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BLUR SHIFT SPACES: A MULTI-POINT COMPACTIFICATION SCHEME FOR INFINITE-ALPHABET SHIFT SPACES

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O presente trabalho em nível de doutorado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

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"A dúvida é o preço da pureza."
(Sartre, 1939)

RESUMO

Propomos um novo tipo de espaços *shift*, chamados *blur shift spaces*. Estes espaços são construídos a partir de espaços *shift* clássicos, aumentando-se o alfabeto com símbolos que representam subconjuntos infinitos do alfabeto original e então definindo-se uma topologia conveniente. Os espaços obtidos são inspirados e generalizam os espaços *shift* apresentados por Ott, Tomforde e Willis em [15] e por Gonçalves e Royer em [7], os quais foram usados para encontrar isomorfismos entre C^*-álgebras associadas a algumas classes de espaços *shift*. Estudamos algumas propriedades topológicas dos *blur shift spaces*, como axiomas de separação, axiomas de enumerabilidade, metrizabilidade e caracterizamos quando os espaços são compactos ou localmente compactos, dependendo da escolha dos conjuntos utilizados como novos símbolos do alfabeto. Em particular, a construção apresentada dos *blur shif spaces* pode ser utilizada como um esquema multi-pontos para compactificar espaços *shift* clássicos. Finalizamos caracterizando as funções contínuas que comutam com a função *shift*, e os *sliding block codes* generalizados, definidos sobre *blur shift spaces*.

Palavras-chave: Dinâmica simbólica. Dinâmica topológica. Alfabetos infinitos. Teorema de Curtis-Hedlund-Lyndon. Sliding block codes generalizados.

RESUMO EXPANDIDO

Introdução

Em dinâmica simbólica começamos com um conjunto $\mathcal{A}$, chamado alfabeto, cujos elementos são chamados letras, ou símbolos, e consideramos sequências destes símbolos indexadas por $\mathbb{N}$ ou por $\mathbb{Z}$. Chamamos o conjunto de todas sequências possíveis de *full shift* e consideramos subconjuntos fechados que são invariantes por translações (a função *shift*). Estes subconjuntos são chamados *one-sided shift spaces*, no caso de sequências indexadas por $\mathbb{N}$ ou *two-sided shift spaces*, no caso de sequências indexadas por $\mathbb{Z}$. Nesta tese estamos interessados apenas em *one-sided shift spaces*. No caso em que o alfabeto considerado é finito, equipamos o alfabeto com a topologia discreta, e o *full shift* com a topologia produto, obtendo assim um espaço compacto. Além disto, neste caso a função *shift* é contínua.
Quando o alfabeto é infinito não há mais compacidade, sendo mais difícil de se obter resultados. Na tentativa de se contornar esta falta de compacidade, algumas abordagens têm sido propostas. Em [15] os autores propõem uma nova definição para *one-sided shift spaces*, que são chamados aqui *Ott-Tomforde-Willis shift spaces*. Estes espaços são obtidos, no caso de um alfabeto infinito, aplicando-se a compactificação de Alexandroff ao alfabeto, ou seja, adicionando-se um novo símbolo "∞" ao alfabeto e adicionando-se à topologia do alfabeto conjuntos que são complementares de conjuntos finitos. Podemos considerar que o símbolo ∞ representa uma indeterminação no seguinte sentido: se um ponto do espaço *shift* tem ∞ como uma de suas coordenadas, podemos interpretar que as coordenadas deste ponto são conhecidas até a primeira vez que aparece o símbolo ∞, e a partir dali não podemos mais determinar quais são as coordenadas. Assim pode-se definir uma relação de equivalência que identifica sequências que são iguais até a primeira coordenada em que o símbolo ∞ aparece em ambas. Tomando o quociente, obtém-se um novo espaço *shift* que, entre outras características, é compacto.
Outra abordagem que influenciou esta tese foi proposta por Gonçalves e Royer em [7]. Nesta abordagem, define-se um espaço *shift* clássico a partir de caminhos sobre um ultragrafo inicialmente fixado, então se encontra um conjunto de novos símbolos que são adicinados ao alfabeto. Estes novos símbolos são conjuntos infinitos de arestas do ultragrafo, e podemos pensar intuitivamente que uma sequência que tem um destes símbolos entre suas coordenadas representa um caminho finito sobre o ultragrafo, que termina em um vértice com infinitas arestas saindo, e não podemos determinar sobre qual destas arestas o caminho continua. Temos novamente pontos do espaço *shift* com indeterminações, porém nesta abordagem podemos distinguir entre tipos de indeterminações, conforme a estrutura do ultragrafo subjacente. Uma vez definidos estes espaços, que chamaremos de *Gonçalves-Royer ultragraph shift spaces,* e sua topologia, os autores estabelecem em [7], entre outros resultados, condições suficientes para que o espaço seja localmente compacto.
Estas duas abordagens, de Ott, Tomforde e Willes, e de Gonçalves e Royer, foram utilizadas com sucesso para estabelecer correspondências entre conjugação de algumas classes de espaços *shift* e isomorfismo de suas C^*-álgebras.

Objetivos

Inspirados pelas abordagens mencionadas acima, nos lançamos o desafio de propor uma nova classe de *one-sided shift spaces* em que novos símbolos sejam introduzidos ao alfabeto de um espaço *shift* clássico, representando a impossibilidade de se determinar o símbolo em alguma coordenada, porém com a possibilidade de se distinguir entre tipos de indeterminações. Uma vez definidos tais espaços e sua topologia, gostaríamos de estudar algumas propriedades topológicas como axiomas de enumerabilidade e separabilidade, metrizabilizade e compacidade. Algo que também nos despertou interesse foi a caracterização de funções contínuas que comutam com a função *shift* usando o conceito de *sliding block codes* generalizados propostos em [9] e [17], como feito em [8] para *Gonçalves-Royer ultragraph shift spaces.*

Metodologia

Em busca de encontrar uma nova definição adequada de espaços *shift*, inicialmente tentamos encontrar uma maneira de escrever um conjunto de sequências de símbolos com algumas indeterminações, como feito para *Gonçalves-Royer ultragraph shift spaces,* porém sem a rigidez imposta pelo ultragrafo subjacente. Simultaneamente tentamos encontrar uma maneira adequada de definir os cilindros generalizados, que viriam a ser uma base da topologia. Durante este processo, percebemos que também poderíamos englobar os espaços *shift* de Ott, Tomforde e Willis como caso particular. Uma vez definido o espaço e sua topologia, passamos a estudar suas propriedades, conforme mencionamos acima, nos objetivos.

Resultados e Discussão

Obtivemos com sucesso uma nova classe de *one-sided shift spaces*. Estes espaços foram denominados *blur shift spaces* (sem tradução). Os espaços *shift* de Ott, Tomforde e Willis e de ultragrafos de Gonçalves e Royer podem ser vistos como exemplos de *blur shift spaces*. Mostramos que o espaço *full blur shift* é Hausdorff, regular, e caracterizamos quando ele é primeiro contável ou segundo contável. Estabelecemos resultados sobre a metrizabilidade do espaço, exibindo uma família de métricas para o caso particular em que os *blur shift spaces* satisfazem o segundo axioma de enumerabilidade. Estas métricas são semelhantes àquelas apresentadas em [15] e [11], porém são apresentadas aqui mais diretamente e mais intuitivamente.
Sobre a compacidade, caracterizamos quando os espaços propostos são compactos ou localmente compactos, dependendo da escolha dos novos símbolos que são introduzidos no alfabeto original. Em particular, os *blur shift spaces* podem ser utilizados como um esquema multipontos para compactificar *one-sided shift spaces* clássicos.
Para concluir o trabalho, caracterizamos funções contínuas que comutam com a função *shift* e *sliding block codes* generalizados, definidos sobre os espaços aqui introduzidos.

Considerações Finais

Uma vez que esta nova classe de espaços *shift* foi introduzida e algumas características importantes foram estabelecidas, algumas perguntas sobre como podemos utilizar estes espaços surgem naturalmente. Uma questão interessante é se podemos usar estes espaços para encontrar correspondência entre conjugação de classes de espaços *shift* e isomorfismo entre suas C^*-álgebras associadas.
Do ponto de vista de sistemas dinâmicos, os *blur shift spaces* podem servir como fonte de novos exemplos de fenômenos. Em particular, o entendimento de suas propriedades dinâmicas, como por exemplo, caoticidade, ergodicidade, entropia, entre outras, podem ser objeto de estudos futuros.

Palavras-chave: Dinâmica simbólica. Dinâmica topológica. Alfabetos infinitos. Teorema de Curtis-Hedlund-Lyndon. Sliding block codes generalizados.

ABSTRACT

We propose a new type of shift spaces, called blur shift spaces. These spaces are constructed from classical shift spaces, enlarging the alphabet with symbols that represent infinite subsets of the original alphabet, then defining a convenient topology. The resulting spaces are inspired and generalize those shift spaces presented by Ott, Tomforde e Willis in [15], and by Gonçalves e Royer em [7], which were used to find isomorphisms between C^*-algebras associated to some classes of shift spaces. We study some topological properties of the blur shift spaces, such as separation axioms, countability axioms, metrizability and we characterize when the spaces are compact, or locally compact, depending on the choice of the sets used as new symbols of the extended alphabet. In particular, the presented construction of blur shift spaces may be used as a multi-point compactification scheme for classical shift spaces. We close this thesis characterizing continuous shift commuting maps and generalized sliding block codes defined over blur shift spaces.

Keywords: Symbolic dynamics. Topological dynamics. Infinite alphabets. Curtis-Hedlund-Lyndon Theorem. Generalized sliding block codes.

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1 INTRODUCTION

This thesis presents a study on the area called *symbolic dynamics*. Symbolic dynamics is part of a larger area called *dynamical systems*, where, roughly speaking, the interest lies on the long-term evolution of points in a set X under a function $T : X \to X$. The usual framework for a general (X, T) is giving X a topology, and require X to be compact, T to be continuous. The pair (X, T) is then called a *topological dynamical system*.

In symbolic dynamics, given a non-empty set $\mathcal{A}$, called the *alphabet*, with elements called *letters* or *symbols*, we consider the set of all sequences on $\mathcal{A}$ indexed by $\mathbb{N}$, where $\mathbb{N}$ denotes the set of all non-negative integers. This resulting set, $\mathcal{A}^{\mathbb{N}}$ is called the **one-sided full shift over** $\mathcal{A}$:

$$\mathcal{A}^{\mathbb{N}} := \{(x_i)_{i\in\mathbb{N}} : \ x_i \in \mathcal{A} \ \forall \ i \in \mathbb{N}\}.$$

A typical point of $\mathcal{A}^{\mathbb{N}}$ can be written

$$\mathrm{x} = x_0 x_1 x_2 \ldots,$$

where each $x_i \in \mathcal{A}$. Sometimes a point $\mathrm{x} \in \mathcal{A}$ will be referred as a sequence. One could also consider the *two-sided full shift over* $\mathcal{A}$, which is the set of all sequences of letters of $\mathcal{A}$ indexed by the set of integers $\mathbb{Z}$. In this work we will be concerned exclusively with the one-sided case.

The dynamics is given by successive applications of the **shift map**, which is a function $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$, defined by

$$\sigma\big((x_i)_{i\in\mathbb{N}}\big) = (x_{i+1})_{i\in\mathbb{N}}.$$

The shift map transforms a sequence of symbols $\mathrm{x} = (x_0 x_1 \ldots)$ into the sequence $\sigma(\mathrm{x}) = (x_1 x_2 \ldots)$ by erasing the first symbol.

Once we have a pair $(\mathcal{A}^{\mathbb{N}}, \sigma)$, we need a structure to work with. When the alphabet $\mathcal{A}$ is finite, we consider on $\mathcal{A}$ the discrete topology, on $\mathcal{A}^{\mathbb{N}}$ the product topology, and then $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is a topological dynamical system. However, when we turn to infinite alphabets, the compactness is lost. Since compactness is an important feature and many fundamental results rely on it, several approaches have been developed to deal with this lack of compactness, each of them being useful for different purposes. In this thesis, we give details for two of them, which have motivated the construction of the blur shift spaces presented in this thesis. In section 2.2 we present a discussion for the Ott-Tomforde-Willis shift spaces introduced in [15]. The Gonçalves-Royer ultragraph shift spaces introduced in [7] are presented in section 2.3.

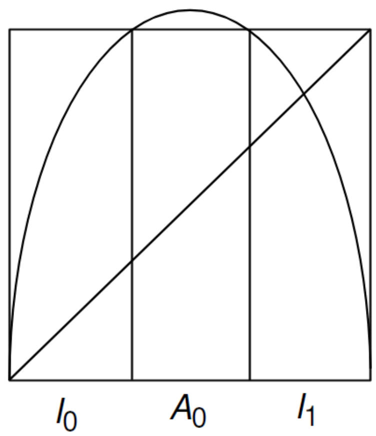
The interest in symbolic dynamics comes from the fact that, besides symbolic dynamical systems provide rich examples of dynamical behaviours, they are also a powerful tool to analyse other dynamical systems. To illustrate this affirmation, we give some examples below, of dynamical systems whose dynamics are topologically conjugate[1] to the dynamics of some shift

[1] Two topological dynamical systems, (X, T) and (Y, S), are said *topologically conjugate* if and only if there exists a homeomorphism $\phi : X \to Y$ such that $S \circ \phi = \phi \circ T$.

space. In these examples we are concerned in giving the reader a taste of this subject without getting deep in details.

Example 1.1. For each μ, consider the one-variable real functions given by $F_\mu(x) = \mu x(1-x)$. This family of functions, called **Quadratic Family**, illustrates many of the crucial phenomena that occur in dynamical systems. Again, we are interested in the behavior of points x under F_μ, i.e., the orbit of $x : F_\mu(x), F(F_\mu(x)) = F_\mu^2(x), F_\mu^3(x), \ldots$. We restrict our attention for $\mu > 2+\sqrt{5}$ and x in the unit interval $I = [0,1]$, where all interesting behaviour happens. Since $\mu > 4$, the maximum value of $F_\mu(x)$ is greater than one. Denote by A_0 the set of points in I that leave I after one iteration of F_μ. A_0 is an open interval centered in $\frac{1}{2}$ and all of its points scape from I after one application of F_μ. Then $I \setminus A_0$ is an union of two closed intervals I_0 and I_1, such that 0 belongs to I_0, 1 belongs to I_1, see Figure 1. Again, by focusing on each of these two closed intervals, we have a centered subinterval where every point scape from I after two iterations of F_μ. In next step the centered subinterval is removed and the process follows. Proceeding recursively, we can define a set K of points in I that never leave I, that is, $F_\mu^i(x) \in I$ for all i. This set K is a Cantor set and each point in K can be identified with a sequence of zeros and ones, by following its itinerary: $x \sim (x_0 x_1 \ldots)$, where $x_n = 0$ if $F_\mu^n(x)$ belongs to the left subinterval, and $x_n = 1$ if $F_\mu^n(x)$ belongs to the right subinterval, after removal of centered subinterval in n^{th} step. Denote by Λ the topological dynamical system (K, F_μ), and by Σ_2 the full shift over $\{0,1\}$. We remark that since the relation $\sim$ is a topological conjugacy, the theory of symbolic dynamics can be applied to find interesting dynamical properties for Σ_2, then assert equivalent properties, or invariants, for Λ, via conjugacy, allowing to state that the shift map is an accurate model for the quadratic map. This example can be found in details in (Sections 1.5 to 1.7 in [3]).

Figure 1 – The quadratic family.



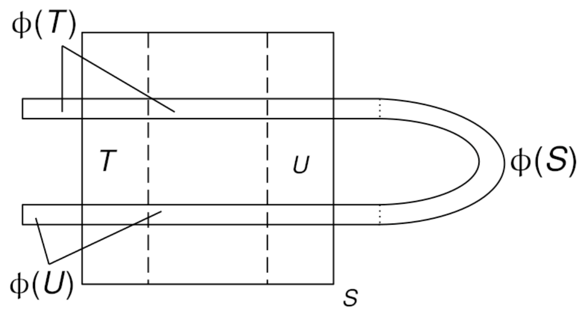
The next example is the *horseshoe mapping*, due to Smale.

Example 1.2. Let S be a square in the plane and $\phi : S \to \mathbb{R}^2$ a continuous one-to-one mapping described as follows. We divide S in three vertical rectangles, call the first T, and the third S. The action of the map is defined geometrically by squishing the square, then stretching the result into a long strip, which is then folded into the shape of a horseshoe. The horseshoe is put back over a copy of the initial square, in such a way both ends of horseshoe exceed the square, as shown in Figure 2. Both restrictions $\phi|_T$ and $\phi|_U$ are linear mappings which vertically contract and horizontally expand. Restricting our attention to the compact set

$$M := \bigcap_{n=-\infty}^{\infty} \phi^{-n}(S)$$

of those points in S whose orbit remain in S, $(M, \phi|_M)$ is a topological dynamical system. Again, if we associate the symbol zero (respectively one) to those points in M whose orbits remain in T (respectively, U), then $(M, \phi|_M)$ is topologically conjugate to the full shift with two symbols.

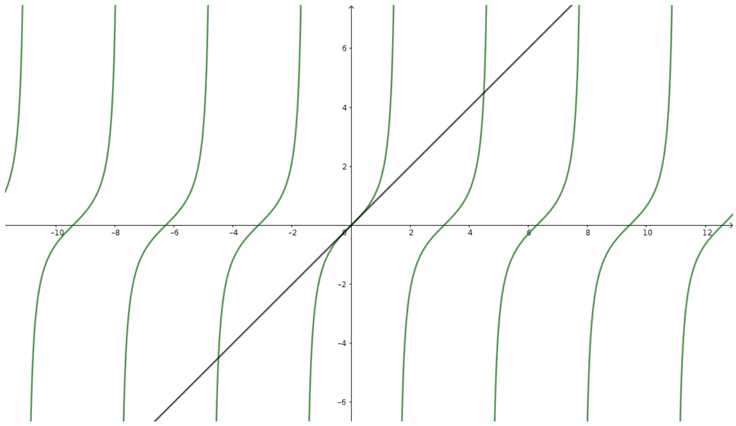
Figure 2 – The horseshoe dynamical system.



Example 1.3. Using ideas presented above for other functions like $T_\infty(x) = \mu \tan(x)$ and $f(x) = \mu x \sin(x)$, it is possible to show the dynamics associated are conjugate (or semi-conjugate) to a shift space with infinitely many symbols. To understand the reason the conjugacy is to a infinite-symbol shift space, we have to observe the itinerary of a point, under the iterations of the underlying function. Remember we are focusing on those points which present interesting behaviour. For the previous examples, under each iteration a point reaches one subinterval (example 1.1) or subrectangle (example 1.2), associating the symbol 0 for one possibility, and symbol 1 for the other possibility. These subintervals or subractangles can be seen as partitions of the interval or the rectangle. For the case of the function $T_\infty(x) = \mu \tan(x)$ on the real line, notice it is undefined at $\frac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$, and that the function maps each interval $(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2})$, $n \in \mathbb{Z}$, onto $\mathbb{R}$, see Figure 3. This generates rich dynamics and a natural partition of $\mathbb{R}$, in particular an infinite partition. Proceeding as previous examples,

for each point $x \in \mathbb{R}$ we associate a symbol n if, and only if $T_\infty^i(x) \in (\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2})$. This way we have defined a correspondence between the points in the real line which have $T_\infty^i(x)$ defined for all $i \in \mathbb{N}$, and a shift space with infinitely many symbols. Again, we can use symbolic dynamics to give crucial information about the system, via conjugacy. See [12, Section 3.1] for a complete discussion.

Figure 3 – y = tan(x).



The main goal of this work is to establish an approach to embed a classical shift space in a new type of shift space, where whole sets of infinitely many symbols are represented by new symbols added to the original alphabet. This new shift space is endowed with a Hausdorff (and regular) topology which makes the original shift space dense in it. In particular, such new shift space can be defined to be compact (or locally compact), thus it can be seen as a compactification of the original shift space.

In this thesis we propose a new class of shift spaces, called *blur shift spaces*, where new symbols are introduced to the original alphabet and used to represent some uncertainties. In our construction, we are free to choose which subsets will be used as symbols, and by this choice, we can turn any classical shift space in a compact or a locally compact new shift space. Thus, a particular feature of our construction is the possibility to serve as a multi-point compactification scheme for one-sided shift spaces.

Once the space and the topology are defined, we give some examples. In particular, Ott-Tomforde-Willis shifts and Gonçalves-Royer ultragraph shifts can be seen as examples of blur shift spaces. We follow by studying some properties of the topology like separation axioms, countability axioms and metrizability. To conclude this work we use the generalized sliding block codes to characterize functions among blur shift spaces which are continuous and shift-commuting.

2 PRELIMINARIES

The purpose of this chapter is to present basic concepts about symbolic dynamics in the classical framework, set notation, and to offer an overview on the state of art for previous works that have influenced the construction of the main object of this thesis, the *blur shift spaces*.

Given a sequence $\mathrm{x} = (x_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, for each $\ell, k \in \mathbb{N}$ with $\ell \leq k$ the finite list $(x_\ell \dots x_k) \in \mathcal{A}^{k-\ell+1}$ will be called **word** or *block*, and denoted as $\mathrm{x}_{[\ell,k]}$. We endow $\mathcal{A}^{\mathbb{N}}$ with the prodiscrete topology, that is, on $\mathcal{A}$ we consider the discrete topology, and on $\mathcal{A}^{\mathbb{N}}$ the product topology. Notice that the prodiscrete topology on $\mathcal{A}^{\mathbb{N}}$ can be generated by the basic open sets called **cylinders**, which are defined, for each choice of letters $a_0,\ a_1,\ \dots,\ a_{n-1} \in \mathcal{A}$ as

$$[a_0a_1\dots a_{n-1}] := \{(x_i)_{i\in\mathbb{N}} :\ x_j = a_j\ \forall\, j = 0, \dots, n-1\}. \tag{1}$$

This is one among many possible choices of cylinders, by choosing a finite number of coordinates to be fixed. We have chosen to fix the first n coordinates. These cylinders are clopen sets.

The classical theory of symbolic dynamics is concerned with finite alphabets. In this case Tychonoff's theorem implies that $\mathcal{A}^{\mathbb{N}}$ is compact. However, when $\mathcal{A}$ is not finite, $\mathcal{A}^{\mathbb{N}}$ is not even locally compact. We prove this affirmation, as done in [15]. Take $\mathcal{A} = \{a_1, a_2, \dots\}$, a countably infinite alphabet, and suppose, for a contradiction, that $\mathcal{A}^{\mathbb{N}}$ is a locally compact space. Choosing any point $\mathrm{x} \in \mathcal{A}^{\mathbb{N}}$ there would be a compact subspace C of $\mathcal{A}^{\mathbb{N}}$ and an open set U with $\mathrm{x} \in U \subset C$. Taking the closure of U, we would have a compact subset $\bar{U}$. On the other hand, the closure of any open set U in $\mathcal{A}^{\mathbb{N}}$ cannot be sequentially compact. Indeed, any open set U must contain a basis element $[x_0x_1\dots x_{m-1}]$. Defining $\mathrm{x}^n := x_0\dots x_{m-1}a_na_na_n\dots$, we have that $\{\mathrm{x}^n\}_{n=1}^{\infty}$ is a sequence in $[x_0x_1\dots x_{m-1}]$ without a convergent subsequence. This is a contradiction, completing the affirmation.

This topology is metrizable. Take any pair $\mathrm{x}, \mathrm{y} \in \mathcal{A}^{\mathbb{N}}$ and define $i(\mathrm{x}, \mathrm{y}) = inf\{i \in \mathbb{N} : x_i \neq y_i\}$, then one possible metric is the following:

$$d(\mathrm{x}, \mathrm{y}) := \begin{cases} 1/2^{i(\mathrm{x},\mathrm{y})} & , \ \text{if } \mathrm{x} \neq \mathrm{y}, \\ 0 & , \ \text{if } \mathrm{x} = \mathrm{y}. \end{cases} \tag{2}$$

Given a set $\mathrm{F} \subset \bigcup_{n\geq 1} \mathcal{A}^n$, called **set of forbidden words**, the **shift space** defined by F is the set $X_{\mathrm{F}} := \{\mathrm{x} \in \mathcal{A}^{\mathbb{N}} : \mathrm{x}_{[\ell,k]} \notin \mathrm{F},\ \forall\, \ell, k \in \mathbb{N}\}$. It is well known that $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ is a shift space (for some $\mathrm{F} \subset \bigcup_{n\geq 1} \mathcal{A}^n$) if and only if it is closed with respect to the topology of $\mathcal{A}^{\mathbb{N}}$ and σ-invariant, that is, $\sigma(\Lambda) \subset \Lambda$. Let $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ be a shift space, then by considering on Λ the topology induced from $\mathcal{A}^{\mathbb{N}}$, i.e., the topology whose basis elements are the sets $[a_0a_1\dots a_{n-1}]_\Lambda := [a_0a_1\dots a_{n-1}] \cap \Lambda$, and by restricting σ to Λ, the pair (Λ, σ) is a topological dynamical system.

We proceed with general notions for shift spaces. Define $B_0(\Lambda) = \{\epsilon\}$, where ϵ represents the empty word (that is, the identity of the free group over $\mathcal{A}$), and for $n \geq 1$ define $B_n(\Lambda)$ as the set of all **words of length** n that appear in some sequence of Λ, that is,

$$B_n(\Lambda) := \{x_{[0,n-1]} \in \mathcal{A}^n : \ x \in \Lambda\}. \tag{3}$$

The **language** of Λ will be

$$B(\Lambda) := \bigcup_{n \geq 0} B_n(\Lambda). \tag{4}$$

It is straightforward that $B_n(\mathcal{A}^{\mathbb{N}}) = \mathcal{A}^n$ for all $n \geq 1$. Given $u = (u_1 \ldots u_m), v = (v_1 \ldots v_n) \in B(\mathcal{A}^{\mathbb{N}})$ we define the *concatenation* of u and v as $uv = (u_1 \ldots u_m v_1 \ldots v_n) \in B(\mathcal{A}^{\mathbb{N}})$. We also notice that $B_1(\Lambda)$ is the set of all letters of $\mathcal{A}$ that are used in some sequence of Λ. In particular, $B_1(\mathcal{A}^{\mathbb{N}}) = \mathcal{A}$.

Given $w \in B(\mathcal{A}^{\mathbb{N}})$ we define the **follower set** of w in Λ as the set

$$\mathcal{F}_\Lambda(w) := \{a \in \mathcal{A} : \ wa \in B(\Lambda)\}. \tag{5}$$

In an analogous way, we define the **predecessor set** of $w \in B(\mathcal{A}^{\mathbb{N}})$ as the set $\mathcal{P}(w) := \{a \in \mathcal{A} : \ aw \in B(\Lambda)\}$. Given $A \subset B(\mathcal{A}^{\mathbb{N}})$ we will denote

$$\mathcal{F}_\Lambda(A) = \bigcup_{w \in A} \mathcal{F}_\Lambda(w) \qquad \text{and} \qquad \mathcal{P}_\Lambda(A) = \bigcup_{w \in A} \mathcal{P}_\Lambda(w).$$

Note that $\mathcal{F}_\Lambda(\epsilon) = \mathcal{P}_\Lambda(\epsilon) = B_1(\Lambda)$, and $\mathcal{F}_\Lambda(w)$ is empty if and only if $w \notin B(\Lambda)$, thus $\mathcal{F}_\Lambda(A) = \mathcal{F}_\Lambda(A \cap B(\Lambda))$.

2.1 SLIDING BLOCK CODES AND THE CURTIS-HEDLUND-LYNDON THEOREM

Definition 2.1. *A map $\Phi : \Lambda \subset \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ is a (classical)* **sliding block code** *if there are $r \in \mathbb{N}$ and a function $\phi : B_{r+1}(\Lambda) \to \mathcal{B}$, for some nonempty set $\mathcal{B}$, such that*

$$(\Phi(x))_i = \phi(x_i x_{i+1} \ldots x_{i+r}), \qquad \forall\, x \in \Lambda, \forall\, i \in \mathbb{N}. \tag{6}$$

The function ϕ is called a *local rule* or a *block map*, r is called the *anticipation*. Intuitively, we can think that to determine the i^{th} coordinate $(\Phi(x))_i$, as in (6), the local rule ϕ "looks at the window" of coordinates of x from i to $i + n$. To compute the next coordinate, the window is slid one position to the right.

When $\mathcal{A}$ is finite, the Curtis-Hedlund-Lyndon Theorem (see [13, Theorem 6.2.9]) states that a map $\Phi : \Lambda \subset \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ is a sliding block code if, and only if, it is continuous and shift commuting, that is, $\Phi \circ \sigma = \sigma \circ \Phi$. However, if $\mathcal{A}$ is infinite, the authors in [17] give the example $\Phi : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$, $(\Phi(x))_i = x_{i+x_i}$, for all $x \in \mathcal{A}^{\mathbb{N}}, i \in \mathbb{N}$. This function Φ is continuous and shift-commuting, but is not a (classical) sliding block code since is impossible to describe Φ by a local rule. Although, this map will be a generalized sliding block code, as defined in [9] and in [17]. This alternative definition for sliding block codes coincides with the classical one, when the alphabet is finite, but enlarges the class of sliding block codes when the alphabet is infinite.

Definition 2.2. *A map* $\Phi : \Lambda \subset \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ *is a* **generalized sliding block code** *if*

$$(\Phi(x))_i = \sum_{b \in \mathcal{B}} b \mathbf{1}_{C_b} \circ \sigma^i(x), \qquad \forall\, x \in \Lambda, \forall\, i \in \mathbb{N} \tag{7}$$

where $\{C_b\}_{b\in\mathcal{B}}$ *is a partition of* Λ *such that each nonempty* C_b *is a union of cylinders of* Λ*,* $\mathbf{1}_{C_b}$ *is the characteristic function of the set* C_b *and* $\sum$ *stands for the symbolic sum.*

Roughly speaking, a generalized sliding block code is such that $(\Phi(x))_i$ depends only on a finite number of coordinates of x, namely $(x_i, \dots, x_{i+r})$, for some $r \geq 0$, which does not depend on i, but depends on the configuration of x, from x_i on. Intuitively, to determine $(\Phi(x))_i$, the local rule "looks at a window" with possibly variable lenght.

Provided with this generalization, the authors obtained the result that a map $\Phi : \Lambda \subset \mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ is a generalized sliding block if, and only if, it is continuous and shift commuting, i.e., a more general version of the Curtis-Hedlund-Lyndon Theorem.

2.2 OTT-TOMFORDE-WILLIS SHIFT SPACES

In [15] Ott, Tomforde and Willis proposed a new type of shift spaces, which will refer here as *Ott-Tomforde-Wills shift spaces*. Such shift spaces are obtained by using the minimal compactification to a given alphabet $\mathcal{A}$. If $\mathcal{A}$ is infinite, the minimal compactification is the Alexandroff compactification, which adds a new point "∞", establishing a new alphabet $\bar{\mathcal{A}} := \mathcal{A} \cup \{\infty\}$ and considering on $\bar{\mathcal{A}}$ the topology given by:

$$\{U : U \subset \mathcal{A}\} \cup \{\bar{\mathcal{A}} \setminus F : F \text{ is a finite set of } \mathcal{A}\}.$$

Since $\mathcal{A}$ is given the discrete topology, $\bar{\mathcal{A}}$ is a compact Hausdorff space, and by Tychonoff's theorem, $\bar{\mathcal{A}}^{\mathbb{N}}$ is compact Hausdorff. In this scheme, we could intuitively think a sequence $(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}}$ where $x_k = \infty$ for some k, as a failure in determining the symbol at position k among the infinite ones possible. In most of the cases it becomes natural to assume that as long as we are not able to determine the symbol in the position k we will not be able to determine the symbol at any position after k. Under such assumption, we can consider the equivalence relation $\sim$ in $\bar{\mathcal{A}}^{\mathbb{N}}$ given by

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} \qquad \Leftrightarrow$$

$$\min\{j : x_j = \infty\} = \min\{j : y_j = \infty\} =: k, \ \textit{and } x_i = y_i, \ \forall\, i < k,$$

and therefore define $\Sigma_{\mathcal{A}} := \bar{\mathcal{A}}^{\mathbb{N}}_{/\sim}$ with the quotient topology. The set $\Sigma_{\mathcal{A}}$ can be identified with the set $\{(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} : \ x_i = \infty \Rightarrow x_{i+1} = \infty\}$, by taking an adequate representative of each equivalence class. If $\mathrm{x} = x_0x_1 \cdots \in \Sigma_{\mathcal{A}}$ has an ∞ occurring first at position k, we can write $\mathrm{x} = x_0x_1 \ldots x_{k-1}$, *a finite sequence*, and denote $\Sigma_{\mathcal{A}}^{fin}$ as the set of all finite sequences of $\Sigma_{\mathcal{A}}$, i.e., $\Sigma_{\mathcal{A}}^{fin} := \bigcup_{k\in\mathbb{N}} \mathcal{A}^k$. Defining $\Sigma_{\mathcal{A}}^{inf} := \mathcal{A}^{\mathbb{N}}$, the set of all *infinite sequences* of $\Sigma_{\mathcal{A}}$, we can write $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}}^{inf} \cup \Sigma_{\mathcal{A}}^{fin}$. We will refer to the constant sequence with all positions equal to ∞ as $\oslash := (\infty, \infty, \ldots)$. The *length* of $\mathrm{x} \in \Sigma_{\mathcal{A}}$, denoted $l(\mathrm{x})$, is defined to be ∞ if x is an infinite sequence, and if $\mathrm{x} = x_0x_1 \ldots x_{k-1}$, $l(\mathrm{x}) = k$. Note that a point $\mathrm{x} \in \Sigma_{\mathcal{A}}$ has finite length if, and only if, $\mathrm{x} \in \Sigma_{\mathcal{A}}^{fin}$.

Next the authors give a basis for the topology on $\Sigma_{\mathcal{A}}$. Given two elements $\mathrm{x} = (x_0x_1 \ldots x_m)$ and $\mathrm{y} = (y_0y_1 \ldots y_n) \in \Sigma_{\mathcal{A}}^{fin}$, we may form their *concatenation* xy :

$$\mathrm{xy} = (x_0x_1 \ldots x_m y_0 y_1 \ldots y_n).$$

Similarly we can concatenate an element $\mathrm{x} \in \Sigma_{\mathcal{A}}^{fin}$ with an element $\mathrm{y} \in \Sigma_{\mathcal{A}}^{inf}$ to obtain $\mathrm{xy} \in \Sigma_{\mathcal{A}}^{inf}$. We interpret the concatenation $\mathrm{x}\oslash = \mathrm{x}$ for any $\mathrm{x} \in \Sigma_{\mathcal{A}}^{fin}$.

Definition 2.3. *Given* $\mathrm{x} \in \Sigma_{\mathcal{A}}^{fin}$ *and a finite set* $F \subset \mathcal{A}$, *a* **generalized cylinder** *of* $\Sigma_{\mathcal{A}}$ *is the set defined as:*

$$Z(\mathrm{x}, F) := \begin{cases} \{\mathrm{y} \in \Sigma_{\mathcal{A}} : y_i = x_i \ \forall\ i = 0, \ldots, l(x)-1, y_{l(x)} \notin F\} & , \ \textit{if}\ \mathrm{x} \neq \oslash, \\ \{\mathrm{y} \in \Sigma_{\mathcal{A}} : y_0 \notin F\} & , \ \textit{if}\ \mathrm{x} = \oslash. \end{cases} \tag{8}$$

If $F = \emptyset$, the set $Z(\mathrm{x}, F)$ is simply denoted by $Z(\mathrm{x})$, which are the usual cylinders, as defined in (1). Furthermore, the collection of generalized cylinders, as in (8), is a basis for the topology of Σ_A, consisting of compact open subsets.

Since the quotient map is continuous, $\Sigma_{\mathcal{A}}$ is a compact Hausdorff space (Proposition 2.5 in [15]). Moreover it is a totally disconnected (since the collection of generalized cylinders is a clopen basis) metrizable space. In Section 3.3.2 we make use of the ideas from [11] and [15] to define a family of metrics for the particular case of second-countable blur shift spaces. The family of metrics defined for the Ott-Tomforde-Willis shift spaces in [15], when the alphabet is countable, can be seen as a particular case.

Definition 2.4. *The* **shift map** *is the function* $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ *defined by:*

$$\sigma(\mathrm{x}) = \begin{cases} x_1x_2 \ldots & , \ \textit{if}\ \mathrm{x} = x_0x_1 \cdots \in \mathcal{A}^{\mathbb{N}} \\ x_1x_2 \ldots x_{n-1} & , \ \textit{if}\ \mathrm{x} = x_0x_1 \ldots x_{n-1} \in \bigcup_{k=2}^{\infty} \mathcal{A}^k \\ \oslash & , \ \textit{if}\ \mathrm{x} \in \mathcal{A}^1 \cup \{\oslash\}. \end{cases} \tag{9}$$

The shift map is continuous at all points in $\Sigma_{\mathcal{A}} \setminus \{\oslash\}$ and discontinuous at the point $\oslash = (\infty, \infty, \ldots)$ (see [15, Proposition 2.23]). The pair $(\Sigma_{\mathcal{A}}, \sigma)$ is called the *one-sided Ott-Tomforde-Willis full shift*, where $\Sigma_{\mathcal{A}}$ is the topological space defined above, and σ is the shift map.

Before defining shift spaces as subspaces of the full shift, we need some auxiliary notations and to define the *infinite-extension property*.

Given $\Lambda \subset \Sigma_{\mathcal{A}}$, let $\Lambda^{fin} := \Lambda \cap \Sigma_{\mathcal{A}}^{fin}$ and $\Lambda^{inf} := \Lambda \cap \Sigma_{\mathcal{A}}^{inf}$ be the set of all finite sequences of Λ and the set of all infinite sequences of Λ, respectively. Let $L_\Lambda := B_1(\Lambda) \setminus \{\infty\}$, the set of all symbols used by elements of Λ.

Definition 2.5. *We say that $\Lambda \subset \Sigma_{\mathcal{A}}$ satisfies the* **infinite-extension property** *if*

i. $\oslash \in \Lambda^{fin}$ *if and only if* $|L_\Lambda| = \infty$;

ii. $\mathsf{x} \in \Lambda^{fin} \setminus \{\oslash\}$ *if and only if* $|\mathcal{F}_\Lambda(\mathsf{x})| = \infty$.

This definition, as given in [9] page 3, is equivalent to the following one, as given in [15]: If $\Lambda \subset \Sigma_{\mathcal{A}}$, we say that Λ satisfies the **infinite-extension property** if for all $\mathsf{x} \in \Lambda$ with $l(\mathsf{x}) < \infty$, there are infinitely many $a \in \mathcal{A}$ such that $Z(\mathsf{x}a) \cap \Lambda \neq \emptyset$. In other words, Λ has the infinite-extension property if and only if whenever $\mathsf{x} \in \Lambda$ with $l(\mathsf{x}) < \infty$, then

$$|\{a \in \mathcal{A} : \mathsf{x}a\mathsf{y} \in \Lambda \text{ for some } \mathsf{y} \in \Sigma_{\mathcal{A}}\}| = \infty.$$

Definition 2.6. *A subset $\Lambda \subset \Sigma_{\mathcal{A}}$ is called an* **Ott-Tomforde-Willis shift space** *over $\mathcal{A}$ if the three conditions hold:*

i. Λ *is closed with respect to the topology of* $\Sigma_{\mathcal{A}}$;

ii. Λ *is invariant under the shift map, that is,* $\sigma(\Lambda) \subset \Lambda$;

iii. Λ *satisfies the infinite-extension property.*

As mentioned before, when defining a shift space over a finite alphabet, one requires only conditions 1 and 2 above. In the context of O-T-W (Ott-Tomforde-Willis) shift spaces, condition 3 is then added and ensures that there are always infinite sequences in a nonempty shift space. Moreover, Λ^{inf} is dense in Λ, as proved in [15] Proposition 3.8. Considering the subspace topology, the shifts Λ inherits the topology of $\Sigma_{\mathcal{A}}$. We write its basic elements:

$$Z_\Lambda(\mathsf{x}, F) := Z(\mathsf{x}, F) \cap \Lambda = \{\mathsf{xy} : \mathsf{xy} \in \Lambda \text{ and } y_0 \notin F\}$$

for any $\mathsf{x} \in \Sigma_{\mathcal{A}}^{fin}$ and finite subsets $F \subset \mathcal{A}$.

Remark 2.7. For the particular case of an infinite alphabet, the minimal compactification of $\mathcal{A}$ is the Alexandroff compactification, however when $\mathcal{A}$ is finite, it is already compact with the discrete topology, thus the minimal compactification of $\mathcal{A}$ is itself, i.e., $\bar{\mathcal{A}} = \mathcal{A}$. Hence all statements about the symbol ∞ are vacuous and the usual definition of the full shift $\Sigma_{\mathcal{A}} := \mathcal{A}^{\mathbb{N}}$ with the prodiscrete topology is recovered. Moreover $\Sigma_{\mathcal{A}}$ contains no finite sequences and any subset of $\Sigma_{\mathcal{A}}$ vacuously satisfies the infinite-extension property. In this case of finite alphabet, a subset $\Lambda \subset \Sigma_{\mathcal{A}}^{inf}$ is a shift space if, and only if, it is closed and σ-invariant ($\sigma(\Lambda) \subset \Lambda$), the "classical theory" of shift spaces is then recovered. We also single out that $\Lambda^{inf} = \Lambda$, $\Lambda^{fin} = \emptyset$ and that the collection of cylinder sets

$$\{Z(x_0x_1\dots x_{n-1}) : n \in \mathbb{N} \text{ and } x_i \in \mathcal{A} \text{ for } 0 \leq i \leq n-1\}$$

form a basis for the topology on $\Sigma_{\mathcal{A}}$. □

One more feature of these shift spaces to be mentioned is the equivalent definition in terms of the forbidden words. Given $\mathbf{F} \subset \Sigma_{\mathcal{A}}^{fin}$, define

$$X_{\mathbf{F}}^{inf} := \{\mathsf{x} \in \Sigma_{\mathcal{A}}^{inf} : B(\{\mathsf{x}\}) \cap \mathbf{F} = \emptyset\},$$

which is the set of all infinite sequences that do not have a subblock in **F**, and define

$$X_{\mathbf{F}}^{fin} := \{\mathsf{x} \in B(X_{\mathbf{F}}^{inf}) : |\mathcal{F}_{X_{\mathbf{F}}^{inf}}(\mathsf{x})| = \infty\},$$

the set of all finite sequences which satisfy the "infinite-extension property". We can, alternatively, write

$$X_{\mathbf{F}}^{fin} = \begin{array}{l}\{\mathsf{x} \in \Sigma_{\mathcal{A}}^{fin} : \text{ there are infinitely many } a \in \mathcal{A} \\ \text{ for which there exists } \mathsf{y} \in \Sigma_{\mathcal{A}}^{inf} \text{ such that } \mathsf{x}a\mathsf{y} \in \Sigma_{\mathcal{A}}^{inf}\}.\end{array} \tag{10}$$

We can then define $X_F := X_{\mathbf{F}}^{inf} \cup X_{\mathbf{F}}^{fin}$, and prove that $\Lambda \subset \Sigma_{\mathcal{A}}$ is a shift space if, and only if, $\Lambda = X_F$, for some $\mathbf{F} \subset \Sigma_{\mathcal{A}}^{fin}$, see Theorem 3.16 in [15].

The authors in [15] continue with their beautiful exposition, dealing with shifts of finite type, M-Step, row-finite shifts, and so on. We shall remark they used this compactification to prove that two Ott-Tomforde-Willis edge shifts over countable alphabets will have the groupoids and the C^*-algebras of their associated graphs being isomorphic, whenever they are topologically conjugate through a map Φ such that $\Phi^{-1}(\oslash) = \{\oslash\}$.

2.3 GONÇALVES-ROYER ULTRAGRAPH SHIFT SPACES

In [7] the authors define a notion of shift spaces associated to ultragraphs, which we will refer here as *Gonçalves-Royer ultragraph shift spaces*. In this section we discuss ultragraphs, shift spaces associated, the topology and its properties. We follow closely the notation as in [8].

Definition 2.8. *An* **ultragraph** *is a quadruple $\mathfrak{G} = (V, E, s, r)$ consisting of a countable set of vertices V, a countable set of edges E, a function $s : E \to V$, called* source, *and a function $r : E \to 2^V$, called* range, *where 2^V stands for the power set of V.*

Definition 2.9. *Let $\mathfrak{G}$ be an ultragraph. Define $\mathcal{V}_0$ to be the smallest subset of 2^V that contains $\{v\}$ for all $v \in V$, contains $r(e)$ for all $e \in E$, and is closed under finite unions and finite intersections.*

Taking $E = \mathcal{A}$, the shift space defined from an ultragraph $\mathfrak{G}$ is the set $X_{\mathfrak{G}} \subset \mathcal{A}^{\mathbb{N}}$ given by

$$X_{\mathfrak{G}} := \{(x_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : \ s(x_{i+1}) \in r(x_i) \ \forall i \in \mathbb{N}\},$$

which can be seen as an infinite walk on the ultragraph $\mathfrak{G}$.

Definition 2.10. *We say that a set H in $\mathcal{V}_0$ is an* **infinite emitter** *whenever the set $s^{-1}(H) = \{e \in E : s(e) \in H\}$ is infinite, and denote by $\mathcal{V}_1$ the set of all infinite emitters. In symbols:*

$$\mathcal{V}_1 := \{H \in \mathcal{V}_0 : \ |s^{-1}(H)| = \infty\}.$$

Definition 2.11. *A set $H \in \mathcal{V}_1$ is called a* **minimal infinite emitter** *if it contains no proper subsets in $\mathcal{V}_1$. We denote the set of all minimal infinite emitters by $\hat{\mathcal{V}}$.*

Definition 2.12. *The* **Gonçalves-Royer ultragraph shift space***, denoted by $\Sigma_{\mathfrak{G}}$, is defined as*

$$\Sigma_{\mathfrak{G}} := X_{\mathfrak{G}} \cup \{(w_i)_{i\in\mathbb{N}} : \ \exists\, n \in \mathbb{N} \text{ s. t. } w_0 \ldots w_{n-1} \in B(X_{\mathfrak{G}}), \ w_i = H \ \forall\, i \geq n,$$
$$\text{where } H \in \hat{\mathcal{V}} \text{ and } H \subset r(w_{n-1})\}.$$

Notice that if we identify a point $(w_0 \ldots w_{n-1} HHH \ldots) \in \Sigma_{\mathfrak{G}}$ with $(w_0 \ldots w_{n-1} H)$, we can interpret an Gonçalves-Royer ultragraph shift space as an union of a set of infinite sequences, denoted $X_{\mathfrak{G}}$, and a set of finite sequences $\{(w_0 \ldots w_{n-1} H) : n \in \mathbb{N}, w_0 \ldots w_{n-1} \in B(X_{\mathfrak{G}}), H \in \hat{\mathcal{V}}, H \subset r(w_{n-1})\}$, similarly to the O-T-W shift spaces. Notice that for the Gonçalves-Royer ultragraph shifts, it is possible to specify the symbol H which comes after w_{n-1} among infinitely many possible ones, and can be thought as an indication that the symbol after w_{n-1} lies in $s^{-1}(H)$. Recall that for O-T-W shifts, if $x \in \Sigma_{\mathcal{A}}^{fin}$, $x = x_0 x_1 \ldots x_{k-1}$ means that the symbol after x_{k-1} is the symbol $\infty \in \bar{\mathcal{A}}$, but there is no need to write it since it is the only possibility for a finite sequence.

2.3.1 The topology of the Gonçalves-Royer ultragraph shift spaces

In this section, given an ultragraph $\mathfrak{G}$, we define paths, ultrapaths and concatenation among them. Then we define the generalized cylinders and the associated topology on $\Sigma_{\mathfrak{G}}$.

Let $\mathfrak{G}$ be an ultragraph. A **finite path** in $\mathfrak{G}$ is either an element of $\mathcal{V}_0$ or a sequence of edges $\alpha = (\alpha_i)_{i=0}^{k-1}$ in E, where $s(\alpha_{i+1}) \in r(\alpha_i)$ for $0 \leq i \leq k-1$. The set of finite paths in $\mathfrak{G}$ is denoted by $\mathfrak{G}^*$.

If we write $\alpha = (\alpha_i)_{i=0}^{k-1}$, then the length $|\alpha|$ of α is just k. The length $|A|$ of a path $A \in \mathcal{V}^0$ is zero. We define $r(\alpha) = r(\alpha_{k-1})$ and $s(\alpha) = s(\alpha_0)$. For $A \in \mathcal{V}^0$, we set $r(A) = A = s(A)$.

An **infinite path** in $\mathfrak{G}$ is an element of $X_{\mathfrak{G}}$, i.e., an infinite sequence of edges $\gamma = (\gamma_i)_{i\geq 0}$ in $\prod E$, such that $s(\gamma_{i+1}) \in r(\gamma_i)$ for all i. The length $|\gamma|$ of $\gamma \in X_{\mathfrak{G}}$ is defined to be ∞, and we define $s(\gamma) = s(\gamma_0)$. A vertex v in $\mathfrak{G}$ is called a *sink* if $\left|s^{-1}(v)\right| = 0$ and is called an **infinite emitter** if $\left|s^{-1}(v)\right| = \infty$. The ultragraphs in [7] are assumed to have no sinks.

We set $\mathfrak{p}^0_{\mathfrak{G}} := \mathcal{V}^0$ and, for $n \geq 1$, we define $\mathfrak{p}^n_{\mathfrak{G}} := \{(\alpha, A) : \alpha \in \mathfrak{G}^*, |\alpha| = n, A \in \mathcal{V}^0, A \subseteq r(\alpha)\}$, and

$$\mathfrak{p}_{\mathfrak{G}} := \bigcup_{n\geq 0} \mathfrak{p}^n_{\mathfrak{G}}.$$

We specify that $(\alpha, A) = (\beta, B)$ if, and only if, $\alpha = \beta$ and $A = B$. We define the length of $(\alpha, A) \in \mathfrak{p}_{\mathfrak{G}}$ as $|(\alpha, A)| := |\alpha|$. We call $\mathfrak{p}_{\mathfrak{G}}$ the **ultrapath space** associated with $\mathfrak{G}$ and the elements of $\mathfrak{p}_{\mathfrak{G}}$ are called **ultrapaths**. Each $A \in \mathcal{V}^0$ is regarded as an ultrapath of length zero and can be identified with the pair (A, A). We embed the set of finite paths $\mathfrak{G}^*$ in $\mathfrak{p}_{\mathfrak{G}}$ by sending α to $(\alpha, r(\alpha))$. We extend the range map r and the source map s to $\mathfrak{p}_{\mathfrak{G}}$ by the formulas, $r((\alpha, A)) = A$, $s((\alpha, A)) = s(\alpha)$ and $r(A) = s(A) = A$.

Given $\alpha = (\alpha_i)_{i=0}^{k-1}$ and $\beta = (\beta_i)_{i=0}^{\ell-1}$ in $\mathfrak{G}^*$ with $s(\beta) \in r(\alpha)$, we define the *concatenation* of α with β as $\alpha\beta := (\alpha_0 \dots \alpha_{k-1}\beta_0 \dots \beta_{\ell-1}) \in \mathfrak{G}^*$. Given $\alpha \in \mathfrak{G}^*$ we say that $\alpha' \in \mathfrak{G}^*$ is a *prefix* of α if either $\alpha' = \alpha$ or $\alpha = \alpha'\beta$ for some $\beta \in \mathfrak{G}^*$.

Given $x \in \mathfrak{p}_{\mathfrak{G}}$ and $y \in \mathfrak{p}_{\mathfrak{G}} \cup \mathfrak{p}^\infty_{\mathfrak{G}}$ such that $s(y) \subset r(x)$ (if $|y| = 0$) or $s(y) \in r(x)$ (if $|y| \geq 1$), we define the *concatenation of x and y*, denoted by xy, as follows:

$$\begin{array}{lcl} x = A & \Rightarrow & xy := y; \\ x = (\alpha, A) \text{ and } y = B & \Rightarrow & xy := (\alpha, B); \\ x = (\alpha, A) \text{ and } y = (\beta, B) & \Rightarrow & xy := (\alpha\beta, B); \\ x = (\alpha, A) \text{ and } y = (y_i)_{i\geq 0}\dots & \Rightarrow & xy := (\alpha_0 \dots \alpha_{|\alpha|-1} y_0 y_1 y_2 \dots) \end{array} \tag{11}$$

Given $x \in \mathfrak{p}_{\mathfrak{G}} \cup \mathfrak{p}^\infty_{\mathfrak{G}}$, we say that x has $x' \in \mathfrak{p}_{\mathfrak{G}}$ as a *prefix* if $x = x'y$, for some $y \in \mathfrak{p}_{\mathfrak{G}} \cup \mathfrak{p}^\infty_{\mathfrak{G}}$.

Definition 2.13. *For a finite path α in $\mathfrak{G}$, we say that H is a* minimal infinite emitter in $r(\alpha)$ *if H is a minimal infinite emitter and $H \subset r(\alpha)$. We denote the set of all minimal infinite emitters in $r(\alpha)$ by M_α, and define*

$$\mathfrak{p}_{\mathfrak{G}min} := \{(\alpha, H) \in \mathfrak{p}_{\mathfrak{G}} : H \in M_\alpha\}.$$

An element $(\alpha, H) \in \mathfrak{p}_{\mathfrak{G}min}$ can be identified with $(\alpha, HHH\dots)$. If we write

$$\Sigma^{fin}_{\mathfrak{G}} := \{(x_i)_{i\geq 0} : (x_i)_{i\geq 0} = (\alpha_0\alpha_1 \dots \alpha_{k-1}HH\dots) \text{ with } (\alpha_0\alpha_1 \dots \alpha_{k-1}, H) \in \mathfrak{p}_{\mathfrak{G}min}\},$$

and $\Sigma^{inf}_{\mathfrak{G}} := X_{\mathfrak{G}} = \mathfrak{p}^\infty_{\mathfrak{G}}$, we can write the Gonçalves-Royer ultragraph shift space as

$$\Sigma_{\mathfrak{G}} = \Sigma^{fin}_{\mathfrak{G}} \cup \Sigma^{inf}_{\mathfrak{G}}.$$

To endow $\Sigma_{\mathfrak{G}}$ with a topology, we define a basis whose basic open sets are the generalized cylinders.

Definition 2.14. *Let* $\mathrm{y} \in \mathfrak{p}_{\mathfrak{G}}$ *and a finite set* $F \subset s^{-1}(r(\mathrm{y}))$*, the sets*

$$D_{\mathrm{y},F} := \{\mathrm{x} \in \Sigma_{\mathfrak{G}} : \mathrm{y} \textit{ is a prefix of } \mathrm{x} \textit{ and } \mathrm{x}_{|\mathrm{y}|+1} \notin F\} \tag{12}$$

are the **generalized cylinders***. If* $F = \emptyset$ *we write* $D_{\mathrm{y}} := D_{\mathrm{y},F}$*.*

The generalized cylinders are clopen. Furthermore, since the alphabet is countable, they form a countable basis for $\Sigma_{\mathfrak{G}}$. Endowed with this topology, $\Sigma_{\mathfrak{G}}$ is a Hausdorff space. From Urysohn's metrization Theorem, $\Sigma_{\mathfrak{G}}$ is metrizable, see [7, Section 3] for details. In [11] the authors define a family of metrics that induces the topology on $\Sigma_{\mathfrak{G}}$. To do so, notice that since $\mathfrak{p}_{\mathfrak{G}}$ is countable, we can order it: $\mathfrak{p}_{\mathfrak{G}} = \{p_1, p_2, p_3, \ldots\}$. The metric is then defined, for each pair $\mathrm{x}, \mathrm{y} \in \Sigma_{\mathfrak{G}}$ as

$$d(\mathrm{x},\mathrm{y}) := \begin{cases} 1/2^i & , \text{ where } i \in \mathbb{N} \text{ is the smallest integer such that} \\ & \ \ p_i \text{ is a prefix either of x or of y,} \\ 0 & , \text{ if x = y.} \end{cases} \tag{13}$$

Notice the metric depends on the order $\{p_1, p_2, p_3, \ldots\}$.

2.3.2 Condition RFUM

A generalized cylinder may be not compact. For instance, if $\mathfrak{G}$ is an ultragraph and e is an edge for which $r(e)$ contains infinite vertices, each of them being an infinite emitter, then $D_e = \{\mathrm{x} \in \Sigma_{\mathfrak{G}} : e \text{ is a prefix of x}\}$ is not (sequentially) compact. To assure that the cylinders are compact, the authors make an additional hypothesis.

Definition 2.15. *An ultragraph* $\mathfrak{G}$ *satisfies* **condition RFUM** *if for each edge* $e \in E$*, its range can be written as*

$$r(e) = \bigcup_{n=1}^{k} A_n, \tag{14}$$

where A_n *is either a minimal infinite emitter or a single vertex.*

In [7] the authors proved that if an ultragraph $\mathfrak{G}$ satisfies the condition RFUM, then each one of the generalized cylinders $D_{\mathrm{y},F}$ is compact, see [7, Prop. 3.12], hence the Gonçalves-Royer ultragraph shift is locally compact. We remark that RFUM is a stronger condition than locally-compactness, as we discuss in Subsection 3.3.3.

Then, it was proved in [7], that two ultragraphs $\mathfrak{G}$ and $\mathfrak{H}$ without sinks, satisfying the RFUM condition, and whose respective Gonçalves-Royer ultragraph shifts are conjugate by a map Φ such that, $\hat{\mathcal{V}}_{\mathfrak{G}} = \bigcup_{H \in \hat{\mathcal{V}}_{\mathfrak{H}}} \Phi^{-1}(H)$, will have associated ultragraph C^*-algebras being isomorphic.

2.3.3 The shift map

To complete these preliminaries, we give the precise definition of the shift map and the Gonçalves-Royer shift space associated to an ultragraph.

Definition 2.16. *We define the* **shift map** $\sigma : \Sigma_{\mathfrak{G}} \to \Sigma_{\mathfrak{G}}$ *by cases:*

$$\sigma(x) = \begin{cases} x_1 x_2 \dots & \textit{if } x = x_0 x_1 \dots \in X_{\mathfrak{G}} \\ (\alpha_1 \dots \alpha_{n-1}, H) & \textit{if } x = (\alpha_0 \dots \alpha_{n-1}, H) \in \Sigma_{\mathfrak{G}}^{fin} \textit{ and } |x| > 1 \\ (H, H) & \textit{if } x = (\alpha_0, H) \in \Sigma_{\mathfrak{G}}^{fin} \\ (H, H) & \textit{if } x = (H, H) \in \Sigma_{\mathfrak{G}}^{fin}. \end{cases}$$

The shift map $\sigma : \Sigma_{\mathfrak{G}} \to \Sigma_{\mathfrak{G}}$ is continuous at all points of $\Sigma_{\mathfrak{G}}$ with length greater than zero.

Definition 2.17. *Let $\mathfrak{G}$ be an ultragraph. The* **Gonçalves-Royer ultragraph shift space** *is the pair $(\Sigma_{\mathfrak{G}}, \sigma)$, where σ is the shift map, and $\Sigma_{\mathfrak{G}}$ is the topological space as defined above.*

3 BLUR SHIFT SPACES

One possible way to compare the shift spaces we have mentioned, with this one we are proposing here, is by looking at finite sequences. For all these shift spaces, each finite sequence represents "an escape to infinity". O-T-W shifts can be seen then as a compactification of the full shift, by adding infinitely many points (all the finite sequences) to the space, and each of them represents an escape to infinity, after a number of steps. From this point of view, a Gonçalves-Royer ultragraph shift space is an improvement, since it adds infinitely many points as well, but here finite sequences represent escaping to different kinds of infinities. In particular, Gonçalves-Royer ultragraph shifts depend strongly on the structure of the underlying ultragraph.

The scheme to be developed allows the addition of points to any given shift space, in a way the escapes may be to different kinds of infinities, but differently from Gonçalves-Royer ultragraph shifts, there is freedom to choose the points to represent these infinities. With this (multi-point) scheme we have not only a method which can be applied to any shift space, but also allows to control which of topological properties (compactness, local-compactness, metrizability, etc) the resulting shift space will have.

In what follows, we reproduce the article "Blur shift spaces" [1] co-authored by M. Sobottka.

3.1 INTRODUCTION

Recently it was proposed two new types of shift spaces: Ott-Tomforde-Willis shift spaces [15] and Gonçalves-Royer ultragraph shift spaces [7]. Inspired by C^*-algebras and the boundary path space theories for graphs, an Ott-Tomford-Willis shift space is, roughly speaking, obtained by adding a single new symbol to a given infinite alphabet and then defining an appropriate topology that turns any shift space over this alphabet a compact space. Gonçalves-Royer ultragraph shift spaces are obtained by firstly fixing a classical shift space generated from the walks on an ultragraph, and then finding a (possibly infinite) set of new symbols which are added to the alphabet. Then by defining the correspondent neighborhood system of sequences with those new symbols, the authors find sufficient conditions under which the shift space becomes locally compact. Both, Ott-Tomforde-Willis shifts and Gonçalves-Royer ultragraph shifts were successfully used to find correspondence between the conjugacy of a subclass of Markovian shift spaces and the isomorphism of their C^*-algebras.

The key idea of both, Ott-Tomford-Willis shifts and Gonçalves-Royer ultragraph shifts, is to use new symbols to represent some sets of infinitely many symbols: in Ott-Tomforde-Willis shifts the entire original alphabet is represented by a single new symbol, while in Gonçalves-Royer ultragraph shift spaces many symbols are introduced to represent the so called minimal infinite emitters of a given countably infinite ultragraph. In spite of sharing the same paradigm, these constructions differ in a fundamental (and apparently uncorrelated) way: Ott-Tomford-

Willis shifts are constructed by firstly adding a new symbol to the alphabet (and defining its neighborhood) what makes compact any shift space with this alphabet, while each Gonçalves-Royer ultragraph shift space is *ad-hoc* constructed from a previously defined classical ultragraph shift space.

One possible interpretation for the constructions of Ott-Tomford-Willis shifts and Gonçalves-Royer ultragraph shifts is that the new symbols added to the alphabet are representing a failure in determining what symbol would follow in some sequence when there were infinitely many ones that could follow. This interpretation leads us to propose a construction that encompasses Ott-Tomford-Willis shifts and Gonçalves-Royer ultragraph shifts constructions. The shift spaces we are proposing here will be called blur shift spaces, alluding to the fact that some subsets of infinitely many symbols can be seen as being blurred in a way that we cannot individualize what symbol would follow in some sequence. Differently than occur in Ott-Tomford-Willis shifts and Gonçalves-Royer ultragraph shifts, we are free to choose the subsets of symbols that will be *blurred* in the alphabet, and so we can turn any classical shift space in a compact or locally compact new shift space.

In the subsections below we shall present the background needed to develop the blur shift spaces and the formal constructions of Ott-Tomforde-Willis and Gonçalves-Royer ultragraph shift spaces. In section 3.2, inspired by the schemes proposed in [7] and [15], we propose a general scheme for constructing shift spaces where is considered the possibility of representing undetermined symbols. In Section 3.3 we present a zero-dimensional topology for blur shifts and its properties, with particular focuses on criteria for its metrizability and (local) compactness. In Section 3.4 we characterize continuous shift-commuting maps and generalized sliding block codes.

3.1.1 Background

Let $\mathbb{N}$ denote the set of all nonnegative integers. Given a nonempty set $\mathcal{A}$ (which is called an alphabet), we define the (one-sided) full shift on $\mathcal{A}$ as the set

$$\mathcal{A}^{\mathbb{N}} := \{(x_i)_{i\in\mathbb{N}} :\ x_i \in \mathcal{A}\ \forall i \in \mathbb{N}\},$$

where $\mathbb{N}$ denotes the set of all non-negative integers.

Given a sequence $\mathrm{x} = (x_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, for each $\ell, k \in \mathbb{N}$ with $\ell \leq k$ we will denote the finite word $(x_\ell \ldots x_k) \in \mathcal{A}^{k-\ell+1}$ as $\mathrm{x}_{[\ell,k]}$. We endow $\mathcal{A}$ with the discrete topology and $\mathcal{A}^{\mathbb{N}}$ with the associated prodiscrete topology. We recall that a basis for the topology of $\mathcal{A}^{\mathbb{N}}$ are the cylinders, that clopen sets defined for each given choice of letters $a_0,\ a_1,\ \ldots,\ a_{n-1} \in \mathcal{A}$ as

$$[a_0a_1\ldots a_{n-1}] := \{(x_i)_{i\in\mathbb{N}} :\ x_j = a_j\ \forall j = 0, \ldots, n-1\}.$$

Note that $\mathcal{A}^{\mathbb{N}}$ is compact if and only if $\mathcal{A}$ is finite. Moreover, when $\mathcal{A}$ is not finite, then $\mathcal{A}^{\mathbb{N}}$ is not even locally compact. In any case the topology is metrizable and cylinders are always clopen sets.

Consider on $\mathcal{A}^{\mathbb{N}}$ the **shift map** $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ given by

$$\sigma\big((x_i)_{i\in\mathbb{N}}\big) = (x_{i+1})_{i\in\mathbb{N}}.$$

Given $\mathsf{F} \subset \bigcup_{n\geq 1} \mathcal{A}^n$ (which we will call the **set of forbidden words**), the **shift space** defined by F is the set $X_{\mathsf{F}} := \{\mathsf{x} \in \mathcal{A}^{\mathbb{N}} : \mathsf{x}_{[\ell,k]} \notin \mathsf{F},\ \forall \ell, k \in \mathbb{N}\}$. It is well known that $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ is a shift space (for some $\mathsf{F} \subset \bigcup_{n\geq 1} \mathcal{A}^n$) if and only if it is closed with respect to the topology of $\mathcal{A}^{\mathbb{N}}$ and σ-invariant (that is, $\sigma(\Lambda) \subset \Lambda$). Let $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ be a shift space, then by considering on Λ the topology induced from $\mathcal{A}^{\mathbb{N}}$, that is, the topology whose basis elements are the sets $[a_0a_1\dots a_{n-1}]_\Lambda := [a_0a_1\dots a_{n-1}] \cap \Lambda$, and by restricting σ to Λ, the pair (Λ, σ) is a topological dynamical system.

Define $B_0(\Lambda) = \{\epsilon\}$, where ϵ represents the empty word (that is, the identity of the free group over $\mathcal{A}$), and for $n \geq 1$ define $B_n(\Lambda)$ as the set of all **words of length** n that appear in some sequence of Λ, that is,

$$B_n(\Lambda) := \{\mathsf{x}_{[0,n-1]} \in \mathcal{A}^n :\ \mathsf{x} \in \Lambda\}. \tag{15}$$

The **language** of Λ will be

$$B(\Lambda) := \bigcup_{n\geq 0} B_n(\Lambda). \tag{16}$$

It is straightforward that $B_n(\mathcal{A}^{\mathbb{N}}) = \mathcal{A}^n$ for all $n \geq 1$. Given $\mathsf{u} = (u_1\dots u_m), \mathsf{v} = (v_1\dots v_n) \in B(\mathcal{A}^{\mathbb{N}})$ we define the concatenation of u and v as $\mathsf{uv} = (u_1\dots u_m v_1\dots v_n) \in B(\mathcal{A}^{\mathbb{N}})$. We also notice that $B_1(\Lambda)$ is the set of all letters of $\mathcal{A}$ that are used in sequence of Λ. In particular, $B_1(\mathcal{A}^{\mathbb{N}}) = \mathcal{A}$.

Given $\mathsf{w} \in B(\mathcal{A}^{\mathbb{N}})$ we define the **follower set** of w in Λ as the set

$$\mathcal{F}_\Lambda(\mathsf{w}) := \{a \in \mathcal{A} :\ \mathsf{w}a \in B(\Lambda)\}. \tag{17}$$

In an analogous way, we define the **predecessor set** of $\mathsf{w} \in B(\mathcal{A}^{\mathbb{N}})$ as the set $\mathcal{P}(\mathsf{w}) := \{a \in \mathcal{A} :\ a\mathsf{w} \in B(\Lambda)\}$. Given $A \subset B(\mathcal{A}^{\mathbb{N}})$ we will denote

$$\mathcal{F}_\Lambda(A) = \bigcup_{\mathsf{w}\in A} \mathcal{F}_\Lambda(\mathsf{w}) \qquad \text{and} \qquad \mathcal{P}_\Lambda(A) = \bigcup_{\mathsf{w}\in A} \mathcal{P}_\Lambda(\mathsf{w}).$$

Note that $\mathcal{F}_\Lambda(\epsilon) = \mathcal{P}_\Lambda(\epsilon) = B_1(\Lambda)$, and $\mathcal{F}_\Lambda(\mathsf{w})$ is empty if and only if $\mathsf{w} \notin B(\Lambda)$, and thus $\mathcal{F}_\Lambda(A) = \mathcal{F}_\Lambda(A \cap B(\Lambda))$ and $\mathcal{P}_\Lambda(A) = \mathcal{P}_\Lambda(A \cap B(\Lambda))$.

3.1.2 Ott-Tomforde-Willis shifts and Gonçalves-Royer ultragraph shifts

Suppose that $\mathcal{A}$ is a infinite alphabet (with the discrete topology) and $\mathcal{A}^{\mathbb{N}}$ is the correspondent full shift (with the product topology). The most classical approach to turn a shift space $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ compact, is through the Alexandroff compactification scheme (also known as one-point compactification - see [14]). The Alexandroff compactification can be applied whenever Λ is locally compact, and it consists of adding a new point "∞" to Λ, whose open neighbourhood system will be conformed by the complement of compact sets. Hence we have a new space $\bar{\Lambda} := \Lambda \cup \{\infty\}$ and we extend the shift map $\sigma : \Lambda \to \Lambda$ continuously on $\bar{\Lambda}$ by defining $\sigma(\infty) = \infty$. We could intuitively consider the point ∞ as a failure in determining the symbol at any position among the infinite possible symbols that could appear at any position. This scheme is presented in [4].

Another way to use Alexandroff compactification consists in applying it on the infinite alphabet $\mathcal{A}$. In this procedure, proposed in [15], a new point ∞ is added to $\mathcal{A}$, and its open neighbourhood system is conformed by the complement of finite sets. Thus, we have a new alphabet $\bar{\mathcal{A}} := \mathcal{A} \cup \{\infty\}$ and can consider shift spaces $\Lambda \subset \bar{\mathcal{A}}^{\mathbb{N}}$ with the correspondent product topology. In this scheme, we could intuitively think a sequence $(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}}$ where $x_k = \infty$ for some k, as a failure in determining the symbol at position k among the infinite possible ones. In most of the cases it becomes natural to assume that as long as we are not able to determine the symbol in the position k we will not be able to determine the symbol in any position after k. Under such assumption, we can consider the equivalence relation $\sim$ in $\bar{\mathcal{A}}^{\mathbb{N}}$ given by

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} \qquad \Leftrightarrow$$

$$\min\{j : x_j = \infty\} = \min\{j : y_j = \infty\} =: k, \textit{ and } x_i = y_i, \ \forall\ i < k,$$

and therefore define $\Sigma_{\mathcal{A}} := \bar{\mathcal{A}}^{\mathbb{N}}_{/\sim}$ with the quotient topology. Such compactification scheme was proposed in [15] and studied in [6, 9]. The set $\Sigma_{\mathcal{A}}$ can be identified with the set $\{(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} :\ x_i = \infty \Rightarrow x_{i+1} = \infty\}$. Furthermore, the quotient topology has a clopen basis whose sets were called generalized cylinders.

This compactification scheme has the advantage that we can apply it directly to $\mathcal{A}^{\mathbb{N}}$ (instead of being applied only for locally compact shift spaces, as the previous one). However, the shift map $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ will be continuous at all points but at the sequence $\oslash := (\infty, \infty, \ldots)$.

An **Ott-Tomford-Willis shift space** of $\Sigma_{\mathcal{A}}$ is the closure in $\Sigma_{\mathcal{A}}$ of a shift space $\Lambda \subset \mathcal{A}^{\mathbb{N}}$. Equivalently, $X \subset \Sigma_{\mathcal{A}}$ will be a shift space if and only if it is: *(i)* closed; *(ii)* shift-invariant; and *(iii)* for $a_0 \ldots a_n \in B(X) \cap \mathcal{A}^{n+1}$ we have $(a_0 \ldots a_n \infty\infty \ldots) \in X$ if and only if $\mathcal{F}_X(a_0 \ldots a_n)$ is infinite, the so called infinite-extension property.

This compactification was used to prove that two Ott-Tomforde-Willis edge shifts over countable alphabets (an special case of Markovian shifts) will have both, the groupoids and the C^*-algebras of their associated graphs, being isomorphic, whenever they are topologically conjugated through a map Φ such that $\Phi^{-1}(\oslash) = \{\oslash\}$.

In [7] it was proposed a scheme to turn locally compact a class of ultragraph shifts over an countably infinite alphabet, that is, shift spaces defined from walks on a class of countably infinite ultragraphs. More specifically, given an countably infinite alphabet $\mathcal{A}$, let V be a nonempty set (the set of vertices), $E := \mathcal{A}$ (the set of edges), $s : E \to V$ (the source function), and $r : E \to 2^V$ (the range function, where 2^V stands for the power set of V). Therefore $\mathfrak{G} = (V, E, s, r)$ will be an ultragraph, and the ultragraph shift space defined from $\mathfrak{G}$ is the shift space $X_{\mathfrak{G}} \subset \mathcal{A}^{\mathbb{N}}$ given by

$$X_{\mathfrak{G}} := \{(x_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : \; s(x_{i+1}) \in r(x_i) \; \forall i \in \mathbb{N}\}.$$

We recall that the class of ultragraph shifts coincides with the class of Markov shifts [16, Theorem 7.3].

Once $X_{\mathfrak{G}}$ is given, the authors in [7] define:

$\mathcal{V}_0 \subset 2^V$ as the smallest family of sets which contains $\{v\}$ for all $v \in V$, contains $r(e)$ for all $e \in E$, and which is closed under finite unions and intersections;

$\mathcal{V}_1 := \{H \in \mathcal{V}_0 : \; |s^{-1}(H)| = \infty\}$;

$\hat{\mathcal{V}} \subset \mathcal{V}_1$ as the family of all sets of $\mathcal{V}_1$ which does not contain a proper subset which also belongs to $\mathcal{V}_1$.

Therefore the authors define the **Gonçalves-Royer ultragraph shift space**

$$\Sigma_{\mathfrak{G}} := X_{\mathfrak{G}} \cup \{(w_i)_{i\in\mathbb{N}} : \; \exists\, n \in \mathbb{N} \; s.\,t. \; w_0 \dots w_{n-1} \in B(X_{\mathfrak{G}}), \; w_i = H \; \forall\, i \geq n,$$
$$\text{where} \;\; H \in \hat{\mathcal{V}} \text{ and } H \subset r(w_{n-1})\}.$$

Then, in $\Sigma_{\mathfrak{G}}$ it were defined generalized cylinders in an analogous way to in [15], which form a clopen basis for a topology on $\Sigma_{\mathfrak{G}}$. Therefore, the shift map $\sigma : \Sigma_{\mathfrak{G}} \to \Sigma_{\mathfrak{G}}$, which is defined on $\Sigma_{\mathfrak{G}}$ in the usual way, is always continuous out of the set $\{(HHH\dots) : \; H \in \hat{\mathcal{V}}\}$.

Note that, here again, a sequence $(w_0 \dots w_{n-1}HHH\dots)$ can be intuitively interpreted as a failure in determining the symbol which comes after w_{n-1} among infinitely many possible ones. However, in this scheme it is possible to specify the subset to which the first undetermined symbol belongs (that is, $(w_0 \dots w_{n-1}HHH\dots)$ can be thought as an indication that the symbol after w_{n-1} lies in $s^{-1}(H)$).

In [7] it was considered the RFUM condition on the ultragraphs, which implies the local compactness of the Gonçalves-Royer ultragraph shift. Then, it was proved that two ultragraphs $\mathfrak{G}$ and $\mathfrak{H}$ without sinks, satisfying the RFUM condition, and whose the respective Gonçalves-Royer ultragraph shifts are conjugate by a map Φ such that $\hat{V}_{\mathfrak{G}} = \bigcup_{H\in\hat{\mathcal{V}}_{\mathfrak{H}}} \Phi^{-1}(H)$ will have associated ultragraph C^*-algebras being isomorphic.

3.2 BLUR SHIFT SPACES

In this section we propose a general scheme to define a new type of shift spaces, called here blur shift spaces, where some uncertainties are represented by special symbols. We start by adding new symbols to a given alphabet which will represent some uncertainties, and then we use this extended alphabet to define a shift space with an appropriated topology. Such construction is inspired by the schemes proposed in [7] and [15], and generalizes them.

CONSTRUCTION

Let $\mathcal{A}$ be an alphabet.

Step 1: Let $\mathcal{V} \subset 2^{\mathcal{A}}$ be any family of subsets of $\mathcal{A}$ such that

$$H \in \mathcal{V} \qquad \Rightarrow \qquad |H| = \infty$$

and

$$G, H \in \mathcal{V} \text{ and } G \neq H \qquad \Rightarrow \qquad |G \cap H| < \infty.$$

The sets in $\mathcal{V}$ will be said to be the **blurred sets** of $\mathcal{A}$.

Label each $H \in \mathcal{V}$ with a symbol $\tilde{H}$, and denote by $\tilde{\mathcal{V}}$ the set of all symbols used to label blurred sets.

Step 2: Let $\bar{\mathcal{A}} := \mathcal{A} \cup \tilde{\mathcal{V}}$;

Step 3: Define the full shift $\bar{\mathcal{A}}^{\mathbb{N}}$ and consider the equivalence relation $\sim$ in $\bar{\mathcal{A}}^{\mathbb{N}}$ given by

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} \qquad \Leftrightarrow$$

$$\min\{j : x_j \in \tilde{\mathcal{V}}\} = \min\{j : y_j \in \tilde{\mathcal{V}}\} =: k, \text{ and } x_i = y_i, \ \forall\, i \leq k.$$

Define

$$\Sigma_{\mathcal{A}^{\mathbb{N}}}^{\mathcal{V}} := \bar{\mathcal{A}}^{\mathbb{N}}_{/\sim}.$$

We recall that, although there is a biunivocal relationship between $\mathcal{V}$ and $\tilde{\mathcal{V}}$, an element in $\mathcal{V}$ is a subset of $\mathcal{A}$ while an element in $\tilde{\mathcal{V}}$ is a symbol of $\bar{\mathcal{A}}$. Furthermore, $\tilde{H} \notin H$ for any $H \in \mathcal{V}$.

Given $H \in \mathcal{V}$ we will denote $\bar{H} := H \cup \{\tilde{H}\}$ which is a subset of $\bar{\mathcal{A}}$ but not of $\mathcal{A}$, and $\tilde{H} \in \bar{H}$. Define

$$\bar{\mathcal{V}} := \{\bar{H} : \ H \in \mathcal{V}\}.$$

Note that $\bar{\mathcal{V}}$ is a family of subsets of $\bar{\mathcal{A}}$ which also satisfies the properties imposed in *Step 1* on the family $\mathcal{V}$.

We remark that, if $x \in \bar{\mathcal{A}}^{\mathbb{N}}$ is such that $x_i \in \mathcal{A}$ for all $i \in \mathbb{N}$, then [x], the equivalence class of x in $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ contains only x. In such a case we shall identify [x] with the point x itself. On the other hand, if $x \in \bar{\mathcal{A}}^{\mathbb{N}}$ is such that $x_i \in \tilde{\mathcal{V}}$ for some $i \in \mathbb{N}$, then [x] contains infinitely many points and to represent it we will pick $(y_i)_{i\in\mathbb{N}} \in [x]$ such that $y_i = x_i$ for all $i < n := \min\{i : x_i \in \tilde{\mathcal{V}}\}$ and $y_i = \tilde{H}$ for $i \geq n$. Thus, we are going to identify

$$\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}} \equiv \{(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} : \ x_i = \tilde{H} \in \tilde{\mathcal{V}} \Rightarrow x_{i+1} = \tilde{H}\} =$$

$$\mathcal{A}^{\mathbb{N}} \cup \{(x_0...x_{n-1}\tilde{H}\tilde{H}\tilde{H}...) : \ x_0...x_{n-1} \in B(\mathcal{A}^{\mathbb{N}}), \tilde{H} \in \tilde{\mathcal{V}}\}.$$

Hence, we can define on it the shift map $\sigma : \Sigma_{\mathcal{A}^{\mathbb{N}}} \to \Sigma_{\mathcal{A}^{\mathbb{N}}}$ in the usual way. Furthermore, given $\mathfrak{X} \subset \Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$, consider the sets $B_n(\mathfrak{X})$ of all words with length $n \geq 0$ in $\mathfrak{X}$, the set $B(\mathfrak{X})$ of all finite words in $\mathfrak{X}$, and the follower sets $\mathcal{F}_{\mathfrak{X}}(a_0 \dots a_{n-1})$ for each $a_0 \dots a_{n-1} \in B(\mathfrak{X})$, as in (15), (16) and (17), respectively. We notice that a word $a_0 \dots a_{n-1}$ stands for the empty word ϵ whenever $n \leq 0$.

Definition 3.1. *The space $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is the* **full blur shift space** *of $\mathcal{A}^{\mathbb{N}}$ with resolution $\mathcal{V}$. We say that $\Lambda' \subset \Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is a* **blur shift space** *with resolution $\mathcal{V}$ if and only if there exists a shift space $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ such that*

i. $\Lambda = \{(x_n)_{n\in\mathbb{N}} \in \Lambda' : \ x_n \in \mathcal{A} \ \forall n \in \mathbb{N}\}$;

ii. $(a_0 \dots a_{n-1}\tilde{H}\tilde{H}\dots) \in \Lambda'$ *for some* $\tilde{H} \in \tilde{\mathcal{V}} \iff$
$a_0 \dots a_{n-1} \in B(\Lambda)$ *and* $|\mathcal{F}_\Lambda(a_0 \dots a_{n-1}) \cap H| = \infty$ *(Λ' verifies the infinite-extension property).*

Under the above notations, we have that Λ' is the blur shift space of Λ with resolution $\mathcal{V}$, and denote $\Lambda' = \Sigma^{\mathcal{V}}_{\Lambda}$.

Note that in the above construction if $\mathcal{V} = \emptyset$ (which always holds if $\mathcal{A}$ is finite), then $\Sigma^{\mathcal{V}}_{\Lambda} = \Lambda$, which means the maximum resolution for a blur shift. On the other hand, $\mathcal{V} = \{\mathcal{A}\}$ corresponds to the minimum resolution.

Example 3.2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint family of countably infinite sets, where $A_0 = \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Consider the alphabet $\mathcal{A} := \bigcup_{n\geq 0} A_n$ and, for each $k \geq 1$, define the set $H_k := A_k \cup \{i \in A_0 : \ i \leq k\}$. Define $\mathcal{V} := \{H_k\}_{k\geq 1}$ which is an countably infinite family of countably infinite sets. Since for all $m, n \geq 1$ with $m < n$ we have $H_m \cap H_n = \{1, ..., m\}$, it follows that $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is a full blur shift.*

Example 3.3. *Let $A_0 := \mathbb{N}^*$, and $\{A_n\}_{n\in\mathbb{N}^*}$ be a disjoint family of non-countable sets. Consider the alphabet $\mathcal{A} := \bigcup_{n\geq 0} A_n$ and, for each integer $k \geq 1$, let H_k be a countably infinite set such that $\{1, ..., k\} \subset H_k \subset A_k \cup \{1, ..., k\}$. Then, $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ with $\mathcal{V} := \{H_k\}_{k\geq 1}$ is a full blur shift.*

Example 3.4. *Let $\mathcal{A} := \mathbb{R}^+ = [0, \infty)$ and, for each $\lambda \in [0, 1)$, define the set $H_\lambda := \{x \in \mathbb{R}^+ : x := \lambda + k,\ k \in \mathbb{N}\}$. Define $\mathcal{V} := \{H_\lambda\}_{\lambda \in [0,1)}$ and so the full blur shift $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$. Note that $\mathcal{V}$ is a non-countable family of countable disjoint sets.*

Example 3.5. *Let $\mathcal{A} := \mathbb{N}$ and define the sets $H_p := \{n \in \mathbb{N} :\ n$ is prime$\}$ and $H_c := \mathcal{A} \setminus H_p$. Hence $\mathcal{V} := \{H_p, H_c\}$ is a finite family of blurred sets, and $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is a full blur shift.*

Example 3.6. *Let $\mathcal{A} := \mathbb{R} \times \mathbb{R}$ and, for each $\lambda \in \mathbb{R}$ define the set $H_\lambda := \{\lambda\} \times \mathbb{R}$. Thus $\mathcal{V} := \{H_\lambda\}_{\lambda \in \mathbb{R}}$ is a non-countable family of non-countable disjoint sets, and $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is a full blur shift.*

Example 3.7. *Let $\mathcal{A} = \mathbb{N}$ and let $\{H_n \subset \mathcal{A} : n \in \mathbb{N}^*\}$ be a countable family of blurred sets. Let $H_0 := \{h_1, h_2, h_3, ...\}$ be any set where $h_k \in H_k \setminus \bigcup_{i<k} H_i$, and consider $\mathcal{V} = \{H_n\}_{n \geq 0}$ and the respective full blur shift $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$.*

Recall that $B_1(\Lambda)$ is the set of symbols of $\mathcal{A}$ that are used in $\Sigma^{\mathcal{V}}_\Lambda$. Given a blur shift space $\Sigma^{\mathcal{V}}_\Lambda$, denote

$$\mathcal{V}_\Lambda := \{H \in \mathcal{V} :\ |B_1(\Lambda) \cap H| = \infty\}, \tag{18}$$

which is the family of all blurred sets that appear in $\Sigma^{\mathcal{V}}_\Lambda$, and denote $\tilde{\mathcal{V}}_\Lambda := \{\tilde{H} :\ H \in \mathcal{V}_\Lambda\}$ and $\bar{\mathcal{V}}_\Lambda := \{\bar{H} :\ H \in \mathcal{V}_\Lambda\}$. Furthermore, denote $\mathcal{L}^{\mathcal{V}}_\infty(\Lambda) := \Lambda$, and for any $n \in \mathbb{N}$ let

$$\mathcal{L}^{\mathcal{V}}_n(\Lambda) := \{(x_i)_{i \in \mathbb{N}} \in \Sigma^{\mathcal{V}}_\Lambda : x_n \in \tilde{\mathcal{V}}_\Lambda \text{ and } x_{n-1} \notin \tilde{\mathcal{V}}_\Lambda\}, \tag{19}$$

and

$$\partial^{\mathcal{V}}\Lambda := \bigcup_{n \in \mathbb{N}} \mathcal{L}^{\mathcal{V}}_n(\Lambda). \tag{20}$$

Note that under this notation it follows that $\Sigma^{\mathcal{V}}_\Lambda = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathcal{L}^{\mathcal{V}}_n(\Lambda) = \Lambda \cup \partial^{\mathcal{V}}\Lambda$. Furthermore, in general $\mathcal{L}^{\mathcal{V}}_n(\Lambda) \subsetneq \mathcal{L}^{\mathcal{V}}_n(\Sigma_{\mathcal{A}^{\mathbb{N}}})$ for any $n \in \mathbb{N} \cup \{\infty\}$ (even when $B_1(\Lambda) = \mathcal{A}$). Finally, recall that $B_1(\Sigma^{\mathcal{V}}_\Lambda) = B_1(\Lambda) \cup \tilde{\mathcal{V}}_\Lambda$.

For simplicity of the notation, whenever the family $\mathcal{V}$ used to define the blur shift space is clear or off topic, we shall omit it and denote just Σ_Λ, $\mathcal{L}_n(\Lambda)$ and $\partial\Lambda$.

Proposition 3.8. *Let $\Sigma_\Lambda \subset \Sigma_{\mathcal{A}^{\mathbb{N}}}$ be a blur shift space. Then*

i. $\sigma(\Sigma_\Lambda) \subset \Sigma_{\sigma(\Lambda)} \subset \Sigma_\Lambda$*;*

ii. $\sigma(\mathcal{L}_n) = \mathcal{L}_{n-1}$, $\forall n \geq 1$ *and* $\sigma(\mathcal{L}_0) = \mathcal{L}_0$.

Proof.

i. It is direct that $\Sigma_{\sigma(\Lambda)} \subset \Sigma_\Lambda$. Note that $(y_1...y_{n-1}\tilde{H}\tilde{H}\tilde{H}...) \in \sigma(\partial\Lambda)$ means that there exists $y_0 \in \mathcal{A}$ such that $(y_0 y_1...y_{n-1}\tilde{H}\tilde{H}\tilde{H}...) \in \partial\Lambda$, that is, such that $|\mathcal{F}_\Lambda(y_0 y_1...y_{n-1}) \cap H| = \infty$. Hence, it follows that $|\mathcal{F}_{\sigma(\Lambda)}(y_1...y_{n-1}) \cap H| = \infty$, which means that $(y_1...y_{n-1}\tilde{H}\tilde{H}\tilde{H}...) \in \partial\sigma(\Lambda)$. Thus we have proved that $\sigma(\partial\Lambda) \subset \partial\sigma(\Lambda)$, we conclude that $\sigma(\Sigma_\Lambda) = \sigma(\Lambda \cup \partial\Lambda) = \sigma(\Lambda) \cup \sigma(\partial\Lambda) \subset \sigma(\Lambda) \cup \partial\sigma(\Lambda) = \Sigma_{\sigma(\Lambda)}$.

ii. It is straightforward.

□

The next example shows that it in general $\Sigma_{\sigma(\Lambda)} \not\subset \sigma(\Sigma_\Lambda)$ even when $\sigma(\Lambda) = \Lambda$.

Example 3.9. *Let $\mathcal{A}$ be any infinite alphabet, and define $\mathcal{V} = \{\mathcal{A}\}$ (the Ott-Tomforde-Willis case). Consider $\Lambda := \{(x_i)_{i\in\mathbb{N}} \subset \mathcal{A}^{\mathbb{N}} : \ x_{i+2} = x_i \ \forall i \in \mathbb{N}\}$. It follows that Σ_Λ is such that $\partial\Lambda = \{(\tilde{\mathcal{A}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}...), \ (a\tilde{\mathcal{A}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}...) : \ a \in \mathcal{A}\}$. Therefore, from part ii. of Proposition 3.8, that $\sigma(\Sigma_\Lambda) = \sigma(\Lambda) \cup \sigma(\partial\Lambda) = \sigma(\Lambda) \cup \{(\tilde{\mathcal{A}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}...)\}$.*

On the other hand, since $\sigma(\Lambda) = \Lambda$, we have $\Sigma_{\sigma(\Lambda)} = \Sigma_\Lambda = \Lambda \cup \partial\Lambda = \sigma(\Lambda) \cup \{(\tilde{\mathcal{A}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}...), \ (a\tilde{\mathcal{A}}\tilde{\mathcal{A}}\tilde{\mathcal{A}}...) : \ a \in \mathcal{A}\}$.

3.2.1 Graph presentation of blur shift spaces

We say that $\mathcal{G} = (V, E, s, r, L)$ is a directed labeled graph, if V and E are nonempty sets (the set of vertexes and the set of edges, respectively) $s : E \to V$, $r : E \to V$ and $L : E \to \mathcal{A}$ are maps (the source map, the range map and the label map, respectively). So it is said that in $\mathcal{G}$ there is an edge $e \in E$ labeled as $a \in \mathcal{A}$ from the vertex $v_s \in V$ to the vertex $v_r \in V$ if and only if $s(e) = v_s$, $r(e) = v_r$ and $L(e) = a$.

A directed labeled graph $\mathcal{G}$ generates a shift $\Lambda_\mathcal{G}$ given by:

$$\Lambda_\mathcal{G} = \left\{ \big(L(e_i)\big)_{i\geq 0} \in \mathcal{A}^{\mathbb{N}} : \ (e_i)_{i\in\mathbb{N}} \in E^{\mathbb{N}}, \ s(e_{i+1}) = r(e_i) \ \forall i \geq 0 \right\}.$$

In [16] it was proved that for any shift space $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ there exists a (possibly infinite) labeled directed graph $\mathcal{G}$ which represent Λ, that is, such that $\Lambda = \Lambda_\mathcal{G}$. An analogous result can be obtained for blur shift spaces:

Theorem 3.10. *Any blur shift space $\Sigma_\Lambda^\mathcal{V} \subset \bar{A}^{\mathbb{N}}$ can be generated by a directed labeled graph $\bar{\mathcal{G}}$.*

Proof. Let $\mathcal{G} = (V, E, s, r, L)$ be the graph given in [16, Theorem 1.1] such that $\Lambda = \mathcal{G}_\Lambda$ where: $V := \{V(\mathrm{w}) : \ \mathrm{w} \in B(\Lambda)\}$, where $V(\mathrm{w}) := \{\mathrm{u} \in B(\Lambda) : \ \mathrm{wu} \in B(\Lambda)\}$; $E := \{e_{V(\mathrm{v})V(\mathrm{v}a)} : \mathrm{v} \in B(\Lambda), \ a \in \mathcal{A}, \text{ and } \mathrm{v}a \in B(\Lambda)\}$; $s(e_{V(\mathrm{v})V(\mathrm{v}a)}) = V(\mathrm{v})$; $r(e_{V(\mathrm{v})V(\mathrm{v}a)}) = V(\mathrm{v}a)$; and $L : E \to \mathcal{A}$ given by $L(e_{V(\mathrm{v})V(\mathrm{v}a)}) = a$. Note that for all $\mathrm{v} \in B(\Lambda)$ we have $\mathcal{F}_\Lambda(\mathrm{v}) = \{a \in \mathcal{A} : a \in V(\mathrm{v})\}$ and so for all $H \in \mathcal{V}$ it follows that $\mathcal{F}_\Lambda(\mathrm{v}) \cap H = V(\mathrm{v}) \cap H$.

We build the graph $\bar{\mathcal{G}} = (\bar{V}, \bar{E}, \bar{s}, \bar{r}, \bar{L})$ from $\mathcal{G}$ as follows:

- $\bar{V} := V \cup \mathcal{L}_0^\mathcal{V}(\Lambda)$;
- $\bar{E} := E \cup \{e_{\tilde{H}\tilde{H}} : \ \tilde{H} \in \mathcal{L}_0^\mathcal{V}(\Lambda)\} \cup \{e_{V(\mathrm{v})\tilde{H}} : \ \tilde{H} \in \mathcal{L}_0^\mathcal{V}(\Lambda) \text{ and } V(\mathrm{v}) \in V \text{ are such that } |V(\mathrm{v}) \cap H| = \infty\}$;

- For all $e_{\mathrm{rs}} \in \bar{E}$ let $\bar{s}(e_{\mathrm{rs}}) := \mathrm{r}$, $\bar{r}(e_{\mathrm{rs}}) := \mathrm{s}$, and $\bar{L}(e_{\mathrm{rs}}) := \begin{cases} a & \text{, if } \mathrm{s} = V(va) \\ \tilde{H} & \text{, if } \mathrm{s} = \tilde{H} \end{cases}$

Hence, it is direct that

$$\Lambda_{\bar{\mathcal{G}}} = \left\{ \left(\bar{L}(e_{\mathrm{r}_i\mathrm{s}_i})\right)_{i\geq 0} \in \bar{\mathcal{A}}^{\mathbb{N}} :\ (e_{\mathrm{r}_i\mathrm{s}_i})_{i\in\mathbb{N}} \in \bar{E}^{\mathbb{N}},\ \bar{s}(e_{\mathrm{r}_{i+1}\mathrm{s}_{i+1}}) = \bar{r}(e_{\mathrm{r}_i\mathrm{s}_i})\ \forall i \geq 0 \right\} = \Sigma_{\Lambda}^{\mathcal{V}}.$$

□

3.3 THE TOPOLOGY OF BLUR SHIFT SPACES

We consider on $\mathcal{A}$ the discrete topology, and on $\bar{\mathcal{A}}$ we consider the same open sets of $\mathcal{A}$ plus the sets $U \subset \bar{\mathcal{A}}$ that have the property that if $\tilde{H} \in U$ then $H \setminus F \subset U$ for some finite $F \subset H$. Note that a basis for the topology on $\bar{\mathcal{A}}$ is the family of all singletons of $\mathcal{A}$ plus the sets of the form $\bar{H} \setminus F$ where $\bar{H} \in \bar{\mathcal{V}}$ and $F \subset H$ is finite. In Appendix A more details are given.

On the full shift $\bar{\mathcal{A}}^{\mathbb{N}}$ we consider the product topology $\tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$. The basic open sets of $\tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$ are the cylinders, which can be written as follows: Let $S := \{\bar{H} \setminus F :\ \bar{H} \in \bar{\mathcal{V}} \text{ and } F \subset H \text{ is finite}\}$, and for given $n \geq 0$ and $a_0, ..., a_{n-1} \in \mathcal{A} \cup S$ define a cylinder as the set

$$[a_0 a_1 \ldots a_{n-1}] := \{(x_i)_{i\in\mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}} :\ \forall j = 0, \ldots, n-1, x_j = a_j \text{ if } a_j \in \mathcal{A}, \text{ and } x_j \in a_j \text{ if } a_j \in S\}.$$

Finally, on $\Sigma_{\mathcal{A}^{\mathbb{N}}}$ we define the **quotient topology** denoted as $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$.

Now we give a characterization of the quotient topology $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$. In particular, we shall prove that the quotient topology has a basis of clopen sets called **generalized cylinders**, which are the sets defined for any $w_0 \ldots w_{n-1} \in B(\mathcal{A}^{\mathbb{N}})$, $\bar{H} \in \bar{\mathcal{V}}$, and $F \subset H$ a finite set, as

$$Z(w_0 \ldots w_{n-1}) := \{\mathrm{x} \in \Sigma_{\mathcal{A}^{\mathbb{N}}} : x_i = w_i, 0 \leq i \leq n-1\} \tag{21}$$

or

$$Z(w_0 \ldots w_{n-1}\bar{H}, F) := \{\mathrm{x} \in \Sigma_{\mathcal{A}^{\mathbb{N}}} : x_i = w_i, 0 \leq i \leq n-1, x_n \in \bar{H} \setminus F\}. \tag{22}$$

Note that $x_n \in \bar{H} \setminus F$ means that either $x_n = \tilde{H}$ or $x_n \in H \setminus F$. For simplicity of notation and further use, given $\alpha = (\alpha_0 \ldots \alpha_{n-1}) \in B(\mathcal{A}^{\mathbb{N}})$ and $\bar{H} \in \bar{\mathcal{V}}$ we shall denote $Z(\alpha_0 \ldots \alpha_{n-1}\bar{H}, F) =: Z(\alpha\bar{H}, F)$. We will denote a generalized cylinder as $Z(\alpha\bar{H})$ whenever $F = \emptyset$, and as $Z(\bar{H}, F)$ whenever $\alpha = \epsilon$, the empty word (recall that $a_0 \ldots a_{n-1} = \epsilon$ whenever $n \leq 0$). Furthermore, we shall consider $Z(\epsilon) = \Sigma_{\mathcal{A}^{\mathbb{N}}}$.

The relationship between cylinders of $\bar{\mathcal{A}}^{\mathbb{N}}$ and generalized cylinders of $\Sigma_{\mathcal{A}^{\mathbb{N}}}$ is given by the canonical projection map $q : \bar{\mathcal{A}}^{\mathbb{N}} \to \Sigma_{\mathcal{A}^{\mathbb{N}}}$ which maps $\tilde{\mathrm{x}} \in \bar{\mathcal{A}}^{\mathbb{N}}$ to its equivalence class

$[\tilde{x}] = x \in \Sigma_{\mathcal{A}^{\mathbb{N}}}$. It is direct that

$$q^{-1}\big(Z(\alpha)\big) = [\alpha_0 \dots \alpha_{n-1}]$$
$$\text{and} \tag{23}$$
$$q^{-1}\big(Z(\alpha\bar{H}, F)\big) = [\alpha_0 \dots \alpha_{n-1}(\bar{H} \setminus F)].$$

Proposition 3.11. *For any $\alpha \in B(\mathcal{A}^{\mathbb{N}})$, $\bar{H} \in \bar{\mathcal{V}}$, finite $F \subset H$, the generalized cylinders $Z(\alpha)$ and $Z(\alpha\bar{H}, F)$ are clopen sets of $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$.*

Proof. To prove that $Z(\alpha)$ and $Z(\alpha\bar{H}, F)$ are open sets of $\tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$, consider the canonical projection map $q : \bar{\mathcal{A}}^{\mathbb{N}} \to \Sigma_{\mathcal{A}^{\mathbb{N}}}$. Recall that $U \in \tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ if and only if $q^{-1}(U) \in \tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$. Thus, from (23) we have that $q^{-1}\big(Z(\alpha)\big)$ and $q^{-1}\big(Z(\alpha\bar{H}, F)\big)$ are both cylinders of $\tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$. Then, $Z(\alpha)$ and $Z(\alpha\bar{H}, F)$ are open sets of $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$.

To prove that $Z(\alpha)$ and $Z(\alpha\bar{H}, F)$ are closed sets of $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$, we will show that their complementary are open sets.

Given $\alpha = \alpha_0 \dots \alpha_{n-1} \in B(\mathcal{A}^{\mathbb{N}})$ and $x \in Z(\alpha)^c$, we take $j := min\{k : x_k \neq \alpha_k\}$. Therefore, if $x_j \in \mathcal{A}$, then $x \in Z(x_0 \dots x_j) \subset Z(\alpha)^c$, while if $x_j = \tilde{G} \in \tilde{\mathcal{V}}$, then $x \in Z(x_0 \dots x_{j-1}\tilde{G}, \{\alpha_j\}) \subset Z(\alpha)^c$. Since x is any point of $Z(\alpha)^c$ and the correspondent generalized cylinder $Z(x_0 \dots x_j)$ or $Z(x_0 \dots x_{j-1}\tilde{G}, \{\alpha_j\})$ is an open set of $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$, it means that $Z(\alpha)^c$ is open in $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ and therefore $Z(\alpha)$ is closed.

Now, suppose $x \in Z(\alpha\bar{H}, F)^c$. We have two possibilities: $x_0 \dots x_{n-1} \neq \alpha_0 \dots \alpha_{n-1}$ or $x_n \notin \bar{H} \setminus F$. If $x_0 \dots x_{n-1} \neq \alpha_0 \dots \alpha_{n-1}$, then we have $x \in Z(\alpha)^c \subset Z(\alpha\bar{H}, F)^c$ and we proceed as before to find a generalized cylinder which contains x and is contained in $Z(\alpha)^c$. If $x_n \notin \bar{H} \setminus F$ we have two sub cases: $x_n \in \mathcal{A}$ or $x_n = \tilde{G} \in \tilde{\mathcal{V}} \setminus \{\tilde{H}\}$. If $x_n \in \mathcal{A}$, then $x \in Z(x_0 \dots x_{n-1}x_n) \subset Z(\alpha\bar{H}, F)^c$; If $x_n = \tilde{G} \neq \tilde{H}$, then we set $F' = H \cap G$ and it follows that $x \in Z(\alpha\bar{G}, F') \subset Z(\alpha\bar{H}, F)^c$. Hence, we conclude that $Z(\alpha\bar{H}, F)^c$ is open in $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$, thus $Z(\alpha\bar{H}, F)$ is closed.

□

The proposition below ensures that the cylinders form a basis for the quotient topology on Σ_Λ.

Proposition 3.12. *The collection $\mathfrak{B} := \{Z(\alpha),\ Z(\alpha\bar{H}, F) : \alpha \in B(\mathcal{A}^{\mathbb{N}}), F \subset \mathcal{A}$ finite $, \bar{H} \in \bar{\mathcal{V}}\}$ is a basis for the quotient topology $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$.*

Proof. Since we already proved that $\mathfrak{B} \subset \tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$, we just need to prove that for any given $U \in \tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ and $x \in U$, there exists $Z \in \mathfrak{B}$ such that $x \in Z \subset U$. We remark $q^{-1}(U)$ is an open set of $\tau_{\bar{\mathcal{A}}^{\mathbb{N}}}$, and if $x \in U$, then $q^{-1}(U)$ contains all points of $q^{-1}(x)$.

In the case $x \in U \cap \mathcal{A}^{\mathbb{N}}$ it follows that $q^{-1}(x) = x$ (recall that x stands for both, the point of $\bar{\mathcal{A}}^{\mathbb{N}}$ and its equivalence class in $\Sigma_{\mathcal{A}^{\mathbb{N}}}$). Since $q^{-1}(U)$ is an open set, there exists a cylinder $[x_0...x_n]$ in $\bar{\mathcal{A}}^{\mathbb{N}}$ such that $x \in [x_0...x_n] \subset q^{-1}(U)$. Therefore $x = q(x) \in q([x_0...x_n]) = Z(x_0...x_n) \subset U$.

In the case $x \in U \cap \partial\mathcal{A}^{\mathbb{N}}$, say $x \in U \cap \mathcal{L}_n(\mathcal{A}^{\mathbb{N}})$ and $x_n = \tilde{G}$, it follows that $q^{-1}(x) = \{y \in \bar{\mathcal{A}}^{\mathbb{N}} : \ y_i = x_i \ \forall i \leq n\} \subset q^{-1}(U)$. Now, by contradiction suppose that for all finite $F \subset G$ we have that the cylinder in $\bar{\mathcal{A}}^{\mathbb{N}}$, $[x_0...x_{n-1}\bar{G} \setminus F] \cap q^{-1}(U)^c \neq \emptyset$. Hence, take any $z^0 \in [x_0...x_{n-1}\bar{G}] \cap q^{-1}(U)^c$, and recursively, for each $\ell \geq 1$ take an arbitrary $z^\ell \in [x_0...x_{n-1}(\bar{G} \setminus \{z^0_n, ..., z^{\ell-1}_n\})] \cap q^{-1}(U)^c$. Note that the sequence $(z^\ell)_{\ell \in \mathbb{N}} \in \bar{\mathcal{A}}^{\mathbb{N}}$ is such that $z^\ell_0...z^\ell_{n-1} = x_0...x_{n-1}$ for all $\ell \in \mathbb{N}$, and for any finite $F \subset G$, there exists $L \in \mathbb{N}$ such that $z^\ell_n \in G \setminus F$ for all $\ell \geq L$. It means that the first $n+1$ coordinates of z^ℓ are converging to $x_0...x_{n-1}\tilde{G}$ as ℓ goes to infinity, and so $q(z^\ell) \to x = (x_0...x_{n-1}\tilde{G}\tilde{G}\tilde{G}...)$ as $\ell \to \infty$. But $z^\ell \notin q^{-1}(U)$ for all $\ell \in \mathbb{N}$, and so $q(z^\ell) \notin U$, which is in contradiction to the fact that it converges to x. Hence, shall exist a finite $F \subset G$ such that $[x_0...x_{n-1}\bar{G} \setminus F] \subset q^{-1}(U)$, and thus $Z(x_0...x_{n-1}\bar{G}, F) = q\big([x_0...x_{n-1}\bar{G} \setminus F]\big) \subset U$.

□

Let us recall the convergence of sequences in the topology of a blur shift. Suppose a sequence $(x^n)_{n \geq 1} \in \Sigma_{\mathcal{A}^N}$ converges to some $\bar{x}$. If $\bar{x} \in \mathcal{A}^{\mathbb{N}}$, then for all $k \in \mathbb{N}$ there is a $N \geq 1$ such that $x^n_0...x^n_k = \bar{x}_0...\bar{x}_k$ for all $n \geq N$. On the other hand, if $\bar{x} \in \partial\mathcal{A}^{\mathbb{N}}$, say $\bar{x} = (\bar{x}_0...\bar{x}_{k-1}\tilde{H}\tilde{H}\tilde{H}...)$, then for any finite $F \subset H$ there is a $N \geq 1$ such that for all $n \geq N$ we have $x^n_0...x^n_{k-1} = \bar{x}_0...\bar{x}_{k-1}$ and $x^n_k \in \bar{H} \setminus F$. We remark that though, in general, blur shift spaces are not first countable (see Proposition 3.22), they are always **sequential spaces** (see Proposition 3.17 and Annex A). Thus, sequences suffice to determine the topology of blur shifts (see [5]).

Example 3.13. *If $\mathcal{A}$ is any infinite set and $\mathcal{V} := \{\mathcal{A}\}$, then the full blur shift $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is the full Ott-Tomforde-Willis shift $\Sigma_{\mathcal{A}}$ defined in [15]. In fact, it is direct that $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}} = \Sigma_{\mathcal{A}}$. Furthermore the basis given in Propostion 3.12 for the topology of $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ coincides with the basis given in [15, Theorem 2.15] for Ott-Tomforde-Willis shifts.*

Example 3.14. *Consider the construction of Gonçalves-Royer ultragraph shifts presented on page 29 onwards. Let $\mathfrak{G} = (V, E, s, r)$ be an ultragraph, and $\Lambda := X_{\mathfrak{G}} \subset E^{\mathbb{N}}$ be the respective classical ultragraph shift. Recall that the family $\hat{\mathcal{V}}$ contains the sets of vertexes of $\mathfrak{G}$ that will represent the new symbols used in $\Sigma_{\mathfrak{G}}$. Note that each $A \in \hat{\mathcal{V}}$ can be associated to the infinite set of symbols $H_A := s^{-1}(A) \subset E$. Since given $A, B \in \hat{\mathcal{V}}$ such that $A \neq B$ it follows that $A \cap B$ does not belong to $\hat{\mathcal{V}}$, it implies that $H_A \cap H_B$ is finite. Hence $\mathcal{V} := \{H_A\}_{A \in \hat{\mathcal{V}}}$ is a family of blurred sets and then by labeling each H_A with the symbol A, we get $\tilde{\mathcal{V}} = \hat{\mathcal{V}}$, and $\Sigma_{\mathfrak{G}} = \Sigma^{\mathcal{V}}_{\Lambda}$. To conclude that Gonçalves-Royer ultragraph shift spaces are blur shifts just observe that the basis of the topology in $\Sigma^{\mathcal{V}}_{\Lambda}$ (Proposition 3.12) coincides with the basis given in [7, Proposition 3.4] for Gonçalves-Royer ultragraph shifts.*

Observation: We notice that due to the construction of $\hat{\mathcal{V}}$ and the fact that the ultragraph has countably many vertexes and edges, we have here $\mathcal{V}$ being a countable family of countable sets.

Now, we shall prove several topological properties of blur shift spaces.

Proposition 3.15. *The full blur shift $\Sigma_{\mathcal{A}^{\mathbb{N}}}$ with the topology $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ is a Hausdorff topological space.*

Proof. To check that $\Sigma_{\mathcal{A}^{\mathbb{N}}}$ is Hausdorff, note that given two distinct points $\mathrm{x}, \mathrm{y} \in \Sigma_{\mathcal{A}^{\mathbb{N}}}$ we have three cases: $\mathrm{x}, \mathrm{y} \in \mathcal{A}^{\mathbb{N}}$; $\mathrm{x} \in \mathcal{A}^{\mathbb{N}}$ and $\mathrm{y} \in \partial\mathcal{A}^{\mathbb{N}}$; or $\mathrm{x}, \mathrm{y} \in \partial\mathcal{A}^{\mathbb{N}}$. If $\mathrm{x}, \mathrm{y} \in \mathcal{A}^{\mathbb{N}}$ just find $k \in \mathbb{N}$ such that $x_k \neq y_k$ and consider the generalized cylinders $Z(x_0...x_k)$ and $Z(y_0...y_k)$. If $\mathrm{x} \in \mathcal{A}^{\mathbb{N}}$ and $\mathrm{y} \in \partial\mathcal{A}^{\mathbb{N}}$, say $\mathrm{y} = (y_0...y_{n-1}\tilde{H}\tilde{H}\tilde{H}...)$, we just need to take the generalized cylinders $Z(x_0...x_n)$ and $Z(y_0...y_{n-1}\bar{H}, \{x_n\})$. Finally, in the case $\mathrm{x}, \mathrm{y} \in \partial\mathcal{A}^{\mathbb{N}}$, say $\mathrm{x} = (x_0...x_{m-1}\tilde{G}\tilde{G}\tilde{G}...)$ and $\mathrm{y} = (y_0...y_{n-1}\tilde{H}\tilde{H}\tilde{H}...)$, if $m \neq n$ we can, without loss of generality, suppose $m < n$ and then we consider the generalized cylinders $Z(x_0...x_{m-1}\bar{G}, \{y_m\})$ and $Z(y_0...y_m)$, while if $m = n$ we take the generalized cylinders $Z(x_0...x_{m-1}\bar{G})$ and $Z(y_0...y_{n-1}\bar{H}, G \cap H)$. □

Proposition 3.16. *The full blur shift $\Sigma_{\mathcal{A}^{\mathbb{N}}}$ with the topology $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ is a regular topological space.*

Proof. Let $C \subset \Sigma_{\mathcal{A}^{\mathbb{N}}}$ be a closed set and $\mathrm{y} \in \Sigma_{\mathcal{A}^{\mathbb{N}}}$ a point not belonging to C. We need to find two disjoint open sets $A, B \in \tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ such that $\mathrm{y} \in A$ and $C \subset B$.

Note that, if $\mathrm{y} \in \mathcal{A}^{\mathbb{N}}$, then there exists $m \geq 0$ such that for all $\mathrm{x} \in C$, we have $\mathrm{x}_{[0,m]} \neq \mathrm{y}_{[0,m]}$. In fact, if it would not hold, that is, if for each $\ell \geq 0$ there would exist $\mathrm{x}^\ell \in C$ such that $\mathrm{x}^\ell_{[0,\ell]} = \mathrm{y}_{[0,\ell]}$, then we would have $\mathrm{x}^\ell \to \mathrm{y}$ as $\ell \to \infty$, which is not possible since C is closed and $\mathrm{y} \notin C$. In this case, we define $A := Z(\mathrm{y}_{[0,m]})$.

On the other hand, if $\mathrm{y} = (y_0...y_{m-1}\tilde{G}\tilde{G}\tilde{G}...) \in \partial\mathcal{A}^{\mathbb{N}}$, then it follows that the set $F := \{g \in G : \exists \mathrm{x} \in C, \text{ with } x_0...x_{m-1}x_m = y_0...y_{m-1}g\}$ is finite. In fact, if such set was not finite, then for each $\ell \geq 1$ we could take a distinct $g^\ell \in F$ and the respective sequence $(\mathrm{x}^\ell)_{\ell\geq 1} \in C$ where $x^\ell_0...x^\ell_{m-1}x^\ell_m = y_0...y_{m-1}g^\ell$. Therefore, it would follows again that $x^\ell \to \mathrm{y}$ which is not possible due to the fact that C is closed and y is not in C. In this case, we define $A := Z(y_0...y_{m-1}\bar{G}, F)$.

By considering that $\mathrm{y} = (y_0y_1y_2...) \in \mathcal{A}^{\mathbb{N}}$ or $\mathrm{y} = (y_0...y_{m-1}\tilde{G}\tilde{G}\tilde{G}...) \in \partial\mathcal{A}^{\mathbb{N}}$, for each

$\mathrm{x} \in C$ we define

$$B_{\mathrm{x}} := \begin{cases} Z(\mathrm{x}_{[0,k]}) & \text{, if } \; x_k \in \mathcal{A} \text{ and } x_k \neq y_k, \text{ for some } 0 \leq k \leq m, \\ \\ Z(y_0 ... y_{k-1}\bar{H}, \{y_k\}) & \text{, if } \; \mathrm{x} = (y_0 ... y_{k-1}\tilde{H}\tilde{H}\tilde{H}...) \text{ for some } 0 \leq k \leq m-1, \\ \\ Z(y_0 ... y_{m-1}\bar{H}, \{y_m\}) & \text{, if } \; \mathrm{x} = (y_0 ... y_{m-1}\tilde{H}\tilde{H}\tilde{H}...) \text{ and } \mathrm{y} \in \mathcal{A}^{\mathbb{N}}, \\ \\ Z(y_0 ... y_{m-1}\bar{H}, H \cap G) & \text{, if } \; \mathrm{x} = (y_0 ... y_{m-1}\tilde{H}\tilde{H}\tilde{H}...) \text{ and } \mathrm{y} \in \partial\mathcal{A}^{\mathbb{N}}. \end{cases}$$

Hence, defining $B := \bigcup_{\mathrm{x} \in C} B_{\mathrm{x}}$ we have that A and B are two open sets that separate C and x.

□

Given a blur shift space $\Sigma_\Lambda \subset \Sigma_{\mathcal{A}^{\mathbb{N}}}$, we define on Σ_Λ the topology τ_{Σ_Λ} induced from $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$. Thus, the basic open sets of Σ_Λ are generalized cylinders in the form $Z_\Lambda(\alpha) := Z(\alpha) \cap \Sigma_\Lambda$ and $Z_\Lambda(\alpha\bar{H}, F) := Z(\alpha\bar{H}, F) \cap \Sigma_\Lambda$ for $\alpha \in B(\Lambda)$, $\bar{H} \in \bar{\mathcal{V}}_\Lambda$ and $F \subset H$ finite.

The next proposition shows that blur shift spaces, with the topology of generalized cylinders, are always **Fréchet-Uryson spaces** and so they are a sequential space[1]:

Proposition 3.17. *Any blur shift $\Sigma_\Lambda \subset \Sigma_{\mathcal{A}^{\mathbb{N}}}$ is a Fréchet-Uryson space.*

Proof. Given a $A \subset \Sigma_\Lambda$, denote its closure in the topology τ_{Σ_Λ} as $\bar{A}$, and define its sequential closure as the set

$$[A]_{seq} := \{\mathrm{x} \in \Sigma_\Lambda : \; \exists (\mathrm{y}^n)_{n \geq 0} \in A \text{ s.t. } \mathrm{y}^n \to \mathrm{x} \text{ as } n \to \infty\}.$$

We need to prove that for all $A \subset \Sigma_\Lambda$ we have $\bar{A} \setminus A \subset [A]_{seq}$.

Let $\mathrm{x} \in \bar{A} \setminus A$. We shall consider two cases separately: $\mathrm{x} \in \Lambda$; and $\mathrm{x} \in \partial\Lambda$. If $\mathrm{x} \in \Lambda$, then for every $n \geq 0$ the generalized cylinder $Z_\Lambda(x_0 ... x_n)$ intersects A. Hence, we can take $\mathrm{y}^n \in Z_\Lambda(x_0 ... x_n) \cap A$, and since $Z_\Lambda(x_0 ... x_n) \supset Z_\Lambda(x_0 ... x_{n+1})$ for all $n \geq 0$, it follows that $(\mathrm{y}^n)_{n \geq 0}$ is a sequence in A converging to x. Thus $\mathrm{x} \in [A]_{seq}$.

If $\mathrm{x} \in \partial\Lambda$, say $\mathrm{x} = (x_0 ... x_{m-1}\tilde{H}\tilde{H}\tilde{H}...)$, then for any finite $F \subset H$ we have that the generalized cylinder $Z(x_0 ... x_{m-1}\bar{H}, F)$ intersects A. We will construct a sequence in A as follows: Choose an arbitrary $\mathrm{y}^0 \in Z(x_0 ... x_{m-1}\bar{H}) \cap A$; for each $n \geq 1$ we will chose any point $\mathrm{y}^n \in Z(x_0 ... x_{m-1}\bar{H}, \{y_m^0, ..., y_m^{n-1}\}) \cap A$. Observe that each y^n is such that $y_0^n ... y_{m-1}^n = x_0 ... x_{m-1}$ and $y_m^n \in H$. Furthermore $y_m^s \neq y_m^t$ for all $s \neq t$. Thus, for any finite $F \subset H$ the

[1] We refer the reader to [5] and [2, Section 1.8] for more details about Fréchet-Uryson spaces and sequential spaces, or to Annex A for the essential parts used here.

generalized cylinder $Z(x_0...x_{m-1}\bar{H}, F)$ contains all but a finite number of terms of $(\mathrm{y}^n)_{n\geq 0}$, which means that $(\mathrm{y}^n)_{n\geq 0}$ converges to $\mathrm{x} \in [A]_{seq}$.

□

The next theorem gives an alternative definition for blur shift spaces, and its proof is direct.

Theorem 3.18. *A set $\Lambda' \subset \Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is a blur shift space with resolution $\mathcal{V}$ if and only if there exists a shift space $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ such that Λ' is the closure of Λ with respect to the topology $\tau_{\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}}$.*

□

As a direct consequence of Proposition 3.8 and Theorem 3.18, we have the following corollary:

Corollary 3.19. *A set $\Lambda' \subset \Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ is a blur shift space with resolution $\mathcal{V}$ if and only if Λ' is σ invariant, closed in $\Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ and verifies the infinite-extension property (Definition 3.1.ii).*

□

We recall that there are infinitely many subfamilies of $\mathfrak{B}_{\Sigma_\Lambda}$ that are basis for τ_{Σ_Λ}. For instance, we could just consider cylinders $Z_\Lambda(\alpha)$ and $Z_\Lambda(\alpha\bar{H}, F)$, where $\alpha \in B_{n_k}(\Lambda)$ for any infinite $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Furthermore, if $\mathcal{F}_\Lambda(\alpha) \cap G$ is countable, then F could be restricted to any prefixed family $\{F^i :\ F^i \text{ is finite and } F^i \nearrow \mathcal{F}_\Lambda(\alpha) \cap G\}$.

The following two results give conditions for the topological separability and countability of blur shifts:

Proposition 3.20. *A blur shift Σ_Λ is separable if and only if $B_1(\Lambda)$ is countable.*

Proof. Let $\mathcal{D} \subset \Sigma_\Lambda$ be a countable dense subset. Then, since $\mathcal{D}$ is dense in Σ_Λ, it follows that for each $a \in B_1(\Lambda)$ there exists $\mathrm{y} \in \mathcal{D} \cap Z_\Lambda(a)$. Since for distinct a we have necessarily a distinct y, then the cardinality of $\mathcal{D}$ is not less than the cardinality of $B_1(\Lambda)$, which implies that $B_1(\Lambda)$ is countable.

Conversely, if $B_1(\Lambda)$ is countable, then $B(\Lambda)$ is countable. For each $\mathrm{w} = w_1...w_m \in B(\Lambda)$ we pick a point $\mathrm{y^w} \in Z(\mathrm{w})$. Thus, $D := \{\mathrm{y^w} \in \Sigma_\Lambda :\ \mathrm{w} \in B(\Lambda)\}$ is a countable set. Furthermore, D is dense, since given any $Z_\Lambda(a_0...a_n\bar{H}, F)$ we can take $\mathrm{w} = a_0...a_nh \in B(\Lambda)$ with $h \in H \setminus F$ and thus $\mathrm{y^w} \in D \cap Z_\Lambda(a_0...a_n\bar{H}, F)$.

□

Proposition 3.21. *A blur shift Σ_Λ is second countable if and only if $B_1(\Lambda)$ and $\mathcal{V}_\Lambda$ are countable.*

Proof. Suppose $B_1(\Lambda)$ and $\mathcal{V}_\Lambda$ are countable. It follows that $B(\Lambda)$ is also countable. Hence, the topological basis of Σ_Λ,

$$\mathfrak{B}_{\Sigma_\Lambda} := \{Z_\Lambda(\mathrm{w}),\ Z_\Lambda(\mathrm{w}\bar{H}, F) :\ \mathrm{w} \in B(\Lambda),\ H \in \mathcal{V}_\Lambda,\ F \subset \mathcal{F}_\Lambda(\mathrm{w}) \cap H \text{ finite}\}$$

is countable.

The converse, comes directly to the fact that if $B_1(\Lambda)$ or $\mathcal{V}_\Lambda$ were not countable, then $\mathfrak{B}_{\Sigma_\Lambda}$ would have an uncountable number of generalized cylinders of the form $Z_\Lambda(\mathrm{w})$. Therefore, $\mathfrak{B}_{\Sigma_\Lambda}$ would not be countable neither would have a countable subfamily which is a basis.

□

Proposition 3.22. *A blur shift Σ_Λ is first countable if and only if for all $H \in \mathcal{V}_\Lambda$ we have $H \cap B_1(\Lambda)$ countable.*

Proof. Just note that if there exists some $G \in \mathcal{V}_\Lambda$ such that $G \cap B_1(\Lambda)$ is not countable, then the point $\mathrm{x} = (\tilde{G}\tilde{G}\tilde{G}\ldots)$ has an uncountable local basis $\mathfrak{B}_\mathrm{x} := \{Z(\bar{G}, F) :\ F \subset G \text{ finite}\}$ which has not any countable subfamily which is a local basis. Thus, Σ_Λ is not first countable.

Conversely, observe that if $\mathrm{x} \in \Lambda$, then $\mathfrak{B}_\mathrm{x} := \{Z_\Lambda(x_0 \ldots x_n) :\ n \in \mathbb{N}\}$ is always a countable local basis of x. On the other hand, given $\mathrm{x} = (x_0 \ldots x_{n-1}\tilde{H}\tilde{H}\tilde{H}\ldots) \in \partial\Lambda$, if $\mathcal{F}_\Lambda(x_0 \ldots x_{n-1}) \cap H$ is countable we can take an enumeration of it, $\mathcal{F}_\Lambda(x_0 \ldots x_{n-1}) \cap H = \{h_1, h_2, \ldots\}$, and then $\mathfrak{B}_\mathrm{x} := \{Z_\Lambda(x_0 \ldots x_{n-1}\bar{H}, \{h_1, \ldots, h_\ell\}) :\ \ell \geq 0\}$ is a countable local basis of x.

□

Define

$$B_1(\mathcal{V}_\Lambda) := \{a \in \mathcal{A} :\ a \in B_1(\Lambda) \cap H \text{ for some } H \in \mathcal{V}_\Lambda\}.$$

We notice that even if Σ_Λ is first countable, that is, if $B_1(\Lambda) \cap H$ is countable for all $H \in \mathcal{V}_\Lambda$, it is possible that $\mathcal{V}_\Lambda$ and $B_1(\mathcal{V}_\Lambda)$ are not countable.

Example 3.23. *The blur shifts of examples 3.14, 3.2, 3.5 and 3.7 are first and second countable, since they have countable alphabets.*

The blur shifts of examples 3.3 and 3.4 are both first countable, but not second countable. Observe that while $B_1(\mathcal{V}_\Lambda)$ is countable in Example 3.3, it is uncountable in Example 3.4.

The blur shift of Example 3.6 is neither first nor second countable.

3.3.1 Metrizability

To find criteria for the metrizability of blur shifts, we shall use the Nagata-Smirnov metrization theorem, which states that a topological space is metrizable if and only if it is regular, Hausdorff and has a **countably locally finite** basis. We recall that a family $\mathcal{U}$ of sets in a topological space is said to be **locally finite** if any point of the space has an open neighborhood which intersects at most a finite number of sets $\mathcal{U}$, while a family $\mathcal{U}$ of sets is said to be **countably locally finite** if it is the countable union of locally finite families.

Since Σ_Λ is always regular and Hausdorff (propositions 3.15 and 3.16), we just need to find conditions under which τ_{Σ_Λ} has countably locally finite basis.

The following lemma translates to the language of generalized cylinders the obvious fact that in a Hausdorff and regular space, a family of closed sets is locally finite if and only if any intersection of infinitely many sets of the family is always empty and any convergent sequence is contained at most in a finite union of sets of the family.

Lemma 3.24. *Let Σ_Λ be a blur shift space. A family of generalized cylinders $\mathfrak{F} \subset \mathfrak{B}_{\Sigma_\Lambda}$ is locally finite if and only if all the following conditions hold:*

i. Any infinite intersection of generalized cylinders of $\mathfrak{F}$ is empty;

ii. For all $k \geq 0$, $\alpha \in B_k(\Lambda)$ and $H \in \mathcal{V}_\Lambda$ there exists a finite set $F \subset H$ such that no word $\gamma \in B(\Lambda)$ with $\gamma_0 ... \gamma_{k-1} = \alpha$ and $\gamma_k \in H \setminus F$ is used in any generalized cylinder of $\mathfrak{F}$.

iii. Any infinite subcollection $\{Z_\Lambda(\gamma\bar{G}^\ell, F^\ell) :\ \ell \in \lambda\} \subset \mathfrak{F}$ is such that for all $H \in \mathcal{V}_\Lambda$ we have $|H \cap \bigcup_{\ell\in\lambda}(G^\ell \setminus F^\ell)| < \infty$.

Proof. Firstly let us check that *i.–iii.* are necessary conditions to $\mathfrak{F}$ be locally finite. In fact, it is direct that if condition *i.* does not hold, then we can take an element x which belongs to the intersection of infinitely many generalized cylinders of $\mathfrak{F}$ and any neighborhood of such x will intersect all of those infinitely many generalized cylinders. Now, suppose there are $\alpha \in B_k(\Lambda)$ and $H \in \mathcal{V}_\Lambda$ and suppose an infinite subset $\{h^i\}_{i\geq 1} \subset H$ with the property that for each $i \geq 1$ there exists $\gamma^i \in B(\Lambda)$ which is used in the definition of some generalized cylinder $Z^i \in \mathfrak{F}$, and which is such that $\gamma^i_0 ... \gamma^i_k = \alpha h^i$. Therefore, any neighborhood $Z_\Lambda(\alpha\bar{H}, F)$ of the point $(\alpha_0 ... \alpha_{k-1}\tilde{H}\tilde{H}\tilde{H}...)$ will intersect infinitely many generalized cylinders Z^i. Finally, suppose there exist $\{Z_\Lambda(\gamma\bar{G}^\ell, F^\ell) :\ \ell \in \lambda\} \subset \mathfrak{F}$ and $H \in \mathcal{V}_\Lambda$ such that we have $|H \cap \bigcup_{\ell\in\lambda}(G^\ell \setminus F^\ell)| = \infty$. It implies that $|H \cap \mathcal{F}_\Lambda(\gamma)| = \infty$ and for all finite $F \subset H \cap \mathcal{F}_\Lambda(\gamma)$ we have $Z_\Lambda(\gamma\bar{H}, F)$ intersecting infinitely many generalized cylinders of $\{Z_\Lambda(\gamma\bar{G}^\ell, F^\ell) :\ \ell \in \lambda\}$.

To check that *i. – iii.* are sufficient conditions for $\mathfrak{F}$ be locally finite, note that for any $x \in \Lambda$, if for all $n \geq 0$ the generalized cylinder $Z_\Lambda(x_0 ... x_n)$ intersects infinitely many generalized cylinders of $\mathfrak{F}$ means that x should belong to infinitely many generalized cylinders of $\mathfrak{F}$, contradicting condition *i.*. For $x \in \partial\Lambda$, say $x = (\alpha_0 ... \alpha_{k-1}\tilde{H}\tilde{H}\tilde{H}...)$, if any neighborhood $Z_\Lambda(\alpha_0 ... \alpha_{k-1}\bar{H}, F)$ intersects infinitely many generalized cylinders of $\mathfrak{F}$, then at least one of the three situations holds: (*a*) x belongs to infinitely many generalized cylinders of $\mathfrak{F}$; (*b*) For any finite $F \subset H$, $Z_\Lambda(\alpha_0 ... \alpha_{k-1}\bar{H}, F)$ intersects infinitely many generalized cylinders of $\mathfrak{F}$, each one of them fixing the k first entries as $x_0 ... x_{k-1}$ and with the $(k+1)^{th}$ entry belonging to

H; (c) $Z_\Lambda(\alpha_0...\alpha_{k-1}\bar{H}, F)$ intersects infinitely many cylinders $Z_\Lambda(\alpha_0...\alpha_{k-1}\bar{G}^\ell, F^\ell)$. However, situation ($a$) is avoided by condition *i.*, situation (b) is avoided by condition *ii.*, and situation (c) cannot occur due to condition *iii.*.

□

Theorem 3.25. *Suppose Σ_Λ is a blur shift which is first countable and such that at least one of the following conditions holds:*

i. $\mathcal{V}_\Lambda$ is countable;

ii. Each $H \in \mathcal{V}_\Lambda$ has just a finite number of elements that appear in some other set of $\mathcal{V}_\Lambda$ (but it is possible that some element appears in infinitely many sets of $\mathcal{V}_\Lambda$).

Then Σ_Λ is metrizable.

Proof. We just need to prove that if Σ_Λ is first countable (that is, for all $H \in \mathcal{V}_\Lambda$ we have $H \cap B_1(\Lambda)$ countable - Proposition 3.22) and *i.* or *ii.* holds, then τ_{Σ_Λ} has a countably locally finite basis.

First, we observe that if $\mathcal{V}_\Lambda$ is empty, then $\Sigma_\Lambda = \Lambda$ which is metrizable (in this case both *i.* and *ii.* hold trivially). Hence, we will assume $\mathcal{V}_\Lambda$ nonempty.

Recall that a basis of τ_{Σ_Λ} is $\mathfrak{B}_{\Sigma_\Lambda} := \mathfrak{C} \cup \mathfrak{G}$, where

$$\mathfrak{C} := \{Z_\Lambda(\alpha) : \alpha \in B(\Lambda)\}$$

and

$$\mathfrak{G} := \{Z_\Lambda(\alpha\bar{H}, F) : \alpha \in B(\Lambda),\ H \in \mathcal{V}_\Lambda,\ F \subset \mathcal{F}_\Lambda(\alpha) \cap H \text{ finite }\}.$$

i. Since Σ_Λ is first countable, we have that $\mathcal{V}_\Lambda$ countable implies $B_1(\mathcal{V}_\Lambda)$ is countable.

Consider an enumeration $B_1(\mathcal{V}_\Lambda) = \{a^1, a^2, a^3, ...\}$ and for each $k \geq 1$ define $P_k := \{a^k, a^{k+1}, a^{k+2}, ...\}$. Now, for each $k, n \geq 1$ define

$$\mathfrak{C}_{k,n} := \{Z_\Lambda(\alpha) :\ \alpha \in B_n(\Lambda) \text{ contains just symbols of } B_1(\Lambda) \setminus P_k\}.$$

It follows that each $\mathfrak{C}_{k,n}$ satisfies all the three conditions of Lemma 3.24 (condition *iii.* is satisfied by vacuity).

Hence,

$$\mathfrak{C} = \bigcup_{k,n \geq 1} \mathfrak{C}_{k,n}$$

is countably locally finite.

Now, for each $H \in \mathcal{V}_\Lambda$ consider an enumeration of $B_1(\Lambda) \cap H = \{h^1, h^2, h^3, ...\}$ and define $F_\ell^H := \{h^1, .., h^\ell\}$. For each $k \geq 1$, $\ell, n \geq 0$ and $H \in \mathcal{V}_\Lambda$, define

$$\mathfrak{G}_{k,\ell,n,H} := \{Z_\Lambda(\alpha\bar{H}, F_\ell^H) : \ \alpha \in B_n(\Lambda) \text{ contains just symbols of } B_1(\Lambda) \setminus P_k,$$
$$|\mathcal{F}_\Lambda(\alpha) \cap H| = \infty\}.$$

We have that $\mathfrak{G}_{k,\ell,n,H}$ satisfies all the three conditions of Lemma 3.24 and so it is locally finite. Thus,

$$\mathfrak{G} = \bigcup_{k\geq 1,\ \ell,n\geq 0,\ H\in\mathcal{V}_\Lambda} \mathfrak{G}_{k,\ell,n,H}$$

is countably locally finite.

Therefore, $\mathfrak{B}_{\Sigma_\Lambda}$ is a countably locally finite basis, and so Σ_Λ is metrizable.

ii. If $\mathcal{V}_\Lambda$ is countable, then the result follows from the case *i.*. Suppose $\mathcal{V}_\Lambda$ is an uncountable family. Given $H \in \mathcal{V}_\Lambda$, define

$$S_H := \{h \in H : \ \exists G \in \mathcal{V}_\Lambda \text{ s.t. } G \neq H \text{ and } h \in G\} = H \cap \bigcup_{G\in\mathcal{V}_\Lambda,\ G\neq H} G, \tag{24}$$

which is finite by hypothesis.

Since Σ_Λ is first countable, for each $H \in \mathcal{V}_\Lambda$ we can take an enumeration of $B_1(\Lambda) \cap H := \{h^1, h^2, h^3, ...\}$ and for $n \geq 1$ we define $F_n^H := \{h^1, h^2, ..., h^n\}$ and $O_n^H := \{h^n, h^{n+1}, h^{n+2}, ...\}$. Now, for each $n \geq 1$, consider the set

$$Q_n := \bigcup_{H\in\mathcal{V}_\Lambda} O_n^H.$$

We remark that Q_n is an uncountable set, since there are uncountable many sets in $\mathcal{V}_\Lambda$.

For each $k, n \geq 1$ define

$$\mathfrak{C}_{k,n} := \{Z_\Lambda(\alpha) : \ \alpha \in B_n(\Lambda) \text{ contains just symbols of } B_1(\Lambda) \setminus Q_k\},$$

which is a locally finite family. In fact, conditions *i.* and *iii.* of Lemma 3.24 follow directly, while condition *ii.* follows from the fact that any word α defining a generalized cylinder $Z_\Lambda(\alpha)$ is such that it can use only a finite number of symbols lying in any $H \in \mathcal{V}_\Lambda$. Indeed, with respect to a given H, α can use at most the $k-1$ symbols of F_{k-1}^H, and the finite number of elements in $\bigcup_{G\in\mathcal{V}_\Lambda} \left(S_H \cap F_{k-1}^G\right)$.

Thus,

$$\mathfrak{C} = \bigcup_{k,n\geq 1} \mathfrak{C}_{k,n}$$

is a countably locally finite family.

Now, for each $k \geq 1$ and $\ell, n \geq 0$ define

$$\mathfrak{G}_{k,\ell,n} := \{Z_\Lambda(\alpha\bar{H}, F_\ell^H \cup S_H) : \ H \in \mathcal{V}_\Lambda, \ \alpha \in B_n(\Lambda) \text{ has just symbols of}$$

$$B_1(\Lambda) \setminus Q_k, \ |\mathcal{F}_\Lambda(\alpha) \cap H| = \infty\}.$$

It follows that though $\mathfrak{G}_{k,\ell,n}$ contains uncountable many sets, it is a locally finite family. In fact, condition *i.* of Lemma 3.24 follows from the fact that we can only have subfamilies of generalized cylinders $\{Z_\Lambda(\alpha^t\bar{H}^t, F_\ell^{H^t} \cup S_{H^t}) : \ t \geq 1\} \subset \mathfrak{G}_{k,\ell,n}$ if and only if $\alpha^t = \alpha^s$ for all $s, t \geq 1$ and $\bigcap_{t\geq 1} \left(H^t \setminus (F_\ell^{H^t} \cup S_{H^t})\right) \neq \emptyset$, which is avoided by the fact that for fixed α, each H^t of $\mathcal{V}_\Lambda$ is used just once in a generalized cylinder of $\mathfrak{G}_{k,\ell,n}$, and $H^t \setminus (F_\ell^{H^t} \cup S_{H^t})$ is disjoint of $H^s \setminus (F_\ell^{H^s} \cup S_{H^s})$ if $t \neq s$. Condition *ii.* of Lemma 3.24 is checked as made for $\mathfrak{C}_{k,n}$. Finally, condition *iii.* of Lemma 3.24 comes from (24).

Hence, we can define

$$\hat{\mathfrak{G}} = \bigcup_{k\geq 1, \ \ell,n\geq 0} \mathfrak{G}_{k,\ell,n}$$

which is a countably locally finite family, and so

$$\hat{\mathfrak{B}}_{\Sigma_\Lambda} := \mathfrak{C} \cup \hat{\mathfrak{G}}$$

is also a countably locally finite family.

We notice that $\hat{\mathfrak{G}}$ is also a basis for the topology τ_{Σ_Λ} in spite of $\hat{\mathfrak{G}} \subsetneq \mathfrak{G}$. In order to check this, it is sufficient to observe that any $Z_\Lambda(\alpha_0 \ldots \alpha_{k-1}\bar{H}, F) \in \mathfrak{G}$ can be written as

$$Z_\Lambda(\alpha_0 \ldots \alpha_{k-1}\bar{H}, F) = Z_\Lambda(\alpha_0 \ldots \alpha_{k-1}\bar{H}, F_\ell^H \cup S_H) \cup \bigcup_{h\in(F_\ell^H \cup S_H)\setminus F} Z_\Lambda(\alpha_0 \ldots \alpha_{k-1}h)$$

where F_ℓ^H is chosen such that $F \subset F_\ell^H \cup S_H$.

□

The next corollary can be obtained as a direct consequence of Proposition 3.21 and Theorem 3.25.i (as well as it could be derived from propositions 3.15 and 3.16, and then applying the Urysohn metrization theorem):

Corollary 3.26. *If a blur shift is second countable, then it is metrizable.*

□

Example 3.27. *The full blur shifts of examples 3.2, 3.5, 3.7 and 3.14 are metrizable due to Corollary 3.26. The blur shift of Example 3.3 is metrizable due to Theorem 3.25.i, while the blur shift of Example 3.4 is metrizable due to Theorem 3.25.ii.*

We remark that contrarily to what happens in the particular contexts of Ott-Tomforde-Willis shifts (where metrizability is equivalent to countability), in the general context of blur shifts it remains open to determine a complete characterization of the metrizable shifts. We remark that Ott-Tomforde-Willis shifts are blur shifts constrained to have at most one blurred set which is all the alphabet. In the general context, when we can consider any quantity of blurred sets and cardinality for the alphabet, Theorem 3.25 shows that such characterization is a bit more complicated since it depends on finding a general strategy under which one can characterize whether or not a topology has a countably locally finite basis. In Subsection 3.3.2 we will construct a metric for blur shifts over countable alphabets, however it also remains an open problem to present an explicit metric for other cases of metrizable blur shifts.

Althought it remains an open problem to give a complete characterization of metrizables blur shifts, for the particular cases of compact blur shifts we can obtain such characterization of metrizability. The following corollary uses two results that will be proved in Subsection 3.3.3 to prove a result analogous to [15, Corollary 2.18]. In particular, it shows a relationship between first countability, metrizability and compactness.

Corollary 3.28. *Let Σ_Λ be a compact blur shift. The following statements are equivalent:*

i. $B_1(\Lambda)$ *is countable;*

ii. Σ_Λ *is first countable;*

iii. Σ_Λ *is second countable;*

iv. Σ_Λ *is separable;*

v. Σ_Λ *is metrizable.*

Proof.

iii. $\Rightarrow$ *ii.* It is direct.

ii. $\Rightarrow$ *i.* Since Σ_Λ is first countable, then from Proposition 3.22 we have that $H \cap B_1(\Lambda)$ is countable for all $H \in \mathcal{V}_\Lambda$. On the other hand, from the compactness of Σ_Λ, applying Theorem 3.34, we get that $\mathcal{V}_\Lambda$ is a finite family that covers $B_1(\Lambda) \setminus \{a_1, ..., a_\ell\}$, for some a finite set $\{a_1, ..., a_\ell\} \subset B_1(\Lambda)$. Therefore $B_1(\Lambda) = \{a_1, ..., a_\ell\} \cup \bigcup_{H \in \mathcal{V}_\Lambda} (H \cap B_1(\Lambda))$ is countable.

i. $\Rightarrow$ *iii.* From Theorem 3.34 the compactness of Σ_Λ implies $\mathcal{V}_\Lambda$ being finite. Thus, from Proposition 3.21, if $B_1(\Lambda)$ is also countable, then Σ_Λ is second countable.

iii. $\Rightarrow$ *v.* From Corollary 3.26.

v. $\Rightarrow$ *i.* By hypothesis Σ_Λ is compact. If it is also metrizable, then from Corollary 3.36 we get that the alphabet shall be countable.

i. $\Leftrightarrow$ *iv*. It is given by Proposition 3.20.

$\square$

3.3.2 Second-countable blur shifts and metrics

In this subsection, we will assume $\Sigma_{\mathcal{A}}$ being second countable and we will construct a family of metrics for the topology $\tau_{\Sigma_{\mathcal{A}^{\mathbb{N}}}}$ (and so for any blur shift over the same alphabet and same blurred sets). Such construction follows ideas of [11] and [15], but uses more direct and intuitive construction than the ones presented in these references.

We recall that the case $\Sigma_{\mathcal{A}}$ being second countable corresponds to the particular case of metrizable blur shifts given by Corollary 3.26. It remains an open problem to construct metrics for other cases.

Let

$$\mathfrak{p} := \{(\alpha), (\alpha\bar{H}) : \ \alpha \in B(\mathcal{A}^{\mathbb{N}}), \ \bar{H} \in \bar{\mathcal{V}}\}. \tag{25}$$

Definition 3.29. *We say that $p \in \mathfrak{p}$ is a* **prefix** *of a sequence* x *if and only if* $x \in Z(p)$. *Given $p, q \in \mathfrak{p}$ we will say that p is prefix of q if and only if $Z(q) \subset Z(p)$.*

Note that, using the language of prefixes, we have that the cylinders $Z(\alpha)$ and $Z(\alpha\bar{H})$ are the sets of all sequences in $\Sigma_{\mathcal{A}^{\mathbb{N}}}$ that have (α) and $(\alpha\bar{H})$ as prefix, respectively. Furthermore, any element of $\mathcal{A}^{\mathbb{N}}$ has infinitely many prefixes in $\mathfrak{p}$, while an element of $\mathcal{L}_n(\Lambda)$, with $n \in \mathbb{N}$, has exactly $n+1$ prefixes in $\mathfrak{p}$.

From Proposition 3.21 both $\mathcal{A}$ and $\mathcal{V}$ are countable, and then $\mathfrak{p}$ is also countable. Consider an enumeration of $\mathfrak{p}$:

$$\mathfrak{p} := \{p_1, p_2, p_3, \ldots\}.$$

For each specific enumeration of $\mathfrak{p}$, we define $d : \Sigma_{\mathcal{A}^{\mathbb{N}}} \times \Sigma_{\mathcal{A}^{\mathbb{N}}} \to [0, +\infty)$ by

$$d(x, y) := \begin{cases} 1/2^i & , \text{ where } i \in \mathbb{N} \text{ is the smallest such that } p_i \text{ is prefix either of x or of y,} \\ 0 & , \text{ if } x = y. \end{cases} \tag{26}$$

Let us check that d is a metric on $\Sigma_{\mathcal{A}^{\mathbb{N}}}$. It is direct that $d(x, y) = d(y, x) \geq 0$ for all $x, y \in \Sigma_{\mathcal{A}^{\mathbb{N}}}$, and $d(x, y) = 0$ if and only if $x = y$. To check the triangular inequality, given $x, y, z \in \Sigma_{\mathcal{A}^{\mathbb{N}}}$, suppose that $d(x, z) = 2^{-i}$, where i is the first index such that p_i is a prefix of only one of x or z. Without loss of generality, assume that p_i is a prefix of x and is not of z. We have then two possibilities: Either p_i is a prefix of y or it is not. In the former case, we have $d(y, z) = 1/2^j$ where $j = \min\{k \geq 1 : \ p_k \text{ is prefix either of y or of z}\} \leq i$, and therefore $d(x, z) \leq d(y, z)$. In the latter case, when p_i is not a prefix of y, then $d(x, y) = 1/2^\ell$, where $\ell = \min\{k \geq 1 : \ p_k \text{ is prefix either of x or of y}\} \leq i$, and so we have that $d(x, z) \leq d(x, y)$. Hence, we conclude that $d(x, z) \leq d(x, y) + d(y, z)$.

Denote as $B_d(x,\varepsilon)$ the open ball for the metric d, centered in x and with radius $\varepsilon > 0$. Given $\varepsilon > 0$, let k be the smallest positive integer such that $1/2^k < \varepsilon$. Observe that a point $y \neq x$ belongs to $B_d(x,\varepsilon)$ if and only if the first $p_j \in \mathfrak{p}$ which is prefix either of x or of y is such that $j \geq k$.

Proposition 3.30. *The metric d, defined from any enumeration of $\mathfrak{p}$, and the generalized cylinders generate the same topology.*

Proof. First we need to show that for any given $x \in \Sigma_{\mathcal{A}^\mathbb{N}}$ and a generalized cylinder of its local basis $\mathfrak{B}_x$, there exists $\varepsilon > 0$ such that $B_d(x,\varepsilon)$ is contained in that generalized cylinder. In fact, if $x \in \mathcal{A}^\mathbb{N}$, and the generalized cylinder is $Z(x_0...x_n)$, then we just need take $k \geq 1$ such that $p_k \in \mathfrak{p}$ is the prefix $p_k = (x_0...x_n)$. Therefore, $y \in B_d(x, 1/2^k)$ implies that the first $p_j \in \mathfrak{p}$ which is prefix either of x or of y is such that $j > k$, and so p_k is also prefix of y, which implies that $B_d(x,1/2^k) \subset Z(p_k) = Z(x_0...x_n)$. If $x \in \partial\mathcal{A}^\mathbb{N}$ and the generalized cylinder is $Z(x_0...x_{n-1}\bar{H}, F)$, take $k \geq 1$ as the greatest index such that $p_k \in \{p \in \mathfrak{p} :\ p = (x_0...x_{n-1}h) \text{ with } h \in F \cup \{\bar{H}\}\}$. Hence, if $y \in B_d(x,1/2^k)$, since the first $p_j \in \mathfrak{p}$ which is prefix either of x or of y is such that $j > k$, and since p_k comes later than $(x_0...x_{n-1}\bar{H})$, prefix of x, we conclude that $(x_0...x_{n-1}\bar{H})$ shall also be prefix of y. Furthermore, p_k comes after all prefixes $(x_0...x_{n-1}h)$ with $h \in F$ which are not prefixes of x and therefore they could not be prefixes of y either. These facts imply that $y \in Z(x_0...x_{n-1}\bar{H}, F)$, and so $B_d(x,1/2^k) \subset Z(x_0...x_{n-1}\bar{H}, F)$.

Now, we will show that for any given $x \in \Sigma_{\mathcal{A}^\mathbb{N}}$ and $\varepsilon > 0$, there exists a generalized cylinder in $\mathfrak{B}_{\Sigma_{\mathcal{A}^\mathbb{N}}}$ which contains x and is contained in $B_d(x,\varepsilon)$. Indeed, if $x \in \mathcal{A}^\mathbb{N}$, we can take $k \geq 1$ such that $1/2^k \leq \varepsilon$, p_k is prefix of x, and p_k is not prefix of any p_j with $j < k$ (we recall it is always possible to find such k since elements of $\mathcal{A}^\mathbb{N}$ have infinitely many prefix in $\mathfrak{p}$). Thus, any $y \in Z(p_k)$ is such that p_k is its prefix and for each $j < k$ a prefix p_j either is prefix of both x and y or is not prefix of any of them. Hence, we have that the first j such p_j is either prefix of x or of y is necessarily greater than k, and so $y \in B_d(x,1/2^k) \subset B_d(x,\varepsilon)$, which means $Z(p_k) \subset B_d(x,\varepsilon)$. On the other hand, if $x = (x_0...x_{n-1}\tilde{H}\tilde{H}\tilde{H}...) \in \partial\mathcal{A}^\mathbb{N}$, then we just need to take $k \geq 1$ such that $1/2^k \leq \varepsilon$ and consider the generalized cylinder $Z(x_0...x_{n-1}\bar{H}, F)$ where $F := \{h \in H :\ \exists j \leq k \text{ s.t. } p_j = (x_0...x_{n-1}h...x_{n_j}) \text{ or } p_j = (x_0...x_{n-1}\bar{G}) \text{ with } G \neq H \text{ and } h \in G\}$. Therefore, if $y \in Z(x_0...x_{n-1}\bar{H}, F)$ it follows that any prefix of x is also a prefix of y, and there is not a $j \leq k$ such that p_j is prefix of y but not of x. Thus, the first index j such p_j is prefix of y but no of x is greater than k, which implies $y \in B(x,1/2^k) \subset B_d(x,\varepsilon)$ which leads to conclude that $Z(x_0...x_{n-1}\bar{H}, F) \subset B_d(x,\varepsilon)$.

$\square$

3.3.3 Compactness and local-compactness criteria

In this subsection we shall present necessary and sufficient conditions under which a blur shift Σ_Λ is (sequentially) compact or locally compact.

Firstly, we shall prove that sequential compactness and compactness are equivalent concepts in the context of blur shifts (in spite of them being metrizable or not).

Recall that the induced topology on some given $\mathfrak{X} \subset \Sigma_{\mathcal{A}^\mathbb{N}}$ has a basis $\mathfrak{B}_\mathfrak{X}$ composed by the sets $Z_\mathfrak{X}(\alpha) := Z(\alpha) \cap \mathfrak{X}$ and $Z_\mathfrak{X}(\alpha\bar{H}, F) := Z(\alpha\bar{H}, F) \cap \mathfrak{X}$ for $\alpha\tilde{H} \in B(\mathfrak{X})$, and a finite set $F \subset \mathcal{F}_\mathfrak{X}(\alpha) \cap H$. Furthermore, $B_n(\mathfrak{X})$ stands for the set of all words with length n in $\mathfrak{X}$, $B(\mathfrak{X})$ stands for the set of all finite words in $\mathfrak{X}$, and $\mathcal{F}_\mathfrak{X}(a_0 \dots a_{n-1})$ stands for the follower set of $a_0 \dots a_{n-1} \in B(\mathfrak{X})$ in $\mathfrak{X}$.

Remark 3.31. *For all $\alpha \in B(\mathcal{A}^\mathbb{N})$, $H \in \mathcal{V}$, and any family of finite sets $\{F_\ell\}_{\ell\in\lambda}$, there exist $F_1, ..., F_t \in \{F_\ell\}_{\ell\in\lambda}$ such that*

$$\bigcup_{\ell\in\lambda} Z(\alpha\bar{H}, F_\ell) = \bigcup_{i=1}^{t} Z(\alpha\bar{H}, F_i).$$

Furthermore, taking $F := \bigcap_{i=1}^t F_i$, we have

$$\bigcup_{\ell\in\lambda} Z(\alpha\bar{H}, F_\ell) = Z(\alpha\bar{H}, F).$$

In order to characterize compactness and local compactness of blur shift spaces we shall first prove that sequential compactness and compactness are equivalent concepts in our topology. To achieve this we need the following auxiliary lemma.

Lemma 3.32. *Let $\mathfrak{X} \subset \Sigma_{\mathcal{A}^\mathbb{N}}$ be any subset (not necessarily a blur shift space). If $\mathfrak{X}$ is sequentially compact, then for all $\alpha \in B(\mathfrak{X}) \cap B(\mathcal{A}^\mathbb{N})$ the sets $V_\alpha := \{H \in \mathcal{V} :\ \tilde{H} \in \mathcal{F}_\mathfrak{X}(\alpha)\}$ and $W_\alpha := \mathcal{F}_\mathfrak{X}(\alpha) \setminus \left(\bigcup_{H\in V_\alpha} \bar{H}\right)$ are finite. Conversely, under the additional assumption that $\mathfrak{X}$ is closed, if for all $\alpha \in B(\mathfrak{X}) \cap B(\mathcal{A}^\mathbb{N})$ the sets V_α and W_α are finite, then $\mathfrak{X}$ is sequentially compact.*

Proof. Suppose $\mathfrak{X}$ sequentially compact and let us show that for each $a_1...a_{n-1} \in B(\mathfrak{X}) \cap B(\mathcal{A}^\mathbb{N})$ there are at most finitely many $\tilde{H} \in \mathcal{V}$ such that $(a_1...a_{n-1}\tilde{H}\tilde{H}\tilde{H}...) \in \mathfrak{X}$. Indeed, if we suppose, by contradiction, that for some $a_1...a_{n-1} \in B(\mathfrak{X})$ there exist an infinite set $\{\tilde{H}^\ell \in \mathcal{V} : \ell \geq 0\}$ such that $(a_1...a_{n-1}\tilde{H}^\ell\tilde{H}^\ell\tilde{H}^\ell...) \in \mathfrak{X}$ for all $\ell \geq 0$, then $(a_1...a_{n-1}\tilde{H}^\ell\tilde{H}^\ell\tilde{H}^\ell...)_{\ell\geq 0}$ is a sequence in $\mathfrak{X}$ which has not any convergent subsequence, contradicting that $\mathfrak{X}$ is sequentially compact. Hence, given $\alpha \in B(\mathfrak{X})$, the set $V_\alpha := \{H \in \mathcal{V} :\ \tilde{H} \in \mathcal{F}_\mathfrak{X}(\alpha)\}$ is finite (possibly empty).

Now, given $\alpha = a_0...a_{n-1} \in B(\mathfrak{X}) \cap B(\mathcal{A}^\mathbb{N})$, define $W_\alpha := \mathcal{F}_\mathfrak{X}(\alpha) \setminus \left(\bigcup_{H\in V_\alpha} \bar{H}\right)$. Note that W_α just contains symbols of the alphabet $\mathcal{A}$. Suppose by contradiction W_α is infinite, and for each $\ell \geq 0$ take $w^\ell \in W_\alpha \setminus \{w^j : j < \ell\}$. Hence, there exists a sequence $(\mathrm{x}^\ell)_{\ell \geq 0} \in \mathfrak{X}$ such that $x_0^\ell ... x_{n-1}^\ell = \alpha$ and $x_n^\ell = \mathrm{w}^\ell$. Since for each distinct ℓ, we have a distinct x_n^ℓ which does not belong to any blurred set, it follows that $(\mathrm{x}^\ell)_{\ell \geq 0}$ does not contain any convergent subsequence, which contradicts the sequential compactness of $\mathfrak{X}$. Thus, W_α shall be finite.

Now suppose $\mathfrak{X}$ closed and that for all $\alpha \in B(\mathfrak{X}) \cap B(\mathcal{A}^\mathbb{N})$ the sets V_α and W_α are finite. Let $(\mathrm{x}^\ell)_{\ell \geq 0} \in \mathfrak{X}$ be any sequence. If $(\mathrm{x}^\ell)_{\ell \geq 0}$ has a subsequence $(\mathrm{x}^{\ell_{0_k}})_{k \geq 0}$ such that $x_0^{\ell_{0_j}} = x_0^{\ell_{0_k}}$ for all $j, k \geq 0$, then we take this subsequence. Now, if $(\mathrm{x}^{\ell_{0_k}})_{k \geq 0}$ has a subsequence $(\mathrm{x}^{\ell_{1_k}})_{k \geq 0}$ such that $x_1^{\ell_{1_j}} = x_1^{\ell_{1_k}}$ for all $j, k \geq 0$, then we take this subsequence. We proceed recursively, either infinitely or until not to be able of finding a subsequence $(\mathrm{x}^{\ell_{n_k}})_{k \geq 0}$ of $(\mathrm{x}^{\ell_{n-1_k}})_{k \geq 0}$ such that $x_n^{\ell_{n_j}} = x_n^{\ell_{n_k}}$ for all $j, k \geq 0$. If there always be such subsequences, then we will have an infinite family of subsequences

$$\left\{(\mathrm{x}^{\ell_{n_k}})_{k \geq 0}\right\}_{n \geq 0}, \tag{27}$$

and it follows that $(\mathrm{x}^{\ell_{n_1}})_{n \geq 0}$ is a subsequence of $(\mathrm{x}^\ell)_{\ell \geq 0}$ which converges in $\mathfrak{X}$ (since $\mathfrak{X}$ is closed).

On the other hand, suppose that it is not possible to construct an infinite family as (27), and let $n \geq 0$ be the first integer for which $(\mathrm{x}^{\ell_{n-1_k}})_{k \geq 0}$ is such that $x_n^{\ell_{n-1_j}} \neq x_n^{\ell_{n-1_k}}$ for all $j, k \geq N$ for some $N \in \mathbb{N}$ (recall that from its construction we have that $x_i^{\ell_{n-1_j}} = x_i^{\ell_{n-1_k}} =: b_i$ for all $j, k \geq 0$ and for all $0 \leq i \leq n-1$). Denote $\beta := b_1 ... b_{n-1}$. Since W_β finite, then for all but finitely many indexes $j \geq 0$ we have $x_n^{\ell_{n-1_j}} \notin W_\beta$. On the other hand, since V_β is a finite family of sets, then infinitely many $x_n^{\ell_{n-1_j}}$ belong to the same $H \in V_\beta$, that is, there exists a subsequence $(\mathrm{x}^{\ell_{n_k}})_{k \geq 0}$ of $(\mathrm{x}^{\ell_{n-1_k}})_{k \geq 0}$ such that $x_n^{\ell_{n_k}} \in H \setminus \{x_n^{\ell_{n_0}}, ..., x_n^{\ell_{n_{k-1}}}\}$ for all $k \geq 0$. Thus, $x_n^{\ell_{n_k}} \to \tilde{H}$ as $k \to \infty$, which implies that $\mathrm{x}^{\ell_{n_k}} \to (\beta \tilde{H}\tilde{H}\tilde{H}...) \in \mathfrak{X}$ (observe that we did not need to use here the hypothesis of $\mathfrak{X}$ being closed, since from the definition of V_β we have that $(\beta \tilde{H}\tilde{H}\tilde{H}...) \in \mathfrak{X}$).

$\square$

Now we can prove the equivalence between compactness and sequential compactness. We remark that as in the previous lemma, the following theorem does not require the set $\mathfrak{X}$ being a blur shift space.

Theorem 3.33. *A subset $\mathfrak{X} \subset \Sigma_{\mathcal{A}^\mathbb{N}}$ is compact if and only if it is sequentially compact.*

Proof. Suppose that $\mathfrak{X}$ is compact and let $(\mathrm{x}^\ell)_{\ell \geq 0} \in \mathfrak{X}$ be a sequence. As done in the second part of Lemma 3.32, if $(\mathrm{x}^\ell)_{\ell \geq 0}$ has a subsequence $(\mathrm{x}^{\ell_{0_k}})_{k \geq 0}$ such that $x_0^{\ell_{0_j}} = x_0^{\ell_{0_k}}$ for all $j, k \geq 0$, we take this subsequence. We proceed recursively, either infinitely or until not to be able of finding a subsequence $(\mathrm{x}^{\ell_{n_k}})_{k \geq 0}$ of $(\mathrm{x}^{\ell_{n-1_k}})_{k \geq 0}$ such that $x_n^{\ell_{n_j}} = x_n^{\ell_{n_k}}$ for all $j, k \geq 0$. As before, if we can proceed infinitely, then we get an infinite family of subsequences (27) and we can define a convergent subsequece. On the other hand, if we cannot proceed infinitely, and $n \geq 0$ is the first integer for which $(\mathrm{x}^{\ell_{n-1_k}})_{k\geq 0}$ is such that $x_n^{\ell_{n-1_j}} \neq x_n^{\ell_{n-1_k}}$ for all $j, k \geq N$ for some $N \in \mathbb{N}$, we consider the open cover of $\mathfrak{X}$,

$$C_n := \{Z_{\mathfrak{X}}(a_0...a_n),\ Z_{\mathfrak{X}}(b_0...b_{n-1}\bar{H}) :\ a_0...a_n,\ b_0...b_{n-1} \in B(\mathfrak{X}) \cap B(\mathcal{A}^{\mathbb{N}}),$$

$$\text{and } \tilde{H} \in \mathcal{F}_{\mathfrak{X}}(b_0...b_{n-1})\}.$$

From the compactness of $\mathfrak{X}$ there is $C'_n \subset C_n$ a finite subcover of $\mathfrak{X}$, and then there is a subsequence of $(\mathrm{x}^{\ell_{n-1_k}})_{k\geq 0}$ which lies in some cylinder $Z_{\mathfrak{X}}(b_0...b_{n-1}\bar{H}) \in C'_n$. Hence, since $x_n^{\ell_{n-1_k}}$ is different for each $k \geq N$, it implies that such subsequence converges to $(b_0...b_{n-1}\tilde{H}\tilde{H}\tilde{H}...) \in \mathfrak{X}$.

Now suppose that $\mathfrak{X}$ is sequentially compact. Firstly, note that if $\mathcal{C}$ is a disjoint open cover, then $\mathcal{C}$ shall be finite. In fact, if by contradiction we suppose that $\mathcal{C}$ is infinite, then we can take a sequence $(\mathrm{x}^{\ell})_{\ell\geq 0}$ such that each x^{ℓ} belongs to a different set of $\mathcal{C}$. Therefore, since $\mathfrak{X}$ is sequentially compact, there exists a subsequence $(\mathrm{x}^{\ell_k})_{k\geq 0}$ which converges to some point $\bar{\mathrm{x}} \in \mathfrak{X}$, which implies that the set $A \in \mathcal{C}$ which contains $\bar{\mathrm{x}}$ also contains infinitely many terms of $(\mathrm{x}^{\ell_k})_{k\geq 0}$, contradicting that $\mathcal{C}$ is a disjoint family and each x^{ℓ} belongs to a distinct set of $\mathcal{C}$.

For the general case, let $\mathcal{C}$ be an open cover, and we can assume, without loss of generality, that its elements are generalized cylinders of $\mathfrak{B}_{\mathfrak{X}}$. Consider the subcover $\mathcal{C}'$ formed by all maximal sets of $\mathcal{C}$, that is,

$$\mathcal{C}' := \{Z \in \mathcal{C} :\ \nexists Y \in \mathcal{C} \text{ s.t. } Z \subset Y\}.$$

Let us prove that $\mathcal{C}'$ is finite. To achieve this, we shall construct an open cover $\tilde{\mathcal{C}}$ which is finite if and only if $\mathcal{C}'$ is finite.

In what follows we shall denote as α a word in $\mathcal{B}(\mathfrak{X}) \cap B(\mathcal{A}^{\mathbb{N}})$. Hence, firstly note that a generalized cylinder of type $Z_{\mathfrak{X}}(\alpha)$ in $\mathcal{C}'$ is disjoint of any other cylinder in $\mathcal{C}'$. Thus, we have that a generalized cylinder of type $Z_{\mathfrak{X}}(\alpha\bar{H}, F_H)$ of $\mathcal{C}'$ can only intersect a generalized cylinder $Z_{\mathfrak{X}}(\alpha\bar{G}, F_G)$ of $\mathcal{C}'$.

Define a cover $\mathcal{C}''$ from $\mathcal{C}'$ as follows: For each $(\alpha\tilde{H}\tilde{H}\tilde{H}...) \in \mathfrak{X}$, replace all of its basic neighborhoods in $\mathcal{C}'$, $\{Z_{\mathfrak{X}}(\alpha\bar{H}, F^{\ell}_{\alpha H})\}_{\ell\in\lambda(\alpha\tilde{H})} = \mathfrak{B}_{(\alpha\tilde{H}\tilde{H}\tilde{H}...)} \cap \mathcal{C}'$, by a single generalized cylinder $Z_{\mathfrak{X}}(\alpha\bar{H}, F_{\alpha H})$ such that

$$Z_{\mathfrak{X}}(\alpha\bar{H}, F_{\alpha H}) = \bigcup_{\ell\in\lambda(\alpha\tilde{H})} Z_{\mathfrak{X}}(\alpha\bar{H}, F^{\ell}_{\alpha H}) \qquad \text{(See Remark 3.31)}.$$

From Lemma 3.32, sequential compactness implies that for each $\alpha \in B(\mathfrak{X}) \cap B(\mathcal{A}^{\mathbb{N}})$ the set $V_\alpha := \{H \in \mathcal{V} :\ \tilde{H} \in \mathcal{F}_{\mathfrak{X}}(\alpha)\}$ is finite. It implies that for each α, the family $V_{\alpha,\mathcal{C}''} := \{H \in \mathcal{V} :\ Z_{\mathfrak{X}}(\alpha\bar{H}, F_{\alpha H}) \in \mathcal{C}''\} \subset V_\alpha$ is finite. Denote $V_{\alpha,\mathcal{C}''} = \left\{H^1, ..., H^{k(\alpha)}\right\}$, and denote $N_{\alpha,\mathcal{C}''} := \left\{ Z_{\mathfrak{X}}\left(\alpha\bar{H}^1, F_{\alpha H^1}\right), ..., Z_{\mathfrak{X}}\left(\alpha\bar{H}^{k(\alpha)}, F_{\alpha H^{k(\alpha)}}\right)\right\}$, which is the family of all generalized cylinders of the type (22) in $\mathcal{C}''$ whose definition uses α.

Note that if $\mathcal{C}'$ is finite, then $\mathcal{C}''$ is finite. On the other hand, since from Remark 3.31 each generalized cylinder $Z_{\mathfrak{X}}(\alpha\bar{H}^i, F_{\alpha H^i})$ in $\mathcal{C}''$ is a finite union of generalized cylinders in $\{Z_{\mathfrak{X}}(\alpha\bar{H}^i, F^{\ell}_{\alpha H^i})\}_{\ell\in\lambda(\alpha\tilde{H}^i)} \subset \mathcal{C}'$, it follows that if $\mathcal{C}''$ is finite, then $\mathcal{C}'$ is finite.

Now, define the cover $\tilde{\mathcal{C}}$ from $\mathcal{C}''$ as follows: For each $\alpha = \alpha_0...\alpha_{n-1} \in B(\mathfrak{X}) \cap B(\mathcal{A}^{\mathbb{N}})$, such that $N_{\alpha,\mathcal{C}''}$ is nonempty, define $\tilde{N}_{\alpha,\mathcal{C}''}$ as the smallest family of disjoint generalized cylinders that refines $N_{\alpha,\mathcal{C}''}$. Since $N_{\alpha,\mathcal{C}''}$ is finite and for any $x \in Z_{\mathfrak{X}}(\alpha\bar{H}^i, F_{\alpha H^i}) \cap Z_{\mathfrak{X}}(\alpha\bar{H}^j, F_{\alpha H^j})$ there are only a finite number of possible symbols for $x_n \in H^i \cap H^j$, it follows that $\tilde{N}_{\alpha,\mathcal{C}''}$ is also finite and can be written as

$$\tilde{N}_{\alpha,\mathcal{C}''} = \left\{ Z_{\mathfrak{X}}\left(\alpha\bar{H}^1, G_{\alpha H^1}\right), ..., Z_{\mathfrak{X}}\left(\alpha\bar{H}^{k(\alpha)}, G_{\alpha H^{k(\alpha)}}\right), Z_{\mathfrak{X}}\left(\alpha g^1\right), ..., Z_{\mathfrak{X}}\left(\alpha g^{l(\alpha)}\right)\right\}.$$

Now, $\tilde{\mathcal{C}}$ is obtained by replacing in $\mathcal{C}''$ the sets of each family $N_{\alpha,\mathcal{C}''}$ by the sets of the family $\tilde{N}_{\alpha,\mathcal{C}''}$.

Observe that, $\tilde{\mathcal{C}}$ is also a cover of $\mathfrak{X}$, and $\tilde{\mathcal{C}}$ is finite if and only if $\mathcal{C}''$ is finite. Hence, we conclude by noting that, since the unique nonempty intersections in $\mathcal{C}''$ could occur between sets of a same family $N_{\alpha,\mathcal{C}''}$, it follows that $\tilde{\mathcal{C}}$ is a disjoint cover, and so it is a finite cover.

□

Now, we are able to characterize compactness and local compactness in blur shifts.

Theorem 3.34. *A blur shift Σ_Λ is compact if and only if $\mathcal{V}_\Lambda$ is a finite family of sets which covers all except a finite number of elements of $B_1(\Lambda)$.*

Proof. Suppose Σ_Λ compact, then from Lemma 3.32 and Theorem 3.33, it follows that for all $\alpha \in B(\Lambda)$ the sets $V_\alpha := \{H \in \mathcal{V}_\Lambda : \tilde{H} \in \mathcal{F}_{\Sigma_\Lambda}(\alpha)\}$ and $W_\alpha := \mathcal{F}_{\Sigma_\Lambda}(\alpha) \setminus \left(\bigcup_{H\in V_\alpha} \bar{H}\right) = \mathcal{F}_\Lambda(\alpha) \setminus \left(\bigcup_{H\in V_\alpha} H\right)$ are finite. Taking $\alpha = \epsilon$, the empty word, it follows $\mathcal{F}_\Lambda(\epsilon) = B_1(\Lambda)$, $V_\epsilon = \mathcal{V}_\Lambda$ and $W_\epsilon = B_1(\Lambda) \setminus \left(\bigcup_{H\in\mathcal{V}_\Lambda} H\right)$, which implies that $\mathcal{V}_\Lambda$ is a finite family which covers all the set of $B_1(\Lambda)$ but the finite set of symbols W_ϵ.

On the other hand, if $\mathcal{V}_\Lambda$ is a finite family of sets which covers all except a finite number of elements of $B_1(\Lambda)$, then any follower set in Σ_Λ can be written as the union of a finite number of blurred sets with a finite set of symbols. Hence, since Σ_Λ is closed, from Lemma 3.32 we conclude that it is sequentially compact and so compact (Theorem 3.33).

□

Proposition 3.35. *A blur shift Σ_Λ over an uncountable alphabet cannot be simultaneously compact and first countable.*

Proof. Just note that if $B_1(\Lambda)$ is uncountable, it is not possible that all three following statement hold: $\mathcal{V}_\Lambda$ is finite; $\mathcal{V}_\Lambda$ covers all but finite number of symbols of $B_1(\Lambda)$; For all $H \in \mathcal{V}_\Lambda$ we have $H \cap B_1(\Lambda)$ countable. Hence, from Proposition 3.22 and Theorem 3.34 we get that either Σ_Λ is compact or it is first countable.

□

As a direct consequence of the previous proposition, we have that:

Corollary 3.36. *A blur shift Σ_Λ over an uncountable alphabet cannot be simultaneously compact and metrizable.*

□

Next, we characterize locally-compact blur shift spaces.

Theorem 3.37. *A blur shift Σ_Λ is locally compact if and only if for all $w \in B_1(\Lambda)$ there exists a family of words $\{\mathsf{v}_\ell\}_{\ell\in\lambda} \subset B(\Lambda)$ such that $\bigcup_{\ell\in\lambda} Z_\Lambda(w\mathsf{v}_\ell) \cap \Lambda = Z_\Lambda(w) \cap \Lambda$, and for all $\ell \in \lambda$ and $\mathsf{u} \in B(\Lambda)$ such that $w\mathsf{v}_\ell\mathsf{u} \in B(\Lambda)$ there is a finite number of sets in $\mathcal{V}_\Lambda$ that cover all except a finite number of elements of $\mathcal{F}_\Lambda(w\mathsf{v}_\ell\mathsf{u})$.*

Proof.

($\Longrightarrow$) Suppose that Σ_Λ is locally compact. Given $w_0 \in B_1(\Lambda)$, define $\lambda := Z_\Lambda(w_0) \cap \Lambda$ and for each $\mathsf{x} \in \lambda$ let $\mathsf{v}_\mathsf{x} = (x_1...x_{k(\mathsf{x})}) \in B(\Lambda)$ such that $\mathfrak{X}_\mathsf{x} := Z_\Lambda(w_0\mathsf{v}_\mathsf{x}) = Z_\Lambda(w_0x_1...x_{k(\mathsf{x})})$ is compact. From Lemma 3.32 and Theorem 3.33 it follows that for any $\alpha = w_0\mathsf{v}_\mathsf{x}\mathsf{u} \in B(\mathfrak{X}_\mathsf{x})$ we have a finite number of blurred sets that cover all except a finite number of letters of $\mathcal{F}_{\mathfrak{X}_\mathsf{x}}(w_0\mathsf{v}_\mathsf{x}\mathsf{u}) \supset \mathcal{F}_\Lambda(w_0\mathsf{v}_\mathsf{x}\mathsf{u})$. Hence, the family $\{\mathsf{v}_\mathsf{x}\}_{\mathsf{x}\in\lambda}$ satisfies the desired property.

($\Longleftarrow$) Given $\mathsf{x} = (x_i)_{i\in\mathbb{N}} \in \Lambda$, consider $w = x_0$ and let $\{\mathsf{v}_\ell\}_{\ell\in\lambda} \subset B(\Lambda)$ such that $\bigcup_{\ell\in\lambda} Z_\Lambda(x_0\mathsf{v}_\ell) \cap \Lambda = Z_\Lambda(x_0) \cap \Lambda$ be the family that verifies the property given in the statement of the theorem. Then, there is v_t such that $\mathsf{x} \in Z_\Lambda(x_0\mathsf{v}_t)$, which means, $\mathsf{v}_t = x_1...x_{N(t)}$ for some $N(t) \geq 0$. By hypothesis, there are a finite number of sets in $\mathcal{V}_\Lambda$ that cover all except a finite number of elements of $\mathcal{F}_\Lambda(w\mathsf{v}_t\mathsf{u}) = \mathcal{F}_\Lambda(x_0...x_{N(t)}\mathsf{u})$.

Denote $\mathfrak{X} := Z_\Lambda(x_0...x_{N(t)})$. It follows that $\mathfrak{X}$ is closed and any $\alpha \in B(\mathfrak{X}) \cap B(\mathcal{A}^\mathbb{N})$ is written as either $\alpha = x_0...x_k$ with $k < N(t)$ or as $\alpha = x_0...x_{N(t)}\mathsf{u}$ for some $\mathsf{u} \in B(\Lambda)$. In the former case $\mathcal{F}_\mathfrak{X}(x_0...x_k) = \{x_{k+1}\}$, while in the later case the sets V_α and W_α given in Lemma 3.32 are also finite. In both cases, from Lemma 3.32 and Theorem 3.33 we conclude that $\mathfrak{X}$ is compact.

□

Example 3.38. *Let $\mathcal{A} := \mathbb{R}^+ = [0, \infty)$ and as in Example 3.4, for each $\lambda \in [0, 1)$ define the set $H_\lambda := \{x \in \mathbb{R}^+ : \ x := \lambda + k, \ k \in \mathbb{N}\}$, and then define $\mathcal{V} := \{H_\lambda\}_{\lambda \in [0,1)}$.*

Let $f : \mathbb{R}^+ \to [0, 1)$ be the function given by

$$f(\lambda) := \left\{ \frac{\lambda - \lfloor \lambda \rfloor}{k} + k : \ 0 < k \leq \lceil \lambda \rceil^\star \right\},$$

where $\lfloor \lambda \rfloor$ denotes the largest integer less than or equal to λ, and $\lceil \lambda \rceil^\star$ denotes the smallest integer strictly greater than λ.

Hence, the blur shift $\Sigma_\Lambda \subset \Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$, where Λ is defined as follows:

$$\mathrm{x} \in \Lambda \quad \iff \quad \forall i \in \mathbb{N}, \ x_{i+1} \in H_{\ell_i} \text{ for some } \ell_i \in f(x_i),$$

is a locally compact blur shift which is not compact.

Recall that for any blur shift $\Sigma^{\mathcal{V}}_{\Lambda}$ there exist directed labeled graphs $\mathcal{G}$ and $\bar{\mathcal{G}}$ such that $\mathcal{G} \subset \bar{\mathcal{G}}$, $\Lambda = \Lambda_{\mathcal{G}}$ and $\Sigma^{\mathcal{V}}_{\Lambda} = \Lambda_{\bar{\mathcal{G}}}$ (Theorem 3.10). Thus, the local compactness condition given in Theorem 3.37, translated for the graph language, means that, independently of the initial vertex, any walk on $\mathcal{G}$ will be eventually enclosed in a subgraph where each vertex has all but a finite number of outgoing labels covered by some finite number of blurred sets. We notice that this condition is more general than the RFUM condition in [7, Proposition 3.12] for Gonçalves-Royer ultragraph shifts. Indeed, RFUM condition was a sufficient but not necessary condition for local compactness, since it supposes that the whole graph has the property that almost all outgoing edges of each vertex are covered by a finite number of blurred sets. In our general context, RFUM condition corresponds to the condition given in Corollary 3.39 below.

Corollary 3.39. *If for any nonempty letter $a \in B_1(\Lambda)$ and $\mathrm{u} \in B(\Lambda)$ there are a finite number of sets in $\mathcal{V}_\Lambda$ that cover all except a finite number of elements of $\mathcal{F}_\Lambda(a\mathrm{u})$, then Σ_Λ is locally compact. If the previous property also holds for the empty word ϵ, then Σ_Λ is compact.*

Proof. Note that for each nonempty word $\mathrm{w} = w_0 \ldots w_n \in B(\Lambda)$ we have $\mathcal{F}_\Lambda(\mathrm{w}) \subset \mathcal{F}_\Lambda(w_n)$. Hence, the first part of the corollary corresponds to the special case in Theorem 3.37 where, for each $w \in B_1(\Lambda)$, the correspondent family $\{\mathrm{v}_\ell\}_{\ell \in \lambda}$ is such that $\mathrm{v}_\ell = \epsilon$ for all $\ell \in \lambda$.

If for the empty word ϵ we have a finite number of sets of $\mathcal{V}_\Lambda$ covering all except a finite number of elements of $\mathcal{F}_\Lambda(\epsilon\mathrm{u})$, then, in particular, a finite number of sets of $\mathcal{V}_\Lambda$ cover all except a finite number of elements of $\mathcal{F}_\Lambda(\epsilon) = B_1(\Lambda)$, and from Theorem 3.34 we conclude that Σ_Λ is compact.

□

3.4 SHIFT COMMUTING MAPS, CONTINUITY AND GENERALIZED SLIDING BLOCK CODES

In this section we present several results that characterize shift-commuting maps, with particular interest on the case where they are continuous or generalized sliding block codes. The results presented here hold for general blur shifts and they are expressed in the general framework, but their proofs share ideas with those given in [8] in the particular context of Gonçalves-Royer ultragraph shifts.

We remark that, differently than in [9] where it was considered only Ott-Tomforde-Willis shift spaces over countable alphabets, or than in [8] where the alphabet was always countable, here we do not impose any restriction on the cardinality of the alphabet, which can be uncountable. Furthermore, even the case where the blur shift is not first countable is contemplated in our results.

Before starting examining the continuity of general shift-commuting maps, we remark that as occurs in the particular cases studied in [7, 15], the shift map itself is not, in general, everywhere continuous. It is encapsulated in the next theorem.

Proposition 3.40. *Let $\Sigma_\Lambda \subset \Sigma_{\mathcal{A}^\mathbb{N}}$ be a blur shift space. Then*

i. The shift map is continuous on $\Sigma_\Lambda \setminus \mathcal{L}_0$;

ii. The shift map is continuous at $(\tilde{H}\tilde{H}\tilde{H}...) \in \mathcal{L}_0(\Lambda)$ if, and only if, there exists a finite $F \subset H$ such that $\mathcal{F}_\Lambda(H \setminus F) \subset H$ and for all $g \in H$ the set $H \cap \mathcal{P}_\Lambda(g)$ is finite.

Proof.

i. Let $\mathrm{x} = (x_0x_1x_2...) \in \Sigma_\Lambda \setminus \mathcal{L}_0$, and let $\mathrm{y} := (y_0y_1...) = \sigma(\mathrm{x}) = (x_1x_2...)$. Recall that, since $\mathrm{x} \notin \mathcal{L}_0$, we have $x_0 \in \mathcal{A}$. We need to prove that for any given generalized cylinder U containing y, there exists a generalized cylinder V containing x such that $\sigma(V) \subset U$. If $\mathrm{y} \in \Lambda$, then it is direct that given any $U := Z_\Lambda(y_0...y_{k-1}) = Z_\Lambda(x_1...x_k)$ it follows $V := Z_\Lambda(x_0x_1...x_k) \ni \mathrm{x}$ is such that $\sigma(V) \subset U$. On the other hand, if $\mathrm{y} \in \mathcal{L}_n$ for some $n \in \mathbb{N}$, we have that a generalized cylinder containing y is in the form $U := Z_\Lambda(y_0...y_{n-1}\bar{H}, F) = Z_\Lambda(x_1...x_n\bar{H}, F)$, and hence $V := Z_\Lambda(x_0x_1...x_n\bar{H}, F)$ contains x and is such that $\sigma(V) \subset U$.

ii. The continuity of the shift map at some point $\mathrm{x} = (\tilde{H}\tilde{H}\tilde{H}...)$ means that given a generalized cylinder neighborhood of $\sigma(\mathrm{x}) = \mathrm{x}$, say $U := Z_\Lambda(\bar{H}, F')$ with $F' \subset H$ finite, there exists a generalized cylinder neighborhood of x, say $V := Z_\Lambda(\bar{H}, F)$ with $F \subset H$ finite, such that $\sigma(V) \subset U$. Note that $\sigma(V) \subset U$ means that $\mathcal{F}_\Lambda(H \setminus F) \subset H \setminus F'$.

Thus, setting $U := Z_\Lambda(\bar{H})$, the continuity of σ at x means that $\mathcal{F}_\Lambda(H \setminus F) \subset H$ for some finite $F \subset H$. On the other hand, setting $U := Z_\Lambda(\bar{H}, F')$ for a nonempty finite $F' \subset H$, the continuity of σ at x means that there is a finite $F \subset H$ such that $\mathcal{F}_\Lambda(H \backslash F) \subset H \backslash F'$, which in its turn means that for each $g \in F'$ we have $\mathcal{P}_\Lambda(g) \cap H$ finite. Since F' can be taken as any finite subset of H, it means that for any $g \in H$ we shall have $\mathcal{P}_\Lambda(g) \cap H$ finite.

$\square$

Note that condition *ii.* in Proposition 3.40 above is just ensuring that if $(x^n)_{n\in\mathbb{N}} \in \Sigma_\Lambda$ is such that $x_0^n \to \tilde{H}$ as $n \to \infty$, then $x_1^n \to \tilde{H}$ as $n \to \infty$ as well.

Example 3.41. *Consider the alphabet $\mathcal{A} := \mathbb{N}^2$ and $\mathcal{V} := \{H_m\}_{m\in\mathbb{N}}$ the family of blurred sets where $H_m := \{(m, k) : k \in \mathbb{N}\}$ defined for each $m \in \mathbb{N}$.*

Let $\Lambda \subset \mathcal{A}^{\mathbb{N}}$ be the shift where $(x_i)_{i\in\mathbb{N}} = (m_i, k_i)_{i\in\mathbb{N}} \in \Lambda$ if and only if for all $i \in \mathbb{N}$ we have (m_{i+1}, k_{i+1}) is any whenever $k_i \leq m_i$, and (m_{i+1}, k_{i+1}) is such that $m_{i+1} = m_i$ and $k_{i+1} \geq k_i$ whenever $k_i > m_i$.

It follows that $\Sigma_\Lambda^{\mathcal{V}}$ satisfies the hypotheses in Proposition 3.40.ii. and then the shift map is continuous on the whole $\Sigma_\Lambda^{\mathcal{V}}$. In fact, for all $m \in \mathbb{N}$ we have $\mathcal{F}_\Lambda(H_m \backslash \{(m, 0), ..., (m, m)\}) = \{(m, k) : k > m\} \subset H_m$, and for any $(m, \ell) \in H_m$ it follows that $\mathcal{P}_\Lambda\big((m, \ell)\big) \cap H_m = \big\{(m, k) : k \leq \max\{m, \ell\}\big\}$.

3.4.1 Finitely defined sets

In this subsection we recall the concepts of pseudo cylinders and finitely defined sets which were introduced in [9, 10] and developed in [8].

Definition 3.42. *A* **pseudo cylinder** *in a blur shift space Σ_Λ is a set of the form*

$$[w]_k^\ell := \{(x_i)_{i\in\mathbb{N}} \in \Sigma_\Lambda : (x_k \dots x_\ell) = w\},$$

where $0 \leq k \leq \ell$ and $w \in B_{\ell-k+1}(\Sigma_\Lambda)$. We also assume that the empty set is a pseudo cylinder.

We recall that the topology of generalized cylinders when restricted to Λ coincides with the product topology of $\mathcal{A}^{\mathbb{N}}$ restricted to Λ, which means that a pseudo cylinder $[w]_k^\ell$ with $w \in B(\Lambda)$ is always an open set of Λ and so of Σ_Λ. However, it will be a closed set of Σ_Λ if and only if $k = 0$ in its definition, in which case $[w]_0^\ell = Z_\Lambda(w)$. On the other hand, a pseudo cylinder $[w]_k^\ell$, with w containing a symbol of $\tilde{V}$, is never an open set of Σ_Λ.

Definition 3.43. *We say that $C \subset \Sigma_\Lambda$ is* **finitely defined** *of Σ_Λ if both C and C^c can be written as unions of pseudo cylinders.*

Clearly from the above definition, if C is a finitely defined set, then C^c is too. Intuitively, a set C is finitely defined in Σ_Λ if we can check whether or not any given $x \in \Sigma_\Lambda$ belongs to C by knowing a finite number of coordinates of x. The empty set and Σ_Λ are examples of finitely defined sets.

The next proposition can be proved using the same approach than in propositions 5 and 6 of [8].

Proposition 3.44. *Finite unions and finite intersections of generalized cylinders are finitely defined sets.*

$\square$

We remark that, in general, infinite unions or intersections of finitely defined sets are not finitely defined sets. Hence, we get that an infinite union of generalized cylinders might not be a finitely defined set.

3.4.2 Shift-commuting maps

In this subsection we shall present a characterization of shift-commuting maps. In what follows, let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets, and $\mathcal{V} \subset 2^{\mathcal{A}}$ and $\mathcal{U} \subset 2^{\mathcal{B}}$ be two families of blurred sets, $\Sigma_\Lambda \subset \Sigma^{\mathcal{V}}_{\mathcal{A}^{\mathbb{N}}}$ and $\Sigma_\Gamma \subset \Sigma^{\mathcal{U}}_{\mathcal{B}^{\mathbb{N}}}$ be two blur shift spaces, and $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ be a map. For each $a \in \bar{\mathcal{B}} = \mathcal{B} \cup \tilde{\mathcal{U}}$ define

$$C_a := \Phi^{-1}([a]_0^0). \tag{28}$$

Note that $\{C_a\}_{a \in \bar{\mathcal{B}}}$ is a partition of Σ_Λ.

Proposition 3.45. *A map $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is shift commuting (i.e. $\Phi \circ \sigma = \sigma \circ \Phi$) if, and only if, for all $x \in \Sigma_\Lambda$ and $n \geq 0$ we have*

$$\big(\Phi(x)\big)_n = \sum_{a \in \bar{\mathcal{B}}} a \mathbb{1}_{C_a} \circ \sigma^n(x), \tag{29}$$

where $\mathbb{1}_{C_a}$ is the characteristic function of the set C_a and $\sum$ stands for the symbolic sum.

$\square$

The above result has similar proof than Proposition 3 in [8], while the next proposition corresponds to Proposition 4 and Corollary 2 in [8].

Proposition 3.46. *Let $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ be a shift-commuting map. It follows that:*

i. If $x \in \Sigma_\Lambda$ is a sequence with period $p \geq 1$ (that is, $\sigma^p(x) = x$) then $\Phi(x)$ also has period p;

ii. For all $\tilde{G} \in \tilde{\mathcal{U}}$, we have that $\sigma(C_{\tilde{G}}) \subset C_{\tilde{G}}$.

□

We will say that a shift-commuting map $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is **length preserving** if it maps sequences from $\mathcal{L}_n(\Lambda)$ to $\mathcal{L}_n(\Gamma)$ for all $n \in \mathbb{N} \cup \{\infty\}$. Length-preserving maps were used in [7] and [15] to prove isomorphism between the C^* -algebras of Ott-Tomforde-Willis edge shifts and Gonçalves-Royer ultragraph shifts. The next proposition gives a characterization of length-preserving maps.

Proposition 3.47. *A shift-commuting map $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is length-preserving if and only if for all $\mathcal{L}_0(\Lambda) = \bigcup_{\tilde{G}\in\tilde{\mathcal{U}}} C_{\tilde{G}}$.*

Proof. Just note that $\bigcup_{\tilde{G}\in\tilde{\mathcal{U}}} C_{\tilde{G}} = \Phi^{-1}(\mathcal{L}_0(\Gamma))$. Hence, if Φ a is length-preserving shift-commuting map, then it is direct that all points of $\mathcal{L}_0(\Lambda)$, and only points of $\mathcal{L}_0(\Lambda)$, can be mapped to points of $\mathcal{L}_0(\Gamma)$. Conversely, suppose Φ is not length preserving, and let $x \in \mathcal{L}_m(\Lambda)$ for some $m \in \mathbb{N} \cup \{\infty\}$, such that $\Phi(x) \in \mathcal{L}_n(\Gamma)$ with $n \neq m$. Since Φ is shift commuting, taking $k := \min\{m, n\}$, we have that $y := \sigma^k(x) \in \mathcal{L}_{m-k}(\Lambda)$ and $\Phi(y) = \Phi(\sigma^k(x)) = \sigma^k(\Phi(x)) \in \mathcal{L}_{n-k}(\Gamma)$. Thus, either y is a point of $\mathcal{L}_0(\Lambda)$ which is not mapped to $\mathcal{L}_0(\Gamma)$ or y is a point not in $\mathcal{L}_0(\Lambda)$ which is mapped to $\mathcal{L}_0(\Gamma)$. In any case, it follows that $\mathcal{L}_0(\Lambda) \neq \Phi^{-1}(\mathcal{L}_0(\Gamma))$.

□

The next theorem characterizes general continuous shift-commuting maps (including those which are not generalized sliding block codes - see Definition 3.49). Its proof follows analogous outline than Theorem 3.7 in [8].

Theorem 3.48. *If $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is continuous and shift commuting, then:*

i. For each $a \in \mathcal{B}$, the set C_a is a (possibly empty) union of generalized cylinders of Σ_Λ;

ii. If $x = (x_0 \dots x_{k-1}\tilde{H}\tilde{H}\tilde{H}\dots) \in \mathcal{L}_k(\Lambda)$ and $\Phi(x) = (y_0, \dots y_{\ell-1}\tilde{G}\tilde{G}\tilde{G}\dots) \in \mathcal{L}_\ell(\Gamma)$, then $\ell \leq k$ and for all $Z_\Gamma(\bar{G}, F)$ there exists a finite $F' \subset H$ such that $\Phi\big(Z_\Lambda(x_{k-\ell} \dots x_{k-1}\bar{H}, F')\big) \subset Z_\Gamma(\bar{G}, F)$.

iii. If $x = (\tilde{H}\tilde{H}\tilde{H}\dots) \in \mathcal{L}_0(\Lambda)$ and $\Phi(x) = (ddd\dots) \in \Gamma$, then for all $M > 0$ there exists a generalized cylinder $Z_\Lambda(\bar{H}, F)$ such that $\sigma^i(Z_\Lambda(\bar{H}, F)) \subseteq C_d$ for all $i = 0, 1, \dots, M$.

Conversely, if Φ is continuous on the points of $\Lambda \cap \Phi^{-1}\big(\mathcal{L}_0(\Gamma)\big)$ and condition $i. - iii.$ hold, then Φ is continuous on the whole Σ_Λ.

□

3.4.3 Generalized sliding block codes and continuous shift-commuting maps

We conclude this section, by stating the analogous of the Curtis-Hedlund-Lyndon theorem in the context of blur shifts.

Definition 3.49. *A map $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is a* **generalized sliding block code** *if, and only if, for all* $\mathsf{x} \in \Sigma_\Lambda$ *and $i \in \mathbb{N}$ we have*

$$\big(\Phi(\mathsf{x})\big)_i = \sum_{a \in \tilde{\mathcal{B}}} a\mathbb{1}_{C_a} \circ \sigma^i(\mathsf{x}),$$

where each C_a is a finitely defined set.

Generalized sliding block codes were proposed in [17] as the natural generalization of the concept of sliding block codes in the context of classical shift spaces over infinite alphabets, where it was proved that the class of generalized sliding block codes coincides with the class of continuous shift-commuting maps. In the contexts of one-sided and two-sided Ott-Tomford-Willys shifts (see [9] and [10], respectively), and Gonçalves-Royer ultragraph shifts [8], the class of generalized sliding block codes and the class of continuous shift-commuting maps do not coincide in general, but it is possible to obtain sufficient and necessary conditions under which those classes coincide.

Proposition 3.50. *If $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is a generalized sliding block code then it is continuous on Λ and shift commuting.*

□

A proof for the previous proposition can be adapted from [8, Corollary 3]. In what follows we state sufficient and necessary conditions under which both classes coincide in the general context of blur shifts. The proof of Theorem 3.51 can be adapted from [8, Theorem 3.8].

Theorem 3.51. *Suppose that $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is a map such that for each $\tilde{G} \in \tilde{\mathcal{U}}$ the set $C_{\tilde{G}}$ is a finitely defined set. Then Φ is continuous and shift commuting if, and only if, Φ is a generalized sliding block code where:*

i. For any $a \in \mathcal{B}$, the set C_a is a (possibly empty) union of generalized cylinders of Σ_Λ;

ii. If $(x_0 \dots x_{n-1}\tilde{H}\tilde{H}\tilde{H}\dots) \in \partial\Lambda$ is such that $\Phi(x_0 \dots x_{n-1}\tilde{H}\tilde{H}\tilde{H}\dots) = (\tilde{G}\tilde{G}\tilde{G}\dots\dots) \in \partial\Gamma$, then:

a) There exists a finite $F \subset H$ such that $\Phi\big(Z_\Lambda(x_0 \dots x_{n-1}\bar{H}, F)\big) \subset Z_\Gamma(\bar{G})$;

b) For each $g \in G$, there are just a finite number of $h \in H$ for which there exists some sequence $(x_0 \dots x_{n-1}h\dots)$ in C_g.

iii. If $\tilde{H} \in \tilde{\mathcal{V}}$ is such that $\Phi(\tilde{H}\tilde{H}\tilde{H}\dots) = (ddd\dots) \in \Gamma$, then for all $M \geq 1$ there exists a generalized cylinder $Z_\Lambda(\bar{H}, F)$ such that $\sigma^i\big(Z_\Lambda(\bar{H}, F)\big) \subseteq C_d$ for all $i = 0, 1, \dots, M$.

□

The next result can be directly proved by combining Proposition 3.47 and Theorem 3.51 above, and corresponds to Corollary 4. in [8].

Corollary 3.52. *A map $\Phi : \Sigma_\Lambda \to \Sigma_\Gamma$ is continuous, shift commuting, and length-preserving, if and only if it is a generalized sliding block code such that:*

i. For each $a \in \mathcal{B}$, the set C_a is a (possibly empty) union of generalized cylinders of Σ_Λ;

ii. $\mathcal{L}_0(\Lambda) = \bigcup_{\tilde{G} \in \tilde{\mathcal{U}}} C_{\tilde{G}}$.

iii. If $\Phi(\tilde{H}\tilde{H}\tilde{H}\ldots) = (\tilde{G}\tilde{G}\tilde{G}\ldots) \in \partial\Gamma$ then:

a) There exists a finite $F \subset H$ such that $\Phi\big(Z_\Lambda(\bar{H}, F)\big) \subset Z_\Gamma(\bar{G})$;

b) For each $g \in G$, there are only a finite number of $h \in H$ for which there exists some sequence $(hx_1x_2...)$ belonging to C_g.

$\square$

4 CONCLUSION

In this work we proposed a new kind of shift space which were named blur shift spaces. Our construction, gave a way to turn any classical shift space in a compact or locally compact new shift space.

We have noticed that Ott-Tomforde-Willis shift spaces and ultragraph shift spaces can be seen as blur shift spaces, by a convenient choice of the blurred sets, for each case. We studied topological properties of blur shift spaces, establishing results on their axioms of separability and countability, results on metrizablility. In particular, for the case of an countable alphabet, we have defined a metric. We have obtained necessary and sufficient conditions under which a blur shift is compact or locally compact. Finally, we presented a complete study of continuous shift-commuting functions and generalized sliding block codes defined on blur shifts.

Besides generalize other constructions previously proposed, blur shift spaces could also be applied to solve important problems as finding correspondence between the conjugacy of some classes of shift spaces and the isomorphism between their C^*-algebras.

From the point of view of dynamical systems, blur shift spaces could provide a source of new examples of phenomena. In particular, the understanding of their behaviour, such as chaoticity, ergodicity, entropy, etc., should be object of further studies.

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Appendix

APPENDIX A – THE TOPOLOGY OF THE EXTENDED ALPHABET

In section 3.3, we have defined a topology on $\bar{\mathcal{A}} = \mathcal{A} \cup \tilde{\mathcal{V}}$ whose open sets are all the subsets of $\mathcal{A}$ plus the sets $U \subset \bar{\mathcal{A}}$ such that if $\tilde{H} \in U$ then $H \setminus F \subset U$ for some finite $F \subset H$. We will verify that these sets define a topology on $\bar{\mathcal{A}}$. First notice that the empty set is open since it is a subset of $\mathcal{A}$, and that $\bar{\mathcal{A}}$ is itself open since for any $\tilde{H} \in \bar{\mathcal{A}}$, $H \setminus F \subset H \subset \mathcal{A} \subset \bar{\mathcal{A}}$, where F is any finite subset of H. To check that the union of open sets is open, let U be the union of a family of open sets $\{U_i\}_{i \in I}$. There are two possibilities here: either $U \subset \mathcal{A}$ or there is $\tilde{H} \in U$. In the first case U is open since it is a subset of $\mathcal{A}$. In the later case $\tilde{H} \in U_i$ for some $i \in I$, what implies $H \setminus F \subset U_i \subset U$ for some finite $F \subset H$, showing that U is open. To complete the verification, let V be the intersection of a finite family of open sets $\{V_i\}_{i \in I}$. As before, there exist two possibilities: either $V \subset \mathcal{A}$ or $\tilde{H} \in V$, for some $\tilde{H}$. The set V in the first case is open, being a subset of $\mathcal{A}$. In the second case, for every i, $\tilde{H} \in V_i$ assures there is a finite $F_i \subset H$ such that $H \setminus F_i \subset V_i$. Taking $F := \cup F_i$, then $H \setminus F \subset V$, showing V is open, as desired.

A basis for this topology on $\bar{\mathcal{A}}$ is given by the proposition below.

Proposition A.1. *The family of all singletons of $\mathcal{A}$, together with the sets $\bar{H} \setminus F$, where $\bar{H} \in \bar{\mathcal{V}}$ and $F \subset H$ is finite, is a basis for the topology on $\bar{\mathcal{A}}$.*

Proof. To prove this proposition we show i. each set of the family is an open set, ii. for each open set U of $\bar{\mathcal{A}}$ and each $x \in U$, there is an element C of the family such that $x \in C \subset U$.

i. Any singleton $\{x\}$ is open, as any subset of $\mathcal{A}$ is open. To check that $\bar{H} \setminus F$ is open, notice that $\tilde{H} \in \bar{H} \setminus F = H \setminus F \cup \{\tilde{H}\}$, hence the contention $H \setminus F \subset \bar{H} \setminus F$ assures $\bar{H} \setminus F$ is open.

ii. Let U be an open set of $\bar{\mathcal{A}}$ and $x \in U$. If $x \in \mathcal{A}$, taking $C = \{x\}$, then $x \in C \subset U$, as desired. By the other hand, if $x = \tilde{H} \in \tilde{V}$, $\tilde{H} \in U$ implies there exists some finite $F \subset H$ such that $H \setminus F \subset U$. Taking $C = \bar{H} \setminus F$, then $x = \tilde{H} \in \bar{H} \setminus F = H \setminus F \cup \{\tilde{H}\} \subset U$.

$\square$

Annex

ANNEX A – SEQUENTIAL SPACES AND FRÉCHET-URYSON SPACES

We reproduce here part of [5] and [2, Section 1.8].

If X is a topological space and $A \subset X$, the *closure of A in X*, denoted $\bar{A}$, is the smallest closed set containing A, while the *sequential closure of A in X*, denoted $[A]_{seq}$, is the set of all points x in X for which there exists a sequence in A converging to x:

$$\bar{A} = \bigcap \{K \subset X \text{ such that } K \text{ is closed and } A \subset K\},$$

$$[A]_{seq} := \{\mathrm{x} \in X : \ \exists (\mathrm{x}^n)_{n \geq 0} \in A \text{ s.t. } \mathrm{x}^n \to \mathrm{x} \text{ as } n \to \infty\}.$$

Definition A.1. *A set A in a topological space X is* sequentially closed *if* $[A]_{seq} = A$*. A topological space X is* **sequential** *if, and only if, each sequentially closed subset of X is closed ($\bar{A} = A$). Equivalently, a topological space X is* sequential *if, and only if, for every set $A \subset X$ not closed in X, there exists a sequence* $(\mathrm{x}^n)_{n \geq 0}$ *in A converging to a point of the set $\bar{A} \setminus A$.*

Definition A.2. *A topological space X is a* **Fréchet-Uryson space** *if the closure of every $A \subset X$ in X coincides with the sequential closure of A: $\bar{A} = [A]_{seq}$.*

It is clear from definitions A.1 and A.2 that every Fréchet-Uryson space is a sequential space. Example 13 in [2] shows that the converse is not true, in general.

Remark A.3. Notice that, to show that a topological space X is Fréchet-Uryson, one needs only to show that $\bar{A} \setminus A \subset [A]_{seq}$ for any $A \subset X$. In fact, this implies $\bar{A} \subset [A]_{seq}$. The reverse contention $[A]_{seq} \subset \bar{A}$ holds even if X is not Fréchet-Uryson, and follows from the fact that if $x \in [A]_{seq}$, x is the limit of a sequence in A, then any open set containing x intersects A, showing that $x \in \bar{A}$.

To conclude this detour, we mention that if X satisfies the first axiom of countability, then it is a Fréchet-Uryson space. Hence each metric space and each discrete space is a Fréchet-Uryson space.