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Free actions on separated graph C*-algebras and generalized Gross-Tucker theorem

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Orientador: Prof. Dr. Alcides Buss

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O presente trabalho em nível de doutorado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

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Florianópolis, 14 de agosto de 2020.

This work is dedicated to my dear parents, colleagues and friends.

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RESUMO

Dada uma ação livre de um grupo discreto sobre um grafo dirigido existe uma relação entre o grafo e o grafo produto skew através do teorema de Gross-Tucker e, além disso, há certos teoremas de dualidade envolvendo suas respectivas C*-álgebras. Neste trabalho mostramos que é possível estender alguns destes resultados para uma classe mais geral: as C*-álgebras de grafos separados. Através de uma abordagem diferente da usual definimos a C*-algebra reduzida de um grafo separado e, dessa forma, estendemos alguns dos teoremas de dualidade vistos até então para esta classe. Finalmente, como aplicação obtemos resultados similares para as C*-álgebras "mansas" de grafos separados.

Palavras-chave: Grafo produto skew. Teorema de Gross-Tucker. Fibrados de Fell. Coações de grupos discretos. C*-algebra de grafos separados. C*-álgebra reduzida de grafos separados. C*-álgebra "mansa" de grafos separados.

RESUMO EXPANDIDO

Introdução

As C*-álgebras de grafos tem sido instrumento de estudo em diversas áreas da matemática, particularmente em K-teoria. Nos últimos anos C*-álgebras de grafos mais gerais, como o caso de C*-álgebras de grafos separados, tem fornecido exemplos e contra-exemplos para muitos campos de pesquisa. O interessante nessa classe específica é que, diferentemente das C*-álgebras de grafos não separados, há ideais "exóticos" não triviais, o que resulta em estudar C*-álgebras mais exóticas, especialmente C*-álgebras cheias e reduzidas. Várias perguntas surgem em relação a propriedades e teoremas que C*-álgebras de grafos não separados possuem que possam ser válidas para estas classes mais gerais. Algumas dessas perguntas envolvendo nuclearidade, exatidão, simplicidade, entre outras estão sendo respondidas e algumas podem ser vistas no trabalhos de P. Ara e K.R Goodearl em [4], [5] e [2] e, recentemente, de M. Lolk em [44]. Porém alguns teoremas de dualidade envolvendo grafos separados são questões ainda em aberto.

Para contextualizar, podemos observar para a classe de grafos não separados é que dada uma ação livre θ de um grupo discreto G em um grafo dirigido E através do teorema de Gross-Tucker é possível mostrar que todo grafo E pode ser visto como um grafo produto skew da forma $E/G \times_c G$ através de uma função c das arestas de E/G para o grupo G, conhecida como função rótulo. Nesse sentido, existem isomorfismos como $C^*(E) \rtimes_\theta G \cong C^*(E/G) \otimes K(l^2(G))$ que já são conhecidos neste contexto. Há outros isomorfismos relacionados que possuem uma forte correlação com maximalidade e normalidade de coações de grupos discretos. Estes isomorfismos são conhecidos como teoremas de dualidade, dentre eles os mais conhecidos são os teoremas de dualidade de Imai-Takai e Katayama vistos mais geralmente em [48].

No entanto, para uma classe mais geral, como é o caso de grafos separados, estes teoremas ainda não tinham sido explorados e nosso objetivo é dar uma resposta afirmativa a essas questões.

Objetivos

Nosso propósito então era simplesmente responder a seguinte pergunta: Será que é possível obter o teorema de Gross-Tucker para grafos separados e estender alguns teoremas de dualidade envolvendo esta classe de C*-algebras?

Inicialmente este era o nosso objetivo porém, ao longo do processo de pesquisa, nos interessamos em estudar as C*-algebras reduzida de grafos separados e nesse sentido surgiram algumas perguntas similares ao objetivo inicial, como por exemplo: Podemos estender estes mesmos teoremas de dualidade para C*-álgebras reduzida de grafos separados e obter resultados similares aos que foram apresentados anteriormente? Será que estes mesmos resultados podem ser vistos para outras classes de C*-álgebras relacionadas a grafos separados, como por exemplo as C*-algebras "mansas" de grafos separados conhecidas na literatura como C*-

algebras "tame" de grafos separados? Responder estas novas perguntas tornaram-se projeto de pesquisa complementar.

Metodologia

Analisando os trabalhos de S. Echterhoff, J. Quigg e S. Kaliszewski em [21] e [20] percebemos que a versão dos teoremas de dualidade para grupos discretos arbitrários necessitava da teoria de coações de grupos e suas relações com fibrados de Fell o que se tornou fundamental. Os estudos de maximalização e normalização de coações nos possibilitaram fazer uma espécie de classificação dessas C*-algebras.

Quando partimos para o estudo das C*-álgebras reduzida de grafos separados através de produtos amalgamados reduzidos reparamos que os resultados estavam muito complicados de serem verificados. Nesse momento surgiu a ideia de desenvolver C*-álgebras reduzidas através de esperanças condicionais e assim os resultados fluíram de forma mais clara e organizada. Esta teoria pode ser vista nos trabalhos de B.K. Kwasniewski e R. Meyer em [39] e parece ser bastante promissora para obter resultados mais gerais pois foi desenvolvida e aplicada para o caso estudado. Para complementar, percebemos que podemos olhar de forma alternativa as C*-álgebras reduzida de grafos separados como a C*-álgebra de um fibrado de Fell quociente o qual foi desenvolvida neste trabalho.

Resultados e Discussão

Ao utilizarmos a teoria de maximalização e normalização de coações de grupos discretos e focarmos nas C*-álgebras reduzidas através de esperanças condicionais foi possível responder as perguntas propostas inicialmente. Portanto nossas suspeitas de que os teoremas de dualidade para C*-algebras de grafos separados foram confirmadas tanto para a C*-álgebra cheia quanto para reduzida. No entanto, a amenabilidade da ação livre, o qual é válida no caso de grafos não separados, não foi confirmada para grafos separados.

Considerações Finais

Conseguimos responder as perguntas iniciais feitas e através do desenvolvimento da teoria podemos nos questionar sobre outras C*-álgebras associadas a grafos separados. Um exemplo explorado no final deste trabalho foram as C*-algebras "mansas" de grafos separados sobre o qual obtemos resultados similares aos apresentados até então. Esperamos que este trabalho contribua de forma significativa no desenvolvimento da teoria e motive o leitor a explorar e responder muitas outras questões que foram debatidas e propostas na conclusão deste projeto.

Palavras-chave: Grafo produto skew. Teorema de Gross-Tucker. Fibrados de Fell. Coações de grupos discretos. C*-álgebra de grafo separados. C*-álgebra reduzida de grafo separados. C*-algebra "mansa" de grafos separados.

ABSTRACT

Given a free action of a discrete group on a directed graph, there is a relationship between the graph and the skew product graph through the Gross-Tucker theorem and, besides that, there are some duality theorems involving their associated C*-algebras. In this work, we show that it is possible to extend some of these results to a more general class: the separated graph C*-algebras. Through a different approach from the usual one, we define the reduced separated graph C*-algebra and, in this way, we extend some duality theorems for this class. Finally, as an application, we obtain similar results for tame separated graph C*-algebras.

Keywords: Skew product graph. Gross-Tucker theorem. Fell bundles. Coactions of discrete groups. Separated graph C*-algebras. Reduced separated graph C*-algebras. Tame separated graph C*-algebras.

CONTENTS

1 INTRODUCTION

Graph C*-algebras provide an important class of C*-algebras that have been used both as a tool to study large classes of C*-algebras supplying models for the classification theory and as a source of an inexhaustible supply of examples and counterexamples for many other areas of mathematics. The first appearance of this type of algebra was developed by Leavitt in [43], who gives us a construction of a class of algebras denoted by $L_{\mathbb{K}}(m, n)$ over an arbitrary field $\mathbb{K}$ for every integer $1 \leq m \leq n$. In 1979, Cuntz independently gives us a construction of the Cuntz algebra $\mathcal{O}_n$ initially present in his paper [15] which is the universal C*-algebra generated by n isometries $S_1, \ldots, S_n$ such that $S_i^* S_j = \delta_{i,j}$ and $\sum_{i=1}^{n} S_i S_i^* = 1$. It is the most basic example of a graph C*-algebra and it is the subject of many studies in several areas. Some years later, Cuntz and Krieger generalized the Cuntz algebras to the Cuntz-Krieger algebras associated to finite square matrices with entries in $\{0, 1\}$ in [16]. Subsequently, the C*-algebra associated with a directed graph E was defined, called the graph C*-algebra and denoted by $C^*(E)$, and was realized that these C*-algebras are direct generalizations of the Cuntz algebras and Cuntz-Krieger algebras initially studied deeply in [38].

Years later a generalization of graph C*-algebras was presented by P. Ara and K.R. Goodearl (see [4] and [5]) called separated graph C*-algebras and denoted by $C^*(E, C)$, which is based on the concept of separated graphs (E, C) which consists of a directed graph E together with a family C that gives partitions of the set of edges departing from each vertex of E. For a particular choice of C, the C*-algebra $C^*(E, C)$ coincides with $C^*(E)$ as expected. This concept of separated graph C*-algebras is also related to C*-algebras of edge-colored graphs introduced by Duncan in [19].

The great motivation to study these general classes of C*-algebras is that they allow for a more complicated ideal structure than in $C^*(E)$. In this sense, we can study more "exotic" C*-algebras like the reduced separated graph C*-algebra denoted by $C_r^*(E, C)$, for example. This fact makes the situation very interesting. Not only that, there are a lot of open problems related to these C*-algebras, especially in the K-theory (as can be seen in section 7 on [4]) but many questions that arise extending results in the non-separated case, for example, nuclearity, simplicity, exactness, and many others. Some of these questions have already been answered. In this work, the goal is to understand more about the relationship between $C^*(E, C)$ and $C^*(E \times_c G, C \times_c G)$, the C*-algebra of skew product graph for a separated case. We are particularly interested in certain duality theorems connecting these algebras and our results will generalize previous results obtained for a non-separated graph seen in [37], [21] and [33].

This work is structured in four main parts as follows: Chapter 2 is dedicated entirely to introducing a compilation of the theory and results for non-separated graphs inspired by the papers [37] and [33]. First, we introduce the notion of skew product graphs denoted by $E \times_c G$ focusing on the Gross-Tucker theorem and explore the basic theory of graph C*-algebras. Besides, we introduce our main object of study, the crossed product by coactions, and explore the connection between the coactions of discrete groups and Fell bundles. At the end, we explore the theory developed so far applying to graph C*-algebras and study the duality theorems involved. All of these results are already known and will be referenced in the course of the work.

In chapter 3 we focus to extend part of the results seen in the previous chapter to separated graph C*-algebras showing the generalization of the Gross-Tucker theorem and proving that the crossed product $C^*(E,C) \times_\theta G$ is isomorphic to $C^*(E/G, C/G) \otimes K(l^2(G))$ for every free action θ of G on a separated graph (E,C). Many other isomorphisms were explored to achieve this result and complement the theory.

In chapter 4, we introduce the notion of reduced separated graph C*-algebra for finitely separated graphs defined through the reduced amalgamated products seen in [4] and [58]. Although this definition is concrete, it is not very easy to use it. We propose to look at the reduced separated graph C*-algebra from another point of view. We will define the reduced C*-algebra associated with a conditional expectation $P : A \to C_0(X)$ from a C*-algebra A containing a commutative C*-subalgebra $C_0(X)$ which we denote by $A_{P,r}$. From this point of view, we are able to show that the reduced C*-algebra $C^*_r(E,C)$ can be viewed as $C^*(E,C)_{P,r}$ and hence the reduced crossed product $C^*_r(E,C) \rtimes_{\theta,r} G$ is isomorphic to $C^*_r(E/G, C/G) \otimes K(l^2(G))$ for every free action θ of G on (E,C) through the canonical conditional expectation that exists in $C^*(E,C)$. Unfortunately, differently from non-separated case, we will see that the action θ is not amenable in the sense that $C^*(E,C) \rtimes_\theta G$ is isomorphic with $C^*(E,C) \rtimes_{\theta,r} G$.

In chapter 5 we will present the tame C*-algebra of a separated graph and show that part of the duality theorems can be obtained for this particular class of C*-algebras.

Lastly, we complete this work with two appendices, not only to remember some concepts used, but also to establish a relationship with what was seen during the work.

2 PRELIMINARY BACKGROUND

In this chapter, besides fixing some notations, we explore some aspects about graph theory, in particular graph C*-algebras, some duality theorems of our interest involving actions and coactions of discrete groups and the relationship between coactions of discrete groups and Fell bundles which will be of great importance for the next chapters. Also, we discuss the maximalization and normalization of coactions which complements the whole theory and will allow us to better understand the duality theorems in the context of graph C*-algebras.

We have developed some sections to organize ideas and further explore the theory. Our main references in this chapter are [37], [21], [33], [30], and many other references will be cited in each section. We refer to [47] and [12] for the basic theory of C*-algebras.

Throughout this work, $\mathbb{C}$ will denote the field of complex numbers and G will be a discrete group with a neutral element denoted by 1. Notice that sometimes 1 might also denote other things, like units of algebras or the integer number 1, but its use will be clear from the context. Before we explore the sections, let us review some definitions of graphs in general.

Definition 2.0.1. A *directed graph* E is a quadruple of the form $E = (E^0, E^1, s, r)$ consisting of two sets E^0,E^1 and two maps $s, r : E^1 \to E^0$. The elements of E^0 and E^1 are called vertices and edges and the maps s, r are called the source and range maps, respectively.

Remark 2.0.2. Most of the time we will mention a directed graph simply by a graph and denote it just by E to simplify notation. The source and range maps are nothing but the start and end of an edge.

Remark 2.0.3. Throughout this work, we do not make any assumptions of the cardinality of our graphs, and in particular, we do not require the set E^0 or E^1 of our graphs to be finite or countable. In the literature, the countability of the graph ensures that the associated C*-algebra is separable, which is a common hypothesis imposed in the C*-algebra theory, especially in K-theory classification. However, in this work all results that will be seen the countability hypothesis is unnecessary and the same proofs go through for uncountable graphs.

The graph is called *row-finite* if every vertex emits only finitely many edges, that is, $s^{-1}(v)$ is finite for every $v \in E^0$. We say that v is a *sink* if it emits no edges, in other words, if $v \notin s(E^1)$. A vertex $v \in E^0$ is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. We call $v \in E^0$ a *singular vertex* if v is either a sink or an infinite emitter. If v is not a singular vertex, we call it a *regular vertex*.

A *finite path* in E is a sequence of edges of the form $\mu := e_1 \dots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i \in \{1, \dots, n-1\}$ and n is the length of μ which we will denote by $|\mu| := n$. Paths with length 0 are identified with the vertices of E and we set $s(v) = r(v) = v$. We denote by E^n the set of all finite paths with length n and $\text{Path}(E) := \bigcup_{n=0}^{\infty} E^n$ denote the set of all paths of E. We can extend the source and range maps to $\text{Path}(E)$ in the obvious way: if

$\mu = e_1 \dots e_n \in \text{Path}(E)$, then $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$. Given two paths $\mu, \nu \in \text{Path}(E)$ with $r(\mu) = s(\nu)$, one obtains a new path $\mu\nu$ by concatenation with $|\mu\nu| = |\mu| + |\nu|$.

We also denote by $(E^1)^*$ the set of ghost edges $\{e^* \mid e \in E^1\}$ and for each finite path $\mu = e_1 \dots e_n$ we define a ghost finite path $\mu^* = e_n^* \dots e_1^*$ with source and range maps defined by $s(e^*) = r(e)$ and $r(e^*) = s(e)$ for all $e \in E^1$. Also, we define $v^* = v$ for all $v \in E^0$.

Definition 2.0.4. Let E and F be two graphs. We say that F is a *subgraph* of E if $F^0 \subseteq E^0$, $F^1 \subseteq E^1$, and s_F and r_F of F are restrictions of the source and range maps s_E and r_E of E, respectively.

Definition 2.0.5. Let E and F be two graphs. A *graph morphism* $f : E \to F$ is a pair of maps (f^0, f^1) where $f^0 : E^0 \to F^0$ and $f^1 : E^1 \to F^1$ which commutes with source and range maps, that is, satisfy

$$f^0(r(e)) = r(f^1(e)) \quad \text{and} \quad f^0(s(e)) = s(f^1(e))$$

for all $e \in E^1$. If f^0 and f^1 are bijective maps then f is called a *graph isomorphism*.

Remark 2.0.6. Many times we will omit the index of f^0 and f^1 to make writing easier.

Definition 2.0.7. A graph morphism $f : E \to F$ is said to have the *unique path lifting property* if given $v \in E^0$ and $e' \in F^1$ with $s(e') = f^0(v)$, then there is a unique $e \in E^1$ such that $e' = f^1(e)$ with $s(e) = v$.

Remark 2.0.8. There is a natural notion of automorphism of graphs and the collection of all automorphisms of a graph E forms a group under composition, denoted by $\text{Aut}(E)$. Consequently, there is a notion of actions of graphs by a group G.

Definition 2.0.9. Let E be a graph and G be a group. An *action* of G on E is a group homomorphism $\alpha : G \to \text{Aut}(E)$. The action α is called *free* if it acts freely on the vertices, in other words, $\alpha_g(v) = v$ for some $v \in E^0$ implies $g = 1$. If G acts freely on the vertices then it also acts freely on the edges. So, freeness on the edges is automatically.

Definition 2.0.10. Let E, F be two graphs endowed with actions α and β of G, respectively and $f : E \to F$ be a graph morphism. We say that f is G-*equivariant* if

$$f(\alpha_g^0(v)) = \beta_g^0(f(v)) \quad \text{and} \quad f(\alpha_g^1(e)) = \beta_g^1(f(e))$$

for all $v \in E^0$ and $e \in E^1$.

2.1 SKEW PRODUCT GRAPHS AND THE GROSS-TUCKER THEOREM

Our goal in this section is to present a definition of skew product graph, some examples to fix ideas, and to bring up the Gross-Tucker theorem. Our proof of the Gross-Tucker theorem presented in this work is slightly different from the original seen in [30], but it follows the same idea. The main motivation to study this particular aspect of graph theory was the article [37]. Other related paper is [29].

From now on, E is an arbitrary graph, G is a discrete group and $c : E^1 \to G$ is a function.

Definition 2.1.1. With notations as above, we define the *skew product graph*

$$E \times_c G := (E^0 \times G, E^1 \times G, s, r)$$

in which the sets of vertices and edges are the Cartesian product of E^0 and E^1 with G, respectively, and the source and range maps are defined by

$$s(e, g) = (s(e), g) \quad \text{and} \quad r(e, g) = (r(e), gc(e))$$

for all $e \in E^1$ and $g \in G$.

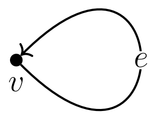
If E is row-finite it follows that $E \times_c G$ is also row-finite since $s^{-1}(v, g) = s^{-1}(v) \times \{g\}$ by definition. In the literature (see [30]) $E \times_c G$ is referred to as the *derived graph* or *voltage graph* and the function $c : E^1 \to G$ is called the *labeling function*. Skew product graphs have many applications, for instance, they are used in the theory of branched covering of surfaces (see [29]). Our definition of skew product graph $E \times_c G$ is not the same as versions $E(c)$ in [37] where $E(c) := (G \times E^0, G \times E^1, s, r)$ with $s(g, e) = (g, s(e))$ and $r(g, e) = (gc(e), r(e))$ and E^c in [33] where $E^c := (E^0 \times G, E^1 \times G, s, r)$ with $s(e, g) = (s(e), c(e)g)$ and $r(e, g) = (r(e), g)$ but all these definitions are equivalent by the isomorphisms below:

$$\begin{array}{ccc} E(c) \to E \times_c G & \qquad & E^c \to E \times_c G \\ (g, v) \to (v, g) & & (v, g) \to (v, g^{-1}) \\ (g, e) \to (e, g) & & (e, g) \to (e, (c(e)g)^{-1}) \end{array}$$

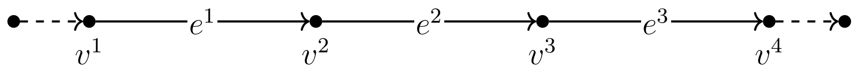
for all $v \in E^0$, $e \in E^1$ and $g \in G$. Our convention is chosen to make the results of our main theorems later more natural.

To fix ideas, let see some examples in this context:

Example 2.1.2. Let E be the graph with $E^0 = \{v\}$ and $E^1 = \{e\}$ as below:

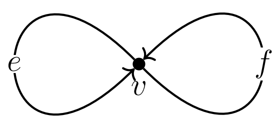


Define $c : E^1 \to \mathbb{Z}$ as $c(e) = 1$. Then we have the skew product graph $E \times_c \mathbb{Z}$ as below:

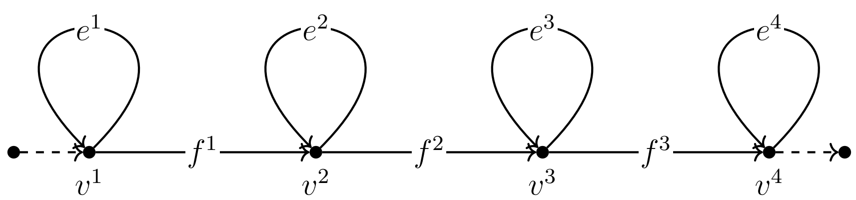


where here we are denoting the vertices and edges in $E \times_c \mathbb{Z}$ as $v^k := (v, k)$ and $e^k := (e, k)$ for every $k \in \mathbb{Z}$. We are going to use this same notation for the next examples for simplicity.

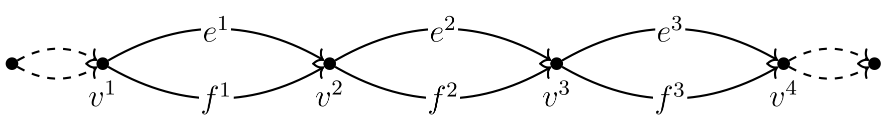
Example 2.1.3. Let E be the graph with $E^0 = \{v\}$ and $E^1 = \{e, f\}$ as below:



Define $c : E^1 \to \mathbb{Z}$ as $c(e) = 0$ and $c(f) = 1$. Then we have the skew product graph $E \times_c \mathbb{Z}$ as below:

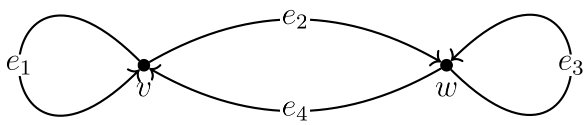


Example 2.1.4. If we change the labeling function in Example 2.1.3 and define $c(e) = c(f) = 1$ we have a different skew product graph $E \times_c \mathbb{Z}$ as below:

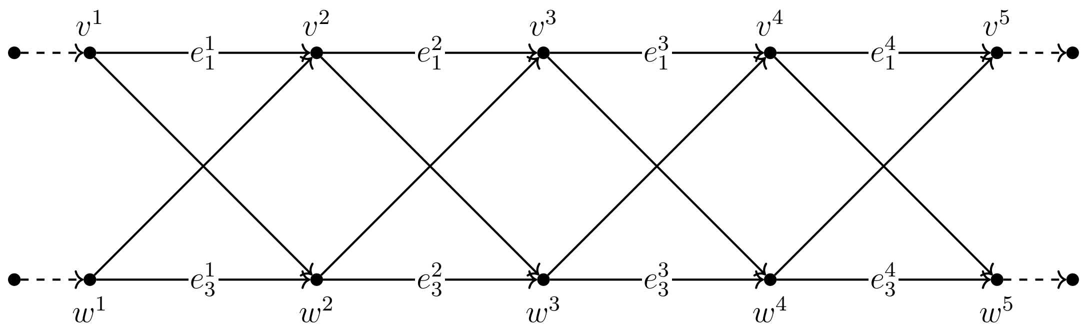


Note that, in both previous examples, if we change c we will notice that when c sends edges to 1 we have edges liking vertices and when c sends edges to 0 we have loops, that is, edges such that source and range are equal.

Example 2.1.5. Another example in this context is the following: Let E be the graph with $E^0 = \{v, w\}$ and $E^1 = \{e_1, e_2, e_3, e_4\}$ as below:

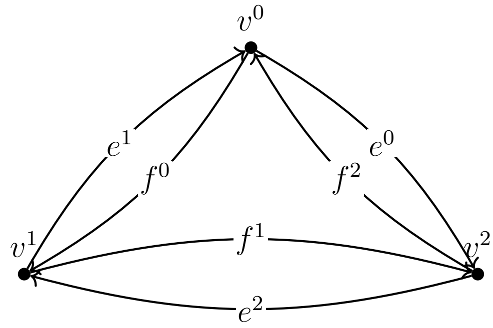


Define $c(e_i) = 1$ for all $i = 1, 2, 3, 4$. Then the skew product graph $E \times_c \mathbb{Z}$ is as picture below:



Observe that the edges $e_1^k := (e_1, k) \in E^1 \times \mathbb{Z}$ are in the top line connecting the vertices v^{k-1} with v^k for every $k \in \mathbb{Z}$. In the same way edges e_3^k for every $k \in \mathbb{Z}$ are on the bottom line. But the edges that connect the vertices diagonally from the top to bottom and from bottom to top are e_2^k and e_4^k for every $k \in \mathbb{Z}$, respectively.

Example 2.1.6. If we consider the graph E in the Example 2.1.3 and use the same labeling function but over $\mathbb{Z}_2$ meaning $c : E^1 \to \mathbb{Z}_2$ such that $c(e) = 0$ and $c(f) = 1$ it is not difficult to see that the resulting skew product graph over $\mathbb{Z}_2$ is isomorphic to the graph E in the Example 2.1.5 identifying $v^0 \to v$, $v^1 \to w$, $e^0 \to e_1$, $e^1 \to e_3$, $f^0 \to e_2$ and $f^1 \to e_4$. But, taking the same graph E in Example 2.1.3 and changing the labeling function c over $\mathbb{Z}_3$ defining $c : E^1 \to \mathbb{Z}_3$ as $c(e) = 2$ and $c(f) = 1$ we have the skew product graph $E \times_c \mathbb{Z}_3$ is as picture below:



We may extend the labeling function c to the set Path(E) by defining $c(v) := 1$ for every $v \in E^0$ and $c(\mu) := c(e_1)\dots c(e_n)$ for every $\mu = e_1 \dots e_n \in$ Path(E). Also, we may define $c(\mu^*) := c(\mu)^{-1}$ for all $\mu \in$ Path(E) and for every $\mu, \nu \in$ Path(E) with $r(\mu) = s(\nu)$ we have $c(\mu\nu) = c(\mu)c(\nu)$.

Moreover, for each path in E we can define a path in $E \times_c G$ in the following way: for each $\mu \in$ Path(E) of the form $\mu = e_1 \dots e_n$ with $r(e_i) = s(e_{i+1})$ for $i \in \{1, \dots, n\}$ and $g \in G$ a path in the skew product graph is of the form:

$$(\mu, g) := (e_1, g)(e_2, gc(e_1)) \dots (e_n, gc(e_1 \dots e_{n-1})). \tag{2.1.7}$$

The reason for having paths of this form is due to the definition of range and source maps in skew product graph. For example, if $e, f \in E^1$ and $g, h \in G$, the short path $(e, g)(f, h)$ makes sense if $r(e, g) = s(f, h)$. But, by definition, $r(e, g) = (r(e), gc(e))$ and $s(f, h) = (s(f), h)$. So, to make sense it is necessary that $r(e) = s(f)$ (that is, ef is a short path in E) and $h = gc(e)$. The notation (μ, g) will be used exclusively for the paths in $E \times_c G$ defined as in 2.1.7. One should not to confuse it with $(\mu, g) = (e_1 \dots e_n, g)$ which is not a path on $E \times_c G$.

Example 2.1.8. Let G be a group with generators $g_1, \dots, g_n$. The *Cayley graph* of G with respect to generators $g_1, \dots, g_n$ is the graph

$$E_G := (E_G^0, E_G^1, s, r)$$

where $E_G^0 = G$, $E_G^1 = G \times \{g_1, \dots, g_n\}$ and the source and range maps are defined by $s(h, g_i) = h$ and $r(h, g_i) = hg_i$ for all $i \in \{1, \dots, n\}$.

Cayley graphs are very interesting examples because they carry a natural action of G by the left multiplication, that is, there is $\beta : G \to$ Aut(E_G) such that $\beta_g^0(h) = gh$ and $\beta_g^1(h, g_i) = (gh, g_i)$ for all $h \in G$ and $i \in \{1, \dots, n\}$. In fact these actions defined in this away are always free.

Note that the graph E in Example 2.1.5 is the Cayley graph for $\mathbb{Z}_2$ with respect to the generating set $\{0, 1\}$. Moreover, the skew product graph in Example 2.1.2 is the Cayley graph of $\mathbb{Z}$ with respect to the generator 1 and the skew product graph in Example 2.1.3 is the Cayley graph of $\mathbb{Z}$ with respect to the generating set $\{0, 1\}$.

These examples show us that Cayley graphs are interesting not only for the existence of a free action but because they are strongly related to skew product graphs as we will see in the next example.

Example 2.1.9. Let E_G be the Cayley graph as in Example 2.1.8. Consider the graph A_n that has only one vertex v and n edges $a_1, \dots, a_n$ with $s(a_i) = r(a_i) = v$, that is, has n loops. This graph is called the *Cuntz graph* and later we will see more clearly the reason for this nomenclature. An interesting fact is that if we define $c : A_n^1 \to G$ such that $c(a_i) = g_i$ for all $i \in \{1, \dots, n\}$ we have an isomorphism of graphs

$$E_G \cong A_n \times_c G.$$

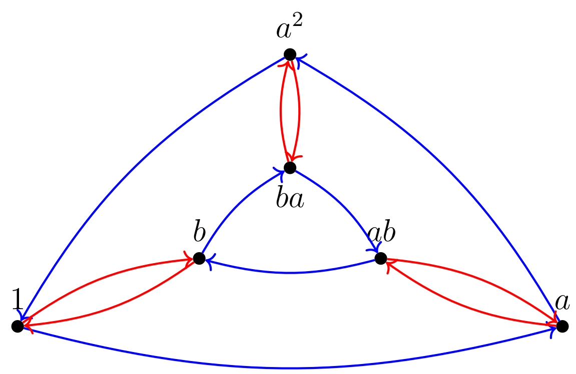
To see this isomorphism we just need to identify the sets of vertices and edges in the following way: Define $\psi^0 : \{v\} \times G \to G$ by $\psi^0(v, g) = g$ and $\psi^1 : \{a_i, \dots, a_n\} \times G \to G \times \{g_1, \dots, g_n\}$ by $\psi^1(a_i, g) = (g, c(a_i))$ for every $g \in G$. These maps are bijective and well defined with the inverse defined in the obvious way. We just need to show that $\psi : A_n \times_c G \to E_G$ is a graph morphism. For each $a_i \in A_n^1$ and $g \in G$, notice that:

$$\begin{aligned} \psi^0(s(a_i, g)) &= \psi^0(s(a_i), g) = \psi^0(v, g) = g \\ &= s(g, g_i) = s(g, c(a_i)) = s(\psi^1(a_i, g)) \end{aligned}$$

and

$$\begin{aligned} \psi^0(r(a_i, g)) &= \psi^0(r(a_i), gc(a_i)) = \psi^0(v, gg_i) = gg_i \\ &= r(g, g_i) = r(g, c(a_i)) = r(\psi^1(a_i, g)). \end{aligned}$$

Example 2.1.10. Consider the group by the presentation $G = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1 \rangle$, known as dihedral group D_3. The Cayley graph E_G can be draw as in the picture below:



where we represent in red the edges which correspond to generator b and in blue the edges which correspond to generator a. For example, the edge that connects a^2 to 1 is (a^2, a) since

$s(a^2, a) = a^2$ and $r(a^2, a) = a^2a = a^3 = 1$. Since $abab = 1$ imply $a^2 = bab$ then the edge that connects a^2 to ba is (a^2, b) because $s(a^2, b) = a^2$ and $r(a^2, b) = a^2b = babb = bab^2 = ba$.

Through the isomorphism seen in Example 2.1.9 we can see that E_G is in fact the skew product graph $A_6 \times_c G$ where A_6 is the graph with one vertex and six loops.

Note that, by Example 2.1.9, every Cayley graph is a skew product graph with some labeling function. After all the examples a pertinent question is: For a group G, how do we identify that a certain graph F is a skew product graph of the form $E \times_c G$?

To answer this question, we need some definitions and results first.

Definition 2.1.11. Let G act on a graph E. We define the *quotient graph*

$$E/G := ((E/G)^0, (E/G)^1, s_G, r_G)$$

where $(E/G)^0 = E^0/G$ and $(E/G)^1 = E^1/G$ are the equivalence classes of vertices and edges respectively under the action of G and the source and range maps are defined to be $s_G([e]) = [s(e)]$ and $r_G([e]) = [r(e)]$.

Remark 2.1.12. It is easy to check that the source and range maps are well defined because the action commutes with both maps. Moreover, the quotient map $q : E \to E/G$ is a surjective graph morphism.

Definition 2.1.13. Let $\alpha : G \to \mathrm{Aut}(E)$ be an action and let $x \in (E/G)^0$. Then a vertex $v_x \in E^0$ is said to be a *base vertex* of x if $q(v_x) = x$, and similarly, for each $y \in (E/G)^1$ we define a *base edge* $e_y \in E^1$ of y if $q(e_y) = y$.

Remark 2.1.14. Of course base vertices always exist but they are usually not unique since for each $x \in E^0/G$ with base vertex v_x of x, $\alpha_g(v_x)$ is also a base vertex of x for any $g \in G$. A similar statement holds for edges.

Next we show that free actions are important in this context.

Proposition 2.1.15. *Let E be a graph and $\alpha : G \to Aut(E)$ be a free action. Then the quotient map $q : E \to E/G$ has the unique path lifting property.*

Proof. Suppose there is $y \in E^1/G$ such that $s(y) = x$ for some $x \in E^0/G$. There is a base edge $e_y \in E^1$ of y such that $s(e_y) = v_x$ for some base vertex v_x associated with x. So for each $w \in E^0$ with $q(w) = x$ there is $g \in G$ such that $w = \alpha_g(v_x)$. We claim that $\alpha_g(e_y) \in E^1$ is the unique edge such that $q(\alpha_g(e_y)) = y$ and $s(\alpha_g(e_y)) = w$.

It is obvious that $q(\alpha_g(e_y)) = y$ because they live in the same orbit and note that $s(\alpha_g(e_y)) = \alpha_g(s(e_y)) = \alpha_g(v_x) = w$. So, $\alpha_g(e_y)$ is the edge that we were looking for. We just need to show the uniqueness. For this, suppose that we have $f \in E^1$ such that $q(f) = y$ and $s(f) = w$. So, there is $h \in G$ such that $f = \alpha_h(e_y)$ because e_y is a base edge. Then, on the one hand, we have

$$w = s(f) = s(\alpha_h(e_y)) = \alpha_h(s(e_y)).$$

But, on the other hand, we have

$$w = s(\alpha_g(e_y)) = \alpha_g(s(e_y)).$$

Therefore, by the freeness of the action, we have $\alpha_g(s(e_y)) = \alpha_h(s(e_y))$ implies $g = h$. Then $f = \alpha_g(e_y)$ as we desired.

□

Proposition 2.1.16. *Let E be a graph and $c : E^1 \to G$ be a labeling function. Then there is an free action of G on $E \times_c G$ such that*

$$(E \times_c G)/G \cong E.$$

Proof. Define $\gamma : G \to \mathrm{Aut}(E \times_c G)$ by the left multiplication, that is,

$$\gamma_g(v, h) := (v, gh) \text{ and } \gamma_g(e, h) := (e, gh)$$

for all $v \in E^0$, $e \in E^1$ and $g \in G$. It is straightforward to check that this defines an action of G on $E \times_c G$ which is free because for every $v \in E^0$ we have $\gamma_g(v, h) = (v, h)$ if and only if $gh = h$ if and only if $g = 1$. This action is designated as the canonical free action on skew product graph.

Now, define $\psi : E \to (E \times_c G)/G$ such that $\psi(v) = [(v, 1)]$ and $\psi(e) = [(e, 1)]$. Note that the classes $[(v, 1)] = [(v, g)]$ for every $g \in G$ because $(v, g) = \gamma_g(v, 1)$ and the same happens on edges. To see that ψ is a graph morphism, for every $e \in E^1$ it is enough to compute

$$\psi(s(e)) = [(s(e), 1)] = s_G([(e, 1)]) = s_G(\psi(e))$$

and

$$\psi(r(e)) = [(r(e), 1)] = [(r(e), c(e))] = r_G([(e, 1)]) = r_G(\psi(e)).$$

So, ψ commutes with source and range maps and consequently it is in fact a graph morphism. The inverse is defined in the obvious way, that is, $\psi^{-1}([(v, g)]) = v$ and $\psi^{-1}([(e, g)]) = e$ for all $v \in E^0$, $e \in E^1$ and $g \in G$.

□

By the last result, if we have some labeling function then the quotient over a skew product graph by the action γ recovers E. Is the converse true? The following result (originally proved by Gross-Tucker in [[30], Theorem 2.2.2]) answer this question.

Theorem 2.1.17 (Gross-Tucker theorem)**.** *Let E be a graph endowed with a free action α of G. Then there is a labeling function $c : E^1/G \to G$ and a G-equivariant isomorphism of graphs*

$$E \cong (E/G) \times_c G.$$

Proof. To begin with, for each $x \in (E/G)^0$ fix a base vertex v_x in E^1 associated to x. Since the quotient map $q : E \to E/G$ has the unique path lifting property, for each $y \in (E/G)^1$ with $s(y) = q(v_x) = x$ there is a unique edge $e_y \in E^1$ such that $q(e_y) = y$ with $s(e_y) = v_x$. Note that, for each $w \in E^0$ with $q(w) = x$ there is $z \in G$ such that $w = \alpha_z(v_x)$. In particular, there is $z \in G$ such that $r(e_y) = \alpha_z(v_{r(y)})$ and it is unique because the action is free. So we can define $c : E^1/G \to G$ such that $c(y) := z$. Hence we can define the maps:

$$\begin{aligned} \phi^0 : (E^0/G)\times_c G &\to E^0 \\ (x,g) &\to \alpha_g(v_x) \end{aligned} \qquad \begin{aligned} \phi^1 : (E^1/G)\times_c G &\to E^1 \\ (y,g) &\to \alpha_g(e_y) \end{aligned}$$

It is obvious that the both maps are well-defined. We compute:

$$\begin{aligned} \phi^0(s(y,g)) &= \phi^0(s(y),g) = \alpha_g(v_{s(y)}) = \alpha_g(s(e_y)) \\ &= s(\alpha_g(e_y)) = s(\phi^1(y,g)) \end{aligned}$$

and

$$\begin{aligned} \phi^0(r(y,g)) &= \phi^0(r(y),gc(y)) = \phi^0(r(y),gz) = \alpha_{gz}(v_{r(y)}) \\ &= \alpha_g(\alpha_z(v_{r(y)})) = \alpha_g(r(e_y)) = r(\alpha_g(e_y)) \\ &= r(\phi^1(y,g)) \end{aligned}$$

for every $y \in E^1/G$ and $g \in G$. Thus ϕ is morphism graph. To prove the injectivity, note that $\phi^0(x,g) = \phi^0(x',g')$ if and only if $\alpha_g(v_x) = \alpha_{g'}(v_{x'})$ if and only if $v_x = \alpha_{g^{-1}g'}(v_{x'})$. Then it is clear that $x = x'$ because they live in the same orbit and since α is free, we see that $g^{-1}g' = 1$, and therefore $g = g'$. Finally the surjectivity, note that for any $v \in E^0$, we can put $x = q(v)$. Now, consider v_x a base vertex to x and note there exists $g \in G$ with $v = \alpha_g(v_x)$. That is, $v = \alpha_g(v_x) = \phi^0(x,g)$. The same ideas work to show injectivity and surjectivity for ϕ^1. Therefore we have an isomorphism of graphs $E \cong (E/G) \times_c G$.

Finally, to see that the isomorphism is G-equivariant remember that skew product graphs are always endowed with the canonical free action γ defined as in Proposition 2.1.16. So, we just compute:

$$\begin{aligned} \alpha_h(\phi^0(x,g)) &= \alpha_h\alpha_g(v_x) \\ &= \alpha_{hg}(v_x) \\ &= \phi^0(x,hg) \\ &= \phi^0(\gamma_h(x,g)) \end{aligned}$$

and

$$\begin{aligned} \alpha_h(\phi^1(y,g)) &= \alpha_h\alpha_g(e_y) \\ &= \alpha_{hg}(e_y) \\ &= \phi^1(e,hg) \\ &= \phi^1(\gamma_h(e,g)) \end{aligned}$$

for all $x \in (E/G)^0$, $y \in (E/G)^1$ and $g, h \in G$.

$\square$

Remark 2.1.18. If, in the proof of Theorem 2.1.17, we choose a different set of base vertices then we obtain a new function $d : E^1/G \to G$ but the results similarly follows.

Example 2.1.19. The Cayley graph has a natural free action and by the Gross-Tucker theorem we recover the same result $E_G \cong A_n \rtimes_c G$ in Example 2.1.9.

2.2 GRAPH C*-ALGEBRAS

This section contains some basic facts on Graph C*-algebras. We mainly follow [37], [9], [36], [38], [31],[8] and [1].

Definition 2.2.1. The *Leavitt path algebra* of E with coefficients in the complex field is the complex *-algebra $L(E)$ with generators $\{P_v\}_{v \in E^0}$ and $\{S_e\}_{e \in E^1}$ subject to following relations:

1. $P_v P_w = \delta_{v,w} P_v$ and $P_v^* = P_v$ for all $v \in E^0$.
2. $P_{s(e)} S_e = S_e P_{r(e)} = S_e$ for all $e \in E^1$.
3. $S_e^* S_f = \delta_{e,f} P_{r(e)}$ for all $e, f \in E^1$.
4. $P_v = \sum_{\substack{e \in E^1 \\ s(e)=v}} S_e S_e^*$ for every regular vertex $v \in E^0$.

Remark 2.2.2. The condition 1 tell us that the generators $\{P_v\}_{v \in E^0}$ are orthogonal projections and conditions 2 and 3 tell us that the generators $\{S_e\}_{e \in E^1}$ are partial isometries.

Definition 2.2.3. The graph C*-algebra $C^*(E)$ is the universal C*-algebra generated by the collections $\{P_v \mid v \in E^0\}$ and $\{S_e \mid e \in E^1\}$ satisfying the relations 1-4. In other words, the graph C*-algebra $C^*(E)$ is the enveloping C*-algebra of $L(E)$.

Remark 2.2.4. This C*-algebra exists because the generating set consists of partial isometries and orthogonal projections. In the literature, the collection of $\{P_v, S_e \mid v \in E^0, e \in E^1\}$ is called a *Cuntz-Krieger E-family*.

Remark 2.2.5. The graph C*-algebra $C^*(E)$ has a universal property in the sense that for every C*-algebra B generated by a Cuntz-Krieger E-family $\{Q_v, T_e \mid v \in E^0, e \in E^1\}$ subject to relations 1-4 as above there is a (unique) surjective *-homomorphism $\Phi : C^*(E) \to B$ such that $\Phi(P_v) = Q_v$ and $\Phi(S_e) = T_e$ for all $v \in E^0$ and $e \in E^1$.

Lemma 2.2.6. *Let E be a graph. Then*

$$C^*(E) = \overline{\mathit{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in \mathit{Path}(E) \text{ such that } r(\mu) = r(\nu)\}.$$

Proof. It is enough to observe using the relations 1-4 in Definition 2.2.1 that

$$(S_\mu S_\nu^*)(S_\eta S_\zeta^*) = \begin{cases} S_{\mu\eta'} S_\zeta^*, & \text{if } \eta = \nu\eta' \text{ for some path } \eta' \\ S_\mu S_{\nu'\zeta}^*, & \text{if } \nu = \eta\nu' \text{ for some path } \nu' \\ 0, & \text{otherwise} \end{cases}.$$

By linearity and continuity, the result follows.

□

For each $z \in \mathbb{T}$, the circle group in the complex field, there is an automorphism $\alpha_z : C^*(E) \to C^*(E)$ such that $\alpha_z(P_v) = P_v$ and $\alpha_z(S_e) = zS_e$ for all $v \in E^0$ and $e \in E^1$. Notice that for each pair $\mu, \nu \in \text{Path}(E)$ the map $z \to \alpha_z(S_\mu S_\nu^*)$ is continuous and it follows from a routine argument that α is a continuous action of $\mathbb{T}$ on $C^*(E)$. This action is called the *gauge action* and the existence of this action characterizes the graph C*-algebra $C^*(E)$ through the *Gauge-invariant uniqueness theorem* as shown in different papers: [9], [31] and [8]. Formal statements of this uniqueness theorem is as follows:

Proposition 2.2.7 ([9], Theorem 2.1)**.** *Let E be a graph and suppose B is a C*-algebra generated by a Cuntz-Krieger E-family $\{Q_v, T_e \mid v \in E^0, e \in E^1\}$. If each Q_v is non-zero, and there is an action β of $\mathbb{T}$ on B such that $\beta_z \circ \Phi = \Phi \circ \alpha_z$ then the canonical *-homomorphism $\Phi : C^*(E) \to B$ is an isomorphism.*

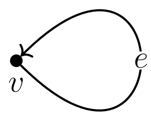
The uniqueness theorems are fundamental results in the study of graph C*-algebras because they provide sufficient conditions for a *-homomorphism from $C^*(E)$ into a C*-algebra to be injective and consequently a huge capacity for examples.

As a next step, we are going to present some basic examples of graphs and their C*-algebras omitting proofs as they can be easily obtained from the universal properties and the above proposition. We refer [36] for further details.

Example 2.2.8. Here are some standard examples of graph C*-algebras:

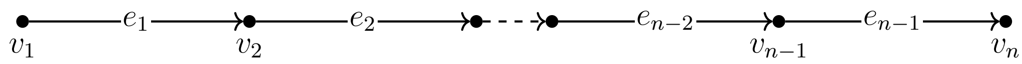
1. Let E be the graph that has only one vertex v and no edges. Then $C^*(E)$ is isomorphic to $\mathbb{C}$.

2. Let E be the graph with $E^0 = \{v\}$ and $E^1 = \{e\}$:



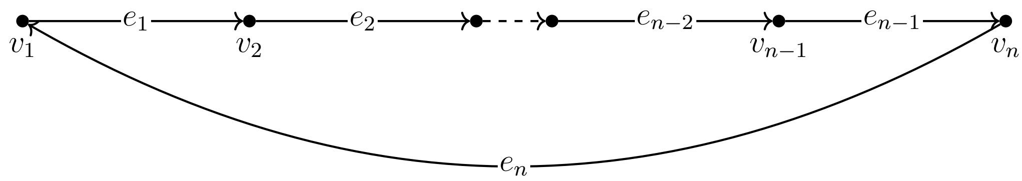
Then $C^*(E)$ is isomorphic to $C(\mathbb{T})$, the space of continuous functions on unit circle $\mathbb{T}$.

3. Let E be the graph with $E^0 = \{v_1, \dots, v_n\}$ and $E^1 = \{e_1, \dots, e_{n-1}\}$:



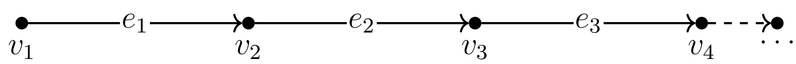
Then $C^*(E)$ is isomorphic to $M_n(\mathbb{C})$.

4. Let E be the graph with $E^0 = \{v_1, \dots, v_n\}$ and $E^1 = \{e_1, \dots, e_n\}$:



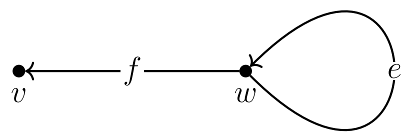
Then $C^*(E)$ is isomorphic to $M_n(C(\mathbb{T}))$.

5. Let E be the graph with $E^0 = \{v_i \mid i \in \mathbb{N}\}$ and $E^1 = \{e_i \mid i \in \mathbb{N}\}$:



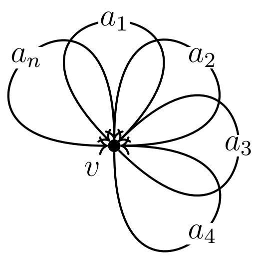
Then $C^*(E)$ is isomorphic to $\mathcal{K}(\mathcal{H})$, the compact operators on some separable infinite-dimensional Hilbert space.

6. Let E be the graph with $E^0 = \{v, w\}$ e $E^1 = \{e, f\}$:



Then $C^*(E)$ is isomorphic to $\mathcal{T}$, the Toeplitz algebra.

7. Consider the Cuntz graph A_n seen in Example 2.1.9 as picture below:



Then $C^*(E)$ is isomorphic to $\mathcal{O}_n$, the Cuntz algebra generated by n isometries. This is the reason why we called this graph the Cuntz graph. In the case when $n = \infty$ then $C^*(E)$ is isomorphic to $\mathcal{O}_\infty$, the Cuntz algebra generated by countably infinite numbers of isometries.

Proposition 2.2.9. *Let α be an action of G on E. Then there is an induced action $\tilde{\alpha}$ of G on $C^*(E)$ such that $\tilde{\alpha}_g(P_v) = P_{\alpha_g(v)}$ and $\tilde{\alpha}_g(S_e) = S_{\alpha_g(e)}$ for all $v \in E^0$ and $e \in E^1$.*

Proof. Fix $g \in G$. We claim that $\{P_{\alpha_g(v)}, S_{\alpha_g(e)}\}$ is a Cuntz-Krieger E-family for $C^*(E)$. It is clear that $\{P_{\alpha_g(v)}\}_{v\in E^0}$ are mutually orthogonal projections because $\alpha_g(v) = \alpha_g(w)$ if and only if $v = w$. Similarly $\{S_{\alpha_g(e)}\}_{e\in E^1}$ are partial isometries that "commute" with the projections. So, they satisfy 1 and 2 in Definition 2.2.1. The condition 4 is automatic since $s(\alpha_g(e)) = \alpha_g(s(e))$ for every $e \in E^1$. Only condition 3 remained to check. But it is not difficult because for $e, f \in E^1$ we have

$$\begin{aligned} S^*_{\alpha_g(e)} S_{\alpha_g(f)} &= \delta_{\alpha_g(e),\alpha_g(f)} P_{r(\alpha_g(e))} \\ &= \delta_{e,f} P_{\alpha_g(r(e))}. \end{aligned}$$

Then, by the universal property we get a *-homomorphism $\tilde{\alpha_g} : C^*(E) \to C^*(E)$ such that $\tilde{\alpha_g}(S_e) = S_{\alpha_g(e)}$ e $\tilde{\alpha_g}(P_v) = P_{\alpha_g(v)}$ for all $e \in E^1$ and $v \in E^0$. The same arguments show that the inverse $\tilde{\alpha_g}^{-1}$ is in fact $\tilde{\alpha}_{g^{-1}}$ and since α is an action shows that $\tilde{\alpha}$ is also an action.

□

Remark 2.2.10. For now on, to simplify the notation we shall use the same symbol α for the action on E and its induced action on $C^*(E)$. Eventually, we shall also use the symbol $\cdot$ for the action on E when necessary.

2.3 FELL BUNDLES

This section introduces one of our main tools to study C*-algebras associated with coactions. We begin with the definition of Fell bundles over discrete groups and the constructions of their full and reduced cross-sectional C*-algebras. Eventually, we will be using some of the inter-relationship with dynamic systems which can be seen in Appendix A of this work, that is, for a C*-algebra A, a discrete group G and an action α of G on A we denote dynamical system as (A, G, α).

Also, we will assume that all representations here are nondegenerate, and so each extends uniquely to a unital representation of $\mathcal{M}(A)$, the multiplier algebra of A. All tensor products $\otimes$ are minimal, and all identity maps are denoted by id with index when necessary. We assume that the reader is familiar with these notions of multiplier algebras and tensor products, especially minimal tensor products which will be widely used.

Our main reference for this section is essentially [25].

Definition 2.3.1. A *Fell bundle* (also known as a *C*-algebraic bundle*) over a discrete group G is a family $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ of Banach spaces $\mathcal{A}_g$ (each of which is called a fiber) endowed with multiplication maps $\cdot : A_g \times A_h \to A_{gh}$ and involution maps $* : \mathcal{A}_g \to A_{g^{-1}}$ satisfying the following properties for all $g, h, k \in G$, $a_g \in \mathcal{A}_g$ and $a_h \in \mathcal{A}_h$:

(i) The multiplication maps are bilinear and associative, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_g \times \mathcal{A}_h \times \mathcal{A}_k & \longrightarrow & \mathcal{A}_g \times \mathcal{A}_{hk} \\ \downarrow & & \downarrow \\ \mathcal{A}_{gh} \times \mathcal{A}_k & \longrightarrow & \mathcal{A}_{(gh)k} = \mathcal{A}_{g(hk)} \end{array}$$

(ii) The involution maps are conjugate-linear and isometric.

(iii) $a_g^{**} = a_g$.

(iv) $(a_g a_h)^* = a_h^* a_g^*$.

(v) $\|a_g^*\| = \|a_g\|$.

(vi) $\|a_g a_h\|_{\mathcal{A}_{gh}} \leqslant \|a_g\|_{\mathcal{A}_g} \|a_h\|_{\mathcal{A}_h}$.

(vii) $\|a_g\|^2_{\mathcal{A}_g} = \|a_g^* a_g\|_{\mathcal{A}_1}$.

(viii) $a^*a \geq 0$ in $\mathcal{A}_1$.

Remark 2.3.2. The properties (i)-(vii) imply that $\mathcal{A}_1$ is a C*-algebra with the restricted operations. We call $\mathcal{A}_1$ the unit fiber algebra. To simplify notation we write $\mathcal{A}_g\mathcal{A}_h$ to mean the closed linear span of $\{a_g a_h \mid a_g \in \mathcal{A}_g \text{ and } a_h \in \mathcal{A}_h\}$. In particular, $\mathcal{A}_g\mathcal{A}_{g^{-1}}$ is a closed two-sided ideal in $\mathcal{A}_1$ for all $g \in G$.

Example 2.3.3. A classical example is given when G is an arbitrary discrete group and we associate a Fell bundle over G, called the *group bundle* and denoted by $\mathbb{C}\times G$, in the canonical way: We define the fibers $\mathcal{A}_g := \mathbb{C}\times\{g\}$ with the structure of Banach space inherited from $\mathbb{C}$, that is, $\mathcal{A}_g$ is a copy of $\mathbb{C}$ for each $g\in G$. The multiplication and involution operations come from $\mathbb{C}$. This results into a Fell bundle $\mathbb{C}\times G=\{\mathcal{A}_g\}_{g\in G}$ over G.

Example 2.3.4. More generally, given a dynamical system (A,G,α) we can define a Fell bundle over G associated to (A,G,α), called the *semi-direct product bundle relative to* α and denoted by $\mathcal{A}^\alpha$ with the fibers $\mathcal{A}_g := A\times\{g\}$, that is, $\mathcal{A}_g$ is a copy of A as a Banach space for each $g\in G$. We write au_g for (a,g) so that $\mathcal{A}_g = Au_g$. The multiplication is defined by

$$(au_g)(bu_h) = a\alpha_h(b)u_{gh}$$

and the involution is defined by

$$(au_g)^* = \alpha_{g^{-1}}(a^*)u_{g^{-1}}$$

for all $a,b\in A$ and $g,h\in G$. These operations turn $\mathcal{A}^\alpha=\{\mathcal{A}_g\}_{g\in G}$ into a Fell bundle over G.

Example 2.3.5. One more general example is when we consider a partial dynamical system (A,G,θ), that is, $\theta=(\{D_g\}_{g\in G},\{\theta_g\}_{g\in G})$ is a partial action of G on C*-algebra A such that D_g are closed two-sided ideals of A and $\theta_g : D_{g^{-1}}\to D_g$ are *-isomorphisms. Also in this case we can define a Fell bundle over G associated to (A,G,θ), also called the *semi-direct product bundle relative to* θ and denoted by $\mathcal{A}^\theta=\{\mathcal{A}_g\}_{g\in G}$ in the following way: The fibers $\mathcal{A}_g := D_g u_g$ where $\mathcal{A}_g$ is a copy of D_g as Banach space for each $g\in G$ while the multiplication and involution maps are given by:

$$(au_g)(bu_h) = \theta_g(\theta_{g^{-1}}(a)b)u_{gh} \quad\text{and}\quad (au_g)^* = \theta_{g^{-1}}(a^*)u_{g^{-1}}$$

for every $a\in D_g$, $b\in D_h$ and $g,h\in G$. More details about partial dynamical systems can also be found in [25].

Example 2.3.6. Another interesting example is given when A is a C*-algebra and we suppose there are closed subspaces $A_g\subseteq A$ such that $A_gA_h\subseteq A_{gh}$ and $A_g^*\subseteq A_{g^{-1}}$, for every $g,h\in G$. In this case $\mathcal{A}=\{A_g\}_{g\in G}$ is a Fell bundle over G with multiplication and involution induced from A.

From now on, we fix a Fell bundle $\mathcal{A}=\{\mathcal{A}_g\}_{g\in G}$ over a discrete group G.

Definition 2.3.7. Consider the algebraic direct sum

$$\bigoplus_{g\in G}\mathcal{A}_g := \left\{\sum_{g\in G}^{finite} a_g u_g \mid a_g\in\mathcal{A}_g\right\}$$

with multiplication and involution given by (linearly extending the following)

$$(au_g)(bu_h) := abu_{gh} \quad\text{and}\quad (au_g)^* := a^*u_{g^{-1}}$$

for all $a \in \mathcal{A}_g$, $b \in \mathcal{A}_h$ and $g, h \in G$. For each $g \in G$ and $a \in \mathcal{A}_g$ we denote by au_g the element of the algebraic sum whose coordinates are equal to zero, expect for the coordinate corresponding to g, which is equal to a. Then it is clearly that any element $a \in \bigoplus_{g\in G} \mathcal{A}_g$ can be written uniquely as a finite sum in the form $a = \sum_{g\in G} a_g u_g$. In this way the algebraic direct sum becomes a *-algebra.

Moreover, we can identify $\bigoplus_{g\in G} \mathcal{A}_g$ with $C_c(\mathcal{A})$, the collection of all compact supported sections ξ from G to $\mathcal{A}$ such that $\xi_g \in \mathcal{A}_g$ for every $g \in G$. From this point of view the multiplication, known as convolution product, and the involution are given by

$$(\xi * \eta)_g := \sum_{h\in G} \xi_h \eta_{h^{-1}g} \quad \text{and} \quad \xi_g^* := \xi_{g^{-1}}^*$$

for all $\xi, \eta \in C_c(\mathcal{A})$ and $g \in G$. With the above operations $C_c(\mathcal{A})$ is therefore a *-algebra.

Remark 2.3.8. With respect to the convolution product note that $\xi_h \eta_{h^{-1}g} \in \mathcal{A}_h \mathcal{A}_{h^{-1}g} \subseteq \mathcal{A}_g$ for every $h \in G$. So all of the summands lie in the same vector space $\mathcal{A}_g$, hence the sum is well defined. The involution map is also well defined since $\xi_{g-1}^* \in \mathcal{A}_{g^{-1}}^* \subseteq \mathcal{A}_g$ for every $g \in G$.

Lemma 2.3.9. *Let B be a C*-algebra and $\pi : C_c(\mathcal{A}) \to B$ be any *-homomorphism. Then*

$$\|\pi(a)\| \leqslant \|a\|_1 := \sum_{g\in G} \|a_g\|.$$

Proof. Let $a = \sum_{g\in G} a_g u_g \in C_c(\mathcal{A})$, $a_g \in \mathcal{A}_g$. Since π is linear then $\pi(a) = \sum_{g\in G} \pi(a_g u_g)$. Note that the map $\mathcal{A}_1 \hookrightarrow C_c(\mathcal{A})$ which sends $x \mapsto xu_1$ is a *-homomorphism for every $x \in \mathcal{A}_1$. Then via the composition with π we get the *-homomorphism $\mathcal{A}_1 \to B$ sending $x \mapsto \pi(xu_1)$. Therefore, $\|\pi(xu_1)\| \leqslant \|x\|_{\mathcal{A}_1}$. Now, for each $a_g \in \mathcal{A}_g$ we compute:

$$\begin{aligned}
\|\pi(a_g u_g)\|^2 &= \|\pi(a_g u_g)^* \pi(a_g u_g)\| \\
&= \|\pi(a_g^* u_{g^{-1}}) \pi(a_g u_g)\| \\
&= \|\pi(a_g^* a_g u_1)\| \\
&\leqslant \|a_g^* a_g\|_{\mathcal{A}_1} \\
&= \|a_g\|_{\mathcal{A}_g}^2,
\end{aligned}$$

that is, $\|\pi(a_g u_g)\| \leqslant \|a_g\|_{\mathcal{A}_g}$. We conclude that

$$\|\pi(a)\| = \|\sum_{g\in G} \pi(a_g u_g)\| \leqslant \sum_{g\in G} \|\pi(a_g u_g)\| \leqslant \sum_{g\in G} \|a_g\| = \|a\|_1$$

as desired.

□

Therefore, given $a \in C_c(\mathcal{A})$, we can define

$$\begin{aligned}
\|a\|_u :&= \sup\{\|\pi(a)\| \mid \pi : C_c(\mathcal{A}) \to B \text{ is a *-homomorphism } \} \\
&= \sup\{p(a) \mid p \text{ is a C*-seminorm on } C_c(\mathcal{A})\}
\end{aligned}$$

where B is a C*-algebra. By the above result $\|.\|_u$ is a well-defined C*-seminorm. Moreover, defining the closed two-sided ideal $\mathcal{N} = \{a \in C_c(\mathcal{A}) \mid \|a\|_u = 0\}$ of $C_c(\mathcal{A})$, this gives rise to the next definition.

Definition 2.3.10. The *(full) C*-algebra of a Fell bundle* $\mathcal{A}$, denote by $C^*(\mathcal{A})$, is the C*-algebra obtained by taking the quotient of $C_c(\mathcal{A})$ by the ideal $\mathcal{N}$ and completing it with respect to $\|\|_u$, that is,

$$C^*(\mathcal{A}) := \overline{C_c(\mathcal{A})/\mathcal{N}}^{\|\cdot\|_u}.$$

Remark 2.3.11. In literature, $C^*(\mathcal{A})$ is also called the (full) cross-sectional C*-algebra of $\mathcal{A}$.

Remark 2.3.12. In other words, $C^*(\mathcal{A})$ is the enveloping C*-algebra of $l^1(\mathcal{A}) := \overline{C_c(\mathcal{A})}^{\|\cdot\|_1}$, that is, $l^1(\mathcal{A})$ is the Banach *-algebra consisting of all sections ξ from G to $\mathcal{A}$ such that $\|\xi\|_1 < \infty$.

Definition 2.3.13. A representation of a Fell bundle $\mathcal{A}$ in a C*-algebra B is a collection $\pi := \{\pi_g\}_{g\in G}$ of linear maps $\pi_g : \mathcal{A}_g \to B$ such that:

(i) $\pi_g(a)\pi_h(b) = \pi_{gh}(ab)$;

(ii) $\pi_g(a)^* = \pi_{g^{-1}}(a^*)$

for all $a \in \mathcal{A}_g$, $b \in A_h$ and $g, h \in G$.

Now, we will present a canonical representation of a Fell bundle $\mathcal{A}$ into $C^*(\mathcal{A})$. For this, for each $g \in G$ we denote by ι_g the natural inclusion maps of $\mathcal{A}_g$ into $C_c(\mathcal{A})$, that is, $\iota_g(a) := au_g$ for all $a \in \mathcal{A}_g$. From the point of view of sections, ι_g is defined by

$$\iota_g(a)_h := \begin{cases} a, & \text{if } h = g \\ 0, & \text{otherwise} \end{cases}.$$

Moreover, we call just by $k : C_c(\mathcal{A}) \to C^*(\mathcal{A})$ the canonical map arising from the completion process. Since k is a *-homomorphism, it is straightforward to see that the composition maps $j_g := k \circ \iota_g$ form a representation $j = \{j_g\}_{g\in G}$ of a Fell Bundle $\mathcal{A}$ into $C^*(\mathcal{A})$, henceforth called the *universal representation* of $\mathcal{A}$.

Remark 2.3.14. We have already seen that $\mathcal{A}_1$ is always a C*-algebra and it is not difficult to see that j_1 is a nondegenerate *-homomorphism.

Proposition 2.3.15. *Let $\pi = \{\pi_g\}_{g\in G}$ be a representation of a Fell Bundle $\mathcal{A}$ in a C*-algebra B. Then there is a unique *-homomorphism $\overline{\pi} : C^*(\mathcal{A}) \to B$ such that $\overline{\pi}(j_g(a)) = \pi_g(a)$ for all $a \in \mathcal{A}_g$ and $g \in G$. We will say that $\overline{\pi}$ is the integrated form of π.*

Proof. A representation π of $\mathcal{A}$ in B produces a representation $\tilde{\pi}$ of $C_c(\mathcal{A})$ in B given by

$$\tilde{\pi}(a) = \sum_{g\in G} \pi_g(a_g) \quad \text{for every } a = \sum_{g\in G} a_g u_g \in C_c(\mathcal{A}).$$

This extends uniquely to a *-homomorphism $\overline{\pi} : C^*(\mathcal{A}) \to B$ by the universal property and by construction it satisfies $\overline{\pi} \circ j_g = \pi_g$ for every $g \in G$ as we asserted.

□

Corollary 2.3.16. *Let B be a C*-algebra. There are canonical bijections*

$$\textit{Rep}(\mathcal{A}, B) \cong \textit{Rep}(C_c(\mathcal{A}), B) \cong \textit{Rep}(C^*(\mathcal{A}), B).$$[1]

Proof. By Proposition 2.3.15, every representation π of $\mathcal{A}$ to B can be extended to $C_c(\mathcal{A})$ and by the universal property extended to $C^*(\mathcal{A})$. In another direction, if $\rho : C_c(\mathcal{A}) \to B$ is a *-homomorphism then we can define $\pi = \{\pi_g\}_{g\in G}$ from $\mathcal{A}$ to B such that $\pi_g(a) := \rho(au_g)$ for every $a \in \mathcal{A}_g$ and $g \in G$. In fact π is a representation since for every $a \in \mathcal{A}_g$, $b \in \mathcal{A}_h$ and $g, h \in G$ we have

$$\pi_g(a)\pi_h(b) = \rho(au_g)\rho(bu_h) = \rho(abu_{gh}) = \pi_{gh}(ab)$$

and

$$\pi_g(a)^* = \rho(au_g)^* = \rho(a^*u_{g^{-1}}) = \pi_{g^{-1}}(a^*).$$

Besides that, $\tilde{\pi} = \rho$ since for every $\xi = \sum_{g\in G} a_g u_g$ we have

$$\tilde{\pi}(\xi) = \sum_{g\in G} \pi_g(a_g) = \sum_{g\in G} \rho(a_g\delta_g) = \rho\left(\sum_{g\in G} a_g\delta_g\right) = \rho(\xi).$$

The same idea works from $C^*(\mathcal{A})$ since it is the universal completion of $C_c(\mathcal{A})$. Thus

$$\begin{gathered} Rep(\mathcal{A}, B) \cong Rep(C_c(\mathcal{A}), B) \cong Rep(C^*(\mathcal{A}), B) \\ \pi \longrightarrow \tilde{\pi} \longrightarrow \overline{\pi} \end{gathered}$$

□

Example 2.3.17. If G is a discrete group then the C*-algebra of the group bundle seen in Example 2.3.3 is the group C*-algebra $C^*(G)$ since $C_c(\mathbb{C} \times G) \cong \mathbb{C}[G]$, the *-algebra of the group G. Most of the time we use the same notation u_g to denote the unitary elements of $C^*(G)$. In fact, $C^*(G)$ is the universal C*-algebra generated by these unitary elements u_g with the same definition of multiplication and involution maps.

Example 2.3.18. More generally, given a dynamical system (A, G, α) the C*-algebra of the semi-direct product bundle $\mathcal{A}^\alpha$ defined in Example 2.3.4 is canonically isomorphic to the crossed product $A \rtimes_\alpha G$ since $C_c(\mathcal{A}^\alpha)$ is isomorphic to $C_c(G, A)$ as *-algebras. Even more general given a partial dynamical system (A, G, θ), the C*-algebra $C^*(\mathcal{A}^\theta)$ is isomorphic to crossed product $A \times_\theta G$ where $\mathcal{A}^\theta$ is the Fell bundle defined in Example 2.3.5.

[1] Rep$(\mathcal{A}, B)$ denotes the set of representations from the Fell bundle $\mathcal{A}$ to B, the sense of Definition 2.3.13, and representations of Rep$(C_c(\mathcal{A}), B)$ and Rep$(C^*(\mathcal{A}), B)$ are just *-homomorphisms.

Example 2.3.19. Let G be an arbitrary group and A be a C*-algebra. If we define

$$\mathcal{A}_g = \begin{cases} A, & \text{if } g = 1 \\ \{0\}, & \text{otherwise} \end{cases}$$

this results into a trivial Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ and it is not difficult to see that $C^*(\mathcal{A}) \cong A$ identifying au_1 to a for all $a \in A$.

Now, the goal is to find a "standard" representation of a Fell bundle $\mathcal{A}$. Until now, j_1 is a *-homomorphism through which we may view $\mathcal{A}_1$ as a *-subalgebra of $C_c(\mathcal{A})$. This makes $C_c(\mathcal{A})$ into a right $\mathcal{A}_1$-module in a standard way: the right action is defined by $(\xi \cdot a)_g := \xi_g a \in \mathcal{A}_g$ for every $\xi \in C_c(\mathcal{A})$ and $a \in \mathcal{A}_1$, and we are going to introduce a $\mathcal{A}_1$-valued inner-product on $C_c(\mathcal{A})$ as follows:

$$\langle \xi, \eta \rangle_{\mathcal{A}_1} := \sum_{g\in G} \xi_g^* \eta_g \qquad \forall \xi, \eta \in C_c(\mathcal{A}).$$

It is not difficult to verify that this is indeed an inner-product. Once this is done $C_c(\mathcal{A})$ becomes a right pre-Hilbert $\mathcal{A}_1$-module.

So, we denote by $l^2(\mathcal{A})$ the right Hilbert $\mathcal{A}_1$-module obtained by completing $C_c(\mathcal{A})$ under the norm $\|.\|_2$ arising from the inner-product defined above. We can see $C_c(\mathcal{A})$ as a dense subspace of $l^2(\mathcal{A})$ and note that ι_g viewed as a map from $\mathcal{A}_g$ to $l^2(\mathcal{A})$ is an isometry because for every $a \in \mathcal{A}_g$ we have

$$\|\iota_g(a)\|_2^2 = \|\langle \iota_g(a), \iota_g(a) \rangle\|_{\mathcal{A}_1} = \|a^*a\|_{\mathcal{A}_1} = \|a\|_{\mathcal{A}_g}^2.$$

We will construct a representation of $\mathcal{A}$ in $\mathcal{L}(l^2(\mathcal{A}))$, the C*-algebra of adjointable operators on $l^2(\mathcal{A})$. More details about Hilbert modules and adjointable operators can be found in Appendix B. Before that, we need the next lemma.

Lemma 2.3.20. *([25], Lemma 17.2) Given $g, h \in G$, $a \in \mathcal{A}_g$ and $b \in \mathcal{A}_h$, we have*

$$b^*a^*ab \leqslant \|a\|_{\mathcal{A}_g}^2 b^*b.$$

Now, for each $g \in G$ and $a \in \mathcal{A}_g$ consider $\lambda_g(a) : C_c(\mathcal{A}) \to C_c(\mathcal{A})$ to be the linear operator defined by

$$\lambda_g(a)(\xi)_h := a\xi_{g^{-1}h}$$

for all $\xi \in C_c(\mathcal{A})$ and $h \in G$. This gives us a representation of $\mathcal{A}$ as we will see in the next proposition.

Proposition 2.3.21. *With notations as above, $\lambda := \{\lambda_g\}_{g\in G}$ is a representation of $\mathcal{A}$ in $\mathcal{L}(l^2(\mathcal{A}))$. This representation is called the regular representation of Fell bundle $\mathcal{A}$.*

Proof. Notice that $\lambda_g(a)$ is well defined because $\mathcal{A}_g\mathcal{A}_{g^{-1}h} \subseteq \mathcal{A}_h$ and it satisfies

$$\begin{aligned}
\|\lambda_g(a)(\xi)\|_2^2 &= \langle \lambda_g(a)(\xi), \lambda_g(a)(\xi)\rangle_{\mathcal{A}_1} \\
&= \sum_{h\in G} (a\xi_{g^{-1}h})^* a\xi_{g^{-1}h} \\
&= \sum_{h\in G} \xi^*_{g^{-1}h} a^* a\xi_{g^{-1}h} \\
&\leq \|a\|^2 \sum_{h\in G} \xi^*_{g^{-1}h}\xi_{g^{-1}h} \qquad \text{(by Lemma 2.3.20)} \\
&= \|a\|^2 \sum_{k\in G} \xi^*_k\xi_k \qquad \text{with } k = g^{-1}h \\
&= \|a\|^2 \langle \xi, \xi\rangle_{\mathcal{A}_1} \\
&= \|a\|^2 \|\xi\|_2^2.
\end{aligned}$$

This implies that $\|\lambda_g(a)(\xi)\|_2 \leqslant \|a\|\|\xi\|_2$ proving that $\lambda_g(a)$ is bounded with $\|\lambda_g(a)\|_2 \leqslant \|a\|$. Therefore $\lambda_g(a)$ extends to a continuous operator on $l^2(\mathcal{A})$ satisfying $\lambda_g(a)(\iota_h(b)) = \iota_{gh}(ab)$ for all $b \in \mathcal{A}_h$ since for each $k \in G$ we have

$$\lambda_g(a)(\iota_h(b))_k = a\iota_h(b)_{g^{-1}k} = \delta_{h,g^{-1}k}ab = \delta_{gh,k}ab = \iota_{gh}(ab)_k.$$

In fact, we can prove that $\|\lambda_g(a)\|_2 = \|a\|$. To do that, take $\iota_{g^{-1}}(a^*)$ an element of $l^2(\mathcal{A})$ via $\iota_{g^{-1}}$. Since $\iota_{g^{-1}}$ is isometric then we have $\|\iota_{g^{-1}}(a^*)\|_2 = \|a\|$. Therefore,

$$\lambda_g(a)(\iota_{g^{-1}}(a^*)) = \iota_1(aa^*).$$

So, we see that

$$\begin{aligned}
\|a\|^2 = \|aa^*\| = \|\iota_1(aa^*)\|_2 = \|\lambda_g(a)(\iota_{g^{-1}}(a^*))\|_2 \leqslant \|\lambda_g(a)\|\|\iota_{g^{-1}}(a^*)\|_2 = \|\lambda_g(a)\|\|a\| \\
\Rightarrow \|\lambda_g(a)\| \geq \|a\|
\end{aligned}$$

That is, $\|\lambda_g(a)\|_2 = \|a\|$. In addition, $\lambda_g(a)$ is adjointable operator with adjoint given by $\lambda_g(a)^* := \lambda_{g^{-1}}(a^*)$. The reason is because

$$\langle \lambda_g(a)(\xi), \eta\rangle_{\mathcal{A}_1} = \sum_{h\in G} (a\xi_{g^{-1}h})^*\eta_h = \sum_{h\in G} \xi^*_{g^{-1}h} a^*\eta_h = \sum_{k\in G} \xi^*_k a^*\eta_{gk} = \langle \xi, \lambda_{g^{-1}}(a^*)(\eta)\rangle_{\mathcal{A}_1}$$

for all $\xi, \eta \in C_c(\mathcal{A})$. Since $C_c(\mathcal{A})$ is dense in $l^2(\mathcal{A})$ we conclude from the above that

$$\langle \lambda_g(a)(\xi), \eta\rangle_{\mathcal{A}_1} = \langle \xi, \lambda_{g^{-1}}(a^*)(\eta)\rangle_{\mathcal{A}_1}$$

for all $\xi, \eta \in l^2(\mathcal{A})$. Therefore, $\lambda_g : \mathcal{A}_g \to \mathcal{L}(l^2(\mathcal{A}))$ is a well defined linear map and isometric. Moreover, for all $a \in \mathcal{A}_g$, $b \in \mathcal{A}_h$, $\xi \in C_c(\mathcal{A})$ and $g, h, k \in G$ we compute

$$\lambda_g(a)(\lambda_h(b)(\xi))_k = a\lambda_h(b)(\xi)_{g^{-1}k} = ab\xi_{h^{-1}g^{-1}k} = ab\xi_{(gh)^{-1}k} = \lambda_{gh}(ab)(\xi)_k.$$

Since $C_c(\mathcal{A})$ is dense in $l^2(\mathcal{A})$ we conclude that $\lambda_{gh}(ab) = \lambda_g(a) \circ \lambda_h(b)$ for all $a \in \mathcal{A}_g$, $b \in \mathcal{A}_h$ and $g, h \in G$. After all these conclusions, $\lambda = \{\lambda_g\}_{g\in G}$ is in fact a representation of $\mathcal{A}$ into $\mathcal{L}(l^2(\mathcal{A}))$ as desired.

□

By the universal property of $C^*(\mathcal{A})$ seen in Proposition 2.3.15 we get a unique *-homomorphism $\Lambda : C^*(\mathcal{A}) \to \mathcal{L}(l^2(\mathcal{A}))$ such that $\Lambda \circ j_g = \lambda_g$ for every $g \in G$. Indeed Λ satisfies $\Lambda(\xi)(\eta) = \xi * \eta$ for all $\xi, \eta \in C_c(\mathcal{A})$.

Proposition 2.3.22. *The *-homomorphism $\Lambda : C^*(\mathcal{A}) \to \mathcal{L}(l^2(\mathcal{A}))$ is injective on $C_c(\mathcal{A})$. Moreover, $C_c(\mathcal{A})$ admits a C*-norm namely $\|\xi\|_r := \|\Lambda(\xi)\|$ for every $\xi \in C_c(\mathcal{A})$.*

Proof. Let $\xi \in C_c(\mathcal{A})$ and suppose that $\Lambda(\xi) = 0$. Thus, $\Lambda(\xi)(\eta) = 0$ for all $\eta \in C_c(\mathcal{A})$. In particular, taking $\eta = \iota_1(a)$ for all $a \in \mathcal{A}_1$ we have

$$0 = \Lambda(\xi)(\iota_1(a))_g = (\xi * \iota_1(a))_g = \sum_{h \in G} \xi_h \iota_1(a)_{h^{-1}g} = \xi_g a$$

for all $g \in G$. Thus $\xi_g a = 0$ for all $a \in \mathcal{A}_1$. Taking a as an approximate identity[2] for $\mathcal{A}_1$ implies that $\xi_g = 0$ for all $g \in G$. Therefore, $\xi = 0$. Because of this, it is straightforward that $\|\xi\|_r := \|\Lambda(\xi)\|$ is a C*-norm for $C_c(\mathcal{A})$.

□

Definition 2.3.23. The *-homomorphism Λ defined above is called *the regular representation* of $C^*(\mathcal{A})$ and we define the *reduced C*-algebra of a Fell bundle* $\mathcal{A}$, denoted by $C^*_r(\mathcal{A})$, as the C*-subalgebra of $\mathcal{L}(l^2(\mathcal{A}))$ generated by the image of the regular representation of $\mathcal{A}$, that is,

$$C^*_r(\mathcal{A}) := \Lambda(C^*(\mathcal{A})) \subseteq \mathcal{L}(l^2(\mathcal{A})).$$

Remark 2.3.24. Observe that $C^*_r(\mathcal{A})$ is isomorphic to the quotient of $C^*(\mathcal{A})$ by the kernel of Λ and some times we will see the regular representation as $\Lambda : C^*(\mathcal{A}) \twoheadrightarrow C^*_r(\mathcal{A})$. In the literature $C^*_r(\mathcal{A})$ is also called the reduced cross-sectional C*-algebra of $\mathcal{A}$.

Remark 2.3.25. In general, Λ is not injective on $C^*(\mathcal{A})$. It is therefore crucial to understand the kernel of Λ and, in particular, to determine conditions under which Λ is injective. We say that the Fell bundle $\mathcal{A}$ is *amenable* if the regular representation $\Lambda : C^*(\mathcal{A}) \twoheadrightarrow C^*_r(\mathcal{A})$ is injective.

Example 2.3.26. In the special case of the group bundle seen in Example 2.3.3 the reduced C*-algebra of the group bundle is precisely $C^*_r(G)$, the reduced C*-algebra of a group. Recall that $C^*_r(G)$ is the image of $C^*(G)$ under the left regular representation λ^G of G on $B(l^2(G))$, that is, $\lambda^G_g(\xi)(h) = \xi(g^{-1}h)$ for every $\xi \in l^2(G)$ and $g, h \in G$. In this case it is well known that the injectivity of Λ is equivalent to the amenability of G by [[22], Theorem 7.3.9]. We will denote by $\Lambda^G : C^*(G) \twoheadrightarrow C^*_r(G) \subseteq B(l^2(G))$ the regular representation of G.

Example 2.3.27. The reduced cross-sectional C*-algebra of the semi-direct product bundle relative to an action α of G on C*-algebra A seen in Example 2.3.4 is naturally isomorphic

[2] An approximate identity for a C*-algebra A is a net $(e_i)_{i \in I}$ of positive elements of A, with $\|e_i\| \leqslant 1$, such that $a = \lim_{i\to\infty} ae_i = \lim_{i\to\infty} e_i a$, for every $a \in A$. Every C*-algebra is known to admit an approximate identity [[47], Theorem 3.1.1].

to the reduced crossed product $A \rtimes_{\alpha,r} G$ and we denote by $\Lambda^{A\rtimes G} : A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$ the regular representation. In this case we know that if G is amenable then $\Lambda^{A\rtimes G}$ is injective by [[60], Theorem 7.13]. However the converse is not true in general. For example, if G acts on $C_0(G)$ by left translation we have $C_0(G) \rtimes_\tau G \cong C_0(G) \rtimes_{\tau,r} G \cong \mathcal{K}(l^2(G))$ for every group G (see in Appendix A).

Even more generally, the reduced C*-algebra of the semi-direct product bundle relative to a partial action θ seen in Example 2.3.5 is naturally isomorphic to the reduced crossed product algebra $A \rtimes_{\theta,r} G$.

Remark 2.3.28. Our slightly unusual choice of notation Λ^G and $\Lambda^{A\rtimes G}$, as opposed to Λ, is due to a potential conflict between this and our previous notation for the regular representation introduced before.

Remark 2.3.29. In this context, the universal representation $j_g : \mathcal{A}_g \to C^*(\mathcal{A})$ is also isometric. The reason is because on the one hand, since k is *-homomorphism and ι_g is isometric, we have

$$\|j_g(a)\| = \|k(\iota_g(a))\| \leqslant \|\iota_g(a)\| = \|a\|.$$

But one the other hand, since Λ is *-homomorphism and λ_g is isometric, we have

$$\|j_g(a)\| \geq \|\Lambda(j_g(a))\| = \|\lambda_g(a)\| = \|a\|$$

for every $a \in A_g$, $g \in G$.

For both full and reduced C*-algebras $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$ there are canonical conditional expectations onto the unit fiber $\mathcal{A}_1$ that will be of great importance for the next chapters. We begin by recalling the concept of a conditional expectation onto a subalgebra.

Definition 2.3.30. Let A be a C*-algebra and let $B \subseteq A$ be a C*-subalgebra. Then we call a linear mapping $P : A \to B$ a *conditional expectation* of A onto B if it is positive, idempotent, contractive and B-bimodule map. Moreover, P is faithful if $P(a^*a) = 0$ implies that $a = 0$.

Lemma 2.3.31 ([25], Lemma 17.8)**.** *For each $g \in G$, there is a unique contractive linear map $E_g : C^*_r(\mathcal{A}) \to \mathcal{A}_g$ such that $E_g(\lambda_h(a)) = \delta_{g,h}a$ for every $a \in A_h$. In the literature, for each $z \in C^*_r(\mathcal{A})$, $E_g(z)$ is called as the g^{th} Fourier coefficient of z.*

Proposition 2.3.32. *With definitions as above, $E_1 : C^*_r(\mathcal{A}) \to \mathcal{A}_1$ is a faithful conditional expectation. Moreover, $\tilde{E}_1 : C^*(\mathcal{A}) \to \mathcal{A}_1$ defined by $\tilde{E}_1 := E_1 \circ \Lambda$ is a conditional expectation and it is faithful if and only if $\Lambda : C^*(\mathcal{A}) \twoheadrightarrow C^*_r(\mathcal{A})$ is an isomorphism. Indeed,*

$$\mathit{Ker}(\Lambda) = \{x \in C^*(\mathcal{A}) \mid \tilde{E}_1(x^*x) = 0\}.$$

Proof. It is straightforward to check that E_1 is a contractive, idempotent and $\mathcal{A}_1$-bimodule map by definition. To become a conditional expectation we need to check that E_1 is a positive map. For this we claim that $a^*E_{gh^{-1}}(x)b = \langle \iota_g(a), x\iota_h(b)\rangle_{\mathcal{A}_1}$ for every $x \in C^*_r(\mathcal{A})$, $a \in \mathcal{A}_g$

and $b \in \mathcal{A}_h$. To see that it is enough to check this for $x = \lambda_k(c)$, $c \in \mathcal{A}_k$ since the closed linear span of the set of all $\lambda_k(c)$, for $c \in \mathcal{A}_k$, is dense in $C_r^*(\mathcal{A})$. So,

$$a^* E_{gh^{-1}}(\lambda_k(c))b = a^*(c\delta_{gh^{-1},k})b = a^*(cb)\delta_{g,kh} = \langle \iota_g(a), \iota_{kh}(cb)\rangle_{\mathcal{A}_1} = \langle \iota_g(a), \lambda_k(c)\iota_h(b)\rangle_{\mathcal{A}_1}.$$

By linearity and continuity, the claim follows. Consequently, for every $a \in A_1$ we have

$$a^* E_1(x^*x)a = \langle \iota_1(a), x^*x\iota_1(a)\rangle_{\mathcal{A}_1} = \langle x\iota_1(a), x\iota_1(a)\rangle_{\mathcal{A}_1} \geq 0$$

which implies that $a^*E_1(x^*x)a \geq 0$. In particular, taking a as an approximate identity in $\mathcal{A}_1$ we conclude that $E_1(x^*x) \geq 0$. Now, for faithfulness, suppose that $E_1(x^*x) = 0$. Using the formula above we have $0 = b^*E_1(x^*x)b = \langle x\iota_g(b), x\iota_g(b)\rangle_{\mathcal{A}_1}$ for every $b \in \mathcal{A}_g$. This is implies that $x\iota_g(b) = 0$ for all $b \in A_g$. Since $C_c(\mathcal{A})$ is dense subspace of $l^2(\mathcal{A})$ we conclude that $x = 0$ proving that E_1 is faithful conditional expectation.

Moreover, by the definition of $\tilde{E}_1$ we have $\tilde{E}_1(x^*x) = 0$ if and only if $E_1(\Lambda(x^*x)) = 0$ if and only if $E_1(\Lambda(x)^*\Lambda(x))) = 0$. Since E_1 is faithful then $\tilde{E}_1(x^*x) = 0$ if and only if $\Lambda(x) = 0$. □

Definition 2.3.33. Let $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ and $\mathcal{B} = \{\mathcal{B}_g\}_{g\in G}$ be a Fell Bundles. We define a Fell bundle morphism $\pi : \mathcal{A} \to \mathcal{B}$ as a family $\pi = \{\pi_g\}_{g\in G}$ of linear maps $\pi_g : \mathcal{A}_g \to \mathcal{B}_g$ such that:

1. $\pi_g(a)\pi_h(b) = \pi_{gh}(ab)$;
2. $\pi_g(a)^* = \pi_{g^{-1}}(a^*)$

for all $a \in \mathcal{A}_g$, $b \in \mathcal{A}_h$ and $g, h \in G$.

Remark 2.3.34. For every $g \in G$, if π_g is bijective then we say that π is an isomorphism with inverse defined naturally, that is, $\pi^{-1} := \{\pi_g^{-1}\}_{g\in G}$. Then $\mathcal{A}$ is isomorphic to $\mathcal{B}$ if there is an isomorphism π as above.

Proposition 2.3.35. *The constructions $\mathcal{A} \mapsto C^*(\mathcal{A})$ and $\mathcal{A} \mapsto C_r^*(\mathcal{A})$ are functorial: if $\pi\colon \mathcal{A} \to \mathcal{B}$ is a morphism of Fell bundles over G, then there are (unique) *-homomorphisms $\bar{\pi} : C^*(\mathcal{A}) \to C^*(\mathcal{B})$ and $\bar{\pi}^r : C_r^*(\mathcal{A}) \to C_r^*(\mathcal{B})$ "extending" π in the sense that $\bar{\pi}(j_g^{\mathcal{A}}(a)) = j_g^{\mathcal{B}}(\pi_g(a))$ and $\bar{\pi}^r(\lambda_g^{\mathcal{A}}(a)) = \lambda_g^{\mathcal{B}}(\pi_g(a))$ for all $a \in \mathcal{A}_g$, where $j_g^{\mathcal{A}}$, $\lambda_g^{\mathcal{A}}$ and $j_g^{\mathcal{B}}$, $\lambda_g^{\mathcal{B}}$ denote the canonical representations of $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover, if π_1 is injective then $\bar{\pi}^r$ is also injective.*

Proof. For the first part, it is straightforward to check that $j_g^{\mathcal{B}}(\pi_g(a)) : \mathcal{A}_g \to C^*(\mathcal{B})$ gives us a representation of $\mathcal{A}$ in $C^*(\mathcal{B})$. By the universal property seen in Proposition 2.3.15 there is a (unique) *-homomorphim $\bar{\pi} : C^*(\mathcal{A}) \to C^*(\mathcal{B})$ such that $\bar{\pi}(j_g^{\mathcal{A}}(a)) = j_g^{\mathcal{B}}(\pi_g(a))$ for all $a \in \mathcal{A}_g$.

For the second part, let $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{B}}$ be the regular representations and $\tilde{E}_1^{\mathcal{A}}$ and $\tilde{E}_1^{\mathcal{B}}$ the conditional expectations of $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$, respectively. So, we define $\bar{\pi}^r : C_r^*(\mathcal{A}) \to C_r^*(\mathcal{B})$

such that $\overline{\pi}^r(\Lambda^{\mathcal{A}}(x)) = \Lambda^{\mathcal{B}}(\overline{\pi}_g(x))$ for all $x \in C^*(\mathcal{A})$. To see that $\overline{\pi}^r$ is a well-defined map we need to check that the null space of $\Lambda_{\mathcal{A}}$ is contained into the null space of $\Lambda^{\mathcal{B}} \circ \overline{\pi}$. Recall from Proposition 2.3.32 that the null space of $\Lambda_{\mathcal{A}}$ is the same as $\{x \in C^*(\mathcal{A}) \mid \tilde{E_1^{\mathcal{A}}}(x^*x) = 0\}$. The same happens for $\Lambda_{\mathcal{B}}$. We claim that

$$\tilde{E_1^{\mathcal{B}}} \circ \overline{\pi}_g(x) = \pi_1 \circ \tilde{E_1^{\mathcal{A}}}(x) \tag{2.3.36}$$

for all $x \in C^*(\mathcal{A})$. To see that it is enough to check it for $x = j_g^{\mathcal{A}}(a)$, $a \in \mathcal{A}_g$ since the closed linear span of the set of all $j_g^{\mathcal{A}}(a)$, for $a \in \mathcal{A}_g$, is dense in $C^*(\mathcal{A})$. We have

$$\begin{aligned}
\tilde{E_1^{\mathcal{B}}} \circ \overline{\pi}_g(j_g^{\mathcal{A}}(a)) &= E_1^{\mathcal{B}} \circ \Lambda_{\mathcal{B}} \circ \overline{\pi}_g(j_g^{\mathcal{A}}(a)) \\
&= E_1^{\mathcal{B}} \circ \Lambda_{\mathcal{B}} \circ j_g^{\mathcal{B}}(\pi_g(a)) \\
&= E_1^{\mathcal{B}} \circ \lambda_g^{\mathcal{B}}(\pi_g(a)) \\
&= \delta_{g,1}\pi_g(a) \\
&= \pi_1(\delta_{g,1}a) \\
&= \pi_1 \circ E_1^{\mathcal{A}} \circ \lambda_g^{\mathcal{A}}(a) \\
&= \pi_1 \circ E_1^{\mathcal{A}} \circ \Lambda_{\mathcal{A}} \circ j_g^{\mathcal{A}}(a) \\
&= \pi_1 \circ \tilde{E_1^{\mathcal{A}}}(j_g^{\mathcal{A}}(a))
\end{aligned}$$

So, the claim is verified. Now, suppose that $\Lambda_{\mathcal{A}}(x) = 0$. Then $\Lambda_{\mathcal{A}}(x^*x) = 0$ and this implies that $\pi_1 \circ \tilde{E_1^{\mathcal{A}}}(x^*x) = \pi_1 \circ E_1^{\mathcal{A}} \circ \Lambda_{\mathcal{A}}(x^*x) = 0$. By 2.3.36, $\tilde{E_1^{\mathcal{B}}} \circ \overline{\pi}_g(x^*x) = E_1^{\mathcal{B}} \circ \Lambda_{\mathcal{B}} \circ \overline{\pi}_g(x^*x) = 0$. Since $E_1^{\mathcal{B}}$ is faithful then $\Lambda_{\mathcal{B}} \circ \overline{\pi}_g(x^*x) = 0$ which implies that $\Lambda_{\mathcal{B}} \circ \overline{\pi}_g(x) = 0$ as we desired. We conclude that $\overline{\pi}^r$ is well-defined *-homomorphism. Moreover, note that for every $x = j_g^{\mathcal{A}}(a)$ with $a \in \mathcal{A}_g$ we have $\overline{\pi}^r(\lambda_g^{\mathcal{A}}(a)) = \overline{\pi}^r(\Lambda_{\mathcal{A}}(x)) = \Lambda^{\mathcal{B}}(\overline{\pi}_g(x)) = \lambda_g^{\mathcal{B}}(\pi_g(a))$.

Finally, if π_1 is faithful then actually the null spaces of $\Lambda_{\mathcal{A}}$ and $\Lambda^{\mathcal{B}} \circ \overline{\pi}$ are equal. To see that, we use the previous formula 2.3.36. Suppose that $\Lambda^{\mathcal{B}}(\overline{\pi}_g(x)) = 0$ and hence $\Lambda^{\mathcal{B}}(\overline{\pi}_g(x^*x)) = 0$ which is equivalent to $\tilde{E_1^{\mathcal{B}}} \circ \overline{\pi}_g(x^*x) = 0$. By 2.3.36, $\pi_1 \circ \tilde{E_1^{\mathcal{A}}}(x^*x) = 0$. Since π_1 is faithful then $\tilde{E_1^{\mathcal{A}}}(x^*x) = 0$ which is equivalent to $\Lambda_{\mathcal{A}}(x) = 0$ and this completes the proof.

□

Definition 2.3.37. Let $\mathcal{A}$ be a Fell bundle. We say that $\mathcal{B}$ is a Fell sub-bundle of $\mathcal{A}$ if $\mathcal{B}_g$ are closed subspaces of $\mathcal{A}_g$ such that $\mathcal{B}_g\mathcal{B}_h \subseteq \mathcal{B}_{gh}$ and $\mathcal{B}_g^* \subseteq \mathcal{B}_{g^{-1}}$ for all $g, h \in G$.

Remark 2.3.38. It is clear that a Fell sub-bundle is itself a Fell bundle with the restricted operations and the inclusion map from $\mathcal{B}$ to $\mathcal{A}$ is a morphism.

Definition 2.3.39. Let $\mathcal{A}$ be a Fell Bundle and $\mathcal{I} = \{\mathcal{I}_g\}_{g\in G}$ a Fell sub-bundle of $\mathcal{A}$. We say that $\mathcal{I}$ is an ideal of $\mathcal{A}$ if $\mathcal{A}_g\mathcal{I}_h \subseteq \mathcal{I}_{gh}$ and $\mathcal{I}_g\mathcal{A}_h \subseteq \mathcal{I}_{gh}$ for all $g, h \in G$.

Given an ideal $\mathcal{I}$ of a Fell bundle $\mathcal{A}$ we can consider the quotient Fell bundle of $\mathcal{A}$ by $\mathcal{I}$. For each $g \in G$, consider the quotient spaces $\mathcal{A}_g/\mathcal{I}_g$ which the multiplication and involution operations included from $\mathcal{A}$ to the quotient, that is, the multiplication and involution are given by

$$\cdot : \mathcal{A}_g/\mathcal{I}_g \times \mathcal{A}_h/\mathcal{I}_h \to \mathcal{A}_{gh}/\mathcal{I}_{gh} \quad \text{and} \quad * : \mathcal{A}_g/\mathcal{I}_g \to \mathcal{A}_{g^{-1}}/\mathcal{I}_{g^{-1}}$$

for every $g, h \in G$. The result is a Fell bundle $\mathcal{A}/\mathcal{I} := \{\mathcal{A}_g/\mathcal{I}_g\}_{g\in G}$ over G called *quotient Fell bundle*.

Also, the canonical quotient maps $q_g : \mathcal{A}_g \to \mathcal{A}_g/\mathcal{I}_g$ gives us a morphism $q^{\mathcal{A}}$ from $\mathcal{A}$ to $\mathcal{A}/\mathcal{I}$ in the obvious way. According to the Proposition 2.3.35 we get surjective *-homomoprhisms $\bar{q}^{\mathcal{A}} : C^*(\mathcal{A}) \to C^*(\mathcal{A}/\mathcal{I})$ and $(\bar{q}^r)^{\mathcal{A}} : C^*_r(\mathcal{A}) \to C^*_r(\mathcal{A}/\mathcal{I})$ that make the diagram below commute:

$$\begin{array}{ccc} C^*(\mathcal{A}) & \xrightarrow{\bar{q}^{\mathcal{A}}} & C^*(\mathcal{A}/I) \\ \downarrow{\scriptstyle \Lambda_{\mathcal{A}}} & & \downarrow{\scriptstyle \Lambda_{\mathcal{A}/\mathcal{I}}} \\ C^*_r(\mathcal{A}) & \xrightarrow{(\bar{q}^r)^{\mathcal{A}}} & C^*_r(\mathcal{A}/\mathcal{I}) \end{array}$$

Moreover, by [[25], Proposition 21.15] $C^*(\mathcal{I})$ is an ideal of $C^*(\mathcal{A})$ and thus gives an exact short sequence of C*-algebras[3]

$$0 \to C^*(\mathcal{I}) \to C^*(\mathcal{A}) \to C^*(\mathcal{A}/\mathcal{I}) \to 0.$$

Also, according to [25] the C*-algebra $C^*_r(\mathcal{I})$ is an ideal of $C^*_r(\mathcal{A})$ and we still have well-defined *-homomorphisms $C^*_r(\mathcal{I}) \to C^*_r(\mathcal{A})$ and $C^*_r(\mathcal{A}) \to C^*_r(\mathcal{A}/\mathcal{I})$, however the short sequence for reduced C*-algebras it is not always exact. If G is exact group then it is exact sequence (see [[25], Theorem 21.18]). However, our interest is the diagram above but it can seen in [25] and [22] many other results involving exact sequences, exact groups and induced ideals in this context.

2.4 TOPOLOGICALLY GRADED C*-ALGEBRAS

A concept which is closely related to Fell bundles is the notion of G-graded C*-algebras. In this section, besides defining G-grading C*-algebras, we are going to define a topologically G-graded C*-algebra which arises naturally from discrete group coactions. Again, we mainly follow [25].

Definition 2.4.1. Let B be a C*-algebra and let G be a group. We say that a linearly independent collection $\{B_g\}_{g\in G}$ of closed subspaces of B is a *grading* if $\oplus_{g\in G} B_g$ is dense in B, and for every $g, h \in G$ one has $B_g B_h \subseteq B_{gh}$ and $B_g^* \subseteq B_{g^{-1}}$. In this case we say that B is a G-graded C*-algebra and each B_g is called a grading subspace.

Remark 2.4.2. Given a G-graded C*-algebra B, the collection of all grading subspaces $\{B_g\}_{g\in G}$ forms a Fell bundle over G with the norm, multiplication and involution operations inherited

[3] Let A, B, C be C*-algebras. We say a short sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{q} C \to 0$ is exact if ι and q are injective and surjective *-homomorphims, respectively and Ker$(q) = Im(\iota)$.

from B. A question is pertinent: Is the converse true? In other words, may every Fell bundle be obtained from a G-graded C*-algebra? The answer is yes but there might be many G-graded C*-algebras with the same (isomorphic) Fell bundle. For example, we will see in Corollary 2.4.7 that the full and reduced C*-algebras $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$ are G-graded, and consequently, the Fell bundle obtained by these two C*-algebras are isomorphic to the original Fell bundle $\mathcal{A}$ by Remark 2.3.29. But there are many situations where $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$ are not isomorphic (For example, the full and reduced C*-algebra of a non-amenable group). Despite the great similarity between Fell bundles over G and G-graded C*-algebra there are important conceptual differences.

Example 2.4.3. The most common example of a graded C*-algebra comes from a graph C*-algebra. For a graph E we have $C^*(E) = \overline{\bigoplus_{n\in\mathbb{Z}} C^*(E)_n}$ with $\mathbb{Z}$-grading given by

$$C^*(E)_n = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in \text{Path}(E) \text{ with } r(\mu) = r(\nu) \text{ and } |\mu| - |\nu| = n\}.$$

Moreover, $C^*(E)_n = \overline{L(E)_n}$ where $L(E)_n$ is defined by the same span above without closure.

Example 2.4.4. Let $\mathcal{O}_n$ be the Cuntz algebra generated by n isometries $S_1, \ldots, S_n$ and let $\beta : \mathbb{T} \to \text{Aut}(\mathcal{O}_n)$ be the gauge action, that is, $\beta_z(S_i) = zS_i$ for every $i \in \{1, \ldots, n\}$. If we define

$$B_m := \{x \in \mathcal{O}_n \mid \beta_z(x) = z^m x\}$$

then this gives us a grading $\{B_m\}_{m\in\mathbb{Z}}$ and consequently a Fell bundle $\mathcal{B} = \{B_m\}_{m\in\mathbb{Z}}$. In fact, we can show that $B_m = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \text{ words with } |\mu| - |\nu| = m\}$.

This is a particular example from the previous one as the Cuntz algebra $\mathcal{O}_n$ can be realized as the graph C*-algebra seen in item 7, Example 2.2.8.

Example 2.4.5. More generally, if α is an action of $\mathbb{T}$ on a C*-algebra A for each $n \in \mathbb{Z}$ we can define a grading

$$A_n := \{a \in A \mid \alpha_z(a) = z^n a\}.$$

Then A is in fact a $\mathbb{Z}$-graded C*-algebra. The previous examples are particular cases considering the gauge action.

In the previous example there is always a faithful conditional expectation of A onto $A_0 = A^\alpha$, the fixed-point algebra concerning the action α. However, not every graded C*-algebra has this property, that is, admits a conditional expectation of A onto unit fiber A_1. As we can see in [25], Exel shows an example of this. So, we consider a stronger condition called topological grading.

Definition 2.4.6. Let A be a G-graded C*-algebra with grading subspaces $\{A_g\}_{g\in G}$. We say that A is a *topological G-graded C*-algebra* if there is a (necessarily unique) conditional expectation of A onto A_1 vanishing on A_g for $g \neq 1$.

Corollary 2.4.7. *If $\mathcal{A}$ is a Fell Bundle over G then both $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$ are topological G-graded C*-algebras.*

Proof. Notice that the conditional expectations $E_1 : C^*_r(\mathcal{A}) \to \mathcal{A}_1$ and $\tilde{E}_1 : C^*(\mathcal{A}) \to \mathcal{A}_1$ given by Proposition 2.3.32 clearly satisfy the required conditions by definition. Moreover, by Remark 2.3.29, the full and reduced C*-algebra of a Fell bundle $\mathcal{A}$ are G-graded C*-algebras with grading subspaces $j_g(\mathcal{A}_g)$ and $\lambda_g(\mathcal{A}_g)$, respectively.

□

Example 2.4.8. A particular example in this context is the conditional expectation for the group C*-algebra. There is a unique continuous linear functional $\tau : C^*_r(G) \to \mathbb{C}$ defined by $\tau(u_1) = 1$ and $\tau(u_g) = 0$ for all $g \in G$. Moreover, τ is a faithful tracial state. Also, the tracial state $\tilde{\tau} : C^*(G) \to \mathbb{C}$ defined by $\tilde{\tau} = \tau \circ \Lambda^G$ is faithful if and only if G is amenable where $\Lambda^G : C^*(G) \twoheadrightarrow C^*_r(G)$ is the regular representation [[12], Theorem 2.6.8]. Indeed, $\mathrm{Ker}(\Lambda^G) = \{x \in C^*(G) \mid \tilde{\tau}(x^*x) = 0\}$. A point to note here is that if we consider the group bundle $\mathbb{C} \times G$ then the conditional expectation E_1 coincides with τ.

Example 2.4.9. Another example is related to the conditional expectation on a crossed product by a dynamical system (A, G, α). There is a unique continuous linear functional $F_g : A \rtimes_{\alpha,r} G \to A$ defined by

$$F_g\left(\sum_{h\in G} a_h u_h\right) = a_g$$

for every $g \in G$. In particular, F_1 is a faithful conditional expectation. Also, the conditional expectation $\tilde{F}_1 : A \rtimes_\alpha G \to A$ defined by $\tilde{F}_1 = F_1 \circ \Lambda^{A\rtimes_\alpha G}$ is faithful if G is amenable [[60], Theorem 7.13] where $\Lambda^{A\rtimes_\alpha G} : A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$ is the regular representation. Again, $\mathrm{Ker}(\Lambda^{A\rtimes_\alpha G}) = \{x \in A \rtimes_\alpha G \mid \tilde{F}_1(x^*x) = 0\}$. If we consider the semi-direct bundle $\mathcal{A}^\alpha$ then the conditional expectation E_1 coincides with F_1.

From now on, fix a topologically G-graded C*-algebra A with associated Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ defined from the grading subspaces.

Theorem 2.4.10. *([25], Theorem 19.1) With notations as above, there is a unique surjective *-homomorphism $\psi : A \to C^*_r(\mathcal{A})$ such that $\psi(a) = \lambda_g(a)$ for all $a \in \mathcal{A}_g$ and $g \in G$.*

Proposition 2.4.11. *If A is a topologically G-graded C*-algebra with associated Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$, then there is a commutative diagram of surjective *-homomorphisms*

$$\begin{tikzcd}
C^*(\mathcal{A}) \arrow[rr, two heads, "\Lambda"] \arrow[dr, two heads, "\overline{\sigma}"'] & & C^*_r(\mathcal{A}) \\
& A \arrow[ur, two heads, "\psi"'] &
\end{tikzcd}$$

Proof. The inclusion maps $\sigma_g : \mathcal{A}_g \hookrightarrow A$ define a representation σ of $\mathcal{A}$ on A and its integrated form yields a *-homomorphism $\overline{\sigma} : C^*(\mathcal{A}) \to A$ such that $\overline{\sigma}(j_g(a)) = \sigma_g(a)$ for all $a \in \mathcal{A}_g$.

The surjectivity is clear because $\oplus_{g\in G}\mathcal{A}_g$ is dense in A. The existence of ψ follows immediately by Theorem 2.4.10. It is enough to show that $\psi\circ\overline{\sigma}=\Lambda$. For this, note for all $a\in\mathcal{A}_g$ we have

$$\psi(\overline{\sigma}(j_g(a)))=\psi(\sigma_g(a))=\lambda_g(\sigma_g(a))=\Lambda(j_g(a)).$$

By linearity and continuity the desired result follows.

□

Proposition 2.4.12. *Let A be a topologically G-graded C*-algebra with grading $\{\mathcal{A}_g\}_{g\in G}$. Then there is a contractive linear map $F_g : A\to\mathcal{A}_g$ with vanishes on $\mathcal{A}_h$ for all $h\neq g$. In particular, F_1 is a conditional expectation which vanishes on $\mathcal{A}_g$ for all $g\neq 1$.*

Proof. For the existence it is enough to define $F_g := E_g\circ\psi$ where E_g is as in Lemma 2.3.31 and ψ is as in Theorem 2.4.10. In particular, F_1 is a conditional expectation that satisfies the desired conditions by Proposition 2.3.32.

□

Proposition 2.4.13. *Given a topologically G-graded C*-algebra A with conditional expectation F_1 as in Proposition 2.4.12 and the *-homomorphism ψ as in Theorem 2.4.10, we have:*

$$\mathit{Ker}(\psi)=\{a\in A\mid F_1(a^*a)=0\}.$$

Moreover, if F_1 is faithful then A is canonically isomorphic to $C_r^*(\mathcal{A})$.

Proof. It is enough to note by definition that $F_1(a^*a)=E_1(\psi(a)^*\psi(a))$. So, $F_1(a^*a)=0$ if and only if $\psi(a)=0$. If F_1 is faithful then the representation $\psi : A\to C_r^*(\mathcal{A})$ will be an isomorphism.

□

Remark 2.4.14. Proposition 2.4.13 tells us that the reduced C*-algebra of a Fell bundle $\mathcal{A}$ associated with the grading subspaces is (up to isomorphism) the unique topologically G-graded C*-algebra having $\mathcal{A}$ as the associated Fell bundle.

2.4.1 FELL'S ABSORPTION PRINCIPLE

In this subsection, we are interested in constructing certain representations of a Fell bundle $\mathcal{A}$ induced by a given representation.

Suppose that π is a representation of a Fell bundle $\mathcal{A}$ into a C*-algebra B and let λ^G be the regular representation of G on $C_r^*(G)\subseteq B(l^2(G))$, that is, $\lambda_g^G(\xi)(h)=\xi(g^{-1}h)$ for every $\xi\in l^2(G)$ and $g,h\in G$. Let us consider another representation of $\mathcal{A}$ in $B\otimes C_r^*(G)$ by putting

$$(\pi\otimes\lambda^G)_g(a):=\pi_g(a)\otimes\lambda_g^G$$

for all $a\in\mathcal{A}_g$ and $g\in G$. Since

$$(\pi\otimes\lambda^G)_g(a)(\pi\otimes\lambda^G)_h(b)=\pi_g(a)\pi_h(b)\otimes\lambda_g^G\lambda_h^G=\pi_{gh}(ab)\otimes\lambda_{gh}^G=(\pi\otimes\lambda^G)_{gh}(ab)$$

and

$$(\pi \otimes \lambda^G)_g(a)^* = \pi_g(a)^* \otimes (\lambda_g^G)^* = \pi_{g^{-1}}(a^*) \otimes \lambda_{g^{-1}}^G = (\pi \otimes \lambda^G)_{g^{-1}}(a^*)$$

for all $a \in \mathcal{A}_g$, $b \in \mathcal{A}_h$ and $g, h \in G$, it follows that $\pi \otimes \lambda^G = \{(\pi \otimes \lambda^G)_g\}_{g \in G}$ is indeed a representation of $\mathcal{A}$ into $B \otimes C_r^*(G)$. The integrated form of $\pi \otimes \lambda^G$ is the representation $\overline{\pi \otimes \lambda^G} : C^*(\mathcal{A}) \to B \otimes C_r^*(G)$ such that $\overline{\pi \otimes \lambda^G}(j_g(a)) = \pi_g(a) \otimes \lambda_g^G$ for all $a \in \mathcal{A}_g$ and $g \in G$. This leads to the next theorem:

Theorem 2.4.15. *(Fell's absorption principle for Fell Bundles) Let π be a representation of the Fell bundle $\mathcal{A}$ in a C*-algebra B and let $\overline{\pi \otimes \lambda^G}$ be the integrated form of the representation $\pi \otimes \lambda^G$ described above. Then $\overline{\pi \otimes \lambda^G}$ factors through $C_r^*(\mathcal{A})$ providing a representation $(\overline{\pi \otimes \lambda^G})^r$ of the latter, such that the diagram below commutes:*

$$\begin{tikzcd}
C^*(\mathcal{A}) \arrow[rr, "\overline{\pi\otimes\lambda^G}"] \arrow[dr, "\Lambda"'] & & B \otimes C_r^*(G) \\
& C_r^*(\mathcal{A}) \arrow[ur, "(\overline{\pi\otimes\lambda^G})^r"'] &
\end{tikzcd}$$

Moreover, if π_1 is faithful then $(\overline{\pi \otimes \lambda^G})^r$ is also faithful.

Proof. Consider $\tau : C_r^*(G) \to \mathbb{C}$ the canonical faithful tracial state. This induces a bounded linear positive map $id_B \otimes \tau : B \otimes C_r^*(G) \to B$ such that $(id_B \otimes \tau)(\sum_{g \in G} b \otimes u_g) = b$. This map is the first appearance of a *slice map*[4]. Moreover, $id_B \otimes \tau$ is faithful because τ is. We claim that $(id_B \otimes \tau)(\overline{\pi \otimes \lambda^G})(x) = \pi_1(\tilde{E}_1(x))$ for all $x \in C^*(\mathcal{A})$. To check that it is enough to apply both on $x = j_g(a)$, $a \in \mathcal{A}_g$. But

$$\begin{aligned}
(id_B \otimes \tau)(\overline{\pi \otimes \lambda^G})(j_g(a)) &= (id_B \otimes \tau)(\pi_g(a) \otimes \lambda_g^G) \\
&= \pi_1(a) \\
&= \pi_1(a\delta_{1,g}) \\
&= \pi_1(E_1(\lambda_g(a))) \\
&= \pi_1(E_1(\Lambda(j_g(a))) \\
&= \pi_1(\tilde{E}_1(j_g(a)))
\end{aligned}$$

Since the closed linear span of the set of all $j_g(a)$, $a \in \mathcal{A}_g$, is a dense subspace of $C^*(\mathcal{A})$ the result follows. Now, we will show that Ker$(\Lambda) \subseteq$ Ker$(\overline{\pi \otimes \lambda^G})$ remembering that Ker$(\Lambda) = \{x \in C^*(\mathcal{A}) \mid \tilde{E}_1(x^*x) = 0\}$. If $x \in$ Ker(Λ) then we have $0 = \pi_1(\tilde{E}_1(x^*x)) = (id_B \otimes \tau)(\overline{\pi \otimes \lambda^G})(x^*x)$. Since $id_B \otimes \tau$ is faithful this implies that $\overline{\pi \otimes \lambda^G}(x^*x) = 0$ concluding that $x \in$ Ker$(\overline{\pi \otimes \lambda^G})$. Therefore $\overline{\pi \otimes \lambda^G}$ vanishes on the kernel of the regular representation Λ and

[4]Given C*-algebras A and B and a linear functional $\varphi \in B^*$, the map $id_A \otimes \varphi : A \otimes_{alg} B \to A \otimes \mathbb{C} \cong A$ defined by $(id_A \otimes \varphi)(a \otimes b) = a\varphi(b)$ extends to a bounded linear map of $A \otimes B$ into A of norm $\|\varphi\|$. Moreover, if φ is faithful then $id_A \otimes \varphi$ is also faithful essentially because we are using minimal tensor product. Such maps are called slices maps. The existence is obtained by view A and B represented on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and φ becomes a functional in $B(\mathcal{H}_2)$. More details can be found in [57] and [40].

then factors through a representation $(\overline{\pi \otimes \lambda^G})^r$ on $C^*_r(\mathcal{A})$ as desired. Finally, if π_1 is faithful then we have the equality Ker$(\Lambda) =$ Ker$(\overline{\pi \otimes \lambda^G})$. To see that let $x \in$ Ker$(\overline{\pi \otimes \lambda^G})$. Then $\overline{\pi \otimes \lambda^G}(x^*x) = 0$ and applying $id_B \otimes \tau$ on both sides we get $\pi_1(\tilde{E}_1(x^*x)) = 0$. Since π_1 is faithful then $\tilde{E}_1(x^*x) = 0$, that is, $x \in$ Ker(Λ). Therefore $(\overline{\pi \otimes \lambda^G})^r$ is faithful, completing the proof.

□

Remark 2.4.16. If we represent $B \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ then the proof is similar to [[25], Proposition 18.4] but we adapted to facilitate the understanding for the next sections. In general we can consider π as a representation of $\mathcal{A}$ in $\mathcal{M}(B)$ for some C*-algebra B and it gives us a *-homomorphism $\overline{\pi \otimes \lambda^G} : C^*(\mathcal{A}) \to \mathcal{M}(B \otimes \mathcal{K}(l^2(G)))$ as long as we use the canonical isomorphism $B(l^2(G)) \cong \mathcal{M}(\mathcal{K}(l^2(G)))$. With certain adaptations the proof is similar to that presented above.

Proposition 2.4.17. *Let $\mathcal{A}$ be a Fell bundle. Then there is an injective *-homomorphism $\varphi : C^*_r(\mathcal{A}) \to C^*_r(\mathcal{A}) \otimes C^*_r(G)$ such that $\varphi(\lambda_g(a)) = \lambda_g(a) \otimes \lambda^G_g$, for all $a \in \mathcal{A}_g$ and $g \in G$.*

Proof. Consider the regular representation λ of $\mathcal{A}$ in $C^*_r(\mathcal{A})$. Then by construction we get another representation of $\mathcal{A}$ in $C^*_r(\mathcal{A}) \otimes C^*_r(G)$ by $(\lambda \otimes \lambda^G)_g : \mathcal{A}_g \to C^*_r(\mathcal{A}) \otimes C^*_r(G)$ such that $(\lambda \otimes \lambda^G)_g(a) = \lambda_g(a) \otimes \lambda^G_g$ for every $a \in \mathcal{A}_g$. By Fell's absorption principle 2.4.15 the integrated form $\overline{\lambda \otimes \lambda^G} : C^*(\mathcal{A}) \to C^*_r(\mathcal{A}) \otimes C^*_r(G)$ factors through an injective *-homomorphism φ from $C^*_r(\mathcal{A})$ as we required.

□

Remark 2.4.18. Also there is a canonical representation $\mathcal{A}_g \to C^*_r(\mathcal{A}) \otimes C^*(G)$ sending a to $\lambda_g(a) \otimes u_g$ for every $a \in \mathcal{A}_g$. The integrated form of this representation gives us a *-homomorphism $\delta^r_{\mathcal{A}} : C^*_r(\mathcal{A}) \to C^*_r(\mathcal{A}) \otimes C^*(G)$ such that the following diagram commute:

$$\begin{tikzcd} C^*_r(\mathcal{A}) \arrow[r, "\delta^r_{\mathcal{A}}"] \arrow[rr, bend left, "\varphi"] & C^*_r(\mathcal{A}) \otimes C^*(G) \arrow[r, "id \otimes \Lambda^G"] & C^*_r(\mathcal{A}) \otimes C^*_r(G) \end{tikzcd}$$

Since φ is injective then $\delta^r_{\mathcal{A}}$ is also injective.

Example 2.4.19. Similarly, there is a canonical representation $\mathcal{A}_g \to C^*(\mathcal{A}) \otimes C^*(G)$ by sending a to $j_g(a) \otimes u_g$ for every $a \in \mathcal{A}_g$. The integrated form of this representation gives us an injective *-homomorphism $\delta_{\mathcal{A}} : C^*(\mathcal{A}) \to C^*(\mathcal{A}) \otimes C^*(G)$ such that $\delta_{\mathcal{A}}(j_g(a)) = j_g(a) \otimes u_g$. But note that $id \otimes \Lambda^G \circ \delta_{\mathcal{A}} : C^*(\mathcal{A}) \to C^*(\mathcal{A}) \otimes C^*_r(G)$ is not injective in general because as we can see from proof of Theorem 2.4.15 we have Ker$(id \otimes \Lambda^G \circ \delta_{\mathcal{A}}) =$ Ker(Λ).

Remark 2.4.20. As we can see in the previous remark and example, in general, there are always an injective *-homomorphisms from $C^*_r(\mathcal{A})$ as below

$$\begin{tikzcd}
 & C^*(\mathcal{A}) \otimes C_r^*(G) \arrow[d, two heads] \\
C_r^*(\mathcal{A}) \arrow[ur, hook, bend left] \arrow[r, hook] \arrow[dr, hook, bend right] & C_r^*(\mathcal{A}) \otimes C_r^*(G) \\
 & C_r^*(\mathcal{A}) \otimes C^*(G) \arrow[u, two heads]
\end{tikzcd}$$

but from $C^*(\mathcal{A})$ the situation change

$$\begin{tikzcd}
C^*(\mathcal{A}) \arrow[r, hook] \arrow[dr, hook, bend right, "/" marking] & C^*(\mathcal{A}) \otimes C^*(G) \arrow[d, two heads] \\
 & C^*(\mathcal{A}) \otimes C_r^*(G)
\end{tikzcd}$$

These canonical injective *-homomorphisms $\delta_{\mathcal{A}}$ and $\delta^r_{\mathcal{A}}$ presented in the examples above are a form of "coaction" of G on $C^*(\mathcal{A})$ and $C_r^*(\mathcal{A})$, respectively. We will study them more deeply in the subsequent section.

2.5 COACTIONS AND THEIR CROSSED PRODUCTS

In this section, we will study part of the theory of coactions and their crossed products over a discrete group. Although we only need this case, we remark that this also generalizes to locally compact groups, see [52]. Again, we will assume that all representations here are nondegenerate and all tensor products $\otimes$ are minimal with identity maps denoted by id with index when necessary. We mainly follow [21], [41], [20], [52], [49], [35], [42], [33] and [48].

To introduce the definition, let us first note that the group C*-algebra $C^*(G)$ carries a natural comultiplication $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ such that $\delta_G(g) = g \otimes g$ (For now on, g means actually u_g in $C^*(G)$). If follows directly that δ_G satisfies the identity $(\delta_G \otimes id_G) \circ \delta_G = (id_G \otimes \delta_G) \circ \delta_G$.

Definition 2.5.1. A *coaction* of a discrete group G on a C*-algebra A is a *-homomorphism $\delta : A \to A \otimes C^*(G)$ that makes the following diagram, called *coaction identity*, commute:

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] & A \otimes C^*(G) \arrow[d, "id_A \otimes \delta_G"] \\
A \otimes C^*(G) \arrow[r, "\delta \otimes id_G"] & A \otimes C^*(G) \otimes C^*(G)
\end{tikzcd}$$

and such that it is *nondegenerate* in the sense that

$$\overline{\delta(A)(1 \otimes C^*(G))} = A \otimes C^*(G).$$

Remark 2.5.2. In fact nondegeneracy for coactions is an apparently stronger condition than nondegeneracy for *-homomorphisms which means $\overline{\delta(A)(A \otimes C^*(G))} = A \otimes C^*(G)$. An open question is whether every coaction is automatically nondegenerate. The affirmative answer to this question is known to be true for amenable groups, see [35] and [41]. In the case of discrete groups, it was believed that it is also true but the proofs for automatic nondegeneracy of discrete groups are unfortunately incorrect, see [34]. So this is still an open question.

Definition 2.5.3. A *dynamical co-system* is a triple (A, G, δ) where A is C*-algebra, G is a discrete group and δ is a coaction of G on A.

Example 2.5.4. It is important to emphasize that the comultiplication δ_G is an example of a coaction of G on $C^*(G)$.

One important case is when G is an abelian group. In this case consider the dual group of G, called *Pontryagin dual* and denoted by $\hat{G}$, which is the set of all homomorphisms from G into the circle group $\mathbb{T}$ in $\mathbb{C}$ with pointwise multiplication. Notice that $\hat{G}$ is a compact group with respect to the topology of pointwise convergence.

Proposition 2.5.5. *If G is an abelian group then there is a one-to-one correspondence between coactions of G and (continuous) actions of the dual group $\hat{G}$.*

Proof. First let us identify $C^*(G) \cong C(\hat{G})$ via the Fourier transformation $\mathcal{F} : C^*(G) \to C(\hat{G})$ given by $\mathcal{F}(g)(\chi) = \chi(g)$, for all $\chi \in \hat{G}$. Also, we can identify $A \otimes C(\hat{G}) \cong C(\hat{G}, A)$ via $a \otimes f \leftrightarrow a \cdot f$ where $a \cdot f \in C(\hat{G}, A)$ is function defined by $(a \cdot f)(\chi) = af(\chi)$ for all $\chi \in \hat{G}$. Moreover, $C(\hat{G}, A)$ is the closed liner span of the functions $a \cdot f$ for $f \in C(\hat{G})$ and $a \in A$.

Identifying in the usual way $C(\hat{G}) \otimes C(\hat{G}) \cong C(\hat{G} \times \hat{G})$ with $f \otimes g$ is identified with $f \cdot g$ where $f \cdot g(\chi, \varsigma) = f(\chi)g(\varsigma)$, we can translate the comultiplication δ_G to the *-homomorphism $\widetilde{\delta_G} : C(\hat{G}) \to C(\hat{G} \times \hat{G})$ given by $\widetilde{\delta_G}(f)(\chi, \varsigma) = f(\chi\varsigma)$. This is because $\delta_G(g) = g \otimes g$ for all $g \in G$. Using Fourier transformation we have $\widetilde{\delta_G}(\mathcal{F}(g))(\chi, \varsigma) = (\mathcal{F}(g).\mathcal{F}(g))(\chi, \varsigma) = \mathcal{F}(g)(\chi)\mathcal{F}(g)(\varsigma) = \chi(g)\varsigma(g) = (\chi\varsigma)(g) = \mathcal{F}(g)(\chi\varsigma)$. Since $\mathcal{F}(g)$ for $g \in G$ generates $C(\hat{G})$ the result follows.

After all the identifications, if α is an action of $\hat{G}$ on C*-algebra A, then we can define a coaction $\delta^\alpha : A \to C(\hat{G}, A)$ by $\delta^\alpha(a)(\chi) := \alpha_\chi(a)$. We are going to check the coaction identity, that is, the diagram below commute:

$$\begin{array}{ccc}
A & \xrightarrow{\qquad\qquad\delta^\alpha\qquad\qquad} & A \otimes C(\hat{G}) \cong C(\hat{G}, A) \\
{\scriptstyle \delta^\alpha}\big\downarrow & & \big\downarrow{\scriptstyle id_A \otimes \delta_G} \\
A \otimes C(\hat{G}) \cong C(\hat{G}, A) & \xrightarrow{\delta^\alpha \otimes id_G} & A \otimes C(\hat{G}) \otimes C(\hat{G}) \cong A \otimes C(\hat{G} \times \hat{G}) \cong C(\hat{G} \times \hat{G}, A)
\end{array}$$

To see this we claim that $(\delta^\alpha \otimes id_G)(f)(\chi,\varsigma) = \alpha_\chi(f(\varsigma))$ for all $f \in C(\hat{G}, A)$. For the elementary tensors we have

$$\begin{aligned}(\delta^\alpha \otimes id_G)(a \otimes \varphi)(\chi,\varsigma) &= (\delta^\alpha(a) \otimes \varphi)(\chi,\varsigma)\\ &= \delta^\alpha(a)(\chi)\varphi(\varsigma)\\ &= \alpha_\chi(a)\varphi(\varsigma)\\ &= \alpha_\chi(\varphi(\varsigma)a)\end{aligned}$$

Since $C(\hat{G}, A)$ is the closed linear span of the set of all functions $a \cdot \varphi$ with $a \in A$ and $\varphi \in C(\hat{G})$, the statement follows.

Similarly we can see that $(id_A \otimes \delta_G)(f)(\chi,\varsigma) = f(\chi\varsigma)$ for all $f \in C(\hat{G}, A)$. Now, we compute:

$$\begin{aligned}((id_A \otimes \delta_G) \circ \delta^\alpha)(a)(\chi,\varsigma) &= (id_A \otimes \delta_G)(\delta^\alpha(a))(\chi,\varsigma)\\ &= \delta^\alpha(a)(\chi\varsigma)\\ &= \alpha_{\chi\varsigma}(a)\end{aligned}$$

On the other hand we compute:

$$\begin{aligned}((\delta^\alpha \otimes id_G) \circ \delta^\alpha)(a)(\chi,\varsigma) &= (\delta^\alpha \otimes id_G)(\delta^\alpha(a))(\chi,\varsigma)\\ &= \alpha_\chi(\delta^\alpha(a)(\varsigma))\\ &= \alpha_\chi(\alpha_\varsigma(a))\end{aligned}$$

This tell us that the coaction identity is directly related to the multiplicativity of the action, that is, $(id_A \otimes \delta_G) \circ \delta^\alpha = (\delta^\alpha \otimes id_G) \circ \delta^\alpha$ if and only if $\alpha_{\chi\varsigma} = \alpha_\chi \circ \alpha_\varsigma$. For the nondegeneracy of δ^α, note that for all $a \in A$ and $\chi \in \hat{G}$, $\delta^\alpha(\alpha_{\chi^{-1}}(a))$ is an element of $\delta^\alpha(A)$ such that $\delta^\alpha(\alpha_{\chi^{-1}}(a))(\chi) = \alpha_\chi(\alpha_{\chi^{-1}}(a)) = a$. Moreover, $\delta^\alpha(A)$ is a C*-subalgebra of $C(\hat{G}, A)$. Hence, we conclude that $\delta^\alpha(A) \cdot C(\hat{G}) = \overline{\text{span}}\{f.g \mid f \in \delta^\alpha(A), g \in C(\hat{G})\}$ is dense in $C(\hat{G}, A)$ by [[10], Lemma 3.2.11]. So, identifying $C(\hat{G}) \cong 1 \otimes C(\hat{G}) \subseteq A \otimes C(\hat{G})$ we have that $\delta^\alpha(A)(1 \otimes C(\hat{G}))$ is dense in $A \otimes C(\hat{G})$ and δ^α is indeed a coaction of G on A.

Conversely, if $\delta : A \to C(\hat{G}, A)$ is a coaction then we obtain an action of $\hat{G}$ on A defining $\alpha_\chi : A \to A$ given by $\alpha_\chi(a) = \delta(a)(\chi)$. Note that $\alpha_{\chi\varsigma} = \alpha_\chi \circ \alpha_\varsigma$ follows from the same argument used above and $\alpha_1 = id_A$ follows from $(id_A \otimes 1_G) \circ \delta^\alpha = id_A$. This completes the proof.

□

Coactions of discrete groups are strongly related to Fell bundles. If $\mathcal{A}$ is a Fell bundle over G, there are canonical coactions on $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$: Consider $\delta_{\mathcal{A}} : C^*(\mathcal{A}) \to C^*(\mathcal{A}) \otimes C^*(G)$ given by $\delta_{\mathcal{A}}(j_g(a_g)) = j_g(a_g) \otimes g$ and $\delta^r_{\mathcal{A}} : C^*_r(\mathcal{A}) \to C^*_r(\mathcal{A}) \otimes C^*(G)$ given by $\delta^r_{\mathcal{A}}(\lambda_g(a_g)) = \lambda_g(a_g) \otimes g$ for all $a_g \in \mathcal{A}_g$ and $g \in G$ the injective *-homomorphisms as we have seen in the previous subsection. It is straightforward to check the coaction identity. Moreover, observe that for each $a \in \mathcal{A}_g$ and $h \in G$ we have

$$j_g(a) \otimes h = (j_g(a) \otimes g)(1 \otimes g^{-1}h) = \delta_{\mathcal{A}}(j_g(a))(1 \otimes g^{-1}h)$$

and since the closed linear span of the set of all $j_g(a)$, with $a \in \mathcal{A}_g$ and $g \in G$, is dense in $C^*(\mathcal{A})$ it follows that $\delta_{\mathcal{A}}$ is nondegenerate in our sense and hence $\delta_{\mathcal{A}}$ is in fact a coaction of G on $C^*(\mathcal{A})$. The same happens for $\delta^r_{\mathcal{A}}$. Eventually we will use the simpler notation a_g to designate $j_g(a_g)$ in $C^*(\mathcal{A})$.

Interestingly, a sort of converse holds: If δ is a coaction of G on a C*-algebra A we can consider the spectral subspaces

$$\mathcal{A}_g := \{a \in A \mid \delta(a) = a \otimes g\}.$$

This gives us a grading for G as we will see in the next proposition and consequently form a Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ over G with operations induced from A.

Before we get there, if D is a C*-algebra and $\rho : D \to B(\mathcal{H})$ is a nondegenerate representation of D, then $x \mapsto \langle \xi, \rho(x)\eta\rangle$ is a continuous linear functional on D for each $\xi, \eta \in \mathcal{H}$, called a matrix coefficient of ρ. Every element of the dual C*-algebra B^* can be written in this way. We denote by B^*_ρ the matrix coefficients of ρ. In particular, for $D = C^*(G)$ then $C^*(G)^*$ can be identified with the Fourier-Stieltjes algebra $B(G)$ of G, the algebra consisting of all bounded functions on G which can be expressed as matrix coefficients of unitary representations of G. The Fourier algebra $A(G)$ is the (closed) *-subalgebra of $B(G)$ consisting of all matrix coefficients of the left regular representation λ^G on $l^2(G)$, that is, $A(G) = C^*(G)^*_{\lambda^G}$. So, every element $f \in A(G)$ is of the form $f(x) = \langle \xi, \lambda^G(x)\eta\rangle$ where $\xi, \eta \in l^2(G)$. The characteristic function χ_g of $\{g\}$ belongs to $A(G)$ because $\chi_g = \langle \varsigma_g, \lambda^G(.)\varsigma_1\rangle_{l^2(G)}$, where ς_g is the standard basis vector corresponding to g. Then λ^G is determined by $\lambda^G_g(\varsigma_h) = \varsigma_{gh}$ for all $g, h \in G$. In particular, $A(G)$ is dense in $C_0(G)$ with respect of the supremum-norm. Moreover, it is not difficult to see that χ_1 is in fact the canonical conditional expectation $\tilde{\tau}$ on $C^*(G)$ since $\tau(a) = \langle \varsigma_1, a\varsigma_1\rangle_{l^2(G)}$ for all $a \in C^*_r(G)$ and $\chi_g = (\chi_1).g^{-1}$ since $(\chi_1).g^{-1} = \langle \varsigma_1, \varsigma_h\rangle_{l^2(G)}.g^{-1} = \langle \varsigma_1, \varsigma_{hg^{-1}}\rangle_{l^2(G)} = \langle \varsigma_g, \varsigma_h\rangle_{l^2(G)} = \chi_g$. More details about the Fourier algebra can be found in [27].

After this short introduction, consider the map $E_g := (id_A \otimes \chi_g) \circ \delta : A \to A$ for each $g \in G$ where χ_g is the characteristic function of $\{g\}$ regarded as an element of the Fourier algebra $A(G)$. In fact, we can see $id_A \otimes \chi_g$ as a slice map of $A \otimes C^*(G)$ to A which "reads" the variable g. It is not difficult to see that E_g is a projection of norm one from A onto $\mathcal{A}_g$ with kernel containing $\mathcal{A}_h$ for all $h \neq g$.

Proposition 2.5.6. *If (A, G, δ) is a dynamical co-system then A is topologically G-graded C*-algebra with grading given by the spectral subspaces $\mathcal{A}_g$ defined above.*

Proof. It is not difficult to see that $\mathcal{A}_g$ are closed subspaces of A such that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ and $\mathcal{A}^*_g = \mathcal{A}_{g^{-1}}$ for every $g, h \in G$ since δ is a *-homomorphism. For the linear independence of the collection $\{\mathcal{A}_g\}_{g\in G}$ suppose a finite sum $\sum_{i=1}^n a_{g_i} = 0$ with $a_{g_i} \in \mathcal{A}_{g_i}$ such that $g_i \neq g_j$ if $i \neq j$. Then we have

$$0 = \delta\left(\sum_{i=1}^n a_{g_i}\right) = \sum_{i=1}^n \delta(a_{g_i}) = \sum_{i=1}^n a_{g_i} \otimes g_i.$$

For $g \in G$, we have

$$0 = E_g\left(\sum_{i=1}^n a_{g_i}\right) = (id_A \otimes \chi_g)\circ\delta\left(\sum_{i=1}^n a_{g_i}\right) = (id_A \otimes \chi_g)\left(\sum_{i=1}^n a_{g_i} \otimes g_i\right) = a_g \otimes g.$$

We conclude that $a_g = 0$, and therefore, we have the desired. We claim that $\oplus_{g\in G}\mathcal{A}_g$ is dense in A. Since δ is nondegenerate, $\overline{\text{span}}\{\delta(a)(1\otimes g) \mid a \in A, g \in C^*(G)\} = A \otimes C^*(G)$. In particular, $a \otimes 1$ can be approximated by $\sum_{i=1}^n \delta(a_i)(1\otimes g_i)$. Then we have

$$\begin{aligned}
a &= (id_A \otimes \chi_1)(a \otimes 1) \\
&\approx (id_A \otimes \chi_1)\left(\sum_{i=1}^n \delta(a_i)(1\otimes g_i)\right) \\
&= \sum_{i=1}^n (id_A \otimes \chi_1)(\delta(a_i)(1\otimes g_i)) \\
&= \sum_{i=1}^n (id_A \otimes \chi_{g_i^{-1}})(\delta(a_i)) \\
&= \sum_{i=1}^n E_{g_i^{-1}}(a_i) \in \bigoplus_{g\in G}\mathcal{A}_g
\end{aligned}$$

Finally, we have already seen that $E_g : A \to \mathcal{A}_g$ is a norm one projection that vanishes on $\mathcal{A}_h$ for all $h \neq g$. In particular, E_1 is norm one projection that vanishes on A_g for all $g \neq 1$. Now, if $x = \sum_{g\in G} a_g$ is a finite sum with $a_g \in \mathcal{A}_g$ then

$$x^*x = \sum_{g,h\in G} a_g^* a_h = \sum_{k\in G}\sum_{g\in G} a_g^* a_{gk}.$$

Note that $a_g^* a_{gk} \in \mathcal{A}_{g^{-1}}\mathcal{A}_{gk} \subseteq \mathcal{A}_k$ and $E_1(x^*x) = \sum_{g\in G} a_g^* a_g$ which is positive. Given $a \in \mathcal{A}_1$ it is straightforward to check that $E_1(ax) = aE_1(x)$ and $E_1(xa) = E_1(x)a$ for every $x \in A$ by checking first on finite sums as above. Therefore, $E_1 : A \to \mathcal{A}_1$ is a conditional expectation that vanishes on A_g for all $g \neq 1$. That is, $\{\mathcal{A}_g\}_{g\in G}$ is a topological grading for A.

□

Remark 2.5.7. According to Proposition 2.5.6 we get an associated Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$. Observe that the conditional expectation E_1 defined above is essentially "the same" conditional expectation F_1 in the context of Fell bundles found in Proposition 2.4.12 since for every $a \in \mathcal{A}_g$

$$F_1(a) = E_1(\psi(a)) = E_1(\lambda_g(a)) = a\delta_{1,g} = (id_A \otimes \chi_1)(a \otimes g) = (id_A \otimes \chi_1)(\delta(a))$$

where ψ is the *-homomorphism seen in Theorem 2.4.10.

Remark 2.5.8. If $\delta : A \to A \otimes C^*(G)$ is *-homomorphism satisfying the coaction identity then δ is injective if and only if $(id_A \otimes 1_G) \circ \delta = id_A$, where 1_G denotes the trivial representation of G integrated to a representation of $C^*(G)$ and $id_A \otimes 1_G : A \otimes C^*(G) \to A$ is a “slice map”. To see it observe that $(id_G \otimes 1_G) \circ \delta_G(g) = (id_G \otimes 1_G)(g \otimes g) = g$ for all $g \in G$, that is, $(id_G \otimes 1_G) \circ \delta_G = id_G$. Now for $a \in A$, we compute:

$$\begin{aligned}
\delta(a) &= (id_A \otimes id_G)\delta(a) \\
&= (id_A \otimes ((id_G \otimes 1_G) \circ \delta_G))\delta(a) \\
&= (id_A \otimes id_G \otimes 1_G)(id_A \otimes \delta_G)\delta(a) \\
&= (id_A \otimes id_G \otimes 1_G)(\delta \otimes id_G)\delta(a) \\
&= \delta \circ (id_A \otimes 1_G) \circ \delta(a)
\end{aligned}$$

Therefore, $\delta = \delta \circ (id_A \otimes 1_G) \circ \delta$ and this completes the prove of the statement above.

Remark 2.5.9. In the literature the injectivity of coactions is usually required but nondegeneracy implies injectivity. To see this, in our discrete case, observe that if $\delta : A \to A \otimes C^*(G)$ is a coaction then by Proposition 2.4.12 A is a topologically G-graded C*-algebra with spectral subspaces $\{\mathcal{A}_g\}_{g \in G}$. Consequently, for each $a \in \mathcal{A}_g$ we have

$$(id_A \otimes 1_G) \circ \delta(a) = (id_A \otimes 1_G)(a \otimes g) = a.$$

By linearity and continuity we have $(id_A \otimes 1_G) \circ \delta = id_A$ implying that δ is injective.

Example 2.5.10. Let H and G be a discrete groups, let $\varphi : H \to G$ be a homomorphism and $\overline{\varphi} : C^*(H) \to C^*(G)$ denote the integrated form of $u \circ \varphi : H \to C^*(G)$ where $u : G \to C^*(G)$ is the canonical representation. If (A, H, ϵ) is any dynamical co-system we can define the inflated co-system $(A, G, \mathsf{Inf}(\epsilon))$ where $\mathsf{Inf}(\epsilon) := (id_A \otimes \overline{\varphi}) \circ \epsilon : A \to A \otimes C^*(G)$.

It is straightforward that $\mathsf{Inf}(\epsilon)$ is a *-homomorphism and it is satisfy the coaction identity since $(\overline{\varphi} \otimes \overline{\varphi}) \circ \delta_H = \delta_G \circ \overline{\varphi}$ combined with the coaction identity of ϵ, that is, we have:

$$\begin{aligned}
(id_A \otimes \delta_G) \circ \mathsf{Inf}(\epsilon) &= (id_A \otimes \delta_G) \circ (id_A \otimes \overline{\varphi}) \circ \epsilon \\
&= (id_A \otimes \delta_G \circ \overline{\varphi}) \otimes \epsilon \\
&= (id_A \otimes (\overline{\varphi} \otimes \overline{\varphi} \circ \delta_H)) \circ \epsilon \\
&= (id_A \otimes \overline{\varphi} \otimes \overline{\varphi}) \circ (id_A \otimes \delta_H) \circ \epsilon \\
&= (id_A \otimes \overline{\varphi} \otimes \overline{\varphi}) \circ (\epsilon \otimes id_H) \circ \epsilon \\
&= (id_A \otimes id_G \otimes id_G) \circ (id_A \otimes \overline{\varphi} \otimes \overline{\varphi}) \circ (\epsilon \otimes id_H) \circ \epsilon \\
&= (id_A \otimes \overline{\varphi} \otimes id_G) \circ (id_A \otimes id_H \otimes \overline{\varphi}) \circ (\epsilon \otimes id_H) \circ \epsilon \\
&= (id_A \otimes \overline{\varphi} \otimes id_G) \circ (\epsilon \otimes id_G) \circ (id_A \otimes \overline{\varphi}) \circ \epsilon \\
&= ((id_A \otimes \overline{\varphi} \circ \epsilon) \otimes id_G) \circ (id_A \otimes \overline{\varphi}) \circ \epsilon \\
&= (\mathsf{Inf}(\epsilon) \otimes id_G) \circ \mathsf{Inf}(\epsilon)
\end{aligned}$$

Finally for the nondegeneracy since $\overline{\overline{\varphi}(C^*(H))C^*(G)} = C^*(G)$ we have

$$\begin{aligned}
\overline{\mathsf{Inf}(\epsilon)(A)(1\otimes C^*(G))} &= \overline{(id_A\otimes\overline{\varphi})\circ\epsilon(A)(1\otimes C^*(G))} \\
&= \overline{(id_A\otimes\overline{\varphi})\circ\epsilon(A)(1\otimes\overline{\varphi}(C^*(H))C^*(G))} \\
&= \overline{(id_A\otimes\overline{\varphi})\circ[\epsilon(A)(1\otimes C^*(H))](1\otimes C^*(G))} \\
&= \overline{(id_A\otimes\overline{\varphi})\circ[A\otimes C^*(H)](1\otimes C^*(G))} \\
&= \overline{(A\otimes\overline{\varphi}(C^*(H)))(1\otimes C^*(G))} \\
&= A\otimes\overline{\overline{\varphi}(C^*(H))C^*(G)} \\
&= A\otimes C^*(G)
\end{aligned}$$

From now on, given a coaction $\delta : A \to A\otimes C^*(G)$ we will consider the Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ associated with the spectral subspaces.

Remark 2.5.11. We have seen that every C*-algebra with a coaction over G is a topologically G-graded C*-algebra with grading given by the spectral subspaces $\mathcal{A}_g$. But the converse is not always true, see [[21], Remark 2.2] for an example of a topologically graded C*-algebra A that does not carry a coaction δ satisfying $\delta(a_g) = a_g\otimes g$ for every $g\in G$.

Now, we can introduce covariant representations and define crossed products associated with coactions.

Definition 2.5.12. Let (A, G, δ) be a dynamical co-system and B a C*-algebra. A *covariant representation* of (A, G, δ) in a multiplier algebra $\mathcal{M}(B)$ is a pair (π, μ) where $\pi : A \to \mathcal{M}(B)$ and $\mu : C_0(G) \to \mathcal{M}(B)$ are nondegenerate *-homomorphisms such that

$$\pi(a)\mu(\chi_h) = \mu(\chi_{gh})\pi(a) \text{ for all } a\in A_g, g, h\in G.$$

Remark 2.5.13. There is a more general equivalent definition that applies to a locally compact group (see [[18], Lemma 3.1] for the equivalence with the discrete case). The definition above is special for discrete groups and makes many computations easier.

Proposition 2.5.14. *Let (π, μ) be a covariant representation of (A, G, δ) into a $\mathcal{M}(B)$. Then*

$$C^*(\pi,\mu) := \overline{\pi(A)\mu(C_0(G))}$$

is a C-algebra.*

Proof. It follows from Proposition 2.5.6 and the fact that (π, μ) is covariant representation.

□

The following proposition shows the existence of covariant representations.

Proposition 2.5.15. *If π is a nondegenerate *-homomorphism of A to $\mathcal{M}(B)$ then the pair $((\pi\otimes\Lambda^G)\circ\delta, 1\otimes M)$ is a covariant representation of (A, G, δ) into $\mathcal{M}(B\otimes\mathcal{K}(l^2(G)))$. We say that $((\pi\otimes\Lambda^G)\circ\delta, 1\otimes M)$ is the covariant representation induced by π.*

Proof. Fist of all, observe that $(\pi \otimes \Lambda^G) \circ \delta(a) = \pi(a) \otimes \lambda_g^G$ and $(1 \otimes M)(\chi_h) = 1 \otimes M_{\chi_h}$ for all $a \in \mathcal{A}_g$, $g, h \in G$. Moreover, using the fact that (M, λ^G) is a covariant representation for the left translation action $\tau : G \to \mathrm{Aut}(C_0(G))$ of dynamical system $(C_0(G), G, \tau)$ which can be seen in Example A.0.8, for each $a \in \mathcal{A}_g$ and $g, h \in G$ we compute:

$$\begin{aligned}(\pi \otimes \Lambda^G) \circ \delta(a)(1 \otimes M)(\chi_h) &= (\pi(a) \otimes \lambda_g^G)(1 \otimes M_{\chi_h}) \\ &= \pi(a) \otimes (\lambda_g^G M_{\chi_h}) \\ &= \pi(a) \otimes (M_{\chi_{gh}} \lambda_g^G) \\ &= (1 \otimes M_{\chi_{gh}}))(\pi(a) \otimes \lambda_g^G) \\ &= (1 \otimes M)(\chi_{gh})(\pi \otimes \Lambda^G) \circ \delta(a)\end{aligned}$$

which completes the proof.

□

Now we are going to define the crossed product by a coaction as the C*-algebra generated by a certain regular representation. For this let (j_A, j_G) be a regular covariant pair induced by $id_A : A \to A$.

Definition 2.5.16. Given a dynamical co-system (A, G, δ) and (j_A, j_G) the regular covariant representation, we define the crossed product by the coaction as the C*-algebra

$$A \rtimes_\delta G := C^*(j_A, j_G).$$

Remark 2.5.17. By definition, the crossed product $A \rtimes_\delta G$ can be seen as a C*-subalgebra of $\mathcal{M}(A \otimes \mathcal{K}(l^2(G)))$.

Remark 2.5.18. Unlike for actions, this time there is no convenient choice of a dense *-subalgebra like $C_c(G, A)$. Also here there is no difference between full and reduced crossed products. The crossed product by a coaction enjoys a universal property as can be seen in the next theorem.

Theorem 2.5.19 ([55], Theorem 4.1(b)). *Let (π, μ) be another covariant representation of (A, G, δ). Then there is a unique nondegenerate *-homomorphism $\pi \times \mu : A \rtimes_\delta G \to \mathcal{M}(B)$ such that $\pi \times \mu \circ j_A = \pi$ and $\pi \times \mu \circ j_G = \mu$. The *-homomorphism $\pi \times \mu$ is called the integrated form of the covariant representation (π, μ) of (A, G, δ).*

Remark 2.5.20. It is not hard to check that there is a one-to-one correspondence between the covariant representations of (A, G, δ) and the *-homomorphisms of $A \rtimes_\delta G$: if we have $\rho : A \rtimes_\delta G \to \mathcal{M}(B)$ a *-homomorphism of $A \rtimes_\delta G$ then $(\rho \circ j_A, \rho \circ j_G)$ is a covariant representation of (A, G, δ) such that $\rho = (\rho \circ j_A) \times (\rho \circ j_G)$.

Remark 2.5.21. We could alternatively define the crossed product by coactions via universal properties as follows: Let (A, G, δ) be a dynamical co-system and B a C*-algebra. If we have a triple (D, k_A, k_G) satisfying:

1. (k_A, k_G) is a covariant representation of (A, G, δ).
2. $D = \overline{\text{span}}\{k_A(a)k_G(f) \mid a \in A, f \in C_0(G)\}$.
3. For every covariant representation (π, μ) of (A, G, δ) into $\mathcal{M}(B)$ there is a unique nondegenerate *-homomorphism $\pi \times \mu : D \to \mathcal{M}(B)$ such that $\pi \times \mu \circ k_A = \pi$ and $\pi \times \mu \circ k_G = \mu$.

Then the C*-algebra D is canonically isomorphic to $A \rtimes_\delta G$ via the integrated form $k_A \times k_G : A \rtimes_\delta G \to D$ which has the inverse $j_A \times j_G : D \to A \rtimes_\delta G$.

Remark 2.5.22. If $((\pi \otimes \Lambda^G) \circ \delta, 1 \otimes M)$ is a covariant representation of (A, G, δ) induced by the faithful representation $\pi : A \to \mathcal{M}(B)$ then the integrated form is a faithful *-homomorphism of $A \rtimes_\delta G$. The reason is because

$$((\pi \otimes \Lambda^G) \circ \delta) \times (1 \otimes M) = (\pi \otimes id_{\mathcal{K}}) \circ (j_A \times j_G)$$

which is faithful since $\pi \otimes id_{\mathcal{K}}$ is faithful (The reason for $\pi \otimes id_{\mathcal{K}}$ being faithful is because of the minimal tensor product that is being used here).

Example 2.5.23. If we consider the comultiplication δ_G as a group coaction of G on $C^*(G)$ we can see $C^*(G) \rtimes_{\delta_G} G \cong \mathcal{K}(l^2(G))$. We claim that (Λ^G, M) is a covariant representation for the coaction δ_G where Λ^G and M are the canonical representations of $C^*(G)$ and $C_0(G)$ into $B(l^2(G))$, respectively. To see that, we are going to use that (M, λ^G) is a covariant representation of $(C_0(G), G, \tau)$ as seen in Example A.0.8. Since for each $a \in \mathcal{A}_g$ and $h \in G$ we have

$$\Lambda^G(a)M_{\chi_h} = \lambda^G_g M_{\chi_h} = M_{\tau_g(\chi_h))}\lambda^G_g = M_{\chi_{gh}}\Lambda^G(a)$$

then (Λ^G, M) is a covariant representation and by the universal property for coactions we get a *-homomorphism $\Lambda^G \times M : C^*(G) \rtimes_{\delta_G} G \to B(l^2(G))$. The injectivity and surjectivity follows from the same reason of Example A.0.8 since $\overline{\text{span}}\{\lambda_g M_f \mid f \in C_0(G), g \in G\}$ is equal to $\mathcal{K}(l^2(G))$, and we have the desired isomorphism. Observe that this result is in fact a dual version of Example A.0.8.

Definition 2.5.24. Let (A, G, δ) and (B, G, ϵ) be two dynamical co-systems. We say that a *-homomorphism $\varphi : A \to B$ is G-equivariant if

$$(\varphi \otimes id_G) \circ \delta = \epsilon \circ \varphi.$$

We say those two co-systems are isomorphic if there is a G-equivariant isomorphism φ.

Remark 2.5.25. Note that the G-equivariance for $\varphi : A \to B$ defined above is equivalent to ask that φ is a graded map[5] of topologically G-graded C*-algebras since for each $a \in \mathcal{A}_g$ we have

$$\epsilon(\varphi(a)) = (\varphi \otimes id_G) \circ \delta(a) = \varphi \otimes id_G(a \otimes g) = \varphi(a) \otimes g$$

[5]A map $\varphi : A \to B$ between to G-graded C*-algebras with grading subspaces $\{\mathcal{A}_g\}_{g \in G}$ and $\{\mathcal{B}_g\}_{g \in G}$, respectively is said to be graded if $\varphi(\mathcal{A}_g) \subseteq \mathcal{B}_g$ for all $g \in G$.

for each $g \in G$. Conversely, if $\varphi(\mathcal{A}_g) \subseteq \mathcal{B}_g$ then for every $a \in A_g$ we have

$$(\varphi \otimes id_G) \circ \delta(a) = (\varphi \otimes id_G)(a \otimes g) = \varphi(a) \otimes g = \epsilon(\varphi(a)).$$

Proposition 2.5.26. *Let (A, G, δ) and (B, G, ϵ) be two dynamical co-systems and $\varphi : A \to B$ a G-equivariant *-homomorphism. Then there is a unique induced *-homomorphism $\varphi \rtimes G : A \rtimes_\delta G \to B \rtimes_\epsilon G$ such that $\varphi \rtimes G(j_A(a)j_G^A(f)) = (j_B \circ \varphi)(a)j_G^B(f)$ for every $a \in A$ and $f \in C_0(G)$. Moreover, $\varphi \rtimes G$ is an isomorphism if φ is so.*

Proof. If φ is G-equivariant then $(j_B \circ \varphi, j_G^B)$ is a covariant representation of (B, G, ϵ) in $\mathcal{M}(B \rtimes_\epsilon G)$. So, $\varphi \rtimes G := (j_B \circ \varphi) \times j_G^B$ is a well-defined *-homomorphism such that $\varphi \rtimes G(j_A(a)j_G^A(f)) = (j_B \circ \varphi)(a)j_G^B(f)$ for every $a \in A$ and $f \in C_0(G)$. If φ is surjective it is straightforward to see that $\varphi \rtimes G$ is also surjective. Finally, if φ is faithful then by Remark 2.5.22 it follows that $\varphi \rtimes G$ is faithful as well since

$$\begin{aligned}
(\varphi \otimes id_{\mathcal{K}})(j_A(a)j_G(f)) &= \varphi \otimes id_{\mathcal{K}}(((id_A \otimes \Lambda^G) \circ \delta(a))(1 \otimes M_f)) \\
&= ((id_B \otimes \Lambda^G) \circ (\varphi \otimes id_{\mathcal{K}}) \circ \delta(a))(1 \otimes M_f) \\
&= ((id_B \otimes \Lambda^G) \circ \epsilon(\varphi(a)))(1 \otimes M_f) \\
&= j_B(\varphi(a))j_G(f) \\
&= \varphi \rtimes G(j_A(a)j_G(f))
\end{aligned}$$

that is, $\varphi \rtimes G = (\varphi \otimes id_{\mathcal{K}}) \circ (j_A \times j_G)$.

$\square$

Proposition 2.5.27. *The maps Λ, $\overline{\sigma}$ and ψ defined in Proposition 2.4.11 are G-equivariant with respect to the coactions $\delta_{\mathcal{A}}$, δ e $\delta_{\mathcal{A}}^r$ and induce a commutative diagram of surjective *-homomorphisms:*

$$\begin{tikzcd}
C^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}} G \arrow[rr, "\Lambda\rtimes G"] \arrow[dr, "\overline{\sigma}\rtimes G"'] & & C^*_r(\mathcal{A}) \rtimes_{\delta^n_{\mathcal{A}}} G \\
& A \rtimes_\delta G \arrow[ur, "\psi\rtimes G"'] &
\end{tikzcd}$$

Proof. It is straightforward to check that Λ, $\overline{\sigma}$ and ψ are G-equivariant with respect to the coactions $\delta_{\mathcal{A}}$, δ e $\delta_{\mathcal{A}}^r$ since it is enough to see it in elementary elements and extend by linearity and continuity. It follows directly from the equivariance and surjectivity that the maps $\Lambda \rtimes G$, $\overline{\sigma} \rtimes G$ and $\psi \rtimes G$ are well-defined surjective maps. Since $\Lambda = \psi \circ \overline{\sigma}$ by Proposition 2.4.11 we also have $\Lambda \rtimes G = \psi \rtimes G \circ \overline{\sigma} \rtimes G$ as desired.

$\square$

2.6 MAXIMAL AND NORMAL COACTIONS

In this section, we are going to introduce the concept of maximal and normal coactions that will play a fundamental role in this work and will complement the theory seen in the previous section. We mainly follow [25], [20] and [52] and many dual relationships will be presented using the Appendix A.

We start the discussion by looking more closely at $A \rtimes_\delta G$ as the crossed product of a dynamical co-system (A, G, δ). In Proposition A.0.16 it was seen that for any dynamical system (A, G, α) there is a coaction $\widehat{\alpha}$ of G on $A \rtimes_\alpha G$ and many duality theorems hold in this context. When we look at dynamical co-systems one of the things that we have to observe is that there is a dual action of G on $A \rtimes_\delta G$ by right translation on $C_0(G)$, that is, there is a *-homomorphism $\widehat{\delta} : G \to Aut(A \rtimes_\delta G)$ defined by

$$\widehat{\delta}_g(j_A(a)j_G(\chi_h)) = j_A(a)j_G(\chi_{hg^{-1}})$$

for all $a \in A$ and $h \in G$. Since the right translation on $C_0(G)$ is continuous it follows that $g \mapsto \widehat{\delta}_g(j_A(a)j_G(\chi_h))$ is continuous.

Now, let λ^G and ρ^G be the left and right regular representations of G on $B(l^2(G))$, respectively. That is, $\lambda^G_g(\xi)(h) = \xi(g^{-1}h)$ and $\rho^G_g(\xi)(h) = \xi(hg)$ for all $\xi \in l^2(G)$ and $g, h \in G$. Consider the canonical *-homomorphism $\Pi := j_A \times j^A_G$ of $A \rtimes_\delta G$ in $\mathcal{M}(A \otimes \mathcal{K}(l^2(G)))$ induced by the universal property. Note that $j_A(a) = (id_A \otimes \Lambda^G) \circ \delta(a) \in A \otimes C^*_r(G) \subseteq \mathcal{M}(A \otimes K(l^2(G)))$. So, $(id_A \otimes \Lambda^G) \circ \delta(a)(1 \otimes k) \in A \otimes K(l^2(G))$ for every k compact operator on $l^2(G)$. Therefore, $j_A \times j^A_G(j_A(a)j^A_G(f)) = (id_A \otimes \Lambda^G) \circ \delta(a)(1 \otimes M_f)$. Since G is discrete, $M_f \in K(l^2(G))$ and consequently $j_A \times j^A_G(j_A(a)j^A_G(f)) \in A \otimes K(l^2(G)) \subseteq \mathcal{M}(A \otimes K(l^2(G)))$. Also, we consider $U := id_A \otimes \rho^G : G \to A \otimes \mathcal{K}(l^2(G))$ defined by $U_g := Id_A \otimes \rho^G_g$ for all $g \in G$. We claim that (Π, U) is a covariant representation of the dynamical system $(A \rtimes_\delta G, G, \hat{\delta})$. Since (M, ρ^G) is a covariant representation of $(C_0(G), G, \tau)$ (see Example A.0.9) and λ^G and ρ^G commute, we are able to compute for all $a \in A_t$ and $g, s \in G$:

$$\begin{aligned}
\Pi(\hat{\delta}_g(j_A(a)j_G(\chi_s)))U_g &= \Pi((j_A(a))j_G(\chi_{sg^{-1}}))U_g = j_A(a)j_G(\chi_{sg^{-1}})U_g \\
&= (a \otimes \lambda_t)(1 \otimes M(\chi_{sg^{-1}}))(1 \otimes \rho_g) = a \otimes (\lambda_t M(\chi_{sg^{-1}})\rho_g) \\
&= a \otimes (\lambda_t \rho_g M(\chi_s)) = a \otimes (\rho_g \lambda_t M(\chi_s)) \\
&= (1 \otimes \rho_g)(a \otimes \lambda_t)(1 \otimes M(\chi_s)) = U_g j_A(a)j_G(\chi_s) \\
&= U_g \Pi((j_A(a)j_G(\chi_s)))
\end{aligned}$$

By the universal property of crossed products by actions we get a surjective *-homomorphism

$$\Pi \times U : A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \twoheadrightarrow A \otimes \mathcal{K}(l^2(G)). \tag{2.6.1}$$

This is surjective because $\overline{\text{span}}\{M_f \rho_g \mid f \in C_0(G), g \in G\}$ is equal to $\mathcal{K}(l^2(G))$ and the fact that δ is nondegenerate. The interesting fact is that this map is not always injective.

Definition 2.6.2. With notations as above, we say that the coaction $\delta : A \to A \otimes C^*(G)$ is *maximal* if $\Pi \times U$ is an isomorphism.

Definition 2.6.3. Let (A, G, δ) and (B, G, ϵ) be two dynamical co-systems. A maximal coaction ϵ is a maximalization of δ if there is a surjective G-equivariant map $\varphi : B \to A$ such that $\varphi \rtimes G : B \rtimes_\epsilon G \to A \rtimes_\delta G$ is an isomorphism.

Proposition 2.6.4 ([20], Theorem 3.3)**.** *Every coaction admits a maximalization and it is unique up to isomorphism.*

Example 2.6.5. If we consider the dynamical co-system $(C^*(G), G, \delta_G)$ we note that the comultiplication δ_G seen as a coaction is always maximal because δ_G is the dual coaction on $C^*(G) \cong \mathbb{C} \rtimes_{\mathrm{tr}} G$ (see Proposition A.0.17).

Example 2.6.6. More generally, for a Fell bundle $\mathcal{A}$ over G if we consider the dynamical co-system $(C^*(\mathcal{A}), G, \delta_{\mathcal{A}})$ the coaction $\delta_{\mathcal{A}}$ is always maximal (see [[20], Proposition 4.2]).

Proposition 2.6.7 ([20], Proposition 4.2)**.** *Let (A, G, δ) be a dynamical co-system. Then $\delta : A \to A \otimes C^*(G)$ is a maximal coaction if and only if the canonical map $\overline{\sigma} : C^*(\mathcal{A}) \twoheadrightarrow A$ seen in Proposition 2.4.11 is an isomorphism.*

Another important concept here is that of normal coactions. To contextualize, if we consider a dynamical system (A, G, α) we see from [60] that the canonical embedding ι_A of A into $\mathcal{M}(A \rtimes_\alpha G)$ is faithful. For coactions, the situation is different because faithfulness of π is not enough to ensure the faithfulness of $(\pi \otimes \Lambda^G) \circ \delta$. Therefore, in general, we can not assume that j_A of A into $\mathcal{M}(A \rtimes_\delta G)$ is faithful. We start with the following definition:

Definition 2.6.8. With notation as above, we say that the coaction $\delta : A \to A \otimes C^*(G)$ is *normal* if $j_A : A \to \mathcal{M}(A \rtimes_\delta G)$ is an injective map.

Remark 2.6.9. Most of the time j_A will be seen as $j_A = (id_A \otimes \Lambda^G) \circ \delta : A \to A \otimes C_r^*(G) \subseteq \mathcal{M}(A \otimes \mathcal{K}(l^2(G)))$.

Remark 2.6.10. Note that for every covariant representation (π, μ) of (A, G, δ) the *-homomorphism π factors through j_A, that is, $\pi \times \mu \circ j_A = \pi$. Thus δ is a normal coaction if and only if it admits a covariant representation (π, μ) with π faithful.

Definition 2.6.11. Let (A, G, δ) and (B, G, ϵ) be two dynamical co-systems. A normal coaction ϵ is a normalization of δ if there is a surjective G-equivariant map $\varphi : A \to B$ such that $\varphi \rtimes G : A \rtimes_\delta G \to B \rtimes_\epsilon G$ is an isomorphism.

Theorem 2.6.12 (Katayama Duality, [20], Proposition 2.2)**.** *A coaction $\delta : A \to A \otimes C^*(G)$ is normal if and only if the *-homomorphism $\Pi \times U$ from 2.6.1 factors through an isomorphism over reduced crossed product:*

$$(\Pi \times U)_r : A \rtimes_\delta G \rtimes_{\hat{\delta}, r} G \twoheadrightarrow A \otimes \mathcal{K}(l^2(G))$$

2.6.1 INVARIANT IDEALS AND CANONICAL NORMALIZATION

In this subsection, we discuss invariant ideals and the connection with normalization for coactions. Also, we present an equivalent criteria for maximal and normal coactions.

Definition 2.6.13. Let (A, G, δ) be a dynamical co-system. An ideal I in A is *strongly δ-invariant* if
$$\overline{\delta(I)(1 \otimes C^*(G))} = I \otimes C^*(G).$$

Remark 2.6.14. We will normally just write invariant to mean strongly invariant. The word "strongly" is emphasized to make a counterpoint with the notion of "weakly" δ-invariant that we will see later.

Remark 2.6.15. Let (A, G, δ) be a dynamical co-system and I be a δ-invariant ideal in A. It is straightforward to check that the map $\delta_I : I \to I \otimes C^*(G)$ defined as restriction of δ to I is a coaction. Moreover, the canonical embedding $I \hookrightarrow A$ is a $\delta_I - \delta$ equivariant map by definition. In fact, δ_I being a coaction is equivalent to say that I is δ-invariant ideal as defined above since $\overline{\delta_I(I)(1 \otimes C^*(G))} = \overline{\delta(I)(1 \otimes C^*(G))} = I \otimes C^*(G)$.

Maybe the most natural way to say that an ideal is "δ-invariant" would be to ask that it satisfies only $\delta(I) \subseteq I \otimes C^*(G)$ which is equivalent of $\delta(I)(1 \otimes C^*(G)) \subseteq I \otimes C^*(G)$. But it is not clear whether this implies the nondegeneracy of δ_I. However, this weaker condition is enough for the existence of a quotient coaction δ^I as we will see in the next proposition.

Proposition 2.6.16. *Let (A, G, δ) be a dynamical co-system and I a δ-invariant ideal in A. Then the map $\delta^I : A/I \to A/I \otimes C^*(G)$ defined by $\delta^I(q(a)) = (q \otimes id_G) \circ \delta(a)$ for all $a \in A$ is a coaction of G on A/I. Moreover, the quotient *-homomorphism $q : A \to A/I$ is a $\delta - \delta^I$ equivariant map.*

Proof. First of all, note that $I = \mathrm{Ker}(q)$. To check that δ^I is a well-defined *-homomorphism it is enough to show that $I \subseteq \mathrm{Ker}((q \otimes id_G) \circ \delta)$. But this follows from the fact that I is an invariant ideal in A since $\delta(\mathrm{Ker}(q)) \subseteq \mathrm{Ker}(q) \otimes C^*(G) \subseteq \mathrm{Ker}(q \otimes id_G)$. Moreover, δ^I takes values in $A/I \otimes C^*(G)$ by definition. The coaction identity for δ^I follows from the coaction identity of δ in this way:

$$\begin{aligned}
(\delta^I \otimes id_G) \circ (\delta^I \circ q) &= (\delta^I \otimes id_G) \circ (q \otimes id_G) \circ \delta \\
&= ((\delta^I \circ q) \otimes id_G) \circ \delta \\
&= ((q \otimes id_G) \circ \delta \otimes id_G) \circ \delta \\
&= ((q \otimes id_G \otimes id_G) \circ (\delta \otimes id_G) \circ \delta \\
&= ((q \otimes id_G \otimes id_G) \circ (id_A \otimes \delta_G) \circ \delta \\
&= (id_{A/I}) \otimes \delta_G) \circ (q \otimes id_G) \circ \delta \\
&= (id_{A/I} \otimes \delta_G) \circ \delta^I \circ q
\end{aligned}$$

Finally, we must to show that δ^I is nondegenerate but since δ it is nondegenerate we see that

$$\begin{aligned} A/I \otimes C^*(G) = (q \otimes id_G)(A \otimes C^*(G)) &= (q \otimes id_G)(\overline{\delta(A)(1 \otimes C^*(G))}) \\ &= \overline{(q \otimes id_G) \circ \delta(A)(1 \otimes C^*(G))} \\ &= \overline{\delta^I(q(A))(1 \otimes C^*(G))} \\ &= \overline{\delta^I(A/I)(1 \otimes C^*(G))} \end{aligned}$$

as desired.

□

Remark 2.6.17. The requirement that I is δ-invariant is stronger than we need to obtain a coaction on the quotient. In Proposition 2.6.16 if we suppose that I satisfies $\delta(I) \subseteq I \otimes C^*(G)$, the result follows in the same way because this was all we needed. This leads to a weaker notation of invariant ideals for coactions:

Definition 2.6.18. Let (A, G, δ) be a dynamical co-system. We say that I is a weakly δ-invariant ideal if $I = \text{Ker}(q \otimes id) \circ \delta$ where $q : A \to A/I$ is the quotient map.

Remark 2.6.19. Notice that q becomes a $\delta - \delta^I$ equivariant map for the unique coaction δ^I of G on A/I. In fact, δ^I being a coaction is equivalent to ask that I is a weakly δ-invariant ideal.

Proposition 2.6.20. *Let (π, μ) be a covariant representation of (A, G, δ) into a $\mathcal{M}(B)$. Then the map $\delta^\pi : \pi(A) \to \pi(A) \otimes C^*(G)$ defined by $\delta^\pi(\pi(a)) := (\pi \otimes id_G) \circ \delta(a)$ for all $a \in A$ is a normal coaction of G on $\pi(A)$. Moreover, $\pi : A \to \pi(A)$ is $\delta - \delta^\pi$ equivariant map.*

Proof. First of all, we need to check that δ^π is a well-defined *-homomorphism. To do that, let denote w_G the function $u : G \to C^*(G)$ viewed as an unitary element of $C_b(G, C^*(G)) \subseteq \mathcal{M}(C_0(G) \otimes C^*(G))$. So, $W := (\mu \otimes id_G)(w_G)$ is a unitary element of $\mathcal{M}(B \otimes C^*(G))$. We claim that $(\pi \otimes id_G) \circ \delta(a) = W(\pi(a) \otimes 1)W^*$, that is, $(\pi \otimes id_G) \circ \delta$ is unitarily equivalent to $\pi \otimes 1$. To see this it is enough to show the equation for each $a \in \mathcal{A}_g$ since $\oplus_{g \in G} \mathcal{A}_g$ is dense in A, that is, $(\pi(a) \otimes u_g)W = W(\pi(a) \otimes 1)$. Since μ is nondegenerate it suffices to prove $(\pi(a) \otimes u_g)(\mu \otimes id_G)(w_G)(\mu(\chi_h) \otimes 1) = (\mu \otimes id_G)(w_G)(\pi(a) \otimes 1)(\mu(\chi_h) \otimes 1)$ for every $h \in G$. But, using the covariant property we have

$$\begin{aligned} (\mu \otimes id_G)(w_G)(\pi(a) \otimes 1)(\mu(\chi_h) \otimes 1) &= (\mu \otimes id_G)(w_G)((\pi(a)\mu(\chi_h)) \otimes 1) \\ &= (\mu \otimes id_G)(w_G)((\mu(\chi_{gh})\pi(a)) \otimes 1) \\ &= (\mu \otimes id_G)(w_G)(\mu(\chi_{gh}) \otimes 1)(\pi(a) \otimes 1) \\ &= (\mu(\chi_{gh}) \otimes u_{gh})(\pi(a) \otimes 1) \\ &= \mu(\chi_{gh})\pi(a) \otimes u_g u_h \\ &= \pi(a)\mu(\chi_h) \otimes u_g u_h \\ &= (\pi(a) \otimes u_g)(\mu(\chi_h) \otimes u_h) \\ &= (\pi(a) \otimes u_g)(\mu \otimes id_G)(w_G)(\mu(\chi_h) \otimes 1) \end{aligned}$$

and the claim follows. Now, it is straightforward to see that δ^π is well defined *-homomorphism. For nondegeneracy, since δ is nondegenerate then we have

$$\overline{\delta^\pi(\pi(A))(1 \otimes C^*(G))} = (\pi \otimes id_G)(\overline{\delta(A)(1 \otimes C^*(G))}) = (\pi \otimes id_G)(A \otimes C^*(G)) = \pi(A) \otimes C^*(G).$$

The coaction identity for δ^π follows from δ in this way:

$$\begin{aligned}(\delta^\pi \otimes id_G) \circ (\delta^\pi \circ \pi) &= (\delta^\pi \otimes id_G) \circ (\pi \otimes id_G) \circ \delta \\ &= ((\delta^\pi \circ \pi) \otimes id_G) \circ \delta \\ &= ((\pi \otimes id_G) \circ \delta \otimes id_G) \circ \delta \\ &= ((\pi \otimes id_G \otimes id_G) \circ (\delta \otimes id_G) \circ \delta \\ &= ((\pi \otimes id_G \otimes id_G) \circ (id_A \otimes \delta_G) \circ \delta \\ &= (id_{\pi(A)}) \otimes \delta_G) \circ (\pi \otimes id_G) \circ \delta \\ &= (id_{\pi(A)} \otimes \delta_G) \circ \delta^\pi \circ \pi\end{aligned}$$

Finally, observe that $(id_{\pi(A)}, \mu)$ is a covariant representation for δ^π because of the definition δ^π and the fact that (π, μ) is a covariant representation for δ. By Remark 2.6.10 the coaction δ^π is a normal. Also, $\pi : A \to \pi(A)$ is $\delta - \delta^\pi$ equivariant by definition of δ^π.

□

Remark 2.6.21. From previous result observe that Ker(π) is a weakly δ-invariant ideal since there is a coaction δ^π of G on $\pi(A) \cong A/\text{Ker}(\pi)$. In particular, Ker$(j_A)$ is a weakly δ-invariant ideal since it forms a covariant pair with j_G.

Corollary 2.6.22. *Let (A, G, δ) be a dynamical system and let $A_n := j_A(A) \cong A/\mathit{Ker}(j_A)$. Then the map $\delta^n : A_n \to A_n \otimes C^*(G)$ defined by $\delta^n(j_A(a)) = (j_A \otimes id_G) \circ \delta(a)$ for all $a \in A$ is a normal coaction of G on A_n. Moreover, $j_A : A \to A_n$ is a $\delta - \delta^n$ equivariant map.*

Proof. It follows directly from Proposition 2.6.20.

□

Remark 2.6.23. An important point here is that not every invariant ideals in the weak sense is invariant in the strong sense. As we can see in [49], Nilsen observes that if I is a strongly δ-invariant ideal then $I \nsubseteq$ Ker(j_A) and hence Ker(j_A) is not strongly invariant unless δ is normal. But notice that Ker(j_A) is a weakly δ-invariant ideal by the previous corollary. Moreover, we do not know whenever the condition $\delta(I) \subseteq I \otimes C^*(G)$ is equivalent to I be invariant in the weak sense. In fact, if $I =$ Ker(q) then $\delta(I) \subseteq I \otimes C^*(G) \subseteq$ Ker$(q \otimes id)$ which means that I is a weakly δ-invariant ideal but the converse is not clear. A sufficient condition would be if Ker$(q \otimes id) =$ Ker$(q) \otimes C^*(G)$ but this not always equal. For example, this last equality is equivalent to the exactness of the short sequence

$$0 \to I \otimes C^*(G) \to A \otimes C^*(G) \to A/I \otimes C^*(G) \to 0.$$

But this is not always exact, if $C^*(G)$ is not exact, which is the case if G is not exact.

This discussion shows that if I is strongly δ-invariant then it also satisfies the condition $\delta(I) \subseteq I \otimes C^*(G)$, which in turn implies that I is weakly δ-invariant. But the converse is not always true. Well, at least if G is amenable, then all these conditions are equivalent, see [42].

What we need in the course of this section is the existence of coactions on the quotient, that is, we only will need weakly δ-invariant ideals.

Corollary 2.6.24. *Let (A, G, δ) be a dynamical co-system. Then the coaction δ^n of G on $A/\mathrm{Ker}(j_A)$ is the normalization of δ, that is, every coaction admits a normalization.*

Proof. Let $A_n := j_A(A) \cong A/\mathrm{Ker}(j_A)$ and consider j_A as a map $j_A : A \to A_n$. Follows from Corollary 2.6.22 that $\delta^n : A_n \to A_n \otimes C^*(G)$ is a normal coaction and consequently $j_A : A \to A_n$ is a $\delta - \delta^n$ equivariant map by definition.

Finally we will see that δ^n is a normalization of δ. By Proposition 2.5.26, $j_A \rtimes G : A \rtimes_\delta G \to A^n \rtimes_{\delta^n} G$ is a surjective *-homomorphism. We must to show that $j_A \rtimes G$ is in fact an isomorphism. Let (j_A, j_G) and (j_{A^n}, j_G) be the canonical covariant representations of (A, G, δ) and (A^n, G, δ^n), respectively, and let ρ be a faithful representation of $A \rtimes_\delta G$ on $\mathcal{M}(C)$ for some C*-algebra C. We know that ρ factors through to representations $\pi = \rho \circ j_A$ and $\mu = \rho \circ j_G$. Since $\mathrm{Ker}(\pi) = \mathrm{Ker}(j_A)$ there is $\tilde{\pi} : A^n \to \mathcal{M}(C)$ such that $\tilde{\pi} \circ j_A = \pi$. Then $(\tilde{\pi}, \mu)$ is a covariant representation of (A^n, G, δ^n) since (π, μ) is a covariant representation of (A, G, δ). Consider the integrated form $\tilde{\pi} \times \mu$ of $(\tilde{\pi}, \mu)$. We claim that $(\tilde{\pi} \times \mu) \circ (j_A \rtimes G) = \rho$. To see that observe

$$\begin{aligned}
\tilde{\pi} \times \mu \circ j_A \rtimes G(j_A(a)j_G(f)) &= \tilde{\pi} \times \mu(j_{A^n} \circ j_A(a)j_G(f)) = \tilde{\pi} \circ j_A(a)\mu(f) \\
&= \pi(a)\mu(f) = \rho(j_A(a)j_G(f))
\end{aligned}$$

Since ρ is a faithful representation it follows that $j_A \rtimes G$ is also faithful, as desired. ∎

Proposition 2.6.25 ([20], Lemma 2.1). *The normalization of a coaction is unique up to isomorphism.*

Remark 2.6.26. A different approach using the reduced C*-algebra of the group is also established in the literature as can be seen in [41] and [35]. These are also called reduced coactions. Basically, the difference is the use of the reduced C*-algebra $C_r^*(G)$ with canonical comultiplication $\delta_G^r : C_r^*(G) \to C_r^*(G) \otimes C_r^*(G)$ such that $\delta_G^r(\lambda_g^G) = \lambda_g^G \otimes \lambda_g^G$. In this case, the definition of covariant representations of reduced coactions is completely analogous to the definition that we use here. In particular, j_A is always faithful for reduced co-systems, so there is no need for a concept of normal reduced co-systems. Normal coactions and reduced coactions are essentially equivalent concepts and because of this the theory of coactions we use here is potentially more richer than the theory of reduced coactions. In this context, Baaj and Skandalis proved in [6] that reduced coactions are automatically nondegenerate for discrete groups.

Example 2.6.27. The normalization of δ_G is the canonical coaction δ_G^r of G on $C_r^*(G)$. So, δ_G is a normal coaction if and only if G is amenable. The same way δ_G^r is a maximal coaction if and only if G is amenable. More generally, given a dynamical system (A, G, α) the normalization of dual coaction $\widehat{\alpha}$ on $A \rtimes_\alpha G$ is the coaction $\widehat{\alpha_r}$ on $A \rtimes_{\alpha,r} G$ (see Proposition A.0.19). Both these results are a special case of the following result.

Proposition 2.6.28. *Let (A, G, δ) be a dynamical co-system and consider the associated Fell bundle $\mathcal{A} = \{\mathcal{A}_g\}_{g\in G}$ given by the spectral subspaces of the coaction. Then the canonical coaction $\delta_{\mathcal{A}}^r$ of G on $C_r^*(\mathcal{A})$ is the normalization of δ_A, that is, $\delta_{\mathcal{A}}^r = \delta_{\mathcal{A}}^n$.*

Proof. By Remark 2.4.18 we know that $\delta_{\mathcal{A}}^r$ is normal coaction. We just need to prove that the *-homomorphism $\Lambda \rtimes G : C^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}} G \to C_r^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}^n} G$ is injective. To start we write $B = C^*(\mathcal{A})$ and $B_r = C_r^*(\mathcal{A})$ and let ρ be a faithful representation of $C^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}} G$ on $\mathcal{M}(C)$ for some C*-algebra C. We know that ρ factors through $\pi = \rho \circ j_B$ and $\mu = \rho \circ j_G$.

Note that for all $a \in \mathcal{A}_g$ we have $j_B(a) = (id_A \otimes \Lambda^G) \circ (\delta_{\mathcal{A}}(a)) = a \otimes \lambda_g^G$. Thus by Fell's absorption principle 2.4.15 there is $\tilde{j_B} : C_r^*(\mathcal{A}) \to \mathcal{M}(C^*(\mathcal{A}) \otimes \mathcal{K}(l^2(G)))$ such that $j_B = \tilde{j_B} \circ \Lambda$. In fact, observe that $\tilde{j_B}$ is exactly j_{B_r} because both act in the same way, that is,

$$j_{B_r}(\Lambda(a)) = (id_A \otimes \Lambda^G)(\delta_{\mathcal{A}}^r(\lambda_g(a))) = \lambda_g(a) \otimes \lambda_g^G = \tilde{j_B}(\lambda_g(a)) = \tilde{j_B}(\Lambda(a))$$

We have $\pi = \rho \circ j_A = \rho \circ j_B \circ \Lambda$. Writing $\tilde{\pi} := \rho \circ j_B$ we see that π factors through $C_r^*(\mathcal{A})$. We claim that $(\tilde{\pi}, \mu)$ is a covariant representation of $(C_r^*(\mathcal{A}), G, \delta_{\mathcal{A}}^r)$. Indeed this is clear because (π, μ) is a covariant representation of $(C^*(\mathcal{A}), G, \delta_{\mathcal{A}}^r)$ and satisfies $\pi = \tilde{\pi} \circ \Lambda$. Consider $\tilde{\rho} := \tilde{\pi} \rtimes \mu$ as a representation of $C_r^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}^r} G$ such that $\tilde{\rho} \circ j_{B_r} = \tilde{\pi}$ and $\tilde{\rho} \circ j_G^{B_r} = \mu$. We just need to observe that

$$\tilde{\rho} \circ \Lambda \rtimes G(j_B(a) j_G^B(f)) = \tilde{\rho}((j_{B_r} \circ \Lambda)(a) j_G^{B_r}(f)) = (\tilde{\pi} \circ \Lambda)(a)\mu(f) = \pi(a)\mu(f) = \rho(j_B(a) j_G^B(f))$$

This implies $\tilde{\rho} \circ \Lambda \rtimes G = \rho$. Since ρ is a faithful representation it follows that $\Lambda \rtimes G$ is also faithful, as desired.

□

Corollary 2.6.29. *The following diagram (as in Proposition 2.5.27)*

$$\begin{tikzcd}
C^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}} G \arrow[rr, "\Lambda \rtimes G"] \arrow[dr, "\overline{\sigma} \rtimes G"'] & & C_r^*(\mathcal{A}) \rtimes_{\delta_{\mathcal{A}}^n} G \\
& A \rtimes_\delta G \arrow[ur, "\psi \rtimes G"'] &
\end{tikzcd}$$

is a diagram of isomorphisms.

Proof. It follows from Proposition 2.6.28 that $\Lambda \rtimes G$ is an isomorphism, which implies that $\varphi \rtimes G$ and $\psi \rtimes G$ are also isomorphisms. In particular, $\mathrm{Ker}(j_A) = \mathrm{Ker}(j_{C_r^*(\mathcal{A})} \circ \psi) = \mathrm{Ker}(\psi)$.
□

Thus we can say that any coaction $\delta : A \to A \otimes C^*(G)$ "lies between" the maximal coaction $\delta_{\mathcal{A}}$ on $C^*(\mathcal{A})$ and the normal coaction $\delta^r_{\mathcal{A}}$ on $C^*_r(\mathcal{A})$. There is a discussion in [54] about coactions which are neither maximal nor normal.

Lemma 2.6.30. *Let $\delta : A \to A \otimes C^*(G)$ be a coaction. Then δ is normal if and only if the conditional expectation $E_1 : A \to \mathcal{A}_1$ defined in Proposition 2.5.6 is faithful.*

Proof. Consider $\tau : C^*_r(G) \to \mathbb{C}$ the canonical faithful condition expectation and j_A seen as $j_A := (id \otimes \Lambda^G) \circ \delta$. We claim that $(id \otimes \tau) \circ j_A = E_1$. To see that it is enough to check the equality on the elements $a \in \mathcal{A}_g$ since $\oplus_{g \in G} \mathcal{A}_g$ is dense in A. So,

$$(id_A \otimes \tau) \circ (id_A \otimes \Lambda^G) \circ \delta(a) = (id_A \otimes \tau)(a \otimes \lambda^G_g) = a\delta_{g,1} = (id \otimes \chi_1)(a \otimes g) = E_1(a).$$

So, if E_1 is faithful by the equality above Ker$(j_A) = 0$ which implies that δ is normal. Conversely, if δ is normal then $j_A = (id \otimes \Lambda^G) \circ \delta$ is faithful. Since $(id \otimes \tau) \circ j_A = E_1$ we conclude that E_1 is faithful because $(id \otimes \tau)$ is faithful on $A \otimes C^*_r(G)$ (The reason $(id \otimes \tau)$ is faithful is because we are considering the minimal tensor product and because τ is faithful). This completes the proof.

□

Proposition 2.6.31. *With notations as above, $\delta : A \to A \otimes C^*(G)$ is a normal coaction if and only if $\psi : A \twoheadrightarrow C^*_r(\mathcal{A})$ seen in Proposition 2.4.11 is an equivariant isomorphism.*

Proof. It is enough to observe that we have a faithful conditional expectation E_1 from A to $\mathcal{A}_1$ seen in Lemma 2.6.30. By Proposition 2.4.13 the result follows.

□

So, to organize the ideas we make a compilation of the results seen so far in the next corollaries:

Corollary 2.6.32. *Let (A, G, δ) be a dynamical co-system. Then the following statements are equivalent:*

1. *$\delta : A \to A \otimes C^*(G)$ is maximal coaction.*
2. *The canonical *-homomorphism $\Pi \times U : A \rtimes_\delta G \rtimes_{\hat{\delta}} G \twoheadrightarrow A \otimes \mathcal{K}(l^2(G))$ is an isomorphism.*
3. *The canonical *-homomorphism $\overline{\sigma} : C^*(\mathcal{A}) \twoheadrightarrow A$ is an equivariant isomorphism.*

Proof. It follows from Definition 2.6.2 and Proposition 2.6.7.

□

Corollary 2.6.33. *Let (A, G, δ) be a dynamical co-system. Then the following statements are equivalent:*

1. *$\delta : A \to A \otimes C^*(G)$ is normal coaction.*

2. $j_A = (id \otimes \lambda) \circ \delta$ *is faithful.*

3. *The canonical *-homomorphism* $\psi : A \twoheadrightarrow C_r^*(\mathcal{A})$ *is an equivariant isomorphism.*

4. *The canonical *-homomorphism* $(\Pi \times U)_r : A \rtimes_\delta G \rtimes_{\hat{\delta},r} G \twoheadrightarrow A \otimes \mathcal{K}(l^2(G))$ *is an isomorphism.*

5. *The canonical conditional expectation* $E_1 := (id_A \otimes \chi_1) \circ \delta : A \to \mathcal{A}_1$ *is faithful.*

Proof. It follows from Definition 2.6.8, Katayama Duality in Theorem 2.6.12, Lemma 2.6.30 and Proposition 2.6.31.

□

We finish this section with some results on maximality and normality of inflated coactions.

Proposition 2.6.34. *Let* H *and* G *be a discrete groups,* $\varphi : H \to G$ *be a homomorphism and let* (A, H, ϵ) *be a dynamical co-system. If* ϵ *is a maximal coaction then so is the inflated coaction Inf*(ϵ).

Proof. Let $\mathcal{A} = \{\mathcal{A}_h\}_{h\in H}$ and $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_g\}_{g\in G}$ be the Fell bundles associated with the spectral subspaces of ϵ and $\mathrm{Inf}(\epsilon)$, respectively. To see this result we are going to prove that the cross-sectional C*-algebra of both Fell bundles are isomorphic. We claim that $\tilde{\mathcal{A}}_g = \overline{\bigoplus_{\substack{h\in H\\ \varphi(h)=g}} \mathcal{A}_h}$.

On the one hand, if we take $a = \sum_{i=1}^n a_{h_i}$ where $a_{h_i} \in \mathcal{A}_{h_i}$ with $\varphi(h_i) = g$ then a belongs to $\tilde{\mathcal{A}}_g$ by definition since

$$\mathrm{Inf}(\epsilon)\left(\sum_{i=1}^n a_{h_i}\right) = \sum_{i=1}^n (id_A \otimes \overline{\varphi})(a_{h_i} \otimes h_i) = \left(\sum_{i=1}^n a_{h_i} \otimes \varphi(h_i)\right) = \left(\sum_{i=1}^n a_{h_i}\right) \otimes g.$$

By continuity we have $\overline{\bigoplus_{\substack{h\in H\\ \varphi(h)=g}} \mathcal{A}_h} \subseteq \tilde{\mathcal{A}}_g$.

On the other hand, let $a \in \tilde{\mathcal{A}}_g$. So, a can be approximated by a finite sum $\sum_{i=1}^n a_{h_i}$ with $a_{h_i} \in \mathcal{A}_{h_i}$. Applying the projection $E_g : A \to \tilde{\mathcal{A}}_g$ we have

$$a = E_g(a) \approx E_g\left(\sum_{i=1}^n a_{h_i}\right) = \sum_{i=1}^n E_g(a_{h_i}).$$

Observe that $E_g(a_{h_i}) = (id_A \otimes \chi_g) \circ (id_A \otimes \varphi) \circ \epsilon(a_{h_i}) = a_{h_i} \otimes \chi_g(\varphi(h_i))$. So, if $g \neq \varphi(h)$ then $E_g(a_{h_i}) = 0$. So, the claim was proved.

Now, φ give us a representation π of the Fell bundle $\mathcal{A}$ into $C^*(\tilde{\mathcal{A}})$, that is, we consider $\pi := j^{\tilde{\mathcal{A}}} \circ \pi_h$ where $\pi_h : \mathcal{A}_h \hookrightarrow \tilde{\mathcal{A}}_{\varphi(h)}$ are the inclusions maps and $j^{\tilde{\mathcal{A}}}$ is the universal representation of $\tilde{\mathcal{A}}$ into $C^*(\tilde{\mathcal{A}})$. By the universal property, we get *-homomorphism $\pi : C^*(\mathcal{A}) \to C^*(\tilde{\mathcal{A}})$ such that $\pi(j_h^{\mathcal{A}}(a)) = j_{\varphi(h)}^{\tilde{\mathcal{A}}}(\pi_h(a))$. From the claim seen above we see that

π is surjective. To see the injectivity we are going to construct the inverse. For this, again by the claim above we can view $\tilde{\mathcal{A}}_g$ in $C^*(\mathcal{A})$ through $j_g^{\mathcal{A}}$ and this gives us a representation of the Fell bundle $\tilde{\mathcal{A}}$ into $C^*(\mathcal{A})$. By the universal property, we get *-homomorphism $\psi : C^*(\tilde{\mathcal{A}}) \to C^*(\mathcal{A})$ such that $\psi(j_g^{\tilde{\mathcal{A}}}(a)) = \sum\limits_{\substack{h \in H \\ \varphi(h)=g}} j_h^{\mathcal{A}}(a_h)$ for all $a \in \tilde{\mathcal{A}}_g$. By construction we have $\psi \circ \pi = id$ since for every $j_h(a)$ with $a \in \mathcal{A}_h$, $h \in H$ we have

$$\begin{aligned}
\psi \circ \pi(j_h^{\mathcal{A}}(a)) &= \psi(j_{\varphi(h)}^{\tilde{\mathcal{A}}}(\pi_h(a))) \\
&= \psi(j_{\varphi(h)}^{\tilde{\mathcal{A}}}(a)) \\
&= j_h^{\mathcal{A}}(a)
\end{aligned}$$

Since the closed linear span of the set of all $j_h(a)$, $a \in \mathcal{A}_h$, is dense in $C^*(\mathcal{A})$ we get $C^*(\mathcal{A}) \cong C^*(\tilde{\mathcal{A}})$ as we desired.

Finally, if ϵ is maximal coaction then $A \cong C^*(\mathcal{A})$ and hence $A \cong C^*(\tilde{\mathcal{A}})$ from the isomorphism seen before which it follows that Inf(ϵ) is maximal coaction.

□

Proposition 2.6.35. *Let H and G be a discrete groups, $\varphi : H \to G$ be a homomorphism and let (A, H, ϵ) be a dynamical co-system. If ϵ is normal coaction then Inf(ϵ) is also normal coaction.*

Proof. As in previous proposition, let $\mathcal{A} = \{\mathcal{A}_h\}_{h \in H}$ and $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_g\}_{g \in G}$ be the Fell bundles associated with the spectral subspaces of ϵ and Inf(ϵ) and E_1^H and E_1^G the conditional expectations of A in $\mathcal{A}_1$ and $\tilde{\mathcal{A}}_1$, respectively. Notice that $\mathcal{A}_1 \subseteq \tilde{\mathcal{A}}_1 = \overline{\bigoplus\limits_{\substack{h \in H \\ c(h)=1}} \mathcal{A}_h}$. Let $Q := E_1^H|_{\tilde{\mathcal{A}}_1}$ then $Q \circ E_1^G = E_1^H$. The reason is because every $a \in A$ can be approximated by a finite sum $\sum_{h \in H} a_h$ an note that $E_1^G(a_h) = \delta_{1,\varphi(h)} a_h$. So,

$$Q \circ E_1^G \left(\sum_{h \in H} a_h \right) = Q \left(\sum_{\substack{h \in H \\ c(h)=1}} a_h \right) = a_1 = E_1^H \left(\sum_{h \in H} a_h \right).$$

By continuity the results follows.

Then if ϵ is normal coaction by Proposition 2.6.31 we have that E_1^H is faithful and hence E_1^G is faithful which implies again by Proposition 2.6.31 then Inf(ϵ) is also normal coaction.

□

2.7 GRAPH C*-ALGEBRA CASE

One of the goals of this work is to study the relationship between $C^*(E \times_c G)$ and $C^*(E)$. What kind of relationship do we have here? We are going to investigate some duality theorems in this context using the theory explored in the previous sections. We assume that the reader is familiar with the notion of crossed products by actions. Again, we have an appendix where we review some important definitions and results about actions and their crossed products.

Our main references for this section are [37] and [33]. In this section, we do not make any assumptions of row-finiteness in our graphs. Despite the references cited here use this hypothesis, in all results that will be seen the row-finite hypothesis is unnecessary and the same proofs go through for general graphs.

Throughout this section let E be an arbitrary graph, G be a discrete group and $c : E^1 \to G$ be a labeling function as in Section 2.1.

Theorem 2.7.1. *[[37], Corollary 3.9] Let E be a graph and let $c : E^1 \to G$ be a labeling function. Then*

$$C^*(E \times_c G) \rtimes_\gamma G \cong C^*(E) \otimes \mathcal{K}(l^2(G))$$

where γ denotes the action of G on $C^*(E \times_c G)$ induced by the canonical action of G on $E \times_c G$.

Corollary 2.7.2. *[[37], Corollary 3.10] Let E be a graph and let θ be a free action of G on E. Then*

$$C^*(E) \rtimes_\theta G \cong C^*(E/G) \otimes \mathcal{K}(l^2(G))$$

Proof. Follows from Theorem 2.7.1 and the Gross-Tucker Theorem 2.1.17.

□

Making a parallel with the abelian case we see that given a labeling function $c : E^1 \to G$, there is an action of $\hat{G}$ on $C^*(E)$ of the form $\beta_\chi(P_v) = P_v$ and $\beta_\chi(S_e) = \chi(c(e))S_e$ for all $v \in E^0$, $e \in E^1$ and $\chi \in \hat{G}$. In this case we can identify naturally $\hat{\hat{G}}$ with G and with this action we are able to show the next result.

Theorem 2.7.3. *Let E be graph and let $c : E^1 \to G$ be a labeling function. Then*

$$C^*(E \times_c G) \cong C^*(E) \rtimes_\beta \hat{G}$$

where γ, the action of G on $C^*(E \times_c G)$, is G-equivariant with the dual action $\hat{\beta}$ of G on $C^*(E) \rtimes_\gamma \hat{G}$.

Proof. This result follows from a more general result, that is, a combination of Proposition 2.5.5 and the Theorem 2.7.7 which will see later.

□

These results can be found in [37] and are proved through the approach of C*-algebras of groupoids associated to directed graphs. In [33] a more direct proof is given through the universal proprieties.

Remark 2.7.4. So far, compiling the results above we have the following diagram of isomorphisms:

$$\begin{tikzcd}
C^*(E \times_c G) \rtimes_\gamma G \arrow[rr, "2.7.3"] \arrow[dr, "2.7.1"] & & C^*(E) \rtimes_\beta \hat{G} \rtimes_{\hat{\beta}} G \arrow[dl] \\
& C^*(E) \otimes \mathcal{K}(l^2(G)) &
\end{tikzcd}$$

The right isomorphism which is a combination of two results seen so far is a consequence of Takai-Takesaki duality (see Corollary A.0.18) for Graph C*-algebras.

Example 2.7.5. Let E be an arbitrary graph and consider the labeling function $c : E^1 \to \mathbb{Z}$ such that $c(e) = 1$ for every $e \in E^1$. Identifying $\hat{\mathbb{Z}} \cong \mathbb{T}$, by Theorem 2.7.3 we have

$$C^*(E \times_c \mathbb{Z}) \cong C^*(E) \rtimes_\alpha \mathbb{T}$$

where α is the gauge action of $\mathbb{T}$ on $C^*(E)$. Note that the skew product graph $E \times_c \mathbb{Z}$ has no loops since for every edge $(e, k) \in E^1 \times \mathbb{Z}$ we have $s(e, k) = (s(e), k) \neq (r(e), k+1) = r(e, k)$. By [[38], Theorem 2.4], $C^*(E \times_c \mathbb{Z})$ is an AF-algebra which implies that it is nuclear C*-algebra by [[47], 6.3.11]. Identifying $\hat{\hat{\mathbb{Z}}} \cong \mathbb{Z}$ it follows that $C^*(E) \rtimes_\alpha \mathbb{T} \rtimes_{\hat{\alpha}} \mathbb{Z}$ is also nuclear by [[12], Theorem 4.2.6]. Since $C^*(E) \rtimes_\alpha \mathbb{T} \rtimes_{\hat{\alpha}} \mathbb{Z} \cong C^*(E) \otimes \mathcal{K}(l^2(G))$ we conclude that $C^*(E)$ is strongly Morita equivalent to a nuclear C*-algebra which implies it is also nuclear. This can be considered as an alternative proof for nuclearity of graph C*-algebras using the duality theorems seen above.

However, what happens if G is not abelian? In this case, we need to replace actions of $\hat{G}$ by coactions of G and hence we are going to use the relationship between coactions of G and Fell bundles in the context of graph C*-algebras. But what kind of coaction do we find here? Now, we will explore this.

Theorem 2.7.6 ([33], Lemma 2.3)**.** *Given a labeling function $c\colon E^1 \to G$ on a graph E, there is a coaction $\delta_c\colon C^*(E) \to C^*(E) \otimes C^*(G)$ satisfying*

$$\delta_c(P_v) = P_v \otimes 1 \quad \text{and} \quad \delta_c(S_e) = S_e \otimes c(e)$$

for all $v \in E^0$ and $e \in E^1$, where $\{P_v, S_e\}$ is the universal Cuntz-Krieger E-family in $C^*(E)$.

Theorem 2.7.7. *[[33], Theorem 2.4] Let E be a graph and let $c : E^1 \to G$ be a labeling function. Then*

$$C^*(E \times_c G) \cong C^*(E) \rtimes_{\delta_c} G.$$

Moreover, the canonical action γ of G on $C^*(E \times_c G)$ is G-equivariant with dual action $\hat{\delta}_c$ of G on $C^*(E) \rtimes_{\delta_c} G$.

Remark 2.7.8. Proposition 2.7.3 now also follows from Theorem 2.7.7 and Proposition 2.5.5.

Example 2.7.9. Let A_n be the Cuntz graph, that is, $A_n^0 = \{v\}$ and $A_n^1 = \{e_1, \dots, e_n\}$. Let $\mathbb{F}_n$ be the free group on n generators $g_1, g_2, \dots, g_n$ and consider the labeling $c\colon A_n^1 \to \mathbb{F}_n$ defined by $c(e_i) = g_i$. This gives a coaction δ_c of $\mathbb{F}_n$ on the Cuntz algebra $\mathcal{O}_n \cong C^*(A_n)$ satisfying $\delta_c(S_i) = S_i \otimes g_i$. This is the usual coaction on $\mathcal{O}_n$ associated to the grading structure on $\mathcal{O}_n$ (that is, a Fell bundle) over $\mathbb{F}_n$ (see [23] and [53]). Of course, the above procedure can be generalized: given any group G and n generators $g_1, \dots, g_n \in G$, define $c\colon A_n^1 \to G$ by $c(e_i) = g_i$. This map induces a coaction δ_c of G on $\mathcal{O}_n \cong C^*(A_n)$ the same way as before.

A particularly interesting case is when G is abelian, so that the coaction δ_c corresponds to an action α of the compact group $\hat{G}$ on $\mathcal{O}_n$. This action sends the generator $S_i \in \mathcal{O}_n$ to $\alpha_\chi(S_i) = \chi_{g_i} S_i$. These types of actions on $\mathcal{O}_n$ are called quasi-free actions and have been studied for a long time, see [28]. In particular, we conclude that

$$C^*(A_n \times_c G) \cong C^*(E_G) \cong \mathcal{O}_n \rtimes_{\delta_c} \mathbb{F}_n$$

where E_G is the Cayley graph discussed in Example 2.1.8.

Remark 2.7.10. So far, compiling the results, we have the following generalized diagram of isomorphisms:

$$\begin{tikzcd}
C^*(E \times_c G) \rtimes_\gamma G \arrow[rr, "2.7.7"] \arrow[dr, "2.7.1"] & & C^*(E) \rtimes_{\delta_c} G \rtimes_{\hat{\delta}_c} G \arrow[dl] \\
& C^*(E) \otimes \mathcal{K}(l^2(G)) &
\end{tikzcd}$$

Corollary 2.7.11. *With notations as above, δ_c is maximal coaction.*

Proof. Using the previous remark, this follows from Theorem 2.7.1 and Theorem 2.7.7. $\square$

Theorem 2.7.12. *With notations as above, δ_c is a normal coaction.*

Proof. It is enough to show that there is a covariant representation (π, μ) such that π is faithful.

Let α be the gauge action of $\mathbb{T}$ on $C^*(E)$ and (π, μ) a covariant representation of $(C^*(E), G, \alpha)$ such that π is faithful. So, $(\pi \otimes \Lambda^G \circ \delta_c, 1 \otimes M)$ is a covariant representation of $(C^*(E), G, \delta_c)$ induced by π. Note that $\{\pi \otimes \Lambda^G \circ \delta_c(P_v), \pi \otimes \Lambda^G \circ \delta_c(S_e)\}$ is a Cuntz-Krieger E-family in which each projection $\pi \otimes \Lambda^G \circ \delta_c(P_v) = \pi(P_v) \otimes \lambda_1^G$ is nonzero.

For each $z \in \mathbb{T}$, the representation $\mu_z \otimes 1 : C^*(E) \otimes C^*(G) \to C^*(E) \otimes C^*(G)$ implements the gauge action in the sense that:

$$\begin{aligned}
(\mu_z \otimes 1)(\pi \otimes \Lambda^G \circ \delta_c(S_e))(\mu_z \otimes 1)^* &= (\mu_z \otimes 1)(\pi(S_e) \otimes \lambda^G_{c(e)})(\mu_z^* \otimes 1)\\
&= \mu_z \pi(S_e)\mu_z^* \otimes \lambda^G_{c(e)}\\
&= \pi(\alpha_z(S_e)) \otimes \lambda^G_{c(e)}\\
&= \pi \otimes \Lambda^G \circ \delta_c(\alpha_z(S_e))
\end{aligned}$$

So, by Proposition 2.2.7 we have $(\pi \otimes \Lambda^G) \circ \delta_c$ is injective, as desired.

$\square$

Remark 2.7.13. The fact that the coaction δ_c is maximal and normal is not a surprise in this context because $C^*(E)$ is nuclear. Indeed, if δ is a coaction of G over a nuclear C*-algebra A then δ is always maximal and normal. The reason is because if we consider the Fell bundle $\mathcal{A}$ associated with the spectral subspaces relative of δ we have a surjective *-homomorphism of A to $C_r^*(\mathcal{A})$ and hence $C_r^*(\mathcal{A})$ is nuclear (because it is a quotient of A under Ker(ψ)) and by [[25], Theorem 25.11] $\mathcal{A}$ is amenable which means that $C^*(\mathcal{A}) \cong C_r^*(\mathcal{A})$.

Remark 2.7.14. Compiling all results that have been seen so far we have the following diagram of isomorphisms:

$$\begin{tikzcd}
C^*(E \times_c G) \rtimes_\gamma G \arrow[rr, "2.7.7"] \arrow[dr, "2.7.1"] \arrow[dd, two heads, "\text{regular representation}"'] & & C^*(E) \rtimes_{\delta_c} G \rtimes_{\hat{\delta}_c} G \arrow[dl] \arrow[dd, two heads, "\text{regular representation}"] \\
& C^*(E) \otimes \mathcal{K}(l^2(G)) & \\
C^*(E \times_c G) \rtimes_{\gamma,r} G \arrow[rr, "2.7.7"] \arrow[ur] & & C^*(E) \rtimes_{\delta_c} G \rtimes_{\hat{\delta}_{c,r}} G \arrow[ul, "2.7.12"']
\end{tikzcd}$$

Now, the question is: can we get a similar diagram above to a more general class of graph C*-algebras? This is the proposal for the next chapters.

3 SEPARATED GRAPH C*-ALGEBRA CASE

In this chapter, our proposal is to extend the duality theorems seen in Chapter 2, specifically in Section 2.7, to a more general class of C*-algebras, the C*-algebras of separated graphs. We introduce the definition of separated graphs based on [4] and [5]. Then we get a Gross-Tucker theorem for separated graphs and some duality theorems involving this class of C*-algebras.

We shall produce results involving group actions and coactions on separated graphs. However, there have been many generalizations of the results in different directions: in [17] the authors work with actions of topological groups on topological graphs. Moreover, in [7] the results have been generalized to labeled graphs, and in [45] part of the results have been generalized to semigroup actions on higher-rank graphs.

We mainly follow [4] and many other references will be cited in each section.

3.1 GROSS-TUCKER THEOREM FOR SEPARATED GRAPHS

In this section we will formally introduce separated graphs based on [4] and [5] and the goal is to extend the Gross-Tucker theorem.

Definition 3.1.1. A *separated graph* is a pair (E, C) where E is a graph and $C = \bigcup_{v \in E^0} C_v$ in which C_v is a partition of $s^{-1}(v)$ into pairwise disjoint nonempty subsets for each vertex v. If all the sets in C are finite then we say that (E, C) is a *finitely separated graph*. This is automatically true when E is row-finite.

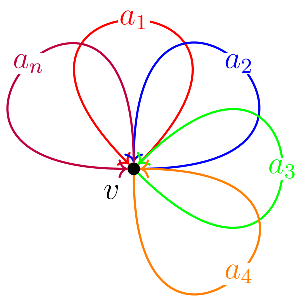
The set C is the *trivial separation* of E if $C_v = \{s^{-1}(v)\}$ for all $v \in E^0$ (in case v is a sink then $s^{-1}(v) = \emptyset$ and therefore we take C_v to be a empty family of subsets). In this case, (E, C) is called a *trivially separated graph* or a *non-separated graph*. Any graph E may be paired with the trivial separation and may thus be viewed as a trivially separated graph.

Definition 3.1.2. Let (E, C) and (F, D) be two separated graphs and $f : E \to F$ a graph morphism. We say that (E, C) is isomorphic to (F, D) if f is a isomorphism and f permutes the separations in the sense that for each $v \in E^0$ and $X \in C_v$ we have $f^1(X) \in D_{f^0(v)}$.

Definition 3.1.3. Let (E, C) be a separated graph and G be a group. An action α of G on (E, C) is an action of G on the graph E which permutes the elements of the separation C meaning that $\alpha_g(X) \in C_{\alpha_g^0(v)}$ whenever $g \in G$ and $X \in C_v$, $v \in E^0$.

Remark 3.1.4. Just as we have seen for graphs in Remark 2.0.8 there is also a natural notion of automorphisms of separated graphs and the collection of all automorphisms of (E, C) forms a group under composition, denoted by Aut(E, C). An action of G on (E, C) is then just a group homomorphism $\alpha : G \to \text{Aut}(E, C)$ and it is said to be free if the underlying action of G on E^0 (and hence also on E^1) is free.

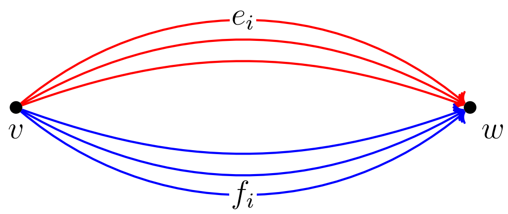
Example 3.1.5. Consider the Cuntz graph A_n seen in Example 2.1.9 and define a separation as $D = D_v := \{X_1, \dots, X_n\}$ where $X_i = \{a_i\}$ are a singleton sets for all $i \in \{1, \dots, n\}$. Thus (A_n, D) is the separated graph, called Cuntz separated graph, that we can see in picture below:



The next example is a classical example of separated graphs.

Example 3.1.6. For all integers $1 \leq m \leq n$, define the separated graph $(E(m,n), C(m,n))$ as follows:

1. $E(m,n)^0 := \{v, w\}$ with $v \neq w$.
2. $E(m,n)^1 := \{e_1, \dots, e_n, f_1, \dots, f_m\}$ ($n+m$ distinct edges).
3. $s(e_i) = s(f_j) = v$ and $r(e_i) = r(f_j) = w$ for all i, j.
4. $C(m,n) = C(m,n)_v := \{X, Y\}$, where $X = \{e_1, \dots, e_n\}$ and $Y = \{f_1, \dots, f_m\}$.



This graph admits no free actions of G unless G is the trivial group. The reason is because if $g \neq 1$ then we must have $\alpha_g(v) = w$. Then we have $s(\alpha_g(e_i)) = v \neq w = \alpha_g(s(e_i))$ regardless how the action acts on edges. We will return to this example later.

Now, preparing for the Gross-Tucker theorem for separated graphs we need to define the quotient and skew product graphs for separated graphs.

An action of G on a separated graph (E, C) yields a quotient separated graph as follows: Keep the usual quotient graph $E/G = (E^0/G, E^1/G, s_G, r_G)$. We are going to define a separation over this graph. For each $X \in C_v$, define $X_G = \{[e] : e \in X\} \subseteq s_G^{-1}([v])$, that is, the equivalence class of edges which belong to the respective set X. The union of X_G is equal to $s_G^{-1}([v])$ since these X form a partition of $s^{-1}(v)$. Note that $X_G = Y_G$ if and only if $[X] = [Y]$. So, these subsets X_G determine a separation

$$C/G := \bigcup_{[v]\in E^0/G} (C/G)_{[v]}$$

for E/G, where $(C/G)_{[v]} := \{X_G : X \in C_v\}$. Then $(E/G, C/G)$ is called *the quotient separated graph*.

Now, let (E, C) to be a separated graph, $c : E^1 \to G$ be a labeling function and keep the usual skew product graph $E \times_c G$. For each $X \in C_v$ and $g \in G$ define $X_g := X \times \{g\} = \{(e, g) \mid e \in X\}$ which is a partition of $s^{-1}(v, g)$. The subsets X_g determine a separation

$$C \times_c G := \bigcup_{(v,g)\in E^0\times G} C \times_c G_{(v,g)}$$

with $C \times_c G_{(v,g)} := \{X_g \mid X \in C_v\}$. Then $(E \times_c G, C \times_c G)$ is called *the skew product separated graph*.

Proposition 3.1.7 (Gross-Tucker theorem for separated graphs)**.** *Suppose a group G acts freely on a separated graph (E, C) by α. Then there is a function $c : E^1/G \to G$ and a G-equivariant isomorphism of separated graphs:*

$$(E, C) \cong (E/G \times_c G, C/G \times_c G).$$

Proof. From the Gross-Tucker Theorem 2.1.17 for non-separated graphs, we already have a labeling function $c : E^1/G \to G$ and a G-equivariant isomorphism $\phi : E/G \times_c G \to E$. We only need to show this isomorphism permutes the separations.

First of all, consider $x \in E^0/G$ and v_x a base vertex of E^0. We just need to prove that for each $(x, g) \in E^0/G \times_c G$ and $Y \in (C/G \times_c G)_{(x,g)}$ we have $\phi(Y) \in C_{\alpha_g(v_x)}$. Now, by definition of the separations we have $(C/G \times_c G)_{(x,g)} = (C/G)_x \times \{g\}$ and $(C/G)_x$ is a set of all $y \in E^1/G$ such that $e_y \in X$ with $X \in C_{v_x}$ (e_y is base edge of y). Therefore, for all $(y, g) \in Y$ we have $\phi(y, g) = \alpha_g(e_y)$. Note that if $e_y \in X$ with $X \in C_{v_x}$ then $\alpha_g(e_y) \in C_{\alpha_g(v_x)}$ as required. So, we have an isomorphism of separated graphs as desired.

$\square$

The above result extends the original Gross-Tucker theorem 2.1.17 for ordinary graphs and, as already mentioned, it is strongly related to a similar result obtained for labeled graphs in [7].

Example 3.1.8. Consider the Cayley graph E_G seen in Example 2.1.8 and define a separation C_G as follows: $C_G = \bigcup_{g\in G}(C_G)_g$ where $(C_G)_g := \{X_1^g, \ldots, X_n^g\}$ in which each $X_i^g := \{g\}\times\{g_i\}$ for every $i \in \{1,\ldots,n\}$. This yields a separated graph (E_G, C_G), called the Cayley separated graph.

As can be seen in Example 2.1.9 we know that E_G carries a free action β and consequently E_G is isomorphic to the skew product graph $A_n \times_c G$ in which $c : A_n^1 \to G$ is defined as $c(a_i) = g_i$. Note that this action in fact permutes the separations since for every $X_i^g \in (C_G)_g$ we have $\beta_h(g, g_i) = (gh, g_i) \in (C_G)_{gh} = (C_G)_{\beta_h(g)}$, that is, $\beta_h(X_i) \in (C_G)_{\beta_h(g)}$ whenever $h \in G$. By Proposition 3.1.7 we have (E_G, C_G) is isomorphic to $(A_n \times_c G, D \times_c G)$ where $D \times_c G$ is the skew separation of D seen in Example 3.1.5.

3.2 C*-ALGEBRAS OF SEPARATED GRAPHS AND DUALITY THEOREMS

In this section, we are going to formally define the separated graph C*-algebras based on [4] and [5], and extend some duality theorems seen in Section 2.7, Chapter 2. As already mentioned, we do not make any assumptions of row-finiteness in our graphs.

Definition 3.2.1. For a separated graph (E, C), the *Leavitt path algebra* of separated graph (E, C) is the complex *-algebra $L(E, C)$ with generators $\{P_v\}_{v\in E^0}$ of mutually orthogonal projections and $\{S_e\}_{e\in E^1}$ of partial isometries subject to following relations:

1. $P_{s(e)}S_e = S_eP_{r(e)} = S_e$ for all $e \in E^1$.
2. $S_e^*S_f = \delta_{e,f}P_{r(e)}$ for all $e, f \in X$, $X \in C$.
3. $P_v = \sum_{e\in X} S_eS_e^*$ for every finite subset $X \in C_v$.

Definition 3.2.2. The graph C*-algebra of a separated graph (E, C) is the universal C*-algebra $C^*(E, C)$ with generators $\{P_v, S_e \mid v \in E^0, e \in E^1\}$ subject to the relations 1-3 of Definition 3.2.1. The collection $\{P_v, S_e \mid v \in E^0, e \in E^1\}$ is called a *Cuntz-Krieger* (E, C)*-family*. In other words, $C^*(E, C)$ is the enveloping C*-algebra of $L(E, C)$.

Remark 3.2.3. The C*-algebra $C^*(E, C)$ exists because the generating set consists of projections and partial isometries.

Definition 3.2.4. For two paths $\mu, \nu \in \text{Path}(E)$ with $s(\mu) = s(\nu) = v$ we say that μ and ν are C*-separated* if the initial edges of μ and ν belong to different sets $X, Y \in C_v$.

Definition 3.2.5. For each finite $X \in C$, we select an edge $e_X \in X$. Let $\mu, \nu \in \text{Path}(E)$ be two paths such that $r(\mu) = r(\nu)$ and let e and f be the terminal edges of μ and ν, respectively. Then the path $\mu\nu^*$ is said to be *reduced* in case we have $(e, f) \neq (e_X, e_X)$ for every finite $X \in C$. In case either μ or ν has length zero then $\mu\nu^*$ is automatically reduced.

Remark 3.2.6. Every time we use the notion of reduced path the choice above is applied.

Lemma 3.2.7. *Let (E, C) be a separated graph and let μ, ν, η and ζ be paths in E with $r(\mu) = r(\nu)$ and $r(\eta) = r(\zeta)$ such that ν and η are C-separated paths and $\mu\nu^*$, $\eta\zeta^*$ are reduced paths. Then we have*

$$(S_\mu S_\nu^*)(S_\eta S_\zeta^*) = \begin{cases} S_\mu S_{\nu'}^* S_{\eta'} S_\zeta^*, & \textit{if } \nu = \tau\nu' \textit{ and } \eta = \tau\eta' \textit{ with } \nu', \eta' \textit{ being } C\textit{-separated} \\ S_\mu S_{\eta'} S_\zeta^*, & \textit{if } \eta = \nu\eta' \textit{ for some path } \eta' \\ S_\mu S_{\nu'}^* S_\zeta^*, & \textit{if } \nu = \eta\nu' \textit{ for some path } \nu' \\ 0, & \textit{otherwise} \end{cases}$$

Proof. First of all, if $\nu = \tau\nu'$ and $\eta = \tau\eta'$ for some $\tau \in \text{Path}(E)$ with ν', η' being C-separated paths then we have

$$\begin{aligned} (S_\mu S_\nu^*)(S_\eta S_\zeta^*) &= S_\mu (S_\tau S_{\nu'})^* S_\tau S_{\eta'} S_\zeta^* \\ &= S_\mu S_{\nu'}^* S_\tau^* S_\tau S_{\eta'} S_\zeta^* \\ &= S_\mu S_{\nu'}^* P_{r(\tau)} S_{\eta'} S_\zeta^* \\ &= S_\mu S_{\nu'}^* S_{\eta'} S_\zeta^* \end{aligned}$$

because $P_{r(\tau)} S_{\eta'} = P_{s(\eta')} S_{\eta'} = S_{\eta'}$. In the same way, if $\eta = \nu\eta'$ for some $\eta' \in \text{Path}(E)$ we have

$$\begin{aligned} (S_\mu S_\nu^*)(S_\eta S_\zeta^*) &= S_\mu S_\nu^* S_\nu S_{\eta'} S_\zeta^* \\ &= S_\mu P_{(r(\nu))} S_{\eta'} S_\zeta^* \\ &= S_\mu S_{\eta'} S_\zeta^* \end{aligned}$$

because $P_{r(\nu)} S_{\eta'} = P_{s(\eta')} S_{\eta'} = S_{\eta'}$. The argument is analogous for the case $\nu = \eta\nu'$.

□

Remark 3.2.8. Note that if $\mu\nu^*$ is not reduced, we can replace it by a linear combination of reduced ones as follows: for each finite $X \in C_v$ choose $e_X \in X$ and define $X' = X \setminus \{e_X\}$. Suppose that $\mu = \mu_1 \dots \mu_n e_X$ and $\nu = \nu_1 \dots \nu_m e_X$ with $\mu_i, \nu_i \in X$. Then we have

$$\begin{aligned} S_\mu S_\nu^* &= S_{\mu_1} \dots S_{\mu_n} S_{e_X} S_{e_X}^* S_{\nu_m}^* \dots S_{\nu_1}^* \\ &= S_{\mu_1} \dots S_{\mu_n} \left(P_{s(e)} - \sum_{e \in X'} S_e S_e^* \right) S_{\nu_m}^* \dots S_{\nu_1}^* \\ &= S_{\mu_1} \dots S_{\mu_n} \left(P_{s(e)} S_{\nu_m}^* \dots S_{\nu_1}^* - \sum_{e \in X'} S_e S_e^* S_{\nu_m}^* \dots S_{\nu_1}^* \right) \\ &= S_{\mu_1} \dots S_{\mu_n} S_{\nu_m}^* \dots S_{\nu_1}^* - \sum_{e \in X'} S_{\mu_1} \dots S_{\mu_n} S_e S_e^* S_{\nu_m}^* \dots S_{\nu_1}^*. \end{aligned}$$

It could happen that $(\mu_n, \nu_m) = (f_Y, f_Y)$ for some $f_Y \in Y$. In this case we just repeat the same argument.

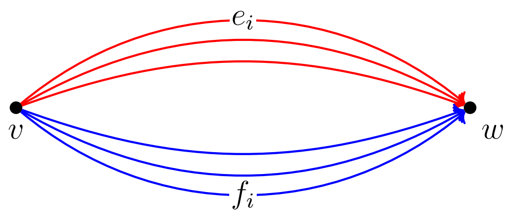
Proposition 3.2.9. *Let (E,C) be a separated graph. Then the set of those elements of the form*

$$S_{\mu_1}S_{\nu_1}^*S_{\mu_2}S_{\nu_2}^*\dots S_{\mu_n}S_{\nu_n}^*, \quad \mu_i,\nu_i \in \mathit{Path}(E)$$

such that ν_i and μ_{i+1} are C-separated paths for all $i \in \{1,\dots,n-1\}$ and $\mu_i\nu_i^*$ is reduced for all $i \in \{1,\dots,n\}$ form a linear basis of $L(E,C)$. We call $\mu_1\nu_1^*\dots\mu_n\nu_n^*$ a C-separated reduced path.

Proof. Follows immediately from Lemma 3.2.7 and Remark 3.2.8. □

Example 3.2.10. Consider the separated graph $(E(m,n),C(m,n))$ seen in Example 3.1.6 as in the picture below:



In this example we can see more clearly that an element of the form $S_{e_i}^*S_{f_j}$ is not zero because e_i and f_j are in different sets for every i,j. In the context of non-separated graphs, all these elements are required to be zero.

Besides that, from [4] we have

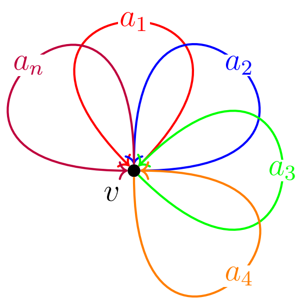
$$C^*(E(m,n),C(m,n)) \cong M_{n+1}(U_{m,n}) \cong M_{m+1}(U_{m,n})$$

where $U_{m,n}$ is the universal C*-algebra generated by the entries of a unitary $m\times n$ matrix, originally studied by Brown in [11] and more generally by McClanahan in [46].

An interesting case is when $m = n = 1$. Then it is straightforward to see that $U_{1,1} \cong C(\mathbb{T})$ and consequently we have $C^*(E(1,1),C(1,1)) \cong M_2(C(\mathbb{T}))$.

Another interesting case is when $m=1$ and $n\geq 1$. Then it is not difficult to see that $U_{1,n} \cong \mathcal{O}_n$ and consequently we have $C^*(E(1,n),C(1,n)) \cong M_2(\mathcal{O}_n)$.

Example 3.2.11. Another important canonical example is when we consider the Cuntz separated graph (A_n,D) seen in Example 3.1.5 draw as picture below:



On can show that $C^*(A_n, D) \cong C^*(\mathbb{F}_n)$ sending $S_{a_i} \mapsto u_{a_i}$, where $\mathbb{F}_n$ is the free group generated by the edges a_i.

Lemma 3.2.12. *If G acts on a separated graph (E, C), then there is an induced action $\alpha : G \to Aut(C^*(E, C))$ such that $\alpha_g(S_e) = S_{g \cdot e}$ e $\alpha_g(P_v) = P_{g \cdot v}$ for all $e \in E^1$ and $v \in E^0$.*

Proof. Fix $g \in G$ and define $P'_v := P_{g \cdot v}$ and $S'_e := S_{g \cdot e}$ for all $v \in E^0$ and $e \in E^1$, where here we use the notation $g \cdot v$ and $g \cdot e$ for the action of G on the separated graph (E, C) to make calculations easier. We claim that $\{P'_v, S'_e\}$ is a Cuntz-Krieger (E, C)-family in $C^*(E, C)$. Note that for all $v, w \in E^0$ we have $g \cdot v = g \cdot w$ if and only if $v = w$. Similarly for edges. It is clear that $\{P'_v\}$ is a family of mutually orthogonal projections satisfying 1 in Definition 3.2.1. To see the condition 2, note that for $e, f \in X$, $X \in C_v$ we have

$$\begin{aligned} S'^*_e S'_f &= S^*_{g \cdot e} S_{g \cdot f} \\ &= \delta_{g \cdot e, g \cdot f} P_{r(g \cdot e)} \\ &= \delta_{e,f} P_{gr(e)} \\ &= \delta_{e,f} P'_{r(e)}. \end{aligned}$$

To see condition 3 note that for all finite $X \in C_v$ we have $g \cdot e \in C_{g \cdot v}$ for all $e \in X$. Then we have

$$\sum_{e \in X} S'_e S'^*_e = \sum_{e \in X} S_{g.e} S^*_{g.e} = P_{g.v} = P'_v.$$

The universal property of $C^*(E, C)$ yields a *-homomorphism $\alpha_g : C^*(E, C) \to C^*(E, C)$ such that $\alpha_g(S_e) = S_{g \cdot e}$ e $\alpha_g(P_v) = P_{g \cdot v}$ for all $e \in E^1$ and $v \in E^0$. It is straightforward to check that $\alpha_g^{-1} = \alpha_{g^{-1}}$. Checking on generators it also follows that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for every $g, h \in G$. Since $\alpha_1 = id$ it follows that α is an action, as desired.

□

From now on, we are ready to extend the results seen in Section 2.7, Chapter 2 for the class of separated graph C*-algebras.

Theorem 3.2.13. *Given a labeling $c\colon E^1 \to G$ on a separated graph (E, C), there is a coaction $\delta_c\colon C^*(E, C) \to C^*(E, C) \otimes C^*(G)$ satisfying*

$$\delta_c(P_v) = P_v \otimes 1 \quad \textit{and} \quad \delta_c(S_e) = S_e \otimes c(e)$$

for all $v \in E^0$ and $e \in E^1$ where $\{S_e, P_v\}$ is the universal Cuntz-Krieger (E, C)-family in $C^*(E, C)$.

Proof. To begin with we prove that the family $\{P_v \otimes 1, S_e \otimes c(e)\}$ satisfies the conditions 1-3 of Definition 3.2.1. It is straightforward to check that $\{P_v \otimes 1\}$ are mutually orthogonal projections satisfying condition 1. To see condition 2, note that for $e, f \in X$, $X \in C_v$, $v \in E^0$ we have

$$\begin{aligned}(S_e \otimes c(e))^*(S_f \otimes c(f)) &= (S_e^* \otimes c(e)^{-1})(S_f \otimes c(f))\\ &= S_e^* S_f \otimes c(e)^{-1}c(f)\\ &= \delta_{e,f} P_{r(e)} \otimes 1\end{aligned}$$

For the last condition, note that for all finite $X \in C_v$ we have

$$\begin{aligned}\sum_{e\in X}(S_e \otimes c(e))(S_e \otimes c(e))^* &= \sum_{e\in X}(S_e \otimes c(e))(S_e^* \otimes c(e)^{-1})\\ &= \sum_{e\in X} S_e S_e^* \otimes c(e)c(e)^{-1}\\ &= (\sum_{e\in X} S_e S_e^*) \otimes 1\\ &= P_v \otimes 1\end{aligned}$$

So, the universal property yields a *-homomorphism $\delta_c : C^*(E, C) \to C^*(E, C) \otimes C^*(G)$ satisfying the properties of the statement. To check the coaction identity just note that on the generators P_v, S_e we have

$$\begin{aligned}(\delta_c \otimes id_G) \circ \delta_c(P_v) &= \delta_c \otimes id_G(P_v \otimes 1)\\ &= \delta_c(P_v) \otimes 1\\ &= (P_v \otimes 1) \otimes 1\\ &= P_v \otimes (1 \otimes 1)\\ &= P_v \otimes \delta_G(1)\\ &= id \otimes \delta_G(P_v \otimes 1)\\ &= (id \otimes \delta_G) \circ \delta_c(P_v)\end{aligned}$$

and

$$\begin{aligned}(\delta_c \otimes id_G) \circ \delta_c(S_e) &= \delta_c \otimes id_G(S_e \otimes c(e)) \\ &= \delta_c(S_e) \otimes c(e) \\ &= (S_e \otimes c(e)) \otimes c(e) \\ &= S_e \otimes (c(e) \otimes c(e)) \\ &= S_e \otimes \delta_G(c(e)) \\ &= id \otimes \delta_G(S_e \otimes c(e)) \\ &= (id \otimes \delta_G) \circ \delta_c(S_e)\end{aligned}$$

So, the coaction identity holds on generators and extends by linearity and continuity to all of $C^*(E,C)$.

Finally, for nondegeneracy, we have to see that $\delta_c(C^*(E,C))(1 \otimes C^*(G))$ is dense in $C^*(E,C) \otimes C^*(G)$. But note that the elements of the form $S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes g$ generate the C*-algebra $C^*(E,C) \otimes C^*(G)$ where $\nu_i, \mu_{i+1} \in$ Path(E) are C-separated paths and $\mu_i\nu_i^*$ are reduced paths like in Proposition 3.2.9. Let us denote by $h := c(\mu_1)c(\nu_1)^{-1} \dots c(\mu_n)c(\nu_n)^{-1}$ and observe that

$$\begin{aligned}S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes g &= (S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes h)(1 \otimes h^{-1}g) \\ &= \delta_c(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*)(1 \otimes h^{-1}g)\end{aligned}$$

That is, $\delta_c(C^*(E,C))(1 \otimes C^*(G))$ contains all elements of these form $S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes g$ and since these elements generate the C*-algebra $C^*(E,C) \otimes C^*(G)$ we are done.

□

Let (E,C) be a separated graph and for each labeling function $c : E^1 \to G$ define

$$L(E,C)_g := \text{span}\{S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \mid \varsigma = \mu_1\nu_1^* \dots \mu_n\nu_n^* \text{ is a } C\text{-separated reduced path with } c(\varsigma) = g\}$$

It is straightforward to observe that this gives $L(E,C)$ a natural algebraic G-grading structure meaning that we have a direct sum decomposition $L(E,C) = \oplus_{g\in G}^{\text{alg}} L(E,C)_g$ with grading property: $L(E,C)_g \cdot L(E,C)_h \subseteq L(E,C)_{gh}$ and $L(E,C)_g^* = L(E,C)_{g^{-1}}$ for all $g,h \in G$. Essentially this follows from Proposition 3.2.9. With this, we get a description of the spectral subspaces of $C^*(E,C)$:

Proposition 3.2.14. *Given a labeling function $c : E^1 \to G$ on a separated graph (E,C) then*

$$C^*(E,C)_g = \overline{L(E,C)_g}$$

where $C^*(E,C)_g$ is the spectral subspace associated with the coaction δ_c.

Proof. It is immediate that $L(E,C)_g$ is contained in $C^*(E,C)_g$ since

$$\delta_c(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*) = S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes c(\varsigma) = S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes g$$

where $\varsigma = \mu_1\nu_1^* \dots \mu_n\nu_n^*$ is a C-separated reduced path with $c(\varsigma) = g$. By continuity we have $\overline{L(E,C)_g} \subseteq C^*(E,C)_g$. Conversely, suppose that $x \in C^*(E,C)$ such that $\delta_c(x) = x \otimes g$. So, x can be approximated by an element $\tilde{x} \in L(E,C)$ since $L(E,C)$ is dense in $C^*(E,C)$. Consider the projection of norm one $E_g : C^*(E,C) \to C^*(E,C)_g$ seen in Proposition 2.5.6, that is, $E_g = (id \otimes \chi_g) \circ \delta_c$. Since $E_g(x) = x$ we have

$$\|x - E_g(\tilde{x})\| = \|E_g(x) - E_g(\tilde{x})\| \leq \|x - \tilde{x}\|.$$

By Proposition 3.2.9 we can consider $\tilde{x}$ as a sum of basis elements of the form $S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*$ with $\varsigma = \mu_1\nu_1^* \dots \mu_n\nu_n^*$ is a C-separated reduced path. For each sum factor observe that

$$\begin{aligned} E_g(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*) &= (id \otimes \chi_g) \circ \delta_c(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*) \\ &= (id \otimes \chi_g)(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes c(\varsigma)) \\ &= S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes g \end{aligned}$$

By linearity we have $E_g(\tilde{x}) \in L(E,C)_g$. Therefore, every element x of $C^*(E,C)_g$ can be approximated by $E_g(\tilde{x}) \in L(E,C)_g$ as desired. This completes the proof.

□

Theorem 3.2.15. *Given a labeling function $c \colon E^1 \to G$ on a separated graph (E,C), the coaction δ_c of G on $C^*(E,C)$ is maximal, that is, there is a canonical isomorphism*

$$C^*(E,C) \rtimes_{\delta_c} G \rtimes_{\widehat{\delta_c}} G \cong C^*(E,C) \otimes \mathcal{K}(l^2(G)).$$

Proof. Let $A := C^*(E,C)$ and $\mathcal{A}_g := C^*(E,C)_g$ be the spectral subspaces. As observed in Proposition 2.6.7 it is enough to show that the canonical surjective map $\overline{\sigma} : C^*(\mathcal{A}) \twoheadrightarrow A$ is an isomorphism. To prove this, we are going to construct the inverse of $\overline{\sigma}$ by using the universal property of $C^*(E,C)$.

In fact, consider the inclusion maps $\kappa_g : L(E,C)_g \hookrightarrow A_g$ arising from the completion process as seen in Proposition 3.2.14. Since $L(E,C)_g$ and $\mathcal{A}_g$ are G-grading subspaces for $L(E,C)$ and A, respectively (for $L(E,C)$ is a algebraic G-grading structure) it is direct to observe that $\kappa := \{\kappa_g\}_{g\in G}$ is a *-morphism of G-graded *-algebras in the sense that $\kappa_g(x)\kappa_h(y) = \kappa_{gh}(xy)$ and $\kappa_g(x^*) = \kappa_{g^{-1}}(x)^*$ for all $x \in L(E,C)_g$, $y \in L(E,C)_h$ and $g, h \in G$. Therefore we can extend it to a *-homomorphism of *-algebras

$$\kappa : L(E,C) = \oplus_{g\in G} L(E,C)_g \to \oplus_{g\in G}\mathcal{A}_g = C_c(\mathcal{A}).$$

Since A and $C^*(\mathcal{A})$ are the enveloping C*-algebras of $L(E,C)$ and $C_c(\mathcal{A})$, this *-homomorphism further extends to $K : A \to C^*(\mathcal{A})$, which is the identity on the fibers $\mathcal{A}_g$.

Finally to see that this map is the inverse of $\overline{\sigma}$ it is enough to compute the composition $K \circ \overline{\sigma}$ on $j_g(a)$, $a \in \mathcal{A}_g$ since the closed linear span of these elements is dense in $C^*(\mathcal{A})$. But for each $a \in \mathcal{A}_g$, $K \circ \overline{\sigma}(j_g(a)) = K(\sigma_g(a)) = a$ by construction, as desired.

□

Before we go to the next result, we need the following lemma:

Lemma 3.2.16. *Let (E, C) be a separated graph and let B be a C*-algebra generated by a Cuntz-Krieger (E, C)-family $\{Q_v, T_e \mid v \in E^0, e \in E^1\}$, and let $\{x_j\}_j$ be a bounded net in B. If $x_j T_{\mu_1} T^*_{\nu_1} \ldots T_{\mu_n} T^*_{\nu_n}$ converges for every $\mu_i, \nu_i \in$ Path(E) such that ν_i and μ_{i+1} are C-separated paths for all $i \in \{1, \ldots, n-1\}$ and $\mu_i \nu_i^*$ is reduced for all $i \in \{1, \ldots, n\}$, then $\{x_j\}_j$ converges strictly to an element $x \in \mathcal{M}(B)$.*

Proof. By the universal property of $C^*(E, C)$ there is a unique surjective *-homomorphism $\Phi : C^*(E, C) \to B$, and hence, we can approximate any $b \in B$ by a linear combination of $T_{\mu_1} T^*_{\nu_1} \ldots T_{\mu_n} T^*_{\nu_n}$, and by $\epsilon/3$-argument shows that $\{x_j b\}_j$ is a Cauchy for every $b \in B$. We define $x : B \to B$ by $x(b) = \lim_{j \to \infty} x_j b$. It is straightforward to check that x defines (by left multiplication) a multiplier x of B. Taking ajdoints shows that $\{b x_j\}_j$ converge to bx for every $b \in B$, so $x_j \to x$ strictly as we desired.

□

Theorem 3.2.17. *With notations as above, there is a canonical isomorphism*

$$C^*(E \times_c G, C \times_c G) \cong C^*(E, C) \rtimes_{\delta_c} G.$$

Under this isomorphism, the action γ on $C^*(E \times_c G, C \times_c G)$ induced by the translation action on $(E \times_c G, C \times_c G)$ corresponds to the dual action $\widehat{\delta_c}$ on $C^*(E, C) \rtimes_{\delta_c} G$.

Proof. Let $B := C^*(E, C)$ and $D := C^*(E \times_c G, C \times_c G)$, (j_B, j_G^B) and (j_D, j_G^D) be the canonical covariant representations of $(B, C_0(G))$ in $\mathcal{M}(B \rtimes_{\delta_c} G)$ and $(D, C_0(G))$ in $\mathcal{M}(D \rtimes_{\delta_c} G)$, respectively and also $\{P_v, S_e\}$ and $\{P_{(v,g)}, S_{(e,g)}\}$ be the Cuntz-Krieger E-families for B and D, respectively. To define a *-homomorphism from the C*-algebra of the separated skew product graph to the crossed product as above let us define

$$p_{(v,g)} := j_B(P_v) j_G(\chi_{g^{-1}}) \text{ and } s_{(e,g)} := j_B(S_e) j_G(\chi_{(gc(e))^{-1}})$$

for all $v \in E^0$, $e \in E^1$ and $g \in G$. We claim that $\{p_{(v,g)}, s_{(e,g)}\}$ is a Cuntz-Krieger family for $C^*(E \times_c G, C \times_c G)$.

Before we continue, note that by the covariance condition in Definition 2.5.12 and the fact that each $P_v \in B_1$ and $S_e \in B_{c(e)}$ we have

$$j_B(P_v) j_G(\chi_{g^{-1}}) = j_G(\chi_{g^{-1}}) j_B(P_v) \text{ and } j_B(S_e) j_G(\chi_{(gc(e))^{-1}}) = j_G(\chi_{g^{-1}}) j_B(S_e) \quad (3.2.18)$$

Now, to check the conditions in Definition 3.2.1 we use the covariance condition and the formula 3.2.18. Note that

$$\begin{aligned}
p_{(v,g)}p_{(w,h)} &= j_B(P_v)j_G^B(\chi_{g^{-1}})j_B(P_w)j_G^B(\chi_{h^{-1}}) \\
&= j_B(P_v)j_B(P_w)j_G^B(\chi_{g^{-1}})j_G^B(\chi_{h^{-1}}) \text{ by 3.2.18} \\
&= j_B(P_vP_w)j_G^B(\chi_{g^{-1}}\chi_{h^{-1}}) \\
&= j_B(P_v)j_G^B(\chi_{g^{-1}}) \quad \text{ if } v = w \text{ and } g = h \\
&= p_{(v,g)}.
\end{aligned}$$

Then $p_{(v,g)}$ is a family of mutually orthogonal projections. Now, for the first condition we have

$$\begin{aligned}
p_{s(e,g)}s_{(e,g)} &= j_B(P_{s(e)})j_G^B(\chi_{g^{-1}})j_B(S_e)j_G^B(\chi_{(gc(e))^{-1}}) \\
&= j_B(P_{s(e)})j_B(S_e)j_G^B(\chi_{(gc(e))^{-1}})j_G^B(\chi_{(gc(e))^{-1}}) \text{ by 3.2.18} \\
&= j_B(P_{s(e)}S_e)j_G^B(\chi_{(gc(e))^{-1}}) \\
&= j_B(S_e)j_G^B(\chi_{(gc(e))^{-1}}) \\
&= s_{(e,g)}.
\end{aligned}$$

For the second condition for each $(e,g),(f,h) \in Y$, $Y \in C \times_c G$ we have

$$\begin{aligned}
s^*_{(f,h)}s_{(e,g)} &= (j_B(S_f)j_G^B(\chi_{(hc(f))^{-1}}))^*j_B(S_e)j_G^B(\chi_{(gc(e))^{-1}}) \\
&= j_G^B(\chi_{(hc(f))^{-1}})j_B(S_f^*)j_B(S_e)j_G^B(\chi_{(gc(e))^{-1}}) \\
&= j_B(S_f^*)j_G^B(\chi_{h^{-1}})j_B(S_e)j_G^B(\chi_{(gc(e))^{-1}}) \text{ by 3.2.18} \\
&= j_B(S_f^*)j_B(S_e)j_G^B(\chi_{(hc(e))^{-1}})j_G^B(\chi_{(gc(e))^{-1}}) \text{ by 3.2.18} \\
&= j_B(S_f^*S_e)j_G^B(\chi_{(hc(e))^{-1}}\chi_{(gc(e))^{-1}}) \text{ if } f = e \text{ and } g = h \\
&= j_B(P_{r(e)})j_G^B(\chi_{(gc(e))^{-1}}) \\
&= p_{(r(e),gc(e))} \\
&= p_{r(e,g)}
\end{aligned}$$

For the third and last condition for each finite $X \in C_v$, $v \in E^0$ the subset $X_g \in C \times_c G_{(v,g)}$

is finite too and we have

$$\begin{aligned}
\sum_{(e,g)\in X_g} s_{(e,g)} s^*_{(e,g)} &= \sum_{(e,g)\in X_g} j_B(S_e) j_G^B(\chi_{(gc(e))^{-1}}) (j_B(S_e) j_G^B(\chi_{(gc(e))^{-1}}))^* \\
&= \sum_{(e,g)\in X_g} j_B(S_e) j_G^B(\chi_{(gc(e))^{-1}}) j_G^B(\chi_{(gc(e))^{-1}}) j_B(S_e^*) \\
&= \sum_{(e,g)\in X_g} j_B(S_e) j_G^B(\chi_{(gc(e))^{-1}}) j_B(S_e^*) \text{ by 3.2.18} \\
&= \sum_{(e,g)\in X_g} j_B(S_e) j_B(S_e^*) j_G^B(\chi_{g^{-1}}) \\
&= j_B\left(\sum_{e\in X} S_e S_e^*\right) j_G^B(\chi_{g^{-1}}) \\
&= j_B(P_v) j_G^B(\chi_{g^{-1}}) \\
&= p_{(v,g)}
\end{aligned}$$

By the universal property there is a *-homomorphism

$$\phi : C^*(E \times_c G, C \times_c G) \to C^*(E,C) \rtimes_{\delta_c} G$$

satisfying $\phi(P_{(v,g)}) = p_{(v,g)}$ and $\phi(S_{(e,g)}) = s_{(e,g)}$.

To see the surjectivity, first of all note that for every path $\mu \in \mathrm{Path}(E)$ and $g \in G$ we have a unique path (μ, g) in $E \times_c G$ such that $\phi(S_{(\mu,g)}) = s_{(\mu,g)} = j_G^B(\chi_{g^{-1}}) j_B(S_\mu)$. By Proposition 3.2.9 we have basis elements in $L(E \times_c G, C \times_c G)$ of the form:

$$S_{(\mu_1,g)} S^*_{(\nu_1,z_1)} S_{(\mu_2,z_1)} S^*_{(\nu_1,z_2)} \dots S_{(\mu_n,g_n)} S^*_{(\nu_n,z_n)}$$

where ν_i and μ_{i+1} are C-separated for every $i \in \{1,\dots,n-1\}$ and $\mu_i \nu_i^*$ is reduced for every $i \in \{1,\dots,n\}$. Note that $z_1 = gc(\mu_1)c(\nu_1)^{-1}$ and $z_i = z_{i-1}c(\mu_i)c(\nu_i)^{-1}$ for all $i \in \{2,\dots,n\}$ because we must have $r(\mu_i, g) = r(\nu_i, z_i)$.

Now, we are going to compute ϕ on the basis elements of the form $S_{(\mu,g)} S^*_{(\nu,z)} S_{(\mu',z)} S^*_{(\nu',z')}$ with $z = gc(\mu)c(\nu)^{-1}$ and $z' = gc(\mu)c(\nu)^{-1}c(\mu')c(\nu')^{-1}$ to see what we get. Basically using the formula 3.2.18 we have:

$$\begin{aligned}
&\phi(S_{(\mu,g)} S^*_{(\nu,z)} S_{(\mu',z)} S^*_{(\nu,z')}) = s_{(\mu,g)} s^*_{(\nu,z)} s_{(\mu',z)} s^*_{(\nu',z')} \\
&= j_B(S_\mu) j_G^D(\chi_{(gc(\mu))^{-1}}) (j_B(S_\nu) j_G^D(\chi_{(zc(\nu))^{-1}}))^* j_B(S_{\mu'}) j_G^D(\chi_{(zc(\mu'))^{-1}}) (j_B(S_{\nu'}) j_G^D(\chi_{(z'c(\nu'))^{-1}}))^* \\
&= j_B(S_\mu) j_G^D(\chi_{(gc(\mu))^{-1}}) j_G^D(\chi_{(zc(\nu))^{-1}}) j_B(S_\nu^*) j_B(S_{\mu'}) j_G^D(\chi_{(zc(\mu'))^{-1}}) j_G^D(\chi_{(z'c(\nu'))^{-1}}) j_B(S^*_{\nu'}) \\
&= j_B(S_\mu) j_G^D(\chi_{(gc(\mu))^{-1}}) j_B(S_\nu^*) j_G^D(\chi_{z^{-1}}) j_B(S_{\mu'}) j_G^D(\chi_{(zc(\mu'))^{-1}}) j_B(S^*_{\nu'}) j_G^D(\chi_{(z')^{-1}}) \\
&= j_B(S_\mu) j_B(S_\nu^*) j_G^D(\chi_{c(\nu)(gc(\mu))^{-1}}) j_G^D(\chi_{z^{-1}}) j_B(S_{\mu'}) j_B(S^*_{\nu'}) j_G^D(\chi_{c(\nu')(zc(\mu'))^{-1}}) j_G^D(\chi_{(z')^{-1}}) \\
&= j_B(S_\mu S_\nu^*) j_G^D(\chi_{z^{-1}}) j_B(S_{\mu'} S^*_{\nu'}) j_G^D(\chi_{(z')^{-1}}) \\
&= j_B(S_\mu S_\nu^*) j_B(S_{\mu'} S^*_{\nu'}) j_G^D(\chi_{c(\nu')c(\mu')^{-1}z^{-1}}) j_G^D(\chi_{(z')^{-1}}) \\
&= j_B(S_\mu S_\nu^* S_{\mu'} S^*_{\nu'}) j_G^D(\chi_{(z')^{-1}}) \\
&= j_G^D(\chi_{g^{-1}}) j_B(S_\mu S_\nu^* S_{\mu'} S^*_{\nu'})
\end{aligned}$$

If we extend linearly to $L(E\times_c G, C\times_c G)$ it follows from the calculation above that the image of ϕ contains all elements of the form $j_G^B(\chi)j_B(b)$ with $\chi\in C_0(G)$ and $b\in B$. Since the span of these elements is a dense subspace of the crossed product $B\rtimes_{\delta_c} G$ this shows that ϕ is surjective.

To prove that ϕ is injective and hence an isomorphism we cannot follow the same idea that we used for non-separated graphs because here we do not have an injectivity theorem as we used for graphs. So the way to show this is to construct the inverse.

For this, we are going to define a covariant representation (π,σ) of $(C^*(E,C),G,\delta_c)$ into $\mathcal{M}(C^*(E\times_c G, C\times_c G))$ which will be given by:

$$\pi(P_v)=\sum_{g\in G}P_{(v,g)},\quad \pi(S_e)=\sum_{g\in G}S_{(e,g)}\quad \text{and}\quad \sigma(\chi_g)=\sum_{v\in E^0}P_{(v,g^{-1})} \tag{3.2.19}$$

for all $v\in E^0$, $e\in E^1$ and $g\in G$.

First of all, we need to make sure that all *-homomorphisms defined above in fact exist. Fixed $v\in E^0$, we claim that the sum $\sum_{g\in G}P_{(v,g)}$ belongs to $\mathcal{M}(C^*(E\times_c G, C\times_c G))$. Since the net of all finite sums of projections have norm uniformly bounded by 1, then by Lemma 3.2.16 it is enough to check that $(\sum_g P_{(v,g)})S_{(\mu,h)}S^*_{(\nu,hc(\mu))}$ converges for each pair of paths $(\mu,h),(\nu,hc(\mu))$ in $E^1\times G$ with $\mu,\nu\in \text{Path}(E)$. But, the sums reduce to a single term, that is,

$$\left(\sum_{g\in G}P_{(v,g)}\right)S_{(\mu,h)}S^*_{(\nu,hc(\mu))}=\begin{cases}S_{(\mu,g)}S^*_{(\nu,gc(\mu))}, & \text{if } g=h \text{ and } v=s(\mu)\\ 0, & \text{otherwise}\end{cases}.$$

So, we have a well-defined element $\pi(P_v)\in\mathcal{M}(C^*(E\times_c G, C\times_c G))$ which is the limit of the net considered above. A similar argument shows that, for each $e\in E^1$, $\pi(S_e)$ belongs to $\mathcal{M}(C^*(E\times_c G, C\times_c G))$. It is straightforward to check that $\{\pi(P_v),\pi(S_e)\}$ is a Cuntz-Krieger family for $(E\times_c G, C\times_c G)$. By the universal property there exists a π with the desired properties 3.2.19.

Moreover, the formula for σ seen in 3.2.19 really determines a *-homomorphism $\sigma: C_0(G)\to\mathcal{M}(C^*(E\times_c G, C\times_c G))$ since for each $g\in G$ the sum $\sum_{v\in E^0}P_{(v,g)}$ belongs to $\mathcal{M}(C^*(E\times_c G, C\times_c G))$ using the same argument seen before. In particular, $\sum_{v\in E^0}P_{(v,g^{-1})}$ is a well-defined element of $\mathcal{M}(C^*(E\times_c G, C\times_c G))$ and it is important to define it in this way to make later calculations work. Alternatively we can see that σ determines a *-homomorphism as follows: we have the canonical *-homomorphism $C_0(E^0\times G)\to\mathcal{M}(C^*(E\times_c G, C\times_c G))$. Extending to the multiplier algebra yields a *-homomorphism $\mathcal{M}(C_0(E^0\times G))\to\mathcal{M}(C^*(E\times_c G, C\times_c G))$. Composed with the canonical *-homomorphism $C_0(G)\to\mathcal{M}(C_0(E^0)\otimes C_0(G))\cong\mathcal{M}(C_0(E^0\times G))$, this gives a *-homomorphism $C_0(G)\to\mathcal{M}(C^*(E\times_c G, C\times_c G))$ which sends χ_g to $\sum_{v\in E^0}P_{(v,g)}$ instead of $\sum_{v\in E^0}P_{(v,g^{-1})}$. But it is enough to composed with the map $C_0(G)\to C_0(G)$ which send χ_g to $\chi_{g^{-1}}$ to make it equal to σ.

Now, we are going to check that (π, σ) is a covariant representation, that is, we prove the relation in Definition 2.5.12. To see this, fix $\mu \in \mathrm{Path}(E)$ and $g \in G$. Since $S_\mu \in B_{c(\mu)}$, on the one hand we have

$$\begin{aligned}
\pi(S_\mu)\sigma(\chi_g) &= \left(\sum_{h\in G} S_{(\mu,h)}\right)\left(\sum_{v\in E^0} P_{(v,g^{-1})}\right) \\
&= \sum_{h\in G, v\in E^0} S_{(\mu,h)}P_{(v,g^{-1})} \\
&= S_{(\mu,(c(\mu)g)^{-1})}
\end{aligned}$$

where in the last step we have used that the summand $S_{(\mu,h)}P_{(v,g^{-1})}$ is non-zero if and only if $(v, g^{-1}) = r(\mu, h) = (r(\mu), hc(\mu))$, that is, $v = r(\mu)$ and $h = (c(\mu)g)^{-1}$. On the other hand,

$$\begin{aligned}
\sigma(\chi_{c(\mu)g})\pi(S_\mu) &= \left(\sum_{v\in E^0} P_{(v,(c(\mu)g)^{-1})}\right)\left(\sum_{h\in G} S_{(\mu,h)}\right) \\
&= \sum_{h\in G, v\in E^0} P_{(v,(c(\mu)g)^{-1})}S_{(\mu,h)} \\
&= S_{(\mu,(c(\mu)g)^{-1})}
\end{aligned}$$

where again we used that the only non-zero summand is for $(v, g^{-1}) = s(\mu, h) = (s(\mu), h)$, that is, $v = s(\mu)$ and $h = (c(\mu)g)^{-1}$. So, this verifies the covariant relation in Definition 2.5.12 for special elements in $B_{c(\mu)}$. Analogous computations also show for elements in $B_{c(\mu)^{-1}}$ of the form S_μ^* and more generally elements in B_g of the form $S_\mu S_\nu^*$ which are products of generators S_μ and S_ν^* with $c(\mu)c(\nu)^{-1} = g$. By linearity this proves the covariance relation for the pair (π, σ) and therefore by the universal property we get a nondegenerate *-homomorphism

$$\psi := \pi \times \sigma : C^*(E, C) \rtimes_{\delta_c} G \to \mathcal{M}(C^*(E \times_c G, C \times_c G))$$

such that $\psi \circ j_B = \pi$ and $\psi \circ j_G^B = \sigma$. Now, we compute:

$$\begin{aligned}
\psi \circ \phi(P_{(w,g)}) &= \psi(p_{(w,g)}) \\
&= \psi(j_B(P_w)j_G^B(\chi_{g^{-1}})) \\
&= \pi(P_w)\sigma(\chi_{g^{-1}}) \\
&= \left(\sum_{h\in G} P_{(w,h)}\right)\left(\sum_{v\in E^0} P_{(v,g)}\right) \\
&= \sum_{h\in G, v\in E^0} P_{(w,h)}P_{(v,g)} \\
&= P_{(w,g)}
\end{aligned}$$

and

$$\begin{aligned}
\psi \circ \phi(S_{(\mu,g)}) &= \psi(s_{(\mu,g)}) \\
&= \psi(j_B(S_\mu) j_G^B(\chi_{(gc(\mu))^{-1}})) \\
&= \pi(S_\mu)\sigma(\chi_{(gc(\mu))^{-1}}) \\
&= \left(\sum_{h\in G} S_{(\mu,h)}\right)\left(\sum_{v\in E^0} P_{(v,gc(\mu))}\right) \\
&= \sum_{h\in G, v\in E^0} S_{(\mu,h)} P_{(v,gc(\mu))} \\
&= S_{(\mu,g)}
\end{aligned}$$

Since the elements $P_{(v,g)}$ and $S_{(\mu,g)}$ generate the C*-algebra $C^*(E\times_c G, C\times_c G)$, it follows that $\psi\circ\phi = id$ and hence ϕ is injective, therefore an isomorphism. In particular, the image of ψ is inside of $C^*(E\times_c G, C\times_c G)$. Finally, we are going to check the G-equivariance of the actions. Note that

$$\begin{aligned}
\phi(\gamma_z(P_{(v,g)})) &= \phi(P_{v,zg}) \\
&= p_{(v,zg)} \\
&= j_B(P_v) j_G^B(\chi_{zg}) \\
&= (\widehat{\delta_c})_z(j_B(P_v) j_G^B(\chi_g)) \\
&= (\widehat{\delta_c})_z(\phi(P_{(v,g)}))
\end{aligned}$$

and

$$\begin{aligned}
\phi(\alpha_z(S_{(\mu,g)})) &= \phi(S_{(\mu,zg)}) \\
&= s_{(\mu,zg)} \\
&= j_B(S_\mu) j_G^B(\chi_{zg}) \\
&= (\widehat{\delta_c})_z(j_B(S_\mu) j_G^B(\chi_g)) \\
&= (\widehat{\delta_c})_z(\phi(S_{(\mu,g)}))
\end{aligned}$$

□

Corollary 3.2.20. *For separated graph (E, C) and labeling $c : E \to G$, there is a canonical isomorphism*

$$C^*(E\times_c G, C\times_c G) \rtimes_\gamma G \cong C^*(E, C) \otimes \mathcal{K}(l^2(G))$$

where γ is the action of G on $C^*(E\times_c G, C\times_c G)$ induced by the translation action on $(E\times_c G, C\times_c G)$.

Proof. Follows from the Theorems 3.2.17 and 3.2.15.

□

Corollary 3.2.21. *For a free action θ of a group on a separated graph (E, C), there is a canonical isomorphism*

$$C^*(E, C) \rtimes_\theta G \cong C^*(E/G, C/G) \otimes \mathcal{K}(l^2(G)).$$

Proof. Follows from Corollary 3.2.20 and Gross-Tucker Theorem for separated graphs 3.1.7. □

Remark 3.2.22. Compiling all results that we have seen so far we obtain the following commutative diagram of isomorphisms:

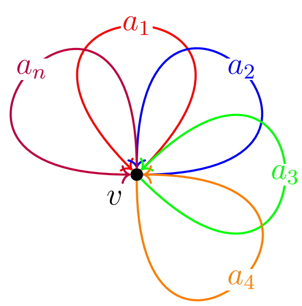
$$\begin{tikzcd} C^*(E \times_c G, C \times_c G) \rtimes_\gamma G \arrow[rr, "3.2.17"] \arrow[dr, "3.2.20"'] & & C^*(E, C) \rtimes_{\delta_c} G \rtimes_{\hat{\delta}_c} G \arrow[dl, "3.2.15"] \\ & C^*(E, C) \otimes \mathcal{K}(l^2(G)) & \end{tikzcd}$$

Example 3.2.23. If we consider the Cayley separated graph (E_G, C_G) seen in Example 3.1.8 we know that (E_G, C_G) carries a free action β of G and hence by Corollary 3.2.21 we can see that

$$C^*(E_G, C_G) \times_\beta G \cong C^*(A_n, D) \otimes \mathcal{K}(l^2(G)) \cong C^*(\mathbb{F}_n) \otimes \mathcal{K}(l^2(G))$$

recalling that $\mathbb{F}_n$ is the free group generated by the n edges.

Remark 3.2.24. The coaction δ_c is not always a normal coaction on $C^*(E, C)$. Indeed, for example, consider the Example 3.2.11 draw as



We have $C^*(A_n, D) \cong C^*(\mathbb{F}_n)$ and δ_c coincides with the canonical coaction $\delta_{\mathbb{F}_n}$. We know that $\delta_{\mathbb{F}_n}$ is always maximal but it is not a normal coaction since $\mathbb{F}_n$ is not amenable.

Remark 3.2.25. Also the normalization of $\delta_{\mathbb{F}_n}$ is the canonical coaction on $C_r^*(\mathbb{F}_n)$ which is isomorphic to $C_r^*(A_n, D)$, the reduced C*-algebra of separated graph to be defined in the next chapter. This suggests the following question: Is the normalization of δ_c always a coaction on $C^*_r(E, C)$? Unfortunately not always: consider (E, C) a separated graph, G a discrete group and the labeling function being $c(e) = 1$ for all $e \in E^1$. Then the spectral subspaces related to δ_c are given by

$$\mathcal{A}_g = \begin{cases} C^*(E, C) & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$$

Then $C^*(\mathcal{A}) \cong C^*(E, C) \cong C^*_r(\mathcal{A})$. So, the coaction δ_c is maximal and normal, indeed, δ_c is the trivial coaction in this case.

This issue will be more clear in the next chapter.

4 REDUCED SEPARATED GRAPH C*-ALGEBRA CASE

In this chapter, our proposal is to get the same results as seen in the previous chapter for the reduced C*-algebra of separated graphs. For this, we need to restrict our attention to finitely separated graphs. The original definition of reduced C*-algebras uses reduced amalgamated free products as can be seen in [58] and [4]. Our strategy here is to give an alternative definition for reduced C*-algebras of separated graphs using the canonical conditional expectation $C^*_r(E,C) \to C_0(E^0)$ that we will see later. For this, it is important to provide a characterization of $C^*(E,C)$ as an amalgamated free coproduct. This will then give us a conditional expectation and enable us to define the reduced C*-algebra $C^*_r(E,C)$.

4.1 REDUCED AMALGAMATED FREE PRODUCTS

Definition 4.1.1. Let $(A_i)_{i\in I}$ be a family of C*-algebras and let A_0 be a common C*-subalgebra of all A_i via embeddings $f_i : A_0 \hookrightarrow A_i$. Then *the amalgamated free coproduct* of $(A_i)_{i\in I}$ over A_0 is a pair $(*_{A_0}A_i, g_i)$ where $*_{A_0}A_i$ is C*-algebra together with a family of *-homomorphisms $g_i : A_i \to *_{A_0}A_i$ such that $g_i \circ f_i = g_{i'} \circ f_{i'}$ for all $i, i' \in I$. This is required to satisfy the following universal property: Given another pair (D, h_i) where D is a C*-algebra together with a family of *-homomorphisms $h_i : A_i \to D$ with $h_i \circ f_i = h_{i'} \circ f_{i'}$ for all $i, i' \in I$, there is a unique $h : *_{A_0}A_i \to D$ such that $h_i = g \circ g_i$ for all $i \in I$.

Remark 4.1.2. More details about the existence and representations of amalgamated free coproduct can be seen in [59].

We need the notion of subgraphs in the separated case, called complete subgraphs.

Definition 4.1.3. Let (E,C) and (F,D) be two finitely separated graphs. A morphism from (E,C) to (F,D) is a graph morphism $f : E \to F$ such that

1. $\phi^0 : E^0 \to F^0$ is injective;
2. For each $v \in E^0$ and each $X \in C_v$ there is $Y \in D_{\phi^0(v)}$ such that ϕ^1 induces a bijection from X to Y.

Remark 4.1.4. Condition 2 does not imply that ϕ^1 is injective since we can map two elements in different sets on C_v to the same set of $D_{\phi^0(v)}$.

Definition 4.1.5. A complete subgraph of (F,D) is a finitely separated graph (E,C) such that E is a subgraph of F and $C_v = \{Y \in D_v \mid Y \cap E^1 \neq \emptyset\}$ for each $v \in E^0$, that is, C is a subset of D.

Remark 4.1.6. If (E,C) is a complete subgraph of (F,D) then the inclusion map $E \hookrightarrow F$ implies a morphism from (E,C) to (F,D).

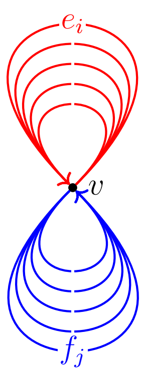
Remark 4.1.7. Any morphism $f : (E, C) \to (F, D)$ induces a unique *-homomorphism from $C^*(E, C) \to C^*(F, D)$ sending $P_v \mapsto P_{f^0(v)}$ and $S_e \mapsto S_{f^1(e)}$ for every $v \in E^0$ and $e \in E^1$ since $\{P_{f^0(v)}, S_{f^1(e)}\}$ give us a Cuntz-Krieger (F, D)-family in natural way.

For each $X \in C$, define the subgraph E_X of E with $(E_X)^0 := E^0$ and $(E_X)^1 := X$ and the source and range maps are the restricted ones. Then set $A_0 = C_0(E^0) = C^*(E^0, \emptyset)$ and $A_X = C^*(E_X)$. Also, we have the induced maps $A_0 \hookrightarrow A_X \hookrightarrow C^*(E, C)$ via the canonical inclusions $(E^0, \emptyset) \hookrightarrow (E_X, X) \hookrightarrow (E, C)$. Denote by $f_X : A_X \hookrightarrow C^*(E, C)$.

Proposition 4.1.8 ([4], Proposition 3.1)**.** *Let (E, C) be a separated graph and consider $A_0 = C_0(E^0)$ and $A_X = C^*(E_X)$ as above. Then $C^*(E, C)$ together with the inclusions $f_X : A_X \hookrightarrow C^*(E, C)$ is the amalgamated free coproduct of the family $(A_X)_{X \in C}$ over A_0.*

This proposition provides some examples of C*-algebras of separated graphs.

Example 4.1.9. Let (E, C) be a separated graph with $E^0 = \{v\}$, $E^1 = \{e_1, \dots, e_n, f_1, \dots, f_m\}$ and $C = C_v := \{X, Y\}$ with $X = \{e_1, \dots, e_n\}$ and $Y = \{f_1, \dots, f_m\}$ as in the picture below:



Consider G a group with generators $g_1, \dots, g_n, h_1, \dots, h_m$ and $c : E_1 \to G$ a labeling function defined by $c(e_i) := g_i$ and $c(f_j) := h_j$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. In this way we get the skew product separated graph $(E \times_c G, C \times_c G)$. By Proposition 4.1.8 we have $C^*(E, C) \cong C^*(E_X) *_{\mathbb{C}} C^*(E_Y) \cong \mathcal{O}_n *_{\mathbb{C}} \mathcal{O}_m$, and consequently

$$C^*(E \times_c G, C \times_c G) \cong C^*(E, C) \rtimes_{\delta_c} G \cong (\mathcal{O}_n *_{\mathbb{C}} \mathcal{O}_m) \rtimes_{\delta_c} G$$

where $\delta_c(s_i) = s_i \otimes g_i$ and $\delta_c(t_j) = t_j \otimes h_j$ for every i, j.

Definition 4.1.10. Let $(A_i)_{i \in I}$ be a family of unital C*-algebras containing a unital C*-subalgebra A_0 with conditional expectations $\phi_i : A_i \to A_0$. Then the *reduced amalgamated free product* of $(A_i)_{i \in I}$ over A_0 is the pair (A, ϕ) uniquely determinated by the following conditions:

1. A is a unital C*-algebra and there are unital *-homomorphisms $\sigma_i : A_i \to A$ such that $\sigma_i \mid_{A_0} = \sigma_{i'} \mid_{A_0}$ for all $i, i' \in I$. Moreover, $\sigma_i \mid_{A_0}$ is injective and we can identify A_0 with its image in A through this map.

2. A is generated by $\bigcup_{i \in I} \sigma_i(A_i)$.

3. $\Phi : A \to A_0$ is conditional expectation such that $\Phi \circ \sigma_i = \phi_i$ for all $i \in I$.

4. For $(i_1, \dots, i_n) \in \Lambda(I)$ and $a_j \in \mathrm{Ker}(\phi_{i_j})$ we have
$$\phi(\sigma_{i_1}(a_1) \dots \sigma_{i_n}(a_n)) = 0$$
where here, $\Lambda(I)$ denotes the set of all finite tuples $(i_1, \dots, i_n) \in \bigcup_{n=1}^{\infty} I^n$ such that $i_1 \neq i_2 \neq \dots \neq i_n$.

5. If $c \in A$ is such that $\Phi((ca)^* ca) = 0$ for all $a \in A$, then $c = 0$.

Now, we are going to define the reduced C*-algebra $C_r^*(E, C)$ of a finitely separated graph (E, C). In the above definition the reduced amalgamated free product is defined only for unital C*-algebras but considering unitalizations this can also be carried over to the non-unital case. More precisely, denote by $\tilde{A}$ the smallest unital C*-algebra containing A which is the C*-subalgebra of the multiplier algebra $\mathcal{M}(A)$ generated by A and $1_{\mathcal{M}(A)}$. Using the previous notations we set $B_0 = \tilde{A}_0$ and $B_X = \tilde{A_X}$ for each $X \in C$.

The following result is essential in this context and it is proved in [4]:

Proposition 4.1.11 ([4], Theorem 2.1)**.** *If E is a row-finite graph then there is a unique faithful conditional expectation*
$$\phi_E : C^*(E) \to C_0(E^0)$$
such that, for all paths $\mu, \nu \in E^*$ we have
$$\phi_E(S_\mu S_\nu^*) = \begin{cases} n_\mu P_{s(\mu)} & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

The exact value of n_μ is $n_\mu := \left(\prod_{i=1}^n |s^{-1}s(\mu_i)|\right)^{-1}$ if $\mu = \mu_1 \dots \mu_n \in \mathrm{Path}(E)$. Also, if the length of μ is zero, then we set $n_\mu = 1$. This number will be not very relevant for our purposes but it is important to be described.

For each $X \in C$, the canonical conditional expectation $\phi_X := \phi_{E_X} : A_X \to A_0$ above can be extended to a conditional expectation $\phi_X : B_X \to B_0$ by [[12], Proposition 2.2.1] and since ϕ_X is faithful it straightforward to see that the extension to B_X is also faithful. Now, consider the reduced amalgamated free product (B, Φ) of the family $(B_X, \phi_X)_{X \in C}$. In general, the condition 5 in Definition 4.1.10 tell us that Φ is almost faithful (see Definition 4.2.3) but since all ϕ_X are faithful conditional expectations by Proposition 4.1.11, it follows from [[32], Theorem 2.1] that the canonical conditional expectation $\Phi : B \to B_0$ is also faithful.

Definition 4.1.12. Let (E,C) be a finitely separated graph and let A_0, B_0, A_X, B_X be as defined above, for each $X \in C$. Consider the reduced amalgamated free product (B,ϕ) of the family $(B_X,\phi_X)_{X\in C}$. Then we define the reduced C*-algebra of the separated graph $C_r^*(E,C)$ as the C*-subalgebra of B generated by $\bigcup\limits_{X\in C} A_X$ in B.

Observe that each A_X can be identified with its image in B. Observe that there is a faithful conditional expectation $\Phi : C_r^*(E,C) \to C_0(E_0)$ such that $\Phi\mid_{A_X} = \phi_X$ for every $X \in C$.

We are going to use the same notations for projections and partial isometries as in $C^*(E,C)$ for their canonical images in $C_r^*(E,C)$. Note that these natural images in $C_r^*(E,C)$ satisfy the relations of $L(E,C)$, and hence there is a unique *-homomorphism $L(E,C) \to C_r^*(E,C)$ sending all projections and partial isometries to their canonical images. By the universal property there is a unique *-homomorphism $\Lambda : C^*(E,C) \to C_r^*(E,C)$ and by condition 2 of Definition 4.1.10 this map is surjective. So, we get a canonical map $C^*(E,C) \twoheadrightarrow C_r^*(E,C)$ and the canonical map $L(E,C) \to C_r^*(E,C)$ is the composition of $L(E,C) \to C^*(E,C)$ and $C^*(E,C) \twoheadrightarrow C_r^*(E,C)$.

Moreover, in [[4], Theorem 3.8] it is shown that the canonical map $L(E,C) \to C_r^*(E,C)$ is injective and for E non-separated graph, we have $C^*(E) \cong C_r^*(E)$ from this point of view.

With notations as above, we want to understand how the conditional expectation $\Phi : C_r^*(E,C) \to C_0(E^0)$ acts on elements of the canonical basis in $L(E,C)$, especially in the products of these elements. To clarify the ideas, let us begin with an short element of the form $S_\mu S_\nu^*$ in $L(E,C)$ such that $\mu\nu^*$ is reduced path and μ,ν are not necessary in the same partition. In the case where $\mu,\nu \in X$ for some $X \in C_v$, $v \in E^0$, it is clear that $\Phi(S_\mu S_\nu^*) = \phi_X(S_\mu S_\nu^*) = \delta_{\mu,\nu}(n_\mu P_{s(\mu)})$. Now, if $\mu \in X$ and $\nu \in Y$ for $X, Y \in C$, by the condition 4 in Definition 4.1.10 we observe that $\Phi(S_\mu S_\nu^*) = 0$ since $S_\mu \in \mathrm{Ker}(\phi_X)$ and $S_\nu^* \in \mathrm{Ker}(\phi_Y)$.

More generally, let $S_{\mu_1}S_{\nu_1}^*S_{\mu_2}S_{\nu_2}^*$ be an element of a basis in $L(E,C)$ which means that ν_1 and μ_2 are C-separated paths, lets say $\nu_1 \in X_1$ and $\mu_2 \in X_2$ with $X_1, X_2 \in C_v$, and $\mu_1\nu_1^*$ and $\mu_2\nu_2^*$ are reduced paths.

Note that, if all μ_i and ν_i "live" in different sets, then by condition 4 in Definition 4.1.10 we have $\Phi(S_{\mu_1}S_{\nu_1}^*S_{\mu_2}S_{\nu_2}^*) = 0$ since each isometry belongs to the kernel of conditional expectation relative to the set it belongs to.

Now, if $\mu_1 \in X_1$ with $\mu_1 \neq \nu_1$ we have $S_{\mu_1}S_{\nu_1}^* \in \mathrm{Ker}(\phi_{X_1})$ and hence, by the condition 4 in Definition 4.1.10, we still have $\Phi(S_{\mu_1}S_{\nu_1}^*S_{\mu_2}S_{\nu_2}^*) = 0$. Analogously for $\nu_2 \in X_2$ with $\mu_2 \neq \nu_2$. Now, the interesting case is when we have $\mu_1 \in X_1$, $\nu_2 \in X_2$, and $\mu_1 = \nu_1 = \mu$ and $\mu_2 \neq \nu_2$. Then we can write $S_\mu S_\mu^* = n_\mu P_{s(\mu)} + \underbrace{(S_\mu S_\mu^* - n_\mu P_{s(\mu)})}_{x_\mu}$ where $x_\mu \in \mathrm{Ker}(\phi_{X_1})$. Note that we still have $S_{\mu_2}S^*_{\nu_2} \in \mathrm{Ker}(\phi_{X_2})$ because $\mu_2 \neq \nu_2$. Since $s(\mu) = s(\mu_2)$, therefore, again by the condition 4 in Definition 4.1.10, we have

$$\begin{aligned}\Phi(S_\mu S^*_\mu S_{\mu_2}S^*_{\nu_2}) &= \Phi((n_\mu P_{s(\mu)} + x_\mu)S_{\mu_2}S^*_{\nu_2}) \\ &= \Phi(n_\mu P_{s(\mu)}S_{\mu_2}S^*_{\nu_2}) + \Phi(x_\mu S_{\mu_2}S^*_{\nu_2}) \\ &= n_\mu \phi_{X_2}(S_{\mu_2}S^*_{\nu_2}) + \Phi(x_\mu S_{\mu_2}S^*_{\nu_2}) \\ &= 0.\end{aligned}$$

Analogously for $\mu_1 \in X_1$, $\nu_2 \in X_2$, and $\mu_1 \neq \nu_1$ and $\mu_2 = \nu_2$. Finally, in the case when $\mu_1 \in X_1$, $\nu_2 \in X_2$ and $\mu_i = \nu_i$ for all $i = 1, 2$, then by the same arguments above we can write $S_{\mu_1}S^*_{\mu_1} = n_{\mu_1}P_{s(\mu_1)} + x_{\mu_1}$ and $S_{\mu_2}S^*_{\mu_2} = n_{\mu_2}P_{s(\mu_2)} + x_{\mu_2}$ where $x_{\mu_1} \in \mathrm{Ker}(\phi_{X_1})$ and $x_{\mu_2} \in \mathrm{Ker}(\phi_{X_2})$. Since $s(\mu_1) = s(\mu_2)$, therefore, by the condition 4 in Definition 4.1.10, we have

$$\begin{aligned}\Phi(S_{\mu_1}S^*_{\mu_1}S_{\mu_2}S^*_{\mu_2}) &= \Phi((n_{\mu_1}P_{s(\mu_1)} + x_{\mu_1})(n_{\mu_2}P_{s(\mu_2)} + x_{\mu_2})) \\ &= \Phi(n_{\mu_1}n_{\mu_2}P_{s(\mu_1)}P_{s(\mu_2)}) + \Phi(n_{\mu_1}P_{s(\mu_1)}x_{\mu_2}) + \Phi(x_{\mu_1}n_{\mu_2}P_{s(\mu_2)}) + \Phi(x_{\mu_1}x_{\mu_2}) \\ &= n_{\mu_1}n_{\mu_2}\phi_{X_1}(P_{s(\mu_1)}) + n_{\mu_1}\phi_{X_2}(x_{\mu_2}) + n_{\mu_2}\phi_{X_1}(x_{\mu_1}) + \Phi(x_{\mu_1}x_{\mu_2}) \\ &= n_{\mu_1}n_{\mu_2}P_{s(\mu_1)}.\end{aligned}$$

After all this calculations we can conclude that for any basic element of $L(E, C)$ the conditional expectation Φ behave as follows:

$$\Phi(S_{\mu_1}S^*_{\nu_1}\dots S_{\mu_n}S^*_{\nu_n}) = \begin{cases} N_\mu P_{s(\mu)}, & \text{if } \mu_i = \nu_i \text{ for all } i \in \{1\dots n\} \\ 0, & \text{otherwise}\end{cases}$$

where $\mu := \mu_1\nu_1^* \dots \mu_n\nu_n^*$ is a C-separated reduced path and $N_\mu := \prod_{i=1}^n n_{\mu_i}$.

Now we are able to define the conditional expectation $P : C^*(E, C) \to C_0(E^0)$ as $P := \Lambda \circ \Phi$ where $\Lambda : C^*(E, C) \twoheadrightarrow C^*_r(E, C)$.

As we have seen earlier, the original definition of the reduced C*-algebras uses reduced amalgamated free products. Now, let us look at the reduced C*-algebra from another perspective using the canonical expectation $P : C^*(E, C) \to C_0(E^0)$ defined above.

4.2 REDUCED C*-ALGEBRA ASSOCIATED WITH A CONDITIONAL EXPECTATION

In this section, suppose we have a general C*-algebra A, a commutative C*-subalgebra $C_0(X) \subseteq A$ where X is a locally compact Hausdorff space and a conditional expectation $P : A \to C_0(X)$. The dual map $P^* : C_0(X)^* \to A^*$ induces a map between the states spaces of $C_0(X)$ and A. Identifying X with pure states of $C_0(X)$ where each $x \in X$ identifies with the pure state $\omega_x(f) := f(x)$, we get a map $X \to S(A)$ sending $x \longmapsto \varphi_x$ with $\varphi_x := \omega_x \circ P$. So, for each pure state $\varphi_x \in S(A)$, we may assign its GNS-representation $\Lambda_x : A \to B(\mathcal{H}_x)$. From this we get a representation $\Lambda := \oplus_x \Lambda_x$ on $\mathcal{H} = \oplus_x \mathcal{H}_x$, the direct sum of all representations Λ_x.

Definition 4.2.1. With notations as above, we define $A_{P,r} := \Lambda(A)$ to be a reduced C*-algebra associated to (A, P). The *-homomorphism $\Lambda : A \twoheadrightarrow A_{P,r}$ is called the regular representation.

Notice that $A_{P,r}$ contains a copy of $C_0(X)$ as a C*-subalgebra. More precisely, Λ restricts to an injective *-homomorphism $C_0(X) \hookrightarrow A_{P,r}$. In fact, if $f \in C_0(X)$ and $\Lambda(f) = 0$, then $\Lambda_x(f) = 0$ for all $x \in X$ and, in particular, $\varphi_x(f) = w_x(P(f)) = f(x) = 0$ for all $x \in X$.

As can be seen in [39] we have to be careful with faithfulness here. Generally, let $P : A \to B$ be a conditional expectation onto a C*-subalgebra $B \subseteq A$ and let $\mathcal{N}_P$ be the closed linear span of all ideals I in A with $I \subseteq \mathrm{Ker}(P)$. This is the largest two-sided ideal in A that is contained in $\mathrm{Ker}(P)$. Moreover, $\mathcal{L}_P = \{a \in A \mid P(a^*a) = 0\}$ and $\mathcal{R}_P = \{a \in A \mid P(aa^*) = 0\}$ are the largest left and right ideals in A contained in $\mathrm{Ker}(P)$, respectively. The reason that $\mathcal{L}_P$ is the largest left ideal, for example, is essentially due to the Schwartz inequality. In other words, if $a \in \mathcal{L}_P$ and $b \in A$ we have $P((ba)^*ba) \leq \|b\|^2 P(a^*a)$. Then $ba \in \mathcal{L}_P$ and is in fact a left ideal. Also, by the Schwartz inequality we have $0 \leqslant P(a)^*P(a) \leqslant P(a^*a)$. So, if $P(a^*a) = 0$ then $P(a) = 0$. So, $\mathcal{L}_P \subseteq \mathrm{Ker}(P)$. Finally $\mathcal{L}_P$ is the largest left ideal because if we have a left ideal $I \subseteq \mathrm{Ker}(P)$ and $x \in I$ then $x^*x \in I$ which implies that $P(x^*x) = 0$, that is, $I \subseteq \mathcal{L}_P$. A similar fact holds for $\mathcal{R}_P$. Also, it is not difficult to see that $\mathcal{N}_P \subseteq \mathcal{L}_P \cap \mathcal{R}_P$ and $(\mathcal{L}_P)^* = \mathcal{R}_P$. So, we have the following proposition:

Proposition 4.2.2. *Let $P : A \to B$ be a conditional expectation. Then*

$$\mathcal{N}_P = \{a \in A \mid P((ab)^*(ab)) = 0 \textit{ for all } b \in A\}.$$

Proof. If $a \in \mathcal{N}_P$ then $ab \in \mathcal{N}_P \subseteq \mathcal{L}_P$ for all $b \in A$. Thus $P((ab)^*(ab)) = 0$ for all $b \in B$. Conversely, if $P((ab)^*(ab)) = 0$ for all $b \in A$, then $\overline{aA} \subseteq \mathcal{L}_P$ where $\overline{aA} = \overline{\mathrm{span}}\{ab \mid b \in A\}$. So, $\overline{AaA} \subseteq \mathcal{L}_P$ because $\mathcal{L}_P$ is a left ideal. Since $\overline{AaA}$ is a two-sided ideal that contains a we have $a \in \mathcal{N}_P$.

$\square$

Definition 4.2.3. Let $P : A \to B$ be a conditional expectation. We say that:

1. P is *faithful* if $P(a^*a) = 0$ for some $a \in A$ implies $a = 0$.
2. P is *almost faithful* if $P((ab)^*ab) = 0$ for all $b \in A$ and some $a \in A$ implies $a = 0$.
3. P is *symmetric* if $P(a^*a) = 0$ for some $a \in A$ implies $P(aa^*) = 0$.

Corollary 4.2.4. *Let $P : A \to B$ be a conditional expectation. Then:*

1. *P is symmetric if and only if $\mathcal{L}_P = \mathcal{R}_P = \mathcal{N}_P$.*
2. *P is faithful if and only if $\mathcal{L}_P = \mathcal{R}_P = \mathcal{N}_P = 0$.*
3. *P is almost faithful if and only if $\mathcal{N}_P = 0$.*
4. *P is faithful if and only if P is almost faithful and symmetric.*

Proof. This follows from Proposition 4.2.2. ☐

Example 4.2.5. Not every almost faithful conditional expectation is faithful. Consider the subalgebra $B = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{C} \right\} \subseteq \mathcal{M}_2(\mathbb{C})$ and $P : M_2(\mathbb{C}) \to B$ such that $P\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. It is straightforward to check that P is a well-defined conditional expectation. Moreover, P is almost faithful. To see that consider $X = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathcal{M}_2(\mathbb{C})$ with $x, y \neq 0$. Making the simple calculations we have that $0 \neq |xy|^2 = P(B^*X^*XB)$. But P is not faithful because if we take $X = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ for any $x \in \mathbb{C}$, $x \neq 0$ we have $P(X^*X) = 0$. In fact, observe for the same X that $P(XX^*) \neq 0$, so P is not faithful because it is not symmetric.

Now, back to the conditional expectation $P : A \to C_0(X)$ we have the next lemma:

Lemma 4.2.6. *With notations as above, for all $a \in A$, we have*

$$\Lambda(a) = 0 \Leftrightarrow P((ab)^*ab) = 0, \forall b \in A.$$

Proof. For each $a \in A$, notice that $\Lambda(a) = 0$ if and only if $\Lambda_x(a) = 0$ for all $x \in X$ if and only if $\Lambda_x(a^*a) = 0$ for all $x \in X$ because $\|\Lambda_x(a)\|^2 = \|\Lambda_x(a^*a)\|$. Let ξ_x be the cyclic vector associated to the GNS-representation Λ_x, that is, $\varphi_x(a) = \langle \xi_x, \Lambda_x(a)\xi_x \rangle$ where $\varphi_x = \omega_x \circ P$. Note that for all $b \in A$ we have

$$\varphi_x((ab)^*ab) = \langle \xi_x, \Lambda_x((ab)^*ab)\xi_x \rangle = \langle \Lambda_x(b)\xi_x, \Lambda_x(a^*a)\Lambda_x(b)\xi_x \rangle$$

Since $\mathcal{H}_x$ is the closed linear span of $\Lambda_x(A)\xi_x$ we have $\Lambda_x(a^*a) = 0$ if and only if $\varphi_x((ab)^*ab) = 0$ for all $b \in A$. Therefore we conclude that

$$\begin{aligned} \Lambda(a) = 0 &\Leftrightarrow \Lambda_x(a^*a) = 0 \text{ for all } x \in X \\ &\Leftrightarrow \varphi_x((ab)^*ab) = 0 \text{ for all } x \in X \text{ and } b \in A \\ &\Leftrightarrow \omega_x \circ P((ab)^*ab) = 0 \text{ for all } x \in X \text{ and } b \in A \\ &\Leftrightarrow P((ab)^*ab) = 0 \text{ for all } b \in A \end{aligned}$$

☐

Theorem 4.2.7. *The conditional expectation $P : A \to C_0(X)$ factors through an almost faithful conditional expectation $P_r : A_{P,r} \to C_0(X)$. Moreover, P_r is faithful if and only if P is symmetric.*

Proof. We define $P_r(\Lambda(a)) := P(a)$ for all $a \in A$. Therefore, P_r is a well-defined almost faithful conditional expectation by Lemma 4.2.6 and because P is also a conditional expectation. Moreover, P_r is a faithful if and only if P is symmetric by the item 4 of Corollary 4.2.4.

☐

Theorem 4.2.8. *The C*-algebra $A_{P,r}$ is the unique C*-algebra containing $C_0(X)$ which factors the conditional expectation P to an almost faithful conditional expectation. In other words, if we have another C*-algebra D containing $C_0(X)$, a surjective *-homomorphism $\pi : A \twoheadrightarrow D$ and an almost faithful conditional expectation $Q : D \to C_0(X)$ which factors P, that is, the diagram below commutes*

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"] \arrow[d, "P"'] & D \arrow[dl, "Q"'] \\ C_0(X) & \end{tikzcd}$$

Then, $D \cong A_{P,r}$

Proof. To begin with, define $\psi : A_{P,r} \to D$ such that $\psi(\Lambda(a)) = \pi(a)$ for all $a \in A$. It is not difficult to show that ψ is a surjective *-homomorphism once we prove that it is well defined. We claim that Ker(π) = Ker(Λ). By Lemma 4.2.6, $a \in$ Ker(Λ) if and only if $P((ab)^*ab) = 0$ for all $b \in A$. Since the diagram commutes we have $P((ab)^*ab) = 0$ if and only if $Q(\pi((ab)^*ab)) = 0$ if and only if $Q((\pi(a)\pi(b))^*\pi(a)\pi(b)) = 0$ for all $b \in A$. Since Q is almost faithful we conclude that $\pi(a) = 0$. Conversely, if $\pi(a) = 0$ then $Q((\pi(a)x)^*\pi(a)x) = 0$ for all $x \in D$. Since π is surjective for each $x \in D$ there is $b \in A$ such that $x = \pi(b)$. Then $Q((\pi(a)\pi(b))^*\pi(a)\pi(b)) = 0$ implies $P((ab)^*ab) = 0$ and, consequently, $a \in$ Ker(Λ). Therefore ψ is an isomorphism, as required.

□

Example 4.2.9. Consider the canonical tracial states $\tau : C_r^*(G) \to \mathbb{C}$ and $\tilde{\tau} : C^*(G) \to \mathbb{C}$ with $\tilde{\tau} = \tau \circ \Lambda^G$ where $\Lambda^G : C^*(G) \twoheadrightarrow C_r^*(G)$ is the regular representation. It is clear that $C_r^*(G) \cong C^*(G)_{\tilde{\tau},r}$.

Example 4.2.10. Let (A, G, α) be a dynamical system where $A = C_0(X)$ and consider the canonical conditional expectation $F_1 : A \rtimes_{\alpha,r} G \to A$ such that $P(\sum_{g\in G} a_g u_g) = a_1$ and $\tilde{F}_1 : A \rtimes_\alpha G \to A$ as $\tilde{F}_1 = F_1 \circ \Lambda^{A\rtimes G}$ where $\Lambda^{A\rtimes G} : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$ is the regular representation. Then $A \rtimes_{\alpha,r} G \cong (A \rtimes_\alpha G)_{\tilde{F}_1,r}$ because $A \rtimes_\alpha G$ contains A as C*-subalgebra and F_1 is a faithful conditional expectation which factors $\tilde{F}_1$ in canonical way.

More general, if (A, G, α) is a dynamical system which A containing $C_0(X)$ and carries a faithful conditional expectation $P : A \to C_0(X)$ then we have $A \rtimes_{\alpha,r} G \cong (A \rtimes_\alpha G)_{P\circ\tilde{F}_1,r}$.

Lemma 4.2.11. *Let A and B be C*-algebras with conditional expectations $P : A \to C_0(X)$ and $Q : B \to C_0(Y)$ and suppose that $\pi : A \to B$ is a *-homomorphism commuting with these conditional expectations, that is, the diagram below commute:*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "P"] & B \arrow[d, "Q"] \\ C_0(X) \arrow[r, "\pi|"] & C_0(Y) \end{tikzcd}$$

*Then π factors through a *-homomorphism $\pi_r : A_{P,r} \to B_{Q,r}$. Moreover, π_r is an isomorphism if π is so.*

Proof. Let $\Lambda_A : A \to A_{P,r}$ and $\Lambda_B : B \to B_{Q,r}$ denote the regular representations of A and B, respectively. Define $\pi_r : A_{P,r} \to B_{Q,r}$ such that $\pi_r(\Lambda_A(a)) = \Lambda_B(\pi(a))$ for all $a \in A$. For the well-definedness we must show that $\Lambda_A(a) = 0$ implies $\Lambda_B(\pi(a)) = 0$. But this follows from Lemma 4.2.6 since $\Lambda_A(a) = 0$ if and only if $P(c^*a^*ac) = 0$ for all $c \in A$ and this implies that $0 = \pi(P(c^*a^*ac)) = Q(\pi(c^*a^*ac)) = Q(\pi(c)^*\pi(a)^*\pi(a)\pi(c))$ which is equivalent (again by Lemma 4.2.6) to $\Lambda_B(\pi(a)) = 0$. So, this shows that π_r exists. If π is surjective then so is π_r. Now, suppose that π is faithful and $\Lambda_B(\pi(a)) = 0$. We must show that $\Lambda_A(a) = 0$. But again by Lemma 4.2.6 it is enough to show that $P(c^*a^*ac) = 0$ for all $c \in A$. Note that $\Lambda_B(\pi(a)) = 0$ if and only if $Q(\pi(c^*a^*ac)) = 0$ for all $c \in A$. But $0 = Q(\pi(c^*a^*ac)) = \pi(P(c^*a^*ac))$ which implies that $P(c^*a^*ac) = 0$, as desired.

□

Remark 4.2.12. Actually in above Lemma for π_r to be injective it is enough that π is injective on $C_0(X)$.

For this final part we assume that $P : A \to C_0(X)$ is an almost faithful and symmetric conditional expectation, that is, it is faithful.

Lemma 4.2.13. *Let A and B be C*-algebras with conditional expectations $P : A \to C_0(X)$ and $Q : B \to C_0(Y)$, respectively. Then*

$$(A \otimes B)_{P\otimes Q,r} \cong A_{P,r} \otimes B_{Q,r}.$$

Proof. First of all, it is not difficult to see that $P \otimes Q : A \otimes B \to C_0(X) \otimes C_0(Y)$ is a well-defined conditional expectation defined by $(P \otimes Q)(a \otimes b) = P(a) \otimes Q(b)$ with $\|P \otimes Q\| = \|P\|\|Q\|$. By Theorem 4.2.8 it is enough to check that $A_{P,r} \otimes B_{Q,r}$ has a faithful conditional expectation which factors $P \otimes Q$. But the conditional expectation $P_r \otimes Q_r : A_{P,r} \otimes B_{Q,r} \to C_0(X) \otimes C_0(Y)$ is the perfect candidate because by definition $P_r \otimes Q_r(\Lambda_A \otimes \Lambda_B) = P \otimes Q$. Since P_r and Q_r are faithful then so is $P_r \otimes Q_r$, as required.

□

Proposition 4.2.14. *Let (A, G, δ) be a dynamical co-system and $P : A \to B$ a G-equivariant conditional expectation* [1]*. Then there is a conditional expectation $P \rtimes G : A \rtimes_\delta G \to B \rtimes_\delta G$ such that $P \rtimes G(j_A(a)j_G^A(f)) = (j_B \circ P)(a)j_G^B(f)$ for every $a \in A$ and $f \in C_0(G)$. Moreover, if P is faithful then so is $P \rtimes G$.*

Proof. To begin with, observe that $B \rtimes_\delta G$ is a C*-subalgebra of $A \rtimes_\delta G$ because $B \otimes \mathcal{K}(l^2(G)) \subseteq A \otimes \mathcal{K}(l^2(G))$ and hence also $\mathcal{M}(B \otimes \mathcal{K}(l^2(G))) \subseteq \mathcal{M}(A \otimes \mathcal{K}(l^2(G)))$. Since $B \rtimes_\delta G \subseteq \mathcal{M}(B \otimes \mathcal{K}(l^2(G)))$ via the regular representation $j_B \times j_G^B$ we get $B \rtimes_\delta G \subseteq A \rtimes_\delta G$.

Now, $P \rtimes G$ is a conditional expectation because P is so. Actually $P \rtimes G$ coincides with $P \otimes id_{\mathcal{K}}$, if we identify $B \rtimes_\delta G \subseteq A \rtimes_\delta G \subseteq \mathcal{M}(A \otimes \mathcal{K}(l^2(G)))$. Thus, if P is faithful then $P \otimes id_{\mathcal{K}}$ is faithful too. □

[1] A condition expectation $P : A \to B$ is G-equivariant with respect a coaction δ in the sense of Definition 2.5.24, that is, if $(P \otimes id_G) \circ \delta = \delta \circ P$ where δ on the right side of equality means a restrict δ on B.

Lemma 4.2.15. *Let (A, G, δ) be a dynamical co-system and let $P : A \to C_0(X)$ be a conditional expectation. Then*

$$(A \rtimes_\delta G)_{P\rtimes G,r} \cong A_{P,r} \rtimes_\delta G.$$

Proof. Let $P_r : A_{P,r} \to C_0(X)$ be the faithful conditional expectation that factors P. So, by Proposition 4.2.14 we have conditional expectations $P \rtimes G : A \rtimes_\delta G \to C_0(X) \rtimes_\delta G$ and $P_r \rtimes G : A_{P,r} \rtimes_\delta G \to C_0(X) \rtimes_\delta G$ over $A \rtimes_\delta G$ and $A_{P,r} \rtimes_\delta G$, respectively. Since P_r is faithful then so is $P_r \rtimes G$ and the following diagram commutes:

$$\begin{tikzcd} A \rtimes_\delta G \arrow[r, "\Lambda\rtimes G"] \arrow[d, "P\rtimes G"] & A_{P,r} \rtimes_\delta G \arrow[dl, "P_r\rtimes G"] \\ C_0(X) \rtimes_\delta G & \end{tikzcd}$$

To see this note that

$$\begin{aligned} P_r \rtimes G \circ \Lambda \rtimes G(j_A(a)j_G^A(f)) &= P_r \rtimes G((j_B \circ \Lambda)(a)j_G^B(f)) \\ &= (j_B \circ P_r \circ \Lambda)(a)j_G^B(f) \\ &= (j_B \circ P)(a)j_G^B(f) \\ &= P \rtimes G(j_A(a)j_G^A(f)) \end{aligned}$$

□

4.3 REDUCED SEPARATED GRAPH C*-ALGEBRAS AND EXTEND RESULTS

An immediate consequence of these results is the next corollary that can be used as an alternative definition for the reduced C*-algebra of a finitely separated graph. In this section, through this alternative, we are able to show the duality theorems for the reduced C*-algebra $C_r^*(E, C)$.

Corollary 4.3.1. *Let (E, C) be a finitely separated graph and $P : C^*(E, C) \to C_0(E^0)$ be the canonical conditional expectation. Then*

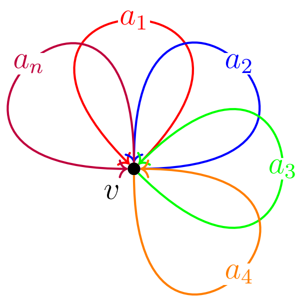
$$C^*(E, C)_{P,r} \cong C_r^*(E, C).$$

Proof. According to the definition of the conditional expectation $P : C^*(E, C) \to C_0(E^0)$, it factors through a faithful conditional expectation $\Phi : C_r^*(E, C) \to C_0(E^0)$. By Theorem 4.2.8 the result follows trivially.

□

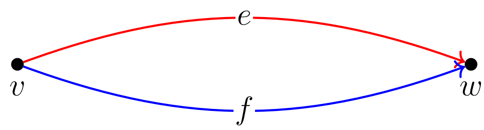
Remark 4.3.2. In our case, the conditional expectation $\Phi : C_r^*(E, C) \to C_0(E^0)$ is faithful, that is, it is symmetric.

Example 4.3.3. A canonical example is when we consider the Cuntz separated graph (A_n, D) seen in Example 3.1.5 draw as the picture below:



Since by Example 4.2.9 $C^*(\mathbb{F}_n)_{\tilde{\tau},r} \cong C_r(\mathbb{F}_n)$ we have $C^*_r(A_n, D) \cong C^*_r(\mathbb{F}_n)$ for the uniqueness, where $\mathbb{F}_n$ is the free group generated by the edges.

Example 4.3.4. If we consider the separated graph $(E(1,1), C(1,1))$ seen in Example 3.2.10 as in the picture below:

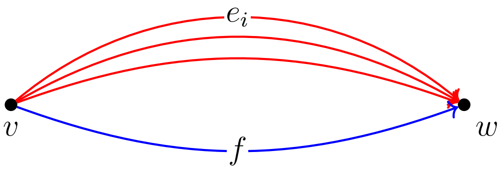


We know that $C^*(E(1,1), C(1,1)) \cong M_2(C(\mathbb{T}))$ which can be viewed isomorphically as $M_2(\mathbb{C}) \otimes C(\mathbb{T})$. Identifying $C_0(E^0) = \mathbb{C}_v \oplus \mathbb{C}_w$ as a $\begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix} \in M_2(C(\mathbb{T}))$ there is a faithful conditional expectation

$$\phi_1 \otimes \phi_2 : M_2(\mathbb{C}) \otimes C(\mathbb{T}) \to \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix},$$

where ϕ_1 is a faithful conditional expectation such that $\phi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and ϕ_2 is the canonical faithful trace on $C(\mathbb{T})$. Since $\phi_1 \otimes \phi_2$ is faithful consequently we have $C^*_r(E(1,1), C(1,1)) = C^*(E(1,1), C(1,1))$.

Example 4.3.5. Another example which is related to the previous example is the separated graph $(E(1,n), C(1,n))$ seen in Example 3.2.10 as in the picture below:



We know that $C^*(E(1,n),C(1,n)) \cong M_2(\mathcal{O}_n) \cong M_2(\mathbb{C}) \otimes \mathcal{O}_n$. As in the previous example using the correct identifications there is also a faithful conditional expectation

$$\phi_1 \otimes \phi_2 : M_2(\mathbb{C}) \otimes \mathcal{O}_n \to \begin{bmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{bmatrix},$$

where ϕ_1 is the same faithful conditional expectation in $M_2(\mathbb{C})$ and ϕ_2 is the canonical conditional expectation on $\mathcal{O}_n$ (just consider $\mathcal{O}_n$ as a graph C*-algebra). As a consequence we have $C_r^*(E(1,n),C(1,n)) \cong C^*(E(1,n),C(1,n))$.

Let (E,C) be a finitely separated graph and let $c : E^1 \to G$ be a labeling function. We have a coaction $\delta_c : C^*(E,C) \to C^*(E,C) \otimes C^*(G)$ defined before and we would like to prove that this coaction factors through the reduced C*-algebra $C_r^*(E,C)$ using this context. The next proposition give us exactly that:

Proposition 4.3.6. *Let (E,C) be a finitely separated graph and $c : E^1 \to G$ be a labeling function. Then the coaction $\delta_c : C^*(E,C) \to C^*(E,C) \otimes C^*(G)$ factors through the reduced C*-algebra $C_r^*(E,C)$, that is, there is a coaction $\delta_c^r : C_r^*(E,C) \to C_r^*(E,C) \otimes C^*(G)$. Moreover, δ_c^r is normal coaction.*

Proof. For the existence, it is enough to check that the coaction δ_c commutes with respect to the conditional expectations. In other words, we are going to prove that the following diagram commutes:

$$\begin{tikzcd} C^*(E,C) \arrow[r, "\delta_c"] \arrow[d, "P"'] & C^*(E,C) \otimes C^*(G) \arrow[d, "P\otimes\tilde{\tau}"'] \\ C_0(E^0) \arrow[r, "id\otimes 1"] & C_0(E^0) \otimes \mathbb{C} \end{tikzcd}$$

where $\tilde{\tau}$ is the tracial state of $C^*(G)$ defined in Example 4.2.9. Let $S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \in L(E,C)$ be an elementary element. Then we have

$$\begin{aligned}(P \otimes \tilde{\tau}) \circ \delta_c(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*) &= (P \otimes \tilde{\tau})(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^* \otimes c(\mu_1)c(\nu_1)^{-1} \dots c(\mu_n)c(\nu_n)^{-1}) \\ &= P(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*) \otimes \tilde{\tau}(c(\mu_1)c(\nu_1)^{-1} \dots c(\mu_n)c(\nu_n)^{-1}) \\ &= N_\mu P_{s(\mu)} \otimes 1\end{aligned}$$

where in the last step we used that $P(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*)$ is non-zero if $\mu_i = \nu_i$ for all i and as a consequence $c(\mu_1)c(\nu_1)^{-1} \dots c(\mu_n)c(\nu_n)^{-1} = 1$. On the other hand, we have

$$(id \otimes 1) \circ P(S_{\mu_1}S_{\nu_1}^* \dots S_{\mu_n}S_{\nu_n}^*) = (id \otimes 1)(N_\mu P_{s(\mu)}) = N_\mu P_{s(\mu)} \otimes 1$$

where again we used that $P(S_{\mu_1}S^*_{\nu_1}\dots S_{\mu_n}S^*_{\nu_n})$ is non-zero if $\mu_i=\nu_i$ for all i. Since $L(E,C)$ is dense in $C^*(E,C)$, by linearity and continuity the diagram above commute. It follows from Lemma 4.2.11 that δ_c factors through

$$(\delta_c^r)':C^*(E,C)_{P,r}\to(C^*(E,C)\otimes C^*(G))_{P\otimes\tilde{\tau},r}$$

such that $(\delta_c^r)'(\Lambda(a))=\Lambda_{P\otimes\tilde{\tau}}\circ\delta_c(a)$ for all $a\in C^*(E,C)$. Since $C^*(E,C)_{P,r}\cong C^*_r(E,C)$ by Corollary 4.3.1 and $(C^*(E,C)\otimes C^*(G))_{P\otimes\tau,r}\cong C^*_r(E,C)\otimes C^*_r(G)$ by Lemma 4.2.13 this can be translated to a *-homomorphism $(\delta_c^r)':C^*_r(E,C)\to C^*_r(E,C)\otimes C^*_r(G)$ such that $(\delta_c^r)'(\Lambda(a))=(\Lambda\otimes\Lambda^G)\circ\delta_c(a)$ for all $a\in C^*(E,C)$. Observe that if we replace $\tilde{\tau}$ by id_G viewed as a trivial conditional expectation the same computation as above also works and we get a well-defined *-homomorphism $\delta_c^r:C^*_r(E,C)\to C^*_r(E,C)\otimes C^*(G)$ such that the diagram below commutes:

$$\begin{tikzcd} C^*_r(E,C) \arrow[r, "\delta_c^r"] \arrow[rr, bend left, "(\delta_c^r)'"] & C^*_r(E,C)\otimes C^*(G) \arrow[r, "id\otimes\Lambda^G"] & C^*_r(E,C)\otimes C^*_r(G) \end{tikzcd}$$

That is, $(\delta_c^r)'(\Lambda(a))=\Lambda_{P\otimes\tilde{\tau}}(\delta_c(a))=(\Lambda\otimes\Lambda^G)\circ\delta_c(a)=(id\otimes\Lambda^G)\circ(\Lambda\otimes id_G)\circ\delta_c(a)=(id\otimes\Lambda^G)\circ\delta_c^r(a)$ for all $a\in C^*(E,C)$.

We claim that $\delta_c^r:C^*_r(E,C)\to C^*_r(E,C)\otimes C^*(G)$ is in fact a coaction. The coaction identity and nondegeneracy is provided from δ_c since $\delta_c^r(\Lambda(a))=(\Lambda\otimes id_G)\circ\delta_c$ and

$$\begin{aligned} C^*_r(E,C)\otimes C^*(G) &= (\Lambda\otimes id_G)(C^*(E,C)\otimes C^*(G))\\ &= (\Lambda\otimes id_G)(\overline{\delta_c(C^*(E,C))(1\otimes C^*(G))})\\ &= \overline{(\Lambda\otimes id_G)\circ\delta_c(C^*(E,C))(1\otimes C^*(G))}\\ &= \overline{\delta_c^r(\Lambda(C^*(E,C)))(1\otimes C^*(G))}\\ &= \overline{\delta_c^r(C^*_r(E,C))(1\otimes C^*(G))} \end{aligned}$$

Similarly, it can be seen that $(\delta_c^r)'$ is nondegenerate too. Consequently, $(\delta_c^r)'$ is faithful which means that δ_c^r is a normal coaction.

□

As a consequence we are able to prove the following corollary:

Corollary 4.3.7. *Let (E,C) be a finitely separated graph and $c:E^1\to G$ be a labeling function. Then*

$$C^*_r(E,C)\rtimes_{\delta_c^r}G\rtimes_{\tilde{\delta_c^r},r}G\cong C^*_r(E,C)\otimes\mathcal{K}(l^2(G)).$$

Proof. It follows from the fact that δ_c^r is a normal coaction.

□

Lemma 4.3.8. *For a free action α of a group G on a finitely separated graph (E, C), the induced action α of G on $C^*(E, C)$ factors through an action $\tilde{\alpha}$ of G on the reduced C*-algebra $C_r^*(E, C)$ such that $\Phi(\tilde{\alpha}_g(x)) = \alpha_g(\Phi(x))$ for all $x \in C_r^*(E, C)$.*

Proof. In order to facilitate the notation we are using $g \cdot v$ and $g \cdot e$ for the action of G on the separated graph (E, C) and the induced action α is given by $\alpha_g(P_v) = P_{g \cdot v}$ and $\alpha_g(S_e) = S_{g \cdot e}$ for every $v \in E^0$, $e \in E^1$ and $g \in G$. For each $g \in G$, it is enough to show that the *-homomorphism $\alpha_g : C^*(E, C) \to C^*(E, C)$ commutes with respect to the conditional expectations, that is, the following diagram commutes:

$$\begin{array}{ccc} C^*(E,C) & \xrightarrow{\alpha_g} & C^*(E,C) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\ C_0(E^0) & \xrightarrow{\alpha_g|_{C_0(E^0)}} & C_0(E^0) \end{array}$$

It is enough to check the commutativity on basic elements of the form $S_{\mu_1} S_{\nu_1}^* \dots S_{\mu_n} S_{\nu_n}^* \in L(E, C)$ since $L(E, C)$ is dense of $C(E, C)$ and α_g is linear and continuous map. So, we have

$$\begin{aligned} P(\alpha_g(S_{\mu_1} S_{\nu_1}^* \dots S_{\mu_n} S_{\nu_n}^*)) &= P(S_{g \cdot \mu_1} S_{g \cdot \nu_1}^* \dots S_{g \cdot \mu_n} S_{g \cdot \nu_n}^*) \\ &= N_\mu P_{g \cdot s(\mu)} \\ &= \alpha_g(N_\mu P_{s(\mu)}) \\ &= \alpha_g(P(S_{\mu_1} S_{\nu_1}^* \dots S_{\mu_n} S_{\nu_n}^*)) \end{aligned}$$

where here we used that $P(S_{\mu_1} S_{\nu_1}^* \dots S_{\mu_n} S_{\nu_n}^*)$ is non-zero whenever $\mu_i = \nu_i$ for all i in which is equivalent to $g \cdot \mu_i = g \cdot \nu_i$ for all i. So, we conclude that the diagram above in fact commutes. According to Lemma 4.2.11, α_g factors through an action $\tilde{\alpha_g} : C_r^*(E, C) \to C_r^*(E, C)$ such that $\tilde{\alpha_g}(\Lambda(x)) = \Lambda(\alpha_g(x))$ for all $x \in C^*(E, C)$. Finally observe that

$$\begin{aligned} \Phi(\tilde{\alpha_g}(\Lambda(x))) &= \Phi(\Lambda(\alpha_g(x))) \\ &= P(\alpha_g(x)) \\ &= \alpha_g(P(x)) \\ &= \alpha_g(\Phi(\Lambda(x))). \end{aligned}$$

This implies that $\Phi \circ \tilde{\alpha_g} = \alpha_g \circ \Phi$, as desired.

□

Theorem 4.3.9. *With notations as above, there is a canonical isomorphism*

$$C_r^*(E \times_c G, C \times_c G) \cong C_r^*(E, C) \rtimes_{\delta_c^r} G.$$

Under this isomorphism, the action $\tilde{\gamma}$ on $C_r^*(E \times_c G, C \times_c G)$ induced by the translation action γ on $C^*(E \times_c G, C \times_c G)$ corresponds to the dual action $\widehat{\delta_c^r}$ on $C_r^*(E, C) \rtimes_{\delta_c^r} G$.

Proof. It is enough to show that the isomorphism $\phi : C^*(E \times_c G, C \times_c G) \to C^*(E, C) \rtimes_{\delta_c} G$ seen in Theorem 3.2.17 commutes with the respective conditional expectations, that is, the following diagram commutes:

$$\begin{array}{ccc} C^*(E \times_c G, C \times_c G) & \xrightarrow{\phi} & C^*(E, C) \rtimes_{\delta_c} G \\ \Big\downarrow P & & \Big\downarrow P \rtimes G \\ C_0(E^0 \times G) & \xrightarrow{\phi|_{C_0(E \times G)}} & C_0(E^0) \rtimes_{\delta_c} G \end{array}$$

To make things simpler we will show that the diagram commutes on basic elements of the form $S_{(\mu,g)}S^*_{(\nu,z)}S_{(\mu',z)}S^*_{(\nu',z')}$ with $z = gc(\mu)c(\nu)^{-1}$ and $z' = gc(\mu)c(\nu)^{-1}c(\mu')c(\nu')^{-1}$ where ν,μ' are C-separated paths and $\mu\nu^*$ and $\mu'\nu'^*$ are reduced paths. We already calculated ϕ on these elements, that is, $\phi(S_{(\mu,g)}S^*_{(\nu,z)}S_{(\mu',z)}S^*_{(\nu,z')}) = j_G(\chi_{g^{-1}})j_B(S_\mu S^*_\nu S_{\mu'} S^*_{\nu'})$ (see proof of Theorem 3.2.17). So, we compute:

$$\begin{aligned} P \rtimes G(\phi(S_{(\mu,g)}S^*_{(\nu,h)})) &= P \rtimes G(j_G(\chi_{g^{-1}})j_B(S_\mu S^*_\nu S_{\mu'} S^*_{\nu'})) \\ &= j_G(\chi_{g^{-1}})(j_B \circ P)(S_\mu S^*_\nu S_{\mu'} S^*_{\nu'}) \\ &= j_G(\chi_{g^{-1}})j_B(N_\mu P_{s(\mu)}) \\ &= \phi|_{C_0(E \times G)}(N_\mu P_{(s(\mu),g)}) \\ &= \phi(P(S_{(\mu,g)}S^*_{(\nu,z)}S_{(\mu',z)}S^*_{(\nu',z')})) \end{aligned}$$

Note that here we used that $P(S_\mu S^*_\nu S_{\mu'} S^*_{\nu'})$ is non-zero if $\mu = \nu$ and $\mu' = \nu'$ and this implies $(\mu, g) = (\nu, z)$ and $(\mu', z) = (\nu', z')$. Since $L(E \times_c G, C \times_c G)$ is dense in $C^*(E \times_c G, C \times_c G)$ by linearity and continuity we conclude that the diagram commutes.

With the correct identifications, $C^*(E \times_c G, C \times_c G)_{P,r} \cong C^*_r(E \times_c G, C \times_c G)$ by Corollary 4.3.1 and $(C^*(E, C) \rtimes_{\delta_c} G)_{P \rtimes G, r} \cong C^*_r(E, C) \rtimes_{\delta^r_c} G$ by Lemma 4.2.15, we conclude that ϕ factors through

$$\phi^r : C^*_r(E \times_c G, C \times_c G) \to C^*_r(E, C) \rtimes_{\delta^r_c} G$$

such that $\phi^r(\Lambda_A(x)) = (\Lambda_B \rtimes G) \circ (\phi(x))$ for all $x \in C^*(E \times_c G, C \times_c G)$ where $\Lambda_A : C^*(E \times_c G, C \times_c G) \twoheadrightarrow C^*_r(E \times_c G, C \times_c G)$ and $\Lambda_B : C^*(E, C) \twoheadrightarrow C^*_r(E, C)$. Since ϕ is an isomorphism so is ϕ^r.

To finish we need to check the G-equivariance. To do that observe that for all $x \in C^*(E \times_c G, C \times_c G)$ we have:

$$\begin{aligned} \phi^r \circ \tilde{\gamma}_g \circ \Lambda_A(x) &= \phi^r \circ \Lambda_A \circ \gamma_g(x) \\ &= (\Lambda_B \rtimes G) \circ \phi \circ \gamma_g(x) \\ &= (\Lambda_B \rtimes G) \circ (\widehat{\delta_c})_g \circ \phi(x) \\ &= (\widehat{\delta^r_c})_g \circ (\Lambda_B \rtimes G) \circ \phi(x) \\ &= (\widehat{\delta^r_c})_g \circ \phi^r \circ \Lambda_A(x) \end{aligned}$$

This completes the proof.

$\square$

Corollary 4.3.10. *Let (E, C) be a finitely separated graph and $c : E^1 \to G$ be a labeling function. Then*

$$C_r^*(E \times_c G, C \times_c G) \rtimes_{\tilde{\gamma},r} G \cong C_r^*(E, C) \otimes \mathcal{K}(l^2(G)).$$

Proof. This follows from Corollary 4.3.7 and Theorem 4.3.9.

□

Corollary 4.3.11. *For a free action θ of a group G on a finitely separated graph (E, C), there is a canonical isomorphism*

$$C_r^*(E, C) \rtimes_{\tilde{\theta},r} G \cong C_r^*(E/G, C/G) \otimes \mathcal{K}(l^2(G)).$$

Proof. Follows from Corollary 4.3.10 and Gross-Tucker Theorem for separated graphs seen in Theorem 3.1.7.

□

Remark 4.3.12. Compiling all results that we have seen so far in this section we get the following diagram of isomorphisms:

$$\begin{tikzcd}
C_r^*(E \times_c G, C \times_c G) \rtimes_{\gamma,r} G \arrow[rr, "4.3.9"] \arrow[dr, "4.3.10"] & & C_r^*(E, C) \rtimes_{\delta_c^r} G \rtimes_{\hat{\delta}_c^r,r} G \arrow[dl, "4.3.7"] \\
& C_r^*(E, C) \otimes \mathcal{K}(l^2(G)) &
\end{tikzcd}$$

Remark 4.3.13. Unlike what happens with non-separated graphs, in the separated case neither γ nor $\tilde{\gamma}$ are amenable in general, in the sense that $C^*(E, C) \rtimes_\gamma G \cong C^*(E, C) \rtimes_{\gamma,r} G$ or $C_r^*(E, C) \rtimes_{\tilde{\gamma}} G \cong C_r^*(E, C) \rtimes_{\gamma,r} G$. Fortunately, we have the following diagrams:

$$\begin{tikzcd}
C^*(E \times_c G, C \times_c G) \rtimes_\gamma G \arrow[rr, "3.2.17"] \arrow[dr, "3.2.20"] & & C^*(E, C) \rtimes_{\delta_c} G \rtimes_{\hat{\delta}_c} G \arrow[dl, "3.2.15"] \\
& C^*(E, C) \otimes \mathcal{K}(l^2(G)) \arrow[d, two heads] & \\
& C_r^*(E, C) \otimes \mathcal{K}(l^2(G)) & \\
C_r^*(E \times_c G, C \times_c G) \rtimes_{\gamma,r} G \arrow[rr, "4.3.9"] \arrow[ur, "4.3.10"] & & C_r^*(E, C) \rtimes_{\delta_c^r} G \rtimes_{\hat{\delta}_c^r,r} G \arrow[ul, "4.3.7"']
\end{tikzcd}$$

4.4 ALTERNATIVE APPROACH TO $C_r^*(E, C)$

In this section, we will present an alternative approach to the definition of reduced separated graphs via Fell bundles. From this point a view, the C*-algebra structure involved is more clear. With certain conditions, we are going to prove that the reduced C*-algebra $C^*(\mathcal{A})_{P,r}$ is isomorphic to a reduced C*-algebra of some quotient Fell bundle and use that to get the desired definition.

Definition 4.4.1. Let $\mathcal{A}$ a Fell bundle over a group G and I be an ideal of $\mathcal{A}_1$. We say that I is an $\mathcal{A}$-invariant ideal if it is invariant for every Hilbert bimodule $\mathcal{A}_g$, that is, $I \cdot \mathcal{A}_g = \mathcal{A}_g \cdot I$ for all $g \in G$.

Lemma 4.4.2. *Let $\mathcal{A}$ be a Fell bundle over a group G . Then I is $\mathcal{A}$-invariant ideal if and only if $\mathcal{A}_g \cdot I \cdot A_{g^{-1}} \subseteq I$ for all $g \in G$.*

Proof. If I is $\mathcal{A}$-invariant it is immediate that $\mathcal{A}_g \cdot I \cdot \mathcal{A}_{g^{-1}} \subseteq I$ for all $g \in G$. Now, if $\mathcal{A}_g \cdot I \cdot \mathcal{A}_{g^{-1}} \subseteq I$ for all $g \in G$ then multiplying both sides by $\mathcal{A}_g$ we get $\mathcal{A}_g \cdot I \cdot \mathcal{A}_{g^{-1}}\mathcal{A}_g \subseteq I\mathcal{A}_g$. Since $\mathcal{A}_{g^{-1}}\mathcal{A}_g$ is an ideal in $\mathcal{A}_1$ then I commutes with $\mathcal{A}_{g^{-1}}\mathcal{A}_g$ and we know that $\mathcal{A}_g\mathcal{A}_{g^{-1}}\mathcal{A}_g = \mathcal{A}_g$ for all $g \in G$. Then

$$\mathcal{A}_g \cdot \mathcal{A}_{g^{-1}} \cdot \mathcal{A}_g \cdot I = \mathcal{A}_g \cdot I \subseteq I \cdot \mathcal{A}_g$$

for all $g \in G$. The other containment is analogous.

□

Now, let $\mathcal{A}$ be a Fell bundle over a group G and consider I an ideal of $C^*(\mathcal{A})$. It is straightforward to see that $I \cap \mathcal{A}_1$ is an ideal of $\mathcal{A}_1$. Set $J := I \cap \mathcal{A}_1$.

Lemma 4.4.3. *With notations as above, J is always an $\mathcal{A}$-invariant ideal.*

Proof. For each $g \in G$, $\mathcal{A}_g \cdot \mathcal{A}_1 \cdot \mathcal{A}_{g^{-1}} \subseteq \mathcal{A}_g \cdot \mathcal{A}_{g^{-1}} \subseteq \mathcal{A}_1$. Since I is an ideal of $C^*(\mathcal{A})$ then $\mathcal{A}_g \cdot I \subseteq I$ and $I \cdot \mathcal{A}_{g^{-1}} \subseteq I$. Therefore, $\mathcal{A}_g \cdot J \cdot \mathcal{A}_{g^{-1}} \subseteq J$ and by the lemma above J is $\mathcal{A}$-invariant. □

Since J is an $\mathcal{A}$-invariant ideal by the lemma above we may restrict the Fell bundle structure on $\mathcal{A}$ to one on $\mathcal{J}_g := J \cdot \mathcal{A}_g = \mathcal{A}_g \cdot J$ and it induces a Fell bundle structure on the quotient $\mathcal{A}/\mathcal{J} := \{\mathcal{A}_g/\mathcal{J}_g\}_{g\in G}$.

Corollary 4.4.4. *With notations as above, $\mathcal{J}_g = I \cap \mathcal{A}_g$ for every $g \in G$.*

Proof. Since I is an ideal of $C^*(\mathcal{A})$ we have $I\mathcal{A}_g \subseteq I$ and obvious $\mathcal{A}_1\mathcal{A}_g \subseteq \mathcal{A}_g$ which this imply directly that $\mathcal{J}_g \subseteq I \cap \mathcal{A}_g$. On the other hand it follows from the fact that every $x \in I \cap \mathcal{A}_g$ can be viewed as $x = \lim_{i\to\infty} e_i x = \lim_{i\to\infty} x e_i$ where $(e_i)_i$ is an approximate identity for J since $I \cap \mathcal{A}_g$ is naturally a J-bimodule and this completes the proof.

□

Now, let $\mathcal{A}$ be a Fell bundle, $C_0(X)$ be a C*-subalgebra of $\mathcal{A}_1$ and $P : C^*(\mathcal{A}) \to C_0(X)$ be a conditional expectation which we assume to be symmetric, that is, $\mathcal{L}_P = \mathcal{R}_P = \mathcal{N}_P$ and, vanishes on $\mathcal{A}_g$ for $g \neq 1$. Let us denote by $P_1 := P|_{\mathcal{A}_1}$ restriction of the conditional expectation P to $\mathcal{A}_1$ and $\mathcal{L}_P = \{x \in C^*(\mathcal{A}) \mid P(x^*x) = 0\}$ the ideal of $C^*(\mathcal{A})$ as in the discussion after Definition 4.2.1. So, $J := \mathcal{L}_P \cap \mathcal{A}_1$ is an ideal of $\mathcal{A}_1$. In fact, the ideal J coincides to $\mathcal{L}_{P_1}$. Since J is an $\mathcal{A}$-invariant ideal by the lemma above we may restrict the Fell bundle structure on $\mathcal{A}$ to one on $\mathcal{J}_g := J \cdot \mathcal{A}_g = \mathcal{A}_g \cdot J$ and it induces a Fell bundle structure on the quotient $\mathcal{A}/\mathcal{J} := \{\mathcal{A}_g/\mathcal{J}_g\}_{g\in G}$. Moreover, $\mathcal{J}_g = \mathcal{L}_P \cap \mathcal{A}_g$. So, we have the next result:

Theorem 4.4.5. *With notations as above, we have*

$$C_r^*(\mathcal{A}/\mathcal{J}) \cong C^*(\mathcal{A})_{P,r}.$$

Proof. First of all, note that $\tilde{P}_1 : \mathcal{A}_1/\mathcal{J}_1 \to C_0(X)$ defined by $\tilde{P}_1(q_1(a)) = P(a)$ for every $a \in \mathcal{A}_1$ is a faithful conditional expectation. Notice the $C_r^*(\mathcal{A}/\mathcal{J})$ carries a faithful conditional expectation onto $C_0(X)$ and the diagram below commutes:

$$\begin{tikzcd}
C_r^*(\mathcal{A}/\mathcal{J}) \arrow[ddd, two heads, bend right, "E_1^{\mathcal{A}/\mathcal{J}}"'] & & \\
& C_r^*(\mathcal{A}) \arrow[ul, two heads, "q_r"'] \arrow[d, "E_1^{\mathcal{A}}"] & C^*(\mathcal{A}) \arrow[l, two heads, "\Lambda^{\mathcal{A}}"] \arrow[d, "P"] \arrow[ull, two heads, bend right, "q_r\circ\Lambda^{\mathcal{A}}=\Lambda^{\mathcal{A}/\mathcal{J}}\circ\overline{q}"'] \\
& \mathcal{A}_1 \arrow[r, "P_1"] \arrow[d, two heads, "q_1"] & C_0(X) \\
& \mathcal{A}_1/\mathcal{J}_1 \arrow[ur, "\tilde{P}_1"] &
\end{tikzcd}$$

Indeed, if $a \in \mathcal{A}_g$ we have:

$$\begin{aligned}
\tilde{P}_1 \circ E_1^{\mathcal{A}/\mathcal{J}} \circ q_r \circ \Lambda^{\mathcal{A}}(j_g(a)) &= \tilde{P}_1 \circ E_1^{\mathcal{A}/\mathcal{J}} \circ \Lambda^{\mathcal{A}/\mathcal{J}} \circ \overline{q}(j_g(a)) \\
&= \tilde{P}_1 \circ q_1 \circ E_1^{\mathcal{A}} \circ \Lambda^{\mathcal{A}}(j_g(a)) \\
&= P_1 \circ E_1^{\mathcal{A}} \circ \Lambda^{\mathcal{A}}(j_g(a)). \\
&= P(j_g(a))
\end{aligned}$$

where $\overline{q}$ is the induced *-homomorphism from $C^*(\mathcal{A})$ to $C^*(\mathcal{A}/\mathcal{J})$ and here we use that P vanishes on $\mathcal{A}_g$ for every $g \neq 1$. Since the closed linear span of the set of all $j_g(a)$, with $a \in \mathcal{A}_g$ and $g \in G$, is dense in $C^*(\mathcal{A})$, the result follows. So, $\tilde{P}_1 \circ E_1^{\mathcal{A}/J}$ is a faithful conditional expectation on $C_r^*(\mathcal{A}/\mathcal{J})$ onto $C_0(X)$ such that factors P through $q_r \circ \Lambda^{\mathcal{A}}$. Then by Theorem 4.2.8 we are done.

□

Theorem 4.4.6. *Let (E, C) be a finitely separated graph and $c : E^1 \to G$ be a labeling function. Then*

$$C_r^*(E, C) \cong C_r^*(\mathcal{A}/\mathcal{J}).$$

Moreover, the induced coaction on $C_r^*(E, C)$ correspond to the coaction on $C_r^*(\mathcal{A}/\mathcal{J})$.

Proof. Consider $P : C^*(E,C) \to C_0(E^0)$ the canonical conditional expectation and the ideal $\mathcal{L}_P = \{x \in C^*(E,C) \mid P(x^*x) = 0\}$ of $C^*(E,C)$. With the construction above, $\mathcal{J} = \{\mathcal{J}_g\}_{g\in G}$ is the ideal of the Fell bundle $\mathcal{A}$ associated to spectral subspaces. Remember that P is symmetric and vanishes on $\mathcal{A}_g$ for $g \neq 1$ and since δ_c is a maximal coaction we have $C^*(\mathcal{A}) \cong C^*(E,C)$. Therefore

$$C_r^*(E,C) \cong C^*(E,C)_{P,r} \cong C^*(\mathcal{A})_{P,r} \cong C_r^*(\mathcal{A}/\mathcal{J})$$

as desired.

□

Example 4.4.7. As we can see in Remark 3.2.24, for the separated graph (A_n, D) we have $C^*(A_n, D) \cong C^*(\mathbb{F}_n)$ where $\mathbb{F}_n$ is the free group generated by the n edges. Remember that $\tilde{\tau} : C^*(\mathbb{F}_n) \to \mathbb{C}$ denotes the canonical trace that we here view as a conditional expectation. In this context, $\mathcal{L}_{\tilde{\tau}} = \mathrm{Ker}(\Lambda^G) \neq 0$ since $\mathbb{F}_n$ is not amenable but $\mathcal{L}_{\tilde{\tau}} \cap \mathcal{A}_1 = 0$ because $\mathcal{A}_1$ coincides with $\mathbb{C}$. Then the reduced C*-algebra $C_r^*(\mathcal{A}/\mathcal{J})$ constructed above coincides with $C_r^*(\mathcal{A}) \cong C_r^*(\mathbb{F}_n)$ since in this case $\mathcal{J} = 0$.

Remark 4.4.8. Note that $J = \mathcal{L}_P \cap \mathcal{A}_1$ depends on the choices of the labeling function c and the group as well. Here is an example where $J \neq 0$. Consider the separated graph $(E(n,n), C(n,n))$ as we can seen in Example 3.1.6, consider $\mathbb{F}_n$ the free group generated by the n edges and $c : E^1 \to \mathbb{F}_n$ as $c(e_i) = g_i = c(f_i)$ for all $i \in \{1, \dots, n\}$. In this situation elements of the form $S_{e_i}S_{f_i}^*S_{e_j}S_{f_j}^* \in J$. The reason is because on the one hand $S_{e_i}S_{f_i}^*S_{e_j}S_{f_j}^* \in \mathcal{A}_1$ since $c(e_i)c(f_i)^{-1}c(e_j)c(f_j)^{-1} = g_i g_i^{-1} g_j g_j^{-1} = 1$. On the other hand, set $x = S_{e_i}S_{f_i}^*S_{e_j}S_{f_j}^*$ and note that

$$P(x^*x) = P(S_{f_j}S_{e_j}^*S_{f_i}S_{e_i}^*S_{e_i}S_{f_i}^*S_{e_j}S_{f_j}^*) = P(S_{f_j}S_{e_j}^*S_{f_i}S_{f_i}^*S_{e_j}S_{f_j}^*) = 0$$

since $e_i \neq f_j$ for every i, j. So, $x \in \mathcal{L}_P$ and hence J is non-zero ideal.

Remark 4.4.9. Note that the normalization of δ_c is in fact something between the full and reduced separated graph C*-algebras since $C_r^*(\mathcal{A})$ is the normalization of $C^*(\mathcal{A})$ as we can see below:

$$\begin{tikzcd}
C^*(\mathcal{A}) \arrow[r, two heads] \arrow[d, "\cong"] & C_r^*(\mathcal{A}) \arrow[r, two heads] \arrow[l, bend right, "\text{normalization}"'] & C_r^*(\mathcal{A}/\mathcal{J}) \arrow[d, "\cong"] \\
C^*(E,C) \arrow[r, dashed] & C_?^*(E,C) \arrow[r, dashed] \arrow[l, bend left, "\text{normalization}"] & C_r^*(E,C)
\end{tikzcd}$$

The same happens for δ_c^r where the maximization of this normal coaction is in fact $C^*(\mathcal{A}/\mathcal{J})$:

$$\begin{tikzcd}
C^*(\mathcal{A}) \arrow[r, two heads] \arrow[d, "\cong"] & C^*(\mathcal{A}/\mathcal{J}) \arrow[r, two heads] \arrow[r, bend left, "\text{maximalization}"] & C_r^*(\mathcal{A}/\mathcal{J}) \arrow[d, "\cong"] \\
C^*(E,C) \arrow[r, dashed] & C_?^*(E,C) \arrow[r, dashed] \arrow[r, bend right, "\text{maximalization}"'] & C_r^*(E,C)
\end{tikzcd}$$

5 TAME C*-ALGEBRAS OF SEPARATED GRAPHS

In this chapter, we present another C*-algebra associated with a separated graph called the tame separated graph C*-algebra. The motivation to study this particular C*-algebra is because the C*-algebra $C^*(E,C)$ seen before is quite different compared to the usual graph C*-algebras for non-separated graphs. One of the reasons is that the final projections of partial isometries coming from different sets of the partitions of C_v do not need to commute. In order to remedy this problem, a different C*-algebra was considered in [3], denoted by $\mathcal{O}(E,C)$. The goal of this chapter is to extend some results seen in Chapter 3 to the tame C*-algebra $\mathcal{O}(E,C)$.

Definition 5.0.1. A set S of partial isometries in a *-algebra A is said to be *tame* if every element of $U = \langle S \cup S^* \rangle$, the multiplicative semigroup generated by $S \cup S^*$, is a partial isometry.

Remark 5.0.2. As can be seen from [[26], Lemma 5.3], the product uv is a partial isometry if and only if u^*u and vv^* commute for u,v partial isometries. In fact, by [[26], Proposition 5.4], $u \in U$ is a partial isometry if and only if for all $u,u' \in U$ the elements uu^* and $u'u'^*$ commute.

Now, let (E,C) be a separated graph, $S = \{S_e\}_{e\in E^1}$ be the generating family of partial isometries of $C^*(E,C)$ and let U be the multiplicative semigroup of $C^*(E,C)$ generated by $S \cup S^*$. For $u \in U$ we write $e(u) := uu^*$.

Definition 5.0.3. With notations as above, the *tame graph C*-algebra* of (E,C) is defined as the C*-algebra

$$\mathcal{O}(E,C) := C^*(E,C)/J$$

where J is the closed two-sided ideal of $C^*(E,C)$ generated by all the commutators $[e(u),e(u')]$ for $u,u' \in U$.[1]

Remark 5.0.4. By Remark 5.0.2, the ideal $J = \langle [e(u),e(u')] \mid u,u' \in U\rangle = \langle e(u)u - u \mid u \in U\rangle$ where the last one is the closed two sided ideal generated by $e(u)u - u$ for every $u \in U$.

Also, considering J^{ab} as the algebraic (two-sized) ideal of $L(E,C)$ generated by all commutators $[e(u),e(u')]$ for $u,u' \in U$ we define:

$$L^{ab}(E,C) := L(E,C)/J^{ab}.$$

In this way we can consider the C*-algebra $\mathcal{O}(E,C)$ as the enveloping C*-algebra of $L^{ab}(E,C)$.

Observe that J precisely forces the set S to be tame in these quotients. To understand better let us compare with the case of non-separated graphs. In this case every element of the multiplicative semigroup U is a partial isometry, that is, the ideal J is equal to zero. In fact,

[1] Remember that $[e(u),e(u')] := e(u)e(u') - e(u')e(u)$.

$U = \{S_\mu S_\nu^* \mid \mu, \nu \in \mathsf{Path}(E) \text{ with } r(\mu) = r(\nu)\}$ essentially by Lemma 2.2.6. More precisely, if we take $u = S_\mu S_\nu^*$ and $u' = S_{\mu'} S_{\nu'}^*$ with $r(\mu) = r(\nu)$ and $r(\mu') = r(\nu')$, we have

$$\begin{aligned} e(u) = uu^* &= S_\mu S_\nu^* S_\nu S_\mu^* \\ &= S_\mu P_{r(\nu)} S_\mu^* \\ &= S_\mu S_\mu^* \end{aligned}$$

Similarly, $e(u') = u'u'^* = S_{\mu'} S_{\mu'}^*$. By Lemma 2.2.6 the commutator is zero. For example, for the case where $\mu' = \mu\eta$ for some $\eta \in \mathsf{Path}(E)$ we have

$$\begin{aligned} e(u)e(u') = uu^*u'u'^* &= S_\mu S_\mu^* S_{\mu'} S_{\mu'}^* \\ &= S_\mu S_\mu^* S_\mu S_\eta S_\eta^* S_\mu^* \\ &= S_\mu S_\eta S_\eta^* S_\mu^* \\ &= S_\mu S_\eta S_\eta^* S_\mu^* S_\mu S_\mu^* \\ &= S_{\mu'} S_{\mu'}^* S_\mu S_\mu^* \\ &= uu^*uu'^* = e(u')e(u) \end{aligned}$$

Others cases are analogous. However, in the case of separated graphs, the situation changes. We can see that not every element of U is a partial isometry because, for example, if $e, f \in E^1$ with $e \in X$ and $f \in Y$ for $X, Y \in C_v$, then $S_e^* S_f$ is not always a partial isometry because $S_e S_e^*$ does not need to commute with $S_f S_f^*$.

Now, let $c : E^1 \to G$ be a labeling function and consider the associated coaction δ_c of G on $C^*(E, C)$. The natural question for us is: Can we factor the coaction δ_c of $C^*(E, C)$ through $\mathcal{O}(E, C)$?

The following will give an answer to this question.

Proposition 5.0.5. *With definitions as above, J is a strongly δ_c-invariant ideal. Moreover, the generating commutators of J are fixed by δ_c, that is, $\delta_c(x) = x \otimes 1$ for all commutators $x = [e(u), e(u')]$ with $u, u' \in U$.*

Proof. It is straightforward to check that for every $u \in U$ we have $\delta_c(u) = u \otimes c(u)$ by definition. There is a slight abuse of notation here when we write $c(u)$. For example, if $u = S_\mu S_\nu^*$ then $c(u) := c(\mu)c(\nu)^{-1}$. We continue to use this notation for simplicity. Observe that for all $u \in U$ we have $\delta_c(e(u)) = \delta_c(uu^*) = \delta_c(u)\delta_c(u)^* = e(\delta_c(u))$ since δ_c is a *-homomorphism. We claim that all commutators are fixed by the coaction δ_c, that is, $\delta_c([e(u), e(u')]) = [e(u), e(u')] \otimes 1$

for all $u, u' \in U$. We compute:

$$\begin{aligned}
\delta_c([e(u), e(u')]) &= \delta_c(e(u)e(u') - e(u')e(u)) \\
&= \delta_c(e(u))\delta_c(e(u')) - \delta_c(e(u'))\delta_c(e(u)) \\
&= e(\delta_c(u))e(\delta_c(u')) - e(\delta_c(u'))e(\delta_c(u)) \\
&= \delta_c(u)\delta_c(u)^*\delta_c(u')\delta_c(u')^* - \delta_c(u')\delta_c(u')^*\delta_c(u)\delta_c(u)^* \\
&= uu^*u'u'^* \otimes \underbrace{c(u)c(u)^{-1}c(u')c(u')^{-1}}_{1} - u'u'^*uu^* \otimes \underbrace{c(u')c(u')^{-1}c(u)c(u)^{-1}}_{1} \\
&= (uu^*u'u'^* - u'u'^*uu^*) \otimes 1 \\
&= [e(u), e(u')] \otimes 1
\end{aligned}$$

Therefore, $\delta_c(x) = x \otimes 1$ for all commutators x in J. Now, to see that J is a strongly δ_c-invariant ideal we are going to use the nondegeneracy of δ_c. Set $A := C^*(E, C)$, consider X the set of all commutators and thus we have $J = \overline{AXA}$, meaning the closed linear span of elements axb for $a, b \in A$ and $x \in X$. First of all, note that

$$\delta_c(axb) = \delta_c(a)(x \otimes 1)\delta_c(b) \in (A \otimes C^*(G))(X \otimes 1)(A \otimes C^*(G)) \subseteq AXA \otimes C^*(G).$$

By linearity and continuity we have $\overline{\delta_c(J)(1 \otimes C^*(G))} \subseteq J \otimes C^*(G)$. On the other hand, for every $y \in J$, y can be approximately by an element of the form $\sum_i a_i x_i b_i$ where $a_i, b_i \in A$ and $x_i \in X$. Then,

$$\begin{aligned}
\sum_i a_i x_i b_i \otimes g &= \sum_i (a_i x_i \otimes g)(b_i \otimes 1) = \sum_i \sum_j (a_i x_i \otimes g)\delta_c(b_i^j)(1 \otimes g_j) \\
&= \sum_i \sum_j (a_i \otimes g)(x_i \otimes 1)\delta_c(b_i^j)(1 \otimes g_j) \\
&= \sum_i \sum_j (a_i \otimes g)\delta_c(x_i)\delta_c(b_i^j)(1 \otimes g_j) \\
&= \sum_i \sum_j (a_i \otimes g)\delta_c(x_i b_i^j)(1 \otimes g_j) \\
&= \sum_i \sum_j \sum_k (1 \otimes g_k)\delta_c(a_i^k)\delta_c(x_i b_i^j)(1 \otimes g_j) \\
&= \sum_i \sum_j \sum_k \delta_c(a_i^k x_i b_i^j)(1 \otimes g_k g_j)
\end{aligned}$$

where here for each i we used that $b_i \otimes 1$ can be approximated by $\sum_j \delta_c(b_i^j)(1 \otimes g_j)$ and $a_i \otimes g$ by $\sum_k (1 \otimes g_j)\delta_c(a_i^k)$ since δ_c is nondegenerate. With this, we conclude that $\overline{\delta_c(J)(1 \otimes C^*(G))} = J \otimes C^*(G)$ as desired.

□

It follows from Proposition 2.6.16 that δ_c factors through a coaction δ_c^J on the quotient $\mathcal{O}(E, C)$. Therefore we can consider the crossed product $\mathcal{O}(E, C) \rtimes_{\delta_c^J} G$. Moreover, the quotient map $q : C^*(E, C) \twoheadrightarrow \mathcal{O}(E, C)$ is $\delta_c - \delta_c^J$ equivariant and, consequently, we get a surjective *-homomorphism $q \rtimes G : C^*(E,C) \rtimes_{\delta_c} G \twoheadrightarrow \mathcal{O}(E,C) \rtimes_{\delta_c^J} G$ such that $q \rtimes G(j_B(a)j_G^B(f)) = j_B \circ q(a)j_G(f)$ for all $a \in C^*(E,C)$ and $f \in C_0(G)$.

Theorem 5.0.6. *Let (E,C) be a separated graph and let $c : E^1 \to G$ be a labeling function. Then*

$$\mathcal{O}(E \times_c G, C \times_c G) \cong \mathcal{O}(E,C) \rtimes_{\delta_c^J} G.$$

Under this isomorphism, the action γ on $C^*(E \times_c G, C \times_c G)$ induced by the translation action on $(E \times_c G, C \times_c G)$ factors through $\mathcal{O}(E \times_c G, C \times_c G)$ and corresponds to the dual action $\widehat{\delta_c^J}$ on $\mathcal{O}(E,C) \rtimes_{\delta_c^J} G$.

Proof. Consider $(q \rtimes G) \circ \phi : C^*(E \times_c G, C \times_c G) \twoheadrightarrow \mathcal{O}(E,C) \rtimes_{\delta_c^J} G$ where ϕ is the isomorphism seen in Theorem 3.2.17 and denote by $Q := q \rtimes G \circ \phi$. Also, denote it by J' the (closed two-sided) ideal of $C^*(E \times_c G, C \times_c G)$ generated by all commutators to not cause confusion with J. To start with, we need to check that $J' \subseteq \mathrm{Ker}(Q)$ and for this it is enough to show that for every commutator $[e(u), e(u')] \in J'$ we have $Q([e(u), e(u')]) = 0$. For simplicity, let us assume that $u = S_{(\mu,g)}S^*_{(\nu,z)}$ and $u' = S_{(\mu',g')}S^*_{(\nu',z')}$ with $r(\mu) = r(\nu)$, $r(\mu') = r(\nu')$, $\mu, \nu, \mu', \nu' \in \mathrm{Path}(E)$ possibly in different subsets of C and $z = gc(\mu)c(\nu)^{-1}$, $z' = gc(\mu')c(\nu')^{-1}$. The proof for $u = S^*_{(\mu,g)}S_{(\nu,g)}$ with $s(\mu) = s(\nu)$ and μ, ν possibly in different sets of C_v is more extensive but follows the same ideas. Then

$$e(u) = uu^* = S_{(\mu,g)}S^*_{(\nu,z)}S_{(\nu,z)}S^*_{(\mu,g)} = S_{(\mu,g)}P_{r(\nu,z)}S^*_{(\mu,g)} = S_{(\mu,g)}S^*_{(\mu,g)}$$

since $r(\nu, z) = (r(\nu), zc(\nu)) = (r(\mu), gc(\mu)) = r(\mu, g)$. Similarly we get $e(u') = u'u'^* = S_{(\mu',g')}S^*_{(\mu',g')}$. Note that:

$$\begin{aligned}\phi(e(u)) &= \phi(S_{(\mu,g)}S^*_{(\mu,g)}) \\ &= s_{(\mu,g)}s^*_{(\mu,g)} \\ &= j_B(S_\mu S^*_\mu)j_G^B(\chi_{g^{-1}}).\end{aligned}$$

Hence, using the same conditions 3.2.18 seen in the proof of Theorem 3.2.17 we get

$$\begin{aligned}\phi([e(u), e(u')]) &= \phi(uu^*u'u'^* - u'u'^*uu^*) \\ &= \phi(uu^*)\phi(u'u'^*) - \phi(u'u'^*)\phi(uu^*) \\ &= j_B(S_\mu S^*_\mu)j_G^B(\chi_{g^{-1}})j_B(S_{\mu'}S^*_{\mu'})j_G^B(\chi_{g'^{-1}}) - j_B(S_{\mu'}S^*_{\mu'})j_G^B(\chi_{g'^{-1}})j_B(S_\mu S^*_\mu)j_G^B(\chi_{g^{-1}}) \\ &= j_B(S_\mu S^*_\mu S_{\mu'}S^*_{\mu'})j_G^B(\chi_{g^{-1}})j_G^B(\chi_{g'^{-1}}) - j_B(S_{\mu'}S^*_{\mu'}S_\mu S^*_\mu)j_G^B(\chi_{g'^{-1}})j_G^B(\chi_{g^{-1}}) \\ &= j_B(S_\mu S^*_\mu S_{\mu'}S^*_{\mu'} - S_{\mu'}S^*_{\mu'}S_\mu S^*_\mu)j_G^B(\chi_{g^{-1}}) \\ &= j_B([e(w), e(w')])j_G^B(\chi_{g^{-1}})\end{aligned} \tag{5.0.7}$$

where we used here that $\chi_{g^{-1}} \cdot \chi_{g'^{-1}}$ is non-zero if and only if $g = g'$ and $[e(w), e(w')] \in J$ since we can write $w = S_\mu S_\nu^*$ and $w' = S_{\mu'} S_{\nu'}^*$. Therefore

$$\begin{aligned} Q([e(u), e(u')]) &= q \rtimes G \circ \phi([e(u), e(u')]) \\ &= q \rtimes G(j_B([e(w), e(w')]) j_G^B(\chi_{g^{-1}})) \\ &= j_B(q([e(w), e(w')])) j_G^B(\chi_{g^{-1}}) \\ &= 0 \end{aligned}$$

This shows that ϕ factors through a surjective *-homomorphism

$$\tilde{\phi} : \mathcal{O}(E \times_c G, C \times_c G) \twoheadrightarrow \mathcal{O}(E, C) \rtimes_{\delta_c^J} G$$

such that $\tilde{\phi}(P_{(v,g)}) = p_{(v,g)}$ and $\tilde{\phi}(S_{(e,g)}) = s_{(e,g)}$ for all $v \in E^0$, $e \in E^1$ and $g \in G$.

To see that $\tilde{\phi}$ is injective, we are going to use the same idea as in Theorem 3.2.17, that is, we need to get a covariant representation of $(\mathcal{O}(E, C), C_0(G))$ to $\mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$. For this, consider the covariant representation (π, σ) of $(C^*(E, C), C_0(G))$ to $\mathcal{M}(C^*(E \times_c G, C \times_c G))$ defined in Theorem 3.2.17 and composed with the canonical map $\mathcal{M}(C^*(E \times_c G, C \times_c G)) \to \mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$. We need to prove that the covariant representation (π, σ) of $(C^*(E, C), C_0(G))$ to $\mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$ factors through on $\mathcal{O}(E, C)$. Again, for simplicity let us assume that $u = S_\mu S_\nu^*$ and $u' = S_{\mu'} S_{\nu'}^*$ with $r(\mu) = r(\nu)$, $r(\mu') = r(\nu')$ and μ, μ' eventually in different sets of C. The general case is analogous. Then

$$e(u) = uu^* = S_\mu S_\nu^* S_\nu S_\mu^* = S_\mu P_{r(\nu)} S_\mu^* = S_\mu S_\mu^*.$$

Similarly we get $e(u') = u'u'^* = S_{\mu'} S_{\mu'}^*$. Hence we have

$$\begin{aligned} \pi(e(u)) &= \pi(S_\mu S_\mu^*) \\ &= \left(\sum_{g \in G} S_{(\mu,g)} \right) \left(\sum_{h \in G} S_{(\mu,h)}^* \right) \\ &= \sum_{g \in G} S_{(\mu,g)} S_{(\mu,g)}^* \end{aligned}$$

where here we used that the last summand is non-zero if and only if $g = h$. Therefore

$$\begin{aligned} \pi([e(u), e(u')]) &= \pi(uu^*u'u'^* - u'u'^*uu^*) \\ &= \pi(uu^*)\pi(u'u'^*) - \pi(u'u'^*)\pi(uu^*) \\ &= \left(\sum_{g \in G} S_{(\mu,g)} S_{(\mu,g)}^* \right) \left(\sum_{h \in G} S_{(\mu',h)} S_{(\mu',h)}^* \right) - \left(\sum_{h \in G} S_{(\mu',h)} S_{(\mu',h)}^* \right) \left(\sum_{g \in G} S_{(\mu,g)} S_{(\mu,g)}^* \right) \\ &= \left(\sum_{g \in G} S_{(\mu,g)} S_{(\mu,g)}^* S_{(\mu',g)} S_{(\mu',g)}^* - S_{(\mu',g)} S_{(\mu',g)}^* S_{(\mu,g)} S_{(\mu,g)}^* \right) \\ &= [e(w), e(w')] \end{aligned}$$

where the products of summands above are non-zero if $h = g$ and μ, μ' possibly belongs to different sets in the same C_v, $v \in E^0$. So, $[e(v), e(v')]$ is in fact a commutator when we consider $w = \sum_{g\in G} S_{(\mu,g)}S_{(\nu,z)}$ and $w' = \sum_{h\in G} S_{(\mu',h)}S_{(\nu',z')}$ with $z = gc(\mu)c(\nu)^{-1}$ and $z' = hc(\mu')c(\nu')^{-1}$. Therefore, since $[e(v), e(v')] \in J'$ we conclude that $J \subseteq \mathrm{Ker}(\pi)$ and in this way π factors through a *-homomorphism $\pi' : \mathcal{O}(E, C) \to \mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$ satisfying

$$\pi'(P_v) = \sum_{g\in G} P_{(v,g)} \quad \text{and} \quad \pi'(S_e) = \sum_{g\in G} S_{(e,g)}$$

for all $v \in E^0$, $e \in E^1$ and $g \in G$. For the representation of $C_0(G)$ we consider the same σ as in Theorem 3.2.17 by looking into $\mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$, that is, define $\sigma' : C_0(G) \to \mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$ such that

$$\sigma'(\chi_g) = \sum_{v\in E^0} P_{(v,g^{-1})}$$

for all $g \in G$. Similarly as in Theorem 3.2.17 it is straightforward to check that (π', σ') is a covariant representation for $(\mathcal{O}(E, C), G, \delta_c^J)$ into $\mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$. Therefore by the universal property there is a *-homomorphism

$$\psi' := \pi' \times \sigma' : \mathcal{O}(E, C) \rtimes_{\delta_c^J} G \to \mathcal{M}(\mathcal{O}(E \times_c G, C \times_c G))$$

such that $\pi' \times \sigma' \circ j_B = \pi'$ and $\pi' \times \sigma' \circ j_G^B = \sigma'$. The same argument seen in Theorem 3.2.17 shows that $\psi' \circ \phi' = id$ as desired.

Finally, to see the action γ factors through $\mathcal{O}(E \times_c G, C \times_c G)$ it is enough to show that J is γ-invariant ideal. But, it is clear that $\gamma_g(J) = J$ since $\gamma_g([e(u), e(u')]) = [e(\gamma_g(u)), e(\gamma_g(u'))]$ because γ_g is *-homomorphism and the action is given by the left multiplication.

The proof of G-equivariance of the actions is the same as in Theorem 3.2.17 since on generators the covariant representation does not change.

□

Remark 5.0.8. In [49] Nilsen shows that for any dynamical co-system (A, G, δ) and I a strongly δ-invariant ideal, $I \rtimes_{\delta_I} G$ is an ideal of $A \rtimes_\delta G$ and there is a short exact sequence

$$0 \to I \rtimes_{\delta_I} G \to A \rtimes_\delta G \to A/I \rtimes_{\delta^I} G \to 0.$$

Since J is a strongly δ_c-invariant ideal of $C^*(E, C)$ then using Nilsen's results it follows that $J \rtimes_{(\delta_c)_J} G$ is an ideal of $C^*(E, C) \rtimes_{\delta_c} G$ and we get

$$\mathcal{O}(E, C) \rtimes_{\delta_c^J} G \cong (C^*(E, C) \rtimes_{\delta_c} G)/(J \rtimes_{(\delta_c)_J} G).$$

The calculation 5.0.7 seen in proof of the theorem above tell us that $\phi(J')$ is in fact the ideal $J \rtimes_{\delta_J} G$. So, we could give an alternative proof of the theorem by observing that the diagram below commutes:

$$\begin{array}{ccc} C^*(E \times_c G, C \times_c G) & \xrightarrow{\phi} & C^*(E,C) \rtimes_{\delta_c} G \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q \rtimes G} \\ \mathcal{O}(E \times_c G, C \times_c G) & \xrightarrow{\tilde{\phi}} & \mathcal{O}(E,C) \rtimes_{\delta_c^J} G \end{array}$$

Since ϕ is an isomorphism and $\phi(J')$ is equal to $J \rtimes_{\delta_J} G$ it follows that $\tilde{\phi}$ is an isomorphism, as desired.

Now, the ideal J of $C^*(E,C)$ is a special ideal in this context. In Proposition 5.0.5 we observe that all commutators that generate J are contained on the unit fiber $\mathcal{A}_1 = C^*(E,C)_1$ since $\delta_c(x) = x \otimes 1$ for all $x \in X$. Now we are going to use the same idea seen in Section 4.4. Consider the ideal of $\mathcal{A}_1$ as the intersection of J with the unit fiber $\mathcal{A}_1$ and denote it by I. Because of this, I is an $\mathcal{A}$-invariant ideal and we consider $\mathcal{I}_g = \mathcal{A}_g \cdot I = I \cdot \mathcal{A}_g$ for every $g \in G$. Moreover, $\mathcal{I}_g = J \cap \mathcal{A}_g$ for every $g \in G$. Thus we get the quotient Fell bundle $\mathcal{A}/\mathcal{I} = \{\mathcal{A}_g/\mathcal{I}_g\}_{g \in G}$.

Lemma 5.0.9. *With notations as above, we have*

$$\mathcal{O}(E,C) \cong C^*(\mathcal{A}/\mathcal{I}).$$

Moreover, J is isomorphic to $C^*(\mathcal{I})$ and the coaction δ_c^J of G on $\mathcal{O}(E,C)$ coincides with the canonical coaction $\delta_{\mathcal{A}/\mathcal{I}}$ of G on $C^*(\mathcal{A}/\mathcal{I})$.

Proof. To begin with, consider the canonical morphism $q^J = \{q_g^J\}_{g \in G}$ from $\mathcal{A}$ to $\mathcal{A}/\mathcal{I}$ where $q_g^J : \mathcal{A}_g \to \mathcal{A}_g/\mathcal{I}_g$ and the induced map $\overline{q^J} : C^*(\mathcal{A}) \to C^*(\mathcal{A}/\mathcal{I})$ such that $\overline{q^J} \circ j_g^{\mathcal{A}} = j_g^{\mathcal{A}/\mathcal{I}} \circ q_g^J$ for all $g \in G$. Since K acts trivially on the fibers by construction, to define K' such that the diagram below commutes

$$\begin{array}{ccc} C^*(E,C) & \xrightarrow{K} & C^*(\mathcal{A}) \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle \overline{q^J}} \\ \mathcal{O}(E,C) & \xrightarrow{K'} & C^*(\mathcal{A}/\mathcal{I}) \end{array}$$

it is enough to show that $J \subseteq \mathrm{Ker}(\overline{q^J} \circ K)$. For this, consider a generating commutator $x \in J$ and note that $x \in \mathcal{A}_1$. So, $\overline{q^J} \circ K(x) = \overline{q^J}(j_1^{\mathcal{A}}(x)) = j_1^{\mathcal{A}/\mathcal{I}} \circ q_1(x) = 0$. Then by linearity and continuity we have $J \subseteq \mathrm{Ker}(\overline{q^J} \circ K)$ and K' is a well-defined map. The surjectivity is directly.

Now, to complete the proof we need to show that there is a well-defined inverse left to K'. To do that, recall that we have the canonical representation σ of $\mathcal{A}$ in $C^*(E,C)$ given by the inclusions $\sigma_g : \mathcal{A}_g \hookrightarrow C^*(E,C)$ and therefore consider $q \circ \sigma_g : \mathcal{A}_g \to \mathcal{O}(E,C)$ as a new representation of $\mathcal{A}$ on $\mathcal{O}(E,C)$. In fact, these morphisms factors through to a representation $\mathcal{A}_g/\mathcal{I}_g \to \mathcal{O}(E,C)$ and denote by l_g, that is, $l_g \circ q_g = q \circ \sigma_g$ for every $g \in G$ by construction. Indeed, $q \circ \sigma_g$ factors because q cancel any element of J which consequently cancel any element of $\mathcal{I}_g$. Now, we can consider the induced map $\bar{l} : C^*(\mathcal{A}/\mathcal{I}) \to \mathcal{O}(E,C)$ such that $\bar{l} \circ j_g^{\mathcal{A}/\mathcal{I}} = l_g$ for every $g \in G$. We claim that $\bar{l}$ in the inverse of K'. To see that it is enough to show that $\bar{l} \circ K'(x) = x$ for every $x = q(y)$ for some $y \in L(E,C)_g$. But observe that

$$\begin{aligned}
\bar{l} \circ K'(q(y)) &= \bar{l} \circ \overline{q^J} \circ K(y) \\
&= \bar{l} \circ \overline{q^J} \circ j_g^{\mathcal{A}}(y) \\
&= \bar{l} \circ j_g^{\mathcal{A}/\mathcal{I}} \circ q_g(y) \\
&= l_g \circ q_g(y) \\
&= q \circ \sigma_g(y) \\
&= q(y)
\end{aligned}$$

Therefore, K' is an isomorphism as desired.

Since by [[25], Proposition 21.15] we have the exact short sequence of C*-algebras $0 \to C^*(\mathcal{J}) \to C^*(\mathcal{A}) \to C^*(\mathcal{A}/\mathcal{J}) \to 0$, by the isomorphism below we have $J \cong C^*(\mathcal{I})$.

Finally, for the coincidence of coactions we just need to show that the isomorphism is G-equivariant with respect to δ_c^J and $\delta_{\mathcal{A}/\mathcal{I}}$. In other words, we need to show $(\bar{l} \otimes id_G) \circ \delta_{\mathcal{A}/\mathcal{I}} = \delta_c^J \circ \bar{l}$. To see that it enough to show for $j_g^{\mathcal{A}/\mathcal{I}}(a)$, $g \in G$ since the closed linear span of these elements is dense in $C^*(\mathcal{A}/\mathcal{I})$. Observe that

$$\begin{aligned}
(\bar{l} \otimes id_G) \circ \delta_{\mathcal{A}/\mathcal{I}}(j_g^{\mathcal{A}/\mathcal{I}}(a)) &= (\bar{l} \otimes id_G)(j_g^{\mathcal{A}/\mathcal{I}}(a) \otimes g) \\
&= (\bar{l} \circ j_g^{\mathcal{A}/\mathcal{I}}(a)) \otimes g \\
&= l_g(a) \otimes g \\
&= q \circ \sigma_g(a) \otimes g \\
&= (q \otimes id_G)(\sigma_g(a) \otimes g) \\
&= (q \otimes id_G) \circ \delta_c(\sigma_g(a)) \\
&= \delta_c^J(q \circ \sigma_g(a)) \\
&= \delta_c^J(l_g(a)) \\
&= \delta_c^J(\bar{l} \circ j_g^{\mathcal{A}/\mathcal{I}}(a))
\end{aligned}$$

This completes the proof.

□

Remark 5.0.10. An immediate consequence of the lemma above is that the fibers $O(E,C)_g$ is isomorphic to $\mathcal{A}_g/\mathcal{I}_g$.

Remark 5.0.11. Since $J \cong C^*(\mathcal{I})$ this shows us that $\oplus_{g \in G}\mathcal{I}_g$ is dense in J which means that J is an induce ideal in the Exel's sense seen in [25].

Corollary 5.0.12. *With notations as above, δ_c^J is a maximal coaction, that is,*

$$\mathcal{O}(E,C) \rtimes_{\delta_c^J} G \rtimes_{\hat{\delta_c^J}} G \cong \mathcal{O}(E,C) \otimes \mathcal{K}(l^2(G)).$$

Proof. This follows from the previous lemma.

□

Corollary 5.0.13. *For any separated graph (E, C) and any labeling $c : E \to G$, there is a canonical isomorphism*

$$\mathcal{O}(E \times_c G, C \times_c G) \rtimes_\gamma G \cong \mathcal{O}(E, C) \otimes \mathcal{K}(l^2(G))$$

where γ is the action of G on $C^*(E \times_c G, C \times_c G)$ induced by the translation action on $(E \times_c G, C \times_c G)$.

Proof. Follows from the Theorem 5.0.6 and Corollary 5.0.12.

□

Corollary 5.0.14. *For a free action θ of a group G on a separated graph (E, C), there is a canonical isomorphism*

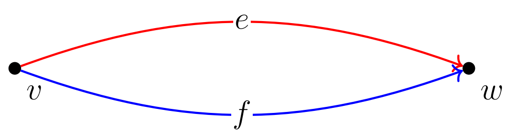
$$\mathcal{O}(E, C) \rtimes_\theta G \cong \mathcal{O}(E/G, C/G) \otimes \mathcal{K}(l^2(G)).$$

Proof. Follows from Corollary 5.0.13 and the Gross-Tucker theorem for separated graphs 3.1.7.
□

Remark 5.0.15. Compiling all results that we have seen so far we get have the following diagram of isomorphisms:

$$\begin{tikzcd}
\mathcal{O}(E \times_c G, C \times_c G) \rtimes_\gamma G \arrow[rr, "5.0.6"] \arrow[dr] & & \mathcal{O}(E, C) \rtimes_{\delta_c^r} G \rtimes_{\hat{\delta}_c^r, r} G \arrow[dl, "5.0.12"] \\
& \mathcal{O}(E, C) \otimes \mathcal{K}(l^2(G)) &
\end{tikzcd}$$

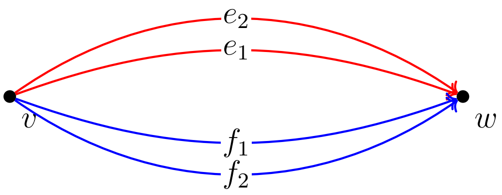
Example 5.0.16. Consider the separated graph $(E(1,1), C(1,1))$ seen in Example 3.1.6 as in the picture below:



According to [[5], Proposition 2.12], we have an isomorphism

$$C^*(E(1,1), C(1,1)) \cong M_2(U_{1,1}) \cong M_2(C(\mathbb{T})).$$

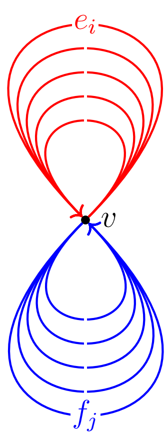
In this case it not difficult to see that $\mathcal{O}(E(1,1), C(1,1)) = C^*(E(1,1), C(1,1))$, that is, $J = 0$. The reason is because every "complicated" product of the form $S_e^* S_f$ commutes because $S_e S_e^* S_f S_f^* = P_w P_w = S_f S_f^* S_e S_e^*$. But if we change the graph a little bit, considering $(E(2,2), C(2,2))$ as in the picture below:



Then $C^*(E(2,2), C(2,2)) \cong M_2(U_{2,2})$ and in this case it is clear that the elements $S_e^* S_f$ do not commute. The ideal J is non zero but it is not trivial to describe.

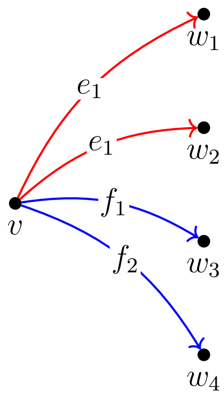
Example 5.0.17. Another example is the Cuntz separated graph seen in Example 3.2.11. We have already seen that $C^*(A_n, C) \cong C^*(\mathbb{F}_n)$, where $\mathbb{F}_n$ is the free group generated by the edges. In this case $J = 0$ and $\mathcal{O}(A_n, C) \cong C^*(A_n, C) \cong C^*(\mathbb{F}_n)$.

Example 5.0.18. Consider the separated graph seen in Example 4.1.9 as in picture below:



Here we have $C^*(E, C) \cong C^*(E_X) *_{\mathbb{C}} C^*(E_Y) \cong \mathcal{O}_n *_{\mathbb{C}} \mathcal{O}_m$. We know that $\mathcal{O}_n$ is simple but we can observe that the free product is not simple since the ideal $J \neq 0$.

Example 5.0.19. Another non-trivial example of a separated graph where there is a huge difference between $C^*(E, C)$ and $\mathcal{O}(E, C)$ is the following separated graph (E, C) draw as in picture below:



By [[3], Example 9.4] the corner $vC^*(E,C)v$ is isomorphic to the free product $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ but the corner $v\mathcal{O}(E,C)v$ is isomorphic to $\mathbb{C}^4$. Here we can see the impressive reduction from $C^*(E,C)$ to $\mathcal{O}(E,C)$.

5.1 ALTERNATIVE APPROACH TO $\mathcal{O}(E,C)$

Described as crossed products, the tame C*-algebras were developed by Ara and Exel in [3] for finite bipartite separated graphs and more generally by Lolk in [44] for finitely separated graphs as the following result:

Theorem 5.1.1 ([44], Theorem 2.10)**.** *For any finitely separated graph (E,C), there is canonical isomorphism*

$$\mathcal{O}(E,C) \cong C_0(\Omega(E,C)) \rtimes_\theta \mathbb{F}_n$$

where $\Omega(E,C)$ is a zero-dimensional locally compact Hausdorff space, θ is a topological partial action of $\mathbb{F}_n$ over $\Omega(E,C)$ and $C_0(\Omega(E,C)) \rtimes_\theta \mathbb{F}_n$ denotes the full crossed product.

Using this result we can give the following definition:

Definition 5.1.2. If (E,C) is a finitely separated graph, then the reduced tame C*-algebra is the reduced crossed product

$$\mathcal{O}^r(E,C) := C_0(\Omega(E,C)) \rtimes_{\theta,r} \mathbb{F}_n.$$

Following this alternative point of view in [3], [2] and [44] one can find many examples of tame separated graph C*-algebras described as crossed products and this facilitates the understanding of these C*-algebras. In this section, we focus to show that there is a maximal and normal coaction on $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$, respectively. Fist of all, let (E,C) be a finitely separated graph and let $c : E^1 \to G$ be a labeling function. It is straightforward to see that c extends to $\mathbb{F}_n$ in the obvious way: $c(w) := c(w_1)\ldots c(w_n)$ and $c(w^*) := c(w)^{-1}$ for any reduce word $\mu = w_1 \ldots w_n \in \mathbb{F}_n$. We use the abuse notation and denote also by $c : C^*(\mathbb{F}_n) \to C^*(G)$ its integrated form.

Proposition 5.1.3. *Let (E, C) be a finitely separated graph and $c : E^1 \to G$ be a labeling function. Then there is a maximal coaction ϵ on $\mathcal{O}(E, C)$ such that $\epsilon(P_v) = P_v \otimes 1$ and $\epsilon(S_e) = S_e \otimes c(e)$ for every $v \in E^0$ and $e \in E^1$.*

Proof. Consider $C_0(\Omega(E, C)) \rtimes_\theta \mathbb{F}_n$ as a C*-algebra $C^*(\mathcal{A}^\theta)$ where $\mathcal{A}^\theta$ is the semi-direct product bundle relative to the partial action θ as seen in Example 2.3.5. Set $\mathcal{B} := \mathcal{A}^\theta$ to make notation easier. We know that $\delta_\mathcal{B} : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*(\mathbb{F})$ is always a maximal coaction and the inflated coaction $\mathrm{Inf}(\delta_\mathcal{B}) : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*(G)$ is also maximal by Proposition 2.6.34. So, the candidate for ϵ is the maximal coaction $\mathrm{Inf}(\delta_\mathcal{B})$ over $\mathcal{O}(E, C)$ provided from the isomorphisms above.

In fact, $\delta_\mathcal{B} : C^*(\mathcal{B}) \to C^*(\mathcal{B}) \otimes C^*(\mathbb{F})$ acts on generators as $\delta_\mathcal{B}(a_w) = a_w \otimes w$ where $a_w \in D_w$, $w \in \mathbb{F}_n$ for any reduced word and the inflated coaction $\mathrm{Inf}(\delta_\mathcal{B})$ just applies $id \otimes c$, that is, $\mathrm{Inf}(\delta_\mathcal{B})(a_w) = a_w \otimes c(w)$ for every $w \in \mathbb{F}_n$. When we are looking at $C^*(\mathcal{A}^\theta)$ as the crossed product $C_0(\Omega(E, C)) \rtimes_\theta \mathbb{F}_n$ we identify a_w with $1_w u_w$ where 1_w denotes the indicator function on $\Omega_w(E, C)$. In this way the coaction has the same behavior on generators, that is, $\mathrm{Inf}(\delta_\mathcal{B})(1_w u_w) = 1_w u_w \otimes c(w)$.

Now, we can see from the proof of [[44], Theorem 2.10] that the isomorphism $\mathcal{O}(E, C) \cong C_0(\Omega(E, C)) \rtimes_\theta \mathbb{F}_n$ identifies P_v with $1_v u_1$ and S_μ with $1_\mu u_\mu$ where again here $1_v, 1_\mu$ denote the indicators functions on $\Omega_v(E, C)$ and $\Omega_\mu(E, C)$ for every reduced word $\mu \in \mathbb{F}_n$. So, $\mathrm{Inf}(\delta_\mathcal{B})(1_v u_1) = 1_v u_1 \otimes 1$ and $\mathrm{Inf}(\delta_\mathcal{B})(1_\mu u_\mu) = 1_\mu u_\mu \otimes c(\mu)$ and this coaction behaves in the same way as we desired.

□

Proposition 5.1.4. *Let (E, C) be a finitely separated graph, $c : E^1 \to G$ be a labeling function and ϵ the coaction of G on $\mathcal{O}(E, C)$ seen in Proposition 5.1.3. Then ϵ factors through a coaction ϵ^r over $\mathcal{O}^r(E, C)$. Moreover, ϵ^r is the normalization of ϵ.*

Proof. Consider $\mathcal{O}^r(E, C) \cong C_0(\Omega(E, C)) \rtimes_{\theta, r} \mathbb{F}_n$ as the C*-algebra $C^*_r(\mathcal{B})$ where $\mathcal{B} = \mathcal{A}^\theta$ is the same semi-direct product bundle relative to partial action θ seen in previous proposition. So, $\delta^r_\mathcal{B} : C^*_r(\mathcal{B}) \to C^*_r(\mathcal{B}) \otimes C^*(\mathbb{F})$ is a normal coaction and the inflated coaction $\mathrm{Inf}(\delta^r_\mathcal{B}) : C^*_r(\mathcal{B}) \to C^*_r(\mathcal{B}) \otimes C^*(G)$ is also normal by Proposition 2.6.35. By the way, $\mathrm{Inf}(\delta^r_\mathcal{B})$ is a normalization of $\mathrm{Inf}(\delta_\mathcal{B})$. Therefore it is enough to consider ϵ^r as the normal coaction $\mathrm{Inf}(\delta^r_\mathcal{B})$ over $\mathcal{O}^r(E, C)$ provided from the isomorphisms above and this finishes the proof.

□

6 CONCLUSION

The main tasks that inspired this project were (1) to obtain a version of the Gross-Tucker theorem allowing us to characterize exactly the separated graphs that carry a free group action and (2) prove some duality results involving separated graph C*-algebras, generalizing previous works on ordinary graph C*-algebras by A. Kumjian and D. Pask in [37] and also by S. Kaliszewski, J. Quigg, and I. Raeburn in [21] and [33].

The Gross-Tucker theorem says that every separated graph carrying a free group action is (isomorphic to) a skew product of another graph by the underlying group via a labeling function. On the other hand, a labeling function yields a coaction on the universal C*-algebra $C^*(E,C)$ of a separated graph (E,C). It became then more and more clear that the key to understanding this whole process was to realize the structure of C*-algebras of separated graphs and to study the correlation between coactions and the associated Fell bundles. One of our main results show that the coaction associated to a labeling function on the universal C*-algebra of a separated graph is always a maximal coaction. This is proved by using the universal properties of these C*-algebras and the associated Fell bundle C*-algebras.

After finishing the work on the universal separated graph C*-algebras $C^*(E,C)$, the second main task was to study similar problems and duality theorems for the corresponding reduced C*-algebras $C^*_r(E,C)$ of separated graphs. The original definition of these reduced C*-algebras is via amalgamated free products, but it soon became clear that this definition would make our work very hard since it is usually difficult to deal with free products. However, these products usually carry certain faithful conditional expectations and we then had the idea to realize reduced separated graph C*algebra as a certain reduced quotient of a C*-algebra associated with a canonical conditional expectation. Using this tool it was then easier to prove that our duality results on universal graph C*-algebra also factor through the reduced ones. More than that, we were also able to describe the Fell bundles associated with a given label on the level of the reduced C*-algebra as some quotient of the Fell bundle on the universal C*-algebra. This description also makes it clear that the coaction on $C^*_r(E,C)$, although always normal, is usually not the normalization of the maximal coaction on $C^*(E,C)$ and vice-versa. Indeed, this picture makes it clear that the normalization of $C^*(E,C)$ and the maximalization of $C^*_r(E,C)$ are certain "exotic" C*-algebras "living between them". Moreover, following similar ideas, it was also possible to show some of these results for the tame separated graph C*-algebras which are special quotients of $C^*(E,C)$.

The whole process during the development of the results of this work made it clear that C*-algebras associated with separated graphs form a very interesting class of C*-algebras and techniques to work with them usually involve theories from different fields of mathematics.

This work opens many new questions and stimulates new projects, some of which we present in the following.

1. Is there an algebraic version analogues of our Theorem 3.2.17?

2. What is the relationship of our results with the theory of generalized fixed-point algebras seen in [13]?

3. We show that the C*-algebras $C^*(E,C)$ and $C^*_r(E,C)$ can be viewed as cross-sectional C*-algebras for some Fell bundle once a labeling function $c : E^1 \to G$ on the graph is fixed. The resulting Fell bundle depends strongly on the group G and the label. This gives rise to the question of whether these C*-algebras can also be intrinsically described as Fell bundles even in the absence of a labeling function? Or maybe one could describe these C*-algebras intrinsically as section C*-algebras of Fell bundles over (inverse) semigroups?

4. Can one generalize the duality results proved in this work to locally compact groups?

Regarding the first question, there is a well-established concept of smash products associated with a G-graded algebra (see [14]) and this should play the algebraic role of the crossed product by a coaction. At least if the group G is finite, the idea should work. But for infinite groups, some technical issues might appear if we want to use the same analytic techniques that we have developed in this work since we have used many infinite sums and certain special convergent nets concerning the strict topology of a multiplier C*-algebra. For the second question, at least in the case of ordinary graph C*-algebras, there is already a relationship between the theory of free actions on graphs and the generalized fixed point algebras. More precisely, if G acts freely on E by α, the generalized fixed-point algebra $C^*(E)^\alpha$ is isomorphic to $C^*(E/G)$, see [50]. Moreover, the reduced crossed product $C^*(E) \rtimes_{\alpha,r} G$ is Morita equivalent to $C^*(E)^\alpha$. Does this remain valid for separated graphs?

Regarding the third question, we suspect that the C*-algebras of separated graphs can be probably described as section C*-algebras of Fell bundles over inverse semigroups. Maybe the theory of noncommutative Cartan pairs [see [24]] is useful for this proposal. This seems to be a very interesting project. Finally, regarding the fourth question, there is a well-established theory of coactions of locally compact groups, but it seems that the theory of separated graph C*-algebra does not trivially relate to the topological setting, maybe a theory of "topological separated graphs" needs to be developed for this.

I hope that this work will contribute significantly to the development of graph C*-algebras, especially the separated graph C*-algebras, and will make it possible to explore and answer many other related questions that were discussed and proposed above.

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Appendix

APPENDIX A – DYNAMICAL SYSTEMS

In this appendix, we will introduce some basic framework for the study of crossed products by actions to complement the similarities and differences between dynamical systems and dynamical co-systems. We decided to emphasize the discreteness of G throughout this appendix to remove any possible doubt regarding this fact. We have omitted some proofs which are widely known and the reader can find all these facts in [60] and [51].

From now one, fix an action α of a discrete group G on a C*-algebra A.

Definition A.0.1. A dynamical system (A, G, α) consists of a discrete group G, a C*-algebra A and an action α of G on A, that is, a homomorphism $\alpha : G \to \text{Aut}(A)$.

For a dynamical system (A, G, α), let $C_c(G, A)$ be the linear span of finitely supported functions on G with values in A, that is,

$$C_c(G,A) = \left\{ \sum_{g \in G} a_g u_g \mid a_g \in A \text{ where only finitely many } a_g' s \text{ are non-zero} \right\}$$

where $au_g : G \to A$ is defined by $au_g(h) = \delta_{g,h} a$. One equips $C_c(G, A)$ with a product and involution defined by

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_{g^{-1}}(a^*)u_{g^{-1}}$$

for all $a, b \in A$ and $g, h \in G$. Linearly extending these operations $C_c(G, A)$ becomes a *-algebra.

Definition A.0.2. Let B a C*-algebra. A covariant representation (π, μ) of (A, G, α) consists of a representation $\pi : A \to \mathcal{M}(B)$ and a unitary representation $\mu : G \to \mathcal{M}(B)$ such that

$$\pi(\alpha_g(a)) = \mu_g \pi(a) \mu_g^*$$

for every $a \in A$ and $g \in G$.

Proposition A.0.3. *Given a C*-algebra B, for every covariant representation (π, μ) of (A, G, α) there is a *-homomorphism $\pi \times \mu : C_c(G, A) \to \mathcal{M}(B)$, called the integrated form of (π, μ), such that*

$$\pi \times \mu \left(\sum_{g \in G} a_g u_g \right) = \sum_{g \in G} \pi(a_g) \mu_g.$$

It is not hard to see that $\pi \times \mu$ is well defined and satisfies $\|(\pi \times \mu)(x)\| \leqslant \|x\|_1$ for all $x \in C_c(G, A)$. The full C*-algebra norm on $C_c(G, A)$ is defined by

$$\|.\| := \sup \|(\pi \times \mu)(.)\|$$

where the supremum is taken over all covariant representations (π, μ) of (A, G, α).

Definition A.0.4. The full crossed product, denoted $A \rtimes_\alpha G$, is the completion of $C_c(G, A)$ with respect to the full C*-algebra norm.

Example A.0.5. A especial case is when $A = \mathbb{C}$ and α is the trivial action of G on $\mathbb{C}$, that is, $\alpha_g(z) = z$, for all $z \in \mathbb{C}$ and $g \in G$. We denote this action by "tr". Then it easy to see that $C^*(G) \cong \mathbb{C} \rtimes_{\text{tr}} G$.

Example A.0.6. More generally, if we have a trivial action of G on A then $A \rtimes_{tr} G$ is isomorphic to $A \otimes_{max} C^*(G)$.

Example A.0.7. Let $\theta \in \mathbb{R}$ and $\mathbb{Z}$ act on $\mathbb{T}$ by rotation through θ. This induces an action of $\mathbb{Z}$ on $C(\mathbb{T})$, that is, $R_\theta : \mathbb{Z} \to Aut(C(\mathbb{T}))$ such that $(R_\theta)_n(f)(z) = f(e^{-2\pi i n\theta} z)$ for every $n \in \mathbb{Z}$, $f \in C(\mathbb{T})$ and $z \in \mathbb{T}$. Therefore $(C(\mathbb{T}, \mathbb{Z}, R_\theta))$ is a dynamical system and the resulting crossed product $C_0(\mathbb{T}) \rtimes_{R_\theta} \mathbb{Z}$ is isomorphic to A_θ, the universal C*-algebra generated by unitaries u and v satisfying $uv = e^{2\pi i\theta} vu$.

Example A.0.8. One of the most interesting example is the dynamical system $(C_0(G), G, \tau)$. Define an action $\tau : G \to \text{Aut}(C_0(G))$ given by the left translation, that is,

$$\tau_g(f)(h) = f(g^{-1}h)$$

for all $f \in C_0(G)$ and $g, h \in G$. We claim that the crossed product $C_0(G) \rtimes_\tau G$ is isomorphic to $\mathcal{K}(l^2(G))$. For this, let $M : C_0(G) \to B(l^2(G))$ be the regular representation of $C_0(G)$ as multiplication operators on $l^2(G)$ and let $\lambda^G : G \to B(l^2(G))$ denote the left regular representation, that is, $M_f(\xi)(g) = f(g)\xi(g)$ and $\lambda^G_g(\xi)(h) = \xi(g^{-1}h)$ for all $f, \xi \in C_0(G)$ and $g, h \in G$. Thus (M, λ^G) is a covariant representation of $(C_0(G), G, \tau)$ since for all $f \in C_0(G)$, $\xi \in C_c(G)$ and $g, h \in G$ we have:

$$\begin{aligned}
(M_{\tau_g(f)}\lambda^G_g)(\xi)(h) &= M_{\tau_g(f)}(\lambda^G_g(\xi))(h) \\
&= \tau_g(f)(h)\lambda^G_g(\xi)(h) \\
&= f(g^{-1}h)\xi(g^{-1}h) \\
&= (f.\xi)(g^{-1}h) \\
&= \lambda^G_g(f.\xi)(h) \\
&= \lambda^G_g(M_f(\xi))(h) \\
&= (\lambda^G_g M_f)(\xi)(h).
\end{aligned}$$

By universal property there is a *-homomorphism $M \times \lambda^G : C_0(G) \rtimes_\tau G \to B(l^2(G))$ such that

$$(M \times \lambda^G)\left(\sum_{g\in G} f u_g\right) = \sum_{g\in G} M_f \lambda^G_g$$

The fact that $M \times \lambda^G$ is faithful is not a trivial result but can be seen as a special case of [[12], Proposition 4.1.7].

Now, it is enough to show that $Im(M \times \lambda^G) := \overline{\text{span}}\{M_f\lambda_g^G \mid f \in C_0(G), g \in G\}$ is equal to $\mathcal{K}(l^2(G))$[1]. To see this, observe that

$$M_{\chi_g}(\xi)(h) = \delta_{g,h}\xi(h) = \chi_g\langle\chi_g, \xi\rangle(h) = |\chi_g\rangle\langle\chi_g|(\xi)(h)$$

for all $\xi \in C_0(G)$ and $g, h \in G$. By linearity and continuity we have M_f is a compact operator for all $f \in C_0(G)$. Since $\mathcal{K}(l^2(G))$ is an ideal of $B(l^2(G))$ we conclude that $Im(M \times \lambda^G) \subseteq \mathcal{K}(l^2(G))$. To prove the otherwise it is enough to observe that for all $\xi \in l^2(G)$ and $g, h, k \in G$ we have

$$\begin{aligned}
|\chi_g\rangle\langle\chi_h|(\xi)(k) &= \chi_g\langle\chi_h, \xi\rangle(k) \\
&= \delta_{g,k}\xi(h) \\
&= \delta_{g,k}\xi(hg^{-1}k) \\
&= M_{\chi_g}\lambda^G_{gh^{-1}}(\xi)(k).
\end{aligned}$$

Again by linearity and continuity we conclude $\mathcal{K}(l^2(G)) \subseteq Im(M \times \lambda^G)$, as desired.

Example A.0.9. If we consider the action $\tau : G \to \text{Aut}(C_0(G))$ by the right translations then the result of Example A.0.8 is the same since the proof is similar considering ρ^G the right regular representation of G. Alternatively one can consider the automorphism $f \mapsto \tilde{f}$, where $\tilde{f}(g) = f(g^-1)$ of $C_0(G)$ and check that this commutes the left and right regular representation, so that both dynamical systems are isomorphic.

Now, we will present a definition of reduced crossed products by actions. For this, let $\pi : A \to B(H)$ be a faithful representation. Consider the Hilbert space $l^2(G, H)$ as a completion of $C_c(G, H)$ via the inner-product defined by $\langle\xi, \eta\rangle := \sum_{g\in G}\langle\xi_g, \eta_g\rangle_H$. Using the canonical isomorphism $l^2(G, H) \cong H \otimes l^2(G)$ we are able to define $\psi : A \to B(l^2(G, H))$ such that $\psi(a)(\xi)_g = \pi(\alpha_{g^{-1}}(a))(\xi_g)$ for all $a \in A$, $\xi \in C_c(G, H)$ and $g \in G$. Note that ψ is a bounded *-homomorphism with $\|\psi(a)(\xi)\| \leqslant \|a\|\|\xi\|$. Also, define $U : G \to B(l^2(G, H))$ as $U_g(\xi)_h = \xi_{g^{-1}h}$. Actually $U_g \in \mathcal{U}(l^2(G, H))$ so that U is a unitary representation of G. We claim that (ψ, U) is a covariant representation. In fact note that

$$\begin{aligned}
(\psi(\alpha_g(a))U_g)(\xi)_h &= (\psi(\alpha_g(a))(U_g(\xi))_h \\
&= \pi(\alpha_{h^{-1}}(\alpha_g(a)))(U_g(\xi)_h) \\
&= \pi(\alpha_{h^{-1}g}(a))\xi_{g^{-1}h} \\
&= \psi(a)(\xi)_{g^{-1}h} \\
&= U_g(\psi(a)(\xi))_h
\end{aligned}$$

Therefore, define $\Lambda^{A\rtimes_\alpha G} := \psi \rtimes U$ as a integrated form for (ψ, U) in $B(l^2(G, H))$ and call this the regular representation associated to π. Note that $\Lambda^{A\rtimes_\alpha G}$ is faithful on $C_c(G, A)$.

[1] $\mathcal{K}(l^2(G))$ can be seen as $\overline{\text{span}}\{|\xi\rangle\langle\eta| \mid \xi, \eta \in l^2(G)\}$ where $|\xi\rangle\langle\eta|$ is defined by $|\xi\rangle\langle\eta|(\varsigma) = \xi\langle\eta, \varsigma\rangle$ for all $\xi, \eta, \varsigma \in l^2(G)$.

Definition A.0.10. The reduced crossed product, denoted by $A \rtimes_{\alpha,r} G$, is a completion of $C_c(G, A)$ with respect to reduced norm defined by $\|f\|_r := \|\Lambda^{A\rtimes_\alpha G}\|$.

Remark A.0.11. One can show that $\|.\|_r$ does not depend on the representation π.

Example A.0.12. If we consider the trivial action of G on A we have that $\mathbb{C} \rtimes_{tr,r} G = C^*_r(G)$.

Example A.0.13. In the special case of Example A.0.8 $C_0(G) \rtimes_\tau G = \mathcal{K}(l^2(G))$ is simple so that $C_0(G) \rtimes_{\tau,r} G \cong C_0(G) \rtimes_\tau G \cong \mathcal{K}(l^2(G))$ even "G is not amenable".

Example A.0.14. If we have a trivial action of G on A then $A \rtimes_{tr,r} G \cong A \otimes C^*_r(G)$.

The following gives us a universal property for dynamical systems.

Theorem A.0.15. *Let (A, G, α) be a dynamical system and B a C*-algebra. Then there is a universal covariant representation (ι_A, ι_G) of (A, G) in $\mathcal{M}(A \rtimes_\alpha G)$ and for any covariant representation (π, μ) of (A, G) in $\mathcal{M}(B)$ there is a representation $\pi \times \mu : A \rtimes_\alpha G \to \mathcal{M}(B)$ such that $\pi \times \mu \circ \iota_A = \pi$ and $\pi \times \mu \circ \iota_G = \mu$. Moreover, $A \rtimes_\alpha G = \overline{span}\{\iota_A(a)\iota_G(f) \mid a \in A \text{ and } f \in C_c(G)\}$.*

*Conversely, if $\rho : A \rtimes_\alpha G \to \mathcal{M}(B)$ be a *-homomorphism then $\pi := \rho \circ \iota_A$ and $\mu := \rho \circ \iota_G$ define a covariant representation of (A, G, α) into $\mathcal{M}(B)$.*

Proposition A.0.16. *Let (A, G, α) be a dynamical system, (ι_A, ι_G) be the universal covariant representation of (A, G) into $\mathcal{M}(A \rtimes_\alpha G)$ and u denote the canonical map from G into $C^*(G)$. Then there is a coaction of G on $A \rtimes_\alpha G$ defined as the integrated form:*

$$\widehat{\alpha} := (\iota_A \otimes 1) \times (\iota_G \otimes u) : A \rtimes_\alpha G \to (A \rtimes_\alpha G) \otimes C^*(G).$$

Proof. To begin with, let $\iota_A : A \to \mathcal{M}(A \rtimes_\alpha G)$ and $\iota_G : G \to \mathcal{M}(A \rtimes_\alpha G)$ the universal covariant representation. Define $\iota_A \otimes 1 : A \to \mathcal{M}(A \rtimes_\alpha G \otimes C^*(G))$ such that $(\iota_A \otimes 1)(a) = \iota(a) \otimes 1$ and $\iota_G \otimes u : G \to \mathcal{M}(A \rtimes_\alpha G \otimes C^*(G))$ such that $(\iota_G \otimes u)_g = \iota_G(g) \otimes u_g$. We claim that $(\iota_A \otimes 1, \iota_G \otimes u)$ is a covariant representation of (A, G, α). Since (ι_A, ι_G) is the universal covariant representation of (A, G, α) we have

$$\begin{aligned}(\iota_A \otimes 1)(\alpha_g(a))(\iota_G \otimes u)_g &= (\iota_A(\alpha_g(a)) \otimes 1)(\iota_G(g) \otimes u_g) \\ &= \iota_A(\alpha_g(a))\iota_G(g) \otimes u_g \\ &= \iota_G(g)\iota_A(a) \otimes u_g \\ &= (\iota_G(g) \otimes u_g)(\iota_A(a) \otimes 1) \\ &= (\iota_G \otimes u)_g(\iota_A \otimes 1)(a)\end{aligned}$$

for all $a \in A$ and $g \in G$. Therefore, by the universal property of (A, G, α) we get a *-homomorphism $\widehat{\alpha} := (\iota_A \otimes 1) \times (\iota_G \otimes u) : A \rtimes_\alpha G \to \mathcal{M}((A \rtimes_\alpha G) \otimes C^*(G))$ such that $\widehat{\alpha} \circ \iota_A = \iota_A \otimes 1$ and $\widehat{\alpha} \circ \iota_G = \iota_G \otimes u$. This is the integrated form of $(\iota_A \otimes 1, \iota_G \otimes u)$. In fact, $\widehat{\alpha}$ take values in $(A \rtimes_\alpha G) \otimes C^*(G)$ since $\widehat{\alpha}(\iota_A(a)\iota_G(g)) = (\iota_A(a) \otimes 1)(\iota_G(g) \otimes u_g) = \iota_A(a)\iota_G(g) \otimes u_g$. To see the coaction identity we have to note that

$$\begin{aligned}(\widehat{\alpha} \otimes id_G) \circ \widehat{\alpha}(i_A(a)) &= (\widehat{\alpha} \otimes id_G)(\iota_A(a) \otimes 1)) \\ &= i_A(a) \otimes 1 \otimes 1 \\ &= (id \otimes \delta_G)(\iota_A(a) \otimes 1) \\ &= (id \otimes \delta_G) \circ \widehat{\alpha}(i_A(a))\end{aligned}$$

and

$$\begin{aligned}(\widehat{\alpha} \otimes id_G) \circ \widehat{\alpha}(i_G(g)) &= (\widehat{\alpha} \otimes id_G)(\iota_G(g) \otimes u_g)) \\ &= i_G(g) \otimes u_g \otimes u_g \\ &= (id \otimes \delta_G)(\iota_G(g) \otimes u_g) \\ &= (id \otimes \delta_G) \circ \widehat{\alpha}(i_G(g))\end{aligned}$$

So, by linearity and continuity we get the coaction identity. Finally, for nondegeneracy, observe that for each $a \in A$ and $g, k \in G$ we have

$$\begin{aligned}\iota_A(a)\iota_G(g) \otimes k &= (\iota_A(a)\iota_G(g) \otimes g)(1 \otimes g^{-1}k) \\ &= ((\iota_A \otimes 1)(a)(\iota_G \otimes u)(g))(1 \otimes g^{-1}k) \\ &= \widehat{\alpha}(\iota_A(a)\iota_G(g))(1 \otimes g^{-1}k)\end{aligned}$$

and since the closed linear span of $\iota_A(a)\iota_G(g)$ is dense in $A \rtimes_\alpha G$ it follows that $\widehat{\alpha}$ is nondegenerate as desired.

□

Proposition A.0.17 (Imai-Takai duality, [48], Corollary 2.12)**.** *Let $\alpha : G \to Aut(A)$ be an action. Then the coaction $\widehat{\alpha}$ is always maximal, that is,*

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} G \cong A \otimes \mathcal{K}(l^2(G)).$$

Remark A.0.18*.* In the case G is abelian it is straightforward to check that the coaction $\widehat{\alpha}$ defined above corresponds to the action $\widehat{\alpha}$ of $\hat{G}$ on $A \rtimes_\alpha G$ given by $\widehat{\alpha}_\chi(\iota_A(a)\iota_G(g)) = \iota_A(a)\iota_G(g)\chi(g)$ and consequently we have

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \hat{G} \cong A \otimes \mathcal{K}(l^2(G)).$$

In the literature this result is known as the Takai-Takesaki duality.

Proposition A.0.19. *The dual coaction $\widehat{\alpha}$ can be factored through a coaction $\widehat{\alpha}_r$ on $A \rtimes_{\alpha,r} G$. Indeed, $\widehat{\alpha}_r$ is the normalization of $\widehat{\alpha}$.*

APPENDIX B – HILBERT MODULES

This appendix contains some basic facts on Hilbert modules and adjointable operators. We omit any demonstration but the reader can find all these facts in [40], [12] and [56].

Definition B.0.1. Let B be a C*-algebra. A (right) pre-Hilbert B-module is a complex vector space $\mathcal{E}$ which is a right B-module equipped with a B-inner product $\langle\cdot,\cdot\rangle_B : \mathcal{E}\times\mathcal{E}\to B$, that is, a linear map in the second variable and conjugate-linear in the first satisfying for all $\xi,\eta\in\mathcal{E}$ and $b\in B$,

1. $\langle\xi,\eta b\rangle_B = \langle\xi,\eta\rangle_B b$;
2. $\langle\xi,\eta\rangle_B^* = \langle\eta,\xi\rangle_B$;
3. $\langle\xi,\xi\rangle_B \geq 0$ in B;
4. $\langle\xi,\xi\rangle_B = 0$ implies $\xi = 0$.

A (left) pre-Hilbert B-module is defined in a similar way but in this case, we require the inner product, usually denoted by ${}_B\langle\cdot,\cdot\rangle$, to be B-linear in the first variable and thus conjugate-linear in the second.

Remark B.0.2. The axioms (1) and (3) imply that $\langle\xi b,\eta\rangle_B = b^*\langle\xi,\eta\rangle_B$. Consequently, we have $\langle\mathcal{E},\mathcal{E}\rangle_B :=$ span$\{\langle\xi,\eta\rangle_B \mid \xi,\eta\in\mathcal{E}\}$ is a closed ideal in B.

Lemma B.0.3. *Let $\mathcal{E}$ be a pre-Hilbert B-module and $\xi,\eta\in\mathcal{E}$. Then*

$$\langle\xi,\eta\rangle_B^*\langle\xi,\eta\rangle_B \leq \|\langle\xi,\xi\rangle_B\|\langle\eta,\eta\rangle_B.$$

Corollary B.0.4. *Let $\mathcal{E}$ be a pre-Hilbert B-module. Then $\|\xi\| := \langle\xi,\xi\rangle_B^{1/2}$ is a norm on $\mathcal{E}$.*

Definition B.0.5. A (right) Hilbert B-module is a pre-Hilbert B-module $\mathcal{E}$ that is complete in the norm coming from the B-inner product defined above. We say that $\mathcal{E}$ is *full* if $\langle\mathcal{E},\mathcal{E}\rangle_B$ is dense in B.

Example B.0.6. Every complex Hilbert space $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module in the canonical way using the convention that inner products on Hilbert spaces are linear in the second variable.

Example B.0.7. A C*-algebra B can be viewed as a right Hilbert B-module with right module action given by the multiplication and B-inner product defined by $\langle a,b\rangle_B := a^*b$ for every $a,b\in B$. Similarly, if we define ${}_B\langle a,b\rangle := ab^*$ as an inner product, B becomes a left Hilbert B-module. Consequently, a closed ideal I in B can be viewed as a left and right Hilbert B-module. In fact an ideal I in B is a Hilbert B-module if only if I is closed in B.

Example B.0.8. Let be a family of Hilbert B-modules $(\mathcal{E}_i)_{i\in I}$, then the algebraic direct sum $\oplus_{i\in I}\mathcal{E}_i$ is a pre-Hilbert B-module with the module action defined coordinate-wise and B-inner product defined as $\langle \xi, \eta\rangle := \sum_{i\in I}^{\text{finite}} \langle \xi_i, \eta_i\rangle_{\mathcal{E}_i}$ where $\xi = (\xi_i)_{i\in I}$ and $\eta = (\eta_i)_{i\in I}$. Its completion is a Hilbert B-module.

Definition B.0.9. Given Hilbert B-modules $\mathcal{E}$,$\mathcal{F}$, we say that a map $T : \mathcal{E} \to \mathcal{F}$ is adjointable if there exists a (necessarily unique) map $T^* : \mathcal{F} \to \mathcal{E}$ satisfying

$$\langle T(\xi), \eta\rangle_B = \langle \xi, T^*(\eta)\rangle_B$$

We say that T^* is the adjoint of T and we denote by $\mathcal{L}(\mathcal{E}, \mathcal{F})$ the set of adjointable maps.

Proposition B.0.10. *Let $\mathcal{E}$,$\mathcal{F}$ be Hilbert B-modules. Then adjointable maps are automatically B-linear and bounded, and $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is a Banach space with the operator norm. Moreover, $\mathcal{L}(\mathcal{E}) := \mathcal{L}(\mathcal{E}, \mathcal{E})$ is a C*-algebra.*

Remark B.0.11. Not every B-linear bounded map between Hilbert B-modules is adjointable.

Example B.0.12. Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules. For each $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$ there is an adjointable operator $|\xi\rangle\langle\eta| : \mathcal{E} \to \mathcal{F}$ defined by $|\xi\rangle\langle\eta|(\zeta) = \xi\langle\eta, \zeta\rangle_B$ for every $\zeta \in \mathcal{E}$ which its adjoint is $|\eta\rangle\langle\xi|$. The closed linear span of operators of this form is denoted by $\mathcal{K}(\mathcal{E}, \mathcal{F})$.

The $\mathcal{K}(\mathcal{E}, \mathcal{F})$ is also a Banach space with the operator norm, and $\mathcal{K}(\mathcal{E})$ is a closed ideal of $\mathcal{L}(\mathcal{E})$. We recall that $\mathcal{L}(\mathcal{E})$ has a natural structure of Hilbert $\mathcal{L}(\mathcal{E})$-module given by composition.

Proposition B.0.13. *If $\mathcal{E}$ is a Hilbert B-module then $\mathcal{M}(\mathcal{K}(\mathcal{E}))$ is isomorphic to $\mathcal{L}(\mathcal{E})$.*

Definition B.0.14. Let A and B be C*-algebras. An imprimitivity A, B-bimodule is an A, B-bimodule $\mathcal{E}$ such that

1. $\mathcal{E}$ is a left full Hilbert A-module and a full right Hilbert B-module;

2. ${}_A\langle \xi, \eta\rangle\zeta = \xi\langle\eta, \zeta\rangle_B$ for every $\xi, \eta, \zeta \in \mathcal{E}$.

Example B.0.15. A C*-algebra B has a canonical structure of imprimitivity B, B-bimodule with left inner product ${}_B\langle a, b\rangle = ab^*$ and right inner product $\langle a, b\rangle_B = a^*b$.

Example B.0.16. A full Hilbert B-bimodule is an imprimitivity $\mathcal{K}(\mathcal{E}), B$-bimodule.

Definition B.0.17. Given C*-algebras A and B, we say that A is a strongly *Morita equivalent* or *Morita-Rieffel-equivalent* to B if there exists an imprimitivity A, B-bimodule.