && \longrightarrow \text{infinitesimal “rotation”}^{*}\,, \end{aligned}$$

expression (1.3.10) gives

$$\xi_{\mu}(x) = a_{\lambda}\xi^{(\lambda)}{}_{\mu}(x) + b_{\alpha\beta}\xi^{(\alpha\beta)}{}_{\mu}(x) \,.$$

Therefore the flat spacetime is indeed maximally symmetric.

We can also extend the notion of form invariant tensors now that we know MS spacetimes are homogeneous and isotropic at every point. Just like in (1.3.2), we define a form invariant tensor if

$$T'^{\mu\nu\cdots}(x) = T^{\mu\nu\cdots}(x) \qquad \forall x \tag{2.1.4}$$

is satisfied. In a similar fashion to (1.3.6), this condition is encoded as the null Lie derivative under the infinitesimal transformation (1.3.3), namely,

*Note that those Killing vectors will only correspont to proper rotations for a flat space. If we are instead working with spacetimes with no defined signature, those “rotations” will correspond to boost components as well.

$$\mathcal{L}_\xi[T_{\mu\nu\ldots}] \equiv 0 = \xi^\lambda T_{\mu\nu\ldots,\lambda} + \xi^\lambda{}_{,\mu} T_{\lambda\nu\ldots} + \xi^\lambda{}_{,\nu} T_{\mu\lambda\ldots} + \ldots \,. \tag{2.1.5}$$

We say T is *Maximally Form Invariant* if it has $\frac{1}{2}N(N+1)$ Killing vectors satisfying (2.1.5). In that case, we are free to choose Killing vectors $\xi_\mu(x_0)$ and $\xi_{\mu;\nu}(x_0)$ such that

$$\xi^\lambda(x_0) = 0 \tag{2.1.6}$$

$$\xi^\sigma{}_{;\nu}(x_0) = \xi^\sigma{}_{,\nu}(x_0) \qquad \Longrightarrow \qquad \xi^\lambda{}_{,\nu}(x_0) = g^{\lambda\mu}(x_0)\xi_{\mu;\nu}(x_0)\,. \tag{2.1.7}$$

This choice os Killing vectors represent the Geodesic frame of reference; along the geodesic line there will be no translation (first condition) and the spacetime is flat (second condition).

The isometry condition (2.1.5) at x_0 then becomes

$$\mathcal{L}_\xi[T_{\mu\nu\ldots}]\bigg|_{x_0} = 0 = \ \xi_{\sigma;\tau}(x_0)\bigg(\delta^\tau_\mu T^\sigma{}_{\nu\ldots}(x_0) + \delta^\tau_\nu T_\mu{}^\sigma{}_{\ldots}(x_0) + \ldots\bigg)\,.$$

For this to be consistent for any x_0, we remember the anti-symmetry of the Killing condition (1.3.8) and impose that the term inside the brackets is *symmetric* in $\sigma\tau$. Consequently,

$$\delta^\tau_\mu T^\sigma{}_{\nu\ldots} + \delta^\tau_\nu T_\mu{}^\sigma{}_{\ldots} + \ldots = \delta^\sigma_\mu T^\tau{}_{\nu\ldots} + \delta^\sigma_\nu T_\mu{}^\tau{}_{\ldots} + \ldots \qquad , \forall x_0\,. \tag{2.1.8}$$

If this condition is satisfied, T will be a maximally form invariant tensor. Although this permits us to construct maximally form invariant tensors of any order, we shall be only interested in tensors of rank ≤ 2, except for the Riemann tensor.

✦ Rank 1:

$$
\begin{aligned}
\delta^\tau_\mu A^\sigma &= \delta^\sigma_\mu A^\tau \\
N A^\sigma &= A^\sigma \\
A^\sigma &= \begin{cases} 0 & , N > 1 \, , \\ \text{any} & , N = 1 \, . \end{cases}
\end{aligned}
$$

- Rank 2:

$$
\delta^\tau_\mu B^\sigma{}_\nu + \delta^\tau_\nu B_\mu{}^\sigma = \delta^\sigma_\mu B^\tau{}_\nu + \delta^\sigma_\nu B_\mu{}^\tau \, ;
$$

contracting $\tau\mu$ yields

$$
(N-1) B_{\sigma\nu} + B_{\nu\sigma} = g_{\sigma\nu} B^\lambda{}_\lambda \, . \tag{2.1.9}
$$

Anti-symmetrizing this expression,

$$
(N-2)(B_{\sigma\nu} - B_{\nu\sigma}) = 0 \, ,
$$

so that

$$
\Longrightarrow \quad B_{\sigma\nu} = \begin{cases} B_{\nu\sigma} & , N \neq 2 \\ \text{any} & , N = 2 \end{cases} \, .
$$

Let us consider cases for which $N > 2$, so plugging it back to (2.1.9) gives

$$
\begin{aligned}
N B_{\sigma\nu} &= g_{\sigma\nu} B^\lambda{}_\lambda \\
\therefore B_{\sigma\nu} &= f g_{\sigma\nu} \, ,
\end{aligned} \tag{2.1.10}
$$

where f is determined if we impose the form invariant condition again,

$$\begin{aligned}\mathcal{L}_\xi[B_{\mu\nu}] \equiv 0 = \mathcal{L}_\xi[f g_{\mu\nu}] \\ = g_{\mu\nu}\mathcal{L}_\xi[f] + f \underbrace{\mathcal{L}_\xi[g_{\mu\nu}]}_{=0} \\ = g_{\mu\nu}\xi^\lambda f_{,\lambda}\end{aligned}$$

$$\therefore\ f_{,\lambda} = 0 \implies f(x) = \text{const} \equiv \kappa\ .$$

Therefore the only maximally form invariant rank 2 tensor is the one and the same metric tensor apart from a multiplicative constant.

Now, remembering the integrability conditions (1.3.11) and setting the purely isotropical Killing vectors

$$\begin{cases}\xi_\mu(x) & = 0 \\ \xi_{\mu;\nu}(x) & \neq 0\end{cases}, \tag{2.1.11}$$

we see that

$$\left(-R^\lambda{}_{\rho\sigma\nu}\delta^\kappa_\mu + R^\lambda{}_{\mu\sigma\nu}\delta^\kappa_\rho - R^\lambda{}_{\nu\mu\rho}\delta^\kappa_\sigma + R^\lambda{}_{\sigma\mu\rho}\delta^\kappa_\nu\right)\xi_{\lambda;\kappa} = 0\,;$$

As a matter of fact, it is a maximally form invariant tensor, so, by (2.1.8),

$$\begin{aligned}-R^\lambda{}_{\rho\sigma\nu}\delta^\kappa_\mu + R^\lambda{}_{\mu\sigma\nu}\delta^\kappa_\rho - R^\lambda{}_{\nu\mu\rho}\delta^\kappa_\sigma + R^\lambda{}_{\sigma\mu\rho}\delta^\kappa_\nu \\ = -R^\kappa{}_{\rho\sigma\nu}\delta^\lambda_\mu + R^\kappa{}_{\mu\sigma\nu}\delta^\lambda_\rho - R^\kappa{}_{\nu\mu\rho}\delta^\lambda_\sigma + R^\kappa{}_{\sigma\mu\rho}\delta^\lambda_\nu\,.\end{aligned} \tag{2.1.12}$$

Contracting $\mu\kappa$ gives

$$(N-1)R_{\lambda\rho\sigma\nu} = R_{\sigma\rho}g_{\lambda\nu} - R_{\nu\rho}g_{\lambda\sigma}\,. \tag{2.1.13}$$

Since $R_{\lambda\rho\sigma\nu}$ is anti-symmetrical in $\lambda\rho$,

$$R_{\sigma\rho}g_{\lambda\nu} - R_{\nu\rho}g_{\lambda\sigma} = -\left(R_{\sigma\lambda}g_{\rho\nu} - R_{\nu\lambda}g_{\rho\sigma}\right);$$

contracting $\lambda\nu$

$$NR_{\sigma\rho} - \cancel{R_{\sigma\rho}} = -\left(\cancel{R_{\sigma\rho}} - R^{\lambda}{}_{\lambda}g_{\rho\sigma}\right)$$

$$\therefore\ R_{\sigma\rho} = \frac{1}{N}R^{\lambda}{}_{\lambda}g_{\sigma\rho}\,. \tag{2.1.14}$$

Back to (2.1.13),

$$R_{\lambda\rho\sigma\nu} = \frac{R^{\lambda}{}_{\lambda}}{N(N-1)}\left(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}\right), \tag{2.1.15}$$

we get the curvature tensor only in terms of the Ricci scalar and the metric tensor. Since MS spacetimes are isotropic at every point, the Bianchi identities will be valid everywhere, which allows us to find the Ricci scalar by returning (2.1.14) in

$$
\begin{aligned}
0 &= \left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R^{\lambda}{}_{\lambda}\right)_{;\mu} \\
&= \left(\frac{1}{N}\delta^{\mu}_{\nu}R^{\lambda}{}_{\lambda} - \frac{1}{2}\delta^{\mu}_{\nu}R^{\lambda}{}_{\lambda}\right)_{;\mu} \\
&= \left(\frac{1}{N} - \frac{1}{2}\right)\delta^{\mu}_{\nu}R^{\lambda}{}_{\lambda;\mu} \\
&= \left(\frac{1}{N} - \frac{1}{2}\right)R^{\lambda}{}_{\lambda,\mu} = 0
\end{aligned}
$$

$$
\therefore\ R^{\lambda}{}_{\lambda} \equiv R = \text{const} =: N(N-1)K\ ; \quad N > 2\,, \tag{2.1.16}
$$

where K is the *curvature constant*, $N(N-1)$ was put by hand to clear (2.1.15) up a bit and it is valid for $N > 2$.

Finally, putting this back on (2.1.15) and (2.1.14) gives

$$
\begin{cases}
R_{\lambda\rho\sigma\nu} &= K\Big(g_{\nu\lambda}g_{\rho\sigma} - g_{\nu\rho}g_{\lambda\sigma}\Big) \\
R_{\mu\nu} &= (N-1)Kg_{\mu\nu}
\end{cases} . \tag{2.1.17}
$$

Spacetimes that have those properties are called *spacetimes of constant curvature**. Furthermore, this solution is also quite *unique* so two “different” spaces with (2.1.17) are connected if and only if there exists a diffeomorphism that takes $x \to x'$, preserving the curvature constant K, if both have the same metric signature.

To prove that, we consider two distinct curvature tensors

*Another elegant way to determine this constant curvature tensor is to notice that the curvature tensor itself is an isometry, that is, $\mathcal{L}_{\xi}[R_{\mu\nu\alpha\beta}] = 0$, and solve the equation to find (2.1.17).

$$\begin{cases} R_{\lambda\rho\sigma\nu}(x) & = K\Big(g_{\nu\lambda}(x)g_{\rho\sigma}(x) - g_{\nu\rho}(x)g_{\lambda\sigma}(x)\Big) \\ R'_{\lambda\rho\sigma\nu}(x') & = K'\Big(g'_{\nu\lambda}(x')g'_{\rho\sigma}(x') - g'_{\nu\rho}(x')g'_{\lambda\sigma}(x')\Big) \end{cases} .$$

Since the metric tensors have the same signature, there exists a coordinate transformation that takes $x \longrightarrow x'$:

$$\begin{aligned} R'_{\lambda\rho\sigma\nu}(x') &= \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\rho}\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\delta}{\partial x'^\nu}R_{\alpha\beta\gamma\delta}(x) \\ &= \frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\beta}{\partial x'^\rho}\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial x^\delta}{\partial x'^\nu}K\Big(g_{\delta\alpha}(x)g_{\beta\gamma}(x) - g_{\delta\beta}(x)g_{\alpha\gamma}(x)\Big) \\ &= K\Bigg[\left(\frac{\partial x^\delta}{\partial x'^\nu}\frac{\partial x^\alpha}{\partial x'^\lambda}g_{\delta\alpha}(x)\right)\left(\frac{\partial x^\beta}{\partial x'^\rho}\frac{\partial x^\gamma}{\partial x'^\sigma}g_{\beta\gamma}(x)\right) \\ &\qquad - \left(\frac{\partial x^\delta}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\rho}g_{\delta\beta}(x)\right)\left(\frac{\partial x^\alpha}{\partial x'^\lambda}\frac{\partial x^\gamma}{\partial x'^\sigma}g_{\alpha\gamma}(x)\right)\Bigg] \\ &= K\Big[g'_{\nu\lambda}(x')g'_{\rho\sigma}(x') - g'_{\nu\rho}(x')g'_{\lambda\sigma}(x')\Big] \\ &= K\frac{1}{K'}R'_{\lambda\rho\sigma\nu}(x') \end{aligned}$$

$$\Longrightarrow \quad K' = K\,.$$

To construct the line element and determine the MS metric tensor, we shall employ the method of *embedded spaces*, in which we insert the actual N-dimensional spacetime as the surface of a $N-$sphere (or of an $N-$dimensional hyperboloid if $K < 0$) in a $(N+1)-$dimensional space. By doing that, the intrinsic curvature properties will emerge as *external* curvature, just as we see the inherit curvature of a 3D sphere, for instance. That is done by adding a z coordinate that is constrained by the unitary $(N+1)-$sphere

$$K h_{\mu\nu} x^\mu x^\nu + z^2 = 1\,, \tag{2.1.18}$$

giving the line element

$$\begin{aligned} ds^2 &= g_{AB} dx^A dx^B\,, \qquad A, B = 0, \ldots, N \\ &= K h_{\mu\nu} dx^\mu dx^\nu + dz^2 \\ &= h_{\mu\nu} dx^\mu dx^\nu + K^{-1} dz^2\,, \qquad \mu, \nu = 0, \ldots, N-1 \end{aligned} \tag{2.1.19}$$

where a simple redefinition of x^μ and z enables us to rearrange K and $h_{\mu\nu}$ is some $N \times N$ constant metric tensor.

Differentiating (2.1.18),

$$0 = 2K h_{\mu\nu} x^\mu dx^\nu + 2z dz$$

$$\begin{aligned} dz^2 &= \frac{K^2 (h_{\mu\nu} x^\mu dx^\nu)^2}{z^2} \\ &= \frac{K^2 (h_{\mu\nu} x^\mu dx^\nu)^2}{1 - K h_{\alpha\beta} x^\alpha x^\beta} \end{aligned}$$

and plugging it back to (2.1.19), we obtain

$$ds^2 = \left[h_{\mu\nu} + \frac{K}{1 - K h_{\alpha\beta} x^\alpha x^\beta} h_{\mu\sigma} x^\sigma h_{\nu\rho} x^\rho \right] dx^\mu dx^\nu \tag{2.1.20}$$

$$\therefore\ g_{\mu\nu} = h_{\mu\nu} + \frac{K}{1 - K h_{\alpha\beta} x^\alpha x^\beta} h_{\mu\sigma} x^\sigma h_{\nu\rho} x^\rho\,, \tag{2.1.21}$$

which is MS since $h_{\mu\nu}$ has $N^2/2$ independent components and (2.1.18) gives the other $N/2$.

As a freebie we get rotation symmetries around the $(N+1)-$sphere embedding, so the spacetime interval above has to be invariant under transformations of the $\mathcal{SO}(N+1)$ rotation group

$$\begin{cases} x^\mu & \longrightarrow x'^\mu = R^\mu{}_\nu x^\nu + R^\mu{}_z z \\ z & \longrightarrow z' = R^z{}_\nu x^\nu + R^z{}_z z \end{cases} , \tag{2.1.22}$$

where $R^A{}_B$ are constants. Thus, from (2.1.19),

$$\begin{aligned} ds'^2 &= h_{\alpha\beta} dx'^\alpha dx'^\beta + K^{-1} dz'^2 \\ &= \left(h_{\alpha\beta} R^\alpha{}_\mu R^\beta{}_\nu + K^{-1} R^z{}_\mu R^z{}_\nu \right) dx^\mu dx^\nu \\ &\qquad + \left(2h_{\alpha\beta} R^\alpha{}_\sigma R^\beta{}_z + 2K^{-1} R^z{}_\sigma R^z{}_z \right) dx^\sigma dz \\ &\qquad + \left(h_{\alpha\beta} R^\alpha{}_z R^\beta{}_z + K^{-1} (R^z{}_z)^2 \right) dz^2 , \end{aligned}$$

or,

$$\begin{aligned} h_{\alpha\beta} R^\alpha{}_\mu R^\beta{}_\nu + K^{-1} R^z{}_\mu R^z{}_\nu &= h_{\mu\nu} , \\ h_{\alpha\beta} R^\alpha{}_\sigma R^\beta{}_z + K^{-1} R^z{}_\sigma R^z{}_z &= 0 , \\ h_{\alpha\beta} R^\alpha{}_z R^\beta{}_z + K^{-1} (R^z{}_z)^2 &= K^{-1} . \end{aligned} \tag{2.1.23}$$

We split those transformations in two distinct simpler classes to facilitate our understanding of them.

(1) Rigid rotations on the actual coordinates

$$\begin{cases} R^\mu{}_\nu & = \mathcal{R}^\mu{}_\nu \qquad ; \qquad \mathcal{R}^\mu{}_\nu \in \mathcal{SO}(N) \\ R^\mu{}_z & = R^z{}_\nu = 0 \\ R^z{}_z & = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x'^{\mu} &= \mathcal{R}^{\mu}{}_{\nu} x^{\nu} \\ z' &= z \end{cases} \quad \Longrightarrow \quad h_{\alpha\beta} \mathcal{R}^{\alpha}{}_{\mu} \mathcal{R}^{\beta}{}_{\nu} = h_{\mu\nu} \,. \tag{2.1.24}$$

(2) Quasi-translations ($a^{\mu} \in \mathbb{R}$)

$$\begin{cases} R^{\mu}{}_{z} &= a^{\mu} \\ R^{z}{}_{\mu} &= -K h_{\mu\nu} a^{\nu} \\ R^{z}{}_{z} &= (1 - K h_{\alpha\beta} a^{\alpha} a^{\beta})^{\frac{1}{2}} \\ R^{\mu}{}_{\nu} &= \delta^{\mu}_{\nu} - b K h_{\nu\rho} a^{\mu} a^{\rho} \end{cases} \quad ; \quad b = \frac{1-(1-K h_{\alpha\beta} a^{\alpha} a^{\beta})^{\frac{1}{2}}}{K h_{\sigma\rho} a^{\sigma} a^{\rho}} \,,$$

with $R^{z}{}_{z} \in \mathbb{R} \implies K h_{\alpha\beta} a^{\alpha} a^{\beta} \leq 1$

$$\Longrightarrow \; x'^{\mu} = (\delta^{\mu}_{\nu} - b K h_{\nu\rho} a^{\mu} a^{\rho}) x^{\nu} + a^{\mu} z$$

$$\begin{aligned} x'^{\mu} &= x^{\mu} + a^{\mu} \left[(1 - K h_{\alpha\beta} a^{\alpha} a^{\beta})^{\frac{1}{2}} - b K h_{\nu\rho} a^{\rho} x^{\nu} \right] \\ z' &= -K h_{\mu\nu} a^{\nu} x^{\mu} + \left[(1 - K h_{\alpha\beta} a^{\alpha} a^{\beta})(1 - K h_{\sigma\rho} x^{\sigma} x^{\rho}) \right]^{\frac{1}{2}} . \end{aligned} \tag{2.1.25}$$

The associated Killing vectors come from the infinitesimal group elements of the transformations above

(1) Infinitesimal transformation of $\mathcal{SO}(N)$

$$\begin{aligned} \mathcal{R}^{\mu}{}_{\nu} &= \delta^{\mu}_{\nu} + \varepsilon \Omega^{\mu}{}_{\nu} \,, \\ h_{\mu\sigma} \Omega^{\mu}{}_{\rho} + h_{\mu\rho} \Omega^{\mu}{}_{\sigma} &= 0 \,, \end{aligned}$$

where $|\varepsilon| \ll 1$, $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$

$$\Longrightarrow \begin{cases} x'^{\mu} = \mathcal{R}^{\mu}{}_{\nu} x^{\nu} \\ \qquad = x^{\mu} + \varepsilon \Omega^{\mu}{}_{\nu} x^{\nu} \\ \\ x'^{\mu} = x^{\mu} + \varepsilon \xi^{\mu} \end{cases}$$

$$\therefore\ \xi^{\mu}{}_{iso}(x) = \Omega^{\mu}{}_{\nu} x^{\nu}\ . \tag{2.1.26}$$

(2) Infinitesimal transformation of $\mathcal{T}_N$

$$a^{\mu} \to \varepsilon \alpha^{\mu}\,, \qquad |\varepsilon| \ll 1$$

$$\Longrightarrow \begin{cases} x'^{\mu} = x^{\mu} + \varepsilon \alpha^{\mu} (1 - K h_{\alpha\beta} x^{\alpha} x^{\beta})^{\frac{1}{2}} + \mathcal{O}(\varepsilon^2) \\ \\ x'^{\mu} = x^{\mu} + \varepsilon \xi^{\mu} \end{cases}$$

$$\therefore\ \xi^{\mu}{}_{homo}(x) = \alpha^{\mu} (1 - K h_{\sigma\rho} x^{\sigma} x^{\rho})^{\frac{1}{2}}\ . \tag{2.1.27}$$

If we count all the free parameters we verify the maximum number of Killing vectors. Indeed

- $\Omega^{\mu}{}_{\nu}$: $N \times N$ anti-symmetric matrix $\Longrightarrow$ $\frac{1}{2}N(N-1)$ independent entries;
- α^{μ}: N-dimensional vector $\Longrightarrow$ N independent entries;

$\therefore\ \frac{1}{2}N(N+1)$ symmetries $\Longrightarrow$ MS.

We shift our focus now from the symmetry analysis to some physical results

Geodesic equations

From (1.1.47), with

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(g_{\alpha\nu,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right),$$
$$g_{\mu\nu} = h_{\mu\nu} + \frac{K}{1 - K h_{\alpha\beta} x^{\alpha} x^{\beta}} h_{\mu\sigma} x^{\sigma} h_{\nu\rho} x^{\rho} ,$$

we get

$$\Gamma^{\mu}_{\alpha\beta} = K g_{\alpha\beta} x^{\mu} , \tag{2.1.28}$$

where we used

$$g^{\mu\nu} h_{\nu\sigma} = \delta^{\mu}_{\sigma} - \frac{K}{1 - K h_{\alpha\beta} x^{\alpha} x^{\beta}} g^{\mu\nu} h_{\nu\alpha} h_{\sigma\beta} x^{\alpha} x^{\beta}$$

to discover the inverse metric $g^{\mu\nu}$ contration with the original metric $h_{\nu\sigma}$.

When put back into the geodesic equation, it takes the simple shape

$$\frac{d^2 x^{\mu}}{d\tau^2} + K g_{\alpha\beta} x^{\mu} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau} = 0$$
$$\frac{d^2 x^{\mu}}{d\tau^2} + K x^{\mu} = 0 , \tag{2.1.29}$$

where we used the fact that $u^{\alpha} u_{\alpha} = 1$, and has the familiar solutions

$$x^\mu = \begin{cases} \{\sin(\sqrt{K}\tau), \cos(\sqrt{K}\tau)\} & , K > 0 \\ \{\sinh(\sqrt{-K}\tau), \cosh(\sqrt{-K}\tau)\} & , K < 0 \\ a^\mu + b^\mu \tau & , K = 0 \end{cases} . \tag{2.1.30}$$

Here we verify once again that for flat spacetimes, the geodesics are indeed straight lines.

Next we verify that the curvature tensor constructed with (2.1.21) corresponds to the MS (2.1.17) found previously. Starting from

$$\begin{aligned} R^\mu{}_{\nu\alpha\beta} &= \Gamma^\mu_{\nu\alpha,\beta} - \Gamma^\mu_{\nu\beta,\alpha} + \Gamma^\varepsilon_{\nu\beta}\Gamma^\mu_{\varepsilon\alpha} - \Gamma^\varepsilon_{\nu\alpha}\Gamma^\mu_{\varepsilon\beta} \\ &= (Kx^\mu g_{\nu\alpha})_{,\beta} - (Kx^\mu g_{\nu\beta})_{,\alpha} \\ &\quad + (Kx^\varepsilon g_{\nu\beta})(Kx^\mu g_{\varepsilon\alpha}) - (Kx^\varepsilon g_{\nu\alpha})(Kx^\mu g_{\varepsilon\beta}) \\ &= K\left[(g_{\nu\beta,\alpha} - g_{\nu\alpha,\beta})x^\mu - (\delta^\mu_\alpha g_{\nu\beta} - \delta^\mu_\beta g_{\nu\alpha})\right] \\ &\quad + K^2\left[g_{\nu\alpha}g_{\varepsilon\beta} - g_{\nu\beta}g_{\varepsilon\alpha}\right]x^\varepsilon x^\mu \end{aligned} \tag{2.1.31}$$

we obtain

$$R_{\mu\nu\alpha\beta} = K\Big(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\Big), \tag{2.1.32}$$

as desired. Therefore, the constant K put quite artificially into (2.1.19) does in fact coincide with the curvature constant of MS spaces. Moreover, since $h_{\mu\nu}$ is an arbitrary metric tensor, after all, we can always change it through diffeomorphisms. Due to that reason, we can impose a particular form for that metric tensor with a Riemannian signature, which shall be useful in the future, when we start dealing exclusively with the space sector.

Enforcing

$$h_{\mu\nu} \equiv \begin{cases} |K|^{-1}\delta_{\mu\nu} & , K \neq 0 \\ \delta_{\mu\nu} & , K = 0 \end{cases} \tag{2.1.33}$$

into (2.1.21), we obtain

$$ds^2 = \begin{cases} K^{-1}\left(d\boldsymbol{x}^2 + \dfrac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{1-\boldsymbol{x}^2}\right) & , K > 0 \\ |K|^{-1}\left(d\boldsymbol{x}^2 - \dfrac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{1+\boldsymbol{x}^2}\right) & , K < 0 \\ d\boldsymbol{x}^2 & , K = 0 \end{cases} . \tag{2.1.34}$$

By a suitable change of coordinates, we are able to condense the above conditions to

$$ds^2 = d\boldsymbol{x}^2 + k\frac{(\boldsymbol{x}\cdot d\boldsymbol{x})^2}{1-k\boldsymbol{x}^2} , \tag{2.1.35}$$

where k is the curvature parameter and is normalized to the three possible values $k = 0, \pm 1$, without loss of generality.

As a function of z, we get from (2.1.35)

$$ds^2 = \begin{cases} K^{-1}\left(d\boldsymbol{x}^2 + dz^2\right) & , K > 0 \\ |K|^{-1}\left(d\boldsymbol{x}^2 - dz^2\right) & , K < 0 \end{cases} . \tag{2.1.36}$$

It is evident that it represents the surface of the $(N+1)-$sphere (hyperbole) with radius $K^{-\frac{1}{2}}$ and thus represents finite (infinite) spaces, even when approaching the aparent singularity at x^2 as seen in (2.1.35). This last bit deserves a more commentary: a particle travelling around the $(N+1)-$sphere will swap to the other z solution when it passes through the singularity, all thanks to an ambiguity of this coordinate in the quadratic interval (2.1.18).

2.1.1 Maximally Symmetric Subspaces

We have seen so far many of the great qualities a spacetime have if it is MS, but although it is very tempting, most of the times the whole space being MS does not represent the reality. Still, its valuable properties are assured if subsets of the space are MS instead, which in turn represents real physical conditions. In fact, the vast majority of maximal symmetry is only applied to a restrict subspace of lower dimension; in particular, in a usual 4-dimensional spacetime, the 3-dimensional spatial sector might be MS whereas the time sector decouples from the whole description.

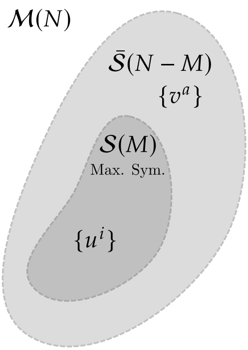


Figure 13: Subspaces diagram. Here we omitted the metric from the notation for clarity.

This decomposition is actually a critical theorem of this whole formalism and the proof of it will be the focus of this subsection. The *subdivision theorem* is stated, mathematically, as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(v)dv^a dv^b + f(v)\tilde{g}_{ij}(u)du^i du^j \,, \quad (2.1.37)$$

with $f(v)$ being a parametric scale factor of the MS metric tensor

$\tilde{g}_{ij}(u)$.

So let $\mathcal{M}$ be a $N-$dimensional spacetime and $\mathcal{S}$ a $M-$dimensional MS subspace with $M < N$, such that it admits $\frac{1}{2}M(M+1)$ independent Killing vectors. We label by u^i the coordinates of the MS subspace $\mathcal{S}$ and by v^a the coordinates of the complementary $(N-M)-$dimensional subspace $\bar{\mathcal{S}}$, where $i, j, k, \ldots$ and $a, b, c, \ldots$ label the respective coordinates. By hypothesis, we assume the full space to be invariant under the following infinitesimal transformations:

$$\begin{aligned} u^i &\longrightarrow u'^i = u^i + \varepsilon \xi^i(u;v)\,, \\ v^a &\longrightarrow v'^a = v^a\,, \end{aligned} \tag{2.1.38}$$

where the Killing vectors $\xi^i(u;v)$ depends on the u-coordinates of $\mathcal{S}$ and may depend parametrically on v^a.

Next, we break the metric tensor in three pieces – one purely function of the $u-$coordinates, g_{ij}, other only on the $v-$coordinates,

g_{ab}, and a mixed one, g_{ia}, – and apply the isometry conditions (1.3.6):

$$\begin{aligned}\mathcal{L}_\xi[g_{ij}] \equiv 0 = \xi^k(u,v)\frac{\partial g_{ij}(u,v)}{\partial u^k} \\ + g_{kj}(u,v)\frac{\partial \xi^k(u,v)}{\partial u^i}\,, \\ + g_{ik}(u,v)\frac{\partial \xi^k(u,v)}{\partial u^j}\end{aligned} \tag{2.1.39a}$$

$$\begin{aligned}\mathcal{L}_\xi[g_{ia}] \equiv 0 = \xi^k(u,v)\frac{\partial g_{ia}(u,v)}{\partial u^k} \\ + g_{ka}(u,v)\frac{\partial \xi^k(u,v)}{\partial u^i}\,, \\ + g_{ik}(u,v)\frac{\partial \xi^k(u,v)}{\partial v^a}\end{aligned} \tag{2.1.39b}$$

$$\begin{aligned}\mathcal{L}_\xi[g_{ab}] \equiv 0 = \xi^k(u,v)\frac{\partial g_{ab}(u,v)}{\partial u^k} \\ + g_{kb}(u,v)\frac{\partial \xi^k(u,v)}{\partial v^a}\,. \\ + g_{ak}(u,v)\frac{\partial \xi^k(u,v)}{\partial v^b}\end{aligned} \tag{2.1.39c}$$

Immediately we see that (2.1.39a) represents precisely the MS condition of $\mathcal{S}$, which is to be expected from our construction. Since there are no $v-$coordinate derivatives, we conclude that this metric depends *parametrically* on v^a,

$$g_{ij}(u,v) \longrightarrow g_{ij}(u;v) \longrightarrow \text{ homogeneous and isotropic } \forall v\,.$$

The other conditions mix up both derivatives with respect to the $\mathcal{S}-$ and $\bar{\mathcal{S}}-$coordinates, that is, u and v respectively. To disentangle them we can find a coordinate system such that $g'_{ia} = 0$, which exists if there is a change of coordinates such that

$$\begin{aligned} g'_{ja}(u',v') \equiv 0 = &\frac{\partial u^l}{\partial u'^j}\frac{\partial u^k}{\partial v'^a} g_{lk}(u,v) + \frac{\partial u^l}{\partial u'^j}\frac{\partial v^b}{\partial v'^a} g_{lb}(u,v) \\ &+ \overset{0}{\cancel{\frac{\partial v^b}{\partial u'^j}}}\frac{\partial u^k}{\partial v'^a} g_{bk}(u,v) + \overset{0}{\cancel{\frac{\partial v^b}{\partial u'^j}}}\frac{\partial v^c}{\partial v'^a} g_{bc}(u,v) \\ = &\frac{\partial u^l}{\partial u'^j}\frac{\partial u^k}{\partial v'^a} g_{lk}(u,v) + \frac{\partial u^l}{\partial u'^j} g_{la}(u,v) \\ = &\frac{\partial u^l}{\partial u'^j}\left(\frac{\partial u^k}{\partial v'^a} g_{lk}(u,v) + g_{la}(u,v)\right), \end{aligned}$$

where we used the fact that $v = v(v')$ and $u = u(u', v')$ from (2.1.38). Then if

$$\frac{\partial u^k}{\partial v'^a} g_{lk}(u,v) = -g_{la}(u,v)\,, \tag{2.1.40}$$

respecting the initial condition $u(u_0, v_0) = u_0$, we guarantee that $g'_{ia} = 0$. However, we still have to demonstrate that the above differential equation always has a solution, which boils down to verify if it satisfies the integrability conditions.

For clarity, we shall change u^k by its uppercase counterpart U^k and define the new quantity

$$\begin{aligned} \frac{\partial U^k}{\partial v^a} &:= -F^k{}_a(U,v) \qquad (2.1.41) \\ \overset{(2.1.40)}{\Longrightarrow} F^k{}_a(U,v) &= g^{ik}(U,v) g_{ka}(U,v)\,. \end{aligned}$$

To demonstrate what is desired, we will work with a Taylor expansion of U^k in the neighbourhood of v_0, proving by induction

$$\begin{aligned} U^k &= c^k + c^k{}_a (v - v_0)^a + c^k{}_{ab} (v - v_0)^a (v - v_0)^b + \ldots \\ &= \sum_n \frac{1}{n!} c^k{}_{a_1 \ldots a_n} (v - v_0)^{a_1} \ldots (v - v_0)^{a_n} \,, \end{aligned}$$

where the symbol $c^k{}_{a_1 \ldots a_n}$ is a shorthand collection of all the lower rank constants with $c^k \equiv c^k{}_{0 \ldots 0}$, $c^k{}_a \equiv c^k{}_{a0 \ldots 0}$, and so on.

The initial conditions take the form

$$\begin{cases} c^k &= u_0^k \\ c^k{}_a &= -F^k{}_a (u_0, v_0) \,. \end{cases}$$

We can also represent $F^k{}_a$ as a series expansion

$$F^k{}_a (U(u_0, v), v) = \sum_n \frac{1}{n!} f^k{}_{ab_1 \ldots b_n} (v - v_0)^{b_1} \ldots (v - v_0)^{b_n} \,.$$

If $F^k{}_a$ is valid up to the nth order, then, by (2.1.41), U^k must be valid up to the $(n+1)$th order. This term is given by

$$U^k_{[n+1]} = \frac{1}{(n+1)!} f^k{}_{ab_1 \ldots b_n} (v - v_0)^a (v - v_0)^{b_1} \ldots (v - v_0)^{b_n} \,.$$

From this, we get that the constants, $f^k{}_{ab\ldots}$ *have* to be symmetric in all of their lower indices (a and b's). This condition can be stated as

$$\frac{\partial F^k{}_a (U(u_0, v), v)}{\partial v^b} = \frac{\partial F^k{}_b (U(u_0, v), v)}{\partial v^a} \,,$$

up to the $(n-1)$th order. One of these derivatives can be written as

$$\frac{\partial F^k{}_a(U(u_0,v),v)}{\partial v^b} = \left[\frac{\partial F^k{}_a(u,v)}{\partial u^l}\frac{\partial u^l}{\partial v^b} + \frac{\partial F^k{}_a(u,v)}{\partial v^b}\right]_{u=U(u_0,v)}$$
$$\overset{(2.1.41)}{=} \left[-F^l{}_b(u,v)\frac{\partial F^k{}_a(u,v)}{\partial u^l} + \frac{\partial F^k{}_a(u,v)}{\partial v^b}\right]_{u=U(u_0,v)};$$

thus

$$\begin{aligned} &-F^l{}_b(u,v)\frac{\partial F^k{}_a(u,v)}{\partial u^l} + \frac{\partial F^k{}_a(u,v)}{\partial v^b} \\ &\qquad = -F^l{}_a(u,v)\frac{\partial F^k{}_b(u,v)}{\partial u^l} + \frac{\partial F^k{}_b(u,v)}{\partial v^a} \end{aligned}. \tag{2.1.42}$$

The last equation is always valid. Indeed, by the isometry relations (2.1.39a,2.1.39b), one can show that

$$\begin{aligned} 0 = &\left\{F^k{}_a\frac{\partial F^m{}_b}{\partial u^k} - F^k{}_b\frac{\partial F^m{}_a}{\partial u^k} + \frac{\partial F^m{}_a}{\partial v^b} - \frac{\partial F^m{}_b}{\partial v^a}\right\}\frac{\partial \xi^l}{\partial u^m} \\ &+ \left\{-F^k{}_a\frac{\partial^2 F^l{}_b}{\partial u^k \partial u^m} + F^k{}_b\frac{\partial^2 F^l{}_a}{\partial u^k \partial u^m}\right. \\ &\qquad \left. + \frac{\partial F^k{}_b}{\partial u^m}\frac{\partial F^l{}_a}{\partial u^k} - \frac{\partial F^k{}_a}{\partial u^m}\frac{\partial F^l{}_b}{\partial u^k} - \frac{\partial^2 F^l{}_a}{\partial v^b \partial u^m} - \frac{\partial^2 F^l{}_b}{\partial v^a \partial u^m}\right\}\xi^m \end{aligned}.$$

Since the Killing vectors are arbitrary and can be determined for the entirety of the MS spacetime, we choose x_0 such that

$$\begin{cases} \xi^k &= 0\,, \\ \xi_{k;i} &= \text{arbitrary}\,, \end{cases}$$

therefore

$$0 = \left\{ F^k{}_a \frac{\partial F^m{}_b}{\partial u^k} - F^k{}_b \frac{\partial F^m{}_a}{\partial u^k} + \frac{\partial F^m{}_a}{\partial v^b} - \frac{\partial F^m{}_b}{\partial v^a} \right\} \frac{\partial \xi^l}{\partial u^m}$$

$$\Longrightarrow \; F^k{}_a \frac{\partial F^m{}_b}{\partial u^k} - F^k{}_b \frac{\partial F^m{}_a}{\partial u^k} = \frac{\partial F^m{}_b}{\partial v^a} - \frac{\partial F^m{}_a}{\partial v^b} \,, \tag{2.1.43}$$

which is exactly (2.1.42), so we can always find $F^k{}_a$ satisfying (2.1.40) such that

$$\boxed{g'_{ia} = 0} \,.$$

In this frame of reference, we can go back to (2.1.39) to determine the last two isometry conditions we have when decoupling the MS subspace $\mathcal{S}$ from the whole space. Imposing

$$\cdot \; \mathcal{L}_\xi[g_{ia}] = 0 = g_{ik}(u,v) \frac{\partial \xi^k(u,v)}{\partial v^a} \,, \qquad \det \hat{g} \neq 0$$

$$\Longrightarrow \; \frac{\partial \xi^k}{\partial v^a} = 0 \;\Longrightarrow\; \boxed{\xi^k = \xi^k(u)} \tag{2.1.44}$$

and

$$\cdot \; \mathcal{L}_\xi[g_{ab}] = 0 = \xi^k(u) \frac{\partial g_{ab}}{\partial u^k} \,;$$

since ξ^k are arbitrary,

$$\frac{\partial g_{ab}}{\partial u^k} = 0 \;\Longrightarrow\; \boxed{g_{ab} = g_{ab}(v)} \,. \tag{2.1.45}$$

In other words, the Killing vectors are in fact function only of the MS $u-$coordinates and the complementary $\bar{\mathcal{S}}$ metric g_{ab} is a function only of its $v-$coordinates.

It only remains to be shown what dependency the MS metric g_{ij} has on u and v. That can be easily done when we recall (2.1.10) from the discussion of maximally form invariant tensors, writing

$$g_{ij} = f(v)\tilde{g}_{ij}(u)\,. \tag{2.1.46}$$

Putting everything back together, we can finally construct the spacetime interval

$$ds^2 = g_{ab}(v)dv^a dv^b + f(v)\tilde{g}_{ij}(u)du^i du^j\,,$$

which is the desired result, thus demonstrating the theorem.

Moreover, since $\tilde{g}_{ij}$ is MS, we can use the MS interval (2.1.34) we previously found,

$$ds^2 = g_{ab}(v)dv^a dv^b + f(v)\left[d\boldsymbol{u}^2 + k\frac{(\boldsymbol{u}\cdot d\boldsymbol{u})^2}{1-k\boldsymbol{u}^2}\right], \tag{2.1.47}$$

where $k = 0, \pm 1$. To finalize this study, we shall unwrap a few particular cases for some recurring scenarios, backed by two physical scenarios of interest.

2.1.2 Particular Maximally Symmetric Subspaces

Whole $3+1-$spacetime MS

From (2.1.21), imposing

$$h_{\mu\nu} \equiv \eta_{\mu\nu}\,,$$

where η is the Minkowski metric tensor, we get the following spacetime interval

$$ds^2 = dt^2 - d\boldsymbol{x}^2 + \frac{K}{1 - K h_{\alpha\beta} x^\alpha x^\beta} \left(t dt - \boldsymbol{x} \cdot d\boldsymbol{x} \right)^2 . \qquad (2.1.48)$$

If $K > 0$ we can change the coordinates by

$$\begin{cases} t & \rightarrow \frac{1}{\sqrt{K}} \left[\frac{1}{2} K \boldsymbol{x}'^2 \cosh(\sqrt{K} t') + \left(1 + \frac{1}{2} K \boldsymbol{x}'^2 \right) \sinh(\sqrt{K} t') \right] , \\ \boldsymbol{x} & \rightarrow \boldsymbol{x}' \exp(\sqrt{K} t') , \end{cases}$$

giving

$$ds^2 = dt'^2 - \exp(2\sqrt{K} t') d\boldsymbol{x}'^2 .$$

Transforming once again by

$$\begin{cases} t'' & = t' - \frac{1}{2\sqrt{K}} \ln \left(1 - K \boldsymbol{x}'^2 \exp(2\sqrt{K} t') \right) , \\ \boldsymbol{x}'' & = \boldsymbol{x}' \exp(\sqrt{K} t') , \end{cases}$$

we get rid of all the time dependency on the spatial sector:

$$ds^2 = (1 - K \boldsymbol{x}''^2) dt''^2 - d\boldsymbol{x}''^2 - \frac{K}{1 - K \boldsymbol{x}''^2} \left(\boldsymbol{x}'' \cdot d\boldsymbol{x}'' \right)^2 . \qquad (2.1.49)$$

This is the well known (anti-)*de Sitter* universe for $K > 0$ ($K < 0$) expressed in hyperbolic coordinates.

MS 3-subspace with spherical symmetry and constant curvature

Assuming $N = 3$ dimensions in spherical coordinates and a positive curvature $k = +1$. The angular $\mathcal{S}$-coordinates $\{\theta, \phi\}$ are MS whereas r is of $\bar{S}$. So the spacetime interval (2.1.47) breaks down to

$$\begin{cases} v & \equiv r \\ u & \equiv \{\theta, \phi\} = \begin{cases} u^1 & = \sin\theta\cos\phi \\ u^2 & = \sin\theta\sin\phi \end{cases} \end{cases}$$

so that

$$\begin{cases} du^1 = d\theta\cos\theta\cos\phi - d\phi\sin\theta\sin\phi \\ du^2 = d\theta\cos\theta\sin\phi + d\phi\sin\theta\cos\phi \end{cases}$$

$$\rightarrow \boldsymbol{u}^2 = \sin^2\theta$$

$$\rightarrow d\boldsymbol{u}^2 = d\theta^2\cos^2\theta + d\phi^2\sin^2\theta$$

$$\rightarrow \boldsymbol{u}\cdot d\boldsymbol{u} = d\theta\sin\theta\cos\theta$$

$$\begin{aligned} \therefore\ g_{ij}du^idu^j &= \left[d\boldsymbol{u}^2 + k\frac{(\boldsymbol{u}\cdot d\boldsymbol{u})^2}{1-k\boldsymbol{u}^2}\right] \\ &= d\theta^2 + \sin^2\theta d\phi^2\,, \end{aligned}$$

yielding

$$ds^2 = g(r)dr^2 + f(r)\Big(d\theta^2 + \sin^2\theta d\phi^2\Big)\,,$$

where $g(r), f(r)$ are compatible with metrics with positive signature.

MS spacetime with spherical symmetry and constant curvature

Similarly to the previous example, we consider a spacetime of $N = 3 + 1$ dimensions, positive curvature $k = +1$ and a Lorenzian metric signature $(+ - --)$. The angular coordinates $\{\theta, \phi\}$ are again MS in $\mathcal{S}$ but now both $\{t, r\}$ live in $\bar{\mathcal{S}}$.

$$\begin{cases} v &= \{t, r\} \\ u &= \{\theta, \phi\} \end{cases}$$

$$\therefore ds^2 = g_{ab} dv^a dv^b + f(r) \tilde{g}_{ij} dx^i dx^j \,,$$

where g_{ab} has a $(+-)$ signature and

$$\begin{aligned} ds^2 = g_{tt}(t, r) dt^2 + 2 g_{tr}(t, r) dt dt + g_{rr}(t, r) dr^2 \\ - f(t, r) \left(d\theta^2 + \sin^2\theta d\phi^2 \right) . \end{aligned}$$

Spacetime with a MS spatial sector, spherical symmetry and arbitrary curvature

Now, the entirety of the spatial sector is MS, so $\{r, \theta, \phi\}$ live in $\mathcal{S}$ and only the time coordinate in $\bar{\mathcal{S}}$, with arbitrary constant curvature k:

$$\begin{cases} v &= t \\ u &= \{r, \theta, \phi\} = \begin{cases} u^1 = r \sin\theta \cos\phi \\ u^2 = r \sin\theta \sin\phi \\ u^3 = r \cos\theta \end{cases} \end{cases}$$

so that

$$\begin{cases} du^1 &= dr\sin\theta\cos\phi + d\theta r\cos\theta\cos\phi - d\phi r\sin\theta\sin\phi \\ du^2 &= dr\sin\theta\sin\phi + d\theta r\cos\theta\sin\phi - d\phi r\sin\theta\cos\phi \\ du^1 &= dr\cos\theta - d\theta r\sin\theta + 0 \end{cases}$$

giving

$$ds^2 = g(t)dt^2 + f(t)\left(d\boldsymbol{u}^2 + k\frac{(\boldsymbol{u}\cdot d\boldsymbol{u})^2}{1-k\boldsymbol{u}^2}\right).$$

Defining

$$t' \equiv \int dt\,\sqrt{g(t)} \longrightarrow f(t) \equiv -S^2(t)\,,$$

we get

$$ds^2 = dt^2 - S^2(t)\left(d\boldsymbol{u}^2 + k\frac{(\boldsymbol{u}\cdot d\boldsymbol{u})^2}{1-k\boldsymbol{u}^2}\right).$$

Next, opening the scalar products inside the brackets,

$$\boldsymbol{u}^2 = r^2$$

$$\longrightarrow d\boldsymbol{u}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

$$\longrightarrow \boldsymbol{u}\cdot d\boldsymbol{u} = rdr\,,$$

results in

$$
\begin{aligned}
g_{ij}du^i du^j &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + k\frac{r^2 dr^2}{1-kr^2} \\
&= \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2
\end{aligned}
$$

so that

$$
ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (2.1.50)
$$

which is none other than the Friedmann-Lemaître-Robertson-Walker spacetime interval, the standard model of modern Cosmology.

It is remarkable that we could construct this line element from the ground up in a solid and robust way by using just premises of symmetry, encoded within the Killing vectors and inherit to the isometries of the metric tensor. We recall that by virtue of being MS, the spatial sector is automatically homogeneous and isotropic about every point, which are the basic heuristic ingredients frequently used in the literature of gravitation and Cosmology.

In the next section we will explore the main consequences and results that we can extract from (2.1.50), comparing with the observational data, that makes this model the standard adopted by the cosmologists.

2.2 The Friedmann-Lemaître-Robertson-Walker Cosmology

Recalling the FLRW line element (2.1.50) derived above

$$ds^2 = dt^2 - S^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{2.2.1}$$

where we interpret $S^2(t)$ as the conformal factor related to the scale factor of the Universe (the Hubble constant), $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the element of solid angle, and

$$k = \begin{cases} -1, & \text{open space} \\ 0, & \text{flat space} \\ +1, & \text{closed space} \end{cases}, \tag{2.2.2}$$

is the curvature constant, remembering that any other value of it can be brought to $\{0, \pm 1\}$ by a suitable coordinate transformation.

From this spacetime interval, we infer the metric

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{S^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -S^2 r^2 & 0 \\ 0 & 0 & 0 & -S^2 r^2 \sin^2\theta \end{bmatrix}, \tag{2.2.3}$$

and its inverse

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kr^2}{S^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{S^2 r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{S^2 r^2 \sin^2\theta} \end{bmatrix}. \tag{2.2.4}$$

By tediously calculating all the non-vanishing Christoffel symbols, we get

$$
\begin{aligned}
\Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^3{}_{03} &= \frac{\dot{S}}{S} \\
\Gamma^0{}_{11} = \frac{S\dot{S}}{1-kr^2} \quad &; \quad \Gamma^0{}_{22} = r^2 S\dot{S} \\
\Gamma^0{}_{33} = r^2 \sin^2\theta S\dot{S} \quad &; \quad \Gamma^1{}_{11} = \frac{kr}{1-kr^2} \\
\Gamma^1{}_{22} = -r(1-kr^2) \quad &; \quad \Gamma^1{}_{33} = -r\sin^2\theta(1-kr^2) \\
\Gamma^2{}_{12} = \frac{1}{r} \quad &; \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta \\
\Gamma^3{}_{23} &= \cot\theta
\end{aligned}
\tag{2.2.5}
$$

With those in hand, we are able to compute the Ricci tensors (1.1.52) and then the Einstein equations (1.1.59):

$$
R_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu,\lambda} - \Gamma^\lambda{}_{\mu\lambda,\nu} - \Gamma^\varepsilon{}_{\mu\lambda}\Gamma^\lambda{}_{\varepsilon\nu} + \Gamma^\varepsilon{}_{\mu\nu}\Gamma^\lambda{}_{\varepsilon\lambda}\,, \tag{2.2.6}
$$

those resulting in

$$
\begin{aligned}
R^0{}_0 &= -\frac{\ddot{S}}{S}\,, \\
R^1{}_1 = R^2{}_2 = R^3{}_3 &= -\frac{\ddot{S}}{S} - \frac{2\dot{S}^2 + 2k}{S^2}\,, \\
R^\mu{}_\nu &= 0\ ; \quad \mu \neq \nu\,,
\end{aligned}
$$

$$
R = -6\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + k}{S^2}\right) \tag{2.2.7}
$$

and

$$G^0{}_0 = 3\frac{\dot{S}^2 + k}{S^2} = 8\pi G T^0{}_0\,, \tag{2.2.8a}$$

$$G^0{}_j = 0 = -8\pi G T^0{}_j\,, \tag{2.2.8b}$$

$$G^i{}_j = \delta^i{}_j\left(\frac{\dot{S}^2 + k}{S^2} + 2\frac{\ddot{S}}{S}\right) = 8\pi G T^i{}_j\,. \tag{2.2.8c}$$

These are the Einstein field equations for the FLRW Universe model. To study these equations more deeply, we have to specify an energy-momentum tensor $T^\mu{}_\nu$ consistent with the actual observations. One of such is the fluid energy-momentum tensor in equilibrium,

$$T^{\mu\nu} = (p + \varepsilon)u^\mu u^\nu - p g^{\mu\nu}\,, \tag{2.2.9}$$

in the perfect fluid approximation with constant pressure among particles, described by the conditions

$$T^0{}_0 = \varepsilon\,,$$
$$T^1{}_1 = T^2{}_2 = T^3{}_3 = \text{const} = -p\,,$$
$$T^\mu{}_\nu = 0\,,\quad \mu \neq \nu\,,$$

where ε is the energy density and p the pressure.

In this approximation, each galaxy of our Universe is, in fact, treated as a "particle" while the constant pressure components are all equal due to the spatial homogeneity and isotropy. While apparently preposterous, this approximation is quite good; the observational data indicate that this is the approximately the case, at large scale.

We also have to impose the Weyl postulate. It states that *all* galactic motions are given in the time direction, mathematically expressed by the 4-velocity

$$u^{\mu} = (1, 0, 0, 0)\,, \tag{2.2.10}$$

which is also supported by observational data, up to a mean spatial velocity of the order $\tilde{v} \sim 1000\ km \cdot s^{-1}$. We shall discuss shortly the validity of such approximations.

The energy-momentum tensor (2.2.9) gives the necessary equations of state for the field equations (2.2.8) above. Indeed, from (2.2.8a) and using (2.2.8c), we get

$$\therefore\ \frac{d}{dS}(T^{0}{}_{0} S^3) - 3T^{i}{}_{i} S^2 = 0\,. \tag{2.2.11}$$

Imposing the condition of *non-interacting free-falling dust*, given by $T^{0}{}_{0} = \varepsilon \equiv \rho$ and null pressures $p = 0$, gives

$$\begin{aligned} T^{\mu\nu} &= \varepsilon u^{\mu} u^{\nu} \\ \Longrightarrow\ T^{00} = \varepsilon \equiv \rho \qquad &; \qquad T^{ij} = -p = 0\,; \end{aligned} \tag{2.2.12}$$

we further simplify the energy-momentum tensor, which enables us to study the validity of those approximations at the present, past and future.

Let us consider a spatial pertubation to Weyl's postulate, introducing a 4-velocity of the order $\tilde{\beta} \sim 10^{-2}$ and comparing it to the order of magnitude of the pressure with relation to the energy density:

$$\begin{aligned} p &\sim \tilde{\beta}^2 \varepsilon \\ &\sim \left(10^{-2}\right)^2 \varepsilon \\ &\sim 10^{-4} \varepsilon\,, \end{aligned}$$

which is insignificant at the present epoch. To analize the validity of this approximation in the past and future, we make use of the geodesic equation (1.1.47) to find a conservation law relating the 4-velocity u^μ and the scale factor S, for $u^i \ll 1$,

$$
\begin{aligned}
0 &= \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta \\
&= \frac{du^\mu}{d\tau} + 2\delta^\mu_i \frac{\dot{S}}{S} u^i \,.
\end{aligned}
$$

For the time and space components we have, respectively,

$$
\begin{aligned}
\frac{du^0}{d\tau} &\equiv 0 = 0 \\
d(\ln u^i) &= d(\ln S^{-2})
\end{aligned}
$$

$$
\Longrightarrow \quad u^i S^2 = \text{const}\,. \tag{2.2.13}
$$

Since S is present in the spatial sector of (2.2.1), the 4-velocity in the *comoving frame** also carries along a multiplicative factor so that the proper 4-velocity, u^i_{proper}, has to decrease with S^{-1} in order to satisfy (2.2.13) at all epochs,

$$
u^i_{\text{proper}} = S u^i \propto S^{-1} \,.
$$

Therefore, this pertubation tends to diminish in the future, guaranteeing the validity of the perfect fluid approximation. Nevertheless, things were not so bright in the past, and we need another way to assess it.

Consider then two distinct regimes, which we call *radiation*

*A comoving frame is such that all the particles of the system are dragged along the same time-like geodesic everywhere, being described by the spacetime interval $ds^2 = dt^2 - \zeta_{ij}(\boldsymbol{x};t)dx^i dx^j$. In the present context, the space metric ζ, parametrized by t, reduces to the scale factor $S(t)$.

dominated era ($p \propto \frac{1}{3}\varepsilon$) and the *matter dominated era* ($p = 0$; $\varepsilon = \rho$), inevitably passing through a phase transition at some point in the past, when $S = S_0$.

- Matter dominated era:

  This era is characterized by

$$p = 0 \quad ; \quad \varepsilon = \rho \quad ; \quad S > S_0 \; .$$

From (2.2.11),

$$\frac{d}{dS}(\rho S^3) = 0$$

and, integrating it over dS from S_0 to $S(t)$, where $S_0 = S(t_0)$ is the scale factor at the present time t_0, results

$$\rho_0 S_0^3 - \rho S^3(t) = 0 \tag{2.2.14}$$

$$\Longrightarrow \quad \rho(t) = \rho_0 \left(\frac{S_0}{S(t)} \right)^3 , \tag{2.2.15}$$

meaning that $\rho \propto S^{-3}$ from now to the future $S > S_0$.

- Radiation dominated era:

  The radiation era is fully described by significative radiation pressure, equiparted due to isotropy

$$p = \frac{1}{3}\varepsilon \quad ; \quad S < S_0 \; .$$

Hence, from (2.2.11),

$$\frac{d}{dS}(\varepsilon S^3) + 3(\frac{1}{3}\varepsilon)S^2 = 0\,;$$

multiplying by S,

$$S\frac{d}{dS}(\varepsilon S^3) + \varepsilon S^3 = 0\,,$$
$$\frac{d}{dS}(\varepsilon S^4) = 0\,,$$

and integrating over dS yields

$$\varepsilon(t) = \varepsilon_0\left(\frac{S_0}{S(t)}\right)^4 . \qquad (2.2.16)$$

Therefore we get for the past $S < S_0$ an order of magnitude of $\varepsilon \propto S^{-4}$.

Observational data points to energy and matter densities of the order $\varepsilon_0 \sim 10^{-13}\ erg \cdot cm^{-3}$ and $\rho_0 \sim 10^{-10} erg \cdot cm^{-3} \cdot c^{-2}$, so

$$\frac{\varepsilon_0}{\rho_0 c^2} \sim 10^{-3}\ .$$

Comparing both densities above ($c = 1$),

$$\begin{aligned}\frac{\varepsilon}{\rho} &= \frac{\varepsilon_0 (S_0/S)^4}{\rho_0 (S_0/S)^3} \\ &= \frac{\varepsilon_0}{\rho_0}\frac{S_0}{S} \\ &\sim 10^{-3}\frac{S_0}{S}\ ,\end{aligned}$$

we conclude that the approximation in question is valid if the scale ratio $\frac{S_0}{S} \lesssim 10^3$, otherwise both radiation and matter will be indistinguishable, breaking the premises of the approximation and thus, the validity of Weyl's postulate.

By virtue of that, we have to keep in mind that the results henceforward are valid for epochs such that $\frac{S_0}{S}$ is small enough for matter to be decoupled from radiation.

Before inserting the energy-momentum tensor (2.2.12) into the Einstein equations (2.2.8), we need first to derive a necessary result, to enable us to compare with real data; the redshift relations from the FLRW line element (2.2.1).

Redshift

The *redshift*, of the Doppler effect of light, is characterized by the broadening of electromagnetic wavelengths, i.e. light, due to the constancy of the speed of light by the relative motion of the object and its observer. If λ_e is the actual emitted wavelength and λ_o is the observed quantity, then the redshift z is then given by

$$1 + z := \frac{\lambda_o}{\lambda_e} \, .$$

This is of particular interest, since Edwin Hubble's[26] discovery in 1929 that extragalactic objects are all distancing themselves away from us, pointing to a Universe expansion; the greater the distance of a given object, the greater is its redshift. His relation, coined *Hubble's law*, is a linear relation between the redshift itself and the luminous distance D of the objects

$$z = H_0 D \, ,$$

where the constant of proportionality H_0 is the so-called *Hubble con-*

stant.

Although empirically found by Hubble, we can show that it can be deduced from the FLRW spacetime interval (2.2.1) quite nicely.

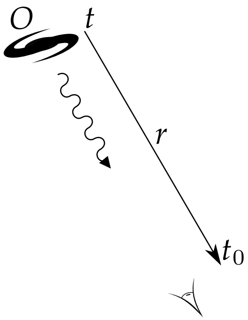


Figure 14: Redshift observation.

Considering radial light rays ($ds = 0$; θ and ϕ constant) coming from some galaxy located at a distance r, emitted at an instant t and detected at t_0. Since the space is homogeneous and isotropic everywhere, we can choose the origin at the emission point, in the galaxy, as shown in Fig. 14. We can then compute the null-path the light travels,

$$ds^2 \equiv 0 = dt^2 - \frac{S^2(t)}{1 - kr^2} dr^2$$
$$dt = \pm \frac{S}{\sqrt{1 - kr^2}} dr \; .$$

Integrating both sides and choosing the positive sign,

$$\int_t^{t_0} \frac{dt}{S} = \int_0^r \frac{dr}{\sqrt{1 - kr^2}} \; . \tag{2.2.17}$$

When the law was first discovered, the maximum observed redshift was of the order $z \approx 0.003$, small enough to justify an expansion of the integrands above. In this regime, S will barely vary, taking the value S_0 instead.

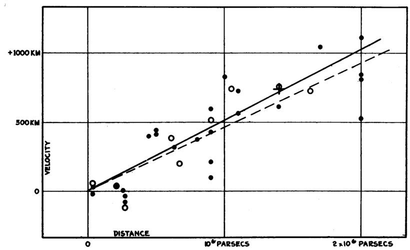


Figure 15: Hubble's law. The velocity on the y-axis can be encoded as the redshift z (From E. Hubble[26]).

$$\int_t^{t_0} \frac{dt}{S} \approx \frac{1}{S_0} \int_t^{t_0} dt \approx \frac{t_0 - t}{S_0} \ .$$

The righthand side of (2.2.17) can be expanded for small r as well,

$$\int_0^r \frac{dr}{\sqrt{1 - kr^2}} \approx \int_0^r dr(1 + O(r)) \approx r$$

Comparing both expansions, we arrive at

$$D := rS_0 \approx t_0 - t \ , \tag{2.2.18}$$

where $D := rS_0$ is the luminous distance of an object corrected by the expansion factor.

Now, expanding $S(t)$ in a Taylor series up to the first order,

$$
\begin{aligned}
S(t) &\approx S_0 + \left.\frac{dS}{dt}\right|_{t=t_0} (t - t_0) \\
&\approx S_0 - \left.\frac{\dot{S}}{S}\right|_{t=t_0} (t_0 - t)\, S_0\,,
\end{aligned}
$$

and considering the redshift correction

$$
S(t) = \frac{S_0}{1+z} \approx S_0(1 - z)\,,
$$

both have to be consistent with each other, namely,

$$
S_0(\not{1} - z) = S_0\left(\not{1} - \left.\frac{\dot{S}}{S}\right|_{t=t_0} (t_0 - t)\right)
$$

$$
\therefore\ z = \left.\frac{\dot{S}}{S}\right|_{t=t_0} (t_0 - t)\,.
$$

Then, by (2.2.18), it is immediate that

$$
\begin{aligned}
z &= \left.\frac{\dot{S}}{S}\right|_{t=t_0} D &\quad (2.2.19)\\
&= H_0 D\,, &\quad (2.2.20)
\end{aligned}
$$

recovering the empirical law, as desired. Modern measurements of the Hubble constant give $H_0 = h_0 \times 100\ km \cdot s^{-1} \cdot Mpc^{-1}$ with $0.5 \leq h_0 \leq 1.0$.

Let us resume to the specific calculations of the Friedmann solutions.

Flat sector $k = 0$

For $k = 0$, we get from (2.2.8) the following Einstein field equations, considering the limit of the matter dominated era (2.2.15)

$$\begin{aligned}\left(\frac{\dot{S}}{S}\right)^2 &= \frac{8\pi G}{3}\rho \\ &= \frac{8\pi G}{3}\rho_0\left(\frac{S_0}{S}\right)^3 ,\end{aligned}$$

or yet

$$\dot{S}^2 = \frac{8\pi G}{3}\rho_0\frac{S_0^3}{S} . \tag{2.2.21}$$

Evaluating at the present epoch $t = t_0$,

$$\begin{aligned}\dot{S}^2\Big|_{t_0} &= \frac{8\pi G}{3}\rho_0 S_0^2 , \\ \left(\frac{\dot{S}}{S}\right)^2_{t_0} &= \frac{8\pi G}{3}\rho_0 ,\end{aligned}$$

and noting that $\left.\frac{\dot{S}}{S}\right|_{t_0} = H_0$,

$$\rho_c := \rho_0 = \frac{3H_0^2}{8\pi G} , \tag{2.2.22}$$

where we define the critical density of matter ρ_c, which will become clearer for the $k = \pm 1$ solutions.

Returning it to (2.2.21), gives

$$S\dot{S}^2 = H_0^2 S_0^3 ,$$

or, square rooting,

$$S^{\frac{1}{2}}\dot{S} = H_0 S_0^{\frac{3}{2}} ,$$

and integrating over dt:

$$\underbrace{\int_t^{t_0} dt S^{\frac{1}{2}}\dot{S}}_{I} = H_0 S_0^{\frac{3}{2}} \int_t^{t_0} dt .$$

We compute the integral I apart.

$$\begin{aligned} I &= \int_t^{t_0} dt S^{\frac{1}{2}}\dot{S} \\ &= \int_t^{t_0} dt \frac{d}{dt}(S^{\frac{1}{2}} S) - \int_t^{t_0} dt \frac{1}{2}\dot{S} S^{-\frac{1}{2}} S \\ &= S^{\frac{3}{2}}\Big|_t^{t_0} - \underbrace{\frac{1}{2}\int_t^{t_0} dt \dot{S} S^{\frac{1}{2}}}_{=I} \end{aligned}$$

$$\therefore I = \frac{2}{3}\left(S_0^{\frac{3}{2}} - S^{\frac{3}{2}}\right).$$

Thus,

$$\begin{aligned} \frac{2}{3}\left(S_0^{\frac{3}{2}} - S^{\frac{3}{2}}\right) &= H_0 S_0^{\frac{3}{2}}(t_0 - t) , \\ H_0 S_0^{\frac{3}{2}} t - \frac{2}{3} S^{\frac{3}{2}} &= H_0 S_0^{\frac{3}{2}} t_0 - \frac{2}{3} S_0^{\frac{3}{2}} \equiv C \\ \therefore S(t) &= S_0 \left(\frac{3H_0}{2} t\right)^{\frac{2}{3}} , \end{aligned} \tag{2.2.23}$$

assuming that $t \rightarrow 0 \implies S \rightarrow 0$, so $C = 0$. This is the *Big-Bang* scenario.

Inverting the relation above, we are capable of estimating the age of the Universe, which at the present epoch gives

$$t_0 = \frac{2}{3H_0} \,. \tag{2.2.24}$$

2.2.0.1 Closed sector $k = +1$

Similarly, for $k = +1$, we have from (2.2.8) the following Einstein system of equations

$$\frac{\dot{S}^2 + 1}{S^2} - \frac{8\pi G}{3}\rho_0\left(\frac{S_0}{S}\right)^3 = 0\,, \tag{2.2.25a}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + 1}{S^2} = 0\,. \tag{2.2.25b}$$

By defining the Hubble "constant" $H = H(t)$, not necessarily at the present, and the *deceleration factor* $q = q(t)$ associated with the deceleration (or acceleration) of the Universe, respectively, as

$$H = \frac{\dot{S}}{S}\,,$$

$$\frac{\ddot{S}}{S} = -qH^2\,, \tag{2.2.26}$$

(2.2.25) simplifies to

$$(2q - 1)H^2 = \frac{1}{S^2}\,. \tag{2.2.27}$$

On the other hand, inserting the (2.2.25a) into (2.2.25b),

$$2qH^2 = \frac{8\pi G}{3}\rho_0\left(\frac{S_0}{S}\right)^3 ,$$

and evaluating at the present epoch, we get

$$\rho_0 = \frac{3H_0^2}{8\pi G}2q_0 , \tag{2.2.28}$$

which when compared to (2.2.22) can be brought in the form

$$\rho_0 = \rho_c\Omega_0 \qquad ; \ \Omega_0 := 2q_0 , \tag{2.2.29}$$

where Ω_0 is called *density parameter*; it assumes values in the range $\Omega_0 > 1$ if $q_0 > \frac{1}{2}$, for $k = 1$, when (2.2.27) is evaluated at the present epoch, since $S_0 > 0$.

Inserting into (2.2.25a),

$$\frac{\dot{S}^2 + 1}{S^2} = \frac{8\pi G}{3}\rho_0\left(\frac{S_0}{S}\right)^3 ,$$

$$\dot{S}^2 + 1 = 2q_0H_0^2\frac{S_0^3}{S} ,$$

and substituting with (2.2.27) evaluated at the present, gives

$$\begin{aligned}\dot{S}^2 + 1 &= \frac{2q_0 H_0^2}{S} \frac{1}{(2q_0 - 1)^{\frac{3}{2}} H_0^3} \\ &= \frac{2q_0}{(2q_0 - 1)^{\frac{3}{2}} H_0} \frac{1}{S}\end{aligned}$$

$$\therefore \dot{S}^2 = \frac{Q_0^+}{S} - 1 \, , \tag{2.2.30}$$

where the modified expansion factor $Q_0^+ := \dfrac{2q_0}{(2q_0 - 1)^{\frac{3}{2}} H_0}$ assists us to solve the equations. The last equation then can be solved by the simple trigonometric substitution

$$\begin{aligned} S &= Q_0^+ \sin^2 \Theta \\ dS &= 2Q_0^+ \cos\Theta \sin\Theta d\Theta \, , \end{aligned} \tag{2.2.31}$$

so that (2.2.30) can be rewritten as

$$\frac{dS}{dt} \left(\frac{S}{Q_0^+ - S} \right)^{\frac{1}{2}} = 1 \, ,$$

so, when integrated over t, using (2.2.31),

$$t_0 - t = Q_0^+ \left(\Theta - \frac{1}{2} \sin(2\Theta) \right) \Bigg|_{\Theta}^{\Theta_0} . \tag{2.2.32}$$

At the present epoch, (2.2.31) can be rewritten in the form

$$\sin^2\Theta_0 = \frac{1}{2}\Big(1-\cos(2\Theta_0)\Big) = \frac{S_0}{Q_0^+}$$
$$\cos(2\Theta_0) = \frac{1-q_0}{q_0}\,, \tag{2.2.33}$$

so that

$$\begin{aligned}\sin(2\Theta_0) &= \sqrt{1-\cos^2(2\Theta_0)}\\ &= \sqrt{1-\left(\frac{1-q_0}{q_0}\right)^2}\\ &= \frac{(2q_0-1)^{\frac{1}{2}}}{q_0}\,.\end{aligned} \tag{2.2.34}$$

Thus, evaluating (2.2.32) under the big-bang condition $t \to 0$ and $S \to 0 \implies \Theta \to 0$, we get our estimation for the age t_+ of the Universe for a positive curvature $k = +1$,

$$\begin{aligned} t_+ &= Q_0^+\left(\Theta_0 - \frac{1}{2}\sin(2\Theta_0)\right)\\ &= Q_0^+\left(\frac{1}{2}\arccos\left(\frac{1-q_0}{q_0}\right) - \frac{1}{2}\frac{(2q_0-1)^{\frac{1}{2}}}{q_0}\right)\\ \therefore\ t_+ &= \frac{q_0}{(2q_0-1)^{\frac{3}{2}}H_0}\left(\arccos\left(\frac{1-q_0}{q_0}\right) - \frac{(2q_0-1)^{\frac{1}{2}}}{q_0}\right)\,. \end{aligned} \tag{2.2.35}$$

It is interesting to remark that there is a limit to the age of the Universe in the closed space. This is due to the oscillatory behaviour of (2.2.31). The universe reaches its peak size when $\Theta_0 = \frac{\pi}{2}$, which, if normalized by $q_0 = 1$, corresponds to

$$
\begin{aligned}
S_{\max} = Q_0^+ &= \frac{2q_0}{(2q_0 - 1)^{\frac{3}{2}} H_0} \\
&= \frac{2}{H_0} \, ,
\end{aligned}
$$

so the maximum size the Universe predicted by the $k = +1$ FLRW model is double the current observed size. After reaching this apex, the sign of $\dot{S}$ flips and the Universe begins a great contraction until the so-called *big-crunch* $S = 0$, which happens when $\Theta_0 = \pi$. Under these conditions, the *limit age* t_L of the Universe is estimated as the full cycle

$$
\begin{aligned}
t_L &= Q_0^+ \left(\Theta - \frac{1}{2} \sin 2\Theta \right) \Bigg|_{\Theta=\pi} \\
&= \pi Q_0^+ \\
&= \frac{2\pi q_0}{(2q_0 - 1)^{\frac{3}{2}} H_0} \, ,
\end{aligned}
$$

where, for $q_0 = 1$,

$$
t_L = \frac{2\pi}{H_0} \, .
$$

2.2.0.2 Open sector $k = -1$

The open sector solution is given in an analogous fashion to the previous section. From (2.2.8), with $k = -1$, we have

$$\frac{\dot{S}^2 - 1}{S^2} - \frac{8\pi G}{3}\rho_0\left(\frac{S_0}{S}\right)^3 = 0\,, \tag{2.2.36a}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 - 1}{S^2} = 0\,. \tag{2.2.36b}$$

Introducing (2.2.26) again into (2.2.36b),

$$\begin{aligned} -2qH^2 - \frac{1}{S^2} + H^2 &= 0 \\ \frac{1}{S^2} &= (1 - 2q)H^2\,, \end{aligned} \tag{2.2.37}$$

and evaluating (2.2.36a) at the present, using (2.2.37),

$$\begin{aligned} H_0^2 - \frac{1}{S_0^2} &= \frac{8\pi G}{3}\rho_0\,, \\ H_0^2 - (1 - 2q_0)H_0^2 &= \frac{8\pi G}{3}\rho_0\,, \end{aligned}$$

or,

$$\rho_0 = \frac{3H_0^2}{8\pi G}2q_0 =: \rho_C\Omega_0\,. \tag{2.2.38}$$

We define the other range of validity for (2.2.29), now with the restrictions

$$q_0 < \frac{1}{2} \qquad \Longrightarrow \qquad 0 \leq \Omega_0 < 1\,, \tag{2.2.39}$$

imposed by (2.2.37). Here it is evident why we call ρ_C the critical density of matter, for any value smaller than it defines an open space whereas values greater than it defines the closed section, as (2.2.39) and (2.2.29) respectively shows. The critical density is the value of the density of matter that separetes the three distinct regimes.

Now, substituting (2.2.38) into (2.2.36a), results in

$$\dot{S} = \left(\frac{Q_0^-}{S} + 1\right)^{\frac{1}{2}},$$

where we similarly defined $Q_0^- := \dfrac{2q_0}{(1-2q_0)^{\frac{3}{2}} H_0}$ in order to perform the substitution

$$\begin{aligned} S &= Q_0^- \sinh^2 \psi\,, \\ dS &= Q_0^- \sinh\psi \cosh\psi d\psi\,. \end{aligned} \tag{2.2.40}$$

Next, integrating over t, we get

$$\begin{aligned} t_0 - t &= \int_S^{S_0} dS \frac{S^{\frac{1}{2}}}{(Q_0^- + S)^{\frac{1}{2}}} \\ &= Q_0^- \left(\frac{1}{2}\sinh 2\psi - \psi\right)\Big|_{\psi}^{\psi_0}\,. \end{aligned} \tag{2.2.41}$$

At the present epoch, the hyperbolic substitution (2.2.40) reads

$$\begin{aligned} \sinh^2 \psi_0 = \frac{S_0}{Q_0^-} &= \frac{1}{2}(\cosh 2\psi - 1) \\ \rightarrow \cosh 2\psi &= 2\frac{S_0}{Q_0^-} + 1\,; \end{aligned}$$

inserting (2.2.40) and (2.2.37) evaluated at the present, yields

$$\cosh 2\psi = \frac{1-q_0}{q_0}\,. \tag{2.2.42}$$

Using hyperbolic identities, we can rewrite the above expression in the desired form

$$\begin{aligned}\sinh 2\psi &= \sqrt{\cosh^2 2\psi - 1}\\ &= \sqrt{\frac{(1-q_0)^2}{q_0^2} - 1}\\ &= \frac{(1-2q_0)^{\frac{1}{2}}}{q_0}\,.\end{aligned} \tag{2.2.43}$$

When the big-bang conditions $t \to 0,\ S \to 0 \implies \psi \to 0$ are imposed on (2.2.41), we determine the age t_- of the Universe for $k = -1$, only in terms of the Hubble constant H_0 and the deceleration parameter q_0,

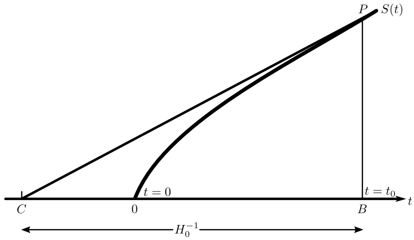
$$t_- = \frac{q_0}{(1-2q_0)^{\frac{3}{2}} H_0}\left(\frac{(1-2q_0)^{\frac{1}{2}}}{q_0} - \operatorname{arccosh}\left(\frac{1-q_0}{q_0}\right)\right), \tag{2.2.44}$$

or, remembering that $\operatorname{arccosh}(a) = \ln(a + \sqrt{a^2+1})$,

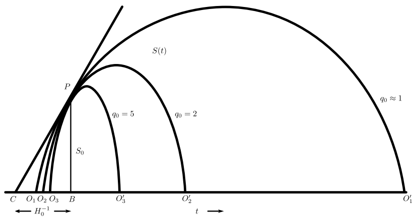
$$t_- = \frac{q_0}{(1-2q_0)^{\frac{3}{2}} H_0}\left(\frac{(1-2q_0)^{\frac{1}{2}}}{q_0} - \ln\left(\frac{1}{q_0}(1 - q_0 + \sqrt{1-2q_0})\right)\right). \tag{2.2.45}$$

The name “open” given to this Universe is specifically due to the divergent hyperbolic functions; th hyperbolic Universe thus originates from the big-bang singularity and expands indefinitely, never collapsing back to the primordial singularity, as seen in the closing section.

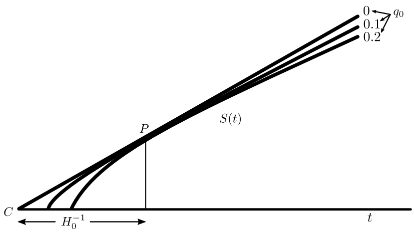
We finish by showing the general aspect the plots of the Universe age have for different values of q_0, according to relations (2.2.22), (2.2.29) and (2.2.39):



(a) $k = 0$



(b) $k = +1$



(c) $k = -1$

Figure 16: The Universe ages for different values of q_0 and for the different sectors $k = 0, -1$ and $+1$, normalized

2.3 Discussion

In this chapter we showed that the standard cosmological model can be derived in an *ab initio* manner from symmetry groups for spaces admiting maximal symmetry, thus being both homogeneous and isotropic about every point, instead of resorting to the usual heuristic construction found in most literatures. This noteworthy approach, established on a solid mathematical ground, reveals how powerful the symmetry group formalism can be when building up a theory and also shows how Friedmann and Lemaître were in tune with their physical intuitions.

In addition, as briefly seen on (2.1.49), the Einstein-de Sitter model is another important result that can also be derived on the same footing, having merits of its own.

One cannot forgo mentioning that although adopted as the standard model of cosmology, the FLRW is not self-contained and it does not fully describe the Universe history.

First of all, the observations of the *Cosmic Microwave Background* (CMB) done so far shows that the Universe is not perfectly isotropic, as it can be seen in Fig. 1, so it cannot possible be fully described by a maximal symmetric model that assumes isotropy about every point. However, for all intents and purposes, it serves as an effective model, blurring all those local anisotropies out to give a somewhat isotropic aspect.

Secondly, the FLRW solutions require the validity of the perfect fluid approximation, which in turn stem from the Weyl postulate. As discussed in the previous section, we can only guarantee the matter to satisfy such approximation for scale ratios of $\frac{S_0}{S} \lesssim 10^3$, when matter and radiation decouples. Thus, we cannot possibly have the whole picture with this model; this restrict us to naïve speculations by extrapolating the age back to early epochs.

These two problems give rise to one of the main shortcommings of FLRW models: without being able to properly access the Universe state at early epochs, how does it evolved with an exceptional degree of homogeneity and isotropy? Moreover, it is possible to show by simple extrapolation that different parts of the very early Universe, in the radiation dominated era, were causally disconnected, hence it is quite improbable for it to evolve to what it is today. This issue is the so-called *Horizon problem*. To address that and other drawbacks, Guth proposed the *Inflationary scenario*[34], characterized by a very hot rapid expansion at a very early stage.

Thence, the problem is posed: is there a way to bypass these difficulties or are there other models which could better describe the Universe? To the former we presume that there are no trivial solutions; fortunately, there still are models which does not require spatial isotropy about every point and might be all compatible with the CMB on their own, taking into account the possible local anisotropies. Renouncing spatial isotropy, while still preserving homogeneity, is precisely the focus of the next chapter.

CHAPTER 3

Biancbi Classification

As great as the formalism of Maximally Symmetric spaces built in the previous chapter reveals to be, giving fantastic results such as the ab initio construction of the standard model of Cosmology, the FLRW model, it still is very idealistically symmetric, and so, quite a restritive theory, valid only to some remarkably special cases. There exists situations where renouncing isotropy at every point has physical utility. One of such is the mathematical modelling of the small, yet present, anisotropies found in the Universe as, for instance, the *Cosmic Microwave Background* (CMB).

By using the framework of N-Tuples elaborated in Section 1.2, in the synchronous frame of reference introduced in Section 1.4.2, while imposing isometry conditions in a similar fashion as in the previous chapter, we shall notice that a more fundamental structure emerges: Lie groups and their associated group of motions inherited from the corresponding isometries. Such groups will not be generic at all, and it will be shown that there exists only a few non-redundant different groups associated with the imposed homogeneity. Those distinct families compose what we call the *Bianchi classification* of homogeneous spaces.

Finally, we will expand a bit upon each type, listing their principal algebraic qualities, and then constrast some of them to well-known cosmologies, such as the flat Euclidian space and the FLRW models.

3.1 Homogeneous spaces

As briefly said in the introduction, we shall construct the theory exclusively in the syncronous frame of reference, that can be summed up from Section 1.4.2 as a frame in which the clocks are syncronized at all points, represented by the proper time τ, effectively decoupling the time sector of the metric from the spatial counterpart. Thus,

$$\begin{aligned} ds^2 &= d\tau^2 - \zeta_{ij} dx^i dx^j \\ &= d\tau^2 - d\ell^2 \,, \end{aligned} \tag{3.1.1}$$

where $\zeta_{ij} = \zeta_{ij}(x;\tau) := -g_{ij}(x;\tau)$ is the positively defined spatial metric, possibly parametrized by τ, and $d\ell$ is the purely spatial line element.

Then, we evoke the isometry condition (1.3.2) that preserves the functional form of the metric when evaluated in a neighboring point,

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x') \,, \tag{3.1.2}$$

remembering that it formally represents an automorphism from $\mathcal{M}$ back into itself.

The 3−dimensional spatial sector can be brought into the local inertial frame of reference given by the 3-Tuples or *triads* $e^{(a)}{}_i(x)$ discussed in Section 1.2. The choice of those triads is not unique; we can always choose another set by means of a “coordinate transformation”, so that the basis elements $\boldsymbol{e}_{(a)}$ in (1.2.1) can transform into one-another by a relation of the type

$$\boldsymbol{e}_{(a)} = A^b{}_a \boldsymbol{e}_{(b)} \,, \tag{3.1.3}$$

with constant $A^b{}_a$.

So, by virtue of (1.2.6), the isometry condition (3.1.2) is expressed as

$$\zeta'_{ij}(x') = \zeta_{ij}(x')\,,$$
$$\eta_{ab}e'^{(a)}{}_i(x')e'^{(b)}{}_j(x') = \eta_{ab}e^{(a)}{}_i(x')e^{(b)}{}_j(x')$$

$$\therefore\ e'^{(a)}{}_i(x') = e^{(a)}{}_i(x')\,. \tag{3.1.4}$$

By the invariance of the line element (3.1.1), we have

$$d\ell^2 = \zeta'_{ij}(x')dx'^i dx'^j = \zeta_{ij}(x)dx^i dx^j\,,$$
$$\left(\eta_{ab}e'^{(a)}{}_i(x')e'^{(b)}{}_j(x')\right)dx'^i dx'^j = \left(e^{(a)}{}_i(x)e^{(b)}{}_j(x)\right)dx^i dx^j\,,$$

which, by (3.1.4), results in

$$e^{(a)}{}_i(x')dx'^i = e^{(a)}{}_i(x)dx^i\,. \tag{3.1.5}$$

Contracting it with $e_{(a)}{}^j(x')$, gives

$$(e^{(a)}{}_i(x')\,dx'^i)e_{(a)}{}^j(x') = (e^{(a)}{}_i(x)\,dx^i)e_{(a)}{}^j(x')\,,$$
$$\delta_i^j dx'^i = e^{(a)}{}_i(x)e_{(a)}{}^j(x')\,dx^i$$
$$\therefore\ dx'^j = e^{(a)}{}_i(x)e_{(a)}{}^j(x')\,dx^i\,, \tag{3.1.6}$$

which is valid if

$$\frac{\partial x'^j}{\partial x^i} = e^{(a)}{}_i(x)\,e_{(a)}{}^j(x')\,, \tag{3.1.7}$$

or, equivalently,

$$\frac{\partial x^i}{\partial x'^j} = e^{(a)}{}_j(x')\, e_{(a)}{}^i(x)\,, \tag{3.1.8}$$

considering its reciprocal.

The set of equations (3.1.7) above only has a solution if it satisfies integrability conditions

$$\frac{\partial^2 x'^j}{\partial x^i \partial x^k} = \frac{\partial^2 x'^j}{\partial x^k \partial x^i}\,, \tag{3.1.9}$$

namely

$$\begin{aligned}
&\left(\frac{\partial e^{(a)}{}_k}{\partial x^i}(x) - \frac{\partial e^{(a)}{}_i}{\partial x^k}(x)\right) e_{(a)}{}^j(x') \\
&\quad = \left(\frac{\partial e_{(b)}{}^j}{\partial x'^l}(x')e_{(a)}{}^l(x') - \frac{\partial e_{(a)}{}^j}{\partial x'^l}(x')e_{(b)}{}^l(x')\right) e^{(a)}{}_k(x)e^{(b)}{}_i(x)\,,
\end{aligned}$$

and, upon contracting both sides by $e_{(c)}{}^k(x)e_{(d)}{}^i(x)e^{(f)}{}_j(x')$, using the N-Tuple properties (1.2.13),

$$\begin{aligned}
&\overbrace{\left(\frac{\partial e^{(f)}{}_k}{\partial x^i}(x) - \frac{\partial e^{(f)}{}_i}{\partial x^k}(x)\right) e_{(c)}{}^k(x)e_{(d)}{}^i(x)}^{\text{Function only of } x} \\
&\quad = \underbrace{\left(\frac{\partial e_{(d)}{}^j}{\partial x'^l}(x')e_{(c)}{}^l(x') - \frac{\partial e_{(c)}{}^j}{\partial x'^l}(x')e_{(d)}{}^l(x')\right) e^{(f)}{}_j(x')}_{\text{Function only of } x'}\,.
\end{aligned}$$

Since the lefthand side depends exclusively on x and righthand side on x', both must reduce to constants, independent of the frame of reference,

$$\left(\frac{\partial e^{(f)}{}_{k}}{\partial x^{i}}(x) - \frac{\partial e^{(f)}{}_{i}}{\partial x^{k}}(x)\right) e_{(c)}{}^{k}(x) e_{(d)}{}^{i}(x) \equiv C^{f}{}_{cd}\,, \tag{3.1.10}$$

which are precisely identical to (1.2.18). Promptly, we see that the constants are symmetrical in their covariant indices,

$$C^{f}{}_{cd} = C^{f}{}_{dc}\,. \tag{3.1.11}$$

We conclude that the constants $C^{f}{}_{cd}$ do represent the structure constants of the algebra associated with a *Lie group*. This will be evidenced next. Multiplying (3.1.10) by* $e_{(f)}{}^{m}$

$$\begin{aligned}\left(e^{(f)}{}_{k,i} - e^{(f)}{}_{i,k}\right) e_{(c)}{}^{k} e_{(d)}{}^{i} e_{(f)}{}^{m} &= C^{f}{}_{cd} e_{(f)}{}^{m} \\ \rightarrow e_{(d)}{}^{m,k} e_{(c)}{}^{k} - e_{(c)}{}^{m,i} e_{(d)}{}^{i} &= C^{f}{}_{cd} e_{(f)}{}^{m}\,,\end{aligned} \tag{3.1.12}$$

we can define the directional derivative operator X_a along a triad direction (a)

$$X_a := e_{(a)}{}^{i} \partial_i\,. \tag{3.1.13}$$

The above-defined operator X_a also represents a Lie derivative of a scalar field. This is expected since we are working in the N-Tuple frame, so all the fields expressed in it are reduced to scalars, as discussed in Section 1.2.

When taking a commutator,

*We dropped the point dependency for clarity.

$$[X_a, X_b] = e_{(a)}{}^i \partial_i \left(e_{(b)}{}^j \partial_j \right) - [a \leftrightarrow b]$$
$$\therefore [X_a, X_b] = C^d{}_{ab} X_d \,. \tag{3.1.14}$$

we see that it indeed satisfies a Lie algebra with structure constants $C^d{}_{ab}$.

As a consequence, the *Jacobi identity* is automatically satisfied:

$$[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0 \,, \tag{3.1.15a}$$
$$[C^d{}_{ab} X_d, X_c] + [C^d{}_{bc} X_d, X_a] + [C^d{}_{ca} X_d, X_b] = 0 \,,$$
$$\left(C^d{}_{ab} C^f{}_{dc} + C^d{}_{bc} C^f{}_{da} + C^d{}_{ca} C^f{}_{db} \right) X_f = 0 \,. \tag{3.1.15b}$$

Instead of working with the $3 \times 3 \times 3$ constants $C^d{}_{ab}$, we can simplify our life by working with its dual counterpart, by introducing

$$C^c{}_{ab} := \varepsilon_{abd} C^{dc} \,, \tag{3.1.16}$$

where ε_{abd} is the totally anti-symmetric pseudo-tensor with $\varepsilon_{123} \equiv 1$. Using that, the commutators and the Jacobi identity reduce to

$$\varepsilon^{abc} X_b X_c = C^{ad} X_d \tag{3.1.17}$$
$$\varepsilon_{bcd} C^{cd} C^{ba} = 0 \,. \tag{3.1.18}$$

The biggest advantage to work with the dual quantities is that it becomes much easier to determine the constants, by decomposing the two-indexed dual constants C^{ab} into a symmetrical part, n^{ab}, and an anti-symmetrical one, $\varepsilon^{abc} a_c$, along with the arbitrary vector a_c to be determined

$$C^{ab} = n^{ab} + \varepsilon^{abc} a_c \,. \tag{3.1.19}$$

These constants must satisfy (3.1.15b) in order to represent a Lie algebra, which gives

$$n^{ab} a_b = 0 \,. \tag{3.1.20}$$

This is the *characteristic equation* that enables us to verify n^{ab} and a_c, thus fully determining C^{ab} and, by (3.1.16), $C^a{}_{bc}$. Since the triads are not unique, by means of (3.1.3), we can bring n^{ab} to a diagonal form, so a_c corresponds to a nil-eigenvalue. Without loss of generality, we set

$$a_b \equiv (a, 0, 0) \quad ; \quad n_i \equiv n^{ii} \,, \tag{3.1.21}$$

given that either $a = 0$ or $n_1 = 0$, due to (3.1.20). Those two cases define two classes of solutions: the former is called class A, while the latter, B.

These constraints restricts the number of solutions, which can be classified into all the possible combinations of a and n_i. Plugging these back into (3.1.14), one gets

$$[X_1, X_2] = -aX_2 + n_3 X_3 \,, \tag{3.1.22a}$$

$$[X_2, X_3] = n_1 X_1 \,, \tag{3.1.22b}$$

$$[X_3, X_1] = n_2 X_2 + aX_3 \,. \tag{3.1.22c}$$

Furthemore, all the n_i can be normalized to ± 1, by rescaling the associated triads by means of a basis transformation (3.1.3). All in all, there are nine different classes of solutions, which shall be called *Types* is allusion to the original papers of Bianchi.[4] All those types can be arranged as

Class	Type	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$
A	I	0	0	0	0
	II	0	1	0	0
	VII_0	0	1	1	0
	VI_0	0	1	-1	0
	IX	0	1	1	1
	VIII	0	1	1	-1
B	V	1	0	0	0
	IV	1	0	0	1
	VII_a	a	0	1	1
	III ($a = 1$) VI_a ($a \neq 1$)	a	0	1	-1

Table 1: Bianchi Classification

and constitute the so-called *Bianchi Classification* of homogeneous spaces.

The isometry condition (3.1.2) defines the group of automorphisms (or group of motions) with respect to the actions defined by the scalar Lie derivative X_a (3.1.13), which represents nothing less than the tangent vectors to a point x, which, in turn, are precisely the generators of translations. The invariance inherited from this kind of isometry corresponds to the homogeneity condition (2.1.1), which defines the whole spatial sector as a 3-surface of transitivity of the associated Lie groups.

Furthermore, it is interesting to notice that when compared to the Killing vector description, this formalism, in the local inertial frame, has some interesting links. Consider for a moment the infinitesimal transformations (1.3.3) and (3.1.7). Inserting the former into the latter, results in

$$\xi^j{}_{,i} = \xi^k \left(e^{(a)}{}_i e_{(a)}{}^j{}_{,k} \right) ,$$

which, by (1.1.33), can be expressed as

$$\xi^j{}_{,i} + \Gamma^j_{ik}\xi^k \equiv \xi^j{}_{;i} = 0$$

so the Killing vectors are parallel-transported along geodesics, as expected due to the very definition of local isometries.

It remains to be shown how the Einstein equations are written in the locally inertial frame of reference. For this purpose, we recall the projection rule (1.2.13e) and apply it to the Ricci tensor,

$$R_{(a)(b)} = R_{\mu\nu}e_{(a)}{}^{\mu}e_{(b)}{}^{\nu} ,$$

and remember also the "synchronous metric time-derivatives" (1.4.7)

$$\chi_{ab} := \dot{\zeta}_{ab} , \tag{3.1.23a}$$

$$\chi_a{}^b = \zeta^{bc}\dot{\zeta}_{ac} . \tag{3.1.23b}$$

Picking (1.2.16a) up and returning it into (1.2.24), we can express the spatial Ricci tensor $P_{(a)(b)}$ only as a function of the structure constants,

$$\begin{aligned} P_{(a)(b)} = -\frac{1}{2}\Big(& C^{cd}{}_{b}C_{cda} + C^{cd}{}_{b}C_{dca} \\ & - \frac{1}{2}C_{b}{}^{cd}C_{acd} + C^{c}{}_{cd}C_{ab}{}^{d} + C^{c}{}_{cd}C_{ba}{}^{d}\Big) ; \end{aligned} \tag{3.1.24}$$

by employing (3.1.16), its symmetry (3.1.11) and the transformation law of the pseudo-tensor

$$\zeta_{ad}\zeta_{be}\zeta_{cf}\varepsilon^{def} = \zeta\varepsilon_{abc} , \tag{3.1.25}$$

we can rewrite (3.1.24) as function of the duals C^{dc}. Hence, after tedious algebra,

$$P_{(a)}{}^{(b)} = \frac{1}{2\zeta}\Big[2C_{ac}C^{bc} + C^{bc}C_{ca} + C_{ac}C^{cb} \\ - C^{c}{}_{c}\Big(C^{b}{}_{a} + C_{a}{}^{b}\Big) + \delta^{b}{}_{a}\Big((C^{c}{}_{c})^2 - 2C^{cd}C_{cd}\Big)\Big]. \tag{3.1.26}$$

The other components of the synchronous Ricci tensor (1.4.10) in the triad frame are given by

$$R_0{}^0 = \frac{1}{2}\chi_{(a)}{}^{(a)} + \frac{1}{2}\chi^{(a)}{}_{(b)}\chi^{(b)}{}_{(a)}, \tag{3.1.27a}$$

$$R_{(a)}{}^0 = \frac{1}{2}\chi^{(c)}{}_{(d)}\Big(C^{d}{}_{ca} - \delta^d_a C^{b}{}_{bc}\Big), \tag{3.1.27b}$$

$$R_{(a)}{}^{(b)} = P_{(a)}{}^{(b)} + \frac{1}{2\sqrt{\zeta}}\partial_t(\sqrt{\zeta}\chi_{(a)}{}^{(b)}), \tag{3.1.27c}$$

where we made use of (1.2.21) and

$$P_{(a)}{}^{(b)} = \frac{1}{2\zeta}\Big[2C^{bd}C_{ad} + C^{db}C_{ad} + C^{bd}C_{da} \\ - C^{d}{}_{d}(C^{b}{}_{a} + C_{a}{}^{b}) + \delta^b_a\big((C^{d}{}_{d})^2 - 2C^{df}C_{df}\big)\Big]. \tag{3.1.28}$$

Conformal symmetry

The ability to rescale the constants n_i by rescaling the triads themselves hint to an underlying conformal symmetry* the spaces may possess. We show in this section that the homogeneity condition described by the isometry condition (3.1.2) also entails a *conformal symmetry*. We say a space has a conformal symmetry if it is equipped with an equivalence class of metric tensors differing only by a *conformal factor* $\Omega^2(x)$,

$$\zeta_{ij}(x) \to \Omega^2(x)\, g_{ij} \;\; ; \quad \Omega^2(x) > 0 \,. \tag{3.1.29}$$

In order to show that, we shall employ the same isometry condition mentioned above but we will now impose that it carries the conformal symmetry. We will see that all the results will be much similar to those obtained previously, but with the addition of such conformal factor.

By (3.1.2) in the local inertial frame of reference, we get

$$\begin{gathered} \zeta'_{ij}(x') = \zeta_{ij}(x')\,, \\ \Omega'^2(x')\eta_{ab}e'^{(a)}{}_i(x')e'^{(b)}{}_j(x') = \Omega^2(x')\eta_{ab}e^{(a)}{}_i(x')e^{(b)}{}_j(x') \end{gathered}$$

$$\Longrightarrow \;\; \Omega'^2(x')e'^{(a)}{}_i(x') = \Omega^2(x')e^{(a)}{}_i(x')\,. \tag{3.1.30}$$

Using again the invariance of the line element (3.1.1),

$$\begin{gathered} d\ell^2 = \zeta'_{ij}(x')dx'^i dx'^j = \zeta_{ij}(x)dx^i dx^j\,, \\ \left(\Omega'^2(x')\eta_{ab}e'^{(a)}{}_i(x')e'^{(b)}{}_j(x')\right)dx'^i dx'^j = \left(\Omega^2(x)e^{(a)}{}_i(x)e^{(b)}{}_j(x)\right)dx^i dx^j\,, \end{gathered}$$

*For more information regargin the Conformal symmetry, we refer the reader to Appendix C.

which gives

$$\Omega^2(x')e^{(a)}{}_i(x')dx'^i = \Omega^2(x)e^{(a)}{}_i(x)dx^i \tag{3.1.31}$$

$$\therefore\ dx'^j = \frac{\Omega^2(x)}{\Omega^2(x')}e^{(a)}{}_i(x)e_{(a)}{}^j(x')\,dx^i\ . \tag{3.1.32}$$

This change of coordinates is only valid if

$$\frac{\partial x'^j}{\partial x^i} = \frac{\Omega^2(x)}{\Omega^2(x')}e^{(a)}{}_i(x)\,e_{(a)}{}^j(x')\ . \tag{3.1.33}$$

Imposing again the integrability conditions (3.1.9), we find*

$$\therefore\ \frac{1}{\Omega^2}\left[\frac{\partial e^{(a)}{}_e}{\partial x^f} - \frac{\partial e^{(a)}{}_f}{\partial x^e} + \frac{2}{\Omega}\left(\frac{\partial \Omega}{\partial x^f}e^{(a)}{}_e - \frac{\partial \Omega}{\partial x^e}e^{(a)}{}_f\right)\right]e_{(b)}{}^e e_{(c)}{}^f \equiv C^a{}_{bc}\ , \tag{3.1.34}$$

which are the structure constants of some Lie algebra as well, perfectly consistent with (3.1.10) for $\Omega = 1$. To see this, define instead the operator

$$Y_a := \frac{1}{\Omega^2}e_{(a)}{}^m\partial_m\ , \tag{3.1.35}$$

so its commutator results in

*Refer to Appendix E for the explicit calculations.

$$[Y_a, Y_b] = \frac{1}{\Omega^2}\left\{\frac{1}{\Omega^2}\left[e_{(a)}{}^{d} e_{(b)}{}^{c}{}_{,d} - e_{(b)}{}^{d} e_{(a)}{}^{c}{}_{,d} \right.\right. \\ \left.\left. + 2\frac{\Omega_{,d}}{\Omega}\left(e_{(a)}{}^{c} e_{(b)}{}^{d} - e_{(a)}{}^{d} e_{(b)}{}^{c}\right)\right]\right\}\partial_c \, . \tag{3.1.36}$$

Note that, taking (3.1.34) and multiplying it by, $e_{(a)}{}^{m}$

$$C^{a}{}_{bc} e_{(a)}{}^{m} = \frac{1}{\Omega^2}\left[e_{(b)}{}^{n} e_{(c)}{}^{m}{}_{,n} - e_{(c)}{}^{n} e_{(b)}{}^{m}{}_{,n} \right. \\ \left. + 2\frac{\Omega_{,n}}{\Omega}\left(e_{(b)}{}^{m} e_{(c)}{}^{n} - e_{(b)}{}^{n} e_{(c)}{}^{m}\right)\right] , \tag{3.1.37}$$

that is precisely the last term of (3.1.36) *sans* $\frac{1}{\Omega^2}$. So, using (3.1.35), we are left with

$$[Y_a, Y_b] = \frac{1}{\Omega^2} C^{c}{}_{ab} e_{(c)}{}^{d} \partial_d = C^{c}{}_{ab} Y_c \, , \tag{3.1.38}$$

which does indeed characterizes a Lie algebra.

This means that all the results succeeding (3.1.14) are also valid for the conformal symmetry and thus the Bianchi classification described in Table 1 entails this symmetry as well, which might depend on the point instead of just the constants as discussed in (3.1.3).

Now notice that the covariant derivatives of the Killing vectors no longer vanish for the extended conformal symmetry

$$\xi^{j}{}_{;i} = -2\omega_{,k}\xi^{k}\delta^{j}_{i} \, , \tag{3.1.39}$$

where $\omega = \omega(x)$ comes from writing the conformal factor as an exponential $\Omega(x) = e^{\omega(x)}$. This means that the Killing vectors *are no longer being transported along geodesics*, those being crooked by the non-zero covariant derivative.

Moreover, this gradient term $\omega_{,k}$ is precisely the infinitesimal conformal acceleration parameter b_k (C.37) appearing in the conformal Killing vectors (C.39)

$$\xi^j{}_{;i} = -2b_k \xi^k \delta^j_i \,.$$

Also, in light of the equivalence principle, we can interpret the deviations of the Killing vectors as coming from a gravitational field. This might be a natural way to incorporate the equivalence principle on a fundamental level.

Yet, we can also take notice that the gradient term can be rewritten as the d'Alembertian of a Killing vector (C.33)

$$\partial^2 \xi_k = -2\omega_{,k} \,,$$

which bears a striking resemblance to the Gauge transformation of the electromagnetism. We see this as the most promising way to propose a Gauge principle to the Gravitation. Now, if we return with it to (3.1.39), we obtain

$$\xi^j{}_{;i} = (\partial^2 \xi_k) \xi^k \delta^j_i \,, \tag{3.1.40}$$

purely determined by the Killing vectors themselves.

3.2 Bianchi classification

To further study the Bianchi classification of homogeneous spaces, it will be important to first analyze some of the mathematical and physical properties that each type has, a way to ensure we are indeed safe to proceed in any eventual calculation. Having that in mind, we shall make a list of the main characteristics from the Bianchi algebras $\mathfrak{b}_i$, of the associate groups, and occasionally to which well known algebra they might be isomorphic.

Though we have to proceed with caution, this kind of analysis can easily go very deep into an abyss of formal mathematical definitions and nuances that come with it, which fall off of the scope of this work, so we shall just list those introduced in Appendix A, only detailing some mathematical concepts when necessary, for completeness.

The main reference of this section is the great thesis of Allegra Fowler-Wright.[17] Her classification of finite dimensional Lie algebras admitting three real parameters greatly differs from that of Bianchi's, where she employs a much more modern approach of classification, shining new light and thus providing additional information.

In particular, we shall be interested in the following properties: dimension and nature of the derived algebra, solvability, nilpotency, identification of both the radical and the largest solvable ideal and its center. We also will list the derived metric tensors, the corresponding triads as shown by Taub[58]* and which Thurston geometry each type is associated with, as well as some other physical qualities. The last of those correspond to an alternative way to study homogeneous spaces via topology, which has a direct map to the Bianchi algebras.

Briefly, the Thurston theorem[54] states that every 3-dimensional topology which possesses a *maximal geometry* G, that is, all the possible isometries of the covering space $\tilde{\mathcal{M}}$, denoted by $G := ISO(\tilde{\mathcal{M}})$, and if G has a subgroup Γ that acts on $\mathcal{M}$ as a covering group, i.e. the quotient group $\mathcal{M}/\Gamma$ is compact, then the topology is equivalent to one of the eight minimal topologies listed on Table 2.

*In the article, Taub constructed his theory using the Killing vectors instead of the ortogonal frame triads, which is an equivalent construction. As a consequence, our triads are precisely his Killing vectors and vice-versa.

*Universal covering space of the unit tangent space of $\mathbb{H}^2$.

	$\mathcal{M}$	G	Obs
i.	$\mathbb{E}^3$	$\mathbb{E}^3$	Euclidian geometry
ii.	$\mathbb{S}^3$	SO(4)	Spherical geometry
iii.	Nil	$\mathbb{R} \times \mathbb{E}^2$	Heisenberg group
iv.	Sol	$ISO(Sol)$	Solvable group
v.	$\mathbb{H}^3$	$PSL(2, \mathbb{C}) \times \mathbb{Z}_2$	Hyperbolic geometry
vi.	$\widetilde{SL(2, \mathbb{R})}$	$\mathbb{H}^2 \times \mathbb{R}$	Special linear group in 2D*
vii.	$\mathbb{H}^2 \times \mathbb{E}$	$ISO(\mathbb{H}^2) \times ISO(\mathbb{E})$	–
viii.	$\mathbb{S}^2 \times \mathbb{E}$	$ISO(\mathbb{S}^2) \times ISO(\mathbb{E})$	–

Table 2: Thurston topologies.

Although very interesting, we will treat them just as an additional information in our classification without getting much deeper than this. We recommend the interested reader the literature[19,31,32,54] for further reading.

The classification is done based on the dimension of the derived algebra $\mathfrak{b}'$, where the Bianchi types are divided as follows:

- dim $\mathfrak{b}' = 0$: Type I;
- dim $\mathfrak{b}' = 1$: Types II and III;
- dim $\mathfrak{b}' = 2$: Types IV–VII;
- dim $\mathfrak{b}' = 3$: Types VIII and IX.

We began with Type I for dim $\mathfrak{b}' = 0$.

3.2.1 Bianchi I – $\mathfrak{g} = \mathfrak{b}_I$

$$\begin{aligned} [X_1, X_2] &= 0\,, \\ [X_2, X_3] &= 0\,, \\ [X_3, X_1] &= 0\,. \end{aligned} \tag{3.2.1}$$

- Properties:
  - Unique up to isomorphisms.
  - Abelian ,

    $$X_i X_j = X_j X_i \;\; ; \quad \forall X_i \in \mathfrak{g}\,.$$

  - dim $\mathfrak{g}' = 0$.
  - Solvable and nilpotent,

    $$R(\mathfrak{g}) = Z(\mathfrak{g}) = \mathfrak{g}\,.$$

  - Thurston type: $\mathbb{E}^3$.
- Remarks:
  - Any Abelian commutative algebra of three parameters fit in here.
- Physical qualities:
  - Metric

    $$ds^2 = d\tau^2 - A^2(\tau)dx^2 - B^2(\tau)dy^2 - C^2(\tau)dz^2\,.$$

  - Flat, Euclidian metric up to scale factors.
  - Not necessarely isotropic.
  - $A = B$ or $B = C$ or $A = C \quad \Longrightarrow \quad$ axial symmetry,

    $$ds^2 = d\tau^2 - A^2(\tau)(dx^2 + dy^2) - B^2(\tau)dz^2\,.$$

⁕ $A = B = C \qquad \Longrightarrow \qquad$ isotropic,

$$ds^2 = d\tau^2 - A^2(\tau)(dx^2 + dy^2 + dz^2)\,.$$

⁕ Corresponds to FLRW flat universe ($k = 0$):

$$[e_{(a)}{}^{i}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [e^{(a)}{}_{i}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.2.2 Bianchi II – $\mathfrak{g} = \mathfrak{b}_{II}$

$$\begin{aligned} [X_1, X_2] &= 0\,, \\ [X_2, X_3] &= X_1\,, \\ [X_3, X_1] &= 0\,. \end{aligned} \tag{3.2.2}$$

- Properties:
  - Derived algebra $\mathfrak{g}' \subseteq Z(\mathfrak{g})$.
  - $\dim\ \mathfrak{g}' = 1$.
  - $\mathfrak{g}' = X_3$.
  - Non-abelian.
  - Solvable ,

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = [X_3, X_3] = 0\,.$$

  - $R(\mathfrak{g}) = \mathfrak{g}$.
  - Nilpotent,

$$\mathfrak{g}^3 = [\mathfrak{g}', \mathfrak{g}] = [Z(\mathfrak{g}), \mathfrak{g}] = 0\,.$$

  - $\mathfrak{g} \approx \mathfrak{h}$ – Isomorphic to the Heisenberg algebra (represented by upper-triangular matrices).
  - Thurston type: Nil.
- Abstract example:

$$X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Physical qualities:

⁎ Metric

$$
\begin{aligned}
ds^2 = d\tau^2 &- R^2(\tau)dx^2 \\
&- S^2(\tau)dy^2 - 2S^2(\tau)xdydz \\
&- \Big(S^2(\tau)x^2 - R^2(\tau)\Big)dz^2\,,
\end{aligned}
$$

$$
[e_{(a)}{}^{i}] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^3 \\ 0 & 1 & 0 \end{pmatrix}, \qquad [e^{(a)}{}_{i}] = \begin{pmatrix} -x^3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

3.2.3 Bianchi III (a = 1) – $\mathfrak{g} = \mathfrak{b}_{III}$

$$\begin{aligned} [X_1, X_2] &= -X_2 - X_3\,, \\ [X_3, X_1] &= X_2 + X_3\,, \\ [X_2, X_3] &= 0\,. \end{aligned} \tag{3.2.3}$$

Set

$$\left.\begin{aligned} Y_1 &= X_2 + X_3 \\ Y_2 &= \tfrac{1}{2} X_1 \\ Y_3 &= X_2 - X_3 \end{aligned}\right\} \Longrightarrow \qquad \begin{aligned} [Y_1, Y_2] &= Y_1\,, \\ [Y_3, Y_1] &= 0\,, \\ [Y_2, Y_3] &= 0\,. \end{aligned} \tag{3.2.4}$$

✦ Properties:

- $\mathfrak{g}' \not\subseteq Z(\mathfrak{g})$.
- dim $\mathfrak{g}' = 1$.
- $\mathfrak{g}' = Y_1$.
- $\mathfrak{g} = \mathfrak{s}_2 + Z_{\mathfrak{g}}(\mathfrak{s}_2)$, where $\mathfrak{s}_2$ is the algebra of a 2-parameter group of motions and $Z_{\mathfrak{g}}(\mathfrak{s}_2)$ is the center of it with relation to $\mathfrak{g}$, defined as

$$Z_{\mathfrak{g}}(\mathfrak{s}_2) := \{X \in \mathfrak{g} : [X, Y] = 0 \quad \forall Y \in \mathfrak{s}_2\}\,.$$

- $Z(\mathfrak{g}) = Y_3$ is one dimensional.
- Non-abelian.
- Solvable:

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = [Y_1, Y_1] = 0\,.$$

- $R(\mathfrak{g}) = \mathfrak{g}$.
- Non-nilpotent:

$$\mathfrak{g}^n = Y_1 \neq 0\ , \quad \forall n \in \mathbb{N}\,.$$

- Thurston type: $\mathbb{H}^2 \times \mathbb{E}$.

- Abstract example:

$$Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad Y_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad Y_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

- Physical qualities:

  - Metric

$$ds^2 = d\tau^2 - A^2(\tau)dx^2 \\ - B^2(\tau)e^{-2ax}dy^2 - C^2(\tau)dz^2 \ , \quad a = \text{const}$$

  - Related to Petrov* type D:

$$[e_{(a)}{}^i] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^2 \\ 0 & 1 & 0 \end{pmatrix} , \qquad [e^{(a)}{}_i] = \begin{pmatrix} -x^2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

*We are going to study the Petrov classification shortly on Chapter 4.

The next four Bianchi types are particular cases of the same algebraic structure. We shall expand a bit on it and, next, on how and where they differ among themselves.

Those are defined when the dimension of the derived Algebra is dim $\mathfrak{g}' = 2$, so it constitutes an algebra where, for $X_1, X_2 \in \mathfrak{g}'$,

$$[X_1, X_2] = 0 \,.$$

For the actual algebra $\mathfrak{g}$, we expand this basis further to accomodate $X_3 \notin \mathfrak{g}'$, such that

$$[X_1, X_3] = aX_1 + bX_2 \,,$$
$$[X_2, X_3] = cX_1 + dX_2 \,.$$

Theorem 3.1 from [17] guarantees that the Lie algebra is similar to a unique block-diagonal matrix in its canonical form, formed by the *companion matrices* which are comprised of monic polynomials $p_k = x^k + a_{k-1}x^{k-1} + \ldots + a_0$.

Since the derived algebra has only two dimentions, the only possible rational canonical forms are such that

$$\mathrm{ad}_{X_3} : \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad A_{2,c} = \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix} \quad , \quad A_{3,d} = \begin{pmatrix} 0 & d \\ 1 & 1 \end{pmatrix} \right\} ,$$

where $\mathrm{ad}_{X_3} X = [X, X_3]$, then the only algebras we can construct are

- $\mathrm{ad}_{X_3} : A_1 =: \mathfrak{g}_1$,

$$[X_1, X_2] = 0 \,, \qquad [X_1, X_3] = X_1 \,, \qquad [X_2, X_3] = X_2 \,; \tag{3.2.5}$$

✦ $\mathrm{ad}_{X_3} : A_{2,c} =: \mathfrak{g}_{2,c}$,

$$[X_1, X_2] = 0\,, \qquad [X_1, X_3] = cX_2\,, \qquad [X_2, X_3] = X_1\,; \tag{3.2.6}$$

✦ $\mathrm{ad}_{X_3} : A_{3,d} =: \mathfrak{g}_{3,d}$,

$$[X_1, X_2] = 0\,, \qquad [X_1, X_3] = dX_2\,, \qquad [X_2, X_3] = X_1 + X_2\,. \tag{3.2.7}$$

Those are unique, since, by isomorphisms, the characteristics of the matrices A_i are invariant.

3.2.4 Bianchi IV – $\mathfrak{g} = \mathfrak{b}_{IV}$

$$
\begin{aligned}
&[X_2, X_3] = 0\,, \\
&[X_2, X_1] = aX_2 - X_3\,, \\
&[X_3, X_1] = aX_3\,.
\end{aligned}
\tag{3.2.8}
$$

Set

$$
\left.\begin{aligned}
Y_1 &= \tfrac{1}{2}a^{-1}X_1 \\
Y_2 &= -aX_2 + X_3 \\
Y_3 &= \tfrac{1}{2}aX_2
\end{aligned}\right\} \implies
\begin{aligned}
&[Y_2, Y_3] = 0\,, \\
&[Y_2, Y_1] = Y_2 + Y_3\,, \\
&[Y_3, Y_1] = Y_2\,.
\end{aligned}
\tag{3.2.9}
$$

✦ Properties:

- dim $\mathfrak{g}' = 2$,
- The derived algebra $\mathfrak{g}'$ is abelian.
- The whole algebra $\mathfrak{g}$ is non-abelian.
- Solvable:

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = 0\,.$$

- $R(\mathfrak{g}) = \mathfrak{g}$.
- Non-nilpotent $\mathfrak{g}^n \neq 0\ ,\quad \forall n \in \mathbb{N}$.
- $Z(\mathfrak{g}) = 0$.
- Corresponds to $\mathfrak{g}_{3,1}$ $(d = 1)$ (3.2.7).
- Thurston type: none.

✦ Physical qualities:

- Metric

$$ds^2 = d\tau^2 - A^2(\tau)dx^2 - B^2(\tau)e^{2x}dy^2 - C^2(\tau)e^{2x}(xdy + dz)^2\,,$$

$$[e_{(a)}{}^{i}] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^2 + x^3 \\ 0 & 1 & x^3 \end{pmatrix} \qquad , [e^{(a)}{}_{i}] = \begin{pmatrix} -x^2 - x^3 & 1 & 0 \\ -x^3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

3.2.5 Bianchi V – $\mathfrak{g} = \mathfrak{b}_V$

$$
\begin{aligned}
[X_2, X_1] &= aX_2 \,, \\
[X_2, X_3] &= 0 \,, \\
[X_3, X_1] &= aX_3 \,.
\end{aligned}
\tag{3.2.10}
$$

Set

$$
\left.\begin{aligned}
Y_1 &= aX_1 \\
Y_2 &= X_2 \\
Y_3 &= X_3
\end{aligned}\right\} \implies
\begin{aligned}
[Y_2, Y_3] &= Y_2 \,, \\
[Y_2, Y_3] &= 0 \,, \\
[Y_3, Y_1] &= Y_3 \,.
\end{aligned}
\tag{3.2.11}
$$

- Properties:
  - dim $\mathfrak{g}' = 2$.
  - The derived algebra $\mathfrak{g}'$ is abelian.
  - The whole algebra $\mathfrak{g}$ is non-abelian.
  - Solvable,

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = 0 \,.$$

  - $R(\mathfrak{g}) = \mathfrak{g}$.
  - Non-nilpotent $\mathfrak{g}^n \neq 0 \;, \quad \forall n \in \mathbb{N}$.
  - $Z(\mathfrak{g}) = 0$.
  - Corresponds to $\mathfrak{g}_1$ (3.2.5).
  - Thurston type: $\mathbb{H}^3$.
- Physical qualities:
  - Metric

$$
\begin{aligned}
ds^2 = d\tau^2 - A^2(\tau)dx^2 - B^2(\tau)e^{-2ax}dy^2 \\
- C^2(\tau)e^{-2ax}dz^2 \;, \quad a = \text{const} \,.
\end{aligned}
$$

  - Corresponds to FLRW open universe ($k = -1$).

- Has exact solutions.
- Admits tilted solutions[*]

$$[e_{(a)}{}^{i}] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^2 \\ 0 & 1 & x^3 \end{pmatrix}, \qquad [e^{(a)}{}_{i}] = \begin{pmatrix} -x^2 & 1 & 0 \\ -x^3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

[*] This is a flexibility of Weyl's postulate. We say a solution is tilted if the expansion direction is not purely time-like, that is, it might have a residual spatial component, thus being only approximately described by a synchronous frame of reference.

3.2.6 Bianchi $VI_0 - \mathfrak{g} = \mathfrak{b}_{VI_0}$

$$
\begin{aligned}
[X_1, X_2] &= 0\,, \\
[X_2, X_3] &= X_1\,, \\
[X_1, X_3] &= X_2\,.
\end{aligned} \tag{3.2.12}
$$

- Properties:
  - dim $\mathfrak{g}' = 2$.
  - The derived algebra $\mathfrak{g}'$ is abelian.
  - The whole algebra $\mathfrak{g}$ is non-abelian.
  - Solvable,

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = 0\,,$$

  - $R(\mathfrak{g}) = \mathfrak{g}$.
  - Non-nilpotent $\mathfrak{g}^n \neq 0\,, \quad \forall n \in \mathbb{N}$.
  - $Z(\mathfrak{g}) = 0$.
  - Corresponds to $\mathfrak{g}_{2,1}(\mathrm{c} = 1)$ (3.2.6).
  - Thurston type: Sol.
- Physical qualities:
  - Metric

$$
\begin{aligned}
ds^2 = d\tau^2 - A^2(\tau)dx^2 - B^2(\tau)e^{-2ax}dy^2 \\
- C^2(\tau)e^{2ax}dz^2\,, \quad a = \text{const}\,,
\end{aligned}
$$

$$
[e_{(a)}{}^i] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad [e^{(a)}{}_i] = \begin{pmatrix} -x^2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

3.2.7 Bianchi VI ($a \neq 0$) – $\mathfrak{g} = \mathfrak{b}_{VI}$

$$
\begin{aligned}
[X_2, X_1] &= aX_2 + X_3\,, \\
[X_2, X_3] &= 0\,, \\
[X_3, X_1] &= X_2 + aX_3\,.
\end{aligned}
\tag{3.2.13}
$$

Set

$$
\begin{cases}
Y_1 &= X_1\,, \\
Y_2 &= a(2a-1)X_2 + (2a^2 - a(1+a^2))aX_3\,, \\
Y_3 &= (a-1+a^2)X_2 + (a^2 - (1-a)(1-a^2))X_3\,.
\end{cases}
$$

$$
\Longrightarrow
\begin{aligned}
[Y_2, Y_1] &= aY_3\,, \\
[Y_2, Y_3] &= 0\,, \\
[Y_3, Y_1] &= Y_2 + Y_3\,.
\end{aligned}
\tag{3.2.14}
$$

✦ Properties:

- ⁎ dim $\mathfrak{g}' = 2$.
- ⁎ The derived algebra $\mathfrak{g}'$ is abelian.
- ⁎ The whole algebra $\mathfrak{g}$ is non-abelian.
- ⁎ Solvable,

$$
\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = 0\,,
$$

- ⁎ $R(\mathfrak{g}) = \mathfrak{g}$.
- ⁎ Non-nilpotent $\mathfrak{g}^n \neq 0\;,\quad \forall n \in \mathbb{N}$.
- ⁎ $Z(\mathfrak{g}) = 0$.
- ⁎ Corresponds to $\mathfrak{g}_{3,a}$ ($a \neq 0$) (3.2.7).
- ⁎ Thurston type: $\mathbb{H}^2 \times \mathbb{E}$ (for $a = -1$ only).

✦ Physical qualities:

⁎ Metric

$$ds^2 = d\tau^2 - A^2(\tau)dx^2 - B^2(\tau)e^{2x}dy^2 - C^2(\tau)e^{2ax}dz^2 \quad , \quad a \text{ from the algebra} ,$$

$$[e_{(a)}{}^i] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^2 \\ 0 & 1 & ax^3 \end{pmatrix} , \qquad [e^{(a)}{}_i] = \begin{pmatrix} -x^2 & 1 & 0 \\ -ax^3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

3.2.8 Bianchi VII$_0$ – $\mathfrak{g} = \mathfrak{b}_{VII_0}$

$$\begin{aligned} [X_1, X_2] &= 0\,, \\ [X_2, X_3] &= X_1\,, \\ [X_1, X_3] &= -X_2\,. \end{aligned} \tag{3.2.15}$$

✦ Properties:

- dim $\mathfrak{g}' = 2$.
- The derived algebra $\mathfrak{g}'$ is abelian.
- The whole algebra $\mathfrak{g}$ is non-abelian.
- Solvable,

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = 0\,,$$

- $R(\mathfrak{g}) = \mathfrak{g}$.
- Non-nilpotent $\mathfrak{g}^n \neq 0 \;, \quad \forall n \in \mathbb{N}$.
- $Z(\mathfrak{g}) = 0$.
- Corresponds to $\mathfrak{g}_{2,-1}$ $(c = -1)$ (3.2.6).
- Thurston type: $\mathbb{E}^3$.

✦ Physical qualities:

- Metric

$$\begin{aligned} ds^2 = d\tau^2 - A^2(\tau)dx^2 & \\ &- (B^2(\tau)\cos^2 x - D^2(\tau)\sin^2 x)dy^2 \\ &- 2\cos x \sin x (B^2(\tau) - D^2(\tau))dydz \\ &- (B^2(\tau)\sin^2 x + D^2(\tau)\cos^2 x)dz^2\,. \end{aligned}$$

- Can be isotropic.
- FLRW solution for the flat sector $(k = 0)$.
- Can be tilted,

$$[e_{(a)}{}^i] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -x^3 \\ 0 & 1 & x^2 \end{pmatrix}, \qquad [e^{(a)}{}_i] = \begin{pmatrix} x^3 & 1 & 0 \\ -x^2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

3.2.9 Bianchi VII ($a \neq 0$) – $\mathfrak{g} = \mathfrak{b}_{VII}$

$$\begin{aligned}
[X_1, X_2] &= -aX_2 + X_3\,, \\
[X_2, X_3] &= 0\,, \\
[X_1, X_3] &= -aX_3 - X_2\,.
\end{aligned} \tag{3.2.16}$$

By a suitable change of basis, can be expressed as:

$$\Longrightarrow \begin{aligned}
[Y_2, Y_1] &= aY_3\,, \\
[Y_2, Y_3] &= 0\,, \\
[Y_3, Y_1] &= Y_2 + Y_3\,.
\end{aligned} \tag{3.2.17}$$

✦ Properties:

- dim $\mathfrak{g}' = 2$.
- The derived algebra $\mathfrak{g}'$ is abelian.
- The whole algebra $\mathfrak{g}$ is non-abelian.
- Solvable,

$$\mathfrak{g}^{(2)} = [\mathfrak{g}', \mathfrak{g}'] = 0\,,$$

- $R(\mathfrak{g}) = \mathfrak{g}$.
- Non-nilpotent $\mathfrak{g}^n \neq 0 \;, \quad \forall n \in \mathbb{N}$.
- $Z(\mathfrak{g}) = 0$.
- Corresponds to $\mathfrak{g}_{3,a}$ ($a \neq 0$) (3.2.7).
- Thurston type: $\mathbb{H}^3$.

✦ Physical qualities:

- Metric

$$\begin{aligned}
ds^2 = d\tau^2 &- A^2(t)(\cos\psi dx - \sin\psi dy)^2 \\
&- B^2(t)(\sin\psi dx + \cos\psi dy)^2 \\
&- C^2(t)dz^2 \;, \quad \psi = \psi(\tau)\,.
\end{aligned}$$

- Corresponds to the FLRW open universe ($k = -1$).
- Can be tilted,

$$[e_{(a)}{}^{i}] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -x^3 \\ 0 & 1 & x^2 + ax^3 \end{pmatrix}, \qquad [e^{(a)}{}_{i}] = \begin{pmatrix} x^3 & 1 & 0 \\ -x^2 - ax^3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The two remaining types (VIII and IX) are classified by the author using quaternion algebra, which obviously goes way out of the scope of this text. Having that in mind, we opt to just mention briefly some of the formalism which allow us to classify the desired Bianchi types. If the reader is interested in delving further into the quaternion formalism, he can read the literature.[10]

Similar to what happened to the algebras IV to VII above, both VIII and IX types are also related to a broader algebraic classification, this being an algebra in which its derivation $\mathfrak{g}'$ has the same dimention as the algebra itself, that is

$$\dim\ \mathfrak{g}' = 3 \iff \mathfrak{g}' = \mathfrak{g}\,,$$

constituting a circular, all commutative algebra

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\,.$$

By virtue of $\mathfrak{g} = \mathfrak{g}'$, both the fields of $\mathfrak{g}$, described by $\{x, y, z\}$, and of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\{[x, y], [y, z], [z, x]\}$ can be used as a basis, so a change of basis from on system to the other gives rise to the *structure matrix* M_{xyz}, responsible for connecting both systems. This matrix can be put into the canonical form

$$M_{xyz} = \mathrm{Diag}(\theta, \vartheta, 1)\,,$$

for some θ and ϑ, which is invariant under isomorphisms. Using that, we can define the Killing form (A.12), represented by

$$\begin{aligned}\langle u, v\rangle &:= \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v)\\ &= u^T\,\mathrm{diag}(\theta, \vartheta, \theta\vartheta)v =: \langle \theta, \vartheta, \theta\vartheta\rangle\end{aligned}$$

for some $u, v \in \mathfrak{g}$, defining the algebra $\mathfrak{g}_{\theta,\vartheta}$. This spans the quarternion algebra $(-\theta, -\vartheta)$ with basis $\{1, i, j, ij\}$, such that

$$i^2 = \theta \quad , \quad j^2 = \vartheta \quad , \quad ij = -ji \, ,$$

which in turn gives the Lie brackets

$$[X_2, X_3] = \theta X_1 \, , \qquad [X_3, X_1] = \vartheta X_2 \, , \qquad [X_1, X_2] = X_3 \, . \tag{3.2.18}$$

This algebra belongs to the *Brauer group* $(Br(\mathbb{R}))$. For further detail, see [35].

Some of the general properties of $\mathfrak{g}_{\theta,\vartheta}$ are

- $\mathfrak{g}$ not solvable nor nilpotent; $\mathfrak{g}^{(n)} = \mathfrak{g}$, $\mathfrak{g}^n = \mathfrak{g} \;, \quad \forall n \in \mathbb{N}$.
- $\mathfrak{g}$ is simple.
- $Z(\mathfrak{g}) = 0$.
- $R(\mathfrak{g}) = 0$.
- Isomorphic to $\mathfrak{gl}(V)$, where V is a vector space.

3.2.10 Bianchi VIII – $\mathfrak{g} = \mathfrak{b}_{VIII}$

$$
\begin{aligned}
[X_1, X_2] &= -X_3 \,, \\
[X_2, X_3] &= X_1 \,, \\
[X_3, X_1] &= X_2 \,.
\end{aligned}
\tag{3.2.19}
$$

- Properties:
  - dim $\mathfrak{g}' = 3$.
  - $\mathfrak{g}' = \mathfrak{g}$.
  - $\mathfrak{g}$ is simple.
  - Not solvable, $\mathfrak{g}^{(n)} = \mathfrak{g} \;, \quad \forall n \in \mathbb{N}$.
  - Non-nilpotent, $\mathfrak{g}^{n} = \mathfrak{g} \;, \quad \forall n \in \mathbb{N}$.
  - $Z(\mathfrak{g}) = 0$.
  - $R(\mathfrak{g}) = 0$.
  - $\mathfrak{g} \in \mathfrak{g}_{1,-1}$ $(\theta = 1 \,, \vartheta = -1)$.
  - Isomorphic to $\mathfrak{gl}(V)$, where V is a vector space.
  - Thurston type: $\widetilde{SL(2, \mathbb{R})}$.
- Physical qualities:
  - Metric

$$
\begin{aligned}
ds^2 = d\tau^2 - S^2(\tau)dx^2 & \\
& - R^2(\tau)(dy^2 + \sinh^2 y dz^2) \\
& - S^2(\tau) \cosh y(2dx + \cosh y dz)dz \,,
\end{aligned}
$$

  - May be thought as an "isotropic model with axis reflection".
  - Oscilatory solutions near the fundamental singularity (Mixmaster solution).

$$[e_{(a)}{}^{i}] = \begin{pmatrix} e^{-x^3} & 0 & 0 \\ -(x^2)^2 e^{-x^3} & 0 & e^{x^3} \\ -2x^2 e^{-x^3} & 1 & 0 \end{pmatrix}, \qquad [e^{(a)}{}_{i}] = \begin{pmatrix} e^{x^3} & 0 & 0 \\ 2x^2 & 0 & 1 \\ (x^2)^2 e^{-x^3} & e^{-x^3} & 0 \end{pmatrix}.$$

3.2.11 Bianchi IX – $\mathfrak{g} = \mathfrak{b}_{IX}$

$$
\begin{aligned}
[X_1, X_2] &= X_3\,, \\
[X_2, X_3] &= X_1\,, \\
[X_3, X_1] &= X_2
\end{aligned}
\tag{3.2.20}
$$

$$
\implies [X_i, X_j] = \epsilon_{ijk} X_k\,.
$$

- Properties:
  - dim $\mathfrak{g}' = 3$.
  - $\mathfrak{g}' = \mathfrak{g}$.
  - $\mathfrak{g}$ is simple.
  - Not solvable, $\mathfrak{g}^{(n)} = \mathfrak{g}\ ,\quad \forall n \in \mathbb{N}$
  - Non-nilpotent, $\mathfrak{g}^{n} = \mathfrak{g}\ ,\quad \forall n \in \mathbb{N}$
  - $Z(\mathfrak{g}) = 0$.
  - $R(\mathfrak{g}) = 0$.
  - $\mathfrak{g} \in \mathfrak{g}_{1,1}$ ($\theta = 1\,, \vartheta = 1$).
  - Isomorphic to $\mathfrak{so}(3)$.
  - Thurston type: $\mathbb{S}^3$.
- Physical qualities:
  - Metric

$$
\begin{aligned}
ds^2 = d\tau^2 &- S^2(\tau)dx^2 \\
&- R^2(\tau)(dy^2 + \sin^2 y dz^2) \\
&+ S^2(\tau)\cos y(2dx - \cos y dz)dz\,,
\end{aligned}
$$

  - Corresponds to the FLRW closed universe ($k = +1$).
  - Isotropic $\implies$ Maximally symmetric.
  - Group of motions $SO(3)$.

⁎ Oscilatory solutions near the fundamental singularity (Mixmaster solution).

⁎ The most interesting type studied in the literature,

$$[e_{(a)}{}^{i}] = \begin{pmatrix} 0 & \cos x^2 & -\sin x^2 \\ 1 & -\cot x^1 \sin x^2 & \cot x^1 \cos x^2 \\ 0 & \frac{\sin x^2}{\sin x^1} & \frac{\cos x^2}{\sin x^1} \end{pmatrix},$$

$$[e^{(a)}{}_{i}] = \begin{pmatrix} 0 & 1 & \cos x^1 \\ \cos x^2 & 0 & \sin x^1 \sin x^2 \\ -\sin x^2 & 0 & \sin x^1 \cos x^2 \end{pmatrix}.$$

We finalize this section by revisiting Table 1

Class	Type	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	Thurston	Note
A	I	0	0	0	0	$\mathbb{E}^3$	FLRW $k = 0$, Petrov O
	II	0	1	0	0	Nil	Heisenberg algebra
	VII_0	0	1	1	0	$\mathbb{E}^3$	FLRW $k = 0$, Petrov O
	VI_0	0	1	-1	0	Sol	−−
	IX	0	1	1	1	$\mathbb{S}^3$	FLRW $k = +1$, Petrov O, Mixmaster solution
	VIII	0	1	1	-1	$\widetilde{SL2\mathbb{R}}$	Mixmaster solution
B	V	1	0	0	0	$\mathbb{H}^3$	FLRW $k = -1$, Petrov O
	IV	1	0	0	1	−−	−−
	VII_a	a	0	1	1	$\mathbb{H}^3$	FLRW $k = -1$, Petrov O
	III ($a = 1$)	a	0	1	-1	$\mathbb{H}^2 \times \mathbb{E}$	Petrov D
	VI_a ($a \neq 1$)					$\mathbb{H}^2 \times \mathbb{E}$	Only for $a = -1$

Table 3: Bianchi Classification – Revisited

3.3 Discussion

By just restricting the spatial sector to the constraint imposed by homogeneity, we were able to describe the motions of the space by the action in simply transitive (spatial) 3-surfaces of a (Lie) group of automorphisms,[4] which in turn enabled us to derive the unique Bianchi algebras the space admit. It is clear how varied each type can be when compared to one another. It is also expected that within this classification we recover the maximal FLRW universes, as seen in the $\mathfrak{b}_I$, $\mathfrak{b}_V$, $\mathfrak{b}_{VII_0}$, $\mathfrak{b}_{VII}$ and $\mathfrak{b}_{IX}$ algebras corresponding to the universes with a curvature constant of $k = \{0, -1, 0, -1, +1\}$, respectively. The homogeneous *and* isotropic case *has to be one of those subtypes* after all.

By flexibilizing the isotropic condition, we can, in principle, describe the local anisotropies of the CMB by one of these models. Moreover, the only conditions we implicitly imposed was the Weyl postulate, which allowed us to make use of the syncronous frame of reference (3.1.1) and, quite obviously, the Cosmological Principle, in constrast to the FLRW solutions, where more conditions were imposed. This freedom should then allow us to get a full picture of the Universe, in all epochs, if both of them hold.

In the literature, two were the predominant types of particular interest since the launch of the program by Bianchi: the types I and IX. The latter was thoroughly, but independently, investigated by Belinskii-Khalatnikov-Lifshitz[2,30] (BKL) and Misner,[38] the first looking for dynamical solutions whereas the second, the potentials.

They found out that the type IX model has oscilatory behaviour near the fundamental singularity, at the very beginning of times ($\tau \to 0$), independently but erratically switching axial expansion to contraction, never fully collapsing to a point. To study this, the authors noticed that the metric tensor can be expanded in a power series around the singularity, where matter was so dense that it was approximately homogeneous, such that an empty-space solution is valid; this solution was first found by Kasner.[29] This model was coined by Misner as the *Mixmaster universe*, or simply the *BKL* model.

On the other hand, the former model, of type I, has the simplest and most friendly of the algebras to work with, the Abelian algebra, corresponding to a flat expanding universe, compatible with the de-Sitter universe and very common nowadays, since observations indicate that the Universe is practically flat.

MacCallum found out later,[36] by studying the asymptotic behaviour of solutions of Einstein's equations, that only the Bianchi types corresponding to FLRW models (I, V, VII_0, VII and IX) can evolve into approximately FLRW solutions, with the gross resamblance of the isotropy we witness today in our evolved Universe. All those types also manifests the oscillatory behaviour of the Mixmaster universe.

Another advantageous aspect about the inheritely algebraic structure of these models is the easy straightforward way to incorporate pertubations in the symmetry, that is, local inhomogeneities, by just deforming the algebras by the inclusion of a small parameter $|\varepsilon| \ll 1$ into the commutation relations (3.1.22). This might lead to interesting results, like the description of a universe without the need of dark matter/energy.

Finally, the *ab initio* construction done in this chapter is the natural pathway towards a local quantum field theory of gravitation, since Lie algebras play a prominent role in the Standard Model of electroweak and strong interactions. Furthermore, the extension of the Bianchi classification to the conformal group presented in (3.1.38) suggests that all those models lies, in fact, inside this 15-parameter group*, which might shed light on how to incorporate the gauge principle to different classes of cosmological models, relating the equivalence prin-

*See Appendix C.

ciple to the conformal acceleration. This will be further investigated in the future. Some attempts to quantize Gravitation has been done by Misner himself[39] on types I and IX and by Friedrichsen,[18] among others.

CHAPTER 4

Petrov Classification

INSTEAD of imposing symmetries to the spacetimes directely and classify all the unique kinds of solutions one gets from them, one can work with the most general spacetime characterized by a generic curvature tensor and investigate the possible solutions it inheritely possesses. Developing this kind of classification is precisely the focus of this chapter.

By analyzing the called *bi-vector* structure of the Riemann tensor, we will find that this tensor can only belong to one of three classes of solutions, which in turn are associated with the number of eigenvectors of the characteristic equation that emerges naturally from the algebraic properties of such tensor. This strongly constrains the types of gravitational fields we can obtain from the theory.

4.1 Fundamentals

As seen in Section 1.1, the Riemman-Christoffel tensor $R_{\mu\nu\alpha\beta}$ has the following symmetry relations

$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}\,, \tag{4.1.1a}$$

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}\,, \tag{4.1.1b}$$

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}\,, \tag{4.1.1c}$$

$$R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0\,, \tag{4.1.1d}$$

with $\mu = 0, 1, 2, 4$ in a 4-spacetime. Those symmetries suggest an underlying structure within *pairs of indices*, so we can think the curvature tensor as

$$R_{(\mu\nu)(\alpha\beta)}$$

where each pair $(\mu\nu)$ can be treated as individual entities altogether, anti-symmetric between its constituent indices evidenced (4.1.1a) and (4.1.1b). In this way we can introduce a new index that spans all the possible combinations of $\mu\nu$ that are anti-symmetric

$$A \equiv (\mu\nu) = \{(01), (02), (03), (12), (23), (31)\}\,, \tag{4.1.2}$$

effectively giving rise to an equivalent 6 dimensional vector space composed of *bi-vectors**, which will be labelled by uppercase latin indices spanning $A = 0, \ldots, 5$. Raising and lowering pairs of indices is done with the rank 4 metric tensor

$$g_{AB} \equiv g_{\mu\nu\sigma\rho} := g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} \tag{4.1.3}$$

*Bi-vectors are essentially the most natural way to codify planes. For more information on the underlying theory, refer to Appendix D.

which carries a signature $(- - - + ++)$ due to the original signature and to (4.1.2). We also shall break the 6-index family into two, each one carrying each signed sector of the metric

$$A = \begin{cases} I & := \{0,1,2\} = \{(01),(02),(03)\} \\ X & := \{3,4,5\} = \{(12),(31),(23)\} \end{cases} , \tag{4.1.4}$$

where the former will be represented by letters starting from I $(I, J, K, \cdots)$ and the latter starting from X $(X, Y, Z, \cdots)$. This is completely analogous to how we split the time components "^{0}" from spatial ones "^{i}" in regular 4-spacetimes, which will become evident when we start working with those bi-vector objects.

From here on we will consider a locally inertial frame of reference, so

$$g_{\mu\nu} \longrightarrow \eta_{\mu\nu} ,$$

which implies

$$g_{\mu\nu\alpha\beta} \longrightarrow \eta_{\mu\nu\alpha\beta} \equiv \eta_{AB} .$$

Our interest lies on the *vacuum solutions* such that

$$R_{\mu\nu} = 0 ,$$

but this strongly restricts the class of solutions to a very particular case. We instead will work with the *Weyl tensor* (C.29), which is constructed in such a way that $C_{\mu\nu} \equiv 0$ for any geometry. One nice consequence of using it is that all the results obtained in this chapter will also be valid for the *conformal group of transformations*, expanding the validity of the theory. We leave the details of this symmetry group and its construction to Appendix C, for now it is enough to just use the Weyl tensor given by

$$
\begin{aligned}
C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{2}\Big(R_{\mu\alpha}g_{\nu\beta} + R_{\mu\beta}g_{\nu\alpha} - R_{\nu\alpha}g_{\mu\beta} - R_{\nu\beta}g_{\mu\alpha}\Big) \\
- \frac{1}{6}\Big(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\Big)R
\end{aligned} . \tag{4.1.5}
$$

Instead of working in a 6-dimensional space where the system of equations will have the same dimension, we can complexify it, reducing the dimension by half. That is done by breaking the Weyl tensor into three bits; each with a fixed number of time components. Naturally, those three quantities will be labeled by spatial indices. Thus, we define

$$
\begin{aligned}
X_{ij} &:= C_{0i0j} \\
Y_{ij} &:= \frac{1}{2}\epsilon_{ilm}C_{0jlm} \\
Z_{ij} &:= \frac{1}{4}\epsilon_{ilm}\epsilon_{jrs}C_{lmrs}
\end{aligned} , \tag{4.1.6}
$$

where both X and Z are symmetric and Y can be anything.

The vacuum condition $C_{\mu\nu} = 0$ is then codified as

$$
X_{ii} = 0 \quad ; \quad Y_{ij} = Y_{ji} \quad ; \quad Z_{ij} = -X_{ij} , \tag{4.1.7}
$$

so all of them are symmetric. Not only that, but they also have nil trace. Indeed, X and Z are immediate, while

$$
Y_{ii} = C_{0123} + C_{0231} + C_{0312} \equiv 0
$$

due to the first Bianchi identity (4.1.1d).

Componentwise, we can rewrite (4.1.6) considering the symmetries of the vacuum solution (4.1.7) as

$$[X_{ij}] = \begin{pmatrix} C_{0101} & C_{0102} & C_{0103} \\ C_{0102} & C_{0202} & C_{0203} \\ C_{0103} & C_{0203} & C_{0303} \end{pmatrix} \tag{4.1.8a}$$

$$[Y_{ij}] = \begin{pmatrix} C_{0123} & C_{0131} & C_{0112} \\ C_{0223} & C_{0231} & C_{0212} \\ C_{0323} & C_{0331} & C_{0312} \end{pmatrix} \tag{4.1.8b}$$

$$[Z_{ij}] = \begin{pmatrix} C_{2323} & C_{2331} & C_{2312} \\ C_{2331} & C_{3131} & C_{3112} \\ C_{2312} & C_{3112} & C_{1212} \end{pmatrix} = -[X_{ij}]\,, \tag{4.1.8c}$$

which in the bi-vector space takes the form

$$[X_{ij}] = \begin{pmatrix} C_{00} & C_{01} & C_{02} \\ C_{01} & C_{11} & C_{12} \\ C_{02} & C_{12} & C_{22} \end{pmatrix} \iff [C_{IJ}]\,, \tag{4.1.9a}$$

$$[Y_{ij}] = \begin{pmatrix} C_{05} & C_{04} & C_{03} \\ C_{15} & C_{14} & C_{13} \\ C_{25} & C_{24} & C_{23} \end{pmatrix} \iff [C_{IX}]\,, \tag{4.1.9b}$$

$$[Z_{ij}] = \begin{pmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{43} \\ C_{35} & C_{34} & C_{33} \end{pmatrix} \iff [C_{XY}]\,. \tag{4.1.9c}$$

Finally, the complexification is done by defining the complex tensor

$$W_{ij} = \frac{1}{2}\Big(X_{ij} - Z_{ij} + 2iY_{ij}\Big)$$

such that, in the vacuum,

$$W_{ij} = X_{ij} + iY_{ij}\,, \tag{4.1.10}$$

which is also nil traced, i.e. $W_{ii} = 0$.

We also shall be needing the invariants associated with the curvature tensor, but before doing anything, let us first digress a bit on the electromagnetic case. There are only two unique invariants in this case, which are given by

$$I_1 = F_{\mu\nu}F^{\mu\nu} = E^2 - B^2\,,$$
$$I_2 = F_{\mu\nu} * F^{\mu\nu} = 2\boldsymbol{E}\cdot\boldsymbol{B}$$

or in the bi-vector form

$$I_1 = F_A F^A \qquad ; \qquad I_2 = F_A * F^A\,,$$

where

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

is the Maxwell tensor and

$$*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho} \longrightarrow *F^A = \frac{1}{2}\epsilon^{AB}F_B$$

is the Hodge dual Maxwell tensor, with the bi-vector Levi-Civita symbol assuming the only possible values*

*It is interesting to note that in the bi-vector space the Levi-Civita is symmetric.

$$\epsilon^{AB} = \left\{ \epsilon^{05} = \epsilon^{50} = 1,\ \epsilon^{14} = \epsilon^{41} = 1,\ \epsilon^{23} = \epsilon^{32} = 1 \right\} .$$

Now, if we construct the complex vector

$$\boldsymbol{F} = \boldsymbol{E} + i\boldsymbol{B}$$

and take its square

$$\begin{aligned} \boldsymbol{F}^2 = \boldsymbol{F} \cdot \boldsymbol{F} &= (E^2 - B^2) + 2i\boldsymbol{E} \cdot \boldsymbol{B} \\ &= I_1 + iI_2 , \end{aligned}$$

so the complex vector carries both invariants within it. Returning to the point in question, we expect the same behaviour to manifest with the complex tensor (4.1.10). The curvature tensor has four invariants:

$$\begin{aligned} I_1 &= R_{AB} R^{AB} & ; \qquad I_2 &= R_{AB} * R^{AB} \\ I_3 &= R_A{}^B R_B{}^C R_C{}^A & ; \qquad I_4 &= R_A{}^B R_B{}^C * R_C{}^A , \end{aligned} \tag{4.1.11}$$

with

$$*R_{AB} = \frac{1}{2} \epsilon_{AC} R^C{}_B ,$$

and so do the Weyl tensor, since it has the same symmetries. The complex invariants is then given by*

$$\begin{aligned} \mathcal{I}_1 &= I_1 - iI_2 \\ \mathcal{I}_2 &= I_3 + iI_4 , \end{aligned} \tag{4.1.12}$$

*This is a construction, so the choice of $\pm i$ is arbitrary, but defining these invariants in this form simplifies the results later on.

which using (4.1.9) gives*

$$\begin{aligned} \mathcal{I}_1 &= 2\,\mathrm{tr}(\mathbb{X}^2 - \mathbb{Y}^2) + 4i\,\mathrm{tr}(\mathbb{Y}\mathbb{X}) \\ \mathcal{I}_2 &= -2\,\mathrm{tr}(\mathbb{X}^3 - 3\mathbb{X}\mathbb{Y}^2) + 2i\,\mathrm{tr}(3\mathbb{X}^2\mathbb{Y} - \mathbb{Y}^3)\,. \end{aligned} \tag{4.1.13}$$

Back to the main focus. Since the complex tensor (4.1.10) fully describes the curvature in vacuum, we can analize its algebraic structure in terms of the characteristic system associated with it,

$$\mathbb{W}\boldsymbol{n}^{(k)} = \lambda^{(k)}\boldsymbol{n}^{(k)}\,, \tag{4.1.14}$$

for some complex 3-eigenvector $n_i^{(k)}$ associated with complex eigenvalues λ, corresponding to the invariants of $\mathbb{W}$ (and, by extension, of $C_{\mu\nu\alpha\beta}$). Petrov[49] calls the eigenvectors the *stationary directions* associated with the *stationary curvatures*.

It will be useful to separate the real and imaginary parts from the eigenvalues

$$\lambda^{(k)} = \lambda'^{(k)} + i\lambda''^{(k)}\,.$$

Since $\mathbb{W}$ has nil trace, the eigenvalues are constrained among themselves, that is,

$$\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = 0\,. \tag{4.1.15}$$

There are only three possible classes of solutions of the characteristic equation (4.1.14), depending on the number of independent eigenvectors. Those classes define the types I-III of the *Petrov classification*. There are also the subclasses D, N and O which are degenerated cases (one or more identical eigenvalues) of the former classes.

*If $z = a + ib$ and $z' = -a + ib$, we see that $\mathcal{I}_1$ and $\mathcal{I}_2$ correspond to z^2 and z'^3, respectively. We would expect this due to the complex structure of this representation.

4.2 The classification

In this section we will derive all three canonical forms of (4.1.14), what form $\mathbb{W}$ takes, which are the invariants, and what are the eigenvectors associated with them. We also will study if the characteristics of the independent components $\mathbb{X}$ and $\mathbb{Y}$ recovers the complex solutions of $\mathbb{W}$.

Finally, we shall realify $\mathbb{W}$ to obtain the original 6−dimensional system of equations, similar to those Petrov worked with, by constructing the real matrix

$$\mathbb{W}^{\mathbb{R}} = \left(\begin{array}{c|c} \mathbb{X} & -\mathbb{Y} \\ \hline \mathbb{Y} & \mathbb{X} \end{array}\right),$$

which is pretty much the same as seen on Appendix A. There is also another way to realify the equations by considering instead the matrix

$$\mathbb{W}^{\mathbb{R}} = \left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Y} & -\mathbb{X} \end{array}\right).$$

This gives the results obtained by Petrov.

4.2.1 Type I

The first possibility is if all the eigenvectors are linearly independent (LI), so $\mathbb{W}$ is non-singular. That means there exists a non-singular unitary matrix $\mathbb{P}$ that brings $\mathbb{W}$ into a diagonal form

$$\mathbb{P}^{-1}\mathbb{W}\mathbb{P} = \widetilde{\mathbb{W}},$$

where $\widetilde{\mathbb{W}}$ is diagonal, preserving trace and determinant. Then,

$$\widetilde{\mathbb{W}} = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix} . \tag{4.2.1}$$

Its eigenvalues and eigenvectors are evidently

$$\mathrm{Eig}\mathbb{W} : \begin{cases} \lambda^{(1)} = \lambda'^{(1)} + i\lambda''^{(1)} & \ni \quad \boldsymbol{n}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \lambda^{(2)} = \lambda'^{(2)} + i\lambda''^{(2)} & \ni \quad \boldsymbol{n}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \lambda^{(3)} = -\lambda^{(1)} - \lambda^{(2)} & \ni \quad \boldsymbol{n}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{cases} , \tag{4.2.2}$$

which are obviously diagonal.

Since $\mathbb{W} = \mathbb{X} + i\mathbb{Y}$, both $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ will also be diagonal

$$\widetilde{\mathbb{X}} = \begin{pmatrix} \lambda'^{(1)} & 0 & 0 \\ 0 & \lambda'^{(2)} & 0 \\ 0 & 0 & \lambda'^{(3)} \end{pmatrix} \quad ; \quad \widetilde{\mathbb{Y}} = \begin{pmatrix} \lambda''^{(1)} & 0 & 0 \\ 0 & \lambda''^{(2)} & 0 \\ 0 & 0 & \lambda''^{(3)} \end{pmatrix} \tag{4.2.3}$$

Both of them have the same eigenvectors as in (4.2.2).

The realified $\widetilde{\mathbb{W}}$ has the following form

$$\widetilde{\mathbb{W}}^{\mathbb{R}} = \left(\begin{array}{ccc|ccc} \lambda'^{(1)} & 0 & 0 & -\lambda''^{(1)} & 0 & 0 \\ 0 & \lambda'^{(2)} & 0 & 0 & -\lambda''^{(2)} & 0 \\ 0 & 0 & -\lambda'^{(1)} - \lambda'^{(2)} & 0 & 0 & \lambda''^{(1)} + \lambda''^{(1)} \\ \hline \lambda''^{(1)} & 0 & 0 & \lambda'^{(1)} & 0 & 0 \\ 0 & \lambda''^{(2)} & 0 & 0 & \lambda'^{(2)} & 0 \\ 0 & 0 & -\lambda''^{(1)} - \lambda''^{(2)} & 0 & 0 & -\lambda'^{(1)} - \lambda'^{(2)} \end{array}\right) . \tag{4.2.4}$$

With $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ we are able to compute the invariants (4.1.13), which results in

$$\begin{aligned} \mathcal{I}_1^I &= 2\left\{(\lambda^{(1)})^2 + (\lambda^{(2)})^2 + (\lambda^{(3)})^2\right\} \\ \mathcal{I}_2^I &= 6\lambda^{*(1)}\lambda^{*(2)}\left(\lambda^{*(1)} + \lambda^{*(2)}\right) \end{aligned} .$$

If $\lambda^{(1)} = \lambda^{(2)} =: \lambda$, then

$$\widetilde{\mathbb{X}} = \begin{pmatrix} \lambda' & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & -2\lambda' \end{pmatrix} \quad ; \quad \widetilde{\mathbb{Y}} = \begin{pmatrix} \lambda'' & 0 & 0 \\ 0 & \lambda'' & 0 \\ 0 & 0 & -2\lambda'' \end{pmatrix} ,$$

corresponding to the invariants

$$\begin{aligned} \mathcal{I}_1^D &= 12\lambda^2 \\ \mathcal{I}_2^D &= 12(\lambda^*)^2 \end{aligned} . \tag{4.2.6}$$

This represents the degenerate Petrov type D.

4.2.2 Type II

Here we have two LI eigenvectors (one independent and two linearly dependent (LD)), so one of the eigenvalues has multiplicity of two associated with those two LD eigenvectors*. Say $\lambda^{(1)} = \lambda^{(2)} \equiv \lambda$, then by (4.1.15)

$$\lambda^{(3)} = -2\lambda \; .$$

Without loss of generality, we say $\lambda^{(3)}$ corresponds to the eigenvector

$$n^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ,$$

allowing us to break $\mathbb{W}$ into smaller bits,

$$\mathbb{W} = \begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{D} & 0 \\ \hline 0 & -2\lambda \end{array}\right) .$$

Since both eigenvalues of $\mathbb{D}$ are the same and, with the trace constraint,

$$\begin{aligned} \operatorname{tr} \mathbb{D} \equiv a + b &= 2\lambda \, , \\ b &= 2\lambda - a \, , \end{aligned} \tag{4.2.7}$$

we can determine two of the three unknowns, leaving one of them free. So,

*The proof of this claim is straightforward. Assume that $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$ are LD, i.e. $\boldsymbol{n}^{(1)} = \alpha \boldsymbol{n}^{(2)}$ for some $\alpha \neq 0$. Return to the characteristic equation $\mathbb{W}\boldsymbol{n}^{(2)} = \lambda^{(2)}\boldsymbol{n}^{(2)}$ and verify that $\lambda^{(2)} = \lambda^{(1)}$.

$$\det(\mathbb{D} - \lambda \mathbb{1}) \equiv 0 = (a - \lambda)(b - \lambda) - c^2$$

$$\Longrightarrow \quad a = \lambda \pm ic$$
$$\overset{(4.2.7)}{} b = \lambda \mp ic \;.$$

Choosing the lower sign, we then can write the complete complex matrix $\mathbb{W}$ as

$$\mathbb{W} = \begin{pmatrix} \lambda - ic & c & 0 \\ c & \lambda + ic & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}, \tag{4.2.8}$$

which decomposes into

$$\mathbb{X} = \begin{pmatrix} \lambda' & c & 0 \\ c & \lambda' & 0 \\ 0 & 0 & -2\lambda' \end{pmatrix} \quad ; \quad \mathbb{Y} = \begin{pmatrix} \lambda'' - c & 0 & 0 \\ 0 & \lambda'' + c & 0 \\ 0 & 0 & -2\lambda'' \end{pmatrix}. \tag{4.2.9}$$

The eigenvectors of $\mathbb{W}$ are

$$\mathrm{Eig}\mathbb{W} : \begin{cases} \lambda & \ni \boldsymbol{n}_{\mathrm{W}}^{(1)} = \boldsymbol{n}_{\mathrm{W}}^{(2)} = \begin{pmatrix} \alpha \\ i\alpha \\ 0 \end{pmatrix} \qquad \text{(multiplicity 2)} \\ -2\lambda & \ni \boldsymbol{n}_{\mathrm{W}}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} \end{cases} . \tag{4.2.10}$$

Note that

$$(\boldsymbol{n}^{(1)})^2 = (\boldsymbol{n}^{(2)})^2 \equiv 0\,, \tag{4.2.11}$$

so linearly dependent complex eigenvectors *have nil square*. This seems to be a inherit property of complex eigenvectors for a complexified system of equations.

On the other hand, the Type II invariants are

$$\begin{aligned} \mathcal{I}_1^{II} &= 12\lambda^2 \\ \mathcal{I}_2^{II} &= 12\lambda^3\,, \end{aligned} \tag{4.2.12}$$

and the corresponding realification is

$$\mathbb{W}^{\mathbb{R}} = \left(\begin{array}{ccc|ccc} \lambda' & c & 0 & -\lambda'' + c & 0 & 0 \\ c & \lambda' & 0 & 0 & -\lambda'' - c & 0 \\ 0 & 0 & -2\lambda' & 0 & 0 & 2\lambda'' \\ \hline \lambda'' - c & 0 & 0 & \lambda' & c & 0 \\ 0 & \lambda'' + c & 0 & c & \lambda' & 0 \\ 0 & 0 & -2\lambda'' & 0 & 0 & -2\lambda' \end{array}\right). \tag{4.2.13}$$

In the degenerate case $\lambda' = \lambda'' = 0$, both invariants vanish

$$\mathcal{I}_1 = \mathcal{I}_2 = 0$$

in such a manner that we lose the means to detect the curvature. This particular case is called the Petrov type N.

4.2.3 Type III

In the last case, all the eigenvectors are linearly dependent and therefore all the eigenvalues; so, by (4.1.15), they are all equal to zero.

Following the reasoning of nil squared LD eigenvectors (4.2.11), from the previous type, we can assume it to be true in pairs. But since all of them are LD among themselves, all the $\boldsymbol{n}^{(i)}$ eigenvectors have nil square. Thus,

$$\boldsymbol{n} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix},$$
$$\boldsymbol{n}^2 = \alpha^2 + \beta^2 + \gamma^2 \equiv 0 .$$

Choosing $\gamma = 0$ leads to

$$\beta = i\alpha ,$$

where we chose the positive sign. For a generic $\mathbb{W}$ we have

$$\mathbb{W}\boldsymbol{n} = 0\boldsymbol{n}$$

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} \alpha \\ i\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{cases} a\alpha + i\alpha d &= 0 \\ d\alpha + i\alpha b &= 0 \\ e\alpha + i\alpha f &= 0 \\ a + b + c &= 0 \qquad \text{(From (4.1.15))} \end{cases}$$

$$\implies a = b = c = d = 0 ,$$
$$f = ie .$$

Therefore,

$$\mathbb{W} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & ie \\ e & ie & 0 \end{pmatrix} , \tag{4.2.14}$$

with the following eigenvectors

$$\text{Eig}\,\mathbb{W} : \begin{cases} 0 \quad \ni \boldsymbol{n}^{(1)}_{\mathbb{W}} = \boldsymbol{n}^{(2)}_{\mathbb{W}} = \boldsymbol{n}^{(3)}_{\mathbb{W}} = \begin{pmatrix} \alpha \\ i\alpha \\ 0 \end{pmatrix} & \text{(multiplicity 3)} \end{cases} , \tag{4.2.15}$$

and decomposed as

$$\mathbb{X} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ e & 0 & 0 \end{pmatrix} \qquad ; \qquad \mathbb{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ie \\ 0 & ie & 0 \end{pmatrix} . \tag{4.2.16}$$

Since all the eigenvalues are zero, both Type III invariants vanish

$$\mathcal{I}^{III}_1 = \mathcal{I}^{III}_2 = 0 . \tag{4.2.17}$$

The lack of invariants means that even though the space has a curvature, there are no ways, mathematically speaking, to detect it. Finally, the realified $\mathbb{W}$ is

$$\mathbb{W}^{\mathbb{R}} = \left(\begin{array}{ccc|ccc} 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e \\ e & 0 & 0 & 0 & -e & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & e & 0 & 0 & 0 \\ 0 & e & 0 & e & 0 & 0 \end{array}\right) . \tag{4.2.18}$$

4.2.4 Type O

There is yet another rather different type that happens when the full Weyl tensor vanishes altogether

$$C_{\mu\nu\alpha\beta} = C_{AB} \equiv 0 \,.$$

In this case, $\mathbb{W}$ is identically equal to zero and the characteristic equation (4.1.14) stop making much sense, being automatically satisfied for any vector with null eigenvalue. The Petrov of type O is the most degenerated case there is.

All FLRW models are of type O.

We see that the Petrov types are related between themselves by the invariants, since the same invariants can correspond to different Petrov types. This degeneracy trickles down to the level of degeneracy of the eigenvectors and/or eigenvalues, so one type can be obtained by another by the following degeneracy hierarchy:

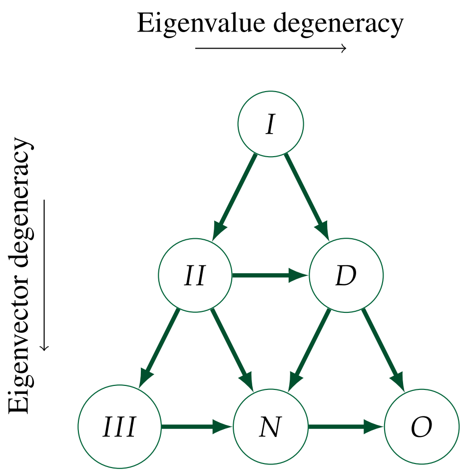


Figure 17: Petrov degeneracy hierarchy

4.3 Discussion

This kind of classification follows directly from the Weyl tensor, which in turn is fully dependents upon the curvature tensor and its contractions. Therefore, *every gravitational model has to belong to one of the six possible Petrov types*, be it a purely gravitational model or, in our case, a cosmological one. This means that the Bianchi classification (and hence the FLRW models) is within the Petrov classification.

In fact, we can promptly infer that, for example, the self-evident case of the flat type O, related to the FLRW $k = 0$ Universe, is equivalent to Bianchi type I.

Since the conception of this classification was completely analogous to the usual way to determine normal modes of oscillation in classical mechanics, it is not surprising that the Petrov types naturally encode within themselves the modes of gravitational radiation. Those modes of oscillation correspond to the different polarizations, related to discontinuities of the curvature tensor. This is one of Pirani's criteria.[50] In the article, the author noticed that the discontinuities only appear when null-directions are present, thus only the types II, III, N and O can comport such modes.

L. Witten[63] in 1959 noticed that the gravitational theory could be described together with electromagnetism in an unified picture if they are geometricized into their principal null directions, linked to the Petrov types by none other than Penrose[44] a year later, when he further developed Witten's null vectors into what became the spinor formalism of gravitation. In this formalism, the tensor quantities are described in terms of the *null tetrads*, which encapsulates the Petrov classification in a natural manner and where the complexified quantities we studied above (such as (4.1.10)) seamlessly belong in it.

Conclusions and Future Outlooks

THIS work focused on the construction of a Classical theory of Gravity from the ground up and the possible catalogations this might admit through a symmetry group approach, which is, in our view, essential for any physical theory and is the natural construct towards a gauge theory, whilst never forgetting the true motivation of a solidly formulated description of cosmological models that could describe our Universe up to lower orders of local pertubation theory, both at classical and quantum level, where open questions still demand for a more careful analysis.

We committed ourselves to re-visit the mainly forgotten literature on the topic, reviewing and compiling the essential pioneering works, which are all scattered throughout the literature and are, nowadays, getting dust, laying those out in a comprehensive and self-contained monograph. We also expect that this dissertation to be a helpful entry point for anyone interested to begin working in this topic. In this aspect we think our objective has been duly fulfilled.

In the first chapter we began outlining the fundamental background indispensable for the subsequent chapters. There we discussed Differential Geometry, the formalism of the locally inertial frame of reference consisting of the N-Tuple basis vectors, following by the general theory underlying Killing vectors and their association with spacetime symmetries, ending with the Synchronous frame of reference, essential for the codification of the Weyl postulate. Minor topical fundamentals were left out in Appendices A – E.

The first catalogation class was done in chapter 2, for spacetimes that admit a maximum number of symmetries. Using the Killing vectors as the main ingredient responsible for all the possible motions of the space, we deduced that such a class of spacetimes are both homogeneous and isotropic about every point and are described by a constant curvature K. Afterwards we demonstrated the decomposition theorem (2.1.37) which allows us to use the same toolbox developed to maximally symmetric *subspaces* as well. As an immediate consequence of this, we derived both the Einstein-de Sitter metric and the standard model of cosmology, the Friedmann-Lemaître-Robertson-Walker metric, those totally constructed from symmetry considerations only, not heuristically as usually done in the literature.

Next, in chapter 3, we loosen the symmetry assumptions up and considered the case of spaces manifesting only *spatial* homogeneity. Through the lens of the Weyl postulate, we considered that the Universe essentially "moves" along time-like geodesics so the synchronous frame is valid. By describing the geometrical objects in the N-Tuple frame, we were able to associate the Lie derivatives to a Lie algebra acting on the spatial 3-surface of transitivity. From this, we showed that only nine unique Lie groups can be conceived, arriving at the *Bianchi classification of homogeneous spaces*, all laid out in Table 3. We finalized our discussion by listing the major qualities each Bianchi type exhibit, in a systematic manner, and by discussing the main implications and where the FLRW models fit it.

We also found out that the Bianchi classification holds for transformations under the conformal group C, a new result that extends said classification to a bigger symmetry group. We believe this discovery is relevant, since the four parameter conformal aceleration might offer the possibility of incorporating gravitation via a gauge principle in the light of the equivalence principle. A paper containing this result and its main implications will be elaborated and submitted in the near future.

We end in chapter 4 by classifying the absolute fundamental symmetries of any gravitation theory. By studying the bi-vector structure of the Weyl tensor (thus indirectely the Riemann curvature tensor), the algebraic decomposition into the "principal axes" took place, where the Petrov classification was devised. The six different classes contrast between themselves by the number of admissible independent eigenvectors and by the degree of degeneracy of their eigenvalues, i.e., the invariants. We then showed the degeneracy hierarchy between the Petrov types and how each type is related to the others by regarding their invariants.

This work prepared us with the necessary pre-requisites for studying particular cases of the Bianchi and/or Petrov classifications with proper depth. In a future study, we shall focus on the Bianchi types I, V, VII_0, VII and IX, all FLRW models, taking advantage of their conformal structure, and what implications it might have on the standard cosmological model.

We also left for the future a more in-depth study of Thurston topologies. We believe that this alternative formlation contains valuable nuggets of information that are invisible in the usual algebraic/geometric way presented in here. This can possibly be of great help to our overall understanding as a tool in the search for solutions to the desired Bianchi types mentioned above.

Finally, after exausting all this, we shall be able to further develop the conformal Bianchi types to then study, and maybe even propose, new ways to incorporate the gauge principle to a theory of gravitation, looking at deviations of the standard cosmology from an algebraic standpoint, where the generators of the isometries play a central role. We expect the conformal symmetry to be of great revelancy, since there exists no axiomatic quantization program which encompasses such symmetry, so it is necessary to revisit Wightman's axioms to include this symmetry to the respective vacuum states of the quantized fields. For that, we require space homogeneity in order to preserve the particle content of the theory, though it may not be a necessary condition, if we impose the Wightman functions to be invariant.[62] Furthermore, the breaking of the space homogeneity has as a direct consequence the violation of microcausality, the main cornerstone of the relativistic field theories, so we do not expect any theory to let go of this symmetry.

Appendices

APPENDIX A

Group Theory

The concept of *groups* in physics can be roughly thought as a set of operations that leave something or some quality invariant. With that in mind we can readily see how that is a fundamental ingredient to incorporate in most, if not all, physical theories constructed *ab initio*. For instance, the principle of covariance may be understood in that light, with the set of operations being a general transformation of coordinates that leave the equations of motion invariant, that is, the geodesics themselves, which are closely related with the associated field equations and particle dynamics.

Group theory is a vast topic and treatises can be written about it, so we will just focus on the concepts necessary to comprehend the main subject of this monograph.

A.1 Abstract Group Theory

A set G of elements g endowed with a bilinear product operation $G \cdot G \to G$ is defined as a *group* if all elements $\{g_i\} \in G$ (not necessarely discrete) satisfy:

$$g_1 \cdot g_2 \in G \,, \qquad \text{(closure)} \tag{A.1a}$$

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \,, \qquad \text{(associativity)} \tag{A.1b}$$

$$\exists\, e \in G \;; \quad g \cdot e = e \cdot g = g \,, \qquad \text{(identity)} \tag{A.1c}$$

$$\exists\, g^{-1} \in G \;; \quad g \cdot g^{-1} = g^{-1} \cdot g = e \,. \qquad \text{(inverse)} \tag{A.1d}$$

A subset of elements $h \in G$ which satisfy all the group axioms (A.1) by themselves and inherits the same multiplication operation is called a *subgroup* of G, where evidently $H \subseteq G$. For example, $\mathrm{SO}(n;\mathbb{R})$ is a subgroup of $\mathrm{O}(n;\mathbb{R})$, which in turn is a subgroup of $\mathrm{GL}(n;\mathbb{R})$ ($\mathrm{O}(n;\mathbb{R}) \subseteq \mathrm{SO}(n;\mathbb{R}) \subseteq \mathrm{GL}(n;\mathbb{R})$). Every group defines its own subgroups.

A map from a group G into another G' preserving the multiplication operation defines a *homomorphism* $\Phi : G \longrightarrow G'$. If the mapping is one-to-one (both injective and surjective, admiting inverse), then it gets called an *isomorphism* ($G' \approx G$). If $G' = G$, the mapping is called an *endomorphism*. If it is one-to-one, then it is called an *automorphism*. The set of elements $g \in G$ which maps onto $e' \in G'$ is called the *kernel* of the homomorphism.

The *center* Z of a group G is the set of all elements $z \in G$ that commutes with all the other group elements, i.e., $zg = gz \; \forall g \in G$. If all the elements of G commute between themselves, the group is called *Abelian*. Thus, by (A.1c), the identity e is always in the center of any group.

For any given element s_0 of any set S, we define the *orbit* of s_0 with respect to G as the subset of all elements that can be obtained from s_0 by the action of G, denoted by $O_G(s_0)$, where the action is the multiplication operation of the group in S.

One example to illustrate the last two concepts naturally emerge from the particular group of rigid rotations* $\mathrm{SO}(n)$ acting on $\mathbb{R}^n$: the center of the group is the identity element e whereas the orbit are the surfaces of the spheres of radius r, $0 \leq r < \infty$ in Euclidian spaces.

The *surface of transitivity* is defined such that any element $s \in O_G(s_0)$ can be obtained from any other group transformation of another element of the orbit, that is, $s_1, s_2 \in O_G(s_0)\,;\, gs_1 = s_2\,,\, \forall g \in G$.

The transformation of the element g given by $g \longrightarrow hgh^{-1}$ for $g, h \in G$ is called a *conjugation*. Similarly, a *group conjugation* occurs when such transformation is applied to all elements of G, that is, $g \longrightarrow hgh^{-1}\,,\, \forall g \in G$ (symbolically $G \longrightarrow hGh^{-1}$ for some $h \in G$). This preserves the group multiplication

*$\mathrm{SO}(n)$ is to be undestood as $\mathrm{SO}(n; \mathbb{R})$ henceforward, unless explicitly stated.

$$\begin{aligned} gg' &\to h(gg')h^{-1} \\ &\to h(geg')h^{-1} \\ &\to hg(h^{-1}h)g'h^{-1} \\ &\to (hgh^{-1})(hg'h^{-1}), \end{aligned}$$

for some $g, g', h \in G$.

The conjugation of a fixed $h \in G$ by the entirety of G, ghg^{-1} , $\forall g \in G$, defines the *conjugation class* of h, which clearly is an orbit of h $O_G(h)$.

The conjugation permits us to finally define the *normal subgroup* (or *invariant subgroup*) H of G, whose elements are invariant under conjugation, i.e., $h \subseteq ghg^{-1}$, $\forall h \in H$ and $\forall g \in G$, symbolically denoted by $H \subseteq gHg^{-1}$. Now if the conjugation itself is in H, that is, $gHg^{-1} \subseteq H$ for all $g \in G$, then the the equality is in place $h = ghg^{-1}$ for some $h \in H$; this implies that h is in the center $h \in Z(G)$.

Yet still, we define the *left* (right) *coset* as the subgroup $H \subset G$ with respect to the left (right) multiplication if g_1 and g_2 are in the same left (right) coset and if there exists $h \in H$ such that $g_1 = hg_2$ $(= g_2h)$. The cosets of a groups are, by definition, orbits of H.

Now if H is a normal subgroup of G, the set of all cosets of G with respect to H makes up a group. This group, denoted by $Q := G/H$, is called the *quotient group* of G and has H as its identity element and its multiplation law is said to be *induced* by G. In general the quotient group *is neither* a subgroup of G nor isomorphic to it.

A.2 Lie Group Theory

Lie groups are those which contains a continuous "amount" of elements and are locally Euclidian. This enables us to parametrize any element by a finite number of continuous variables, $g = g(a_1, \ldots, a_r)$, where r is called the *order* of the group, corresponding to the minimum number of parameters necessary to fully describe the group; $\{a_k\}, k = 1, \ldots, r$ are continuous and, by convention, $e \equiv g(0, \ldots, 0)$. Every Lie group is isomorphic to a matrix group near the identity, to what Hall[24] calls the *Matrix Lie Group*.

As a consequence of the continuous parametrization, the Lie groups also defines a differentiable manifold G (or $\mathcal{M}$ as used in the text) such that the bilinear group product $G \cdot G \to G$, the elements and their inverses are all differentiable. Even stronger than that, the following theorem guarantees that.

Theorem. *Every matrix group is a smooth embedded submanifold of* $\mathcal{M}(N)$ *and is thus a Lie group.*

The smoothness of Lie groups is also preserved by the various morphisms between Lie groups.

Some of the most common Lie groups in a matrix representation, with ordinary matrix multiplication as the composition law, are given below.

For a given *field** $\mathbb{F}$

i. The *General Linear Group* $\mathrm{GL}(n; \mathbb{F})$ of all invertible $n \times n$ matrices over $\mathbb{F}$.

   This corresponds to the group of matrices and their usual multiplication.

*Field is just the set of elements where the usual operations are defined, that is $\{+, -, \times, \div\}$, satisfying the field axioms. For example $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (real or complex numbers).

ii. The *Special Linear Group* $\mathrm{SL}(n;\mathbb{F})$ of all the elements $M \in \mathrm{GL}(n;\mathbb{F})$ with $\det M = +1$.

iii. The *Orthogonal Group* $\mathrm{O}(n;\mathbb{F})$ of all $n \times n$ matrices M over $\mathbb{F}$ with $M^T M = \mathbb{1}$.

iv. The *Special Orthogonal Group* $\mathrm{SO}(n;\mathbb{F})$ of all the elements $M \in \mathrm{O}(n;\mathbb{F})$ with $\det M = +1$,

This corresponds to the group of rotations over $\mathbb{F}$.

v. The *Unitary Group* $\mathrm{U}(n)$ of all the $n \times n$ complex ($\mathbb{F}$ is necessarily $\mathbb{C}$) matrices M where $M^\dagger M = \mathbb{1}$.

vi. The *Special Unitary Group* $\mathrm{SU}(n)$ of all the elements $M \in \mathrm{U}(n)$ with $\det M = +1$.

vii. The *Symplectic Group* $\mathrm{Sp}(n;\mathbb{F})$ of all $2n \times 2n$ matrices M over $\mathbb{F}$ with $M^T J M = J$, where $J := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$. The condition imposed already implies $\det M = +1$.

While it apparently looks like an uncommon group, it naturally arises in the hamiltonian formalism of Classical Mechanics.

viii. The *Lorentz Group** $\mathrm{O}(3,1)$ of all real 4×4 matrices Λ with $\Lambda^T \hat{\eta} \Lambda = \hat{\eta}$, i.e., the group of general transformation of coordinates that leave the metric tensor $\hat{\eta}$ (and thus the line element) invariant.

A Lie group G is said to be *connected* if we can transform one element $g_1 \in G$ into another $g_2 \in G$ through a continuous parametrized path $g(t)$, $t_1 \leq t \leq t_2$ where $g(t_1) = g_1$ and $g(t_2) = g_2$. Furthermore, we say a Lie group is *simply connected* if it is connected and every closed loop in G can be shrunk to a point in G.

* $\mathrm{O}(n,k)$ is also called the *Generalized Orthogonal Group* with a metric signature defined by the inner product $[x,y]_{n,k} = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} - \cdots - x_{n+k} y_{n+k}$.

The simply connectedness condition is fundamental to get a one-to-one correspondence between G and of its Lie algebra. In fact, the Lie algebra is precisely the tangent space in the neighbourhood of the identity, thus locally Euclidian. The connection between Lie groups and algebras will soon become clear when we define the exponential map. But first let us properly define what is a Lie algebra.

If V^n is a $n-$dimensional linear vector space endowed with the closed bilinear multiplication $[V^n, V^n] \to V^n$, we say that V^n is a *Lie algebra* $\mathfrak{g}$ if for $X, Y, Z \in V^n$

$$[X, Y] = -[Y, X]\,, \qquad \text{(Skew-symmetry)} \tag{A.1a}$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\,. \qquad \text{(Jacobi Identity)} \tag{A.1b}$$

We call this product a *commutator* or the *Lie bracket* of X and Y. Usually Lie algebras are denoted by lowercase Fraktur letters. If $[X, Y] = 0$, $\forall X, Y \in \mathfrak{g}$, we say that $\mathfrak{g}$ is Abelian, similar to what we have seen before.

If we take a complete basis $\{X_i\}$, $i = 1, \ldots, n$ for $\mathfrak{g}$ we define the *structure constants* c_{ij}^k as

$$[X_i, X_j] = c_{ij}^k X_k\,, \tag{A.2}$$

which are anti-symmetric in their lower indices $c_{ij}^k = -c_{ji}^k$ and

$$c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l = 0\,, \tag{A.3}$$

by virtue of (A.1). Upon a change of basis,

$$X'_a = d^i_a X_i\,, \quad d^i_j \neq 0$$

we see that the structure constants transform like a (1,2) tensor

$$c'^k_{ij} d^a_k = c^a_{bc} d^b_i d^c_j$$
$$\implies c'^k_{ij} = c^a_{bc} d^k_a d^b_i d^c_j\,.$$

The *exponential map* of a one-parameter subgroup G of a Lie algebra $\mathfrak{g}$ associates a continuous element of that subgroup $A(t) \in G$ to its Lie algebra *generator* $X \in \mathfrak{g}$ via

$$A(t) = e^{tX} = \sum_n \frac{(tX)^n}{n!}\,, \tag{A.4}$$

satisfying* for $X, Y \in \mathfrak{g}$, $A(t), B(t) \in G$

$$A(0) = e^0 = I \tag{A.5a}$$
$$A(t+s) = A(t)A(s)\,, \quad \forall t, s \in \mathbb{R} \tag{A.5b}$$
$$A^*(t) = (e^{tX})^* = e^{tX^*} \tag{A.5c}$$
$$A^{-1}(t) = (e^{tX})^{-1} = e^{-tX} \tag{A.5d}$$
$$A(t)B(t) = e^{tX}e^{tY} = e^{tX+tY}\,, \quad \text{if } [X, Y] = 0 \tag{A.5e}$$
$$||A(t)|| = ||e^{tX}|| = e^{t||X||} \tag{A.5f}$$
$$e^{tCXC^{-1}} = Ce^{tX}C^{-1}\,, \quad C \text{ invertible}\,, \tag{A.5g}$$

where $||\cdot||$ is the norm of a $n \times n$ matrix M defined as

*To not confuse the reader, we shall change the notation of the identity element e by I.

$$||M|| = \left(\sum_{i,j=1}^{n} |M_{ij}|^2 \right)^{\frac{1}{2}} .$$

In the case of noncommuting X and Y, with $[X,[X,Y]] = 0$, (A.5e) becomes the famous Baker-Campbell-Hausdorff formula

$$A(t)B(t) = e^{tX}e^{tY} = e^{tX+tY+\frac{t}{2}[X,Y]} . \tag{A.6}$$

Since the Lie algebra defines a smooth submanifold, then by virtue of being analytical everywhere, the exponential map will also define the same smooth submanifold, thus being differentiable. After that, we can finally link Lie groups to Lie algebras by

$$\left.\frac{d}{dt}e^{tX}\right|_{t=0} = X . \tag{A.7}$$

This strong link between Lie groups and algebras guarantees that most of the properties of Lie groups seen in Sec. A.1 have an algebraic analog, as we will see next. But first let us also list the Lie algebras associated with the most common groups

For a given *field* $\mathbb{F}$

i. The *General Linear Lie Algebra* $\mathfrak{gl}(n;\mathbb{F})$ of all invertible $n \times n$ matrices over $\mathbb{F}$ with $[X,Y] = XY - YX$ for $X, Y \in \mathfrak{gl}$.

ii. The *Special Linear Lie Algebra* $\mathfrak{sl}(n;\mathbb{F})$ of all $n \times n$ matrices X over $\mathbb{F}$ with $\operatorname{tr} X = 0$.

iii. The *Orthogonal Lie Algebra* $\mathfrak{o}(n;\mathbb{F})$ of all $n \times n$ matrices X over $\mathbb{F}$, which are anti-symmetric ($X^T = X^{-1}$)

iv. The *Special Orthogonal Lie Algebra* $\mathfrak{so}(n;\mathbb{F})$ of all $n \times n$ matrices X over $\mathbb{F}$, which are anti-symmetric ($X^T = X^{-1}$) and have $\operatorname{tr} X =$

0.

v. The *Unitary Lie Algebra* $\mathfrak{u}(n)$ of all the $n \times n$ complex matrices X with $X^\dagger = X^{-1}$.

vi. The *Special Unitary Lie Algebra* $\mathfrak{su}(n)$ of all the elements $X \in \mathfrak{su}(n)$ with $\operatorname{tr} X = 0$.

vii. The *Symplectic Lie Algebra* $\mathfrak{sp}(n;\mathbb{F})$ of all $2n \times 2n$ matrices X over $\mathbb{F}$ with $X^T J = -JX$, where $J := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$.

The elements of this algebra are of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

with A being an arbitrary $n \times n$ matrix and B and C arbitrary symmetric matrices.

viii. The *Lorentz Lie Algebra* $\mathfrak{o}(3,1)$ (or $\mathfrak{l}_{3,1}$) of all real 4×4 matrices X with the Lorentz inner product and $\hat{\eta} X^T \hat{\eta} = -X$.

A *Lie subalgebra* $\mathfrak{q} \subseteq g$ is a subset of elements in $\mathfrak{g}$ that is closed under the same multiplication operation $[\ ,\ ]$, that is, for $Q_1, Q_2 \in \mathfrak{q}$, then $[Q_1, Q_2] \in \mathfrak{q}$. For example $\mathfrak{su}(n) \subseteq \mathfrak{u}(n)$.

We say $\mathfrak{q} \subseteq \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if, for all $X \in \mathfrak{g}$ and for all $Y \in \mathfrak{q}$, $[X, Y] \subset \mathfrak{q}$ (symbolically $[\mathfrak{g}, \mathfrak{q}] \subset \mathfrak{q}$). Ideals are also called *invariant subalgebras* and are usually denoted by $\mathfrak{q} \triangleleft \mathfrak{g}$.

Algebraic *homomorphisms* are linear maps between algebras $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that the Lie brackets are preserved, so that, for $X, Y \in \mathfrak{g}$,

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]\,. \tag{A.8}$$

If the mapping is one-to-one, then it is called an *isomorphism*. On the other hand, if the mapping takes $\mathfrak{g}$ back into itself $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$, we say it is an *endomorphism*. The set of elements that map into 0, $\varphi^{-1}(0)$, is called the *kernel* of the homomorphism, denoted by $\ker\varphi$.

If an endomorphism is one-to-one, we call it an *automorphism* and it represents the *group of motions* of the space. In this context, the generators are called the *actions* or *motions* of the group.

A connected Lie group $\widetilde{G}$ that has a homomorphic map $\Phi : \widetilde{G} \longrightarrow G$ to a connected Lie group G is called a *covering group* if their associated Lie algebras are isomorphic by the map $\varphi : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$.

The *center* of $\mathfrak{g}$ is the set of elements $X \in \mathfrak{g}$ that commute with all the elements $Y \in \mathfrak{g}$, defined as

$$Z(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \quad \forall Y \in \mathfrak{g}\} . \tag{A.9}$$

The *centralizer* of $\mathfrak{g}$ with respect to $\mathfrak{s}$ are those that commute with all $Y \in \mathfrak{s}$

$$Z_{\mathfrak{s}}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \quad \forall Y \in \mathfrak{s}\} . \tag{A.10}$$

The special homomorphism of a Lie algebra $\mathfrak{g}$ onto a general vector space* V done by the linear map $\varphi(X) : V \longrightarrow V$, $X \in \mathfrak{g}$ (remember that a Lie algebra is, in fact, a vector space), so that $\varphi(aX+Y) = a\varphi(X) + \varphi(Y)$, is named a *representation of* $\mathfrak{g}$. A representation φ is *faithful* if the kernel of it is precisely 0, i.e., $\ker\varphi = 0$.

The most usual representation of a Lie algebra is the *adjoint* representation, denoted by ad_X, for $X \in \mathfrak{g}$, which takes $\mathfrak{g}$ back into itself, i.e. $\mathrm{ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g}$, such that

$$\mathrm{ad}_X Y := [X, Y] .$$

*Technically this homomorphism maps $\mathfrak{g}$ onto the general linear algebra $\mathfrak{gl}(V)$.

This implies that the adjoint with relation to the commutator satisfies

$$\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X \,, \tag{A.11}$$

where "$\circ$" is the composition symbol. The kernel of ad_X is precisely the center of $\mathfrak{g}$.

With that we can define the *Killing form* of $\mathfrak{g}$ as the symmetric bilinear form

$$\langle X, Y \rangle := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \tag{A.12}$$

for $X, Y \in \mathfrak{g}$, which is invariant under all automorphisms, that is, if φ is an automorphism

$$\langle \varphi(X), \varphi(Y) \rangle = \langle X, Y \rangle \,.$$

Sometimes the Killing form is denoted by $\kappa(X, Y)$.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two distinct Lie algebras, we denote a composition of both vector spaces by the pair (X, Y), for $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_2$. The *direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is then defined such that the bracket operation is done componentwise

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]) \,, \tag{A.13}$$

where we identify the components of each subalgebra $\mathfrak{g}_1 : (X, 0)$ and $\mathfrak{g}_2 : (0, Y)$. Moreover, both subalgebras are ideals of the direct sum ($\mathfrak{g}_1 \triangleleft (\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ and $\mathfrak{g}_2 \triangleleft (\mathfrak{g}_1 \oplus \mathfrak{g}_2)$), so they have null "intersection" $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$.

If a Lie algebra $\mathfrak{g}$ has an Abelian subalgebra, then we can always decompose it as

$$\mathfrak{g} = \mathfrak{g}_0 + Z(\mathfrak{g})$$

where $\mathfrak{g}_0$ is the centerless subalgebra comprised of all the remaining, non-commuting elements of $\mathfrak{g}$.

An algebra is called *semi-simple* if it has no Abelian ideals. Similarly, an algebra is called *simple* if it has no ideals at all, except for the trivial 0 and $\mathfrak{g}$.

The *derived* subalgebra $\mathfrak{g}'$ of the algebra $\mathfrak{g}$ is the ideal of the comutator, i.e. $\mathfrak{g}' \lhd [\mathfrak{g}, \mathfrak{g}]$, spanning through the subspace of all $[X, Y]$, $\forall X, Y \in \mathfrak{g}$. Specifically

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] \,.$$

The *derived series* is formed by successive derived subalgebras; commutators within commutators $\mathfrak{g}'' := [[X, Y], [A, B]], \forall X, Y, A, B \in \mathfrak{g}$. Thus, for the r-th order

$$\mathfrak{g}^{(r)} := [\mathfrak{g}^{(r-1)}, \mathfrak{g}^{(r-1)}] \,. \tag{A.14}$$

g is *solvable* if the derived series goes to 0 up to the r-th order, that is, in that order, we reach an Abelian subgroup. All orders of the derived series are ideals of $\mathfrak{g}$

$$\mathfrak{g}^{(r)} \lhd \mathfrak{g}^{(r-1)} \lhd \ldots \lhd \mathfrak{g}' \lhd \mathfrak{g} \,.$$

The *lower central series* is defined inductively by $\mathfrak{g}^1 := \mathfrak{g}'$

$$\begin{aligned}\mathfrak{g}^{r+1} &:= [\mathfrak{g}, \mathfrak{g}^r] \\ &:= [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^{r-1}]] \\ &:= [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, [\ldots [\mathfrak{g}, \mathfrak{g}] \ldots]]]] \,,\end{aligned} \tag{A.15}$$

or

$$\mathfrak{g}^r = \mathrm{ad}_{\mathfrak{g}}^r \, \mathfrak{g} = \underbrace{(\mathrm{ad}_{\mathfrak{g}} \ldots \mathrm{ad}_{\mathfrak{g}})}_{r \text{ times}} \, \mathfrak{g} \,,$$

and are also all ideals of $\mathfrak{g}$. We say g is *Nilpotent* if $g^r = 0$. If $\mathfrak{g}$ is Nilpotent, then it is also solvable.

The *radical* of $\mathfrak{g}$, $R(\mathfrak{g})$, is the ideal of $\mathfrak{g}$ that contains *all* solvable ideals, that is, it is the largest solvable ideal of $\mathfrak{g}$.

Finally, we define the *complexification* of a real Lie algebra $\mathfrak{g}$, denoted by $\mathfrak{g}_{\mathbb{C}}$ by composing a complex algebra as

$$Z = X_1 + iX_2 \;\;, \quad X_1, X_2 \in \mathfrak{g} \tag{A.16}$$

such that the bracket operation on $\mathfrak{g}$ has an unique extension to $\mathfrak{g}_{\mathbb{C}}$ as

$$\begin{aligned}[][Z_1, Z_2] &= [X_1 + iX_2, Y_1 + iY_2] \\ &= ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1])\end{aligned}$$

for $Z_1, Z_2 \in \mathfrak{g}_{\mathbb{C}}$ and $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$. The Jacobi identity evidently is preserved in this extension.

On the other hand, the *realification* or *de-complexification* $\mathfrak{g}_{\mathbb{R}}$ of a complex Lie algebra $\mathfrak{g}$ can be done by observing that an element $Z = A + iB$, $Z \in \mathfrak{g}$ where A, B are real matrices has a matrix representation

$$Z = \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right)$$

of double dimension, identifying who is A and B in that canonical form. This can be thought as a decomposition into the basis of Pauli matrices $Z = \sigma_0 \otimes A - i\sigma_2 \otimes B$, where $\otimes$ is the *Kroenecker product of matrices*, and $\sigma_0 \equiv \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

APPENDIX B

Formalism of Variations

The *Principle of Least Action* is one of the hallmarks of physics which enabled us to formulate and solve numerous mechanical systems of discrete particles in lagrangian and hamiltonian mechanics and, later, to extend the approach to infinite degrees of freedom, introducing the discipline of *Classical Theory of Fields*, thus giving a new brath to the study of the fundamental interations of Nature.

We define the *Action* functional I as a parametric integral of some function of parameter τ, the $N-1$ generalized coordinates q and its velocities $\dot{q}$, denoted by $L = L(q, \dot{q}; \tau)$ between the interval (τ_1, τ_2) in a $(N-1)$-dimensional space,

$$I[q] := \int_{\tau_1}^{\tau_2} d\tau \, L(q, \dot{q}; \tau) \,, \tag{B.1}$$

which define a *trajectory* between said interval in the configuration space defined by the generalized coordinates. L is called the *Lagrangian* of the system. It is also defined the operation of *variation* δ, which represents a small change of the object in question.

Figure 18: Of all possible trajectories, only one represents the actual physical trajectory: the one that exttrimizes the action I.

The principle of least action is stated as: *The physical trajectory of any physical system is such that the action is stationary, i.e.,* $\delta I = 0$.

This condition is sufficient to find the *equations of motion* for pretty much any physical system in its most general form, taking into account constraint forces and whatnot. As we will only be interested in a particular subset of problems that does not have such peculiarities*, we can safely impose the usual extra condition of *fixed endpoints*, that is,

$$\delta q(\tau_1) = 0 = \delta q(\tau_2)\,. \tag{B.2}$$

When this principle is applied to (B.1), using condition (B.2), we obtain the the famous set of $N - 1$ *Euler-Lagrange equations*

$$\sum_i \frac{\partial L}{\partial q^i} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0\,. \tag{B.3}$$

If we consider the Lagrangian in the continuum, the number of "discrete particles" (degrees of freedom) will go to infinity and the usual sum over the i particles will transform into a spatial integral over the whole space, effectively turning such functional into a *density* to be integrated. This limit allows us to rewrite (B.1) as

$$I[T] := \int_{\tau_1}^{\tau_2} d\tau \int_V d^{N-1}x\; \mathcal{L}(T, \partial_i T; \tau) \tag{B.4}$$

where $\mathcal{L}$ is the *Lagrangian Density*, function of some tensor *field* T and the parameter τ, the proper time. However, this definition is not invariant by a change of coordinates. Since there are $(N-1)$ differentials dx^i, to make it so, we have to take into account the Jacobian (1.1.13) that will take care of any coordinate transformation. With all that, (B.4) becomes

$$I[T] = \int_V d^N x\, \sqrt{g}\, \mathcal{L}(T, \partial_\mu T, x^\mu)\,, \tag{B.5}$$

*We will only briefly study one such case in a moment.

with $\mu = 0, 1, \dots, N-1$, $x^0 \equiv \tau$, and now explicitly turns out to be a function of the N coordinates. Furthermore, the condition of fixed endpoints (B.2) gets replaced by a null variation on the border ∂V, that is

$$\delta x^\mu \Big|_{\partial V} = 0 \,. \tag{B.6}$$

So, by taking that into account and applying the principle of least action, one gets the Euler-Lagrange equations for tensor fields

$$\frac{\partial \mathcal{L}}{\partial T^{\alpha\dots}{}_{\beta\dots}} - \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \frac{\partial \mathcal{L}}{\partial(\partial_\mu T^{\alpha\dots}{}_{\beta\dots})} \right) = 0 \,. \tag{B.7}$$

Now let us return to the main chain of thoughts.

Since we are interested in physical trajectories, (1.1.9) is the natural choice for the lagrangian L, because it already describes the path some particle partakes and it is invariant. Hence by choosing

$$\begin{aligned} L d\tau = ds &\equiv \sqrt{g_{ij} dx^i dx^j} \\ &= \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} d\tau \end{aligned}$$

$$\implies L = \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} \,, \tag{B.8}$$

and plugging it back into (B.3), remembering that $g_{ij} \dot{x}^i \dot{x}^j = c^2 \equiv 1$,

$$\frac{\partial L}{\partial \dot{x}^a} = g_{aj}\dot{x}^j$$
$$\rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = g_{aj}\ddot{x}^j + \frac{1}{2}\left(g_{aj,k} + g_{ak,j}\right)\dot{x}^j\dot{x}^k$$

and

$$\frac{\partial L}{\partial x^a} = \frac{1}{2}g_{ij,a}\dot{x}^i\dot{x}^j \, .$$

Therefore,

$$g_{aj}\ddot{x}^j + \frac{1}{2}\left(g_{aj,k} + g_{ak,j} - g_{ik,a}\right)\dot{x}^j\dot{x}^k = 0$$

which multiplied by g^{ia} gives

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \, .$$

This is precisely the geodesic equation (1.1.47) derived in the absence of constraint forces, confirming that the shortest trajectory possible one particle describes in free-fall on a Riemannian manifold is indeed the geodesic curve.

Now, to properly formulate the Euler-Lagrange equations for gravitational fields, we first need to choose a good candidate for the Lagrangian and assure that it satisfies the desirable conditions of invariance, which has to depend on the fundamental quantity in question, the metric tensor itself, of course. So by taking variations of it and imposing the variational principle,

$$\frac{\delta I}{\delta g} = 0 \, , \tag{B.9}$$

we shall get the field equations we are looking for. One such invariant candidate is precisely the Ricci scalar R (1.1.53), which by construction depends solely on the metric and its derivatives, and is of class C^2, it is inherently invariant by virtue of it being a scalar and it is actually a *density* just like the Lagrangian density. We then propose the following

$$\mathcal{L} = R + \alpha \mathcal{L}_M \,, \tag{B.10}$$

where α is a constant and $\mathcal{L}_M$ the Lagrangian density of matter to be determined later on.

Expression (B.5) gets broken down into two separate actions to be minimized,

$$I = I_G + I_M \,; \tag{B.11}$$

the *geometrical* term

$$I_G[g] = \int_V d^N x \, \sqrt{g} \, R \,, \tag{B.12}$$

called *Einstein-Hilbert action*, and the generic *matter* term to be determined

$$I_M[g] = \alpha \int_V d^N x \, \sqrt{g} \, \mathcal{L}_M \,, \tag{B.13}$$

are both functionals of $g_{\mu\nu}$. Let us first address the matter term. Applying a variation to it, we get

$$
\begin{aligned}
\delta I_M &= \alpha\delta \int_V d^N x \sqrt{g}\, \mathcal{L}_M \\
&= \alpha \int_V d^N x \left(\frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu,\sigma}} \delta g_{\mu\nu,\sigma} \right) \\
&= \alpha \int_V d^N x \left[\frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\sigma} \left(\frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu,\sigma}} \right) \right] \delta g_{\mu\nu} \\
&\qquad + \frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu}} \underbrace{\delta g_{\mu\nu} \Big|_{\partial V}}_{=0} \\
&= \alpha \int_V d^N x \sqrt{g}\, T^{\mu\nu} \delta g_{\mu\nu} \,, \qquad \text{(B.14)}
\end{aligned}
$$

where we define the clearly symmetric energy-momentum tensor by

$$
\sqrt{g}\, T^{\mu\nu} := \frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\sigma} \left(\frac{\partial \sqrt{g}\, \mathcal{L}_M}{\partial g_{\mu\nu,\sigma}} \right) . \qquad \text{(B.15)}
$$

Now, for I_G,

$$
\begin{aligned}
\delta I_G &= \delta \int_V d^N x \sqrt{g}\, R \\
&= \int_V d^N x\, \delta(\sqrt{g}\, R) \,,
\end{aligned}
$$

we need to elaborate a wee bit more. The variation in question has the following form

$$
\delta(\sqrt{g}\, R) = \delta(\sqrt{g}\, g^{\mu\nu} R_{\mu\nu}) = \delta(\sqrt{g}\, g^{\mu\nu}) R_{\mu\nu} + \sqrt{g}\, g^{\mu\nu} \delta R_{\mu\nu} \,,
$$

so more terms will pop out from it and shall be pre-computed next:

$$\begin{aligned}\delta g &= \delta(\det \mathbb{G}) \\ &= \delta e^{\log\det\mathbb{G}} \\ &= e^{\operatorname{tr}\log\mathbb{G}}\delta(\operatorname{tr}\log\mathbb{G}) \\ &= g g^{\mu\nu}\delta g_{\mu\nu}\,,\end{aligned}$$

$$\begin{aligned}\delta\delta^{\mu}_{\nu} \equiv 0 &= \delta(g^{\mu\lambda}g_{\lambda\nu}) \\ &= (\delta g^{\mu\lambda})g_{\lambda\nu} + g^{\mu\lambda}(\delta g_{\lambda\nu})\end{aligned}$$

$$\rightarrow \delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\,.$$

Plugging everything back together yields

$$\begin{aligned}\delta(\sqrt{g}\; g^{\mu\nu})R_{\mu\nu} &= \Big[(\delta\sqrt{g})g^{\mu\nu} + \sqrt{g}(\delta g^{\mu\nu})\Big]R_{\mu\nu} \\ &= -\sqrt{g}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\delta g_{\alpha\beta}\,. \end{aligned} \tag{B.16}$$

It remains to be shown that $\delta R_{\mu\nu}$ does not contribute in anything, that is, it can be put on a total derivative so that it vanishes by virtue of (B.6). To demonstrate this, let us again consider a geodesic frame of reference, so that

$$
\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\delta(\Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu}) \\
&= g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu,\lambda} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu,\lambda} \\
&= \frac{\partial}{\partial x^{\lambda}}\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu}\right) \\
&= \frac{\partial w^{\lambda}}{\partial x^{\lambda}} \,,
\end{aligned}
$$

where we temporarily defined the vector w^{λ} as the expression inside the brackets to save us some ink. This represents de divergence of w^{λ} in the geodesic frame, but since we are dealing with tensor equations, the result is valid in all frames given that we use the generalized divergence instead. So,

$$
g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{g}w^{\lambda}\right),
$$

and, when going back to the action integral, we have

$$
\int_V d^N x\,\sqrt{g}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\lambda}}\left(\sqrt{g}w^{\lambda}\right)\right) = \left(\sqrt{g}w^{\lambda}\right)\Big|_{\partial V} \equiv 0\,,
$$

since

$$
w^{\lambda} \propto \delta\Gamma^{\lambda}_{\mu\nu} \propto \delta g_{\mu\nu,\kappa} \propto \frac{\partial}{\partial x^{\kappa}}\delta g_{\mu\nu} = 0\,, on \partial V\,.
$$

Then, the variation of the geometric action becomes

$$
\delta I_G = \int_V d^N x\,\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu}\,. \tag{B.17}
$$

Finally, putting all together and applying the least action principle on the full action (B.11),

$$\delta I = 0 = \int_V d^N x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} + \alpha T^{\mu\nu} \right) \delta g_{\mu\nu}$$

$$\implies R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \alpha T^{\mu\nu} \,, \tag{B.18}$$

which are the *Einstein field equations* in the presence of matter. The only thing left behind was to determine α such that we recover the classic non-relativistic Newton's gravity law

$$\nabla^2 \Phi = 4\pi G \rho_M \,, \tag{B.19}$$

where ρ_M is the density of matter and Φ is the *static* gravitational potential. To do that we first have to note that (B.18) can be put in the form

$$R^{\mu\nu} = \alpha \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right) , \tag{B.20}$$

by simply contracting said equation with $g_{\mu\nu}$* in order to determine R and feed it back to it.

The energy-momentum tensor carries the sources that generate the field. Now it suffices to say that the only relevant component in the non-relativistic regime will be T^{00}, which represents the rest-mass density

$$T_{00} = \rho_M$$
$$T_{ij} \approx 0 \,.$$

*Remember that $g_{\mu\nu} g^{\mu\nu} = \mathrm{tr}\mathbb{1} = 4$ in the (3 + 1)-dimensional spacetime.

In this regime, we still have the almost "Minkowskian" metric $g_{\mu\nu} \longrightarrow \mathrm{diag}(g_{00}, -1, -1, -1)$, so that when returned into (B.20), it gives rise to the only non-null Christoffel symbols, namely,

$$\Gamma^k_{00} = \frac{1}{2} g_{00}{}^{,k}$$
$$\Gamma^0_{0k} = \frac{1}{2} g^{00} g_{00,k} \; .$$

Here we are only interested in the the dominant terms contained in the Ricci tensor; for the $_{00}$ component we have

$$R_{00} \approx \Gamma^k_{00,k} - \cancel{\Gamma^k_{0k,0}}$$
$$\approx \frac{1}{2} g_{00}{}^{,k}{}_{,k} \; ,$$

where all the other terms have a time derivative ∂_0 which vanishes for static fields. This laplacian of g_{00} is still mysterious. To determine it, we return to the geodesic equation (1.1.47) in the classical limit, where

$$u^\mu = (1, v) \, ; \quad v^i \ll 1 \, .$$

So, for spatial components

$$\begin{aligned} 0 &= \frac{dv^k}{d\tau} + \Gamma^k_{\alpha\beta} u^\alpha u^\beta \\ &= \frac{dv^k}{d\tau} + \Gamma^k_{00} u^0 u^0 + 2 \underbrace{\Gamma^k_{0k}}_{=0} u^0 u^k + \Gamma^k_{ij} \underbrace{u^i u^j}_{\approx 0} \\ &= \frac{dv^k}{d\tau} + \Gamma^k_{00} \\ &= \frac{dv^k}{d\tau} - \frac{1}{2} g_{00}{}^{,k} \, . \end{aligned}$$

Since $\boldsymbol{F} = -m\nabla\Phi$, we have

$$\Phi = -\frac{1}{2}g_{00} ,$$

so that

$$R_{00} = -\Phi^{,k}{}_{,k} .$$

Finally, going back to the Einstein equation,

$$\begin{aligned}
R_{00} &= \alpha\left(T_{00} - \frac{1}{2}g_{00}T\right) \\
-\Phi^{,k}{}_{,k} &= \alpha\left(T_{00} - \frac{1}{2}\underbrace{g_{00}g^{00}}_{=1}T_{00}\right) \\
&= \alpha\frac{1}{2}T_{00}
\end{aligned}$$

$$\begin{aligned}
\rightarrow -4\pi G\rho_M &= \frac{\alpha}{2}\rho_M \\
\therefore\ \alpha &= -8\pi G .
\end{aligned}$$

Now we are able to write the full Einstein equation:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G T^{\mu\nu} , \tag{B.21}$$

or

$$R^{\mu\nu} = -8\pi G\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right) . \tag{B.22}$$

APPENDIX C
Conformal Group

In the following years after the publication of Eintsein's Special Relativity paper,[11] Minkowski studying electromagnetic theory, first associated spacetime symmetries with those of the Poincaré group (Lorentz transformation plus translations), which made up the maximal symmetry of the 4−dimensional spacetime. Yet, two years later, in 1910, both Bateman[1] and Cunningham[8] independently found out that electromagnetic phenomema (viz. the Maxwell equations) have five other additional symmetries, extending the associated group to a *fifteen parameter* group. Those symmetries correspond to scale invariance and the so called *conformal acceleration*, a uniform acceleration that leave physical theories invariant. The former transformation is a one-parameter transformation whereas the latter corresponds to four-parameter transformations. This brand new symmetry group is denominated the *Special Conformal group*.

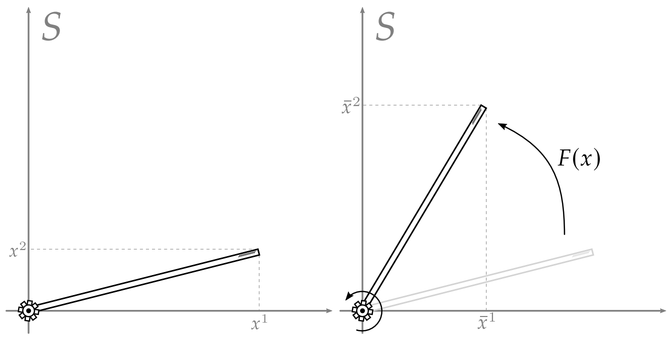
Conformal symmetry is a powerful tool in the study of gravitational effects in the weak field approximation, to the lowest orders of perturbation theory, aiming at infering the next to leading order contributions by setting the renormalization group of equations and then to establish a path towards quantum gravity. Leaving the open problems aside, let us develop the main results of this theory according to Witten.[20]

First, let us begin exploring a rather elementary example. Suppose that we have a rigid metal rod attached to a machine gear by one of its edges, first at rest with a laboratory S and described by a set of coordinates $\{x\}$ (Fig. 19a). When we turn the machine on, the rod starts rotating *relative to the laboratory* and is still described by the $\{x\}$-coordinates by the transformation (Fig. 19)*

$$\bar{x} \to F(x)\,, \tag{C.1}$$

for some F and $\bar{x}$ still in S.

*The space maps back into itself $F : \mathcal{M} \to \mathcal{M}$, i.e., an automorphism.



(a) Initial setup (b) Engine on

Figure 19: The initial conditions consist of a turned off machine on the laboratory frame S (left); then the engine is turned on and its coordinates according to S begin to change through $F(x)$ (right).

Now suppose that we leave the engine turned off and spin the laboratory itself instead, effectively defining a new frame of reference S' described by $\{x'\}$-coordinates. The rod will the start rotating relative to this frame by means of a *change of coordinates* of the form (Fig. 20)*

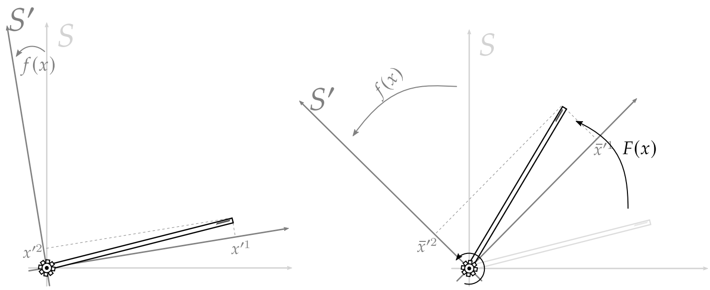
$$x' = f(x)\,, \tag{C.2}$$

for some f and for $x \in S$ and $x' \in S'$.

Finally, let us suppose a mixed situation where the motor is turned on (so (C.1) is in place) and we rotate the laboratory in a way to *cancel out* its effects, i.e., rotate it such that the rod is at rest in S'. In this case, we will have a relationship between f and F in the desired manner (Fig. 20b),

$$\bar{x}' = F(x') = F(f(x)) \equiv x\,, \tag{C.3}$$

*A one-to-one mapping between two spaces $f : \mathcal{M} \to \mathcal{M}'$, i.e., an isomorphism.



(a) Change of frame of reference (b) Motor turned and change of coordinates

Figure 20: Instead of turning the apparus on, we rotate the laboratory itself, defining a spinning frame S' where the bar is rotating relative to it according to $f(x)$ (left); finally we compose both transformations in order to cancel out the effects of one another $F(f(x))$ (right).

where this is *only* valid for this change of frames of reference ($S \to S'$) and it receives a special equality symbol

$$\bar{x}' \doteq x \tag{C.4}$$

to emphasize it. We shall call this a *point equality*. Thus, for this system of coordinates, F and f are inverses to one another.

A transformation in the molds of (C.1) is called an *active* or *point transformation*. On the other hand, a *passive* or simply a *coordinate transformation* is the one where (C.2) takes place. Lastly, transformations like (C.3) does not have a particular name in this form, but will later be called *conformal transformations* when it satisfies some other conditions.

This example lays the groundwork to properly define conformal transformations, essential to work with conformal groups.

Generally speaking, conformal transformations are, by definition, transformations that leave angles invariant. Since this is the only condition to make up this kind of transformation, we are free to change the *magnitude* of the objects under it. This subtle consequence is of utmost importance. So, the angle between two displacements* dx^μ and δx^μ,

$$\cos\theta := \frac{g_{\mu\nu}(x)dx^\mu \delta x^\nu}{\sqrt{g_{\alpha\beta}(x)dx^\alpha dx^\beta}\sqrt{g_{\alpha\beta}(x)\delta x^\alpha \delta x^\beta}},$$

is left invariant whereas its magnitude is not necessarely preserved

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x)\,. \tag{C.5}$$

When plugged back into the angle expression above, we see that the transformation (C.5) does indeed preserve angles for $\Omega \neq 0$. These type of transformations only makes sense when we transform the system itself. In other words, those are *active* transformations (we do not expect a mere change of coordinates to change the magnitude of the objects).

Thus, in this framework, passive and active transformations of the metric tensors are respectively given by†

$$g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'\beta} = h_{\alpha\beta}(x') \tag{C.6}$$

$$g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}\beta} = \Omega^2(\bar{x}) g_{\alpha\beta}(\bar{x})\,. \tag{C.7}$$

The latter is called a *Weyl point transformation* or just *Weyl transformation*. Now like in our little example above, we can finally

*Here we use two different symbols for the differentials just to explicit two different displacements.

†We distinguish the passively transformed metric tensor by $h_{\alpha\beta}$ to remove any sources of confusion. It then becomes evident that this kind of transformation is just a change between spaces.

define a *covariant* conformal transformation by successively applying a Weyl transformation (C.7) and then changing the frame of reference to undo it by means of a passive transformation (C.6). Thus, from (C.3)

$$\bar{x}' \doteq x \implies \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \doteq \frac{\partial \bar{x}^\mu}{\partial \bar{x}'^\alpha} \,. \tag{C.8}$$

Evaluating (C.6) on $\bar{x}$ and using (C.8) and (C.7), one gets

$$h_{\alpha\beta}(\bar{x}') \doteq \Omega^2(x) g_{\alpha\beta}(x) \,. \tag{C.9}$$

With that effect, the line element *is no longer invariant* under conformal transformations

$$ds^2(\bar{x}') \doteq \Omega^2(x) ds^2(x) \,. \tag{C.10}$$

This shows what we have been discussing above: lengths can be changed under conformal transformations. The notation $\bar{x}'$ is rather cumbersome, so we shall be using an upper ^{c} to denote conformal quantities. For example, (C.9) reduces to

$$g^c_{\alpha\beta} = \Omega^2 g_{\alpha\beta} \,,$$

evaluated at the same point in the context of point equality (C.4).

Conversely, it is immediate from orthogonality relations that

$$(g^c)^{\alpha\beta} = \frac{1}{\Omega^2} g^{\alpha\beta} \,,$$

and that the determinant of the metric tensor

$$g := \epsilon^{\alpha\beta\dots} g_{0\alpha} g_{1\beta} \dots$$

for an arbitrary $N-$dimensional spacetime transforms as

$$g \to \Omega^{2n} g \,, \tag{C.11}$$

where n is also called the *conformal weight*. We shall be considering a $4-$spacetime henceforward.

We have seen on (C.9) how the metric tensor transforms under conformal transformations, but what about the other relevant objects of differential geometry, namely the Christoffel symbols, covariant derivatives, geodesics, the Riemann tensor and Ricci tensor and also its scalar? Do they also transform like the metric tensor? Do they stay invariant? The answer is no, those objects are not invariant under such transformations, but knowing how the basic building block transforms (i.e. the metric tensor itself), we are able to find out how all those objects transform and, then, propose a new affine connection that leave them all invariant.

By directly substituting (C.9) into the Christoffel symbol definition (1.1.35), one obtains

$$(\Gamma^c)^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} + \left(\delta^{\mu}_{\alpha} s_{\beta} + \delta^{\mu}_{\beta} s_{\alpha} - g_{\alpha\beta} s^{\mu}\right) \,, \tag{C.12}$$

where $s_{\mu} := \partial_{\mu} \ln \Omega$. With that in our hands, we can evaluate the objects in question:

- Covariant derivatives (1.1.32)

$$(\nabla_{\nu} v^{\mu})^c = \nabla_{\nu} v^{\mu} + \left(\delta^{\mu}_{\nu} v^{\lambda} s_{\lambda} + v^{\mu} s_{\nu} - v_{\nu} s^{\mu}\right) \,. \tag{C.13}$$

- Geodesic equation (1.1.47)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \left(2s_\lambda\frac{dx^\lambda}{d\tau}\frac{dx^\mu}{d\tau} - s^\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right) = 0\,. \tag{C.14}$$

✦ Riemann tensor (1.1.50)

$$\begin{aligned}(R^c)^\mu{}_{\nu\alpha\beta} = R^\mu{}_{\nu\alpha\beta} + \Big[&\delta^\mu_\alpha s_\nu s_\beta - \delta^\mu_\beta s_\nu s_\alpha \\ &+ g_{\nu\beta}s^\mu s_\alpha - g_{\nu\alpha}s^\mu s_\beta \\ &+ g_{\nu\alpha}\left(\delta^\mu_\beta s_\lambda s^\lambda + \partial_\beta s^\mu\right) \\ &- g_{\nu\beta}\left(\delta^\mu_\alpha s_\lambda s^\lambda + \partial_\alpha s^\mu\right)\Big]\,.\end{aligned} \tag{C.15}$$

✦ Ricci tensor (1.1.52)

$$R^c_{\mu\nu} = R_{\mu\nu} + 2\left(s_\mu s_\nu - \partial_\mu s_\nu\right) - g_{\mu\nu}\left(2s_\lambda s^\lambda + \partial_\lambda s^\lambda\right)\,. \tag{C.16}$$

✦ Ricci scalar

For this one we contract (C.16) with the conformal metric $(g^c)^{\mu\nu}$, obtaining

$$R^c = \frac{R}{\Omega^2} - \frac{6}{\Omega^2}\left(s_\lambda s^\lambda + \partial_\lambda s^\lambda\right)\,. \tag{C.17}$$

- Einstein tensor (1.1.59)

$$G^c_{\mu\nu} = G_{\mu\nu} + 2\left(s_\mu s_\nu - \partial_\mu s_\nu\right) + g_{\mu\nu}\left(s_\lambda s^\lambda + 2\partial_\lambda s^\lambda\right) . \tag{C.18}$$

We see that Einstein equations *are not* invariant under conformal transformations. In fact, none of the curvature objects are. Furthermore, all those transformations are function of the logarithmic derivative $s_\mu = \partial_\mu \ln\Omega$, so when it vanishes (constant scale factor $\Omega(x) = \Omega = \text{const}$), we fallback to the usual objects as expected. Moreover, the new term that appears in the geodesic equation (C.14) is associated with a constant acceleration (inertial force if you will), and it is called the *conformal acceleration*. This, in turn, can be interpreted as a constant gravitational field in the light of the equivalence principle.

Despite that, one can easily make all the above objects invariant under conformal transformations simply by re-defining the affine connection such that the spare terms in (C.12) vanish. Instead of being the Christoffel symbols, the new connection is coined as the *Weyl connection*. This is a constructive process, so we first define

$$(\Gamma^c)^\mu_{\alpha\beta} := \Gamma^\mu_{\alpha\beta} - \left(\delta^\mu_\alpha \kappa_\beta + \delta^\mu_\beta \kappa_\alpha - g_{\alpha\beta}\kappa^\mu\right) \tag{C.19}$$

incorporating the desired result, with κ_μ transforming as

$$\kappa^c_\mu = \kappa_\mu + s_\mu \qquad ; \qquad (\kappa^c)^\mu = \frac{1}{\Omega^2}\left(\kappa_\mu + s_\mu\right) . \tag{C.20}$$

Since this makes up a new affine connection, the covariant derivative also changes

$$v^\mu{}_{|\nu} \equiv \nabla^c_\nu v^\mu = \partial_\nu v^\mu + (\Gamma^c)^\mu_{\nu\lambda} v^\lambda , \tag{C.21}$$

where the symbol ${}_{|\nu} \equiv \nabla^c_\nu$ is the indicial notation for the Weyl covariant derivative. One immediate consequence is that (1.1.44) is no longer valid,

$$g_{\mu\nu|\lambda} = 2\kappa_\lambda g_{\mu\nu} \,. \tag{C.22}$$

It only remains to find out what k_μ is. For that, we contract (C.19) in $\mu\beta$

$$\Gamma^c_\alpha \equiv (\Gamma^c)^\mu_{\alpha\mu} = \partial_\alpha \ln \sqrt{|g|} - 4\kappa_\alpha \,.$$

From the covariant derivative of the determinant of the metric,

$$\nabla^c_\mu g = \partial_\mu g - 2g\Gamma^c_\mu \implies \Gamma^c_\mu = \partial_\mu \ln \sqrt{|g|} - \nabla^c_\mu \ln \sqrt{|g|} \,,$$

and substituting it in the above expression,

$$\kappa_\mu = \frac{1}{4}\nabla^c_\mu \ln \sqrt{|g|} \,, \tag{C.23}$$

we finally have the desired quantity that also transforms as expected (C.20):

$$\begin{aligned}
\kappa^c_\mu &= \frac{1}{4}\nabla^c_\mu \ln(\Omega^4 \sqrt{|g|}) \\
&= \frac{1}{4}\nabla^c_\mu \ln \sqrt{|g|} + \nabla^c_\mu \ln \Omega \\
&= \kappa_\mu + s_\mu \,.
\end{aligned}$$

Established the functional form of the Weyl connection, we can rewrite* all the objects listed above which are now conformally invariant

*Effectively we only need to change $s \to -\kappa$ in all of the expressions.

✦ Riemann tensor

$$(R^c)^{\mu}{}_{\nu\alpha\beta} = R^{\mu}{}_{\nu\alpha\beta} + \Big[\delta^{\mu}_{\alpha}\kappa_{\nu}\kappa_{\beta} - \delta^{\mu}_{\beta}\kappa_{\nu}\kappa_{\alpha} + g_{\nu\beta}\kappa^{\mu}\kappa_{\alpha} - g_{\nu\alpha}\kappa^{\mu}\kappa_{\beta} + g_{\nu\alpha}\left(\delta^{\mu}_{\beta}\kappa_{\lambda}\kappa^{\lambda} - \partial_{\beta}\kappa^{\mu}\right) - g_{\nu\beta}\left(\delta^{\mu}_{\alpha}\kappa_{\lambda}\kappa^{\lambda} - \partial_{\alpha}\kappa^{\mu}\right)\Big] . \quad \text{(C.24)}$$

✦ Ricci tensor

$$R^c_{\mu\nu} = R_{\mu\nu} + 2\left(\kappa_{\mu}\kappa_{\nu} + \partial_{\mu}\kappa_{\nu}\right) - g_{\mu\nu}\left(2\kappa_{\lambda}\kappa^{\lambda} - \partial_{\lambda}\kappa^{\lambda}\right) . \quad \text{(C.25)}$$

✦ Ricci scalar

$$R^c = R - 6\left(\kappa_{\lambda}\kappa^{\lambda} - \partial_{\lambda}\kappa^{\lambda}\right) . \quad \text{(C.26)}$$

✦ Einstein tensor

$$G^c_{\mu\nu} = G_{\mu\nu} + 2\left(\kappa_{\mu}\kappa_{\nu} + \partial_{\mu}\kappa_{\nu}\right) + g_{\mu\nu}\left(\kappa_{\lambda}\kappa^{\lambda} - 2\partial_{\lambda}\kappa^{\lambda}\right) . \quad \text{(C.27)}$$

Since the Jacobian is not conformally invariant, the Einstein-Hilbert action (B.12)

$$I_G[g] = \int_V d^N x \sqrt{|g|}\, R$$

is also not invariant, so we need to construct a new action. This can be done if we define an action of the form

$$\begin{aligned} I[C] &:= \int_V d^N x \sqrt{|g|}\, C_{\mu\nu\sigma\rho} C^{\mu\nu\sigma\rho} \\ &= \int_V d^N x \sqrt{|g|}\, C^{\alpha}{}_{\nu\sigma\rho} C^{\mu}{}_{\beta\gamma\delta} g_{\alpha\mu} g^{\beta\nu} g^{\gamma\sigma} g^{\delta\rho}\,, \end{aligned} \tag{C.28}$$

where C is a tensor with all the properties of R and is conformally invariant, and those metric tensors cancel out all the spare factors that appear under a conformal transformation. It only remains to construct C.

For it to hold the conformal symmetry, we require its first contraction to vanish ($C^{\mu}{}_{\alpha\beta\mu} \equiv 0$), inheriting all the indicial symmetries of the curvature tensor, through the ansatz

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + A_{\mu\alpha} g_{\nu\beta} + A_{\nu\beta} g_{\mu\alpha} - A_{\mu\beta} g_{\nu\alpha} - A_{\nu\alpha} g_{\mu\beta}$$

where $A_{\mu\nu}$ is a symmetric rank 2 tensor to be determined. Contracting $\mu\beta$ and *imposing* $C_{\nu\alpha} = 0$,

$$C^{\mu}_{\nu\alpha\mu} \equiv C_{\nu\alpha} = 0 \implies \frac{1}{2}\Big(R_{\nu\alpha} - g_{\nu\alpha} A\Big)\,.$$

Further contracting the remaining indices,

$$A = \frac{1}{6} R\,,$$

and substituting in the expression above gives

$$A_{\nu\alpha} = \frac{1}{2} R_{\nu\alpha} - \frac{1}{12} g_{\nu\alpha} R$$

and, finally,

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{2}\left(R_{\mu\alpha}g_{\nu\beta} + R_{\mu\beta}g_{\nu\alpha} - R_{\nu\alpha}g_{\mu\beta} - R_{\nu\beta}g_{\mu\alpha}\right) - \frac{1}{6}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right)R\,. \tag{C.29}$$

This tensor is the so-called *Weyl tensor* and it holds all the desired symmetries, leaving (C.28) invariant under conformal transformations.

Conformal Group *C*

The set of all conformal transformations that takes $x \longrightarrow x'$ and leaves the metric tensor invariant up to a scalar factor as seen in (C.9) constitutes the *conformal group* C. In the case where such transformation equip Minkowski space with a point dependent metric

$$g^{c}_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu}\,,$$

that is, the point dependency is all inside the conformal factor $\Omega(x) = e^{\omega}(x)$, we define the *special conformal group* C_0.

As briefly discussed in the beginning of this appendix, the Conformal group is an extension of the Poincaré group. This is promptly seen if we take a unitary scale factor $\Omega(x) = 1$, so all the transformations above falls back into the usual forms and are invariant under general Poincaré transformations.

Now, as usual, we shall find the infinitesimal Killing vectors of the conformal transformations, so we consider a conformal transformation (C.9) under $x \longrightarrow x' = x + \varepsilon\xi(x)$ for $\varepsilon \ll 1$

$$
\begin{aligned}
\Omega^2 g_{\mu\nu} &= \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{\alpha\beta} \\
&= \left(\delta^{\alpha}_{\mu} + \varepsilon \xi^{\alpha}{}_{,\mu}\right) \left(\delta^{\beta}_{\nu} + \varepsilon \xi^{\beta}{}_{,\nu}\right) g_{\alpha\beta} \\
&= g_{\mu\nu} + \varepsilon \left(\xi_{\mu;\nu} + \xi_{\mu;\nu}\right) + O(\varepsilon^2) \,.
\end{aligned}
$$

Rewriting the scale factor as $\Omega^2(x) = \exp(2\omega(x)) \approx 1 + 2\varepsilon\omega(x)$ for $\varepsilon \ll 1$, we obtain the conformal Killing equation

$$
\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2\omega g_{\mu\nu} \,, \tag{C.30}
$$

and its trace

$$
\xi^{\alpha}{}_{;\alpha} = 4\omega \,. \tag{C.31}
$$

Assuming that we are on the locally inertial frame of reference ($g \rightarrow \eta$ and $; \rightarrow ,$) and deriving (C.30) once again, we get

$$
\begin{aligned}
\xi_{\mu,\nu,\rho} + \xi_{\nu,\mu,\rho} &= 2\eta_{\mu\nu}\omega_{,\rho} \,, \\
\xi_{\nu,\rho,\mu} + \xi_{\rho,\nu,\mu} &= 2\eta_{\nu\rho}\omega_{,\mu} \,, \\
\xi_{\rho,\mu,\nu} + \xi_{\mu,\rho,\nu} &= 2\eta_{\rho\mu}\omega_{,\nu} \,,
\end{aligned}
$$

which can be put in the form

$$
\xi_{\rho,\mu,\nu} = \eta_{\rho\nu}\omega_{,\mu} + \eta_{\mu\rho}\omega_{,\nu} - \eta_{\mu\nu}\omega_{,\rho} \,, \tag{C.32}
$$

corresponding to

$$
\partial^2 \xi_{\rho} = -2\omega_{,\rho} \,. \tag{C.33}
$$

Next, we derive (C.33) with respect to ρ and apply a d'Alembertian operator to (C.31)

$$\begin{aligned} 4\partial^2\omega &= \partial^2\xi^\alpha{}_{,\alpha} \\ -2\partial^2\omega &= \partial^2\xi^\alpha{}_{,\alpha} \end{aligned} \quad \Longrightarrow \quad \partial^2\omega(x) = 0\,, \tag{C.34}$$

so ω must be linear in x,

$$\omega(x) = A + B_\mu x^\mu\ .$$

Since ξ is related to ω by one derivative, according to (C.30), we infer that it must be of quadratic order,

$$\xi_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\alpha\beta}x^\alpha x^\beta\,, \tag{C.35}$$

given $c_{\mu\alpha\beta} = c_{\mu\beta\alpha}$. We proceed to further determine the constants by comparison with a Taylor expansion

$$\xi_\mu = a_\mu + \xi_{\mu,\nu}x^\nu + \xi_{\mu,\alpha,\beta}x^\alpha x^\beta\ .$$

Substituting (C.31) into (C.30),

$$\begin{aligned} \xi_{\mu,\nu} + \xi_{\nu,\mu} &= \frac{1}{2}\xi^\alpha{}_{,\alpha}\eta_{\mu\nu} \\ \Longrightarrow\ b_{\mu\nu} + b_{\nu\mu} &= \frac{1}{2}b^\alpha{}_\alpha\eta_{\mu\nu}\ . \end{aligned}$$

The simplest solution is given by the decomposition of b into symmetric and anti-symmetric parts

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + \lambda_{\mu\nu}\,, \tag{C.36}$$

where α is a constant infinitesimal dilation parameter and $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ is related to Lorentz transformations. For the remaining quadratic term, we contract (C.32) in $\mu\rho$,

$$\omega_{,\beta} = \frac{1}{4} c^{\alpha}{}_{\alpha\beta} \equiv b_{\beta} \,, \tag{C.37}$$

so that

$$c_{\mu\alpha\beta} = b_{\alpha}\eta_{\mu\beta} + b_{\beta}\eta_{\alpha\mu} - \eta_{\alpha\beta}b_{\mu} \,. \tag{C.38}$$

With all that, the infinitesimal Killing vectors (C.35) become

$$\xi^{\mu} = a^{\mu} + \alpha x^{\mu} + \lambda^{\mu}{}_{\nu}x^{\nu} + 2b_{\alpha}x^{\alpha}x^{\mu} - b^{\mu}x^{2} \,. \tag{C.39}$$

Here the term a^{μ} represents an infinitesimal translation, the α parameter an infinitesimal dilation, the $\lambda^{\mu}{}_{\nu}$ are the infinitesimal Lorentz transformations and $2b_{\alpha}x^{\alpha}x^{\mu} - b^{\mu}x^{2}$ represents the four components of the infinitesimal conformal acceleration.

The finite transformations are, respectively,

$$\begin{aligned} x'^{\mu} &= x^{\mu} + a^{\mu} \,, \\ x'^{\mu} &= \alpha x^{\mu} \,, \\ x'^{\mu} &= \Lambda^{\mu}{}_{\nu}x^{\nu} \,, \\ x'^{\mu} &= \frac{x^{\mu} - b^{\mu}\boldsymbol{x}^{2}}{1 - \boldsymbol{b}\cdot\boldsymbol{x} + \boldsymbol{x}^{2}\boldsymbol{b}^{2}} \,. \end{aligned} \tag{C.40}$$

This set of transformations comprise the conformal group C_0, which is isomorphic to the special ortogonal group $\mathrm{SO}(5,1)$ of fifteen parameters, composed of four translation generators P_{μ}, six of rotations and boosts $M_{\mu\nu}$, one of dilation D and four related to the conformal acceleration K_{μ}. Thus, for a general function $\varphi(x)$, an infinitesimal conformal transformation is given by

$$\begin{aligned}\varphi(x) \to \varphi(x+\xi(x)) \\ &= \left(1 + ia^{\mu}P_{\mu} - \alpha D + \frac{1}{2}\lambda^{\mu\nu}M_{\mu\nu} + ib^{\mu}K_{\mu}\right)\varphi(x)\,,\end{aligned} \tag{C.41}$$

where the generators

$$\begin{aligned} P_{\mu} &= -i\partial_{\mu}\,, \\ D &= -x^{\mu}\partial_{\mu}\,, \\ M_{\mu\nu} &= i(x^{\mu}\partial_{\nu} - x^{\nu}\partial_{\mu})\,, \\ K_{\mu} &= 2ix_{\mu}x^{\nu}\partial_{\nu} - ix^2\partial_{\mu}\,, \end{aligned} \tag{C.42}$$

satisfy the Lie brackets

$$\begin{aligned} [D, P_{\mu}] &= P_{\mu}\,, \\ [D, K_{\mu}] &= -K_{\mu}\,, \\ [K_{\mu}, P_{\nu}] &= 2\eta_{\mu\nu}D - 2iM_{\mu\nu}\,, \\ [K_{\rho}, M_{\mu\nu}] &= i(\eta_{\rho\mu}K_{\nu} - \eta_{\rho\nu}K_{\mu})\,, \\ [P_{\rho}, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_{\nu} - \eta_{\rho\nu}P_{\mu})\,, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right)\,. \end{aligned} \tag{C.43}$$

APPENDIX D

Bi-vectors

From the basic Linear Algebra, we learn the fundamental concept of vector spaces, which is the natural home of entities called vectors that satisfy a set of axioms pretty similar to those of Group Theory, which are, for three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ of a vector space V, endowed with addiction and the multiplication by scalars, and for two scalars a and b

$$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u} \qquad \text{(commutativity)}\,, \tag{D.1a}$$
$$\boldsymbol{v}+(\boldsymbol{u}+\boldsymbol{w})=(\boldsymbol{v}+\boldsymbol{u})+\boldsymbol{w} \qquad \text{(associativity)}\,, \tag{D.1b}$$
$$\boldsymbol{u}+\boldsymbol{0}=\boldsymbol{u} \qquad \text{(identity)}\,, \tag{D.1c}$$
$$\boldsymbol{u}+(-\boldsymbol{u})=\boldsymbol{0} \qquad \text{(inverse)}\,, \tag{D.1d}$$
$$a(\boldsymbol{v}+\boldsymbol{u})=a\boldsymbol{v}+a\boldsymbol{u} \qquad \text{(scalar distributivity)}\,, \tag{D.1e}$$
$$(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u} \qquad \text{(distributivity of scalars)}\,, \tag{D.1f}$$
$$(ab)\boldsymbol{u}=a(b\boldsymbol{u}) \qquad \text{(scalar associativity)}\,, \tag{D.1g}$$
$$1\boldsymbol{u}=\boldsymbol{u} \qquad \text{(scalar identity)}\,. \tag{D.1h}$$

It is also seen two multiplicative operations *between* vectors, the *scalar product* (also called *dot product* or *inner product*), a bi-linear operation that takes two vectors into a number "·": $V \times V \longrightarrow \mathbb{R}$, and the *vector product* (or *cross product*), an anti-commutative bi-linear operation that takes two vectors into another "×": $V \times V \longrightarrow V$. The former allow us to define the *norm* (which is also called the magnitude or length) of a vector by

$$||\boldsymbol{u}|| := \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}} \tag{D.2}$$

and the latter defines a vector perpendicular to the plane defined by two vectors *only in a* 3*−dimensional space*

$$\boldsymbol{w}=\boldsymbol{u}\times\boldsymbol{v}\,; \qquad \boldsymbol{w}\perp\boldsymbol{u}\,,\boldsymbol{w}\perp\boldsymbol{v}\,, \tag{D.3}$$

where its norm

$$||\boldsymbol{u} \times \boldsymbol{v}|| = ||u||\,||v|| \sin\theta \tag{D.4}$$

defines the area of the parallelogram delimited by both vectors and θ is the angle between them. Also, the anti-symmetry defines an orientation for the vector space.

Nevertheless, the vector product *does not* enjoy certain desirable qualities. For instance, it cannot be defined in two dimensions (the orthogonal would pop out into a higher dimentions) and perpendicular vectors are only uniquely defined in three dimensions. The notion of a *bi-vector* emerges to address these issues.

A *bi-vector* is defined by means of a new product of vectors, the *outer product* (or *exterior product*) which is defined as a bi-linear operation that takes two vectors into a bi-vector* "$\wedge$": $V \times V \longrightarrow Bi$ and it has the following properties, for $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w} \in V$,

$$\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u} \qquad \text{(anti-symmetry)} \tag{D.5a}$$

$$\boldsymbol{u} \wedge (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \wedge \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{w} \qquad \text{(distributivity)}\ . \tag{D.5b}$$

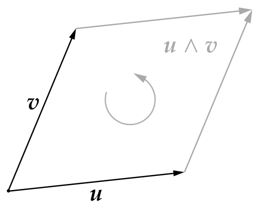


Figure 21: Geometrical representation of the outer product and its orientation.

*The "$\wedge$" (wedge) symbol is used to denote this multiplication and Bi is the vector space containing all the bi-vectors.

Those objects possess a natural way to encode geometrical information. More precisely, we can associate each bi-vector to the *oriented* plane defined by $\boldsymbol{u}$ and $\boldsymbol{v}$, where the orientation can be seen by sweeping one vector onto the other. The bi-vector space can be interpreted as the *set of all planes in the actual space*.

Now, if we take both $\boldsymbol{u}$ and $\boldsymbol{v}$ and decompose them into a $N-$dimensional basis $\{\boldsymbol{e}_i\}$, for $i = 1, \ldots, N$, then

$$\boldsymbol{u} \wedge \boldsymbol{v} = u^i v^j \boldsymbol{e}_i \wedge \boldsymbol{e}_j = U^{(ij)} \boldsymbol{E}_{(ij)} \,, \tag{D.6}$$

where the product $U^{(ij)} := u^i v^j$ is necessarily anti-symmetric and

$$\boldsymbol{E}_{(ij)} := \boldsymbol{e}_i \wedge \boldsymbol{e}_j \tag{D.7}$$

form the $\frac{1}{2}N(N-1)$-dimensional basis of the bi-vector space on the anti-symmetric pairs (ij), consisting of primitive planes defined by the basis vectors $\{\boldsymbol{e}_i\}$. We introduce a new family of indices A that runs through all the possible $\frac{1}{2}N(N-1)$ combinations of the pairs (ij)

$$\begin{aligned} A := \{&(01), (02), \ldots, (0N), \\ &(12), (13), \ldots, (1N), \\ &\vdots \\ &(N1), (N2), \ldots, (N\,N-1)\} \,, \end{aligned}$$

such that

$$\boldsymbol{u} \wedge \boldsymbol{v} = U^A E_A \,. \tag{D.8}$$

To consolidate all these abstract concepts, we finalize by considering a few examples in two, three and four dimensions.

Example: Two dimensions

In this case we have only two basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$:

$$\begin{aligned}\boldsymbol{u} \wedge \boldsymbol{v} &= u^i v^j \boldsymbol{e}_i \wedge \boldsymbol{e}_j \\ &= (u^1 v^2 - u^2 v^1)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \equiv U^1 \boldsymbol{E}_1\end{aligned}$$

with

$$\begin{aligned}U^1 &\equiv (u^1 v^2 - u^2 v^1) \\ \boldsymbol{E}_1 &\equiv \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \,.\end{aligned}$$

We see that $\frac{1}{2}N(N-1)$ for $N = 2$ really gives only one bi-vector, as expected.

Example: Three dimensions

Next, for $N = 3$, we have three basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

$$\begin{aligned}\boldsymbol{u} \wedge \boldsymbol{v} &= u^i v^j \boldsymbol{e}_i \wedge \boldsymbol{e}_j \\ &= (u^1 v^2 - u^2 v^1)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \\ &\qquad + (u^2 v^3 - u^3 v^2)\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \\ &\qquad + (u^3 v^1 - u^1 v^3)\boldsymbol{e}_3 \wedge \boldsymbol{e}_1 \\ &\equiv U^A \boldsymbol{E}_A \,,\end{aligned}$$

where $A = 1, 2, 3$ and

$$U^A \equiv \begin{cases} (u^1 v^2 - u^2 v^1)\,, & A = 1 = (12) \\ (u^2 v^3 - u^3 v^2)\,, & A = 2 = (23) \\ (u^3 v^1 - u^1 v^3)\,, & A = 3 = (31) \end{cases}$$

$$\boldsymbol{E}_A \equiv \begin{cases} \boldsymbol{e}_1 \wedge \boldsymbol{e}_2\,, & A = 1 = (12) \\ \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\,, & A = 2 = (23) \\ \boldsymbol{e}_3 \wedge \boldsymbol{e}_1\,, & A = 3 = (31) \end{cases} \quad .$$

Example: Four dimensions

Since we have not imposed anything about metric signatures in this formalism, we will consider a basis containing a ^{0} component $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. We make this choice instead of the usual all-spatial vector spaces to illustrate the quantities we dealt with in Chapter 4, but rest assured that this formalism is valid *for all* kinds of vector spaces, which includes the Minkowski case $\mathbb{M}^4$ of the aforementioned chapter. Thus, we have six primitive basis planes and

$$\begin{aligned} \boldsymbol{u} \wedge \boldsymbol{v} &= u^i v^j \boldsymbol{e}_i \wedge \boldsymbol{e}_j \\ &= (u^0 v^1 - u^1 v^0)\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \\ &\qquad + (u^0 v^2 - u^2 v^0)\boldsymbol{e}_0 \wedge \boldsymbol{e}_2 \\ &\qquad + (u^0 v^3 - u^3 v^0)\boldsymbol{e}_0 \wedge \boldsymbol{e}_3 \\ &\qquad + (u^1 v^2 - u^2 v^1)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \\ &\qquad + (u^2 v^3 - u^2 v^3)\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \\ &\qquad + (u^3 v^1 - u^3 v^1)\boldsymbol{e}_3 \wedge \boldsymbol{e}_1 \\ &\equiv U^A \boldsymbol{E}_A\,, \end{aligned}$$

where evidently

$$U^A \equiv \begin{cases} (u^0 v^1 - u^1 v^0)\,, & A = 0 = (01) \\ (u^0 v^2 - u^2 v^0)\,, & A = 1 = (02) \\ (u^0 v^3 - u^3 v^0)\,, & A = 2 = (03) \\ (u^1 v^2 - u^2 v^1)\,, & A = 3 = (12) \\ (u^2 v^3 - u^2 v^3)\,, & A = 4 = (23) \\ (u^3 v^1 - u^3 v^1)\,, & A = 5 = (31) \end{cases} ,$$

$$\boldsymbol{E}_A \equiv \begin{cases} \boldsymbol{e}_0 \wedge \boldsymbol{e}_1\,, & A = 0 = (01) \\ \boldsymbol{e}_0 \wedge \boldsymbol{e}_2\,, & A = 1 = (02) \\ \boldsymbol{e}_0 \wedge \boldsymbol{e}_3\,, & A = 2 = (03) \\ \boldsymbol{e}_1 \wedge \boldsymbol{e}_2\,, & A = 3 = (12) \\ \boldsymbol{e}_2 \wedge \boldsymbol{e}_3\,, & A = 4 = (23) \\ \boldsymbol{e}_3 \wedge \boldsymbol{e}_1\,, & A = 5 = (31) \end{cases} .$$

The formulation of the outer product generalization to arbitrary dimensions, as seen in the examples above, alongside with the already established inner product, enabled Grassmann to devise a whole new branch of algebra, the *Geometric Algebra*,[10,22,25,35] mostly centered around yet another kind of product, the *geometric product*, that is nothing but adding together both inner and outer products.

While not taking flight at first, Grassmann's pioneering works were the rudiments for the theory of exterior algebra and, later on, Clifford Algebra, no less, the latter being nowadays the backbone of many fundamental theories, such as the Quantum Field Theory.

APPENDIX E

Conformal Bianchi Symmetry – Explicit calculations

In this appendix we open up all the calculations involved in the demonstration of the conformal symmetry for Bianchi spaces, discussed in 3.1. There, we found out that

$$\frac{\partial x'^j}{\partial x^i} = \frac{\Omega^2(x)}{\Omega^2(x')} e^{(a)}{}_i(x)\, e_{(a)}{}^j(x')\,, \tag{E.1}$$

similar to what is done in the usual process. Enforcing the integrability conditions

$$\frac{\partial^2 x'^j}{\partial x^i \partial x^k} = \frac{\partial^2 x'^j}{\partial x^k \partial x^i}\,, \tag{E.2}$$

we get

$$
\begin{aligned}
\frac{\partial}{\partial x^i}\left(\frac{\partial x'^j}{\partial x^k}\right) &= \frac{\partial}{\partial x^i}\left[\left(\frac{\Omega(x)}{\Omega(x')}\right)^2 e_{(a)}{}^j(x') e^{(a)}{}_k(x)\right] \\
&= \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left[\frac{\partial e_{(a)}{}^j}{\partial x^i}(x') e^{(a)}{}_k(x) + e_{(a)}{}^j(x')\frac{\partial e^{(a)}{}_k}{\partial x^i}(x)\right] \\
&\quad + \frac{\partial}{\partial x^i}\left(\frac{\Omega(x)}{\Omega(x')}\right)^2 e_{(a)}{}^j(x') e^{(a)}{}_k(x) \\
&= \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left[\frac{\partial e_{(a)}{}^j}{\partial x'^m}(x')\frac{\partial x'^m}{\partial x^i} e^{(a)}{}_k(x) + e_{(a)}{}^j(x')\frac{\partial e^{(a)}{}_k}{\partial x^i}(x)\right] \\
&\quad + 2\left(\frac{\Omega(x)}{\Omega(x')}\right)\left[\frac{1}{\Omega(x')}\frac{\partial \Omega}{\partial x^i}(x)\right. \\
&\qquad \left. - \frac{\Omega(x)}{\Omega^2(x')}\frac{\partial \Omega}{\partial x^i}(x')\right] e_{(a)}{}^j(x') e^{(a)}{}_k(x) \\
&= \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left\{\left[\left(\frac{\Omega(x)}{\Omega(x')}\right)^2 e_{(b)}{}^m(x') e^{(b)}{}_i(x)\right]\frac{\partial e_{(a)}{}^j}{\partial x'^m}(x') e^{(a)}{}_k(x)\right. \\
&\qquad \left. + e_{(a)}{}^j(x')\frac{\partial e^{(a)}{}_k}{\partial x^i}(x)\right\} \\
&\quad + 2\left(\frac{\Omega(x)}{\Omega(x')}\right)\left[\frac{1}{\Omega(x')}\frac{\partial \Omega}{\partial x^i}(x)\right. \\
&\qquad \left. - \frac{\Omega(x)}{\Omega^2(x')}\frac{\partial x'^m}{\partial x^i}\frac{\partial \Omega}{\partial x'^m}(x')\right] e_{(a)}{}^j(x') e^{(a)}{}_k(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \Bigg\{ \frac{\partial e^{(a)}{}_k}{\partial x^i}(x) e_{(a)}{}^j(x') \\
&\qquad\qquad + \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \frac{\partial e_{(a)}{}^j}{\partial x'^m}(x') e_{(b)}{}^m(x') e^{(b)}{}_i(x) e^{(a)}{}_k(x) \Bigg\} \\
&\quad + 2\Bigg\{ \frac{\Omega(x)}{\Omega^2(x')} \frac{\partial \Omega}{\partial x^i}(x) - \frac{\Omega(x)}{\Omega(x')} \frac{\Omega(x)}{\Omega^2(x')} \left[\left(\frac{\Omega(x)}{\Omega(x')}\right)^2 e_{(b)}{}^m(x') e^{(b)}{}_i(x) \right] \cdot \\
&\qquad \cdot \frac{\partial \Omega}{\partial x'^m}(x') \Bigg\} e_{(a)}{}^j(x') e^{(a)}{}_k(x) \\
&= \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \Bigg\{ \frac{\partial e^{(a)}{}_k}{\partial x^i}(x) e_{(a)}{}^j(x') \\
&\qquad\qquad + \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \frac{\partial e_{(a)}{}^j}{\partial x'^m}(x') e_{(b)}{}^m(x') e^{(b)}{}_i(x) e^{(a)}{}_k(x) \\
&\qquad\qquad + 2 \frac{1}{\Omega(x)} \frac{\partial \Omega}{\partial x^i}(x) e_{(a)}{}^j(x') e^{(a)}{}_k(x) \\
&\qquad\qquad - 2 \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \frac{1}{\Omega(x')} \frac{\partial \Omega}{\partial x'^m}(x') \cdot \\
&\qquad\qquad\qquad \cdot e_{(b)}{}^m(x') e^{(b)}{}_i(x) e_{(a)}{}^j(x') e^{(a)}{}_k(x) \Bigg\}
\end{aligned}
$$

$$\rightarrow \frac{\partial}{\partial x^i}\left(\frac{\partial x'^j}{\partial x^k}\right) = \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left\{\left[\frac{\partial e^{(a)}{}_k}{\partial x^i}(x) + \frac{2}{\Omega(x)}\frac{\partial \Omega}{\partial x^i}(x) e^{(a)}{}_k(x)\right] e_{(a)}{}^j(x')\right.$$
$$+ \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left[\frac{\partial e_{(a)}{}^j}{\partial x'^m}(x') - \frac{2}{\Omega(x')}\frac{\partial \Omega}{\partial x'^m}(x') e_{(a)}{}^j(x')\right] \cdot$$
$$\left. \cdot\, e_{(b)}{}^m(x') e^{(b)}{}_i(x) e^{(a)}{}_k(x)\right\} \qquad \text{(i)}$$

$$\rightarrow \frac{\partial}{\partial x^k}\left(\frac{\partial x'^j}{\partial x^i}\right) = \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left\{\left[\frac{\partial e^{(a)}{}_i}{\partial x^k}(x) + \frac{2}{\Omega(x)}\frac{\partial \Omega}{\partial x^k}(x) e^{(a)}{}_i(x)\right] e_{(a)}{}^j(x')\right.$$
$$+ \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left[\frac{\partial e_{(a)}{}^j}{\partial x'^m}(x') - \frac{2}{\Omega(x')}\frac{\partial \Omega}{\partial x'^m}(x') e_{(a)}{}^j(x')\right] \cdot$$
$$\left. \cdot\, e_{(b)}{}^m(x') e^{(b)}{}_k(x) e^{(a)}{}_i(x)\right\}$$
$$= \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left\{\left[\frac{\partial e^{(a)}{}_i}{\partial x^k}(x) + \frac{2}{\Omega(x)}\frac{\partial \Omega}{\partial x^k}(x) e^{(a)}{}_i(x)\right] e_{(a)}{}^j(x')\right.$$
$$+ \left(\frac{\Omega(x)}{\Omega(x')}\right)^2 \left[\frac{\partial e_{(b)}{}^j}{\partial x'^m}(x') - \frac{2}{\Omega(x')}\frac{\partial \Omega}{\partial x'^m}(x') e_{(b)}{}^j(x')\right] \cdot$$
$$\left. \cdot\, e_{(a)}{}^m(x') e^{(a)}{}_k(x) e^{(b)}{}_i(x)\right\} \qquad \text{(ii)}$$

Equating both terms

$$
\begin{aligned}
\text{(i)} = \text{(ii)} \quad & \left[\frac{\partial e^{(a)}{}_k}{\partial x^i}(x) - \frac{\partial e^{(a)}{}_i}{\partial x^k}(x) \right. \\
& \left. + \frac{2}{\Omega(x)} \left(\frac{\partial \Omega}{\partial x^i}(x) e^{(a)}{}_k(x) - \frac{\partial \Omega}{\partial x^k}(x) e^{(a)}{}_i(x) \right) \right] e_{(a)}{}^j(x') \\
= & \left(\frac{\Omega(x)}{\Omega(x')} \right)^2 \left[\frac{\partial e_{(b)}{}^j}{\partial x'^m}(x') e_{(a)}{}^m(x') - \frac{\partial e_{(a)}{}^j}{\partial x'^m}(x') e_{(b)}{}^m(x') \right. \\
& - \frac{2}{\Omega(x')} \left(\frac{\partial \Omega}{\partial x'^m}(x') e_{(b)}{}^j(x') e_{(a)}{}^m(x') \right. \\
& \left. \left. - \frac{\partial \Omega}{\partial x'^m}(x') e_{(a)}{}^j(x') e_{(b)}{}^m(x') \right) \right] \cdot \\
& \cdot e^{(b)}{}_i(x) e^{(a)}{}_k(x)
\end{aligned}
$$

and multiplying by $\frac{1}{\Omega^2(x)} e_{(c)}{}^k(x) e_{(f)}{}^i(x) e^{(e)}{}_j(x')$

$$
\underbrace{\begin{aligned}
& \frac{1}{\Omega^2(x)} \left[\frac{\partial e^{(e)}{}_k}{\partial x^i}(x) - \frac{\partial e^{(e)}{}_i}{\partial x^k}(x) \right. \\
& \qquad \left. + \frac{2}{\Omega(x)} \left(\frac{\partial \Omega}{\partial x^i}(x) e^{(e)}{}_k(x) - \frac{\partial \Omega}{\partial x^k}(x) e^{(e)}{}_i(x) \right) \right] e_{(c)}{}^k(x) e_{(f)}{}^i(x)
\end{aligned}}_{\text{Function only of } x}
$$

$$
=
$$

$$
\underbrace{\begin{aligned}
& \frac{1}{\Omega^2(x')} \left[\frac{\partial e_{(f)}{}^j}{\partial x'^m}(x') e_{(c)}{}^m(x') - \frac{\partial e_{(c)}{}^j}{\partial x'^m}(x') e_{(f)}{}^m(x') \right. \\
& \qquad \left. - \frac{2}{\Omega(x')} \frac{\partial \Omega}{\partial x'^m}(x') \left(e_{(f)}{}^j(x') e_{(c)}{}^m(x') - e_{(c)}{}^j(x') e_{(f)}{}^m(x') \right) e^{(e)}{}_j(x') \right]
\end{aligned}}_{\text{Function only of } x'}
$$

$$\therefore\ \frac{1}{\Omega^2}\left[\frac{\partial e^{(a)}{}_k}{\partial x^i} - \frac{\partial e^{(a)}{}_i}{\partial x^k} + \frac{2}{\Omega}\left(\frac{\partial \Omega}{\partial x^i} e^{(a)}{}_k - \frac{\partial \Omega}{\partial x^k} e^{(a)}{}_i\right)\right] e_{(b)}{}^k e_{(c)}{}^i \equiv C^a{}_{bc}\ . \tag{E.3}$$

Now, the commutator of

$$Y_a := \frac{1}{\Omega^2} e_{(a)}{}^m \partial_m\ , \tag{E.4}$$

results in

$$\begin{aligned}
[Y_a, Y_b] &= \left(\frac{1}{\Omega^2} e_{(a)}{}^i \partial_i\right)\left(\frac{1}{\Omega^2} e_{(b)}{}^j \partial_j\right) - [a \leftrightarrow b] \\
&= \frac{1}{\Omega^4} e_{(a)}{}^i \partial_i \left(e_{(b)}{}^j \partial_j\right) + \frac{1}{\Omega^2} \partial_i \left(\frac{1}{\Omega^2}\right) e_{(a)}{}^i e_{(b)}{}^j \partial_j - [a \leftrightarrow b] \\
&= \frac{1}{\Omega^4} e_{(a)}{}^i e_{(b)}{}^j \cancel{\partial_i \partial_j} + \frac{1}{\Omega^4} e_{(a)}{}^i e_{(b)}{}^j{}_{,i} \partial_j \\
&\qquad\qquad - \frac{2}{\Omega^2} \frac{\Omega_{,i}}{\Omega^3} e_{(a)}{}^i e_{(b)}{}^j \partial_j - \cancel{[a \leftrightarrow b]} \\
&= \frac{1}{\Omega^2}\left\{\frac{1}{\Omega^2}\left[e_{(a)}{}^k e_{(b)}{}^j{}_{,k} - e_{(b)}{}^k e_{(a)}{}^j{}_{,k}\right.\right. \\
&\qquad\qquad \left.\left. - 2\frac{\Omega_{,k}}{\Omega}\left(e_{(a)}{}^k e_{(b)}{}^j - e_{(b)}{}^k e_{(a)}{}^j\right)\right]\right\} \partial_j \\
&= \frac{1}{\Omega^2}\left\{\frac{1}{\Omega^2}\left[e_{(a)}{}^k e_{(b)}{}^j{}_{,k} - e_{(b)}{}^k e_{(a)}{}^j{}_{,k}\right.\right. \\
&\qquad\qquad \left.\left. + 2\frac{\Omega_{,k}}{\Omega}\left(e_{(a)}{}^j e_{(b)}{}^k - e_{(a)}{}^k e_{(b)}{}^j\right)\right]\right\} \partial_j \\
&= \frac{1}{\Omega^2} C^c{}_{ab} e_{(c)}{}^j \partial_j \\
&= C^c{}_{ab} Y_c\ .
\end{aligned} \tag{E.5}$$

References

[1] H. Bateman. The transformation of the electrodynamical equations. *Proceedings of the London Mathematical Society*, s2-8(1):223–264, 1910.

[2] V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov. Oscillatory Approach to the Singular Point in Relativistic Cosmology. *Soviet Physics Uspekhi*, 13:745, July 1970.

[3] C. L. Bennett, D. Larson, J. L. Weiland, N. Jarosik, G. Hinshaw, N. Odegard, K. M. Smith, R. S. Hill, B. Gold, M. Halpern, E. Komatsu, M. R. Nolta, L. Page, D. N. Spergel, E. Wollack, J. Dunkley, A. Kogut, M. Limon, S. S. Meyer, G. S. Tucker, and E. L. Wright. Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results. *The Astrophysical Journal Supplement Series*, 208:20, Oct. 2013.

[4] L. Bianchi. On the Three-Dimensional Spaces Which Admit a Continuous Group of Motions. *General Relativity and Gravitation*, 33:2171–2253, Dec. 2001.

[5] M. Blau. Lecture notes on general relativity. 01 2000.

[6] H. Bondi and T. Gold. The Steady-State Theory of the Expanding Universe. *Monthly Notices of the Royal Astronomical Society*, 108:252, Jan. 1948.

[7] C. Brans and R. H. Dicke. Mach's Principle and a Relativistic Theory of Gravitation. *Physical Review*, 124:925–935, Nov. 1961.

[8] E. Cunningham. The principle of relativity in electrodynamics and an extension thereof. *Proceedings of the London Mathematical Society*, s2-8(1):77–98, 1910.

[9] W. de Boer. Grand unified theories and supersymmetry in particle physics and cosmology. *Progress in Particle and Nuclear Physics*, 33:201–301, Jan. 1994.

[10] C. Doran and A. Lasenby. *Geometric Algebra for Physicists*. 2003.

[11] A. Einstein. Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, 322:891–921, Jan. 1905.

[12] A. Einstein. Die Grundlage der allgemeinen Relativitätstheorie. *Annalen der Physik*, 354:769–822, Jan. 1916.

[13] L. P. Eisenhart. *Riemannian Geometry*. Princeton University Press, 1997.

[14] G. F. R. Ellis and M. A. H. MacCallum. A class of homogeneous cosmological models. *Communications in Mathematical Physics*, 12:108–141, June 1969.

[15] G. F. R. Ellis and H. van Elst. Cosmological Models (Cargèse lectures 1998). In M. Lachièze-Rey, editor, *NATO Advanced Science Institutes (ASI) Series C*, volume 541 of *NATO Advanced Science Institutes (ASI) Series C*, pages 1–116, Jan. 1999.

[16] F. B. Estabrook, H. D. Wahlquist, and C. G. Behr. Dyadic analysis of spatially homogeneous world models. *Journal of Mathematical Physics*, 9:497–504, Jan. 1968.

[17] A. Fowler-Wright. *The Classification of Three-dimensional Lie Algebras*. PhD thesis, The University of Warwick, 4 2014.

[18] I. Friedrichsen, James Edward. *Quantization of Bianchi type cosmological models*. PhD thesis, The University of Texas at Austin, Jan. 2000.

[19] Y. Fujiwara, H. Ishihara, and H. Kodama. Comments on closed Bianchi models. *Classical and Quantum Gravity*, 10:859–867, May 1993.

[20] T. Fulton, F. Rohrlich, and L. Witten. Conformal Invariance in Physics. *Reviews of Modern Physics*, 34:442–457, July 1962.

[21] P. Ginsparg. Applied Conformal Field Theory. *arXiv e-prints*, pages hep–th/9108028, Nov 1988.

[22] H. Grassmann and L. Society. *Ausdehnungslehre*. History of Mathematics, 19. American Mathematical Soc.

[23] A. H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23:347–356, Jan. 1981.

[24] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Springer, 2004.

[25] D. Hestenes. *Clifford algebra to geometric calculus : a unified language for mathematics and physics*. D. Reidel Distributed in the U.S.A. and Canada by Kluwer Academic Publishers, Dordrecht Boston Hingham, MA, U.S.A, 1984.

[26] E. Hubble. A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae. *Proceedings of the National Academy of Science*, 15:168–173, Mar. 1929.

[27] F. Iachello. *Lie Algebras and Applications (Lecture Notes in Physics)*. Springer, 2006.

[28] R. T. Jantzen. Spatially Homogeneous Dynamics - a Unified Picture. In *Ganow Cosmology*, page 62, Jan. 1986.

[29] E. Kasner. Geometrical theorems on einstein's cosmological equations. *American Journal of Mathematics*, 43(4):217–221, 1921.

[30] I. M. Khalatnikov and E. M. Lifshitz. General Cosmological Solution of the Gravitational Equations with a Singularity in Time. *Physical Review Letters*, 24:76–79, Jan. 1970.

[31] H. Kodama. Canonical Structure of Locally Homogeneous Systems on Compact Closed 3-Manifolds of Types E^3, Nil and Sol. *Progress of Theoretical Physics*, 99:173–236, Feb. 1998.

[32] M. Lachieze-Rey and J. Luminet. Cosmic topology. *Physical Reports*, 254:135–214, Mar. 1995.

[33] L. D. Landau and E. Lifshitz. *The Classical Theory of Fields: Volume 2 (Course of Theoretical Physics Series)*. Butterworth-Heinemann, 1980.

[34] A. D. Linde. *Particle physics and inflationary cosmology*. 1990.

[35] P. Lounesto. *Clifford algebras and spinors*. Cambridge University Press, Cambridge New York, 2001.

[36] M. A. H. MacCallum. A class of homogeneous cosmological models III: Asymptotic behaviour. *Communications in Mathematical Physics*, 20:57–84, Mar. 1971.

[37] H. Minkowski. Die grundgleichungen für die elektromagnetischen vorgänge in bewegten körpern. *Mathematische Annalen*, 68(4):472–525, Dec 1910.

[38] C. W. Misner. Mixmaster Universe. *Physical Review Letters*, 22:1071–1074, May 1969.

[39] C. W. Misner. Quantum Cosmology. I. *Physical Review*, 186:1319–1327, Oct. 1969.

[40] G. L. Naber. *The Geometry of Minkowski Spacetime*. 1992.

[41] J. V. Narlikar. *Introduction to Cosmology*. Cambridge University Press, 1993.

[42] H. C. Ohanian and R. Ruffini. *Gravitation and Spacetime (Second Edition)*. W. W. Norton & Company, 1994.

[43] L. O'Raifeartaigh. *Group Structure of Gauge Theories (Cambridge Monographs on Mathematical Physics)*. Cambridge University Press, 1986.

[44] R. Penrose. A spinor approach to general relativity. *Annals of Physics*, 10:171–201, June 1960.

[45] R. Penrose and W. Rindler. *Spinors and space-time. Vol. 1: Two-spinor calculus and relativistic fields.* 1984.

[46] A. A. Penzias and R. W. Wilson. A Measurement of Excess Antenna Temperature at 4080 Mc/s. *Astrophysical Journal*, 142:419–421, July 1965.

[47] A. Z. Petrov. *Einstein spaces*. 1969.

[48] A. Z. Petrov. Editor's Note: The Classification of Spaces Defining Gravitational Fields. *General Relativity and Gravitation*, 32:1661–1664, Aug. 2000.

[49] A. Z. Petrov. The Classification of Spaces Defining Gravitational Fields. *General Relativity and Gravitation*, 32:1665–1685, Aug. 2000.

[50] F. A. Pirani. Invariant Formulation of Gravitational Radiation Theory. *Physical Review*, 105:1089–1099, Feb. 1957.

[51] W. Rossmann. *Lie Groups: An Introduction through Linear Groups (Oxford Graduate Texts in Mathematics)*. Oxford University Press, 2002.

[52] R. K. Sachs. Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time. *Proceedings of the Royal Society of London Series A*, 270:103–126, Oct. 1962.

[53] H. Samelson. *Notes on Lie Algebras (Universitext).* Springer, 1990.

[54] P. Scott. The geometries of 3-manifolds. *Bulletin of the London Mathematical Society*, 15(5):401–487, sep 1983.

[55] A. A. Starobinskii. *Spectrum of relict gravitational radiation and the early state of the universe*, pages 767–770. 1996.

[56] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt. *Exact Solutions of Einstein's Field Equations (Cambridge Monographs on Mathematical Physics).* Cambridge University Press, 2003.

[57] H. Stephani and H. Stephani. *Relativity : an introduction to special and general relativity*. 2004.

[58] A. H. Taub. Empty Space-Times Admitting a Three Parameter Group of Motions. *General Relativity and Gravitation*, 36:2699–2719, Dec. 2004.

[59] R. M. Wald. *General Relativity*. University of Chicago Press, 1984.

[60] S. Walter. *Minkowski, Mathematicians, and the Mathematical Theory of Relativity*, volume 7, page 45. Birkäuser, 1999.

[61] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. John Wiley & Sons, Inc., 1972.

[62] A. S. Wightman. Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields. In M. Lévy, editor, *High Energy Electromagnetic Interactions and Field Theory*, page 171, Jan. 1967.

[63] L. Witten. Geometry of Gravitation and Electromagnetism. *Physical Review*, 115:206–214, July 1959.

[64] V. D. Zakharov and R. N. Sen. *Gravitational waves in Einstein's theory.* 1973.