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Semigrupos Inversos Quânticos e Bisseções generalizadas para algebroides de Hopf

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Francielle Kuerten Boeing

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RESUMO

Nesse trabalho é introduzida a noção de um semigrupo inverso quântico como uma generalização linearizada de semigrupos inversos. Além da álgebra de um semigrupo inverso, que é o exemplo natural de semigrupo inverso quântico, são apresentados vários outros exemplos dessa nova estrutura em diferentes contextos, relacionados a álgebras de Hopf, álgebras de Hopf fracas e categorias de Hopf. Finalmente, uma noção generalizada de bisseções locais é definida para algebroides de Hopf comutativos sobre uma álgebra de base comutativa, gerando novos exemplos de semigrupos inversos quânticos associados a algebroides de Hopf da mesma maneira que semigrupos inversos estão relacionados com grupoides.

Palavras-chave: Semigrupos Inversos Quânticos. Algebroides de Hopf. Grupoides. Semigrupos Inversos. Birretrações.

ABSTRACT

In this work, the notion of a quantum inverse semigroup is introduced as a linearized generalization of inverse semigroups. Beyond the algebra of an inverse semigroup, which is the natural example of a quantum inverse semigroup, several other examples of this new structure are presented in different contexts, those are related to Hopf algebras, weak Hopf algebras and Hopf categories. Finally, a generalized notion of local bisections is defined for commutative Hopf algebroids over a commutative base algebra giving rise to new examples of quantum inverse semigroups associated to Hopf algebroids in the same sense that inverse semigroups are related to groupoids.

Keywords: Quantum Inverse Semigroups. Hopf Algebroids. Groupoids. Inverse Semigroups. Biretractions.

RESUMO EXPANDIDO

Introdução

A noção básica de grupo já recebeu muitas generalizações em diferentes contextos, gerando uma miríade de novas estruturas matemáticas. Como os grupos apresentam umas importante relação com as simetrias, pode-se considerar que essas novas estruturas são novas ferramentas para entender aspectos mais profundos e sutis de simetrias. De início, é possível generalizar grupos enfraquecendo suas operações. Por exemplo, se a propriedade dos elementos inversíveis do grupo é enfraquecida, pode-se encontrar semigrupos inversos. Se além disso, deixa-se de exigir a unicidade do elemento inverso, pode-se encontrar semigrupos regulares. Pelo conhecido Teorema de Wagner e Preston (PRESTON, 1954; WAGNER, 1952), todo semigrupo inverso pode ser visto como um semigrupo de bijeções parcialmente definidas em um conjunto, com operação dada pela composição. Essas bijeções parcialmente definidas também lembram outra estrutura que generaliza a noção de grupo: a estrutura de grupoide. No caso dos grupoides, o que o torna mais geral que o grupo é a sua operação, que não é globalmente definida.

A relação entre semigrupos inversos e grupoides vem sendo estudada de diversas maneiras. Por exemplo, sendo S um semigrupo inverso, pode-se associá-lo ao grupoide indutivo cujo espaço de unidades é o conjunto $E(S)$ dos elementos idempotentes de S e operação sendo a restrição da operação em S. Por outro lado, dado um grupoide indutivo, pode-se associá-lo a um novo semigrupo inverso. Essa relação entre semigrupos inversos e grupoides é dada pelo teorema Ehresmann-Nambooripad-Schein, que estabelece um isomorfismo de categorias entre a categoria dos semigrupos inversos com pré-homomorfismos e a categoria de grupoides indutivos e funtores ordenados (EHRESMANN, 1960; NAMBOORIPAD, 1979; SCHEIN, 1979).

Também pode-se observar a relação entre semigrupos inversos e grupoides étale. Essa relação foi primeiramente explorada no contexto de álgebras de operadores (PATERSON, 1999). Um grupoide étale é um grupoide topológico cujas funções *source* e *target* são homeomorfismos locais (MATSNEV; RESENDE, 2010). Dado um grupoide étale $\mathcal{G}$, o conjunto de suas bisseções locais $\mathcal{B}(\mathcal{G})$ é um semigrupo inverso (EXEL, 2008). Por outro lado, dado um semigrupo inverso S, pode-se definir uma ação desse semigrupo sobre o conjunto dos caracteres do seu conjunto de idempotentes e, dessa ação, associar seu grupoide de germes $Gr(S)$, que é um grupoide étale (MATSNEV; RESENDE, 2010).

Por fim, outra maneira completamente diferente de generalizar grupos é pelas álgebras de Hopf, que podem ser consideradas como um tipo de "versão linearizada de grupos". Álgebras de Hopf possuem boas propriedades com relação a dualidade e a teoria de representações. Diversas generalizações de álgebras de Hopf já foram estudadas. Aqui mencionamos três estruturas que generalizam álgebras de Hopf e grupoides: as álgebras de Hopf fracas (BÖHM; NILL; SZLACHÁNYI, 1999), os algebroides de Hopf (BÖHM, 2009; BRZEZINSKI; MILITARU, 2002) e as categorias de Hopf (BATISTA; CAENEPEEL; VERCRUYSSE, 2016). Dentre as estruturas mencionadas, os algebroides de Hopf são, em certo sentido, a opção mais rica e promissora para generalizar grupoides no contexto de Hopf.

Dessa forma, é possível que as relações entre semigrupos inversos e grupoides possam ser generalizadas usando os algebroides de Hopf como generalização de grupoides.

Objetivos

Nesse momento surge a questão: podemos encontrar uma boa generalização de semigrupos inversos que trabalhe junto aos algebroides de Hopf da mesma maneira que os semigrupos inversos e os grupoides se relacionam? Nosso objetivo nesse trabalho é começar a responder essa pergunta introduzindo os semigrupos inversos quânticos. Mais especificamente, vamos generalizar a relação de que o conjunto das bisseções de um grupoide é um semigrupo inverso. Para isso, vamos generalizar a definição de bisseções locais para algebroides de Hopf (satisfazendo condições específicas) e mostrar que essa versão de bisseções gera um semigrupo inverso quântico.

Metodologia

O estudo de ações parciais de álgebras de Hopf e alguns aspectos da teoria de algebroides de Hopf motivaram exemplos do que deveria ser um semigrupo inverso quântico. A próxima ideia foi tentar generalizar a definição de bisseções locais para algebroides de Hopf. Começamos trabalhando com exemplos conhecidos de algebroides de Hopf comutativos e, tentando encontrar de maneira natural como deveria ser definida a generalização da bisseção local, chamada aqui de birretração local. Nos exemplos trabalhados, tentamos dualizar a definição de bisseção local tomando como birretração uma função partindo do algebroide de Hopf para a álgebra de base, e dessa maneira, a birretração local aparecia sempre como uma função multiplicativa, sendo morfismo de módulos à direita, e uma bijeção parcialmente definida quando composta com a função *target*. Dessa forma, chegamos à nossa primeira definição de birretrações locais e com essa definição provamos que as birretrações locais de um algebroide Hopf comutativo sobre uma álgebra base comutativa formam um monoide regular.

Um dos exemplos mais importantes nessa parte do trabalho foi o algebroide de Hopf das funções representativas de um grupoide. Um dos objetivos a ser atingido por esse exemplo era o de relacionar as bisseções locais de um grupoide com as birretrações do algebroide de Hopf de suas funções representativas. É possível construir, de maneira natural, uma função entre os dois conjuntos. O problema encontrado nesse passo da pesquisa foi que essa função não era, necessariamente, um morfismo de monoides regulares. Analisando esse exemplo mais profundamente foi possível ajustar a definição de birretração local, associando um elemento idempotente da álgebra base a cada birretração local.

Resultados

Com o ajuste na definição de birretrações locais, obtivemos um morfismo de monoides regulares entre as bisseções do grupoide e o algebroide de Hopf das suas funções representativas, que se torna um isomorfismo quando consideramos apenas grupoides transitivos finitos.

Além disso, a demonstração de que as birretrações locais formam um monoide regular continua valendo e finalmente mostramos que as birretrações locais geram uma álgebra que é um semigrupo inverso quântico.

Por fim, como as demonstrações não dependiam muito da comutatividade do algebroide de Hopf mas sim da comutatividade da álgebra de base e das relações entre as funções *source* e *target*, foi possível estender os resultados para algebroides de Hopf não necessariamente comutativos sobre uma álgebra de base comutativa, com as funções *source* e *target* satisfazendo condições especiais.

Considerações finais

Dessa forma, começamos a responder a pergunta inicial, encontrando no semigrupo inverso quântico um bom candidato para a generalização de semigrupos inversos no sentido de se relacionar com algebroides de Hopf da mesma maneira que semigrupos inversos se relacionam com grupoides. Como objetivos de trabalhos futuros temos definir as birretrações locais para quaisquer algebroides de Hopf e tentar encontrar mais relações entre os semigrupos inversos quânticos e os algebroides de Hopf.

LIST OF FIGURES

Figure 1 – Representation of the curve g 82

CONTENTS

1 INTRODUCTION 13
2 PRELIMINARIES 16
2.1 GROUPOIDS AND INVERSE SEMIGROUPS 16
2.2 HOPF ALGEBROIDS 23
2.2.1 Some Hopf algebroid's properties 24
2.2.2 Commutative Hopf Algebroids 27
2.2.2.1 The Hopf algebroid of the representative functions 28
3 QUANTUM INVERSE SEMIGROUPS 39
3.1 DEFINITION 39
3.2 EXAMPLES 41
3.2.1 Partial representations 46
3.2.2 Hopf Categories 50
4 GENERALIZED BISECTIONS ON HOPF ALGEBROIDS 53
4.1 BIRETRACTIONS 53
4.1.1 Biretractions and the representative functions of a discrete groupoid 66
4.1.2 The noncommutative case 71
5 CONCLUSION 83
BIBLIOGRAPHY 86

1 INTRODUCTION

The very basic notion of a group has undergone several generalizations in different contexts, giving rise to a myriad of new mathematical structures. Since groups are inherently related to symmetries, one can consider these new structures arising from groups as new tools to understand the deep and subtle aspects of symmetries. In one direction, it is possible to extend groups by weakening their operations. For example, when someone weakens the group inversion, also giving up the uniqueness of units, one ends up with regular semigroups and inverse semigroups. By the widely known theorem due to Wagner and Preston (PRESTON, 1954; WAGNER, 1952), every inverse semigroup can be viewed as a semigroup of partially defined bijections in a set, with the operation given by the composition. These partially defined bijections also evoke another mathematical structure which generalizes the notion of a group, namely, the groupoid structure. For the case of groupoids, what is weakend is the definition of the operation, which is not globally defined anymore. It is easier to understand why groupoids are generalization of groups if we consider a group as a one object category, whose endomorphisms of that object are the elements of the group. In this case, a groupoid is a “multi-object group", more precisely, a small category in which every morphism is an isomorphism.

The relationship between inverse semigroups and groupoids has been elucidated in the literature in several ways. For example, starting from an inverse semigroup S, one can naturally associate a groupoid whose unit space is the set of units $E(S)$ and the operation is the restriction of the operation in S. This groupoid has a partial order induced by the partial order of the semigroup itself, in fact, it is an inductive groupoid, meaning that its set of units is a meet semilattice. On the other hand, given an inductive groupoid, one can associate to it a new inverse semigroup. This exchange between inverse semigroups and groupoids composes the content of the Ehresmann-Nambooripad-Schein theorem, which stablishes a categorical isomorphism between the category of inverse semigroups with prehomomorphisms and the category of inductive groupoids and ordered functors (EHRESMANN, 1960; NAMBOORIPAD, 1979; SCHEIN, 1979).

One can also observe the interchange between inverse semigroups and groupoids considering the case of étale groupoids. This connection was first explored in the context of operator algebras (PATERSON, 1999). An étale groupoid is a topological groupoid in which the source and target maps are local homeomorphisms (MATSNEV; RESENDE, 2010). Given an étale groupoid $\mathcal{G}$, the set of its local bisections $\mathcal{B}(\mathcal{G})$ constitutes an inverse semigroup (EXEL, 2008). In turn, given an inverse semigroup S, one can define an action of this semigroup on the set of characters of its unit space and, from this action, associate its germ groupoid $Gr(S)$, which is an étale groupoid

(MATSNEV; RESENDE, 2010). More precisely, considering the category of inverse semigroups with semigroup morphisms and the category of étale groupoids with algebraic morphisms[1], the functor which associates to each inverse semigroup the germ groupoid of the canonical action on the characteres of its unit space is left adjoint to the functor which associates to each étale groupoid its semigroup of bisections (A.BUSS; EXEL; MEYER, 2012).

Another completely different direction in which it is possible to generalize groups is through Hopf algebras, which can be considered as a kind of "linearized version of groups". Hopf algebras have nice properties relative to duality and representation theory and, due to the emergence of the quantum groups (DRINFEL'D, 1988) became more popular in the nineties, even among the physicists, when quantum groups started to be considered seriously as symmetries of quantum systems, for example, as symmetries of the spectrum of diatomic molecules (CHANG; H.Y. GUO, 1992) or symmetries of Landau states in the quantum Hall effect (SATO, 1995). There are several different generalizations of Hopf algebras in the literature. Here we mention only three structures which generalize both Hopf algebras and groupoids: weak Hopf algebras (BÖHM; NILL; SZLACHÁNYI, 1999), Hopf algebroids (BÖHM, 2009; BRZEZINSKI; MILITARU, 2002) and Hopf categories (BATISTA; CAENEPEEL; VERCRUYSSE, 2016). Among the aforementioned structures, Hopf algebroids are, in a certain sense, the richest and most promising option to generalize groupoids in the Hopf context.

The question that arises in this moment is: can we find a good generalization of inverse semigroups and Hopf algebras which can play the same role with Hopf algebroids as inverse semigroups do with groupoids? In this work our aim is to start filling this gap by introducing the quantum inverse semigroups. This subject appeared as a collection of examples in search of a theory. The lessons coming from the study of partial actions of Hopf algebras and some aspects of the theory of Hopf algebroids motivated examples of what should be a quantum inverse semigroup. We generalize the concept of local bisections for Hopf algebroids and prove that the these "generalized bisections" generate a quantum inverse semigroup.

This work is structured in four parts. In chapter 2, we recall the definitions of inverse semigroups, groupoids and Hopf algebroids to stablish the notations, aside from the proof that the bisections of a groupoid form an inverse semigroup and some properties of Hopf algebroids that will be used throughout the work. Moreover, we present an alternate and more algebraic definition for local bisections for groupoids. Under this new definition, we prove that the statement that the local bisections form an inverse semigroup still holds. Lastly, we give some examples of Hopf algebroids with special attention to the Hopf algebroid of the representative functions of a groupoid,

[1] An algebraic morphism between the groupoids $\mathcal{G}$ and $\mathcal{H}$ is a left action of $\mathcal{G}$ over the arrows of $\mathcal{H}$ commuting with the right action of $\mathcal{H}$ over itself by the multiplication in $\mathcal{H}$ (BUNECI, 2008).

introduced by (KAOUTIT, 2013).

The third chapter is dedicated to the definition and examples of quantum inverse semigroups.

Chapter 4 introduces the generalized bisections for commutative Hopf algebroids over a commutative algebra, that are called *local biretractions*. Then we prove that the set of all local biretractions is a regular monoid with a convolution product and that the free vector space generated by them with the extended linearly convolution product is in fact a quantum inverse semigroup. After that, we recall the Hopf algebroid examples from the first chapter and find their biretractions. Moreover, we present a morphism between the local bisections of a groupoid and the local biretractions of the Hopf algebroid of its representative functions.

Finally, we define local biretractions for not necessarily commutative Hopf algebroids over a commutative algebra with a special condition for the bialgebroids' structures.

The last chapter concludes this work showing some of the difficulties found in the process of the construction of the best definition for local biretractions and what we can expect from future works.

2 PRELIMINARIES

This chapter contains the definition of inverse semigroups, groupoid, bisections of a groupoid and Hopf algebroids, together with some properties that are used in chapters 3 and 4. Then of all relations between inverse semigroups and groupoids, we elucidate the one about the bisections of a groupoid being an inverse semigroup.

Also, since our aim is to generalize the definition of bisections for Hopf algebroids, we introduce in the second section of this chapter some examples of Hopf algebroids that helped us find the best definition of local biretractions. We give special attention to the Hopf algebroid of the representative functions of a groupoid, so we can later work on the relations between the bisections of a groupoid and the biretractions of the Hopf algebroid of its representative functions.

2.1 GROUPOIDS AND INVERSE SEMIGROUPS

Definition 2.1.1 (Groupoid) *A groupoid is a set $\mathcal{G}$ together with a subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a product $\mathcal{G}^{(2)} \to \mathcal{G}$, $(g,h) \mapsto gh$ and an inverse map $i : \mathcal{G} \to \mathcal{G}$, $g \mapsto g^{-1}$ (in the sense that $(g^{-1})^{-1} = g$) such that:*
(G1) *if $(g,h),(h,l) \in \mathcal{G}^{(2)}$, then $(gh,l), (g,hl) \in \mathcal{G}^{(2)}$ and*

$$(gh)l = g(hl).$$

(G2) *$(g,g^{-1}) \in \mathcal{G}^{(2)}$ for every $g \in \mathcal{G}$ and if $(g,h) \in \mathcal{G}^{(2)}$, then*

$$g^{-1}(gh) = h \qquad (gh)h^{-1} = g.$$

If, in addition, $\mathcal{G}$ is a groupoid with a topology and the multiplication and the inversion are continuous, we say that $\mathcal{G}$ is a topological groupoid*.*

Remark 2.1.2 *We also define the unit space $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ as the image of the* source *and* target *maps $s,t : \mathcal{G} \to \mathcal{G}^{(0)}$*

$$s(g) = g^{-1}g \qquad t(g) = gg^{-1},$$

which are well defined, because $(g,g^{-1}),(g^{-1},g) \in \mathcal{G}^{(2)}$ and

$$s(g) = g^{-1}g = g^{-1}(g^{-1})^{-1} = t(g^{-1})$$

for every $g \in \mathcal{G}$, thus $\mathrm{Im}(s) = \mathrm{Im}(t)$.

If $\mathcal{G}$ is a topological groupoid whose unit space $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, with s and t local homeomorphisms, then we say $\mathcal{G}$ is an étale groupoid*.*

Remark 2.1.3 *From the definition of the source and target maps, we have the following consequences:*

(1) For every $g,h \in \mathcal{G}$, $(g,h) \in \mathcal{G}^{(2)}$ if, and only if, $s(g) = t(h)$. Indeed, if $(g,h) \in \mathcal{G}^{(2)}$, then

$$s(g) = g^{-1}g = g^{-1}((gh)h^{-1}) = (g^{-1}(gh))h^{-1} = hh^{-1} = t(h).$$

On the other hand, $s(g) = t(h)$ implies that $g^{-1}g = hh^{-1}$. Then, since $(g,g^{-1}g)$ and (hh^{-1},h) are in $\mathcal{G}^{(2)}$, we have that

$$(g,h) = ((gh)h^{-1},h) = (g(hh^{-1}),h) \in \mathcal{G}^{(2)}.$$

(2) If $(g,h) \in \mathcal{G}^{(2)}$ then $(h^{-1},g^{-1}) \in \mathcal{G}^{(2)}$ and $(gh)^{-1} = h^{-1}g^{-1}$.

Indeed, $s(h^{-1}) = t(h) = s(g) = t(g^{-1})$ implies that $(h^{-1},g^{-1}) \in \mathcal{G}^{(2)}$. Then $(h,h^{-1}g^{-1})$, $(h^{-1}g^{-1},g)$ are in $\mathcal{G}^{(2)}$ and since $(g,h) \in \mathcal{G}^{(2)}$ we have from (G2) *that $(gh,h^{-1}g^{-1})$, $(h^{-1}g^{-1},gh)$ are in $\mathcal{G}^{(2)}$. Then*

$$h^{-1}g^{-1}((gh)(h^{-1}g^{-1})) = (h^{-1}(g^{-1}(gh)))\,h^{-1}g^{-1} = ((h^{-1}h)\,h^{-1})g^{-1} = h^{-1}g^{-1}$$
$$((gh)(h^{-1}g^{-1}))\,gh = (((gh)h^{-1})\,g^{-1})\,gh = ((gg^{-1})\,g)h = gh,$$

respectively. Therefore, $(gh)^{-1} = h^{-1}g^{-1}$.

(3) If $(g,h) \in \mathcal{G}^{(2)}$ then $s(gh) = s(h)$ and $t(gh) = t(g)$:

$$s(gh) = (gh)^{-1}(gh) = h^{-1}(g^{-1}(gh)) = h^{-1}h = s(h)$$

$$t(gh) = (gh)(gh)^{-1} = ((gh)h^{-1})g^{-1} = gg^{-1} = t(g).$$

(4) The maps source *and* target *are the identity when restrict to the unit space $\mathcal{G}^{(0)}$:*

$$s(s(g)) = s(g^{-1}g) = s(g)$$
$$t(s(g)) = t(g^{-1}g) = t(g^{-1}) = s(g).$$

Definition 2.1.4 (Inverse Semigroup) *An inverse semigroup S is a semigroup in which every $s \in S$ has a unique* pseudoinverse *$s^* \in S$ in the sense that $s = ss^*s$ and $s^* = s^*ss^*$.*

Example 2.1.5 *Let X be a set. The set $\mathcal{I}(X)$ formed by all bijections between subsets of X, that is*

$$\mathcal{I}(X) = \{f : Dom(f) \subseteq X \to Im(f) \subseteq X \mid f \text{ bijective}\}$$

is an inverse semigroup. The semigroup operation is given by the composition:

$$fg = f \circ g : g^{-1}(Dom(f) \cap Im(g)) \to f(Dom(f) \cap Im(g)).$$

This inverse semigroup is a monoid, because it contains the identity map $Id_X : X \to X$. Also, $\mathcal{I}(X)$ has a zero element, given by the empty map $\emptyset : \emptyset \subseteq X \to \emptyset \subseteq X$.

Example 2.1.6 *Let A be an algebra. Similarly to the previous example, the set $\mathcal{I}(A)$ formed by all isomorphisms between ideals of A is an inverse semigroup.*

Remark 2.1.7 *Denote by $E(S)$ the set of all the idempotent elements of an inverse semigroup S. Observe that if $e \in E(S)$, then $e^* = e$.*

Proposition 2.1.8 *Let S be a regular semigroup, that is, a semigroup in which every element of S has a pseudoinverse. Then the idempotents of S commute if, and only if, every element of S has a unique pseudoinverse. In other words, a regular semigroup S is an inverse semigroup if, and only if, its idempotents commute.*

Proof. First, suppose that $E(S)$ is commutative. Let s' and s'' both be pseudoinverses of an element s in S. Then we have that $s' = s'ss'$, $s'' = s''ss''$ and $ss's = s = ss''s$. Observe that the elements $s's, ss', s''s, ss''$ are all idempotents, hence

$$s' = s'ss' = s'ss''ss' = s''ss'ss' = s''ss' = s''ss''ss' = s''ss'ss'' = s''ss'' = s''.$$

Now, suppose that the pseudoinverse is unique. Let $e, f \in E(S)$. Being $(ef)^*$ the pseudoinverse of ef, we have

$$(f(ef)^*e)(f(ef)^*e) = f((ef)^*ef(ef)^*)e = f(ef)^*e,$$

which implies that $f(ef)^*e$ is an idempotent with

$$(ef)(f(ef)^*e)(ef) = ef(ef)^*ef = ef$$

and

$$(f(ef)^*e)(ef)(f(ef)^*e) = f(ef)^*ef(ef)^*e = f(ef)^*e.$$

By the uniqueness of the pseudoinverse, we have that $(ef)^* = f(ef)^*e$ is an idempotent. Consequently, $ef = (ef)^* \in E(S)$. Similarly, we also have $fe \in E(S)$. So,

$$ef(fe)ef = (ef)(ef) = ef \quad \text{and} \quad fe(ef)fe = (fe)(fe) = fe,$$

and again by the uniqueness of the pseudoinverse, we conclude that

$$fe = (ef)^* = ef.$$

Therefore, $E(S)$ is commutative. □

Remark 2.1.9 *The above result implies that for any inverse semigroup S, the correspondence $s \mapsto s^*$ is an involutive antimorphism of inverse semigroups. Indeed, for every $s, t \in S$,*

$$(st)(t^*s^*)(st) = s\,(tt^*)(s^*s)\,t = s\,(s^*s)(tt^*)\,t = st$$

and

$$(t^*s^*)(st)(t^*s^*) = t^*(s^*s)(tt^*)\,s^* = t^*(tt^*)(s^*s)\,s^* = t^*s^*.$$

Consequently, $(st)^* = t^*s^*$.

Definition 2.1.10 (Bisection) *Let $\mathcal{G}$ be a groupoid. A bisection of $\mathcal{G}$ is a subset $U \subseteq \mathcal{G}$ such that $s|_U$ and $t|_U$ are injective. The set of all the bisections of $\mathcal{G}$ is denoted by $\mathcal{B}(\mathcal{G})$.*

Proposition 2.1.11 *Let $\mathcal{G}$ be a groupoid. Then the set $\mathcal{B}(\mathcal{G})$ is an inverse semigroup with*

$$UV = \{gh \mid (g,h) \in U \times V \text{ and } s(g) = t(h)\}$$
$$U^* = \{g^{-1} \mid g \in U\}$$

defined for any $U,V \in \mathcal{B}(\mathcal{G})$.

Proof. First, observe that the product is associative: for any U, V and W bisections of $\mathcal{G}$, if $g(hl) \in U(VW)$ then (g,h) and (h,l) are in $\mathcal{G}^{(2)}$ and

$$g(hl) = (gh)l \in (UV)W,$$

leading to $U(VW) \subseteq (UV)W$. Analogously, we have that $(UV)W \subseteq U(VW)$.

Also, for any bisection U of $\mathcal{G}$, $s|_{U^*} = t|_U$ and $t|_{U^*} = s|_U$ are both injective. Thus U^* is a bisection. Moreover, to prove that UV is a bisection for any U and V bisections, take $g_1h_1, g_2h_2 \in UV$ with $(g_1,h_1),(g_2,h_2) \in U \times V$ such that $s(g_1h_1) = s(g_2h_2)$. Then

$$s(h_1) = s(g_1h_1) = s(g_2h_2) = s(h_2)$$

and $h_1,h_2 \in V$ implies that $h_1 = h_2$. And since $s(g_1) = t(h_1)$ and $s(g_2) = t(h_2)$, we obtain

$$s(g_1) = t(h_1) = t(h_2) = s(g_2)$$

leading to $g_1 = g_2$ and, consequently, $g_1h_1 = g_2h_2$. Hence $s|_{UV}$ is injective. Analogously, we have that $t|_{UV}$ is also injective. Therefore UV is a bisection of $\mathcal{G}$ and $\mathcal{B}(\mathcal{G})$ is a semigroup with this product.

Now take $g \in U$. Then $g = g(g^{-1}g)$ with $s(g) = t(g^{-1})$, which implies that $g \in UU^*U$, that is, $U \subseteq UU^*U$. On the other hand, take the element $k = g(h^{-1}l) \in UU^*U$ with $g,h,l \in U$, $s(g) = t(h^{-1})$ and $s(h^{-1}) = t(l)$. Then

$$s(g) = t(h^{-1}) = s(h) \qquad \text{and} \qquad t(h) = s(h^{-1}) = t(l)$$

imply that $g = h = l$. Hence

$$k = g(g^{-1}g) = g \in U$$

and, consequently, $UU^*U \subset U$. Therefore $\mathcal{B}(\mathcal{G})$ is a regular semigroup.

In order to prove that $\mathcal{B}(\mathcal{G})$ is an inverse semigroup, we need to prove that the pseudoinverse U^* is unique. Take V a bisection o $\mathcal{G}$ satisfying

$$UVU = U \qquad \text{and} \qquad VUV = V.$$

If $k \in V$ then there exist $g,l \in V$ and $h \in U$ such that $k = ghl$ with $s(g) = t(h)$ and $s(h) = t(l)$. Now observe that

$$s(k) = s(ghl) = s(l) \qquad \text{and} \qquad t(k) = t(ghl) = t(g).$$

Since V is a bisection, we have that $k = g = l$. Thus $k = khk$. Also, note that $hkh \in UVU = U$ and then $s(h) = s(hkh)$ implies that $h = hkh$. Therefore, $k = h^{-1} \in U^*$.

On the other hand, being $g^{-1} \in U^*$, since $UVU = U$ there exist $k \in V$ and $h,l \in U$ such that $g = hkl$ with $s(h) = t(k)$ and $s(k) = t(l)$. Then

$$s(g) = s(hkl) = s(l) \qquad \text{and} \qquad t(g) = t(hkl) = t(h)$$

imply that $g = l = h$ and $g = gkg$. Also, $kgk \in VUV = V$ and $s(kgk) = s(k)$ imply that $kgk = k$. Therefore $g^{-1} = k \in V$ and, consequently, $U^* = V$.

□

The interplay between groupoids and inverse semigroups has been vastly explored in the literature (EHRESMANN, 1960; LAWSON, 1998; NAMBOORIPAD, 1979; PATERSON, 1999; SCHEIN, 1979). One of the most important sources of inverse semigroups associated to groupoids are the bisections of étale topological groupoids (A.BUSS; EXEL; MEYER, 2012; EXEL, 2008; MATSNEV; RESENDE, 2010). As proved in Proposition 2.1.11, the set of all bisections of a groupoid $\mathcal{G}$ defined this way is an inverse semigroup. Let us redefine bisections in a more algebraic way, so we can better generalize this notion for Hopf algebroids.

Definition 2.1.12 *A local bisection of a groupoid $\mathcal{G}$ is a pair (u,X) in which X is a subset of $\mathcal{G}^{(0)}$ and $u : X \to \mathcal{G}$ is a function such that*

(i) $s \circ u = Id_X$.

(ii) $t \circ u : X \to t(u(X))$ is a bijection.

The set X is called the domain of the bissection (u,X). A global bissection is a local bissection whose domain is $X = \mathcal{G}^{(0)}$.

Note that, item (ii) implies that the function $u : X \to \mathcal{G}$ is injective. Denote again by $\mathcal{B}(\mathcal{G})$ the set of the local bisections of the groupoid $\mathcal{G}$ and by $\mathrm{Gl}\mathcal{B}(\mathcal{G})$ the set of its global bisections.

Remark 2.1.13 *The two notions of a bisection, as a subset of the groupoid restricted to what the source map is injective and as a pair of a subset of the unit set and a function are in fact related. On one hand, given a subset $U \subseteq \mathcal{G}$ for which $s|_U : U \to \mathcal{G}^{(0)}$ is injective, define $X = s(U) \subseteq \mathcal{G}^{(0)}$ and $u : X \to \mathcal{G}$ as the inverse of $s|_U$. On the other hand, given a pair (u,X), as in Definition 2.1.12, define $U = u(X)$, as u is already injective, the corestriction $u : X \to U$ is bijective. As the left inverse of u is s, by definition, then it is the inverse of that corestriction, making $s|_U$ injective.*

Several instances of the following result have already appeared in the literature (see (GARNER, 2019), Example 17, for a version closer to our approach).

Proposition 2.1.14 *Let $\mathcal{G}$ be a groupoid, then the set $\mathcal{B}(\mathcal{G})$ of its local bissections is an inverse semigroup and the set $Gl\mathcal{B}(\mathcal{G})$ of its global bissections is a group considering the product between two local bisections (u,X) and (v,Y) of $\mathcal{G}$ as $(u,X)\cdot(v,Y) = (uv, XY)$, in which*

$$XY = (t \circ v)^{-1}(t \circ v(Y) \cap X) \quad \textit{and} \quad (uv)(y) = u(t \circ v(y))v(y).$$

Proof. The product is associative. Indeed, for $(u,X),(v,Y),(w,Z) \in \mathcal{B}(\mathcal{G})$, we have

$$((u,X)\cdot(v,Y))\cdot(w,Z) = ((uv)w,(XY)Z) \qquad \text{and} \qquad (u,X)\cdot((v,Y)\cdot(w,Z)) = (u(vw),X(YZ)),$$

where

$$(XY)Z = (t \circ v)^{-1}(t \circ v(Y) \cap X)\, Z) = (t \circ w)^{-1}(t \circ w(Z) \cap (t \circ v)^{-1}(t \circ v(Y) \cap X)))$$
$$X(YZ) = X\,(t \circ w)^{-1}(t \circ w(Z) \cap Y)) = (t \circ vw)^{-1}(t \circ vw((t \circ w)^{-1}(t \circ w(Z) \cap Y)) \cap X)).$$

In order to show that these bisections are equal, first note that, for any $z \in YZ$

$$t \circ vw(z) = t(v(t \circ w(z))w(z)) = t(v(t \circ w(z))) = t \circ v \circ t \circ w(z).$$

Hence the inverse map of the bijection $t \circ vw : YZ \to t \circ vw(YZ)$ is

$$(t \circ vw)^{-1} = (t \circ w)^{-1}(t \circ v)^{-1} : t \circ v \circ t \circ w(YZ) \to YZ.$$

Then

$$\begin{aligned}
X(YZ) &= (t \circ vw)^{-1}(t \circ vw((t \circ w)^{-1}(t \circ w(Z) \cap Y)) \cap X)) \\
&= (t \circ vw)^{-1}(t \circ v \circ t \circ w((t \circ w)^{-1}(t \circ w(Z)) \cap Y) \cap X) \\
&= (t \circ w)^{-1}(t \circ v)^{-1}(t \circ v(t \circ w(Z) \cap Y) \cap X) \\
&= (t \circ w)^{-1}(t \circ w(Z) \cap (t \circ v)^{-1}(t \circ v(Y) \cap X)) \\
&= (XY)Z.
\end{aligned}$$

Now, for $z \in XYZ$,

$$u(vw)(z) = u(t \circ vw(z))vw(z) = u(t \circ v \circ t \circ w(z))v(t \circ w(z))w(z)$$

and

$$(uv)w(z) = uv(t \circ w(z))w(z) = u(t \circ v \circ t \circ w(z))v(t \circ w(z))w(z).$$

Therefore $((u,X)\cdot(v,Y))\cdot(w,Z) = (u,X)\cdot((v,Y)\cdot(w,Z))$.

For any bisection $(u,X) \in \mathcal{B}(\mathcal{G})$ define $(u,X)^* = (\overline{u}, t \circ u(X))$, in which, for any $x \in X$, $\overline{u}(t \circ u(x)) = u(x)^{-1}$. Then for any $x \in X \subseteq \mathcal{G}^{(0)}$,

$$\overline{u}u(x) = \overline{u}(t \circ u(x))u(x) = u(x)^{-1}u(x) = s(u(x)) = x,$$

We conclude that

$$u\overline{u}u(x) = u(t \circ \overline{u}u(x))\overline{u}u(x) = u(t(x))\, x = u(x)$$

and

$$\begin{aligned}
\overline{u}u\overline{u}(t \circ u(x)) &= \overline{u}u(t \circ \overline{u}(t \circ u(x)))\overline{u}(t \circ u(x)) \\
&= \overline{u}u(t(u(x)^{-1}))\overline{u}(t \circ u(x)) \\
&= \overline{u}u(s \circ u(x))\overline{u}(t \circ u(x)) \\
&= s \circ u(x)(u(x))^{-1} \\
&= (u(x))^{-1} \\
&= \overline{u}(t \circ u(x)).
\end{aligned}$$

And since

$$(u,X)^* \cdot (u,X) = (\overline{u},t \circ u(X)) \cdot (u,X) = (\overline{u}u, (t \circ u)^{-1}(t \circ u(X) \cap t \circ (X)) = (Id_X,X)$$
$$(u\, Id_X)(x) = u(t \circ Id_X(x))\, Id_X(x) = u(x)x = u(x)s \circ u(x) = u(x)$$
$$(Id_X\overline{u})(t \circ u(x)) = Id_X(t \circ \overline{u}(t \circ u(x)))\, \overline{u}(t \circ u(x)) = t((u(x))^{-1})\, (u(x))^{-1} = \overline{u}(t \circ u(x))$$

for every $x \in X$, we have that

$$\begin{aligned}
(u,X) \cdot (u,X)^* \cdot (u,X) &= (u,X) \cdot (\overline{u},t \circ u(X)) \cdot (u,X) \\
&= (u,X) \cdot (Id_X,X) \\
&= (u,X)
\end{aligned}$$

and

$$\begin{aligned}
(u,X)^* \cdot (u,X) \cdot (u,X)^* &= (Id_X,X) \cdot (\overline{u},t \circ u(X)) \\
&= (\overline{u}, (t \circ u)^{-1}(t \circ \overline{u}(t \circ u(X)) \cap X)) \\
&= (\overline{u},t \circ u(X)) \\
&= (u,X)^*.
\end{aligned}$$

It remains to prove that the idempotents in $\mathcal{B}(\mathcal{G})$ commute among themselves. If (u,X) is an idempotent element, then

$$(u,X) = (u,X) \cdot (u,X) = (uu,(t \circ u)^{-1}(t \circ u(X) \cap X)),$$

implying that $t \circ u(X) = X$ and $u(t \circ u(x))u(x) = u(x)$. Multiplying the last equality on the right by $u(x)^{-1}$ we end up with $u(t \circ u(x)) = t \circ u(x)$. And since $t \circ u(X) = X$, we conclude that $u = Id_X$ and $(u,X) = (Id_X,X)$. Hence multiplying two idempotents we have

$$(Id_X,X)\cdot(Id_Y,Y) = (Id_X Id_Y,(t\circ Id_Y)^{-1}(t\circ Id_Y(Y)\cap X)) = (Id_{X\cap Y},X\cap Y) = (Id_Y,Y)\cdot(Id_X,X).$$

Therefore, the idempotents commute and $\mathcal{B}(\mathcal{G})$ is an inverse semigroup.

The global bisections are of the form $(u,\mathcal{G}^{(0)})$ and clearly, global bisections $\mathrm{Gl}\mathcal{B}(\mathcal{G})$ form a subsemigroup of $\mathcal{B}(\mathcal{G})$. But the only idempotent in $\mathrm{Gl}\mathcal{B}(\mathcal{G})$ is the unit $(Id_{\mathcal{G}^{(0)}},\mathcal{G}^{(0)})$. An inverse semigroup with only one idempotent is a group, therefore $\mathrm{Gl}\mathcal{B}(\mathcal{G})$ is a group. □

2.2 HOPF ALGEBROIDS

Now for the definition of Hopf algebroids and further this text, $\Bbbk$ will denote a field of characteristic 0 and unadorned tensor products will denote tensor products over the base field $\Bbbk$.

Definition 2.2.1 *(BÖHM, 2009) Let A be a $\Bbbk$ algebra. A left bialgebroid over A is a quintuple $(\mathcal{H}, s_l, t_l, \Delta_l, \varepsilon_l)$ in which:*

(LB1) *$\mathcal{H}$ is a $\Bbbk$-algebra, $s_l : A \to \mathcal{H}$ is an algebra map and $t_l : A \to \mathcal{H}$ is an antialgebra map such that $s_l(a)t_l(b) = t_l(b)s_l(a)$, for every $a,b \in A$ making $\mathcal{H}$ an A-bimodule with the structure*

$$a \triangleright h \triangleleft b = s_l(a)t_l(b)h.$$

(LB2) *$(\mathcal{H}, \Delta_l, \varepsilon_l)$, is an A-coring with the above mentioned A-bimodule structure.*

(LB3) *$\Delta_l(\mathcal{H}) \subseteq \mathcal{H} \times^l_A \mathcal{H} = \{\sum h_i \otimes k_i \in \mathcal{H} \otimes_A \mathcal{H} \mid \sum h_i t_l(a) \otimes k_i = \sum h_i \otimes k_i s_l(a), \forall a \in A\}$ (Takeuchi's product) and the co-restriction map is an algebra map.*

(LB4) *$\varepsilon_l(hk) = \varepsilon_l(hs_l(\varepsilon_l(k))) = \varepsilon_l(ht_l(\varepsilon_l(k)))$.*

Definition 2.2.2 *(BÖHM, 2009) Let A be a $\Bbbk$ algebra. A right bialgebroid over A is a quintuple $(\mathcal{H}, s_r, t_r, \Delta_r, \varepsilon_r)$ in which:*

(RB1) *$\mathcal{H}$ is a $\Bbbk$-algebra, $s_r : A \to \mathcal{H}$ is an algebra map and $t_r : A \to \mathcal{H}$ is an antialgebra map such that $s_r(a)t_r(b) = t_r(b)s_r(a)$, for every $a,b \in A$ making $\mathcal{H}$ an A-bimodule with the structure*

$$a \blacktriangleright h \blacktriangleleft b = ht_r(a)s_r(b).$$

(RB2) *$(\mathcal{H}, \Delta_r, \varepsilon_r)$, is an A-coring with the above mentioned A-bimodule structure.*

(RB3) *$\Delta_r(\mathcal{H}) \subseteq \mathcal{H} \times^r_A \mathcal{H} = \{\sum h_i \otimes k_i \in \mathcal{H} \otimes_A \mathcal{H} \mid \sum s_r(a)h_i \otimes k_i = \sum h_i \otimes t_r(a)k_i, \forall a \in A\}$ (Takeuchi's product) and the co-restriction map is an algebra map.*

(RB4) *$\varepsilon_r(hk) = \varepsilon_r(s_r(\varepsilon_r(h))k) = \varepsilon_r(t_r(\varepsilon_r(h))k)$.*

Definition 2.2.3 *(BÖHM, 2009) Let A and $\overline{A}$ be $\Bbbk$ algebras. A Hopf algebroid over the base algebras A and $\overline{A}$ is a triple $\mathcal{H} = (\mathcal{H}_l, \mathcal{H}_r, S)$ such that.*

(HA1) *$\mathcal{H}_l = \mathcal{H}$ is a left bialgebroid over A and $\mathcal{H}_r = \mathcal{H}$ is a right bialgebroid over $\overline{A}$.*

(HA2) $s_l \circ \varepsilon_l \circ t_r = t_r, \qquad t_l \circ \varepsilon_l \circ s_r = s_r,$
$s_r \circ \varepsilon_r \circ t_l = t_l \qquad t_r \circ \varepsilon_r \circ s_l = s_l.$

(HA3) $(\Delta_l \otimes_{\overline{A}} \mathcal{H}) \circ \Delta_r = (\mathcal{H} \otimes_A \Delta_r) \circ \Delta_l$ *and* $(\Delta_r \otimes_A \mathcal{H}) \circ \Delta_l = (\mathcal{H} \otimes_{\overline{A}} \Delta_l) \circ \Delta_r$.

(HA4) $S : \mathcal{H} \to \mathcal{H}$ *is a k linear map such that for all* $a \in A$, $b \in \overline{A}$ *and* $h \in \mathcal{H}$, $S(t_l(a) h t_r(b)) = s_r(b) S(h) s_l(a)$.

(HA5) *Denoting by* μ_l *and* μ_r*, respectively, the multiplication in* $\mathcal{H}$ *as left and right bialgebroid, we have*

$$\mu_l(S \otimes_A \mathcal{H}) \circ \Delta_l = s_r \circ \varepsilon_r, \qquad \text{and} \qquad \mu_r(\mathcal{H} \otimes_{\overline{A}} S) \circ \Delta_r = s_l \circ \varepsilon_l.$$

Throughout this work we use the Sweedler notations for Δ_l and Δ_r : for every $h \in \mathcal{H}$, we write

$$\Delta_l(h) = h_{(1)} \otimes h_{(2)} \qquad \Delta_r(h) = h^{(1)} \otimes h^{(2)}.$$

2.2.1 Some Hopf algebroid's properties

The next properties are valid for a general Hopf algebroid $\mathcal{H} = (\mathcal{H}_l, \mathcal{H}_r, S)$ with the maps $s_l, t_l, \Delta_l, \varepsilon_l, s_r, t_r, \Delta_r$ and ε_r.

(P1) For every $a \in \overline{A}$, $b \in A$ and $h \in \mathcal{H}$, we have that

$$S(t_r(a) h\, t_l(b)) = s_l(b)\, S(h)\, s_r(a).$$

Indeed,

$$\begin{aligned}
S(t_r(a)h) &= S(t_r(a)\, h^{(2)}\, t_r \circ \varepsilon_r(h^{(1)})) \\
&= s_r \circ \varepsilon_r(h^{(1)})\, S(t_r(a)\, h^{(2)}) \\
&= S\left(h^{(1)}{}_{(1)}\right)\, h^{(1)}{}_{(2)}\, S(t_r(a)\, h^{(2)}) \\
&= S(h_{(1)})\, h_{(2)}{}^{(1)}\, S\left(t_r(a)\, h_{(2)}{}^{(2)}\right) \\
&= S(h_{(1)})\, s_r(a)\, h_{(2)}{}^{(1)}\, S\left(h_{(2)}{}^{(2)}\right) \\
&= S(h_{(1)})\, s_r(a)\, s_l \circ \varepsilon_l(h_{(2)}) \\
&\overset{(*)}{=} S(h_{(1)})\, s_l \circ \varepsilon_l(h_{(2)})\, s_r(a) \\
&= S(t_l \circ \varepsilon_l(h_{(2)})\, h_{(1)})\, s_r(a) \\
&= S(h)\, s_r(a),
\end{aligned}$$

where $(*)$ comes from the fact that $s_l = t_r \circ \varepsilon_r \circ s_l$ and the images of t_r and s_r

commute by definition. Moreover,

$$\begin{aligned}
S(h\,t_l(b)) &= S(t_l \circ \varepsilon_l(h_{(2)})\, h_{(1)}\, t_l(b)) \\
&= S(h_{(1)}\, t_l(b))\, s_l \circ \varepsilon_l(h_{(2)}) \\
&= S(h_{(1)}\, t_l(b))\, {h_{(2)}}^{(1)} S\left({h_{(2)}}^{(2)}\right) \\
&= S\left({h^{(1)}}_{(1)}\, t_l(b)\right)\, {h^{(1)}}_{(2)} S(h^{(2)}) \\
&= S\left({h^{(1)}}_{(1)}\right)\, {h^{(1)}}_{(2)}\, s_l(b)\, S(h^{(2)}) \\
&= s_r \circ \varepsilon_r(h^{(1)})\, s_l(b)\, S(h^{(2)}) \\
&= s_l(b)\, s_r \circ \varepsilon_r(h^{(1)})\, S(h^{(2)}) \\
&= s_l(b)\, S\left(h^{(2)}\, t_r \circ \varepsilon_r(h^{(1)})\right) \\
&= s_l(b)\, S(h),
\end{aligned}$$

again using that $s_l = t_r \circ \varepsilon_r \circ s_l$.

(P2) S is antimultiplicative: for every $h,k \in \mathcal{H}$,

$$\begin{aligned}
S(hk) &= S(t_l \circ \varepsilon_l(h_{(2)})\, h_{(1)}k) \\
&= S(h_{(1)}k)\, s_l \circ \varepsilon_l(h_{(2)} \\
&= S(h_{(1)}\, t_l \circ \varepsilon_l(k_{(2)})\, k_{(1)})\, {h_{(2)}}^{(1)} S\left({h_{(2)}}^{(2)}\right) \\
&= S\left({h^{(1)}}_{(1)}\, t_l \circ \varepsilon_l(k_{(2)})\, k_{(1)}\right)\, {h^{(1)}}_{(2)}\, S(h^{(2)}) \\
&= S\left({h^{(1)}}_{(1)}\, k_{(1)}\right)\, {h^{(1)}}_{(2)}\, s_l \circ \varepsilon_l(k_{(2)})\, S(h^{(2)}) \\
&= S\left({h^{(1)}}_{(1)}\, k_{(1)}\right)\, {h^{(1)}}_{(2)}\, {k_{(2)}}^{(1)}\, S\left({k_{(2)}}^{(2)}\right)\, S(h^{(2)}) \\
&= S\left({h^{(1)}}_{(1)}\, {k^{(1)}}_{(1)}\right)\, {h^{(1)}}_{(2)}\, {k^{(1)}}_{(2)}\, S(k^{(2)})\, S(h^{(2)}) \\
&= S\left(\left(h^{(1)}k^{(1)}\right)_{(1)}\right)\, \left(h^{(1)}k^{(1)}\right)_{(2)}\, S(k^{(2)})\, S(h^{(2)}) \\
&= s_r \circ \varepsilon_r(h^{(1)}k^{(1)})\, S(k^{(2)})\, S(h^{(2)}) \\
&= S\left(k^{(2)}\, t_r \circ \varepsilon_r(h^{(1)}k^{(1)})\right)\, S(h^{(2)}) \\
&= S\left(k^{(2)}\, t_r \circ \varepsilon_r(s_r \circ \varepsilon_r(h^{(1)})k^{(1)})\right)\, S(h^{(2)}) \\
&= S\left(t_r \circ \varepsilon_r(h^{(1)})\, k^{(2)}\, t_r \circ \varepsilon_r(k^{(1)})\right)\, S(h^{(2)}) \\
&= S(k)\, s_r \circ \varepsilon_r(h^{(1)})\, S(h^{(2)}) \\
&= S(k)\, S(h^{(2)}\, t_r \circ \varepsilon_r(h^{(1)})) \\
&= S(k)\, S(h).
\end{aligned}$$

(P3) S maps unity to unity, because

$$1_{\mathcal{H}} = s_r \circ \varepsilon_r(1_{\mathcal{H}}) = S(1_{\mathcal{H}})\, 1_{\mathcal{H}} = S(1_{\mathcal{H}}).$$

(P4) The composition $\varepsilon_l \circ s_r$ is an antimorphism of algebras: for every $a,b \in \overline{A}$,

$$\begin{aligned}\varepsilon_l \circ s_r(ab) &= \varepsilon_l(s_r(a)\, s_r(b))\\ &= \varepsilon_l(t_l \circ \varepsilon_l \circ s_r(a)\, s_r(b))\\ &= \varepsilon_l(s_r(b) \triangleleft \varepsilon_l \circ s_r(a))\\ &= \varepsilon_l \circ s_r(b)\, \varepsilon_l \circ s_r(a).\end{aligned}$$

Analogously, $\varepsilon_l \circ t_r$, $\varepsilon_r \circ s_l$ and $\varepsilon_r \circ t_l$ are also antimorphisms of algebras.

(P5) The compositions $\varepsilon_l \circ S$ and $\varepsilon_r \circ S$ can be written as

$$\begin{aligned}\varepsilon_l \circ S(h) &= \varepsilon_l \circ S(t_l \circ \varepsilon_l(h_{(2)})\, h_{(1)})\\ &= \varepsilon_l(S(h_{(1)})\, s_l \circ \varepsilon_l(h_{(2)}))\\ &= \varepsilon_l(S(h_{(1)})\, h_{(2)})\\ &= \varepsilon_l \circ s_r \circ \varepsilon_r(h)\end{aligned}$$

and

$$\begin{aligned}\varepsilon_r \circ S(h) &= \varepsilon_r \circ S(h^{(2)}\, t_r \circ \varepsilon_r(h^{(1)}))\\ &= \varepsilon_r(s_r \circ \varepsilon_r(h^{(1)})\, S(h^{(2)}))\\ &= \varepsilon_r(h^{(1)}\, S(h^{(2)}))\\ &= \varepsilon_r \circ s_l \circ \varepsilon_l(h)\end{aligned}$$

for every $h \in \mathcal{H}$.

(P6) S is anticomultiplicative as in

$$\Delta_l \circ S = (S \otimes_A S) \circ \Delta_r^{cop}.$$

Indeed,

$$\begin{aligned}
\Delta_l \circ S(h) &= \Delta_l \circ S(t_l \circ \varepsilon_l(h_{(2)})\, h_{(1)}) \\
&= \Delta_l(S(h_{(1)})\, s_l \circ \varepsilon_l(h_{(2)})) \\
&= (S(h_{(1)}))_{(1)}\, s_l \circ \varepsilon_l(h_{(2)}) \otimes_A (S(h_{(1)}))_{(2)} \\
&= (S(h_{(1)}))_{(1)}\, {h_{(2)}}^{(1)} S({h_{(2)}}^{(2)}) \otimes_A (S(h_{(1)}))_{(2)} \\
&= \left(S({h^{(1)}}_{(1)})\right)_{(1)} {h^{(1)}}_{(2)} S(h^{(2)}) \otimes_A \left(S({h^{(1)}}_{(1)})\right)_{(2)} \\
&= \left(S({h^{(1)}}_{(1)})\right)_{(1)} t_l \circ \varepsilon_l({h^{(1)}}_{(2)(2)})\, {h^{(1)}}_{(2)(1)} S(h^{(2)}) \otimes_A \left(S({h^{(1)}}_{(1)})\right)_{(2)} \\
&= \left(S({h^{(1)}}_{(1)})\right)_{(1)} {h^{(1)}}_{(2)(1)} S(h^{(2)}) \otimes_A \left(S({h^{(1)}}_{(1)})\right)_{(2)} s_l \circ \varepsilon_l({h^{(1)}}_{(2)(2)}) \\
&= \left(S({h^{(1)}}_{(1)})\right)_{(1)} {h^{(1)}}_{(2)(1)} S(h^{(2)}) \otimes_A \left(S({h^{(1)}}_{(1)})\right)_{(2)} {{h^{(1)}}_{(2)(2)}}^{(1)} S({{h^{(1)}}_{(2)(2)}}^{(2)}) \\
&= \left(S({h^{(1)}}_{(1)})\right)_{(1)} {{{h^{(1)}}_{(2)}}^{(1)}}_{(1)} S(h^{(2)}) \otimes_A \left(S({h^{(1)}}_{(1)})\right)_{(2)} {{{h^{(1)}}_{(2)}}^{(1)}}_{(2)} S({{h^{(1)}}_{(2)}}^{(2)}) \\
&= \left(S({h^{(1)(1)}}_{(1)})\right)_{(1)} {h^{(1)(1)}}_{(2)(1)} S(h^{(2)}) \otimes_A \left(S({h^{(1)(1)}}_{(1)})\right)_{(2)} {h^{(1)(1)}}_{(2)(2)} S(h^{(1)(2)}) \\
&= \left(S({h^{(1)(1)}}_{(1)})\, {h^{(1)(1)}}_{(2)}\right)_{(1)} S(h^{(2)}) \otimes_A \left(S({h^{(1)(1)}}_{(1)})\, {h^{(1)(1)}}_{(2)}\right)_{(2)} S(h^{(1)(2)}) \\
&= \left(s_r \circ \varepsilon_r(h^{(1)(1)})\right)_{(1)} S(h^{(2)}) \otimes_A \left(s_r \circ \varepsilon_r(h^{(1)(1)})\right)_{(2)} S(h^{(1)(2)}) \\
&= \; S(h^{(2)}) \otimes_A \; s_r \circ \varepsilon_r(h^{(1)(1)})\, S(h^{(1)(2)}) \\
&= \; S(h^{(2)}) \otimes_A \; S(h^{(1)(2)}\, t_r \circ \varepsilon_r(h^{(1)(1)})) \\
&= \; S(h^{(2)}) \otimes_A \; S(h^{(1)}) \\
&= (S \otimes_A S) \circ \Delta_r^{cop}(h)
\end{aligned}$$

for every $h \in \mathcal{H}$.

2.2.2 Commutative Hopf Algebroids

From now on, unless it is explicitly said otherwise, we will be working only with commutative Hopf algebroids over a commutative base algebra $A = \overline{A}$. In this case, the source and target maps s_l, s_r, t_l and t_r are all morphisms of algebras and because of the commutativity of $\mathcal{H}$, we have $s_l = t_r$ and $s_r = t_l$. Therefore, one can choose arbitrarily one laterality for the bialgebroid structure. Throughout this work we shall denote by s the right source map and by t the right target map. Also in the commutative case, the left and right Takeuchi tensor products, $\mathcal{H} \times_A^l \mathcal{H}$ and $\mathcal{H} \times_A^r \mathcal{H}$ are identified with the tensor product $\mathcal{H} \otimes_A \mathcal{H}$. Indeed, for every $\sum h_i \otimes k_i \in \mathcal{H} \otimes_A \mathcal{H}$, then

$$\begin{aligned}
\sum s_r(a)\, h_i \otimes k_i &= \sum h_i\, s_r(a) \otimes k_i \\
&= \sum h_i \blacktriangleleft a \otimes k_i \\
&= \sum h_i \otimes a \blacktriangleright k_i \\
&= \sum h_i \otimes t_r(a)\, k_i
\end{aligned}$$

for every $a \in A$. Consequently, $\mathcal{H} \times_A^r \mathcal{H} = \mathcal{H} \otimes_A \mathcal{H}$ and, analogously, $\mathcal{H} \times_A^l \mathcal{H} = \mathcal{H} \otimes_A \mathcal{H}$. Thus the left and right comultiplications and counits coincide and the counit turns out to be an algebra morphism. Finally, we can rewrite some of the axioms in a more suitable way: for any $h,k \in \mathcal{H}$ and $a,b \in A$, we have

(1) $s,t : A \to \mathcal{H}$ are both algebra morphisms and $\mathcal{H}$ has the A–bimodule structure

$$a \triangleright h \triangleleft b = t(a)\, h\, s(b);$$

(2) $\varepsilon := \varepsilon_l = \varepsilon_r : \mathcal{H} \to A$ is an algebra morphism and $\varepsilon \circ s = \varepsilon \circ t = \mathrm{Id}_A$;

(3) $S(s(a)\, h) = t(a)\, S(h)$ and $S(t(a)\, h) = s(a)\, S(h)$;

(4) $S(h_{(1)})h_{(2)} = s(\varepsilon(h))$ and $h_{(1)}S(h_{(2)}) = t(\varepsilon(h))$;

(5) $h_{(1)}S(h_{(2)})h_{(3)} = h$ and $S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$;

using $\Delta := \Delta_l = \Delta_r$ and the notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

Remark 2.2.4 *Observe that if $(\mathcal{H},s,t,\Delta,\varepsilon, S)$ is a commutative Hopf algebroid over a commutative algebra A, then $S^2 = Id_{\mathcal{H}}$. Indeed, for every $h \in \mathcal{H}$,*

$$\begin{aligned} S^2(h) &= S\left(S(t \circ \varepsilon(h_{(1)})\, h_{(2)})\right) \\ &= S\left(S(h_{(2)})\, s \circ \varepsilon(h_{(1)})\right) \\ &= t \circ \varepsilon(h_{(1)})\, S^2(h_{(2)}) \\ &= h_{(1)}\, S(h_{(2)})\, S^2(h_{(3)}) \\ &= h_{(1)}\, S\left(h_{(2)}\, S(h_{(3)})\right) \\ &= h_{(1)}\, S\left(t \circ \varepsilon(h_{(2)})\right) \\ &= h_{(1)}\, s \circ \varepsilon(h_{(2)}) \\ &= h. \end{aligned}$$

Example 2.2.5 *Let A be a commutative algebra and consider $\mathcal{H} = A \otimes A$. Then $\mathcal{H}$ is endowed with a Hopf algebroid structure by*

$$s(a) = 1_A \otimes a, \qquad t(a) = a \otimes 1_A, \qquad \Delta(a \otimes b) = a \otimes 1_A \otimes_A 1_A \otimes b,$$

$$\varepsilon(a \otimes b) = ab \qquad \text{and} \qquad \mathcal{S}(a \otimes b) = b \otimes a.$$

Example 2.2.6 *A little generalization of the previous example is the algebra of Laurent polynomials, $\mathcal{H} = (A \otimes A)[x,x^{-1}]$, for A being a commutative algebra. This algebra is also a Hopf algebroid with*

$$s(a) = 1_A \otimes a, \;\; t(a) = a \otimes 1_A, \;\; \Delta((a \otimes b)x^n) = (a \otimes 1_A)x^n \otimes_A (1_A \otimes b)x^n,$$

$$\varepsilon((a \otimes b)x^n) = ab \qquad \text{and} \qquad \mathcal{S}((a \otimes b)x^n) = (b \otimes a)x^{-n}.$$

2.2.2.1 The Hopf algebroid of the representative functions

Given a groupoid $\mathcal{G}$, we can construct a Hopf algebroid of its representative functions (KAOUTIT, 2013). In order to define a representative function of $\mathcal{G}$, we need to understand what is a representation of a groupoid. A representation of a groupoid is called a $\mathcal{G}$-representation and consists on:

- $\mathcal{E} = \bigsqcup_{x\in\mathcal{G}^{(0)}} E_x$ disjoint union of finite dimensional $\Bbbk$-vector spaces E_x such that there exists an n-dimensional $\Bbbk$-vector space V and linear isomorphisms $\varphi_x : V \to E_x$ for every $x \in \mathcal{G}^{(0)}$.
- A family of linear isomorphisms $\rho_g^{\mathcal{E}} : E_{s(g)} \to E_{t(g)}$ for every $g \in \mathcal{G}$ such that for every $x \in \mathcal{G}^{(0)}$ and $(g,h) \in \mathcal{G}^{(2)}$,
$$\rho_x^{\mathcal{E}} = id_{E_x}, \qquad \rho_{gh}^{\mathcal{E}} = \rho_g^{\mathcal{E}}\rho_h^{\mathcal{E}}.$$

For example, $\mathcal{I} = \bigsqcup_{x\in\mathcal{G}^{(0)}} \mathcal{I}_x$, with $\mathcal{I}_x = \Bbbk$ for every $x \in \mathcal{G}^{(0)}$ and $\rho_g^{\mathcal{I}} = \mathrm{Id}_{\Bbbk}$ for every $g \in \mathcal{G}$ is a $\mathcal{G}$-representation.

A morphism λ between $\mathcal{G}$-representations $(\mathcal{E},\rho^{\mathcal{E}})$ and $(\mathcal{F},\rho^{\mathcal{F}})$ is a family of linear maps $\{\lambda_x\}_{x\in\mathcal{G}^{(0)}}$ with $\lambda_x : E_x \to F_x$ such that for every $g \in \mathcal{G}$,
$$\rho_g^{\mathcal{F}}\lambda_{s(g)} = \lambda_{t(g)}\rho_g^{\mathcal{E}}.$$

Denote by $\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})$ the category of the $\mathcal{G}$-representations in $\Bbbk$-vector spaces, with tensor product and duals for $\mathcal{G}$-representations $(\mathcal{E},\rho^{\mathcal{E}})$ and $(\mathcal{F},\rho^{\mathcal{F}})$ given by
$$(\mathcal{E},\rho^{\mathcal{E}}) \otimes (\mathcal{F},\rho^{\mathcal{F}}) := (\mathcal{E}\otimes\mathcal{F},\rho^{\mathcal{E}}\otimes\rho^{\mathcal{F}}) = \left(\bigsqcup_{x\in\mathcal{G}^{(0)}} (E_x \otimes_{\Bbbk} F_x)\,, \{\rho_g^{\mathcal{E}} \otimes_{\Bbbk} \rho_g^{\mathcal{F}}\}_{g\in\mathcal{G}} \right)$$
$$(\mathcal{E},\rho^{\mathcal{E}})^* = \left(\bigsqcup_{x\in\mathcal{G}^{(0)}} {E_x}^*, \{\rho_g^{\mathcal{E}^*}\}_{g\in\mathcal{G}} \right),$$
where $\rho^{\mathcal{E}^*} : {E_{s(g)}}^* \to {E_{t(g)}}^*$ with $\rho_g^{\mathcal{E}^*}(\varphi) = \varphi \circ \rho_{g^{-1}}^{\mathcal{E}}$ for every $g \in \mathcal{G}$ and $\varphi \in {E_{s(g)}}^*$.

Proposition 2.2.7 *(KAOUTIT, 2013) Let $\mathcal{G}$ be a groupoid and $(\mathcal{E},\rho^{\mathcal{E}})$ a $\mathcal{G}$–representation. Setting $A = Fun(\mathcal{G}^{(0)},\Bbbk)$ the commutative $\Bbbk$–algebra of all maps from $\mathcal{G}^{(0)}$ to $\Bbbk$, we have*
$$\Gamma(\mathcal{E}) = \{p : \mathcal{G}^{(0)} \to \mathcal{E} \mid p(x) \in E_x \ \forall x \in \mathcal{G}^{(0)}\}$$
is a finitely generated and projective A-module.

Proof. In order to prove that $\Gamma(\mathcal{E})$ is a finitely generated and projective A-module, it is enough to construct its dual basis. Take V the underlying n–dimensional vector space from $(\mathcal{E},\rho^{\mathcal{E}})$ with the isomorphisms $\varphi_x : V \to E_x$. Fix $\{v_1,\ldots,v_n\}$ a basis for V and consider for each $i = 1,\ldots,n$, the maps $e_i : \mathcal{G}^{(0)} \to \mathcal{E}$, $x \mapsto \varphi_x(v_i)$. Since $\{\varphi_x(v_1),\ldots,\varphi_x(v_n)\}$ is a basis of E_x and $p(x) \in E_x$ for every $x \in \mathcal{G}^{(0)}$, we can write
$$p(x) = \sum_{i=1}^{n} p_i(x)\varphi_x(v_i) = \sum_{i=1}^{n} p_i(x)e_i(x)$$
with $p_i \in A$. We can also define for each $i = 1,\ldots,n$ a map $e_i^* : \Gamma(\mathcal{E}) \to A$, $p \mapsto p_i$, thus
$$p = \sum_{i=1}^{n} e_i^*(p)\, e_i.$$
Therefore, $\{e_i^*,e_i\}_{i=1}^n$ forms a dual basis for the A–module $\Gamma(\mathcal{E})$. □

Now for any $\mathcal{G}$–representation $(\mathcal{E},\rho^{\mathcal{E}})$, let

$$T_{\mathcal{E}} := \mathrm{End}_{\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})}(\mathcal{E},\rho^{\mathcal{E}}) \qquad T_{\mathcal{E},\mathcal{F}} := \mathrm{Hom}_{\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})}\left((\mathcal{E},\rho^{\mathcal{E}}),(\mathcal{F},\rho^{\mathcal{F}})\right)$$

and consider the tensor products

$$\Gamma := \bigoplus_{(\mathcal{E},\rho^{\mathcal{E}})\in\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})} \Gamma(\mathcal{E}^*)\otimes_{T_{\mathcal{E}}}\Gamma(\mathcal{E}).$$

Γ is a commutative $(A\otimes_{\Bbbk} A)$–algebra with the product

$$(\varphi\otimes_{T_{\mathcal{E}}} p)(\psi\otimes_{T_{\mathcal{F}}} q) = (\varphi\otimes_A\psi)\otimes_{T_{\mathcal{E}\otimes\mathcal{F}}}(p\otimes_A q)$$

for every $\varphi\in\Gamma(\mathcal{E}^*)$, $\psi\in\Gamma(\mathcal{F})$, $p\in\Gamma(\mathcal{E})$ and $q\in\Gamma(\mathcal{F})$. Finally, the quotient

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) := \frac{\bigoplus_{(\mathcal{E},\rho^{\mathcal{E}})\in\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})}\Gamma(\mathcal{E}^*)\otimes_{T_{\mathcal{E}}}\Gamma(\mathcal{E})}{\mathscr{J}_{\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})}}$$

of Γ with the ideal

$$\mathscr{J}_{\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})} = \langle \varphi\otimes_{T_{\mathcal{F}}}\lambda p - \varphi\lambda\otimes_{T_{\mathcal{E}}} p \,|\, \varphi\in\Gamma(\mathcal{F}^*), p\in\Gamma(\mathcal{E}), \lambda\in T_{\mathcal{E},\mathcal{F}}\rangle$$

is a $(A\otimes_{\Bbbk} A)$-algebra with the inherited product from Γ and is called the algebra of the representative functions on the groupoid $\mathcal{G}$. The elements of the algebra are denoted by $\overline{\varphi\otimes_{T_{\mathcal{E}}} p}$. $\mathcal{R}_{\Bbbk}(\mathcal{G})$ has a commutative Hopf algebroid structure over the commutative base algebra A : for every $a\in A$, $\overline{\varphi\otimes_{T_{\mathcal{E}}} p}\in\mathcal{R}_{\Bbbk}(\mathcal{G})$ and $x\in\mathcal{G}^{(0)}$,

$$\overline{s}(a) = \overline{1_A\otimes_{T_{\mathcal{I}}} a} \qquad \overline{t}(a) = \overline{a\otimes_{T_{\mathcal{I}}} 1_A} \qquad \Delta(\overline{\varphi\otimes_{T_{\mathcal{E}}} p}) = \sum_{i=1}^{n}\overline{\varphi\otimes_{T_{\mathcal{E}}} e_i}\otimes_A\overline{e_i{}^*\otimes_{T_{\mathcal{E}}} p},$$

$$\varepsilon(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})(x) = \varphi(x)\,(p(x)) \qquad S(\overline{\varphi\otimes_{T_{\mathcal{E}}} p}) = \overline{\tilde{p}\otimes_{T_{\mathcal{E}^*}}\varphi},\ \textit{with}\ \tilde{(\_)}:\mathcal{E}\cong(\mathcal{E}^*)^*,$$

where $\{e_i{}^*, e_i\}$ is the dual basis of the A-module $\Gamma(\mathcal{E})$ and $\mathcal{I}$ is the trivial $\mathcal{G}$–representation $\mathcal{I} = \bigsqcup_{x\in\mathcal{G}^{(0)}}\Bbbk$, with $\rho_g^{\mathcal{I}} = \mathrm{Id}_{\Bbbk}$.

Example 2.2.8 *A group G can be seen as a groupoid $\mathcal{G} = G$ with $\mathcal{G}^{(0)} = \{1_G\}$. A G–representation is a finite dimensional vector space V together with linear isomorphisms $\rho_g^V : V\to V$ such that $\rho_g^V\rho_h^V = \rho_{gh}^V$ for every $g,h\in G$. Hence the representations of the groupoid G are the same as the representations of the group. Also, we have that $A = Fun(\mathcal{G}^{(0)},\Bbbk)\cong\Bbbk$,*

$$\Gamma(V) = \{p:\{1_G\}\to V\}\cong V$$

and $\Gamma(V^*)\cong V^*$. Moreover, an endomorphism for the representation (V,ρ^V) is a linear map $\alpha : V\to V$ such that

$$\alpha\circ\rho_g^V = \rho_g^V\circ\alpha$$

for every $g\in G$. Thus $\alpha = \lambda\,\mathrm{Id}_V$ for some $\lambda\in\Bbbk$ and then $T_V\cong\Bbbk$. Consequently, the ideal $\mathscr{J}_{\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})}(G) = 0$. Then the algebra of representative functions of G is the algebra

$$\mathcal{R}_{\Bbbk}(G)\cong\bigoplus_{(V,\rho^V)\in\mathcal{R}\mathrm{ep}_{\Bbbk}(\mathcal{G})} V^*\otimes_{\Bbbk} V$$

and an element of $\mathcal{R}_{\Bbbk}(G)$ can be written as a triple (φ, v, ρ^V) with $\varphi\in V^*$, $v\in V$ and ρ^V a G–representation, which can be identified as the representative function for the group G

$$\begin{array}{rccl} f: & G & \longrightarrow & \Bbbk \\ & g & \longmapsto & \varphi(\rho^V(g)(v)) \end{array}$$

Therefore $\mathcal{R}_{\Bbbk}(G)$ is exactly the commutative Hopf $\Bbbk$–algebra $R(G)$ of the representative functions on the group G.

Example 2.2.9 *Let $\mathcal{G}$ be the groupoid known as the* unit groupoid *where $\mathcal{G} = \mathcal{G}^{(0)}$ and the source and target maps are $s = t = \mathrm{Id}_{\mathcal{G}^{(0)}}$. Then a $\mathcal{G}$–representation is given by a disjoint union*

$$\mathcal{E} = \coprod_{x \in \mathcal{G}^{(0)}} E_x \cong \coprod_{x \in \mathcal{G}^{(0)}} V,$$

where V is a n–dimensional vector space and the linear isomorphisms $\rho_x^{\mathcal{E}} : E_x \to E_x$ are the identity map for every $x \in \mathcal{G}^{(0)}$. Hence the $\mathcal{G}$–representation is simply the set $V \times \mathcal{G}^{(0)}$. Also, observe that

$$\Gamma(V \times \mathcal{G}^{(0)}) = \{p : \mathcal{G}^{(0)} \to V \times \mathcal{G}^{(0)} \mid p(x) \in V \times \{x\}\} \cong A^n,$$

where $A = Fun(\mathcal{G}^{(0)},\Bbbk)$. Similarly, $\Gamma((V \times \mathcal{G}^{(0)})^*) \cong A^n$ and morphisms between $\mathcal{G}$–representations are $T_{V\times\mathcal{G}^{(0)},W\times\mathcal{G}^{(0)}} \cong M_{n,m}(A)$, where W is a m–dimensional vector space. Therefore, the Hopf algebroid of the representative functions of $\mathcal{G}$ is given by the quotient

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) = \frac{\bigoplus_{n\in\mathbb{N}} A^n \otimes_{M_n(A)} A^n}{\left\langle u \otimes_{M_n(A)} (\lambda_{ij})v - u(\lambda_{ij}) \otimes_{M_m(A)} v \right\rangle_{u\in A^n, v\in A^m, (\lambda_{ij})\in M_{n,m}(A)}}.$$

This quotient, indeed coincides with the algebra A. Consider, for example the following element of $\mathcal{R}_{\Bbbk}(\mathcal{G})$

$$\overline{(f^1,\dots,f^n) \otimes_{M_n(A)} \begin{pmatrix} g^1 \\ \vdots \\ g^n \end{pmatrix}}.$$

The vector $(f^1,\dots,f^n) \in A^n$ can be viewed as the product $1_A(f^1,\dots,f^n)$, in which $1_A : \mathcal{G}^{(0)} \to \Bbbk$ is the constant unit function, and $(f^1,\dots,f^n) \in M_{1\times n}(A)$ then

$$\overline{(f^1,\dots,f^n) \otimes_{M_n(A)} \begin{pmatrix} g^1 \\ \vdots \\ g^n \end{pmatrix}} = \overline{1_A \otimes_{M_1(A)} (f^1,\dots,f^n) \begin{pmatrix} g^1 \\ \vdots \\ g^n \end{pmatrix}} = \overline{1_A \otimes_{\Bbbk} \sum_i f^i g^i}.$$

Lemma 2.2.10 *(Proposition 2.2, (KAOUTIT, 2013)) Let $\mathcal{G}$ be a groupoid, $A = Fun(\mathcal{G}^{(0)},\Bbbk)$ and put $B = Fun(\mathcal{G},\Bbbk)$. The following map*

$$\begin{array}{rcl} \zeta : & \mathcal{R}_{\Bbbk}(\mathcal{G}) & \longrightarrow B \\ & \overline{\varphi \otimes_{T_{\mathcal{E}}} p} & \longmapsto \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p}), \end{array}$$

with $\zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(g) = \varphi(t(g))\left(\rho_g^{\mathcal{E}}(p(s(g)))\right)$ for each $g \in \mathcal{G}$ is a $(A \otimes_{\Bbbk} A)$–algebra map. Moreover, we have

(1) *$i^* \circ \zeta = \varepsilon$, with $i : \mathcal{G}^{(0)} \to \mathcal{G}$ being the inclusion map;*

(2) *$\zeta \circ S(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(g) = \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(g^{-1})$;*

(3) *For every $g,h \in G$ such that $s(g) = t(h)$ and every $F \in \mathcal{R}_{\Bbbk}(\mathcal{G})$, we have that*

$$\zeta(F)(gh) = \zeta(F_{(1)})(g)\zeta(F_{(2)})(h),$$

where $\Delta(F) = F_{(1)} \otimes_A F_{(2)}$.

Proof. First, ζ is well-defined because

$$\begin{aligned}
\zeta(\overline{\varphi \otimes_{T_{\mathcal{F}}} \lambda p})(g) &= \varphi(t(g))\left(\rho_g^{\mathcal{F}}((\lambda p)(s(g)))\right)\\
&= \varphi(t(g))\left(\rho_g^{\mathcal{F}} \circ \lambda_{s(g)}(p(s(g)))\right)\\
&= \varphi(t(g))\left(\lambda_{t(g)} \circ \rho_g^{\mathcal{E}}(p(s(g)))\right)\\
&= (\varphi\lambda)(t(g))\left(\rho_g^{\mathcal{E}}(p(s(g)))\right)\\
&= \zeta(\overline{\varphi\lambda \otimes_{T_{\mathcal{E}}} p})(g)
\end{aligned}$$

for every $(\mathcal{E}\rho^{\mathcal{E}})$ and $(\mathcal{F},\rho^{\mathcal{F}})$ $\mathcal{G}$–representations, $g \in \mathcal{G}$, $\varphi \in \Gamma(\mathcal{F}^*)$, $p \in \Gamma(\mathcal{E})$ and $\lambda \in T_{\mathcal{E},\mathcal{F}}$. Also, $\zeta(\overline{1_A \otimes_{T_{\mathcal{I}}} 1_A}) = 1_B$ and for $\overline{\varphi \otimes_{T_{\mathcal{E}}} p}, \overline{\psi \otimes_{T_{\mathcal{F}}} q} \in \mathcal{R}_{\Bbbk}(\mathcal{G})$ and $g \in \mathcal{G}$,

$$\begin{aligned}
\zeta((\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(\overline{\psi \otimes_{T_{\mathcal{F}}} q}))(g) &= \zeta((\overline{(\varphi \otimes_A \psi) \otimes_{T_{\mathcal{E}\otimes\mathcal{F}}} (p \otimes_A q)})(g)\\
&= (\varphi \otimes_A \psi)(t(g))\left(\rho_g^{\mathcal{E}} \otimes_{\Bbbk} \rho_g^{\mathcal{F}}((p \otimes_A q)(s(g)))\right)\\
&= \varphi(t(g))\left(\rho_g^{\mathcal{E}}(p(s(g))\right)\,\psi(t(g))\left(\rho_g^{\mathcal{F}}(t(s(g))\right)\\
&= \left(\zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})\,\zeta(\overline{\psi \otimes_{T_{\mathcal{F}}} q})\right)(g).
\end{aligned}$$

Hence ζ is multiplicative. Moreover,

(1) For every $x \in \mathcal{G}^{(0)}$ and $\overline{\varphi \otimes_{T_{\mathcal{E}}} p} \in \mathcal{R}_{\Bbbk}(\mathcal{G})$,

$$\begin{aligned}
i^* \circ \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(x) &= \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(i(x))\\
&= \varphi(t(x))\left(\rho_x^{\mathcal{E}}(p(s(x)))\right)\\
&= \varphi(x)(p(x))\\
&= \varepsilon(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(x).
\end{aligned}$$

(2) For every $g \in \mathcal{G}$ and $\overline{\varphi \otimes_{T_{\mathcal{E}}} p} \in \mathcal{R}_{\Bbbk}(\mathcal{G})$,

$$\begin{aligned}
\zeta \circ S(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(g) &= \zeta(\overline{\tilde{p} \otimes_{T_{\mathcal{E}^*}} \varphi})(g)\\
&= \tilde{p}(t(g))\left(\rho_g^{\mathcal{E}^*}(\varphi(s(g)))\right)\\
&= \rho_g^{\mathcal{E}^*}(\varphi(s(g)))(p(t(g))\\
&= \varphi(s(g)) \circ \rho_{g^{-1}}^{\mathcal{E}}(p(t(g)))\\
&= \varphi(t(g^{-1}))\left(\rho_{g^{-1}}^{\mathcal{E}}(p(s(g^{-1})))\right)\\
&= \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(g^{-1}).
\end{aligned}$$

(3) For every $(g,h) \in \mathcal{G}^{(2)}$ and $F = \overline{\varphi \otimes_{T_{\mathcal{E}}} p} \in \mathcal{R}_{\Bbbk}(\mathcal{G})$,

$$\begin{aligned}
\zeta(F_{(1)})(g)\zeta(F_{(2)})(h) &= \sum_{i=1}^{n} \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} e_i})(g)\,\zeta(\overline{e_i^* \otimes_{T_{\mathcal{E}}} p})(h) \\
&= \sum_{i=1}^{n} \varphi(t(g))\,\left(\rho_g^{\mathcal{E}}(e_i(s(g)))\right)\,e_i^*(t(h))\,\left(\rho_h^{\mathcal{E}}(p(s(h)))\right) \\
&= \varphi(t(g))\left(\rho_g^{\mathcal{E}}\left(\sum_{i=1}^{n} e_i(s(g))\,e_i^*(s(g))\,\left(\rho_h^{\mathcal{E}}(p(s(h)))\right)\right)\right) \\
&= \varphi(t(gh))\left(\rho_{gh}^{\mathcal{E}}(p(s(gh)))\right) \\
&= \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(gh) \\
&= \zeta(F)(gh).
\end{aligned}$$

□

Remark 2.2.11 *The original proposition (Proposition 2.2 (KAOUTIT, 2013)) for the previous result also states that the morphism ζ is injective.*

Example 2.2.12 *(KAOUTIT, 2013) Consider the groupoid $\mathcal{G} = X \times G \times X$, where X is a set, G is a group, $(x,g,y)^{-1} = (y,g^{-1},x)$ and*

$$(x,g,y)\cdot(y,h,z) = (x,gh,z)$$

for every $x,y,z \in X$ and $g,h \in G$. Also consider $\mathcal{G}^{(0)} = \{(x,1_G,x) \mid x \in X\} \cong X$ and the source and target maps being the projections on the third and first coordinates, respectively. Let $A = Fun(X,\Bbbk)$ the set of all maps from X to $\Bbbk$. We will see that

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) \cong A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A.$$

Using the ζ map from Lemma 2.2.10, a representative function $\overline{\varphi \otimes_{T_{\mathcal{E}}} p}$ of $\mathcal{G}$ can be seen as a map from $\mathcal{G}$ onto $\Bbbk$ given by

$$\zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(x,g,y) = \varphi(x)\left(\rho_{(x,g,y)}^{\mathcal{E}}(p(y))\right) \tag{1}$$

for every $x,y \in X$ and $g \in \mathcal{G}$.

Now fix $x_0 \in X$. Hence for a n–dimensional $\mathcal{G}$–representation $(\mathcal{E},\rho^{\mathcal{E}})$,

$$\rho_{(x,g,y)}^{\mathcal{E}} = \rho_{(x,1_G,x_0)}^{\mathcal{E}}\rho_{(x_0,g,x_0)}^{\mathcal{E}}\rho_{(x_0,1_G,y)}^{\mathcal{E}}$$

for every $(x,g,y) \in \mathcal{G}$. Let $\left(a_{ij}^{g}\right)_{1\leq i,j\leq n}$ the n–square matrix representing the $\Bbbk$–linear isomorphism $\rho_{(x_0,g,x_0)}^{\varepsilon}$ and denote by $(b_{ij}^{x_0,x})_{1\leq i,j\leq n}$ and $(b_{ij}^{x,x_0})_{1\leq i,j\leq n}$ the matrices representing $\rho_{(x,1_G,x_0)}^{\mathcal{E}} : E_{x_0} \to E_x$ and $\rho_{(x_0,1_G,x)}^{\mathcal{E}} : E_x \to E_{x_0}$, respectively, for every $x \in X$. Then, with $\{e_i^*, e_i\}$ being the dual basis for $\Gamma(\mathcal{E})$ we can write

$$p(x) = \sum_{i=1}^{n} p_i(x)e_i(x)$$

and from (1) we have

$$\begin{aligned}
\zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})(x,g,y) &= \varphi(x)\left(\rho^{\mathcal{E}}_{(x,g,y)}(p(y))\right)\\
&= \varphi(x)\left(\sum_{l=1}^{n} \rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,g,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,y)}\, p_l(y)\, e_l(x_0)\right)\\
&= \sum_{k,l=1}^{n} p_l(y)\,\varphi(x)\left(\rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,g,x_0)}\, b^{y,x_0}_{kl}\, e_k(x_0)\right)\\
&= \sum_{,j,k,l=1}^{n} p_l(y)\, b^{y,x_0}_{kl}\,\varphi(x)\left(\rho^{\mathcal{E}}_{(x,1_G,x_0)} a^{g}_{jk}\, e_j(x_0)\right)\\
&= \sum_{i,j,k,l=1}^{n} p_l(y)\, b^{y,x_0}_{kl}\, a^{g}_{jk}\,\varphi(x)\left(b^{x_0,x}_{ij}\, e_i(x)\right)\\
&= \sum_{i,j,k,l=1}^{n} b^{x_0,x}_{lk}\, a^{g}_{kj}\, b^{y,x_0}_{ji}\, p_i(y)\,\varphi(x)(e_i(x))\\
&= \sum_{i,j,k,l=1}^{n} b^{x_0,x}_{ij}\, a^{g}_{jk}\, b^{y,x_0}_{kl}\, p_l(y)\,\varphi_i(x)
\end{aligned}$$

with $\varphi_i : X \to \Bbbk$ *given by* $x \mapsto \varphi(x)(e_i(x))$. *Defining*

$$\begin{aligned}
\overline{\varphi}_i : X &\longrightarrow \Bbbk & \qquad \overline{p}_l : X &\longrightarrow \Bbbk\\
x &\longmapsto \textstyle\sum_{j=1}^{n} b^{x_0,x}_{ij}\varphi_i(x) & x &\longmapsto \textstyle\sum_{k=1}^{n} b^{x,x_0}_{kl} p_l(x)
\end{aligned}$$

we have that

$$\zeta(\overline{\varphi \otimes_{T_{\varepsilon}} p})(x,g,y) = \sum_{i,j,k,l=1}^{n} \overline{\varphi}_i(x)\, a^{g}_{jk}\, \overline{p}_l(y). \tag{2}$$

Observe that the maps $a_{ij} : G \to \Bbbk$, $g \mapsto a^{g}_{ij}$ *are all representative functions on the group* G, *because for every* $i,j = 1, \ldots n$,

$$a^{g}_{ij} = e^{*}_i(x_0)\left(\rho^{\mathcal{E}}_{(x_0,g,x_0)}(e_j(x_0))\right),$$

for all $g \in \mathcal{G}$, *with* $e_i(x_0) \in E_{x_0}$ *and* $\rho^{\mathcal{E}}_{(x_0,g,x_0)} : G \to GL(E_{x_0})$ *a* G*–representation. Also note that*

$$a^{1_G}_{ij} = e^{*}_i(x_0)\left(\rho^{\mathcal{E}}_{(x_0,1_G,x_0)}(e_j(x_0))\right) = e^{*}_i(x_0)(e_j(x_0)) = \delta_{ij}, \tag{3}$$

which implies that $(a^{1_G}_{ij})_{1\le i,j\le n} = I_n$. *In addition to that, if* $a \otimes f \otimes b$ *is in* $A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$ *with* $R(G)$ *being the Hopf algebra of the representative function on the group* G, *then* $f : G \to \Bbbk$ *can be written as*

$$f(g) = F(\rho(g)(v)) \qquad \forall g \in G$$

with v *being an element of a* n*–dimensional vector space* V, $F : V \to \Bbbk$ *and* $\rho : G \to GL(V)$ *a representation of the group* G. *Thus* $\mathcal{E}^f = \bigsqcup_{x\in X} V$ *and* $\rho^{\mathcal{E}^f}_{(x,g,h)} = \rho(g) : V \to V$ *form a* $\mathcal{G}$*–representation and defining*

$$\begin{aligned}
\varphi^b : X &\longrightarrow V^{*} & \qquad p^a : X &\longrightarrow V\\
x &\longmapsto \varphi^b(x) : w \mapsto b(x)\, F(w) & x &\longmapsto a(x)\, v
\end{aligned},$$

we have that for every $(x,g,y) \in \mathcal{G}$,

$$\begin{aligned} \zeta(\overline{\varphi^b \otimes_{T_{\mathcal{E}^f}} p^a})(x,g,y) &= \varphi^b(y)\left(\rho^{\mathcal{E}^f}_{(x,g,y)}(p^a(x))\right) \\ &= \varphi^b(y)(\rho(g)(a(x)\,v)) \\ &= a(x)\,b(y)\,F(\rho(g)(v)) \\ &= a(x)\,f(g)\,b(y) \\ &= \imath(a \otimes f \otimes b)(x,g,y), \end{aligned} \tag{4}$$

where $\imath$ is the the canonical map

$$A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A \overset{\imath}{\hookrightarrow} Fun(X \times G \times X, \Bbbk).$$

Consequently, $\imath(A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A) \subseteq \zeta(\mathcal{R}_{\Bbbk}(\mathcal{G}))$ and from the expression (2),

$$\zeta(\overline{\varphi \otimes_{T_\varepsilon} p})(x,g,y) = \imath\left(\sum_{i,j,k,l=1}^{n} \overline{\varphi}_i \otimes a_{jk} \otimes \overline{p}_l\right)(x,g,y)$$

for every $(x,g,y) \in \mathcal{G}$, so we have $\zeta(\mathcal{R}_{\Bbbk}(\mathcal{G})) \subseteq \imath(A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A)$. Therefore, the image of $\mathcal{R}_{\Bbbk}(\mathcal{G})$ in $Fun(X \times G \times X, \Bbbk)$ by ζ coincides with the image of $A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$ in $Fun(X \times G \times X, \Bbbk)$ by $\imath$. And since the two maps are injective, we have an isomorphism of A–bimodules

$$\begin{array}{rcl} \mathcal{R}_{\Bbbk}(\mathcal{G}) & \overset{\xi}{\longrightarrow} & A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A \\ \overline{\varphi \otimes_{T_{\mathcal{E}}} p} & \longmapsto & \sum_{i,j,k,l=1}^{n} \overline{\varphi}_i \otimes a_{jk} \otimes \overline{p}_l \end{array}.$$

Moreover, this is an isomorphism of A–Hopf algebroids. Indeed, with the Hopf algebroid structure on $A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$ being

$$\begin{aligned} &s'(a) = 1_A \otimes 1_{R(G)} \otimes a \\ &t'(a) = a \otimes 1_{R(G)} \otimes 1_A \\ &\Delta'(a \otimes f \otimes b) = (a \otimes f_{(1)} \otimes 1_A) \otimes_A (1_A \otimes f_{(2)} \otimes b) \\ &\varepsilon'(a \otimes f \otimes b)(x) = a(x)b(x)f(1_G) \\ &S'(a \otimes f \otimes b)(x \otimes g \otimes y) = a(y)b(x)f(g^{-1}). \end{aligned} \tag{5}$$

with $a,b \in A$, $x,y \in X$ and $f \in R(G)$, we have that:

(i) For every $a \in A$ and $(x,g,y) \in \mathcal{G}$,

$$\begin{aligned} \zeta(\overline{s}(a))(x,g,y) &= \zeta(\overline{1_A \otimes_{T_{\mathcal{I}}} a})(x,g,y) \\ &= 1_A(x)\left(\rho^{\mathcal{I}}_{(x,g,y)}(a(y))\right) \\ &= a(y) \\ &= \imath(1_A \otimes 1_{R(G)} \otimes a)(x,g,y), \end{aligned}$$

which implies that

$$\begin{aligned} \xi(\overline{s}(a))(x \otimes g \otimes y) &= (1_A \otimes 1_{R(G)} \otimes a)(x \otimes g \otimes y) \\ &= s'(a)(x \otimes g \otimes y). \end{aligned}$$

and $\xi \circ \overline{s} = s'$. *On the other hand,*

$$\begin{aligned}
\zeta(\overline{t}(a))(x,g,y) &= \zeta(\overline{a \otimes_{T_{\mathcal{I}}} 1_A})(x,g,y) \\
&= a(x)\left(\rho^{\mathcal{I}}_{(x,g,y)}(1_A(y))\right) \\
&= a(x) \\
&= \imath(a \otimes 1_{R(G)} \otimes 1_A)(x,g,y)
\end{aligned}$$

implies that

$$\begin{aligned}
\xi(\overline{t}(a))(x \otimes g \otimes y) &= (a \otimes 1_{R(G)} \otimes 1_A)(x \otimes g \otimes y) \\
&= t'(a)(x \otimes g \otimes y)
\end{aligned}$$

and consequently, $\xi \circ \overline{t} = t'$.

(ii) Now for every $x,y,z,t \in X$, $g,h \in G$, $a \in A$ *and* $\overline{\varphi \otimes_{T_{\mathcal{E}}} p} \in \mathcal{R}_{\Bbbk}(\mathcal{G})$,

$$\begin{aligned}
&\sum_{i=1}^{n} \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} e_i}) \otimes_A \zeta(\overline{e_i^* \otimes_{T_{\mathcal{E}}} p})((x,g,y) \otimes_A (z,h,t)) \\
&= \sum_{i=1}^{n} \zeta(\overline{\varphi \otimes_{T_{\mathcal{E}}} e_i})(x,g,y)\, \zeta(\overline{e_i^* \otimes_{T_{\mathcal{E}}} p})(z,h,t) \\
&= \sum_{i=1}^{n} \varphi(x)\left(\rho^{\mathcal{E}}_{(x,g,y)}(e_i(y))\right)\, e_i^*(z)\left(\rho^{\mathcal{E}}_{(z,h,t)}(p(t))\right) \\
&= \sum_{i=1}^{n} \varphi(x)\left(\rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,g,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,y)} e_i(y)\, e_i^*(z)\rho^{\mathcal{E}}_{(z,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,h,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,t)}\, p(t)\right) \\
&\overset{(*)}{=} \varphi(x)\left(\rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,g,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,y)}\varphi_y \circ \varphi_z^{-1}\rho^{\mathcal{E}}_{(z,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,h,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,t)}\, p(t)\right) \\
&= \varphi(x)\left(\rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,g,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,y)}\rho^{\mathcal{E}}_{(y,1_G,z)}\rho^{\mathcal{E}}_{(z,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,h,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,t)}\, p(t)\right) \\
&= \varphi(x)\left(\rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,g,x_0)}\rho^{\mathcal{E}}_{(x_0,h,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,t)}\, p(t)\right) \\
&= \sum_{i,j,k,l=1}^{n} \overline{\varphi}_i(x)\, a^{gh}_{jk}\, \overline{p}_l(t) \\
&= \sum_{i,j,k,l=1}^{n} \overline{\varphi}_i(x)\, a_{jk}{}^{g}_{(1)}\, a_{jk}{}^{h}_{(2)}\, \overline{p}_l(t) \\
&= \sum_{i,j,k,l=1}^{n} \imath(\overline{\varphi}_i \otimes a_{jk_{(1)}} \otimes 1_A) \otimes_A \imath(1_A \otimes a_{jk_{(2)}} \otimes \overline{p}_l)((x,g,y) \otimes_A (z,h,t))
\end{aligned}$$

where $(*)$ *comes from* $e_i(y) = \varphi_y \circ \varphi_z^{-1} \circ \varphi_z(v_i) = \varphi_y \circ \varphi_z^{-1}(e_i(z))$.

Hence

$$
\begin{aligned}
&(\xi\otimes\xi)\Delta(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})((x\otimes g\otimes y)\otimes_A(z\otimes h\otimes t)\\
&\qquad=\sum_{r=1}^{n}\xi(\overline{\varphi\otimes_{T_{\mathcal{E}}} e_r})\otimes_A\xi(\overline{e_r^*\otimes_{T_{\mathcal{E}}} p})((x\otimes g\otimes y)\otimes_A(z\otimes h\otimes t))\\
&\qquad=\sum_{i,j,k,l=1}^{n}(\overline{\varphi}_i\otimes a_{jk_{(1)}}\otimes 1_A)\otimes_A(1_A\otimes a_{jk_{(2)}}\otimes\overline{p}_l)((x\otimes g\otimes y)\otimes_A(z\otimes h\otimes t))\\
&\qquad=\sum_{i,j,k,l=1}^{n}\Delta'(\overline{\varphi_i}\otimes a_{ij}\otimes\overline{p}_l)((x\otimes g\otimes y)\otimes_A(z\otimes h\otimes t))\\
&\qquad=\Delta'\circ\xi(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})((x\otimes g\otimes y)\otimes_A(z\otimes h\otimes t)),
\end{aligned}
$$

that is, $(\xi\otimes\xi)\circ\Delta=\Delta'\circ\xi$.

Also,

$$
\begin{aligned}
\varepsilon'\circ\xi(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})(x) &= \sum_{i,j,k,l=1}^{n}\varepsilon'(\overline{\varphi}_i\otimes a_{jk}\otimes\overline{p}_l)(x)\\
&=\sum_{i,j,k,l=1}^{n}\overline{\varphi}_i(x)\overline{p}_l(x)\,a_{jk}^{1_G}\\
&=\sum_{i,j,k,l=1}^{n}\overline{\varphi}_i(x)\overline{p}_l(x)\,\delta_{jk}\\
&=\sum_{i,j,k=1}^{n}b_{ij}^{x_0,x}b_{j,k}^{x,x_0}\varphi_i(x)p_k(x)\\
&\overset{(*)}{=}\varphi(x)\left(I_n(p_j(x))_j(e_i(x))_i\right)\\
&=\varphi(x)(p(x))\\
&=\varepsilon(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})(x),
\end{aligned}
$$

where $(*)$ *comes from the fact that* $(b_{ij}^{x_0,x})_{ij}^{-1}=(b_{i,j}^{x,x_0})_{ij}$, *because*

$$
\rho^{\mathcal{E}}_{(x_0,1_G,x)}\rho^{\mathcal{E}}_{(x,1_G,x_0)}=\rho^{\mathcal{E}}_{x_0}=\mathrm{Id}_{E_{x_0}}\qquad \rho^{\mathcal{E}}_{(x,1_G,x_0)}\rho^{\mathcal{E}}_{(x_0,1_G,x)}=\rho^{\mathcal{E}}_{x}=\mathrm{Id}_{E_x}.
$$

Therefore, $\xi\circ S=S'\circ\xi$ *and* $\varepsilon'\circ\xi=\varepsilon$.

(iii) Finally, for $(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})\in\mathcal{R}_{\Bbbk}(\mathcal{G})$, $g\in G$ *and* $x,y\in X$,

$$
\begin{aligned}
\zeta(S(\overline{\varphi\otimes_{T_{\mathcal{E}}} p}))(x,g,y) &= \zeta(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})(y,g^{-1},x)\\
&=\sum_{i,j,k,l=1}^{n}\overline{\varphi}_i(y)\,a_{jk}^{g^{-1}}\,\overline{p}_l(x)\\
&=\sum_{i,j,k,l=1}^{n}\imath\circ S'(\overline{\varphi}_i\otimes a_{jk}\otimes\overline{p}_l)(x,g,y)
\end{aligned}
$$

implies that

$$
\begin{aligned}
\xi(S(\overline{\varphi\otimes_{T_{\mathcal{E}}} p}))(x\otimes g\otimes y) &= \sum_{i,j,k,l=1}^{n}S'(\overline{\varphi}_i\otimes a_{jk}\otimes\overline{p}_l)(x\otimes g\otimes y)\\
&=S'\circ\xi(\overline{\varphi\otimes_{T_{\mathcal{E}}} p})(x\otimes g\otimes y).
\end{aligned}
$$

Consequently, ξ is an isomorphism of Hopf algebroids.

Remark 2.2.13 *A transitive groupoid $\mathcal{G}$ with* source s *and* target t *is a groupoid such that for every pair $x,y \in X$ there exists an element $g \in \mathcal{G}$ that satisfies $x = s(g)$ and $y = t(g)$. Then, fixing $x_0 \in \mathcal{G}^{(0)}$, we have $\mathcal{G} \cong \mathcal{G}^{(0)} \times G_{x_0} \times \mathcal{G}^{(0)}$ (G_{x_0} is the isotropy group for x_0) with the isomorphism*

$$\psi(g) = \left(t(g), \varphi_{t(g)}^{-1}\, g\, \varphi_{s(g)}, s(g)\right),$$

where for every $y \in \mathcal{G}^{(0)}$, φ_y gives the element of $\mathcal{G}$ that satisfies $x_0 = s(\varphi_y)$ and $y = t(\varphi_y)$ and source *and* target *given by the projections on third and first coordinates, respectively.*

Therefore, we have from Example 2.2.12 that the Hopf algebroid of the representative functions of a transitive groupoid is

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) \cong A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A,$$

where $A = Fun(\mathcal{G}^{(0)},\Bbbk)$ and $R(G)$ is the Hopf algebra of the representative functions on the isotropy group $G = G_{x_0}$ for some fixed $x_0 \in \mathcal{G}^{(0)}$.

Example 2.2.14 *Consider a set X and the groupoid $\mathcal{G} = X \times X$ with*

$$(x,y)(y,z) = (x,z) \qquad (x,y)^{-1} = (y,x).$$

Observe that this groupoid can be seen as a particular case from Example 2.2.12 with G being a unitary group $\{e\}$. Since the Hopf algebra $R(\{e\})$ is isomomorphic to $\Bbbk$ and, consequently,

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) \cong A \otimes_{\Bbbk} A$$

with the same Hopf algebroid structure seen at the Example 2.2.5.

3 QUANTUM INVERSE SEMIGROUPS

We want the definition of quantum inverse semigroups to be a generalization of inverse semigroups in the same sense that Hopf algebras can be thought as a generalization of groups. With this in mind, we ask for a quantum inverse semigroup to have a comultiplication, a pseudo antipode and some commutative idempotents related to the convolution product.

3.1 DEFINITION

Definition 3.1.1 (Quantum Inverse Semigroup) *A quantum inverse semigroup (QISG) is a triple* $(H, \Delta, \mathcal{S})$ *in which*

(QISG1) *H is a (not necessarily unital) $\Bbbk$-algebra.*

(QISG2) *$\Delta : H \to H \otimes H$ is multiplicative.*

(QISG3) *$\mathcal{S} : H \to H$ is a $\Bbbk$-linear map, called pseudo antipode, satisfying*

(i) $\mathcal{S}(hk) = \mathcal{S}(k)\mathcal{S}(h)$, for all $h,k \in H$.

(ii) $Id_H * \mathcal{S} * Id_H = Id_H$ and $\mathcal{S} * Id_H * \mathcal{S} = \mathcal{S}$ in the convolution algebra $End_{\Bbbk}(H)$.

(QISG4) *The subalgebras generated by the images of $Id_H * \mathcal{S}$ and $\mathcal{S} * Id_H$ mutually commute, that is, for every $h,k \in H$,*

$$h_{(1)}\mathcal{S}(h_{(2)})\mathcal{S}(k_{(1)})k_{(2)} = \mathcal{S}(k_{(1)})k_{(2)}h_{(1)}\mathcal{S}(h_{(2)}).$$

A quantum inverse semigroup is unital if H is a unital $\Bbbk$-algebra and $\mathcal{S}(1_H) = 1_H$. A quantum inverse semigroup is counital if H is a $\Bbbk$-coalgebra and $\varepsilon_H \circ \mathcal{S} = \varepsilon_H$.

Remark 3.1.2 *In an inverse semigroup, we have the uniqueness of the pseudo-inverse and, equivalently, the commutativity of the idempotents. In the definition of quantum inverse semigroups, we don't ask for any of these things, and these are not direct consequences of the definition.*

(i) The pseudo antipode is not always unique. In the case where the idempotents of the convolution algebra $End_{\Bbbk}(H)$ commute, then the pseudo antipode is unique. In fact, being S and $\overline{S}$ both pseudo antipodes of the quantum inverse semigroup H, we have that $Id_H * S$, $Id_H * \overline{S}$, $S * Id_H$ and $\overline{S} * Id_H$ are idempotents in the convolution algebra $End_{\Bbbk}(H)$ and then

$$\begin{aligned}\overline{S} &= \overline{S} * Id_H * \overline{S} = \overline{S} * Id_H * S * Id_H * \overline{S} = S * Id_H * \overline{S} * Id_H * \overline{S} = S * Id_H * \overline{S} \\ &= S * Id_H * S * Id_H * \overline{S} = S * Id_H * \overline{S} * Id_H * S = S * Id_H * S = S.\end{aligned}$$

(ii) Let (H,Δ,ε,S) be a coalgebra satisfying (QISG1), (QISG2), (QISG3) *and $\varepsilon \circ S = \varepsilon$. If the idempotents of the convolution algebra $Hom_{\Bbbk}(H \otimes H,H)$ commute, then the axiom* (QISG4) *follows automatically. In fact, let $e, \overline{e} : H \otimes H \to H$ given by*

$$e(h \otimes k) = h_{(1)}S\left(h_{(2)}\right)\varepsilon(k) \quad and \quad \overline{e}(h \otimes k) = \varepsilon(h)S\left(k_{(1)}\right)k_{(2)}$$

for every $h, k \in H$. *Then* e *is idempotent in the convolution algebra* $Hom_{\mathbb{K}}(H \otimes H, H)$, *because*

$$(e * e)(h \otimes k) = h_{(1)} S\left(h_{(2)}\right) \varepsilon\left(k_{(1)}\right) h_{(3)} S\left(h_{(4)}\right) \varepsilon\left(k_{(2)}\right)$$
$$h_{(1)} S\left(h_{(2)}\right) \varepsilon(k) = e(h \otimes k)$$

for every $h, k \in H$. *Similarly, we obtain that* $\overline{e}$ *is also an idempotent. Also for every* $h, k \in H$,

$$\begin{aligned} e * \overline{e}(h \otimes k) &= h_{(1)} S\left(h_{(2)}\right) \varepsilon\left(k_{(1)}\right) \varepsilon\left(h_{(3)}\right) S\left(k_{(2)}\right) k_{(3)} \\ &= h_{(1)} S\left(h_{(2)}\right) S\left(k_{(1)}\right) k_{(2)} \end{aligned}$$

and

$$\begin{aligned} \overline{e} * e(h \otimes k) &= \varepsilon\left(h_{(1)}\right) S\left(k_{(1)}\right) k_{(2)} h_{(2)} S\left(h_{(3)}\right) \varepsilon\left(k_{(3)}\right) \\ &= S\left(k_{(1)}\right) k_{(2)} h_{(1)} S\left(h_{(2)}\right). \end{aligned}$$

Since the idempotents commute, we have (QISG4).

(iii) *In axiom* (QISG3)*, it is imposed that the pseudo antipode is antimultiplicative, even though in most examples of quantum inverse semigroups it is possible to show this property directly from other intrinsic characteristics of those particular examples. On the other hand, it is not required that the pseudo antipode is anticomultiplicative, that is,* $\Delta \circ \mathcal{S} = (\mathcal{S} \otimes \mathcal{S}) \circ \Delta^{cop}$. *Although this is true in most examples, there are cases where this property is not valid.*

(iv) *In reference (LI, 1998), the author introduced a notion somewhat related to our quantum inverse semigroup, called there as "weak Hopf algebras". This notion of a weak Hopf algebra does not correspond to the usual notion of weak Hopf algebra in the literature (BÖHM; NILL; SZLACHÁNYI, 1999), basically because they were bialgebras, while usual weak Hopf algebras don't satisfy the unitality of the comultiplication nor the multiplicativity of the counit. Despite the fact that the notion of pseudo antipode was introduced there, we must highlight some essential differences between a quantum inverse semigroup and the so called "weak Hopf algebras" (WHA for short). First, a quantum inverse semigroup need not to be unital nor counital, while the WHA are bialgebras, then they are unital and counital, therefore, even the algebra of an inverse semigroup could not be, in general, an example of a WHA. In axiom (QISG3) we demanded the antimultiplicativity of the pseudo antipode, while for WHA this condition was not postulated, but it is asumed in many points in order to obtain relevant results. Finally, for WHA there is no similar to axiom (QISG4).*

(v) *We also acknowledge another similar construction in (AUKHADIEV, 2016) (although it was not so far published elsewhere), also called quantum inverse semigroups. The difference is that the notion of a quantum inverse semigroup given there is a* C^**-algebra with a dense bialgebra with a pseudoantipode satisfying (QISG3). Here we do not demand a quantum inverse semigroup to be unital or counital. Also, the author does not demand any condition similar to our axiom (QISG4).*

3.2 EXAMPLES

Example 3.2.1 *Let S be an inverse semigroup. The algebra*

$$\Bbbk S = \left\{ \sum_{s \in S} a_s \delta_s \mid a_s \in \Bbbk \right\}$$

can be endowed with an structure of a counital quantum inverse semigroup with

$$\Delta(\delta_s) = \delta_s \otimes \delta_s, \qquad \varepsilon(\delta_s) = 1, \qquad \underline{\mathcal{S}}(\delta_s) = \delta_{s^*}.$$

This is a fact because the product from $\Bbbk S$ inherits all the properties from the product in S and

$$\varepsilon \circ \underline{\mathcal{S}}(\delta_s) = \varepsilon(\delta_{s^*}) = 1 = \varepsilon(\delta_s)$$

for every $s \in S$. Also, $\underline{\mathcal{S}}$ is anticomultiplicative:

$$(\underline{\mathcal{S}} \otimes \underline{\mathcal{S}}) \circ \Delta^{cop}(\delta_s) = \underline{\mathcal{S}}(\delta_s) \otimes \underline{\mathcal{S}}(\delta_s) = \delta_{s^*} \otimes \delta_{s^*} = \Delta \circ \underline{\mathcal{S}}(\delta_s)$$

for every $s \in S$.

When S is an inverse monoid, then $\Bbbk S$ is a unital and counital quantum inverse semigroup with $1_{\Bbbk S} = \delta_{1_S}$. The axiom (QISG4) is automatically satisfied, because the algebras generated by the images $Id_{\Bbbk S} * \mathcal{S}$ and $\mathcal{S} * Id_{\Bbbk S}$ both coincide with $\Bbbk E(S)$, which is a commutative algebra.

Example 3.2.2 *An affine inverse semigroup scheme is a functor S from the category of commutative $\Bbbk$-algebras to the category of inverse semigroups whose composition with the forgetful functor $U : \underline{InvSgrp} \to \underline{Set}$ becomes an affine scheme, that is, a representable functor from the category of algebras to the category of sets. Let S be an inverse semigroup scheme and H the commutative algebra which represents it, that is,*

$$S(\_) = Hom_{ComAlg}(H, \_\_).$$

The assumption that $S(A)$ is an inverse semigroup and that for any algbra morphism $\varphi : A \to B$ induces a semigroup morphism $S(\varphi) : S(A) \to S(B)$ leads to the conclusion that the multiplications in each semigroup $S(A)$, define a natural transformation, $m : S \times S \Rightarrow S$. As the functor S is representable, one can write the multiplication as

$$m : Hom_{ComAlg}(H, \_\_) \times Hom_{ComAlg}(H, \_\_) \Rightarrow Hom_{ComAlg}(H, \_\_),$$

or yet, via the canonical natural isomorphism

$$Hom_{ComAlg}(H, \_\_) \times Hom_{ComAlg}(H, \_\_) \cong Hom_{ComAlg}(H \otimes H, \_\_)$$

an associated natural transformation

$$\widetilde{m} : Hom_{ComAlg}(H \otimes H, \_\_) \Rightarrow Hom_{ComAlg}(H, \_\_).$$

By Yoneda's lemma, this natural transformation induces a morphism of algebras

$$\Delta : H \to H \otimes H.$$

Such that, for each algebra A and every pair of algebra morphisms $x, y : H \to A$ we have $x.y = m_A(x, y) = (x \otimes y) \circ \Delta$.

In the same way, the pseudoinverse operation can be viewed as a natural transformation $(\_)^* : S^{op} \Rightarrow S$. Again, by Yoneda's lemma, this natural transformation induces a morphism of algebras (as the algebras are commutative, also an antimorphism of algebras) $\mathcal{S} : H \to H$.

Given a commutative algebra A, the identities $ss^*s = s$ and $s^*ss^* = s^*$ for each $s \in Hom_{ComAlg}(H, A)$ are equivalent to the expressions $Id_H * \mathcal{S} * Id_H = Id_H$ and $\mathcal{S} * Id_H * \mathcal{S} = \mathcal{S}$. Indeed, for any $h \in H$ and for any algebra map $s : H \to A$

$$s(h) = ss^*s(h) = s(h_{(1)})s^*(h_{(2)})s(h_{(3)}) = s(h_{(1)})s(\mathcal{S}(h_{(2)}))s(h_{(3)}) = s(h_{(1)}\mathcal{S}(h_{(2)})h_{(3)}).$$

As this equality is valid for every algebra morphism $s : H \to A$ and for every commutative algebra A, we have

$$h = h_{(1)}\mathcal{S}(h_{(2)})h_{(3)}, \forall h \in H.$$

Finally, axiom (QISG4) is trivially verified because all algebras are commutative, then, for every $h, k \in H$ the elements $Id_H * \mathcal{S}(h)$ and $\mathcal{S} * Id_H(k)$ do commute. Therefore, the algebra H, representing the affine inverse semigroup scheme is a quantum inverse semigroup.

Example 3.2.3 *Given an inverse semigroup S, let H_S be the polynomial algebra generated by all the matrix coordinate functions of isomorphism classes of finite dimensional $\Bbbk$-linear representations π of S, that is*

$$H_S = \Bbbk[\pi_{i,j} \mid \pi : S \to M_n(\Bbbk), \quad 1 \leq i, j \leq n],$$

in which $\pi(s) = \left(\pi_{i,j}(s)\right)_{i,j=1}^{n}$. Define the comultiplication on the generators by

$$\Delta(\pi_{i,j}) = \sum_{k=1}^{n} \pi_{i,k} \otimes \pi_{k,j}$$

and extend to an algebra morphism $\Delta : H_S \to H_S \otimes H_S$ by the universal property of the polynomial algebra. Considering the natural embedding of $H_S \otimes H_S$ as a subalgebra of the algebra of functions from $S \times S$ to $\Bbbk$, the comultiplication can be written in the following way:

$$\Delta(\pi_{i,j})(s, t) = \pi_{i,j}(st) = \sum_{k=1}^{n} \pi_{i,k}(s)\pi_{k,j}(t).$$

Also, one can define the pseudoantipode on the generators as

$$\mathcal{S}(\pi_{i,j})(s) = \pi_{i,j}(s^*), \qquad \forall s \in S,$$

and extend it by the universal property of the polynomial algebra to an algebra morphism $\mathcal{S} : H \to H$ (which is also an anti-algebra morphism because of the commutativity).

It is easy to verify that $(H_S, \Delta, \mathcal{S})$ is a unital quantum inverse semigroup. The unit of the polynomial algebra can be seen as the constant function $1_{H_S} : S \to \Bbbk = M_1(\Bbbk)$ which sends every element of the semigroup S into $1_{\Bbbk}$, and the pseudoantipode $\mathcal{S}$, as algebra morphism, *naturally sends 1_{H_S} to 1_{H_S}. It is enough to prove Axiom (QISG3) for the generators, then taking a generator $\pi_{i,j}$ with $1 \leq i,j \leq n$ and any element $s \in S$, we have*

$$\begin{aligned} Id_H * \mathcal{S} * Id_H(\pi_{i,j})(s) &= \sum_{k,l=1}^{n} \pi_{i,k}(s)\mathcal{S}(\pi_{k,l})(s)\pi_{l,j}(s) \\ &= \sum_{k,l=1}^{n} \pi_{i,k}(s)\pi_{k,l}(s^*)\pi_{l,j}(s) \\ &= \pi_{i,j}(ss^*s) \\ &= \pi_{i,j}(s). \end{aligned}$$

Therefore $Id_H * \mathcal{S} * Id_H = Id_H$. Similar reasoning for $\mathcal{S} * Id_H * \mathcal{S} = \mathcal{S}$. Axiom (QISG4) is satisfied because the algebra H_S is commutative.

Example 3.2.4 *Every Hopf algebra $(H,\mu,\eta,\Delta,\varepsilon,S)$ is a unital and counital quantum inverse semigroup. The axiom (QISG4) follows from the antipode axiom in the Hopf algebra, then the images of $Id_H * S = S * Id_H = \eta \circ \varepsilon$ are contained in the commutative subalgebra $\mathbb{k} \cdot 1_H$.*

Example 3.2.5 *Every weak Hopf algebra is a quantum inverse semigroup. A weak Hopf algebra is a sextuple $(H,\mu,\eta,\Delta,\varepsilon,S)$ such that (H,μ,η) is a unital algebra and (H,Δ,ε) is a coalgebra. Moreover, the comultiplication $\Delta : H \to H \otimes H$ is multiplicative and satisfies*

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) = (\Delta \otimes Id) \circ \Delta(1),$$

which can be rewritten as

$$1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1')} \otimes 1_{(1)}1_{(2')} \otimes 1_{(2)} = 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)}$$

and the counit $\varepsilon : H \to \mathbb{k}$ satisfies $\varepsilon(hkl) = \varepsilon(hk_{(1)})\varepsilon(k_{(2)}l) = \varepsilon(hk_{(2)})\varepsilon(k_{(1)}l)$. Lastly, the antipode $S : H \to H$ in a weak Hopf algebra satifies the following axioms:

$$\begin{aligned} h_{(1)}S(h_{(2)}) &= \varepsilon_t(h) = \varepsilon(1_{(1)}h)1_{(2)}, \\ S(h_{(1)})h_{(2)} &= \varepsilon_s(h) = 1_{(1)}\varepsilon(h1_{(2)}), \\ S(h_{(1)})h_{(2)}S(h_{(3)}) &= S(h). \end{aligned}$$

With these axioms, we have that H imediatly satisfies (QISG1) *and* (QISG2)*. H also satisfies* (QISG4)*: for every $h,k \in H$,*

$$\begin{aligned} h_{(1)}S(h_{(2)})S(k_{(1)})k_{(2)} &= \varepsilon(1_{(1)}h)1_{(2)}1_{(1')}\varepsilon(k1_{(2')}) \\ &= \varepsilon(1_{(1)}h)1_{(1')}1_{(2)}\varepsilon(k1_{(2')}) \\ &= S(k_{(1)})k_{(2)}h_{(1)}S(h_{(2)}). \end{aligned} \tag{6}$$

Besides that, S is antimultiplicative, because

$$\begin{aligned}
S(hk) &= S\left(h_{(1)}k_{(1)}\right) h_{(2)}k_{(2)} S\left(h_{(3)}k_{(3)}\right) \\
&= 1_{(1)}\,\varepsilon\left(h_{(1)}k_{(1)}1_{(2)}\right) S\left(h_{(2)}k_{(2)}\right) \\
&= 1_{(1)}\,\varepsilon\left(h_{(1)}k_{(2)}\right)\varepsilon\left(k_{(1)}1_{(2)}\right) S\left(h_{(2)}k_{(3)}\right) \\
&= S\left(k_{(1)}\right) k_{(2)}\,\varepsilon\left(h_{(1)}k_{(3)}\right) S\left(h_{(2)}k_{(4)}\right) \\
&= S\left(k_{(1)}\right) 1_{(1)}k_{(2)}\,\varepsilon\left(h_{(1)}1_{(2)}k_{(3)}\right) S\left(h_{(2)}k_{(4)}\right) \\
&= S\left(k_{(1)}\right) 1_{(1)}k_{(2)}\,\varepsilon\left(h_{(1)}1_{(2)}1'_{(1)}\right)\varepsilon\left(1'_{(2)}k_{(3)}\right) S\left(h_{(2)}k_{(4)}\right) \\
&= S\left(k_{(1)}\right) 1'_{(1)}k_{(2)}\,\varepsilon\left(h_{(1)}1_{(1)}1'_{(2)}\right)\varepsilon\left(1_{(2)}k_{(3)}\right) S\left(h_{(2)}k_{(4)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}1_{(1)}\right) h_{(2)}1_{(2)}k_{(2)}\,\varepsilon\left(1_{(3)}k_{(3)}\right) S\left(h_{(3)}k_{(4)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}1_{(1)}\right) h_{(2)}1_{(2)}1'_{(1)}k_{(2)}\,\varepsilon\left(1'_{(2)}k_{(3)}\right) S\left(h_{(3)}k_{(4)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right) h_{(2)}k_{(2)} S\left(h_{(3)}k_{(3)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right)\varepsilon\left(1_{(1)}h_{(2)}k_{(2)}\right) 1_{(2)} \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right)\varepsilon\left(1_{(1)}h_{(3)}\right)\varepsilon\left(h_{(2)}k_{(2)}\right) 1_{(2)} \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right)\varepsilon\left(h_{(2)}k_{(2)}\right) h_{(3)} S\left(h_{(4)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right)\varepsilon\left(h_{(2)}1_{(1)}k_{(2)}\right) h_{(3)}1_{(2)} S\left(h_{(4)}\right) \\
&= S\left(k_{(1)}\right)\varepsilon\left(h_{(2)}1'_{(1)}\right)\varepsilon\left(1'_{(2)}1_{(1)}k_{(2)}\right) S\left(h_{(1)}\right) h_{(3)}1_{(2)} S\left(h_{(4)}\right) \\
&= S\left(k_{(1)}\right)\varepsilon\left(h_{(2)}1_{(1)}\right)\varepsilon\left(1'_{(1)}1_{(2)}k_{(2)}\right) S\left(h_{(1)}\right) h_{(3)}1'_{(2)} S\left(h_{(4)}\right) \\
&= S\left(k_{(1)}\right)\varepsilon\left(h_{(2)}1_{(1)}\right) S\left(h_{(1)}\right) h_{(3)}1_{(2)}k_{(2)} S\left(1_{(3)}k_{(3)}\right) S\left(h_{(4)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right)\varepsilon\left(h_{(2)}1_{(1)}\right) h_{(3)}1_{(2)}1'_{(1)}k_{(2)} S\left(1'_{(2)}k_{(3)}\right) S\left(h_{(4)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right) h_{(2)}1_{(1)}k_{(2)} S\left(1_{(2)}k_{(3)}\right) S\left(h_{(3)}\right) \\
&= S\left(k_{(1)}\right) S\left(h_{(1)}\right) h_{(2)}k_{(2)} S\left(k_{(3)}\right) S\left(h_{(3)}\right) \\
&\overset{(*)}{=} S\left(k_{(1)}\right) k_{(2)} S\left(k_{(3)}\right) S\left(h_{(1)}\right) h_{(2)} S\left(h_{(3)}\right) \\
&= S(k)S(h)
\end{aligned}$$

for every $h,k \in H$, where $(*)$ comes from the expression (6). *Finally,*

$$h_{(1)}S(h_{(2)})h_{(3)} = \varepsilon(1_{(1)}h_{(1)})1_{(2)}h_{(2)} = \varepsilon(h_{(1)})h_{(2)} = h$$

for any $h \in H$ and $S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$ for any $h \in H$ by definition. Therefore H satisfies (QISG3) *and is a quantum inverse semigroup.*

Moreover, H is a unital and counital quantum inverse semigroup, because

$$\begin{aligned}
S(1) = S(1_{(1)})\,1_{(2)}\,S(1_{(3)}) &= S(1_{(1)})\,\varepsilon(1_{(1')}1_{(2)})\,1_{(2')} \\
&= S(1_{(1)})\,\varepsilon(1_{(2)})\,1_{(3)} \\
&= S(1_{(1)})\,1_{(2)} \\
&= 1_{(1)}\,\varepsilon(1_{(2)}) \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}\varepsilon \circ S(h) &= \varepsilon\left(S(h_{(1)})h_{(2)}S(h_{(3)})\right)\\ &= \varepsilon\left(S(h_{(1)})h_{(2)}\right)\,\varepsilon\left(h_{(3)}S(h_{(4)})\right)\\ &= \varepsilon\left(1_{(1)}\,\varepsilon(h_{(1)}1_{(2)})\right)\,\varepsilon\left(\varepsilon(1_{(1')}h_{(2)})\,1_{(2')}\right)\\ &= \varepsilon(1_{(2')})\,\varepsilon(1_{(1')}h_{(2)})\,\varepsilon(h_{(1)}1_{(2)})\,\varepsilon(1_{(1)})\\ &= \varepsilon(h)\end{aligned}$$

for all $h \in H$.

Example 3.2.6 *A nontrivial example of a quantum inverse semigroup was inspired in the work of Theodor Banica and Adam Skalski (BANICA; SKALSKI, 2015) on Quantum Permutation Groups. Consider the polynomial $\Bbbk$-algebra generated by the set $\{u_{ij} \mid 1 \leq i,j, \leq n\}$ and then consider the quotient*

$$H = \Bbbk[u_{ij}|1 \leq i,j \leq n]/\mathcal{I},$$

in which $\mathcal{I}$ is the ideal generated by elements of the type

1. $u_{ij}u_{ik} - \delta_{j,k}u_{ij}$,
2. $u_{ij}u_{kj} - \delta_{i,k}u_{ij}$.

Defining the function

$$\begin{array}{rccc}\widetilde{\Delta}: & \{u_{ij}\}_{1\leq i,j\leq n} & \longrightarrow & H\otimes H \\ & u_{ij} & \longmapsto & \sum_{k=1}^{n} u_{ik}\otimes u_{kj}\end{array},$$

one can lift it to a morphism of algebras $\overline{\Delta} : \Bbbk[u_{ij}|1 \leq i,j \leq n] \to H \otimes H$ doing the same on generators. We need to check that $\overline{\Delta}(\mathcal{I}) \subseteq \mathcal{I} \otimes \Bbbk[u_{ij}|1 \leq i,j \leq n] + \Bbbk[u_{ij}|1 \leq i,j \leq n] \otimes \mathcal{I}$. Indeed,

$$\begin{aligned}\overline{\Delta}(u_{ij}u_{ik} - \delta_{j,k}u_{ij}) &= \sum_{p,q=1}^{n} u_{ip}u_{iq}\otimes u_{pj}u_{qk} - \sum_{p=1}^{n}\delta_{j,k}u_{ip}\otimes u_{pj}\\ &= \sum_{p,q=1}^{n} u_{ip}u_{iq}\otimes u_{pj}u_{qk} - \sum_{p,q=1}^{n}\delta_{p,q}u_{ip}\otimes u_{pj}u_{qk}\\ &\qquad + \sum_{p,q=1}^{n}\delta_{p,q}u_{ip}\otimes u_{pj}u_{qk} - \sum_{p=1}^{n}\delta_{j,k}u_{ip}\otimes u_{pj}\\ &= \sum_{p,q=1}^{n}(u_{ip}u_{iq} - \delta_{p,q}u_{ip})\otimes u_{pj}u_{qk} + \sum_{p=1}^{n} u_{ip}\otimes(u_{pj}u_{pk} - \delta_{j,k}u_{pj}),\end{aligned}$$

analogous for $\overline{\Delta}(u_{ij}u_{kj} - \delta_{i,k}u_{ij})$. Therefore, there is a well defined algebra map $\Delta : H \to H \otimes H$ defined on generators as $\Delta u_{ij} = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$.

Also, one can define a function

$$\begin{array}{rccc}\widetilde{S}: & \{u_{ij}\}_{1\leq i,j\leq n} & \longrightarrow & H = H^{op} \\ & u_{ij} & \longmapsto & u_{ji}\end{array},$$

also, lifting to an algebra morphism $\overline{S} : \Bbbk[u_{ij}|1 \leq i,j \leq n] \to H^{op}$. It is easy to see that $\overline{S}(\mathcal{I}) \subseteq \mathcal{I}$, then we have a well defined algebra map $S : H \to H = H^{op}$.

Let us verify that (H, Δ, S) defined as above is indeed a quantum inverse semigroup. First note that

$$Id_H * S(u_{ij}) = \sum_{k=1}^{n} u_{ik} S(u_{kj}) = \sum_{k=1}^{n} u_{ik} u_{jk} = \delta_{i,j} \sum_{k=1}^{n} u_{ik}$$

and

$$S * Id_H(u_{ij}) = \sum_{k=1}^{n} S(u_{ik}) u_{kj} = \sum_{k=1}^{n} u_{ki} u_{kj} = \delta_{i,j} \sum_{k=1}^{n} u_{kj}.$$

Then, we have

$$\begin{aligned}
Id_H * S * Id_H(u_{i_1 j_1} \dots u_{i_N j_N}) &= \sum_{k_1, l_1 = 1}^{n} \dots \sum_{k_N, l_N = 1}^{n} u_{i_1 k_1} \dots u_{i_N k_N} S(u_{k_1 l_1} \dots u_{k_N l_N}) u_{l_1 j_1} \dots u_{l_N j_N} \\
&= \sum_{k_1, l_1 = 1}^{n} \dots \sum_{k_N, l_N = 1}^{n} u_{i_1 k_1} \dots u_{i_N k_N} u_{l_N k_N} \dots u_{l_1 k_1} u_{l_1 j_1} \dots u_{l_N j_N} \\
&= \sum_{k_1, l_1 = 1}^{n} \dots \sum_{k_N, l_N = 1}^{n} u_{i_1 k_1} \dots u_{i_N k_N} \delta_{i_1, l_1} \dots \delta_{i_N, l_N} u_{l_1 j_1} \dots u_{l_N j_N} \\
&= \sum_{k_1 = 1}^{n} \dots \sum_{k_N = 1}^{n} u_{i_1 k_1} \dots u_{i_N k_N} u_{i_1 j_1} \dots u_{i_N j_N} \\
&= \sum_{k_1 = 1}^{n} \dots \sum_{k_N = 1}^{n} \delta_{j_1 k_1} \dots \delta_{j_N k_N} u_{i_1 j_1} \dots u_{i_N j_N} \\
&= u_{i_1 j_1} \dots u_{i_N j_N}.
\end{aligned}$$

leading to $Id_H * S * Id_H = Id_H$ and, analogously, $S * Id_H * S = S$. The elements of the form $I * S(h)$, naturally commute wih elements of the form $S * Id_H(k)$ due to the commutativity of H, satisfying (QISG4). Therefore (H, Δ, S) is a quantum inverse semigroup.

Moreover, this quantum inverse semigroup is unital and counital: first, it is unital because H is a unital algebra and, by construction, $S(1_H) = 1_H$. Also, it is counital because one can define a function $\widetilde{\varepsilon} : \{u_{ij}\}_{1 \leq i,j \leq n} \to \Bbbk$ given by $\widetilde{\varepsilon}(u_{ij}) = \delta_{i,j}$, this can be lifted to an algebra morphism $\overline{\varepsilon} : \Bbbk[u_{ij} | 1 \leq i,j \leq n] \to \Bbbk$ doing the same. It is straightforward to verify that $\overline{\varepsilon}(\mathcal{I}) = 0$, therefore, there exists an algebra morphism $\varepsilon : H \to H$, making, in particular, H to be a commutative bialgebra. It is also easy to check that $S \circ \varepsilon = \varepsilon$. Note that H is an example of a quantum inverse semigroup which is not a Hopf algebra, neither a weak Hopf algebra, nor an inverse semigroup algebra.

3.2.1 Partial representations

Definition 3.2.7 *Let H be a Hopf $\Bbbk$-algebra, and let B be a unital $\Bbbk$-algebra. A partial representation of H in B is a linear map $\pi : H \to B$ such that*

(PR1) $\pi(1_H) = 1_B$,

(PR2) $\pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \pi(hk_{(1)})\pi(S(k_{(2)}))$, *for every $h,k \in H$.*

(PR3) $\pi(h_{(1)})\pi(S(h_{(2)}))\pi(k) = \pi(h_{(1)})\pi(S(h_{(2)})k)$, *for every $h,k \in H$.*

(PR4) $\pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \pi(hS(k_{(1)}))\pi(k_{(2)})$, *for every $h,k \in H$.*

(PR5) $\pi(S(h_{(1)}))\pi(h_{(2)})\pi(k) = \pi(S(h_{(1)}))\pi(h_{(2)}k)$, *for every* $h,k \in H$.

Definition 3.2.8 *(ALVES; BATISTA; VERCRUYSSE, 2015) Let H be a Hopf algebra and let $T(H)$ be the tensor algebra of the vector space H. The partial Hopf algebra H_{par} is the quotient of $T(H)$ by the ideal I generated by elements of the form*

(1) $1_H - 1_{T(H)}$;

(2) $h \otimes k_{(1)} \otimes S(k_{(2)}) - hk_{(1)} \otimes S(k_{(2)})$, *for all* $h,k \in H$;

(3) $h_{(1)} \otimes S(h_{(2)}) \otimes k - h_{(1)} \otimes S(h_{(2)})k$, *for all* $h,k \in H$;

(4) $h \otimes S(k_{(1)}) \otimes k_{(2)} - hS(k_{(1)}) \otimes k_{(2)}$, *for all* $h,k \in H$;

(5) $S(h_{(1)}) \otimes h_{(2)} \otimes k - S(h_{(1)}) \otimes h_{(2)}k$, *for all* $h,k \in H$.

Denoting the class of $h \in H$ in H_{par} by $[h]$, it is easy to see that the map

$$\begin{array}{rccc} [\_]: & H & \to & H_{par} \\ & h & \mapsto & [h] \end{array}$$

is a partial representation of the Hopf algebra H on H_{par}.

The partial Hopf algebra H_{par} has the following universal property: for every partial representation $\pi : H \to B$, there is a unique morphism of algebras $\overline{\pi} : H_{par} \to B$ such that $\pi = \overline{\pi} \circ [\_]$. In (ALVES; BATISTA; VERCRUYSSE, 2015), it was shown that H_{par} has the structure of a Hopf algebroid over the base algebra

$$A_{par}(H) = \langle \varepsilon_h = [h_{(1)}][S(h_{(2)})] \mid h \in H \rangle.$$

For H being a cocommutative Hopf algebra, things become much simpler. For example, in order to verify whether a linear map $\pi : H \to B$ is a partial representation, one needs only to check axioms (PR1) (PR2) and (PR5). In this case, the following result is valid for the universal algebra H_{par}.

Theorem 3.2.9 *Let H be a cocommutative Hopf algebra over a field $\Bbbk$. Then the partial Hopf algebra H_{par} has the structure of a unital quantum inverse semigroup.*

Proof. First, one needs to define a comultiplication $\Delta : H_{par} \to H_{par} \otimes H_{par}$ which is multiplicative. For this, define the linear map

$$\begin{array}{rccc} \delta: & H & \to & H_{par} \otimes H_{par} \\ & h & \mapsto & [h_{(1)}] \otimes [h_{(2)}] \end{array}.$$

One can prove that the map δ is a partial representation of H. For example, let us verify axiom (PR2):

$$\begin{aligned} \delta(h)\delta(k_{(1)})\delta(S(k_{(2)})) &= [h_{(1)}][k_{(1)}][S(k_{(4)})] \otimes [h_{(2)}][k_{(2)}][S(k_{(3)})] \\ &= [h_{(1)}][k_{(1)}][S(k_{(4)})] \otimes [h_{(2)}k_{(2)}][S(k_{(3)})] \\ &= [h_{(1)}][k_{(1)}][S(k_{(2)})] \otimes [h_{(2)}k_{(3)}][S(k_{(4)})] \\ &= [h_{(1)}k_{(1)}][S(k_{(2)})] \otimes [h_{(2)}k_{(3)}][S(k_{(4)})] \\ &= [h_{(1)}k_{(1)}][S(k_{(4)})] \otimes [h_{(2)}k_{(2)}][S(k_{(3)})] \\ &= \delta(hk_{(1)})\delta(S(k_{(2)})). \end{aligned}$$

Therefore, there exists a unique algebra map $\Delta : H_{par} \to H_{par} \otimes H_{par}$ given by

$$\Delta([h^1]\cdots[h^n]) = [h^1_{(1)}]\cdots[h^n_{(1)}] \otimes [h^1_{(2)}]\cdots[h^n_{(2)}].$$

In order to define the pseudo antipode, consider the linear map

$$\begin{array}{rccc} \widetilde{S}: & H & \to & H_{par}^{op} \\ & h & \mapsto & [S(h)] \end{array}.$$

For every $h,k \in H$, we have

$$\begin{aligned} \widetilde{S}(h)\cdot_{op}\widetilde{S}(k_{(1)})\cdot_{op}\widetilde{S}(S(k_{(2)})) &= [S(h)]\cdot_{op}[S(k_{(1)})]\cdot_{op}[S(S(k_{(2)}))] \\ &= [S(S(k)_{(1)})][S(k)_{(2)}][S(h)] \\ &= [S(S(k)_{(1)})][S(k)_{(2)}S(h)] \\ &= [S(S(k_{(2)}))][S(hk_{(1)})] \\ &= [S(hk_{(1)})]\cdot_{op}[S(S(k_{(2)}))] \\ &= \widetilde{S}(hk_{(1)})\cdot_{op}\widetilde{S}(S(k_{(2)})), \end{aligned}$$

and the other axioms of partial representations are easily verified in the same way. Therefore $\widetilde{S}$ is a partial representation of H in H_{par}^{op}, inducing a morphism of algebras $\mathcal{S} : H_{par} \to H_{par}^{op}$, or equivalently, an antimorphism of algebras $\mathcal{S} : H_{par} \to H_{par}$ given by

$$\mathcal{S}([h^1]\cdots[h^n]) = [S(h^n)]\cdots[S(h^1)].$$

In order to verify the identities $Id_{H_{par}} * \mathcal{S} * Id_{H_{par}} = Id_{H_{par}}$ and $\mathcal{S} * Id_{H_{par}} * \mathcal{S} = S$, first note that, for any $h,k \in H$

$$\begin{aligned} [h]\,\varepsilon_k &= [h][k_{(1)}][S(k_{(2)})] = [hk_{(1)}][S(k_{(2)})] \\ &= [h_{(1)}k_{(1)}][S(h_{(2)}k_{(2)})][h_{(3)}k_{(3)}][S(k_{(4)})] \\ &= [h_{(1)}k_{(1)}][S(h_{(2)}k_{(2)})][h_{(3)}k_{(3)}S(k_{(4)})] \\ &= [h_{(1)}k_{(1)}][S(h_{(2)}k_{(2)})][h_{(3)}] \\ &= \varepsilon_{h_{(1)}k}[h_{(2)}]. \end{aligned} \tag{7}$$

This implies, in particular, that the elements ε_h do commute when H is cocommutative (ALVES; BATISTA; VERCRUYSSE, 2015). Indeed

$$\begin{aligned} \varepsilon_h\varepsilon_k &= [h_{(1)}][S(h_{(2)})]\,\varepsilon_k \\ &= [h_{(1)}]\varepsilon_{S(h_{(3)})k}[S(h_{(2)})] \\ &= \varepsilon_{h_{(1)}S(h_{(4)})k}[h_{(2)}][S(h_{(3)})] \\ &= \varepsilon_{h_{(1)}S(h_{(2)})k}[h_{(3)}][S(h_{(4)})] \\ &= \varepsilon_k[h_{(1)}][S(h_{(2)})] \\ &= \varepsilon_k\varepsilon_h. \end{aligned}$$

Let us prove the identity $Id_{H_{par}} * \mathcal{S} * Id_{H_{par}}([h^1]\cdots[h^n]) = [h^1]\cdots[h^n]$ by induction on $n \geq 1$. For $n = 1$, we have

$$Id_{H_{par}} * \mathcal{S} * Id_{H_{par}}([h]) = [h_{(1)}][S(h_{(2)})][h_{(3)}] = [h_{(1)}S(h_{(2)})][h_{(3)}] = [h].$$

Suppose valid for n, then

$$\begin{aligned}
Id_{H_{par}} * \mathcal{S} * Id_{H_{par}}([h^1]\cdots[h^{n+1}]) &= [h^1_{(1)}]\cdots[h^{n+1}_{(1)}][S(h^{n+1}_{(2)})]\cdots[S(h^1_{(2)})][h^1_{(3)}]\cdots[h^{n+1}_{(3)}] \\
&= [h^1_{(1)}]\cdots[h^n_{(1)}]\varepsilon_{h^{n+1}_{(1)}}[S(h^n_{(2)})]\cdots[S(h^1_{(2)})][h^1_{(3)}]\cdots[h^n_{(3)}][h^{n+1}_{(3)}] \\
&\overset{(*)}{=} \varepsilon_{h^1_{(1)}\cdots h^n_{(1)}h^{n+1}_{(1)}}[h^1_{(2)}]\cdots[h^n_{(2)}][S(h^n_{(3)})]\cdots[S(h^1_{(3)})][h^1_{(4)}]\cdots[h^n_{(4)}][h^{n+1}_{(3)}] \\
&= \varepsilon_{h^1_{(1)}\cdots h^n_{(1)}h^{n+1}_{(1)}}[h^1_{(2)}]\cdots[h^n_{(2)}][h^{n+1}_{(2)}] \\
&= [h^1]\cdots[h^n]\varepsilon_{h^{n+1}_{(1)}}[h^{n+1}_{(2)}] \\
&= [h^1]\cdots[h^n][h^{n+1}_{(1)}][S(h^{n+1}_{(2)})][h^{n+1}_{(3)}] \\
&= [h^1]\cdots[h^{n+1}]
\end{aligned}$$

where $(*)$ comes from the expression (7). For the identity $\mathcal{S} * Id_{H_{par}} * \mathcal{S} = \mathcal{S}$, consider $[h^1]\cdots[h^n] \in H_{par}$ and use the fact that S is involutive, then

$$\begin{aligned}
&\mathcal{S} * Id_{H_{par}} * \mathcal{S}([h^1]\cdots[h^n]) \\
&= [S(h^n_{(1)})]\cdots[S(h^1_{(1)})][h^1_{(2)}]\cdots[h^n_{(2)}][S(h^n_{(3)})]\cdots[S(h^1_{(3)})] \\
&= [S(h^n_{(3)})]\cdots[S(h^1_{(3)})][S(S(h^1_{(2)}))]\cdots[S(S(h^n_{(2)}))][S(h^n_{(1)})]\cdots[S(h^1_{(1)})] \\
&= [S(h^n)_{(1)}]\cdots[S(h^1)_{(1)}][S(S(h^1)_{(2)})]\cdots[S(S(h^n)_{(2)})][S(h^n)_{(3)}]\cdots[S(h^1)_{(3)}] \\
&= [S(h^n)]\cdots[S(h^1)] \\
&= \mathcal{S}([h^1]\cdots[h^n]).
\end{aligned}$$

Finally, in order to verify Axiom (QISG4), note that

$$\begin{aligned}
I_{H_{par}} * \mathcal{S}([h^1]\cdots[h^n]) &= [h^1_{(1)}]\cdots[h^n_{(1)}][S(h^n_{(2)})]\cdots[S(h^1_{(2)})] \\
&= [h^1_{(1)}]\cdots[h^{n-1}_{(1)}]\varepsilon_{h^n}[S(h^{n-1}_{(2)})]\cdots[S(h^1_{(2)})] \\
&= \varepsilon_{h^1_{(1)}\cdots h^{n-1}_{(1)}h^n}[h^1_{(2)}]\cdots[h^{n-1}_{(2)}][S(h^{n-1}_{(3)})]\cdots[S(h^1_{(3)})] \\
&= \varepsilon_{h^1_{(1)}\cdots h^{n-1}_{(1)}h^n}[h^1_{(2)}]\cdots[h^{n-2}_{(2)}]\varepsilon_{h^{n-1}_{(2)}}[S(h^{n-2}_{(3)})]\cdots[S(h^1_{(3)})] \\
&= \varepsilon_{h^1_{(1)}\cdots h^{n-1}_{(1)}h^n}\varepsilon_{h^1_{(2)}\cdots h^{n-2}_{(2)}h^{n-1}_{(2)}}[h^1_{(3)}]\cdots[h^{n-2}_{(3)}][S(h^{n-2}_{(4)})]\cdots[S(h^1_{(4)})] \\
&\ \vdots \\
&= \varepsilon_{h^1_{(1)}\cdots h^{n-1}_{(1)}h^n}\varepsilon_{h^1_{(2)}\cdots h^{n-2}_{(2)}h^{n-1}_{(2)}}\cdots\varepsilon_{h^1_{(n)}},
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
\mathcal{S} * I_{H_{par}}([h^1]\cdots[h^n]) &= [S(h^n_{(1)})]\cdots[S(h^1_{(1)})][h^1_{(2)}]\cdots[h^n_{(2)}] \\
&= [S(h^n_{(2)})]\cdots[S(h^1_{(2)})][S(S(h^1_{(1)}))]\cdots[S(S(h^n_{(1)}))] \\
&= [S(h^n)_{(1)}]\cdots[S(h^1)_{(1)}][S(S(h^1)_{(2)})]\cdots[S(S(h^n)_{(2)})] \\
&= \varepsilon_{S(h^n)_{(1)}\cdots S(h^2)_{(1)}S(h^1)}\varepsilon_{S(h^n)_{(2)}\cdots S(h^2)_{(2)}}\cdots\varepsilon_{S(h^n)_{(n)}} \\
&= \varepsilon_{S(h^n_{(n)})\cdots S(h^2_{(n)})S(h^1)}\varepsilon_{S(h^n_{(n-1)})\cdots S(h^2_{(n-1)})}\cdots\varepsilon_{S(h^n_{(1)})}.
\end{aligned}$$

As both expressions can be written in terms of combinations of products of elements ε_x, for $x \in H$, then they commute among themselves. Therefore, for a cocommutative Hopf algebra H, the universal Hopf algebra H_{par} is a quantum inverse semigroup. □

3.2.2 Hopf Categories

Hopf categories were introduced in (BATISTA; CAENEPEEL; VERCRUYSSE, 2016) in the context of enriched categories over a strict braided monoidal category $\mathcal{V}$. In this section we will consider the case of $\mathcal{V} = {}_{\mathbb{K}}\mathcal{M}$, the symmetric monoidal category of left $\mathbb{K}$-modules over a commutative ring $\mathbb{K}$.

Let Coalg($\mathbb{K}$) be the category of $\mathbb{K}$-coalgebras. This is a monoidal category with respect to the tensor product of coalgebras, with unit given by the trivial coalgebra $\mathbb{K}$. Hence we may consider enriched categories over Coalg($\mathbb{K}$), or Coalg($\mathbb{K}$)-categories.

Unraveling the definition, a (small) Coalg($\mathbb{K}$)-category H over the set X consists of a family $\{H_{x,y}\}_{x,y\in X}$ of $\mathbb{K}$-coalgebras, with structure morphisms

$$\Delta_{x,y} : H_{x,y} \to H_{x,y} \otimes H_{x,y}, \quad \varepsilon_{x,y} : H_{x,y} \to \mathbb{K}, \tag{8}$$

plus $\mathbb{K}$-linear mappings $\mu_{x,y,z} : H_{x,y} \otimes H_{y,z} \to H_{x,z}$ and $\eta_x : \mathbb{K} \to H_{x,x}$ such that

$$\mu_{x,y,t} \circ (H_{x,y} \otimes \mu_{y,z,t}) = \mu_{x,z,t} \circ (\mu_{x,y,z} \otimes H_{z,t}); \tag{9}$$

$$\mu_{x,x,y} \circ (\eta_x \otimes H_{x,y}) = H_{x,y} = \mu_{x,y,y} \circ (H_{x,y} \otimes \eta_y). \tag{10}$$

Moreover, the coalgebra structure and the multiplications $\mu_{x,y}$ and unit mappings η_x are required to be compatible in the sense of the following equalities: first, Δ is compatible with multiplications and unit mappings by

$$\Delta_{x,z} \circ \mu_{x,y,z} = (\mu_{x,y,z} \otimes \mu_{x,y,z}) \circ (H_{x,y} \otimes \tau_{H_{x,y},H_{y,z}} \otimes H_{y,z}) \circ (\Delta_{x,y} \otimes \Delta_{y,z}), \tag{11}$$

$$\Delta_{x,x} \circ \eta_x = \eta_x \otimes \eta_x, \tag{12}$$

where $\tau_{H_{x,y},H_{y,z}}$ is the twist map

$$\tau_{H_{x,y},H_{y,z}} : H_{x,y} \otimes H_{y,z} \to H_{y,z} \otimes H_{x,y}, \quad h \otimes k \mapsto k \otimes h;$$

the equalities respective to the counit mappings are

$$\varepsilon_{x,y} \otimes \varepsilon_{y,z} = \varepsilon_{x,z} \circ \mu_{x,y,z}, \tag{13}$$

$$\varepsilon_{x,x} \circ \eta_x = \mathbb{K}. \tag{14}$$

So let H be a Coalg($\mathbb{K}$)-category and let

$$\text{alg}(H) = \oplus_{x,y\in X} H_{x,y}.$$

Since alg(H) is a direct sum of coalgebras it has a canonical coalgebra structure as follows, where "$a_{x,y}$" indicates an element of the component $H_{x,y}$:

- $\Delta : \text{alg}(H) \to \text{alg}(H) \otimes \text{alg}(H)$ defined by $\Delta(a_{x,y}) = \Delta_{x,y}(a_{x,y})$;
- $\varepsilon : \text{alg}(H) \to \mathbb{K}$ defined by $\varepsilon(a_{x,y}) = \varepsilon_{x,y}(a_{x,y})$.

We also can define a product on alg(H) by

- $\mu : \text{alg}(H) \otimes \text{alg}(H) \to \text{alg}(H)$,

$$\begin{aligned}\mu(a_{x,y},b_{y,z}) &= \mu_{x,y,z}(a_{x,y},b_{y,z})\\ \mu(a_{x,y},b_{w,z}) &= 0 \qquad\qquad \text{whenever } y\neq w.\end{aligned}$$

The triple $\mathrm{alg}(H)=(\mathrm{alg}(H),\mu,\Delta)$ satisfies conditions (QISG1) and (QISG2). In fact, it follows from equalitities (9)-(14) that Δ and ε are multiplicative, and also that $\mathrm{alg}(H)$ is an algebra, which is unital if and only if X is finite. Moreover, in any case $\mathrm{alg}(H)$ is at least an idempotent ring since it has a system of local units: the idempotents $\eta_x(1)$ commute amongst themselves and the set of finite sums $\eta_{x_1}(1)+\cdots+\eta_{x_n}(1)$, where $n\geq 1$ and the elements $x_1,x_2,\cdots,x_n$ are distinct, is a system of local units for $\mathrm{alg}(H)$.

A **Hopf $\mathbb{K}$-category** is a $\mathrm{Coalg}(\mathbb{K})$-category H with an antipode which, in this context, is a family of $\mathbb{K}$-linear maps $S_{x,y}:H_{x,y}\to H_{y,x}$ such that

$$\mu_{x,y,x}\circ(H_{x,y}\otimes S_{x,y})\circ\Delta_{x,y} = \eta_x\circ\varepsilon_{x,y}:\ H_{x,y}\to H_{x,x}; \tag{15}$$

$$\mu_{y,x,y}\circ(S_{x,y}\otimes H_{x,y})\circ\Delta_{x,y} = \eta_y\circ\varepsilon_{x,y}:\ H_{x,y}\to H_{y,y}, \tag{16}$$

for all $x,y\in X$. This family induces a $\mathbb{K}$-linear map $S:\mathrm{alg}(H)\to\mathrm{alg}(H)$ which satisfies, in Sweedler notation, the equalities

$$(h_{x,y})_{(1)}S((h_{x,y})_{(2)}) = \varepsilon_{x,y}(h_{x,y})\eta_y(1), \tag{17}$$

$$S((h_{x,y})_{(1)})(h_{x,y})_{(2)} = \varepsilon_{x,y}(h_{x,y})\eta_x(1). \tag{18}$$

Let $S:\mathrm{alg}(H)\to\mathrm{alg}(H)$ be the $\mathbb{K}$-linear map induced by the family $(S_{x,y})_{x,y\in X}$. Then

$$\begin{aligned}(Id*S*Id)(h_{x,y}) &= (h_{x,y})_{(1)}S((h_{x,y})_{(2)})(h_{x,y})_{(3)}\\ &= \varepsilon_{x,y}((h_{x,y})_{(1)})\eta_y(1)(h_{x,y})_{(2)} = h_{x,y}\end{aligned}$$

and

$$\begin{aligned}(S*Id*S)(h_{x,y}) &= S((h_{x,y})_{(1)})(h_{x,y})_{(2)}S((h_{x,y})_{(3)})\\ &= \varepsilon_{x,y}((h_{x,y})_{(1)})\eta_x(1)(h_{x,y})_{(2)} = h_{x,y}\end{aligned}$$

In (BATISTA; CAENEPEEL; VERCRUYSSE, 2016, Lemma 3.6) it is proved that

$$S_{x,z}\circ\mu_{x,y,z} = \mu_{z,y,x}\circ(S_{y,z}\otimes S_{x,y})\circ\tau_{H_{x,y},H_{y,z}} \tag{19}$$

$$\Delta_{y,x}\circ S_{x,y} = \tau_{H_{y,x},H_{y,x}}\circ(S_{x,y}\otimes S_{x,y})\circ\Delta_{x,y}. \tag{20}$$

Hence for $h_{x,y},k_{y,z}\in\mathrm{alg}(H)$ we have

$$S(h_{x,y}k_{y,z}) = S(k_{y,z})S(h_{x,y}),$$

$$S(h_{x,y})_{(1)}\otimes S(h_{x,y})_{(2)} = S((h_{x,y})_{(2)})\otimes S((h_{x,y})_{(1)}),$$

which implies that

$$S(hk)=S(k)S(h),\quad S(h_{(1)})\otimes S(h_{(2)}) = S(h_{(2)})\otimes S(h_{(1)})$$

for all $h,k\in\mathrm{alg}(H)$.

Finally,

$$
\begin{aligned}
(h_{x,y})_{(1)}\mathcal{S}((h_{x,y})_{(2)})\mathcal{S}((k_{z,w})_{(1)})(k_{z,w})_{(2)}& \\
&= \varepsilon_{x,y}(h_{x,y})\varepsilon_{z,w}(k_{z,w})\eta_x(1)\eta_w(1) \\
&= \varepsilon_{x,y}(h_{x,y})\varepsilon_{z,w}(k_{z,w})\eta_w(1)\eta_x(1) \\
&= \mathcal{S}((k_{z,w})_{(1)})(k_{z,w})_{(2)}(h_{x,y})_{(1)}\mathcal{S}((h_{x,y})_{(2)}),
\end{aligned}
$$

and it follows that alg(H) is a counital quantum inverse semigroup.

In ((BATISTA; CAENEPEEL; VERCRUYSSE, 2016), Prop 7.1) it is proved that, for the particular case of a Hopf category H with a finite set of objects, the algebra alg(H) is a weak Hopf algebra, which is also a quantum inverse semigroup. One can easily verify that the structure of quantum inverse semigroup of alg(H) obtained here coincides, in the case of finite Hopf categories, with the structure of quantum inverse semigroup for weak Hopf algebras described in Example 3.2.5.

4 GENERALIZED BISECTIONS ON HOPF ALGEBROIDS

In chapter 2, we redefined local bisections for any groupoid $(\mathcal{G},s,t)$ as a pair (u,X) with $u : X \subseteq \mathcal{G}^{(0)} \to \mathcal{G}$ satisfying $s \circ u = Id_X$ and $t \circ u : X \to t \circ u(X)$ being a bijection. Then for any groupoid $\mathcal{G}$, the set of all local bisections of $\mathcal{G}$ is an inverse semigroup. In this chapter, we dualize this definition for commutative Hopf algebroids over a commutative base algebra and create the local biretractions. Lastly, we extend this definition for not necessarily commutative Hopf algebroids over commutative algebras under a special condition.

4.1 BIRETRACTIONS

Here we introduce the notion of a local biretraction of a Hopf algebroid, as a dual version of local bissections in groupoids. First we focus on commutative Hopf algebroids over a commutative base algebra and then we find a morphism between the bisections of a groupoid and the biretractions of the Hopf algebroid of its representative functions.

Definition 4.1.1 *Let $\mathcal{H}$ be a commutative Hopf algebroid over a commutative algebra A. A* local biretraction *in $\mathcal{H}$ is a linear and multiplicative map $\alpha : \mathcal{H} \to A$ such that*
(BRT1) $\alpha \circ s(a) = a\alpha(1_{\mathcal{H}})$ *for every $a \in A$.*
(BRT2) *There exists $e^\alpha \in A$ such that $\alpha \circ t(e^\alpha) = \alpha(1_{\mathcal{H}})$ and*

$$\alpha \circ t|_{A\,e^\alpha} : A\,e^\alpha \longrightarrow A\alpha(1_{\mathcal{H}})$$

is a bijection.
A local biretraction α is global if $\alpha(1_{\mathcal{H}}) = 1_A$. Denote the set of local biretractions of $\mathcal{H}$ by $\mathcal{B}rt(\mathcal{H}, A)$ and the set of global biretractions of $\mathcal{H}$ by $Gl\mathcal{B}rt(\mathcal{H}, A)$.

Remark 4.1.2 *Observe that*

(1) *For a local biretraction $\alpha : \mathcal{H} \to A$, $\alpha(1_{\mathcal{H}})$ is an idempotent in A, since α is multiplicative. Moreover, for every $h \in \mathcal{H}$ and $a \in A$,*

$$\alpha(h) = \alpha(h)\,\alpha(1_{\mathcal{H}}) \in A\alpha(1_{\mathcal{H}}) \quad \text{and} \quad a\alpha(1_{\mathcal{H}}) = \alpha \circ s(a) \in \alpha(\mathcal{H}).$$

Hence the image $\alpha(\mathcal{H})$ coincides with the ideal $A\alpha(1_{\mathcal{H}}) \trianglelefteq A$. Also, note that $\alpha(1_{\mathcal{H}})$ is the unity of the ideal $A\alpha(1_{\mathcal{H}})$.

(2) *The element e^α is idempotent:*

$$\alpha \circ t(e^\alpha) = \alpha(1_{\mathcal{H}}) = \alpha(1_{\mathcal{H}})\,\alpha(1_{\mathcal{H}}) = \alpha \circ t(e^\alpha)\,\alpha \circ t(e^\alpha) = \alpha \circ t(e^\alpha\,e^\alpha).$$

Since $\alpha \circ t|_{A\,e^\alpha}$ is bijective and $e^\alpha\,e^\alpha \in A\,e^\alpha$, we have that

$$e^\alpha\,e^\alpha = e^\alpha.$$

(3) *Suppose that there exist* e^{α} *and* f^{α} *in* A *such that* $\alpha \circ t(e^{\alpha}) = \alpha(1_{\mathcal{H}}) = \alpha \circ t(f^{\alpha})$ *and the maps*

$$\alpha \circ t|_{A\,e^{\alpha}} : A\,e^{\alpha} \longrightarrow A\alpha(1_{\mathcal{H}})$$

and

$$\alpha \circ t|_{Af^{\alpha}} : Af^{\alpha} \longrightarrow A\alpha(1_{\mathcal{H}})$$

are both bijections. Then,

$$\alpha \circ t\,(e^{\alpha} f^{\alpha}) = \alpha \circ t\,(e^{\alpha})\,\alpha \circ t\,(f^{\alpha}) = \alpha(1_{\mathcal{H}}).$$

Since the element $e^{\alpha} f^{\alpha}$ *is in both ideals* $A\,e^{\alpha}$ *and* Af^{α}, *we obtain*

$$e^{\alpha} = e^{\alpha} f^{\alpha} = f^{\alpha}.$$

Therefore, the element e^{α} *from* (BRT2) *is unique.*

(4) *For any local biretraction* $\alpha : \mathcal{H} \to A$ *and* $a \in A$,

$$\alpha \circ t(a) = \alpha \circ t(a)\,\alpha(1_{\mathcal{H}}) = \alpha \circ t(a)\,\alpha \circ t(e^{\alpha}) = \alpha \circ t(a\,e^{\alpha}).$$

(5) *For a local biretraction* $\alpha : \mathcal{H} \to A$, *the map* $\alpha \circ t|_{A\,e^{\alpha}} : A\,e^{\alpha} \longrightarrow A\alpha(1_{\mathcal{H}})$ *is an element of the inverse semigroup* $\mathcal{I}(A)$ *of the isomorphisms between ideals of* A.

Remark 4.1.3 *For a commutative Hopf algebroid over an integral domain* A, *we only have global biretractions, since the only idempotent element in* A *is* 1_A.

As we have seen before, the set of local bisections of a groupoid $\mathcal{G}$ is an inverse semigroup. Let us explore deeply the algebraic structure of the set of biretractions of a commutative Hopf algebroid.

Theorem 4.1.4 *Let* $(\mathcal{H}, s, t, \Delta, \varepsilon, S)$ *be a commutative Hopf algebroid over a commutative algebra* A. *Then the set* $\mathcal{B}rt(\mathcal{H}, A)$ *of local biretractions of* $\mathcal{H}$ *is a regular monoid with the convolution product*

$$(\alpha * \beta)(h) = \beta(\alpha(h_{(1)}) \triangleright h_{(2)}) = \beta \circ t \circ \alpha(h_{(1)})\,\beta(h_{(2)})$$

for every $\alpha, \beta \in \mathcal{B}rt(\mathcal{H}, A)$ *and any* $h \in \mathcal{H}$.

Proof. For every $\alpha,\beta \in \mathcal{B}rt(\mathcal{H}, A)$, $\alpha * \beta$ is a local biretraction, because $\alpha * \beta$ is multiplicative and for each $a \in A$,

$$\begin{aligned}(\alpha * \beta) \circ s(a) &= \beta \circ t \circ \alpha(1_{\mathcal{H}})\,\beta(s(a)) \\ &= a\beta \circ t \circ \alpha(1_{\mathcal{H}})\beta(1_{\mathcal{H}}) \\ &= a\,(\alpha * \beta)(1_{\mathcal{H}}),\end{aligned}$$

hence $\alpha * \beta$ satisfies (BRT1). Also, since t represents the left action, we can use the fact that $\Delta(t(a)) = t(a) \otimes_A 1_{\mathcal{H}}$ for every $a \in A$, which implies that $(\alpha * \beta) \circ t = \beta \circ t \circ \alpha \circ t$ and

$$\begin{aligned}(\alpha * \beta) \circ t\left((\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\right) &= \beta \circ t \circ \alpha \circ t \circ (\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\\ &= \beta \circ t\left(\alpha(1_{\mathcal{H}})\,e^{\beta}\right)\\ &= \beta \circ t \circ \alpha(1_{\mathcal{H}})\beta(1_{\mathcal{H}})\\ &= (\alpha * \beta)(1_{\mathcal{H}}).\end{aligned}$$

Here, we are simplifying the notation by using $(\alpha \circ t)^{-1} = ((\alpha \circ t)|_{A\,e^{\alpha}})^{-1}$. In order to prove that $\alpha * \beta$ satisfies (BRT2), we will prove that the map

$$(\alpha * \beta) \circ t|_{A\,(\alpha\circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)} : A\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right) \longrightarrow A\,(\alpha * \beta)(1_{\mathcal{H}})$$

is a bijection, leading to $\alpha * \beta$ being a biretraction with $e^{\alpha*\beta} = (\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)$. Indeed,

- $(\alpha * \beta) \circ t|_{A\,(\alpha\circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)} : A\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right) \longrightarrow A\,(\alpha * \beta)(1_{\mathcal{H}})$ is surjective: for each $a \in A$,

$$\begin{aligned}&(\alpha * \beta) \circ t\Big((\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}(a\beta(1_{\mathcal{H}}))\,\alpha(1_{\mathcal{H}})\right)\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\Big)\\ &\quad= \beta \circ t \circ \alpha \circ t \circ (\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}(a\beta(1_{\mathcal{H}}))\,e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\\ &\quad= \beta \circ t\left((\beta \circ t)^{-1}(a\beta(1_{\mathcal{H}}))\,e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\\ &\quad= \beta \circ t \circ (\beta \circ t)^{-1}\left(a\beta \circ t\left(\alpha(1_{\mathcal{H}})\,e^{\beta}\right)\,\beta(1_{\mathcal{H}})\right)\\ &\quad= a\beta \circ t\left(\alpha(1_{\mathcal{H}})\,e^{\beta}\right)\\ &\quad= a\beta \circ t \circ \alpha(1_{\mathcal{H}})\beta(1_{\mathcal{H}})\\ &\quad= a\,(\alpha * \beta)(1_{\mathcal{H}}).\end{aligned}$$

- $(\alpha * \beta) \circ t|_{A\,(\alpha\circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)}$ is injective: suppose that, for some $a \in A$,

$$(\alpha * \beta) \circ t\left(a\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\right) = 0.$$

  Since $\alpha \circ t|_{A\,e^{\alpha}}$ and $\beta \circ t|_{A\,e^{\beta}}$ are injective,

$$\begin{aligned}&0 = \beta \circ t \circ \alpha \circ t\left(a\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\right) = \beta \circ t\left(\alpha \circ t(a)\,e^{\beta}\right)\\ \Rightarrow\ &0 = \alpha \circ t(a)\,e^{\beta} = \alpha \circ t\left(a\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)\,e^{\alpha}\right)\\ \Rightarrow\ &0 = a\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right)e^{\alpha} = a\,(\alpha \circ t)^{-1}\left(e^{\beta}\,\alpha(1_{\mathcal{H}})\right).\end{aligned}$$

This convolution product of biretractions of $\mathcal{H}$ is associative. Indeed, consider $\alpha, \beta, \gamma \in \mathcal{B}rt(\mathcal{H}, A)$, then for any $h \in \mathcal{H}$,

$$\begin{aligned}((\alpha * \beta) * \gamma)(h) &= \gamma \circ t \circ (\alpha * \beta)(h_{(1)})\,\gamma(h_{(2)})\\ &= \gamma \circ t(\beta \circ t \circ \alpha(h_{(1)})\,\beta(h_{(2)}))\,\gamma(h_{(3)})\\ &= \gamma \circ t \circ \beta \circ t \circ \alpha(h_{(1)})\,\gamma \circ t \circ \beta(h_{(2)}))\gamma(h_{(3)})\\ &= (\beta * \gamma) \circ t \circ \alpha(h_{(1)})\,(\beta * \gamma)(h_{(2)})\\ &= (\alpha * (\beta * \gamma))(h).\end{aligned}$$

The counit $\varepsilon : \mathcal{H} \to A$ is a global biretraction, because it is linear, multiplicative, $\varepsilon(1_{\mathcal{H}}) = 1_A$ and $\varepsilon \circ t = \varepsilon \circ s = \mathrm{Id}_A$. The counit ε is the unit for the convolution product. Indeed, for any local biretraction α and any $h \in \mathcal{H}$, we have

$$\begin{aligned} \varepsilon * \alpha(h) &= \alpha \circ t \circ \varepsilon(h_{(1)})\, \alpha(h_{(2)}) \\ &= \alpha\left(t(\varepsilon(h_{(1)}))\, h_{(2)}\right) \\ &= \alpha(h), \end{aligned}$$

and

$$\begin{aligned} \alpha * \varepsilon(h) &= \varepsilon \circ t \circ \alpha(h_{(1)})\, \varepsilon(h_{(2)}) \\ &= \alpha(h_{(1)})\, \varepsilon(h_{(2)}) \\ &= \alpha(h_{(1)})\, \alpha \circ s \circ \varepsilon(h_{(2)}) \\ &= \alpha\left(h_{(1)}\, s(\varepsilon(h_{(2)}))\right) \\ &= \alpha(h). \end{aligned}$$

Therefore, the set $\mathcal{B}rt(\mathcal{H}, A)$ is a monoid relative to the above defined convolution product.

Now, we have to define a pseudo-inverse for any biretraction $\alpha \in \mathcal{B}rt(\mathcal{H}, A)$. Define

$$\alpha^* = (\alpha \circ t)^{-1} \circ \alpha \circ S,$$

where we use $(\alpha \circ t)^{-1} = (\alpha \circ t|_{A\, e^{\alpha}})^{-1}$. Since α,t and S are multiplicative, we have that α^* is multiplicative and observe that

$$\alpha^*(1_{\mathcal{H}}) = (\alpha \circ t)^{-1} \circ \alpha \circ S(1_{\mathcal{H}}) = (\alpha \circ t)^{-1} \circ \alpha(1_{\mathcal{H}}) = e^{\alpha}.$$

So, α^* is a biretraction, because

$$\alpha^* \circ s(a) = (\alpha \circ t)^{-1} \circ \alpha \circ S \circ s(a) = (\alpha \circ t)^{-1} \circ \alpha \circ t(a) = a\, e^{\alpha} = a\, \alpha^*(1_{\mathcal{H}})$$

and

$$\alpha^* \circ t(a) = (\alpha \circ t)^{-1} \circ \alpha \circ S \circ t(a) = (\alpha \circ t)^{-1} \circ \alpha \circ s(a) = (\alpha \circ t)^{-1}(a\alpha(1_{\mathcal{H}}))$$

for every $a \in A$, which implies that $\alpha^* \circ t|_{A\,\alpha(1_{\mathcal{H}})} : A\,\alpha(1_{\mathcal{H}}) \to A\, e^{\alpha}$ is a bijection with $e^{\alpha^*} = \alpha(1_{\mathcal{H}})$.

Finally, we need to prove that every biretraction $\alpha : \mathcal{H} \to A$ satisfies $\alpha * \alpha^* * \alpha = \alpha$ and $\alpha^* * \alpha * \alpha^* = \alpha^*$. Observe that for any $h \in \mathcal{H}$,

$$\begin{aligned} (\alpha * \alpha^*)(h) &= \alpha^* \circ t \circ \alpha(h_{(1)})\, \alpha^*(h_{(2)}) \\ &= (\alpha \circ t)^{-1} \circ \alpha \circ S \circ t \circ \alpha(h_{(1)})\, (\alpha \circ t)^{-1} \circ \alpha \circ S(h_{(2)})) \\ &= (\alpha \circ t)^{-1}\left(\alpha \circ s \circ \alpha(h_{(1)})\, \alpha(S(h_{(2)}))\right) \\ &= (\alpha \circ t)^{-1}\left(\alpha(h_{(1)})\, \alpha(S(h_{(2)}))\right) \\ &= (\alpha \circ t)^{-1} \circ \alpha(h_{(1)} S(h_{(2)})) \\ &= (\alpha \circ t)^{-1} \circ \alpha \circ t(\varepsilon(h)) \\ &= \varepsilon(h)\, e^{\alpha} \end{aligned} \tag{21}$$

and

$$\begin{aligned}(\alpha^* * \alpha)(h) &= \alpha \circ t \circ \alpha^*(h_{(1)})\,\alpha(h_{(2)}) \\ &= \alpha \circ t \circ (\alpha \circ t)^{-1} \circ \alpha \circ S(h_{(1)})\,\alpha(h_{(2)}) \\ &= \alpha \circ S(h_{(1)})\,\alpha(h_{(2)}) \\ &= \alpha(S(h_{(1)})\,h_{(2)}) \\ &= \alpha \circ s \circ \varepsilon(h) \\ &= \varepsilon(h)\,\alpha(1_{\mathcal{H}}) \end{aligned} \tag{22}$$

Then,

$$\begin{aligned}\alpha * \alpha^* * \alpha(h) &= \alpha \circ t \circ (\alpha * \alpha^*)(h_{(1)})\,\alpha(h_{(2)}) \\ &= \alpha \circ t(\varepsilon(h_{(1)})\,e^{\alpha})\,\alpha(h_{(2)}) \\ &= \alpha(t \circ \varepsilon(h_{(1)})\,h_{(2)}) \\ &= \alpha(h) \end{aligned} \tag{23}$$

and

$$\begin{aligned}\alpha^* * \alpha * \alpha^*(h) &= \alpha^* \circ t \circ (\alpha^* * \alpha)(h_{(1)})\,\alpha^*(h_{(2)}) \\ &= (\alpha \circ t)^{-1} \circ \alpha \circ S \circ t\,\left(\varepsilon(h_{(1)})\,\alpha(1_{\mathcal{H}})\right)\,(\alpha \circ t)^{-1} \circ \alpha \circ S(h_{(2)}) \\ &= (\alpha \circ t)^{-1} \circ \alpha\left(s \circ \varepsilon(h_{(1)})\,S(h_{(2)})\right) \\ &= (\alpha \circ t)^{-1} \circ \alpha \circ S\left(h_{(2)}\,t \circ \varepsilon(h_{(1)})\right) \\ &= \alpha^*(h). \end{aligned} \tag{24}$$

for every $h \in \mathcal{H}$. Therefore, $\mathcal{B}rt(\mathcal{H}, A)$ is a regular monoid. □

Remark 4.1.5 *We can not prove, in general, that $\mathcal{B}rt(\mathcal{H}, A)$ is an inverse semigroup. Consider an idempotent $E \in \mathcal{B}rt(\mathcal{H}, A)$ and denote its associated idempotent in A by e^E then, for any $a \in A$,*

$$\begin{aligned}E \circ t(a) &= (E * E)(t(a)) = E \circ t \circ E(t(a))\,E(1_{\mathcal{H}}) \\ &= E \circ t \circ E \circ t(a).\end{aligned}$$

Then, $E \circ t : A \to A$ is a linear and multiplicative map in A which is idempotent with respect to the composition. Moreover, for every $a \in A$,

$$\begin{aligned}E \circ t(E \circ t(a) - a\,E \circ t(1_A)) &= E \circ t \circ E \circ t(a) - E \circ t(a)\,E \circ t \circ E \circ t(1_A) \\ &= E \circ t(a) - E \circ t(a) \\ &= 0,\end{aligned}$$

which implies that $E \circ t(a)\,e^E = a\,E(1_{\mathcal{H}})\,e^E$. Thus for every idempotent $E,F \in \mathcal{B}rt(\mathcal{H},A)$, we have that

$$\begin{aligned}(E * F)(h)\,e^E\,e^F &= F \circ t \circ E(h_{(1)})\,F(h_{(2)})\,e^E\,e^F \\ &= E(h_{(1)})\,F(1_{\mathcal{H}})\,e^F\,F(h_{(2)})\,e^E \\ &= E(h_{(1)})\,F(h_{(2)})\,e^E\,e^F\end{aligned}$$

$$\begin{aligned}(F * E)(h)\, e^E\, e^F &= E \circ t \circ F(h_{(1)})\, E(h_{(2)})\, e^E\, e^F \\ &= F(h_{(1)})\, E(1_{\mathcal{H}})\, e^E\, F(h_{(2)})\, e^F \\ &= F(h_{(1)})\, E(h_{(2)})\, e^E\, e^F.\end{aligned}$$

for every $h \in \mathcal{H}$. Therefore, there is no a priori reason to suppose that the idempotents of $\mathcal{B}rt(\mathcal{H},A)$ should commute in general.

Remark 4.1.6 *Let $\alpha, \beta \in \mathcal{B}rt(\mathcal{H}, A)$. Then,*

$$((\alpha * \beta) \circ t|_{A\, e^{\alpha}})^{-1} = \left(\alpha \circ t|_{A\, e^{\alpha*\beta}}\right)^{-1} \circ \left(\beta \circ t|_{A\, e^{\beta}}\right)^{-1} |_{A\,(\alpha*\beta)(1_{\mathcal{H}})} : A\,(\alpha * \beta)(1_{\mathcal{H}}) \longrightarrow A\, e^{\alpha*\beta},$$

or simplifying the notation as before,

$$((\alpha * \beta) \circ t)^{-1} = (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1}.$$

Indeed, we have for every $a \in A$,

$$\begin{aligned}&(\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \circ (\alpha * \beta) \circ t \left(a\, e^{\alpha*\beta}\right) \\ &\quad = (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \circ \beta \circ t \circ \alpha \circ t \left(a\, (\alpha \circ t)^{-1} \left(e^{\beta}\, \alpha(1_{\mathcal{H}})\right)\right) \\ &\quad = (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \circ \beta \circ t \left(\alpha \circ t \left(a\, (\alpha \circ t)^{-1} \left(e^{\beta}\, \alpha(1_{\mathcal{H}})\right)\right)\, e^{\beta}\right) \\ &\quad = (\alpha \circ t)^{-1} \left(\alpha \circ t \left(a\, (\alpha \circ t)^{-1} \left(e^{\beta}\, \alpha(1_{\mathcal{H}})\right)\right)\, e^{\beta}\right) \\ &\quad = (\alpha \circ t)^{-1} \circ (\alpha \circ t) \left(a\, (\alpha \circ t)^{-1} \left(e^{\beta}\, \alpha(1_{\mathcal{H}})\right)\right) \\ &\quad = (\alpha \circ t)^{-1} \circ (\alpha \circ t) \left(a\, (\alpha \circ t)^{-1} \left(e^{\beta}\, \alpha(1_{\mathcal{H}})\right)\, e^{\alpha}\right) \\ &\quad = a\, (\alpha \circ t)^{-1} \left(e^{\beta}\, \alpha(1_{\mathcal{H}})\right) \\ &\quad = a\, e^{\alpha*\beta}\end{aligned}$$

and

$$\begin{aligned}&((\alpha * \beta) \circ t) \circ (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1}\, (a\, (\alpha * \beta)(1_{\mathcal{H}})) \\ &\quad = \beta \circ t \circ \alpha \circ t \circ (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1}\, (a \beta \circ t \circ \alpha(1_{\mathcal{H}})\, \beta(1_{\mathcal{H}})) \\ &\quad = \beta \circ t \circ \alpha \circ t \circ (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \left(a \beta(1_{\mathcal{H}})\, \beta \circ t \left(\alpha(1_{\mathcal{H}})\, e^{\beta}\right)\right) \\ &\quad = \beta \circ t \circ \alpha \circ t \circ (\alpha \circ t)^{-1} \left((\beta \circ t)^{-1} (a \beta(1_{\mathcal{H}}))\, e^{\beta}\, \alpha(1_{\mathcal{H}})\right) \\ &\quad = \beta \circ t \left((\beta \circ t)^{-1} (a \beta(1_{\mathcal{H}}))\, \alpha(1_{\mathcal{H}})\, e^{\beta}\right) \\ &\quad = \beta \circ t \circ (\beta \circ t)^{-1} \left(a \beta \circ t \left(\alpha(1_{\mathcal{H}})\, e^{\beta}\right)\, \beta(1_{\mathcal{H}})\right) \\ &\quad = a \beta \circ t \left(\alpha(1_{\mathcal{H}})\, e^{\beta}\right) \\ &\quad = a\, (\alpha * \beta)(1_{\mathcal{H}}).\end{aligned}$$

Remark 4.1.7 *Observe that the pseudoinverse α^* of a biretraction $\alpha \in \mathcal{B}rt(\mathcal{H}, A)$ satisfies $(\alpha^*)^* = \alpha$. Indeed, for any $h \in \mathcal{H}$,*

$$\begin{aligned}(\alpha^*)^*(h) &= (\alpha^* \circ t)^{-1} \circ \alpha^* \circ S(h)\\ &= \alpha \circ t \circ (\alpha \circ t)^{-1} \circ \alpha \circ S(S(h))\\ &= \alpha \circ S^2(h)\\ &\overset{(*)}{=} \alpha(h),\end{aligned}$$

where $(*)$ comes from Remark 2.2.4. Also, for any $\alpha, \beta \in \mathcal{B}rt(\mathcal{H}, A)$, we have that $(\alpha * \beta)^* = \beta^* * \alpha^*$. Indeed, for every $h \in \mathcal{H}$,

$$\begin{aligned}(\alpha * \beta)^*(h) &= ((\alpha * \beta) \circ t)^{-1} \circ (\alpha * \beta) \circ S(h)\\ &= (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \left(\beta \circ t \circ \alpha \circ S(h_{(2)})\, \beta \circ S(h_{(1)})\right)\\ &= (\alpha \circ t)^{-1} \circ \alpha \circ S(h_{(2)})\, (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \circ \beta \circ S(h_{(1)})\\ &= \alpha^*(h_{(2)})\, (\alpha^* \circ t) \circ (\beta \circ t)^{-1} \circ \beta \circ S(h_{(1)})\\ &= \alpha^* \circ t \circ \beta^*(h_{(1)})\, \alpha^*(h^{(2)})\\ &= (\beta^* * \alpha^*)(h).\end{aligned}$$

Consider now the free vector space generated by the biretractions of $\mathcal{H}$ and extend linearly the convolution product to this space. Then, we have an algebra structure on the space $\Bbbk\mathcal{B}rt(\mathcal{H}, A)$, henceforth denoted by $\mathcal{B}(\mathcal{H})$.

Theorem 4.1.8 *Let $\mathcal{H}$ be a commutative Hopf algebroid over a commutative algebra A. Then the algebra $\mathcal{B}(\mathcal{H})$, generated by the set of all biretractions of $\mathcal{H}$ with the convolution product is a unital quantum inverse semigroup with a comultiplication $\underline{\Delta} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ and a pseudo antipode $\mathcal{S} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined on the basis elements of $\mathcal{B}rt(\mathcal{H}, A)$ as*

$$\underline{\Delta}(\alpha) = \alpha \otimes \alpha \qquad \text{and} \qquad \mathcal{S}(\alpha) = \alpha^* = (\alpha \circ t)^{-1} \circ \alpha \circ S$$

and linearly extended to $\mathcal{B}(\mathcal{H})$.

Proof. As we have already proven in the last theorem, $\mathcal{B}rt(\mathcal{H}, A)$ is a regular monoid, hence the algebra $\mathcal{B}(\mathcal{H})$ is a unital algebra. For proving that the comultiplication is multiplicative with respect to the convolution product, it is enough to check on the biretractions. Being $\alpha, \beta \in \mathcal{B}rt(\mathcal{H}, A)$,

$$\underline{\Delta}(\alpha * \beta) = (\alpha * \beta \otimes \alpha * \beta) = (\alpha \otimes \alpha)(\beta \otimes \beta) = \underline{\Delta}(\alpha)\underline{\Delta}(\beta).$$

Hence $\mathcal{B}(\mathcal{H})$ satisfies (QISG1) and (QISG2).

Again, to prove that $\mathcal{S}$ is antimultiplicative, it is enough to check on the biretractions. Then for $h \in \mathcal{H}$ and $\alpha, \beta \in \mathcal{B}rt(\mathcal{H}, A)$, we have

$$
\begin{aligned}
\mathcal{S}(\alpha * \beta)(h) &= ((\alpha * \beta) \circ t)^{-1} \circ (\alpha * \beta) \circ S(h) \\
&\overset{(*)}{=} ((\alpha * \beta) \circ t)^{-1} \left(\beta \circ t \circ \alpha \circ S(h_{(2)})\, \beta \circ S(h_{(1)})\right) \\
&\overset{(**)}{=} (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \left(\beta \circ t \circ \alpha \circ S(h_{(2)})\, \beta \circ S(h_{(1)})\, \beta \circ t\left(e^{\beta}\right)\right) \\
&= (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \circ \beta \circ t \left(\alpha \circ S(h_{(2)})\, (\beta \circ t)^{-1} \circ \beta \circ S(h_{(1)})\, e^{\beta}\right) \\
&= (\alpha \circ t)^{-1} \left(\alpha \circ S(h_{(2)})\, e^{\beta}\, (\beta \circ t)^{-1} \circ \beta \circ S(h_{(1)})\right) \\
&= (\alpha \circ t)^{-1} \circ \alpha \circ S(h_{(2)})\, (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \left(\beta \circ S(h_{(1)})\, \alpha(1_{\mathcal{H}})\right) \\
&= \alpha^*(h_{(2)})\, (\alpha \circ t)^{-1} \left(\beta^*(h_{(1)})\, \alpha(1_{\mathcal{H}})\right),
\end{aligned}
$$

where we used in $(*)$ the property (P6) of Hopf algebroids, and in $(**)$ the result $((\alpha * \beta) \circ t)^{-1} = (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1}$ from Remark 4.1.6. On the other hand,

$$
\begin{aligned}
(\mathcal{S}(\beta) * \mathcal{S}(\alpha))(h) &= (\beta^* * \alpha^*)(h) \\
&= \alpha^* \circ t \circ \beta^*(h_{(1)})\, \alpha^*(h_{(2)}) \\
&= (\alpha \circ t)^{-1} \circ \alpha \circ S \circ t \circ \beta^*(h_{(1)})\, \alpha^*(h_{(2)}) \\
&= (\alpha \circ t)^{-1} \circ \alpha \circ s \circ \beta^*(h_{(1)})\, \alpha^*(h_{(2)}) \\
&= (\alpha \circ t)^{-1} \left(\beta^*(h_{(1)})\, \alpha(1_{\mathcal{H}})\right)\, \alpha^*(h_{(2)}).
\end{aligned}
$$

Consequently, $\mathcal{S}(\alpha * \beta) = \mathcal{S}(\beta) * \mathcal{S}(\alpha)$ and $\mathcal{S}$ is antimultiplicative. Hence $\mathcal{H}$ satisfies item (i) of (QISG3) and the equations (23) and (24) imply the item (ii).

Finally, for checking axiom (QISG4), we use the equations (21) and (22). Then for $\alpha, \beta \in \mathcal{B}rt(\mathcal{H}, A)$ and $h \in \mathcal{H}$,

$$
\begin{aligned}
\alpha_{(1)} * \mathcal{S}(\alpha_{(2)}) * \mathcal{S}(\beta_{(1)}) * \beta_{(2)}(h) &= (\alpha * \alpha^*) * (\beta^* * \beta)(h) \\
&= (\beta^* * \beta) \circ t \circ (\alpha * \alpha^*)(h_{(1)})\, (\beta^* * \beta)(h_{(2)}) \\
&= (\beta^* * \beta) \circ t\, (\varepsilon(h_{(1)})\, e^{\alpha})\, (\beta^* * \beta)(h_{(2)}) \\
&= \varepsilon \circ t \left(\varepsilon(h_{(1)})\, e^{\alpha}\right) \varepsilon(h_{(2)})\, \beta(1_{\mathcal{H}}) \\
&= \varepsilon(h_{(1)})\, e^{\alpha}\, \varepsilon(h_{(2)})\, \beta(1_{\mathcal{H}}) \\
&= \varepsilon(h)\, e^{\alpha}\, \beta(1_{\mathcal{H}})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{S}(\beta_{(1)}) * \beta_{(2)} * \alpha_{(1)} * \mathcal{S}(\alpha_{(2)})(h) &= (\beta^* * \beta) * (\alpha * \alpha^*)(h) \\
&= (\alpha * \alpha^*) \circ t \circ (\beta^* * \beta)(h_{(1)})\, (\alpha * \alpha^*)(h_{(2)}) \\
&= (\alpha * \alpha^*) \circ t(\varepsilon(h_{(1)})\, \beta(1_{\mathcal{H}}))\, (\alpha * \alpha^*)(h_{(2)}) \\
&= \varepsilon \circ t(\varepsilon(h_{(1)})\, \beta(1_{\mathcal{H}}))\, e^{\alpha}\, \varepsilon(h_{(2)}) \\
&= \varepsilon(h)\, e^{\alpha}\, \beta(1_{\mathcal{H}}).
\end{aligned}
$$

Therefore, $\mathcal{B}(\mathcal{H})$ is a quantum inverse semigroup. More than that, the algebra $\mathcal{B}(\mathcal{H})$ is a unital quantum inverse semigroup. Indeed, for $h \in \mathcal{H}$,

$$\begin{aligned}
\mathcal{S}(\varepsilon)(h) &= (\varepsilon \circ t)^{-1} \circ \varepsilon \circ S(h) \\
&= \varepsilon \circ S(h) \\
&= \varepsilon \circ S\left(t(\varepsilon(h_{(1)}))\, h_{(2)}\right) \\
&= \varepsilon\left(S(h_{(2)})\, s(\varepsilon(h_{(1)}))\right) \\
&= \varepsilon\left(h_{(1)} S(h_{(2)})\right) \\
&= \varepsilon \circ t \circ \varepsilon(h) \\
&= \varepsilon(h).
\end{aligned}$$

□

Example 4.1.9 *Let H be a commutative Hopf algebra, considered as a Hopf algebroid over the field $\Bbbk$ with $s = t : \Bbbk \to H$, $k \mapsto k \cdot 1_H$. Since the only idempotent in $\Bbbk$ is 1, then all biretractions are global and being $\alpha : H \to \Bbbk$ a $\Bbbk$–linear and multiplicative map,*

$$\alpha \circ s(k) = \alpha(k \cdot 1_H) = k\,\alpha(1_H) = k$$

for every $k \in \Bbbk$. Therefore, the set of biretractions coincides with the group of algebra morphisms between H and $\Bbbk$, that is, the group $G(H^\circ)$ of group-like elements of the finite dual Hopf algebra H°.

Example 4.1.10 *Let A be a commutative Hopf algebra and consider the Hopf algebroid $\mathcal{H} = A \otimes A$, from* Example 2.2.5. *Let $M(A)$ be the set of multiplicative functions $\varphi : A \to A$ and*

$$M(A) \times^b E(A) = \{(\varphi, e) \in M(A) \times E(A) \,|\, \varphi|_{Ae} : Ae \longrightarrow A\varphi(e) \text{ is a bijection}\}.$$

Consider the equivalence relation

$$(\varphi, e) \sim (\psi, f) \;\Leftrightarrow\; e = f \text{ and } \varphi|_{Ae} = \psi|_{Ae}.$$

Representing the class of an element $(\varphi, e) \in M(A) \times^b E(A)$ by $[\varphi,e]$, then the biretractions of $\mathcal{H}$ are classified by the set

$$M(A) \ltimes E(A) := \left\{[\varphi,e] :\, (\varphi,e) \in M(A) \times^b E(A)\right\},$$

which is a regular monoid with the multiplication

$$[\varphi,e]\,[\psi,f] = [\varphi \circ \psi, \psi^{-1}(e\psi(f))].$$

Indeed, the multiplication is well defined, because if $[\varphi,e] = [\varphi',e']$ and $[\psi,f] = [\psi',f]$ then $e = e'$, $f = f'$, $\varphi|_{Ae} = \varphi'|_{Ae}$, $\psi|_{Af} = \psi'|_{Af}$ and

$$\psi'^{-1}(e\psi'(f)) = \psi'^{-1}(e\psi(f)) = \psi^{-1}(e\psi(f)),$$

which implies that

$$\begin{aligned}
\varphi' \circ \psi' \left(a\psi'^{-1}(e'\psi'(f')) \right) &= \varphi' \circ \psi' \left(a\psi^{-1}(e\psi(f)) \right) \\
&= \varphi' \circ \psi \left(a\psi^{-1}(e\psi(f)) \right) \\
&= \varphi' \left(\psi(a)\psi(f)e \right) \\
&= \varphi \left(\psi(a)\psi(f)e \right) \\
&= \varphi \circ \psi \left(a\psi^{-1}(e\psi(f)) \right)
\end{aligned}$$

for every $a \in A$. Hence $[\varphi,e][\psi,f] = [\varphi',e'][\psi',f']$.

Also, the function

$$\varphi \circ \psi|_{A\psi^{-1}(e\psi(f))} : A\psi^{-1}(e\psi(f)) \longrightarrow A\varphi(e\psi(f))$$

is bijective, because

- $\varphi \circ \psi|_{A\psi^{-1}(e\psi(f))}$ is injective:

$$\begin{aligned}
& 0 = \varphi \circ \psi(a\psi^{-1}(e\psi(f))) = \varphi(\psi(a)\psi(f)\, e) \\
\Rightarrow\; & 0 = \psi(a\psi^{-1}(e\psi(f))\, e = \psi(a\psi^{-1}(e\psi(f)) = \psi(a\psi^{-1}(e\psi(f)\, f)) \\
\Rightarrow\; & 0 = a\psi^{-1}(e\psi(f))\, f = a\psi^{-1}(e\psi(f)).
\end{aligned}$$

- $\varphi \circ \psi(A\psi^{-1}(e\psi(f))) = A\varphi(e\psi(f))$: given $a\varphi(e\psi(f)) \in A\varphi(e\psi(f))$, we have

$$\begin{aligned}
a\varphi(e\psi(f)) &= \varphi(\varphi^{-1}(a\varphi(e))\,\psi(f)) \\
&= \varphi \circ \psi \circ \psi^{-1}(\varphi^{-1}(a\varphi(e))\,\psi(f) \\
&= \varphi \circ \psi \left(\psi^{-1}(\varphi^{-1}(a\varphi(e)))\,\psi^{-1}(e\psi(f)) \right) \in \varphi \circ \psi(A\psi^{-1}(e\psi(f))).
\end{aligned}$$

Now given the element $[\varphi,e] \in M(A) \ltimes E(A)$, we have that

$$\begin{aligned}
&[\varphi,e][Id_A,1_A] = [\varphi \circ Id_A, (Id_A)^{-1}(e\, Id_A(1_A))] = [\varphi,e] \\
&[Id_A,1_A][\varphi,e] = [Id_A \circ \varphi, \varphi^{-1}(1_A\, \varphi(e))] = [\varphi,e],
\end{aligned}$$

that is, $[Id_A,1_A]$ is the unity element of $M(A) \ltimes E(A)$. On the other hand, denotating by φ^{-1} the multiplicative map from A to A that takes every $a \in A$ to $\varphi^{-1}(a\varphi(e))$, then $\varphi^{-1}|_{A\varphi(e)} = (\varphi|_{Ae})^{-1}$: $A\varphi(e) \to Ae$ is a bijection and

$$\begin{aligned}
[\varphi,e][\varphi^{-1},\varphi(e)][\varphi,e] &= [Id_A, \varphi(e\,\varphi^{-1}(\varphi(e)))][\varphi,e] \\
&= [Id_A,\varphi(e)][\varphi,e] \\
&= [\varphi,\varphi^{-1}(\varphi(e)\varphi(e))] \\
&= [\varphi,e] \\
[\varphi^{-1},\varphi(e)][\varphi,e][\varphi^{-1},\varphi(e)] &= [Id_A,\varphi^{-1}(\varphi(e)\varphi(e))][\varphi^{-1},\varphi(e)] \\
&= [Id_A,e][\varphi^{-1},\varphi(e)] \\
&= [\varphi^{-1},\varphi(e\,\varphi^{-1}(\varphi(e)))] \\
&= [\varphi^{-1},\varphi(e)].
\end{aligned}$$

Therefore, every element $[\varphi, e]$ has a pseudoinverse and $M(A) \ltimes E(A)$ is a regular monoid.

Thus given the element $[\varphi,e] \in M(A) \ltimes E(A)$, we can define the multiplicative map

$$\begin{array}{rccc} \alpha_{[\varphi,e]}: & A\otimes A & \to & A \\ & a\otimes b & \mapsto & \varphi(ae)\, b, \end{array}$$

which is clearly a biretraction with $e^{\alpha_{[\varphi,e]}} = e$, because for every $a \in A$,

$$\alpha_{[\varphi,e]} \circ s(a) = \alpha_{[\varphi,e]}(1_A \otimes a) = a\,\varphi(e) = a\,\alpha_{[\varphi,e]}(1_{A\otimes A}),$$
$$\alpha_{[\varphi,e]} \circ t(e) = \alpha_{[\varphi,e]}(e \otimes 1_A) = \varphi(e) = \alpha_{[\varphi,e]}(1_{A\otimes A})$$

and $\alpha_{[\varphi,e]} \circ t|_{Ae} = \varphi|_{Ae}$ is a bijection. The convolution product between two local biretractions $\alpha_{[\varphi,e]}, \alpha_{[\psi,f]} \in \mathcal{B}rt(A\otimes A,A)$ is given by

$$\begin{aligned} \alpha_{[\varphi,e]} * \alpha_{[\psi,f]}(a\otimes b) &= \alpha_{[\psi,f]} \circ t \circ \alpha_{[\varphi,e]}(a\otimes 1_A)\,\alpha_{[\psi,f]}(1_A\otimes b) \\ &= \alpha_{[\psi,f]}(\varphi(ae)\otimes 1_A)\,\psi(f)\,b \\ &= \psi(\varphi(ae)f)\,\psi(f)\,b \\ &= \psi(\varphi(ae)f)\,b \\ &= \psi\circ\varphi(a\varphi^{-1}(f\varphi(e)))\,b \\ &= \alpha_{[\psi\circ\varphi,\varphi^{-1}(f\varphi(e))]}(a\otimes b) \\ &= \alpha_{[\psi,f]\,[\varphi,e]}(a\otimes b) \end{aligned}$$

for every $a,b \in A$. Then there is an isomorphism of semigroups

$$\begin{array}{rccc} \alpha: & (M(A)\ltimes E(A))^{op} & \to & \mathcal{B}rt(A\otimes A,A) \\ & [\varphi,e] & \mapsto & \alpha_{[\varphi,e]} \end{array},$$

whose inverse is

$$\begin{array}{rccc} \lambda: & \mathcal{B}rt(A\otimes A,A) & \to & (M(A)\ltimes E(A))^{op} \\ & \beta & \mapsto & [\beta\circ t, e^{\beta}] \end{array}.$$

Indeed, for every $[\varphi,e] \in M(A) \ltimes E(A)$,

$$\lambda\circ\alpha([\varphi,e]) = [\alpha_{[\varphi,e]}\circ t, e^{\alpha_{[\varphi,e]}}] = [\varphi, e]$$

and for every $\beta \in \mathcal{B}rt(A\otimes A,A)$ and $a,b \in A$,

$$\alpha\circ\lambda(\beta)(a\otimes b) = \alpha_{[\beta\circ t,e^{\beta}]}(a\otimes b) = \beta\circ t(a e^{\beta})\,b = \beta\circ t(a)\,\beta\circ s(b) = \beta(a\otimes 1_A)\beta(1_A\otimes b) = \beta(a\otimes b).$$

Moreover, this is an isomorphism of regular monoids:

- α maps unity to unity: for every $a,b \in A$,

$$\alpha_{[Id_A,1_A]}(a\otimes b) = Id_A(a)b = ab = \varepsilon(a\otimes b).$$

- α even maps the specific pseudoinverse $[\varphi^{-1},\varphi(e)]$ of $[\varphi,e]$ to the pseudoinverse $\alpha^*_{[\varphi,e]}$: for every $a,b \in A$,

$$\begin{aligned} \alpha^*_{[\varphi,e]}(a\otimes b) &= (\alpha_{[\varphi,e]}\circ t)^{-1}\circ\alpha_{[\varphi,e]}\circ S(a\otimes b) \\ &= (\alpha_{[\varphi,e]}\circ t)^{-1}\circ\alpha_{[\varphi,e]}(b\otimes a) \\ &= \varphi^{-1}(\varphi(be)a) \\ &= \varphi^{-1}(a\varphi(e))\,b \\ &= \alpha_{[\varphi^{-1},\varphi(e)]}(a\otimes b). \end{aligned}$$

Example 4.1.11 *Generalizing slightly the previous example, we can find the local biretractions for the Hopf algebroid* $\mathcal{H} = (A \otimes A)[x, x^{-1}]$ *from* Example 2.2.6*, with A commutative. Consider the set*

$$(M(A) \ltimes E(A)) \times' A = \{([\varphi, e], p) \in (M(A) \ltimes E(A)) \times A \,|\, \exists p' \in A: \; pp' = \varphi(e)\},$$

where $M(A) \ltimes E(A)$ *is the regular monoid from the previous example.*

Observe that if $p', p'' \in A$ *both satisfy* $pp' = \varphi(e) = pp''$*, then*

$$p'\varphi(e) = p'pp'' = p''pp' = p''\varphi(e). \tag{25}$$

Now, considering the equivalence relation

$$([\varphi, e], p) \sim ([\psi, f], q) \;\Leftrightarrow\; [\varphi, e] = [\psi, f] \text{ and } p\varphi(e) = q\varphi(e)$$

and representing by $[[\varphi, e], p]$ *the class of equivalent elements by this relation, we have that* $\mathcal{B}rt(A \otimes A)[x, x^{-1}]$ *can be identified with*

$$(M(A) \ltimes E(A)) \ltimes A := \left\{[[\varphi, e], p] : ([\varphi, e], p) \in (M(A) \ltimes E(A)) \times' A\right\},$$

which is a regular monoid with the product

$$[[\varphi, e], p] \; \big[[\psi, f], q\big] = \big[[\varphi, e][\psi, f], p\varphi(q)\big] = \Big[[\varphi \circ \psi, \psi^{-1}(e\psi(f))], p\varphi(q)\Big],$$

unity $[[\mathrm{Id}_A, 1_A], 1_A]$ *and* $[[\varphi, e], p]^* = \big[[\varphi, e]^*, \varphi^{-1}(p'\varphi(e))\big]$.

Indeed, the product is well defined, because if we take $[[\varphi, e], p] = \big[[\varphi', e'], \overline{p}\big]$ and $\big[[\psi, f], q\big] = \big[[\psi', f'], \overline{q}\big]$, then from the previous example,

$$\psi^{-1}(e\,\psi(f)) = \psi'^{-1}(e'\,\psi'(f')) \qquad\qquad \varphi \circ \psi|_{A\psi^{-1}(e\psi(f))} = \varphi' \circ \psi'|_{A\psi^{-1}(e\psi(f))}.$$

And $p\,\varphi(e) = \overline{p}\,\varphi(e)$ and $q\,\psi(f) = \overline{q}\,\psi(f)$ imply that

$$\begin{aligned} p\,\varphi(q)\,\varphi \circ \psi(\psi^{-1}(e\psi(f))) &= p\,\varphi(q)\varphi(e\,\psi(f)) \\ &= p\,\varphi(e)\,\varphi(q\,\psi(f)) \\ &= \overline{p}\,\varphi(e)\,\varphi(\overline{q}\,\psi(f)) \\ &= \overline{p}\,\varphi(\overline{q}\,\psi(f)\,e) \\ &= \overline{p}\,\varphi'(\overline{q})\,\varphi \circ \psi(\psi^{-1}(e\psi(f))). \end{aligned}$$

Also, we can take $(p\varphi(q))' = p'\varphi(q')$, because

$$\begin{aligned} p\varphi(q)\,p'\varphi(q') &= pp'\varphi(qq') \\ &= \varphi(e)\varphi(\psi(f)) \\ &= \varphi \circ \psi(\psi^{-1}(e\psi(f))). \end{aligned}$$

Now, given $[[\varphi, e], p] \in (M(A) \ltimes E(A)) \ltimes A$, we have

$$\begin{aligned} &[[\varphi, e], p]\,[[Id_A, 1_A], 1_A] = [\varphi, e, p\,\varphi(1_A)] = [[\varphi, e], p] \\ &[[Id_A, 1_A], 1_A]\,[[\varphi, e], p] = [[\varphi, e], 1_A\, Id_A(p)] = [[\varphi, e], p] \end{aligned}$$

and

$$
\begin{aligned}
[\varphi,e],p] \left[[\varphi^{-1},\varphi(e)],\varphi^{-1}(p'\,\varphi(e))\right] [[\varphi,e],p] &= \left[[Id_A,\varphi(e)],\, p\,\varphi(\varphi^{-1}(p'\,\varphi(e)))\right] [[\varphi,e],p] \\
&= [[Id_A,\varphi(e)],\varphi(e)]\,[[\varphi,e],p] \\
&= [[\varphi,e],\,\varphi(e)\; Id_A(p)] \\
&= [[\varphi,e],p]
\end{aligned}
$$

$$
\begin{aligned}
&\left[[\varphi^{-1},\varphi(e)],\varphi^{-1}(p'\,\varphi(e))\right] [[\varphi,e],p] \left[[\varphi^{-1},\varphi(e)],\varphi^{-1}(p'\,\varphi(e))\right] \\
&\quad = \left[[Id_A,e],\varphi^{-1}(p'\,\varphi(e))\,\varphi^{-1}(p)\right] \left[[\varphi^{-1},\varphi(e)],\,\varphi^{-1}(p'\,\varphi(e))\right] \\
&\quad = \; [[Id_A,e],e] \left[[\varphi^{-1},\varphi(e)],\,\varphi^{-1}(p'\,\varphi(e))\right] \\
&\quad = \left[[\varphi^{-1},\varphi(e)],\, e\, Id_A(\varphi^{-1}(p'\,\varphi(e)))\right] \\
&\quad = \left[[\varphi^{-1},\varphi(e)],\varphi^{-1}(p'\,\varphi(e))\right].
\end{aligned}
$$

Therefore, $(M(A) \ltimes E(A)) \ltimes A$ is a regular monoid.

Then, given $[[\varphi,e],p] \in (M(A) \ltimes E(A)) \ltimes A$, we can define for $n \in \mathbb{N}$,

$$
\begin{array}{rccl}
\alpha_{[[\varphi,e],p]}: & (A \otimes A)[x,x^{-1}] & \to & A \\
 & (a \otimes b)x^n & \mapsto & \varphi(ae)bp^n \\
 & (a \otimes b)x^{-n} & \mapsto & \varphi(ae)b(p')^n,
\end{array}
$$

which is also well defined because of (25).

This map is a biretraction in $\mathcal{H}$ just like in the previous example and the convolution product between two local biretractions $\alpha_{[[\varphi,e],p]}, \alpha_{[[\psi,f],q]}$ is given by

$$
\begin{aligned}
\alpha_{[[\varphi,e],p]} * \alpha_{[[\psi,f],q]}((a \otimes b)x^n) &= \alpha_{[[\psi,f],q]} \circ t \circ \alpha_{[[\varphi,e],p]}((a \otimes 1_A)x^n)\, \alpha_{[[\psi,f],q]}((1_A \otimes b)x^n) \\
&= \psi(\varphi(ae)p^n f)\psi(f)bq^n \\
&= \psi \circ \varphi(a\varphi^{-1}(f(\varphi(e))))b\psi(p^n)q^n \\
&= \alpha_{[[\psi\circ\varphi,\varphi^{-1}(f\varphi(e))],q\psi(p)]}((a \otimes b)x^n) \\
&= \alpha_{[[\psi,f],q][[\varphi,e],p]}((a \otimes b)x^n)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_{[[\varphi,e],p]} * \alpha_{[[\psi,f],q]}((a \otimes b)x^{-n}) &= \alpha_{[[\psi,f],q]} \circ t \circ \alpha_{[[\varphi,e],p]}((a \otimes 1_A)x^{-n})\, \alpha_{[[\psi,f],q]}((1_A \otimes b)x^{-n}) \\
&= \psi(\varphi(ae)(p')^n f)\psi(f)b(q')^n \\
&= \psi \circ \varphi(a\varphi^{-1}(f(\varphi(e))))b\psi((p')^n)(q')^n \\
&\overset{(*)}{=} \alpha_{[[\psi\circ\varphi,\varphi^{-1}(f\varphi(e))],q\psi(p)]}((a \otimes b)x^{-n}) \\
&= \alpha_{[[\psi,f],q][[\varphi,e],p]}((a \otimes b)x^{-n})
\end{aligned}
$$

for every $(a \otimes b)x^n \in (A \otimes A)[x,x^{-1}]$, where $(*)$ comes from $(q\psi(p))' = q'\psi(p')$. Therefore, the map

$$
\begin{array}{rccl}
\alpha: & ((M(A) \ltimes E(A)) \ltimes A)^{op} & \to & \mathcal{B}rt((A \otimes A)[x,x^{-1}],A) \\
 & [[\varphi,e],p] & \mapsto & \alpha_{[[\varphi,e],p]}
\end{array}
$$

is an isomorphism of semigroups, whose inverse is given by

$$\begin{array}{rccl} \lambda : & \mathcal{B}rt((A\otimes A)[x,x^{-1}],A) & \to & ((M(A)\ltimes E(A))\ltimes A)^{op} \\ & \beta & \mapsto & [[\beta\circ t, e^{\beta}],\beta(x)] \end{array}.$$

Indeed, for every $[[\varphi,e],p] \in (M(A)\ltimes E(A))\ltimes A$ and every $(a\otimes b)x^n \in (A\otimes A)[x,x^{-1}]$,

$$\begin{aligned} \alpha_{[[\alpha\circ t,e^{\alpha}],\alpha(x)]}((a\otimes b)x^n) &= \alpha\circ t\,(a\,e^{\alpha})\,b\,\alpha(x^n) \\ &= \alpha\circ t(a)\,\alpha\circ s(b)\,\alpha(x^n) \\ &= \alpha((a\otimes b)x^n) \end{aligned}$$

and
$$[\alpha_{[[\varphi,e],p]}\circ t,\, e^{\alpha_{[[\varphi,e],p]}},\alpha_{[[\varphi,e],p]}(x)] = [[\varphi,e],\varphi(e)p] = [[\varphi,e],p].$$

Moreover, this is an isomorphism of regular monoids, because α maps unity to unity:

$$\alpha_{[[\mathrm{Id}_A,1_A],1_A]}((a\otimes b)x^n) = \mathrm{Id}_A(a)\,b\,(1_A)^n = ab = \varepsilon((a\otimes b)x^n).$$

α also maps the specific pseudoinverse $\left[[\varphi^{-1},\varphi(e)],\varphi^{-1}(p'\,\varphi(e))\right]$ of $[[\varphi,e],p]$ to the pseudoinverse $\alpha^*_{[[\varphi,e],p]}$:

$$\begin{aligned} \alpha^*_{[[\varphi,e],p]}((a\otimes b)x^n) &= (\alpha_{[[\varphi,e],p]}\circ t)^{-1}\circ\alpha_{[[\varphi,e],p]}\circ S((a\otimes b)x^n) \\ &= (\alpha_{[[\varphi,e],p]}\circ t)^{-1}\circ\alpha_{[[\varphi,e],p]}((b\otimes a)x^{-n}) \\ &= \varphi^{-1}(\varphi(be)a(p')^n) \\ &= \varphi^{-1}(a(p')^n\varphi(e))\,b \\ &= \varphi^{-1}(a\,\varphi(e))b\left(\varphi^{-1}(p'\varphi(e))\right)^n \\ &= \alpha_{\left[[\varphi^{-1},\varphi(e)],\varphi^{-1}(p'\varphi(e))\right]}((a\otimes b)x^n) \end{aligned}$$

for every $(a\otimes b)x^n \in (A\otimes A)[x,x^{-1}]$ and analogously for $(a\otimes b)x^{-n}$.

4.1.1 Biretractions and the representative functions of a discrete groupoid

Remember from the first chapter that from a groupoid $\mathcal{G}$ we can construct the Hopf algebroid $\mathcal{R}_{\Bbbk}(\mathcal{G})$ of its representative functions. This Hopf algebroid is commutative over the commutative algebra $A = Fun(\mathcal{G}^{(0)},\Bbbk)$, thus we can study its biretractions.

Here we create local biretractions using the local bisections of the groupoid. From Lemma 2.2.10 we have a multiplicative map ζ from $\mathcal{R}_{\Bbbk}(\mathcal{G})$ to $Fun(\mathcal{G},\Bbbk)$ that we can adapt in a natural way to create a morphism between the bisections $\mathcal{B}(\mathcal{G})$ of $\mathcal{G}$ and the biretractions $\mathcal{B}rt(\mathcal{R}_{\Bbbk}(\mathcal{G}),A)$ of $\mathcal{R}_{\Bbbk}(\mathcal{G})$.

Proposition 4.1.12 *Let $\mathcal{G}$ a groupoid, $A = Fun(\mathcal{G}^{(0)},\Bbbk)$ and $\mathcal{H} = \mathcal{R}_{\Bbbk}(\mathcal{G})$ the Hopf algebroid of representative functions of $\mathcal{G}$ from the section* 2.2.2.1*. The map $\alpha : \mathcal{B}(\mathcal{G}) \to \mathcal{B}rt(\mathcal{R}_{\Bbbk}(\mathcal{G}),A)$, $(u,X)\mapsto\alpha_{(u,X)}$ given by*

$$\alpha_{(u,X)}(\overline{\varphi\otimes_{T_{\mathcal{E}}}p})_x = \varphi(t\circ u(x))\left(\rho^{\mathcal{E}}_{u(x)}(p(x))\right)[\![x\in X]\!]$$

for every $\overline{\varphi\otimes_{T_{\mathcal{E}}}p}\in\mathcal{H}$ and $x\in\mathcal{G}^{(0)}$ is well defined and a morphism of regular monoids.

Proof. First, α can be written as

$$\alpha_{(u,X)}(\overline{\varphi \otimes_{T_\mathcal{E}} p})_x = \zeta(\overline{\varphi \otimes_{T_\mathcal{E}} p})(u(x))[\![x \in X]\!], \tag{26}$$

with ζ from Lemma 2.2.10. Hence each $\alpha_{(u,X)}$ is well defined and multiplicative. Also, (BRT1) is valid, because

$$(\alpha_{(u,X)} \circ \overline{s}(a))_x = \alpha_{(u,X)}(\overline{1_A \otimes_{T_\mathcal{I}} a})_x = a(x)[\![x \in X]\!] = a(x)\,\alpha_{(u,X)}(1_\mathcal{H})_x$$

for every $x \in \mathcal{G}^{(0)}$ and $a \in A$. To prove that $\alpha_{(u,X)}$ satisfies (BRT2) for every bisection (u,X) of $\mathcal{G}$, remember that $(u,X)^* = (\overline{u}, t \circ u(X))$, with $\overline{u}(t \circ u(x)) = u(x)^{-1}$. Then,

$$\begin{aligned}
\alpha_{(u,X)} \circ \overline{t}\left(\alpha_{(u,X)^*}(1_\mathcal{H})\right)_x &= \alpha_{(u,X)^*}(1_\mathcal{H})_{t\circ u(x)}[\![x \in X]\!] \\
&= [\![t \circ u(x) \in t \circ u(X)]\!][\![x \in X]\!] \\
&= [\![x \in X]\!] \\
&= \alpha_{(u,X)}(1_\mathcal{H})_x
\end{aligned}$$

and $\alpha_{(u,X)^*}(1_\mathcal{H})$ is our candidate for $e^{\alpha_{(u,X)}}$. Observe that the map

$$\alpha_{(u,X)} \circ \overline{t}|_{A\alpha_{(u,X)^*}(1_\mathcal{H})} : A\alpha_{(u,X)^*}(1_\mathcal{H}) \longrightarrow A\alpha_{(u,X)}(1_\mathcal{H})$$

is injective. Indeed, for any $a \in A$ such that

$$\alpha_{(u,X)} \circ \overline{t}\left(a\alpha_{(u,X)^*}(1_\mathcal{H})\right) = 0,$$

we have that

$$a(t \circ u(y))[\![y \in X]\!] = (a\alpha_{(u,X)^*}(1_\mathcal{H}))_{t\circ u(y)}[\![y \in X]\!] = 0 \tag{27}$$

for every $y \in \mathcal{G}^{(0)}$. Then

$$\begin{aligned}
\left(a\alpha_{(u,X)^*}(1_\mathcal{H})\right)_x &= a(x)[\![x \in t \circ u(X)]\!] \\
&\overset{(*)}{=} a(t \circ u((t \circ u)^{-1}(x)))[\![(t \circ u)^{-1}(x) \in X]\!][\![x \in t \circ u(X)]\!] \\
&= 0
\end{aligned}$$

for every $x \in \mathcal{G}^{(0)}$, where we used the equation (27) in $(*)$ with $y = (t \circ u)^{-1}(x)$.

Now, observe that for every $\overline{\varphi \otimes_{T_\mathcal{E}} p} \in \mathcal{H}$ and $x \in \mathcal{G}^{(0)}$,

$$\begin{aligned}
\alpha_{(u,X)} \circ S(\overline{\varphi \otimes_{T_\mathcal{E}} p})_x &= \zeta \circ S(\overline{\varphi \otimes_{T_\mathcal{E}} p})(u(x))\,[\![x \in X]\!] \\
&= \zeta(\overline{\varphi \otimes_{T_\mathcal{E}} p})((u(x))^{-1})\,[\![x \in X]\!] \\
&= \varphi(t((u(x))^{-1}))\left(\rho^{\mathcal{E}}_{(u(x))^{-1}}\left(p(s(u(x))^{-1}\right)\right)\,[\![x \in X]\!] \\
&= \varphi(t \circ \overline{u}(t \circ u(x)))\left(\rho^{\mathcal{E}}_{\overline{u}(t\circ u(x))}(p(t \circ u(x))\right)\,[\![t \circ u(x) \in t \circ u(X)]\!] \\
&= \alpha_{(u,X)^*}\left(\overline{\varphi \otimes_{T_\mathcal{E}} p}\right)_{t\circ u(x)},
\end{aligned}$$

which implies that $\alpha_{(u,X)} \circ S = \alpha_{(u,X)^*}$. Thus $\alpha_{(u,X)} \circ \overline{t}|_{A\alpha_{(u,X)^*}(1_\mathcal{H})} : A\alpha_{(u,X)^*}(1_\mathcal{H}) \longrightarrow A\alpha_{(u,X)}(1_\mathcal{H})$ is surjective, because for every $a \in A$ and every $x \in \mathcal{G}^{(0)}$,

$$\begin{aligned}
\left(a\alpha_{(u,X)}(1_\mathcal{H})\right)_x &= \left(\alpha_{(u,X)} \circ \overline{s}(a)\right)_x \\
&= \left(\alpha_{(u,X)} \circ S \circ \overline{t}(a)\right)_x \\
&= \alpha_{(u,X)^*}(\overline{t}(a))_{t\circ u(x)} \\
&= \alpha_{(u,X)} \circ \overline{t}\left(\alpha_{(u,X)^*}(\overline{t}(a))\right)_x.
\end{aligned}$$

Therefore, $e^{\alpha_{(u,X)}} = \alpha_{(u,X)^*}(1_{\mathcal{H}})$ satisfies (BRT2) and $\alpha_{(u,X)}$ is a local biretraction.

Now, for (u,X) and (v,Y) local bisections of $\mathcal{G}$ with $(u,X)\cdot(v,Y) = (uv,Z)$,

$$\begin{aligned}
&\alpha_{(u,X)} * \alpha_{(v,Y)}(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})_x \\
&= \sum_{i=1}^{n} \alpha_{(v,Y)} \circ \bar{t} \circ \alpha_{(u,X)}(\overline{\varphi \otimes_{T_{\mathcal{E}}} e_i})_x\, \alpha_{(v,Y)}(\overline{e_i^* \otimes_{T_{\mathcal{E}}} p})_x \\
&= \sum_{i=1}^{n} \alpha_{(v,Y)}(\overline{\alpha_{(u,X)}(\overline{\varphi \otimes_{T_{\mathcal{E}}} e_i}) \otimes_{T_{\mathcal{I}}} 1_A})_x\, \alpha_{(v,Y)}(\overline{e_i^* \otimes_{T_{\mathcal{E}}} p})_x \\
&= \sum_{i=1}^{n} \alpha_{(u,X)}(\overline{\varphi \otimes_{T_{\mathcal{E}}} e_i})_{t\circ v(x)} \left(\rho^{\mathcal{I}}_{v(x)}(1_A(x))\right) [\![x \in Y]\!]\, \alpha_{(v,Y)}(\overline{e_i^* \otimes_{T_{\mathcal{E}}} p})_x \\
&= \sum_{i=1}^{n} \varphi(t\circ u\circ t\circ v(x)) \left(\rho^{\mathcal{E}}_{u\circ t\circ v(x)}(e_i(t\circ v(x)))\right) e_i^*(t\circ v(x)) \left(\rho^{\mathcal{E}}_{v(x)}(p(x))\right) [\![x \in Y]\!][\![t\circ v(x) \in X]\!] \\
&= \sum_{i=1}^{n} \varphi(t\circ uv(x))\, (e_i^*\,(\rho^{\varepsilon}_u)\, e_i)\, (t\circ v(x)) \left(\rho^{\mathcal{E}}_{v(x)}(p(x))\right) [\![x \in Z]\!] \\
&= \varphi(t\circ uv(x))\rho^{\mathcal{E}}_{u\circ t\circ v(x)}\rho^{\mathcal{E}}_{v(x)}(p(x))[\![x \in Z]\!] \\
&= \varphi(t\circ uv(x)) \left(\rho^{\mathcal{E}}_{uv(x)}(p(x))\right) [\![x \in Z]\!] \\
&= \alpha_{(uv,Z)}(\overline{\varphi \otimes_{T_{\mathcal{E}}} p})_x
\end{aligned}$$

for every $\overline{\varphi \otimes_{T_{\mathcal{E}}} p} \in \mathcal{H}$ and every $x \in \mathcal{G}^{(0)}$. Consequently, α is a morphism of semigroups. Finally, with $i : \mathcal{G}^{(0)} \to \mathcal{G}$ being the inclusion map of the groupoid, we have that

$$\begin{aligned}
\alpha_{(i,\mathcal{G}^{(0)})}(\overline{\varphi \otimes_{T_{\varepsilon}} p})_x &= \varphi(t\circ i(x)) \left(\rho^{\varepsilon}_{i(x)}(p(x))\right) [\![x \in \mathcal{G}^{(0)}]\!] \\
&= \varphi(x)(p(x)) \\
&= \varepsilon(\overline{\varphi \otimes_{T_{\varepsilon}} p})_x
\end{aligned}$$

and

$$\begin{aligned}
\alpha^*_{(u,X)}(\overline{\varphi \otimes_{T_{\varepsilon}} p})_x &= (\alpha_{(u,X)} \circ \bar{t})^{-1} \circ \alpha_{(u,X)} \circ S(\overline{\varphi \otimes_{T_{\varepsilon}} p})_x \\
&= (\alpha_{(u,X)} \circ \bar{t})^{-1} \circ \alpha_{(u,X)^*} \left(\overline{\varphi \otimes_{T_{\mathcal{E}}} p}\right)_{t\circ u(x)} \\
&= \alpha_{(u,X)^*} \left(\overline{\varphi \otimes_{T_{\mathcal{E}}} p}\right)_x
\end{aligned}$$

for every $\overline{\varphi \otimes_{T_{\mathcal{E}}} p} \in \mathcal{H}$ and $x \in \mathcal{G}^{(0)}$. Therefore, α is a morphism of regular monoids.

□

The morphism between bisections and biretractions introduced above is not necessarily a bijection. But we can prove it is an isomorphism for some specific groupoids:

Proposition 4.1.13 *Let $\mathcal{G}$ be a finite and transitive groupoid, $A = Fun(\mathcal{G}^{(0)},\Bbbk)$ and $\mathcal{H} = \mathcal{R}_{\Bbbk}(\mathcal{G})$ the Hopf algebroid of representative functions of $\mathcal{G}$. Then there exists an isomorphism of regular monoids between the bisections $\mathcal{B}(\mathcal{G})$ of $\mathcal{G}$ and the set of the biretractions $\mathcal{B}rt(\mathcal{H},A)$ of $\mathcal{H}$.*

Proof. The Proposition 4.1.12 gives a morphism between the regular monoids $\mathcal{B}(\mathcal{G})$ and $\mathcal{B}rt(\mathcal{H},A)$. Hence it is enough to prove that when $\mathcal{G}$ is a finite and transitive groupoid, this morphism is bijective.

We saw on Remark 2.2.13 that the groupoid $\mathcal{G}$ can be seen as the groupoid $\mathcal{G}^{(0)} \times G \times \mathcal{G}^{(0)}$, where G is a group, and that $\mathcal{H} \cong A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$, with $R(G)$ being the Hopf algebra of representative functions of the group G. Recall from Example 2.2.12 the Hopf algebroid structure of $A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$ given by the expressions (5). Besides that, if $(u,X) \in \mathcal{B}(\mathcal{G})$, we can write

$$\begin{aligned} u : X &\longrightarrow \mathcal{G}^{(0)} \times G \times \mathcal{G}^{(0)} \\ x &\longmapsto (\lambda(x),\varphi(x),\mu(x)) \end{aligned}$$

with $\lambda,\mu : X \to \mathcal{G}^{(0)}$ and $\varphi : X \to \mathcal{G}$. By the definition of bisections, we have that $x = s \circ u(x) = \mu(x)$ for all $x \in X$ and $t \circ u = \lambda : X \to \lambda(X)$ is a bijection. Thus u can be written as

$$u(x) = (\lambda(x),\varphi(x),x)$$

for all $x \in X$, with $\varphi : X \to G$ and $\lambda : X \to \lambda(X)$ being a bijection. Now for $a \otimes f \otimes b \in A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$, recall the function ξ and equation (4) from Example 2.2.12 that we can write

$$a \otimes f \otimes b = \xi\left(\overline{\varphi^b \otimes_{T_{\varepsilon f}} p^a}\right).$$

Hence from expression (26), the morphism α from Proposition 4.1.12 can be written for $\mathcal{G}$ as

$$\begin{aligned} \alpha_{(u,X)}(a \otimes f \otimes b)_x &= \zeta(\overline{\varphi^b \otimes_{T_{\varepsilon f}} p^a})\,(\lambda(x),\varphi(x),x)\,[\![x \in X]\!] \\ &= a(\lambda(x))\,f(\varphi(x))\,b(x)\,[\![x \in X]\!] \end{aligned}$$

for every $a \otimes f \otimes b \in A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$ and $x \in \mathcal{G}^{(0)}$. So we want to prove that the morphism $\alpha : \mathcal{B}(\mathcal{G}) \to \mathcal{B}rt(A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A, A)$, $(u,X) \mapsto \alpha_{(u,X)}$ is bijective. From now on, we use $\mathcal{H}$ to denote $A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$.

First, suppose that (u,X) and (v,Y) are both bisections of $\mathcal{G}$ with

$$u(x) = (\lambda(x),\varphi(x),x) \qquad v(y) = (\lambda'(y),\varphi'(y),y)$$

and $\alpha_{(u,X)} = \alpha_{(v,Y)}$. Then

$$[\![x \in X]\!] = \alpha_{(u,X)}(1_{\mathcal{H}})(x) = \alpha_{(v,Y)}(1_{\mathcal{H}})(x) = [\![x \in Y]\!],$$

which implies that $X = Y$. Also, for any $x \in X$,

$$1 = \delta_{\lambda(x)}(\lambda(x)) = \alpha_{(u,X)}(\delta_{\lambda(x)} \otimes 1_{R(G)} \otimes 1_A)_x = \alpha_{(v,Y)}(\delta_{\lambda(x)} \otimes 1_{R(G)} \otimes 1_A)_x = \delta_{\lambda(x)}(\lambda'(x))$$

implies that $\lambda = \lambda'$. Similarly, we have that $\varphi = \varphi'$ and, consequently, $(u,X) = (v,Y)$. Therefore, α is injective.

On the other hand, let $\beta : A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A \to A$ be a local biretraction and $a \otimes f \otimes b$ in $A \otimes_{\Bbbk} R(G) \otimes_{\Bbbk} A$. Then by definintion,

$$\begin{aligned} &\beta(1_A \otimes 1_{R(G)} \otimes b) = \beta \circ s'(b) = b\beta(1_{\mathcal{H}}); \\ &\beta(a \otimes 1_{R(G)} \otimes 1_A) = \beta \circ t'(a) \end{aligned}$$

and there exists $e^\beta \in A$ such that $\beta \circ t'(e^\beta) = \beta(1_{\mathcal{H}})$ and

$$\beta \circ t'|_{A e^\beta} : A e^\beta \longrightarrow A\beta(1_{\mathcal{H}})$$

is a bijection.

Since $\beta(1_{\mathcal{H}})$ and e^{β} are idempotents in $A = Fun(\mathcal{G}^{(0)},\Bbbk)$, we have that $\beta(1_{\mathcal{H}}) = \chi_X$ and $e^{\beta} = \chi_Y$ for some $X, Y \subseteq \mathcal{G}^{(0)}$, where $\chi_X(x) = [\![x \in X]\!]$. Denoting $\chi_x := \chi_{\{x\}}$, we have that

$$\beta(1_A \otimes 1_{R(G)} \otimes b)_x = b(x)\beta(1_{\mathcal{H}})_x = b(x)[\![x \in X]\!]. \tag{28}$$

Also,

$$\chi_X = \beta(1_{\mathcal{H}}) = \sum_{x\in\mathcal{G}^{(0)}} \beta(\chi_x \otimes 1_{R(G)} \otimes 1_A) = \sum_{x\in\mathcal{G}^{(0)}} \beta \circ t'(\chi_x)$$
$$\chi_X = \beta(1_{\mathcal{H}}) = \beta \circ t'(\chi_Y) = \sum_{x\in Y} \beta(\chi_x \otimes 1_{R(G)} \otimes 1_A) = \sum_{x\in Y} \beta \circ t'(\chi_x)$$

and if $x \neq y$, then $\beta \circ t'(\chi_x)\beta \circ t'(\chi_y) = \beta \circ t'(\chi_x\chi_y) = 0$. Thus $\beta \circ t'(\chi_x) = 0$ for all $x \in \mathcal{G}^{(0)} \setminus Y$ and there exists a bijection $\lambda : X \to Y$, $x \mapsto \lambda(x)$ such that

$$\beta \circ t'(\chi_{\lambda(x)}) = \chi_x.$$

Hence we have that for every $x \in \mathcal{G}^{(0)}$,

$$\begin{aligned}\beta(a \otimes 1_{R(G)} \otimes 1_A)_x &= \sum_{y\in\mathcal{G}^{(0)}} a(y)\beta(\chi_y \otimes 1_{R(G)} \otimes 1_A)_x \\ &= \sum_{y\in Y} a(y)\beta \circ t'(\chi_y)_x \\ &= a(\lambda(x))[\![x \in X]\!]. \end{aligned} \tag{29}$$

Finally, since $\mathcal{G}$ is transitive and finite, we have that $R(G) = Fun(G,\Bbbk)$ (SIMON, 1996), hence f can be written as

$$f(g) = \sum_{h\in G} f(h)\, p_h,$$

where $p_h(g) = [\![g = h]\!]$, for all $g \in G$, and all p_h are functions of $R(G)$. Then

$$\chi_X = \beta(1_{\mathcal{H}}) = \sum_{g\in G} \beta(1_A \otimes p_g \otimes 1_A)$$

with $\beta(1_A \otimes p_g \otimes 1_A)\beta(1_A \otimes p_h \otimes 1_A) = 0$ whenever $g \neq h$. Thus we can define a map $\varphi : X \to G$ that takes each $x \in X$ to the unique $g = \varphi(x) \in G$ such that $\beta(1_A \otimes p_{\varphi(x)} \otimes 1_A)_x = 1$. Therefore,

$$\begin{aligned}\beta(1_A \otimes f \otimes 1_A)_x &= \sum_{g\in G} f(g)\beta(1_A \otimes p_g \otimes 1_A)_x \\ &= f(\varphi(x))[\![x \in X]\!] \end{aligned} \tag{30}$$

for all $x \in \mathcal{G}^{(0)}$.

So, from expressions (28), (29) and (30), and considering the bisection $u : X \to \mathcal{G}$, $x \mapsto (\lambda(x),\varphi(x),x)$, we can write

$$\begin{aligned}\beta(a \otimes f \otimes b)_x &= \beta(a \otimes 1_{R(G)} \otimes 1_A)_x \beta(1_A \otimes f \otimes 1_A)_x \beta(1_A \otimes 1_{R(G)} \otimes b)_x \\ &= a(\lambda(x))\, f(\varphi(x))\, b(x)\, [\![x \in X]\!] \\ &= \alpha_{(u,X)}(a \otimes f \otimes b)_x.\end{aligned}$$

Observe that λ and φ do not depend on $a \otimes f \otimes b$. Therefore $\beta = \alpha_{(u,X)}$ and α is surjective. □

Remark 4.1.14 *As a particular case from the finite and transitive groupoids, take the groupoid $\mathcal{G} = X \times X$, with X being a finite set. Thus a bisection $u : Y \subseteq X \to X$ of $\mathcal{G}$ can be written for an element $y \in Y$ as*

$$u(y) = (\lambda(y), y),$$

where $\lambda : Y \to \lambda(Y)$ is a bijection, that is, any bisection of $\mathcal{G}$ is determined by a subset $Y \subseteq X$ and a bijection $\lambda : Y \to \lambda(Y) \subseteq X$, which from the Proposition 4.1.13, also determines the biretractions for the Hopf algebroid of the representative functions of $\mathcal{G}$.

On the other hand, from Example 2.2.14, the representative functions of $\mathcal{G}$ are given by $\mathcal{R}_{\Bbbk}(\mathcal{G}) \cong A \otimes_{\Bbbk} A$. From Example 4.1.10, a biretraction for $A \otimes_{\Bbbk} A$ with $A = Fun(X, \Bbbk)$ is determined by a pair $[\varphi, e]$ such that $\varphi : A \to A$ is multiplicative, $e^2 = e \in A$ and $\varphi|_{Ae} : Ae \to A\varphi(e)$ is a bijection.

These two characterizations of the biretractions are the same, because since e and $\varphi(e)$ are idempotents in A, there exist $Z, Y \subseteq X$ such that $e = \chi_Z$ and $\varphi(e) = \chi_Y$. And since X is finite and φ is multiplicative, there exists a bijection $\lambda : Y \to Z$ such that for each $y \in Y$, $\varphi(\chi_{\lambda(y)}) = \chi_y$. Therefore $[\varphi, e]$ is also determined by a subset $y \subseteq X$ and a bijection $\lambda : Y \to Z \subseteq X$.

4.1.2 The noncommutative case

We can go one step further and work with a not necessarily commutative Hopf algebroid over a commutative algebra. In this case we have only one base algebra, which is commutative, but we still have two different structures of a left-bialgebroid and of a right bialgebroid. The definition of a biretraction for this structure should be an extension of the definition for commutative Hopf algebroids.

Let us consider a Hopf algebroid $\mathcal{H}$ over a commutative algebra A such that $s_l = t_r = t$ and $s_r = t_l = s$. In this case we can use the exact same definition of biretraction that we used in the commutative case: a biretraction for $\mathcal{H}$ is a multiplicative linear map $\alpha : \mathcal{H} \to A$ satisfying

(BRT1) $\alpha \circ s(a) = a\alpha(1_{\mathcal{H}})$ for every $a \in A$.

(BRT2) There exists $e^{\alpha} \in A$ such that $\alpha \circ t(e^{\alpha}) = \alpha(1_{\mathcal{H}})$ and

$$\alpha \circ t|_{A e^{\alpha}} : A e^{\alpha} \longrightarrow A\alpha(1_{\mathcal{H}})$$

is a bijection.

Denote the set of local biretractions of $\mathcal{H}$ by $\mathcal{B}rt(\mathcal{H}, A)$.

Remark 4.1.15 *Exactly like in the commutative case, we have that for a biretraction $\alpha : \mathcal{H} \to A$, $\alpha(1_{\mathcal{H}})$ and e^{α} are idempotent elements of A and e^{α} satisfying* (BRT2) *is also unique.*

Remark 4.1.16 *Since A is commutative and α is multiplicative, we have that $\alpha(hk) = \alpha(kh)$ for every $h, k \in \mathcal{H}$.*

Remark 4.1.17 *With $s_l = t_r = t$ and $s_r = t_l = s$ we have that all maps $\varepsilon_l \circ s$, $\varepsilon_l \circ t$, $\varepsilon_r \circ s$ and $\varepsilon_r \circ t$ are the identity map Id_A. Then from the property* (P5) *of Hopf algebroids, we have the identities*

$\varepsilon_r \circ S = \varepsilon_l$ and $\varepsilon_l \circ S = \varepsilon_r$. Also, for every $a \in A$,

$$\Delta_l \circ t(a) = \Delta_l(a \triangleright 1_{\mathcal{H}}) = a \triangleright (1_{\mathcal{H}} \otimes 1_{\mathcal{H}}) = t(a) \otimes 1_{\mathcal{H}}$$
$$\Delta_l \circ s(a) = \Delta_l(1_{\mathcal{H}} \triangleleft a) = (1_{\mathcal{H}} \otimes 1_{\mathcal{H}}) \triangleleft a = 1_{\mathcal{H}} \otimes s(a).$$

Analogously, we have $\Delta_r \circ t(a) = t(a) \otimes 1_{\mathcal{H}}$ and $\Delta_r \circ s(a) = 1_{\mathcal{H}} \otimes s(a)$.

Remark 4.1.18 *The counits ε_l and ε_r are not always biretractions, because they are not necesssarily multiplicative functions, but given a biretraction α and using the notation $\Delta_r(h) = h^{(1)} \otimes_A h^{(2)}$ we have that*

$$\begin{aligned} e^{\alpha}\varepsilon_l(h) &= (\alpha \circ t)^{-1} \circ \alpha \circ t \circ \varepsilon_l(h) \\ &= (\alpha \circ t)^{-1} \circ \alpha(h^{(1)}\, S(h^{(2)})) \\ &= (\alpha \circ t)^{-1}\alpha(h^{(1)} S(h^{(2)})) \end{aligned}$$

for every $h \in \mathcal{H}$. Since $t = s_l$ is multiplicative by the definition of Hopf algebroid, we still have $\alpha \circ t$ multiplicative. And since Δ_r is multiplicative and S is antimultiplicative, we have from Remark 4.1.16 that $e^{\alpha}\varepsilon_l$ is multiplicative. Then $e^{\alpha}\varepsilon_l$ is a biretraction with $e^{e^{\alpha}\varepsilon_l} = e^{\alpha}$, because $(e^{\alpha}\varepsilon_l) \circ t|_{A\, e^{\alpha}} = Id_{A\, e^{\alpha}}$. On the other hand, for every $h \in \mathcal{H}$,

$$\alpha(1_{\mathcal{H}})\, \varepsilon_r(h) = \alpha \circ s \circ \varepsilon_r(h) = \alpha(S(h_{(1)})\, h_{(2)}),$$

which is also multiplicative. Then $\alpha(1_{\mathcal{H}})\, \varepsilon_r$ is a biretraction with $e^{\alpha(1_{\mathcal{H}})\, \varepsilon_r} = \alpha(1_{\mathcal{H}})$, because $(\alpha(1_{\mathcal{H}})\, \varepsilon_r) \circ t|_{A\,\alpha(1_{\mathcal{H}})} = Id_{A\,\alpha(1_{\mathcal{H}})}$. And using $\varepsilon_l = \varepsilon_r \circ S$ and the property (P6) *from Hopf algebroids*

$$\begin{aligned} \alpha(1_{\mathcal{H}})\, \varepsilon_l(h) &= \alpha(1_{\mathcal{H}})\, \varepsilon_r \circ S(h) \\ &= \alpha(S \circ S(h^{(2)})\, S(h^{(1)})) \\ &= \alpha \circ S(h^{(1)}\, S(h^{(2)})) \end{aligned}$$

for all $h \in \mathcal{H}$, which implies that $\alpha(1_{\mathcal{H}})\, \varepsilon_l$ is multiplicative and hence is a biretraction with $e^{\alpha(1_{\mathcal{H}})\, \varepsilon_l} = \alpha(1_{\mathcal{H}})$.

Theorem 4.1.19 *Let $\mathcal{H}$ be a Hopf algebroid over a commutative algebra A such that $s_l = t_r = t$, $s_r = t_l = s$. Then the set $\mathcal{B}rt(\mathcal{H}, A)$ is a regular semigroup with the convolution product between two biretractions α and β given by*

$$(\alpha * \beta)(h) = \beta(\alpha(h_{(1)}) \triangleright h_{(2)}) = \beta \circ t \circ \alpha(h_{(1)})\beta(h_{(2)}).$$

Proof. Like in the commutative case, this product is associative and well-defined. Now define a pseudoinverse for any biretraction $\alpha \in \mathcal{B}rt(\mathcal{H}, A)$ and $h \in \mathcal{H}$ as

$$\begin{aligned} \alpha^*(h) &:= (\alpha \circ t)^{-1} \circ \alpha \circ S(\varepsilon_l(h^{(1)}) \triangleright h^{(2)}) \\ &= (\alpha \circ t)^{-1} \circ \alpha \circ S(t \circ \varepsilon_l(h^{(1)})\, h^{(2)}) \\ &= (\alpha \circ t)^{-1} \circ \alpha(S(h^{(2)})\, s \circ \varepsilon_l(h^{(1)})) \\ &= (\alpha \circ t)^{-1} \left(\varepsilon_l(h^{(1)})\, \alpha \circ S(h^{(2)})\right). \end{aligned}$$

Observe that all maps $\alpha \circ t$, S, Δ_r and $t \circ \varepsilon_l$ are multiplicative or antimultiplicative, thus from Remark 4.1.16, α^* is multiplicative. Moreover, we have for every $a \in A$ that

$$\alpha^* \circ s(a) = (\alpha \circ t)^{-1}(\varepsilon_l(1_{\mathcal{H}})\,\alpha \circ S \circ s(a)) = (\alpha \circ t)^{-1} \circ \alpha \circ t(a) = ae^{\alpha} = a\alpha^*(1_{\mathcal{H}})$$

and

$$\begin{aligned} \alpha^* \circ t(a\alpha(1_{\mathcal{H}})) &= (\alpha \circ t)^{-1}(\varepsilon_l \circ t(a\alpha(1_{\mathcal{H}}))\,\alpha \circ S(1_{\mathcal{H}})) \\ &= (\alpha \circ t)^{-1}(a\alpha(1_{\mathcal{H}})), \end{aligned}$$

that is, $\alpha^* \circ t|_{A\alpha(1_{\mathcal{H}})} = (\alpha \circ t)^{-1}|_{A\alpha(1_{\mathcal{H}})}$. Then α^* is a biretraction with $e^{\alpha^*} = \alpha(1_{\mathcal{H}})$. Also,

$$\begin{aligned} \alpha * \alpha^*(h) &= \alpha^* \circ t \circ \alpha(h_{(1)})\,\alpha^*(h_{(2)}) \\ &= (\alpha \circ t)^{-1}(\varepsilon_l \circ t \circ \alpha(h_{(1)})\,\alpha \circ S(1_{\mathcal{H}}))\ (\alpha \circ t)^{-1}\left(\varepsilon_l\left({h_{(2)}}^{(1)}\right)\alpha \circ S\left({h_{(2)}}^{(2)}\right)\right) \\ &= (\alpha \circ t)^{-1} \circ \alpha\left(h_{(1)}\, s \circ \varepsilon_l\left({h_{(2)}}^{(1)}\right) S\left({h_{(2)}}^{(2)}\right)\right) \\ &= (\alpha \circ t)^{-1} \circ \alpha\left({h^{(1)}}_{(1)}\, s \circ \varepsilon_l\left({h^{(1)}}_{(2)}\right) S(h^{(2)})\right) \\ &= (\alpha \circ t)^{-1} \circ \alpha\,(h^{(1)}S(h^{(2)})) \\ &= (\alpha \circ t)^{-1} \circ \alpha \circ t \circ \varepsilon_l(h) \\ &= e^{\alpha}\,\varepsilon_l(h) \end{aligned} \tag{31}$$

and

$$\begin{aligned} \alpha^* * \alpha\,(h) &= \alpha \circ t \circ \alpha^*(h_{(1)})\,\alpha(h_{(2)}) \\ &= \alpha \circ t \circ (\alpha \circ t)^{-1}\left(\varepsilon_l\left({h_{(1)}}^{(1)}\right)\alpha \circ S\left({h_{(1)}}^{(2)}\right)\right)\alpha(h_{(2)}) \\ &= \varepsilon_l(h^{(1)})\,\alpha\left(S\left({h^{(2)}}_{(1)}\right){h^{(2)}}_{(2)}\right) \\ &= \varepsilon_l(h^{(1)})\,\alpha \circ s \circ \varepsilon_r(h^{(2)}) \\ &= \varepsilon_l(h^{(1)})\,\varepsilon_r(h^{(2)})\,\alpha(1_{\mathcal{H}}) \\ &= \varepsilon_l(s \circ \varepsilon_r(h^{(2)})h^{(1)})\,\alpha(1_{\mathcal{H}}) \\ &\overset{(*)}{=} \varepsilon_l(h^{(1)} s \circ \varepsilon_r(h^{(2)}))\,\alpha(1_{\mathcal{H}}) \\ &= \alpha(1_{\mathcal{H}})\,\varepsilon_l(h) \end{aligned} \tag{32}$$

for every $h \in \mathcal{H}$. Recall that for a biretraction α, we have $\alpha(hk) = \alpha(kh)$ for every $h, k \in \mathcal{H}$, which was used in $(*)$ for the biretraction $\alpha(1_{\mathcal{H}})\,\varepsilon_l$.

Now using the identities (31) and (32), we get

$$\begin{aligned} \alpha * \alpha^* * \alpha\,(h) &= \alpha \circ t \circ (\alpha * \alpha^*)(h_{(1)})\,\alpha(h_{(2)}) \\ &= \alpha \circ t(e^{\alpha}\varepsilon_l(h_{(1)}))\,\alpha(h_{(2)}) \\ &= \alpha(t \circ \varepsilon_l(h_{(1)})h_{(2)}) \\ &= \alpha(h) \end{aligned} \tag{33}$$

and

$$\begin{aligned}
\alpha^* * \alpha * \alpha^*(h) &= \alpha^* \circ t \circ (\alpha^* * \alpha)(h_{(1)})\, \alpha^*(h_{(2)}) \\
&= (\alpha \circ t)^{-1} \left(\varepsilon_l \circ t \circ (\alpha^* * \alpha)(h_{(1)})\, \alpha \circ S(1_{\mathcal{H}}))\ \varepsilon_l \left({h_{(2)}}^{(1)} \right)\ \alpha \circ S \left({h_{(2)}}^{(2)} \right) \right) \\
&= (\alpha \circ t)^{-1} \left(\alpha(1_{\mathcal{H}})\, \varepsilon_l(h_{(1)})\, \varepsilon_l \left({h_{(2)}}^{(1)} \right)\ \alpha \circ S \left({h_{(2)}}^{(2)} \right) \right) \\
&= (\alpha \circ t)^{-1} \left(\varepsilon_l \left({h^{(1)}}_{(1)} \right) \varepsilon_l \left({h^{(1)}}_{(2)} \right)\ \alpha \circ S(h^{(2)}) \right) \\
&= (\alpha \circ t)^{-1} \left(\varepsilon_l \left(t \circ \varepsilon_l \left({h^{(1)}}_{(1)} \right) {h^{(1)}}_{(2)} \right)\ \alpha \circ S(h^{(2)}) \right) \\
&= (\alpha \circ t)^{-1} (\varepsilon_l(h^{(1)})\, \alpha \circ S(h^{(2)})) \\
&= \alpha^*(h) \qquad (34)
\end{aligned}$$

for all $h \in \mathcal{H}$. Therefore, $\mathcal{B}rt(\mathcal{H}, A)$ is a regular semigroup. □

Remark 4.1.20 *Observe that given a biretraction* $\alpha : \mathcal{H} \to A$,

$$\begin{aligned}
((e^{\alpha}\, \varepsilon_l) * \alpha)\,(h) &= \alpha \circ t(e^{\alpha}\, \varepsilon_l(h_{(1)}))\, \alpha(h_{(2)}) \\
&= \alpha(t \circ \varepsilon_l(h_{(1)}) h_{(2)}) \\
&= \alpha(h)
\end{aligned}$$

and

$$\begin{aligned}
(\alpha * (\alpha(1_{\mathcal{H}})\, \varepsilon_l))\,(h) &= \alpha(1_{\mathcal{H}})\, \varepsilon_l \circ t \circ \alpha(h_{(1)})\, \alpha(1_{\mathcal{H}})\, \varepsilon_l(h_{(2)}) \\
&= \alpha(h_{(1)})\, \varepsilon_l(h_{(2)}) \\
&= \alpha(s \circ \varepsilon_l(h_{(2)}) h_{(1)}) \\
&= \alpha(h)
\end{aligned}$$

for every $h \in \mathcal{H}$. *Also note that*

$$\begin{aligned}
(e^{\alpha}\varepsilon_l)^*(h) &= (e^{\alpha}\varepsilon_l \circ t)^{-1} (\varepsilon_l(h^{(1)})\, e^{\alpha}\varepsilon_l \circ S(h^{(2)})) \\
&= e^{\alpha}\varepsilon_l(h^{(1)})\, \varepsilon_r(h^{(2)}) \\
&= e^{\alpha}\varepsilon_l(h)
\end{aligned}$$

and analogously, $(\alpha(1_{\mathcal{H}})\, \varepsilon_l)^*(h) = \alpha(1_{\mathcal{H}})\, \varepsilon_l(h)$ *for all* $h \in \mathcal{H}$.

Now, just like in the commutative case, consider the free vector space generated by the biretractions of $\mathcal{H}$ and extend linearly the convolution product to this space. Then, we have an algebra structure on the space $\Bbbk\mathcal{B}rt(\mathcal{H}, A)$, henceforth denoted by $\mathcal{B}(\mathcal{H})$.

Theorem 4.1.21 *Let* $\mathcal{H}$ *be a Hopf algebroid over a commutative algebra A such that* $s_l = t_r = t$, $s_r = t_l = s$. *Then the algebra* $\mathcal{B}(\mathcal{H})$, *generated by the set of biretractions of* $\mathcal{H}$ *with the convolution product is a quantum inverse semigroup with the comultiplication* $\underline{\Delta} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *and* $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ *defined on the biretractions as* $\underline{\Delta}(\alpha) = \alpha \otimes \alpha$ *and* $\mathcal{S}(\alpha) = \alpha^*$ *and linearly extended for* $\mathcal{B}(\mathcal{H})$.

Proof. The comultiplication $\underline{\Delta}$ is multiplicative, just like in the commutative case. Also, $\mathcal{S}$ is antimultiplicative: for every $\alpha,\beta \in \mathcal{B}rt(\mathcal{H}, A)$ and $h \in \mathcal{H}$,

$$\begin{aligned}
\mathcal{S}(\alpha * \beta)(h) &= (\alpha * \beta)^*(h) \\
&= ((\alpha * \beta) \circ t)^{-1}(\varepsilon_l(h^{(1)})\,(\alpha * \beta) \circ S(h^{(2)})) \\
&= (\alpha \circ t)^{-1} \circ (\beta \circ t)^{-1} \left(\varepsilon_l(h^{(1)})\beta \circ t \circ \alpha\left(\left(S(h^{(2)})\right)_{(1)}\right) \beta\left(\left(S(h^{(2)})\right)_{(2)}\right)\right) \\
&\overset{(*)}{=} (\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}\left(\varepsilon_l(h^{(1)})\beta \circ S(h^{(2)(1)})\right)\, \alpha \circ S(h^{(2)(2)})\right) \\
&= (\alpha \circ t)^{-1}\left(\beta^*(h^{(1)})\,\alpha \circ S(h^{(2)})\right),
\end{aligned}$$

where in $(*)$ we used the property $\Delta_l \circ S = (S \otimes_l S) \circ \Delta_r^{cop}$, which holds for any Hopf algebroid. Conversely,

$$\begin{aligned}
(\mathcal{S}(\beta) * \mathcal{S}(\alpha))(h) &= (\beta^* * \alpha^*)(h) \\
&= \alpha^* \circ t \circ \beta^*(h_{(1)})\,\alpha^*(h_{(2)}) \\
&= (\alpha \circ t)^{-1}(\varepsilon_l \circ t \circ \beta^*(h_{(1)})\,\alpha \circ S(1_{\mathcal{H}}))\,\alpha^*(h_{(2)}) \\
&= (\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}\left(\varepsilon_l\left({h_{(1)}}^{(1)}\right)\beta \circ S\left({h_{(1)}}^{(2)}\right)\right)\varepsilon_l\left({h_{(2)}}^{(1)}\right)\alpha \circ S\left({h_{(2)}}^{(2)}\right)\right) \\
&= (\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}\left(\varepsilon_l\left(h^{(1)}{}_{(1)}{}^{(1)}\right)\beta \circ S\left(h^{(1)}{}_{(1)}{}^{(2)}\right)\beta \circ t \circ \varepsilon_l\left({h^{(1)}}_{(2)}\right)\right)\alpha \circ S(h^{(2)})\right) \\
&= (\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}\left(\varepsilon_l\left(h^{(1)(1)}\right)\beta\left(S\left({h^{(1)(2)}}_{(1)}\right) t \circ \varepsilon_l\left({h^{(1)(2)}}_{(2)}\right)\right)\right)\alpha \circ S(h^{(2)})\right) \\
&= (\alpha \circ t)^{-1}\left((\beta \circ t)^{-1}\left(\varepsilon_l\left(h^{(1)(1)}\right)\beta \circ S(h^{(1)(2)})\right)\alpha \circ S(h^{(2)})\right) \\
&= (\alpha \circ t)^{-1}\left(\beta^*(h^{(1)})\,\alpha \circ S(h^{(2)})\right)
\end{aligned}$$

for every $h \in \mathcal{H}$. Consequently, $\mathcal{S}(\alpha * \beta) = \mathcal{S}(\beta) * \mathcal{S}(\alpha)$.

Finally, for checking axiom (QISG4) for any $\alpha,\beta \in \mathcal{B}rt(\mathcal{H},A)$, the expressions (31) and (32) imply that for every $h \in \mathcal{H}$,

$$\begin{aligned}
\alpha_{(1)} * \mathcal{S}(\alpha_{(2)}) * \mathcal{S}(\beta_{(1)}) * \beta_{(2)}(h) &= (\alpha * \alpha^*) * (\beta^* * \beta)(h) \\
&= (\beta^* * \beta) \circ t \circ (\alpha * \alpha^*)(h_{(1)})\,(\beta^* * \beta)(h_{(2)}) \\
&= \beta(1_{\mathcal{H}})\,\varepsilon_l \circ t \circ (\alpha * \alpha^*)(h_{(1)})\,\varepsilon_l(h_{(2)}) \\
&= \beta(1_{\mathcal{H}})\,e^{\alpha}\,\varepsilon_l(h_{(1)})\,\varepsilon_l(h_{(2)}) \\
&= \beta(1_{\mathcal{H}})\,e^{\alpha}\,\varepsilon_l(h)
\end{aligned}$$

The same result for $\mathcal{S}(\beta_{(1)}) * \beta_{(2)} * \alpha_{(1)} * \mathcal{S}(\alpha_{(2)})(h)$.

Therefore, $\mathcal{B}(\mathcal{H})$ is a Quantum Inverse Semigroup. □

Remark 4.1.22 *Consider a Hopf algebroid $\mathcal{H}$ over a commutative algebra A with $s = s_l = t_l = s_r = t_r$. A local biretraction for $\mathcal{H}$ is a linear and multiplicative map $\alpha : \mathcal{H} \to A$ that satisfies $\alpha \circ s(a) = a\alpha(1_{\mathcal{H}})$ for every $a \in A$ and there exists $e^{\alpha} \in A$ such that $\alpha \circ s(e^{\alpha}) = \alpha(1_{\mathcal{H}})$ and*

$\alpha \circ s|_{A e^\alpha} : A e^\alpha \longrightarrow A\alpha(1_{\mathcal{H}})$ is a bijection. Combining both conditions we have

$$\begin{aligned}\alpha \circ s(\alpha(1_{\mathcal{H}})\, e^\alpha) &= \alpha \circ s(\alpha(1_{\mathcal{H}}))\, \alpha \circ s(e^\alpha)\\ &= \alpha(1_{\mathcal{H}})\,\alpha(1_{\mathcal{H}})\,\alpha(1_{\mathcal{H}})\\ &= \alpha(1_{\mathcal{H}}).\end{aligned}$$

Since $\alpha(1_{\mathcal{H}})\, e^\alpha \in A\, e^\alpha$ then $\alpha(1_{\mathcal{H}})\, e^\alpha = e^\alpha$. Therefore

$$\alpha(1_{\mathcal{H}}) = \alpha \circ s(e^\alpha) = e^\alpha\, \alpha(1_{\mathcal{H}}) = e^\alpha.$$

Moreover, for every $a \in A$,

$$\alpha \circ s(a\alpha(1_{\mathcal{H}})) = a\alpha(1_{\mathcal{H}})\,\alpha(1_{\mathcal{H}}) = a\alpha(1_{\mathcal{H}}).$$

Consequently, we can describe a local biretraction for $\mathcal{H}$ as a linear and multiplicative map $\alpha : \mathcal{H} \to A$ such that $\alpha \circ s|_{A\alpha(1_{\mathcal{H}})} = \mathrm{Id}_{A\alpha(1_{\mathcal{H}})}$.

Example 4.1.23 *Recall the definition of a weak Hopf algebra from* Example 3.2.5*. A weak Hopf algebra $(H,\mu,\eta,\Delta,\varepsilon, S)$ has a structure of Hopf algebroid over the algebras $H_t = \varepsilon_t(H)$ and $H_s = \varepsilon_s(H)$ given by*

$$s_r(x) = x \qquad t_r(x) = \varepsilon(x1_{(1)})1_{(2)} \qquad \Delta_r = \pi_r \circ \Delta \qquad \varepsilon_r = \varepsilon_s$$

for every $x \in H_s$, where $\pi_r : H \otimes_{\Bbbk} H \to H \otimes_{H_s} H$ and

$$s_l(x) = x \qquad t_l(x) = \varepsilon(1_{(2)}x)1_{(1)} \qquad \Delta_l = \pi_l \circ \Delta \qquad \varepsilon_l = \varepsilon_t$$

for every $x \in H_t$, where $\pi_l : H \otimes_{\Bbbk} H \to H \otimes_{H_t} H$.

Observe that for every $x \in H_s$, x can be written as $x = \varepsilon_s(h) = 1_{(1)}\,\varepsilon(h1_{(2)})$ for some $h \in H$. Then

$$\begin{aligned}\varepsilon_s(x) &= 1_{(1)}\,\varepsilon(x1_{(2)})\\ &= 1_{(1)}\,\varepsilon(1_{(1')}\,\varepsilon(h1_{(2')})1_{(2)})\\ &= 1_{(1)}\,\varepsilon(h1_{(2')})\,\varepsilon(1_{(1')}1_{(2)})\\ &= 1_{(1)}\,\varepsilon(h1_{(2)})\\ &= x.\end{aligned}$$

Similarly, we have that $\varepsilon_t(x) = \varepsilon(1_{(1)}x)1_{(2)} = x$ for every $x \in H_t$.

Now suppose that $H_t = H_s$ and that $A := H_t = H_s$ is commutative. Then, for every $x \in A$, we have that

$$1_{(1)}\varepsilon(x1_{(2)}) = x = \varepsilon(1_{(1)}x)1_{(2)},$$

which implies that

$$\begin{aligned}t_r(x) &= \varepsilon(x1_{(1)})\,1_{(2)}\\ &= \varepsilon(\varepsilon(1_{(1')}x)\,1_{(2')}1_{(1)})\,1_{(2)}\\ &= \varepsilon(1_{(2')}1_{(1)})\,\varepsilon(1_{(1')}x)\,1_{(2)}\\ &= \varepsilon(1_{(1')}1_{(2)})\,\varepsilon(1_{(1)}x)\,1_{(2')}\\ &= \varepsilon(1_{(1)}x)\,1_{(2)}\\ &= x\end{aligned}$$

and

$$\begin{aligned} t_l(x) &= \varepsilon(1_{(2)}x)\,1_{(1)} \\ &= \varepsilon(1_{(2)}1_{(1')}\,\varepsilon(x1_{(2')}))\,1_{(1)} \\ &= \varepsilon(x1_{(2')})\,\varepsilon(1_{(2)}1_{(1')})\,1_{(1)} \\ &= \varepsilon(x1_{(2)})\,\varepsilon(1_{(1)}1_{(2')})\,1_{(1')} \\ &= \varepsilon(x1_{(2)})\,1_{(1)} \\ &= x. \end{aligned}$$

Therefore, we have that $s_l = t_r = s_r = t_l$ are all the inclusion map $A \to H$. We also have that

$$x = \varepsilon(x1_{(1)})1_{(2)} = \varepsilon(x1_{(2)})1_{(1)} = \varepsilon(1_{(2)}x)1_{(1)} = \varepsilon(1_{(1)}x)1_{(2)}$$

for every $x \in A$ and if $h \in H$.

Then by the Remark 4.1.22, a local biretraction for a weak Hopf algebra with $A := H_t = H_s$ commutative is a linear and multiplicative map $\alpha : H \to A$ such that $\alpha|_{A\alpha(1_H)} = \mathrm{Id}_{A\alpha(1_H)}$.

Example 4.1.24 *As a particular case from the previous example, consider a finite groupoid $\mathcal{G}$ and its groupoid algebra $\Bbbk\mathcal{G}$ given by*

$$\Bbbk\mathcal{G} = \left\{ \sum_{g\in\mathcal{G}} a_g\,\delta_g \mid g \in \mathcal{G}, a_g \in \Bbbk \right\}$$

with product $\delta_g\delta_h = \delta_{gh}$ if $(g,h) \in \mathcal{G}^{(2)}$ and $\delta_g\delta_h = 0$, otherwise. $\Bbbk\mathcal{G}$ is an algebra with unity

$$1_{\Bbbk\mathcal{G}} = \sum_{x\in\mathcal{G}^{(0)}} \delta_x,$$

because $(g,h) \in \mathcal{G}^{(2)}$ if, and only if, $s(g) = t(h)$ implies that

$$\left(\sum_{g\in\mathcal{G}} a_g\,\delta_g\right)\left(\sum_{x\in\mathcal{G}^{(0)}} \delta_x\right) = \sum_{\substack{g\in\mathcal{G}\\ x\in\mathcal{G}^{(0)}}} a_g\,\delta_g\delta_x = \sum_{g\in\mathcal{G}} a_g\,\delta_g\delta_{s(g)} = \sum_{g\in\mathcal{G}} a_g\delta_g$$

$$\left(\sum_{x\in\mathcal{G}^{(0)}} \delta_x\right)\left(\sum_{g\in\mathcal{G}} a_g\,\delta_g\right) = \sum_{\substack{g\in\mathcal{G}\\ x\in\mathcal{G}^{(0)}}} a_g\,\delta_x\delta_g = \sum_{g\in\mathcal{G}} a_g\,\delta_{t(g)}\delta_g = \sum_{g\in\mathcal{G}} a_g\delta_g$$

for every $\sum_{g\in\mathcal{G}} a_g\,\delta_g \in \Bbbk\mathcal{G}$. $\Bbbk\mathcal{G}$ is also a coalgebra with structure given in its base elements by $\Delta(\delta_g) = \delta_g \otimes \delta_g$ and $\varepsilon(\delta_g) = 1$. From the Example 3.2.5*, $\Bbbk\mathcal{G}$ is a weak Hopf algebra with*

$$\varepsilon_t(\delta_g) = \varepsilon(1_{(1)}\delta_g)\,1_{(2)} = \sum_{x\in\mathcal{G}^{(0)}} \varepsilon(\delta_x\delta_g)\,\delta_x = \varepsilon(\delta_{t(g)}\delta_g)\,\delta_{t(g)} = \delta_{t(g)},$$

$$\varepsilon_s(\delta_g) = 1_{(1)}\,\varepsilon(\delta_g 1_{(2)}) = \sum_{x\in\mathcal{G}^{(0)}} \delta_x\varepsilon(\delta_g\delta_x) = \delta_{s(g)}\varepsilon(\delta_g\,\delta_{s(g)}) = \delta_{s(g)}$$

and $S(\delta_g) = \delta_{g^{-1}}$ for every $g \in \mathcal{G}$. Finally, $\Bbbk\mathcal{G}$ also has a Hopf algebroid structure over the algebra $A = \langle \delta_x \mid x \in \mathcal{G}^{(0)} \rangle_{\Bbbk}$ given by $s_l = t_l = s_r = t_r$ being the inclusion maps $A \to \Bbbk\mathcal{G}$,

$$\Delta_l = \Delta_r = \pi_A \circ \Delta \qquad \varepsilon_l = \varepsilon_t \qquad \varepsilon_r = \varepsilon_s$$

and the same S. Note that $\delta_x\delta_x = \delta_x$ *and* $\delta_x\delta_y = 0$ *for all* $x \neq y$ *in* $\mathcal{G}^{(0)}$.

Observe that A is a commutative algebra. Hence by the Remark 4.1.22*, a biretraction for* $\Bbbk\mathcal{G}$ *is a linear and multiplicative map* $\alpha : \Bbbk\mathcal{G} \to A$ *such that* $\alpha|_{A\alpha(1_{\Bbbk\mathcal{G}})} = \mathrm{Id}_{A\alpha(1_{\Bbbk\mathcal{G}})}$. *Now we have for any* $\alpha : \Bbbk\mathcal{G} \to A$ *biretraction,*

- $\alpha(1_{\Bbbk\mathcal{G}})$ *is an idempotent. Then,* $\alpha(1_{\Bbbk\mathcal{G}})$ *can be written as*

$$\alpha(1_{\Bbbk\mathcal{G}}) = \sum_{x\in X} \delta_x$$

  for some $X \subseteq \mathcal{G}^{(0)}$. *If* $X = \mathcal{G}^{(0)}$, *we have a global biretraction.*

- *Fix the subset* $X^\alpha \subseteq \mathcal{G}^{(0)}$ *such that* $\alpha(1_{\Bbbk\mathcal{G}}) = \sum_{x\in X^\alpha} \delta_x$. *If* $y \in X^\alpha$,

$$\delta_y = \sum_{x\in X^\alpha} \delta_y\delta_x = \delta_y\alpha(1_{\Bbbk\mathcal{G}}) \overset{(\mathrm{BRT1})}{=} \alpha(\delta_y).$$

  Then

$$\sum_{x\in X^\alpha} \delta_x = \alpha(1_{\Bbbk\mathcal{G}}) = \sum_{y\in\mathcal{G}^{(0)}} \alpha(\delta_y) = \sum_{x\in X^\alpha} \delta_x + \sum_{z\in\mathcal{G}^{(0)}\setminus X^\alpha} \alpha(\delta_z),$$

  which implies that $\sum_{z\in\mathcal{G}^{(0)}\setminus X^\alpha} \alpha(\delta_z) = 0$, *hence for every* $y \in \mathcal{G}^{(0)} \setminus X^\alpha$,

$$\alpha(\delta_y) = \alpha(\delta_y)\left(\sum_{z\in\mathcal{G}^{(0)}\setminus X^\alpha} \alpha(\delta_z)\right) = 0.$$

- *Now for any* $g \in \mathcal{G}$, *we have that*

$$\alpha(\delta_g) = \alpha(\delta_g\delta_{s(g)}) = \alpha(\delta_g)\alpha(\delta_{s(g)})$$
$$\alpha(\delta_g) = \alpha(\delta_{t(g)}\delta_g) = \alpha(\delta_{t(g)})\alpha(\delta_g).$$

  Hence if $s(g) \notin X^\alpha$ *or* $t(g) \notin X^\alpha$, *then* $\alpha(\delta_g) = 0$. *And writing*

$$\alpha(\delta_g) = \sum_{y\in\mathcal{G}^{(0)}} a_y^g\delta_y$$

  with all a_y^g *in* $\Bbbk$, *then* $s(g), t(g) \in X^\alpha$ *imply that*

$$\alpha(\delta_g) = \alpha(\delta_g)\,\alpha(\delta_{s(g)}) = \sum_{y\in\mathcal{G}^{(0)}} a_y^g\delta_y\delta_{s(g)} = a_{s(g)}^g\delta_{s(g)}$$

  and

$$\alpha(\delta_g) = \alpha(\delta_{t(g)})\,\alpha(\delta_g) = \sum_{y\in\mathcal{G}^{(0)}} a_y^g\delta_{t(g)}\delta_y = a_{t(g)}^g\delta_{t(g)}.$$

  Hence for $\alpha(\delta_g)$ *to be nonzero, we need* $s(g) = t(g) \in X^\alpha$. *Moreover, we have that if* $s(g) = t(g) = x \in X^\alpha$ *then* $\alpha(\delta_g) = a_g\,\delta_x$ *with* $a_g \in \Bbbk\setminus\{0\}$. *Indeed, if* $\alpha(\delta_g) = 0$ *then*

$$0 = \alpha(\delta_g)\,\alpha(\delta_{g^{-1}}) = \alpha(\delta_{t(g)}) = \alpha(\delta_x) = \delta_x,$$

  which is a contradiction.

- *$\mathcal{B}rt(\Bbbk\mathcal{G},A)$ is commutative: for any $\alpha,\beta \in \mathcal{B}rt(\Bbbk\mathcal{G},A)$ we have*

$$\alpha(\delta_g) = \begin{cases} a_g\,\delta_x, & \text{if } s(g) = t(g) = x \in X^\alpha \subseteq \mathcal{G}^{(0)} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta(\delta_g) = \begin{cases} b_g\,\delta_y, & \text{if } s(g) = t(g) = y \in Y^\beta \subseteq \mathcal{G}^{(0)} \\ 0, & \text{otherwise} \end{cases}$$

with $a_g \in \Bbbk \setminus \{0\}$, $b_g \in \Bbbk \setminus \{0\}$, $a_x = 1$ and $b_y = 1$ for every $x \in X^\alpha$ and $y \in Y^\beta$. Then

$$\begin{aligned}(\alpha * \beta)(\delta_g) &= \beta \circ t \circ \alpha(\delta_g)\,\beta(\delta_g)\\ &= \beta \circ \alpha(\delta_g)\,\beta(\delta_g)\\ &= \beta(a_g\,\delta_x)\,\beta(\delta_g)\,[\![s(g) = t(g) = x \in X^\alpha]\!]\\ &= a_g b_g\,\delta_x\,[\![s(g) = t(g) = x \in X^\alpha \cap Y^\beta]\!]\\ &= (\beta * \alpha)(\delta_g)\end{aligned}$$

for every $g \in \mathcal{G}$. Observe that this means that $\mathcal{B}rt(\Bbbk\mathcal{G},A)$ is an inverse semigroup, with α^* given by

$$\alpha^*(\delta_g) = (\alpha \circ t)^{-1}\left(\varepsilon_l(\delta_g)\,\alpha \circ S(\delta_g)\right) = \delta_{t(g)}\,\alpha(\delta_{g^{-1}}) = \alpha(\delta_{t(g)}\,\delta_{g^{-1}}) = \alpha(\delta_{g^{-1}}) = \alpha \circ S(\delta_g)$$

for every $g \in \mathcal{G}$. $\mathcal{B}rt(\Bbbk\mathcal{G},A)$ also has a unity $\mathbf{1} : \Bbbk\mathcal{G} \to A$ given by

$$\mathbf{1}(\delta_g) = \delta_x\,[\![s(g) = t(g) = x]\!]$$

for every $g \in \mathcal{G}$.

With these remarks, we can represent the biretractions using the characters from the isotropy groups $G_x = \{g \in \mathcal{G} \mid s(g) = t(g) = x\}$. Being $\mathcal{G}^{(0)} = \{x_1, \ldots, x_n\}$, consider the algebra

$$\mathcal{F} = \prod_{i=1}^{n} \{\varphi_i : G_{x_i} \to \Bbbk \setminus \{0\} \text{ morphism of groups}\} \cup \{0 = \varphi_i : G_{x_i} \to \Bbbk\}$$

with the pointwise product. The elements of $\mathcal{F}$ are n–tuple of characters from the isotropy groups of G or zero maps. $\mathcal{F}$ is also a commutative inverse semigroup with $(\varphi_1, \ldots, \varphi_n)^* = (\varphi_1^*, \ldots, \varphi_n^*)$, where

$$\varphi_i^*(g) = \begin{cases} \varphi_i(g^{-1}), & \text{if } \varphi_i \neq 0 \\ 0, & \text{if } \varphi_i = 0. \end{cases}$$

For each $(\varphi_1, \ldots, \varphi_n) \in \mathcal{F}$ and $g \in \mathcal{G}$, we can define the map $\alpha_{(\varphi_1,\ldots,\varphi_n)} : \Bbbk\mathcal{G} \to A$

$$\alpha_{(\varphi_1,\ldots,\varphi_n)}(\delta_g) = \begin{cases} \varphi_i(g)\,\delta_{x_i}, & \text{if } s(g) = t(g) = x_i \\ 0, & \text{if } s(g) \neq t(g), \end{cases}$$

which is a biretraction, because for every $i = 1, \ldots, n$,

$$\alpha_{(\varphi_1,\ldots,\varphi_n)}(\delta_{x_i}) = \varphi_i(x_i)\,\delta_{x_i} = \begin{cases} \delta_{x_i}, & \text{if } \varphi_i \text{ is a morphism of groups} \\ 0, & \text{if } \varphi_i = 0, \end{cases}$$

hence $\alpha_{(\varphi_1,\dots,\varphi_n)}$ *is a local biretraction with* $X^\alpha = \{x_i \in \mathcal{G}^{(0)} \mid \varphi_i$ *is a morphism of groups*$\}$. *Then for every* $(\varphi_1,\dots,\varphi_n),(\psi_1,\dots,\psi_n) \in \mathcal{F}$ *and* $g \in \mathcal{G}$,

$$\begin{aligned}
&\alpha_{(\varphi_1,\dots,\varphi_n)} * \alpha_{(\psi_1,\dots,\psi_n)}(\delta_g)\\
&\qquad = \alpha_{(\psi_1,\dots,\psi_n)} \circ \alpha_{(\varphi_1,\dots,\varphi_n)}(\delta_g)\,\alpha_{(\psi_1,\dots,\psi_n)}(\delta_g)\\
&\qquad = \alpha_{(\psi_1,\dots,\psi_n)}(\varphi_i(g)\,\delta_{x_i})\,\alpha_{(\psi_1,\dots,\psi_n)}(\delta_g)\,[\![s(g) = t(g) = x_i]\!]\\
&\qquad = \varphi_i(g)\,\psi_i(g)\,\psi_i(x_i)\,\delta_{x_i}\,[\![s(g) = t(g) = x_i]\!]\\
&\qquad = \varphi_i(g)\,\psi_i(g)\,\delta_{x_i}\,[\![s(g) = t(g) = x_i]\!]\\
&\qquad = \alpha_{(\varphi_1\psi_1,\dots,\varphi_n\psi_n)}(\delta_g)\\
&\qquad = \alpha_{(\varphi_1,\dots,\varphi_b)(\psi_1,\dots,\psi_n)}(\delta_g)
\end{aligned}$$

and the map

$$\begin{array}{rccc}
\alpha: & \mathcal{F} & \longrightarrow & \mathcal{B}rt(\Bbbk\mathcal{G},A)\\
& (\varphi_1,\dots,\varphi_n) & \longmapsto & \alpha_{(\varphi_1,\dots,\varphi_n)}
\end{array}$$

is an isomorphism of inverse semigroups.

Observe that $\mathcal{B}rt(\Bbbk\mathcal{G},A)$ *is a commutative inverse semigroup with unity, but is not necessarily a group. Indeed, for a biretraction* $\alpha_{(\varphi_1,\dots,\varphi_n)}$ *with* $\varphi_i \neq 0$ *for every* i *such that* $x_i \in X^\alpha \subseteq \mathcal{G}^{(0)}$,

$$\begin{aligned}
\left(\alpha_{(\varphi_1,\dots,\varphi_n)} * \alpha^*_{(\varphi_1,\dots,\varphi_n)}\right)(\delta_g) &= \varphi_i(g)\varphi_i(g^{-1})\,\delta_{x_i}\,[\![s(g) = t(g) = x_i \in X^\alpha]\!]\\
&= \varphi_i(x_i)\,\delta_{x_i}\,[\![s(g) = t(g) = x_i \in X^\alpha]\!]\\
&= \delta_{x_i}\,[\![s(g) = t(g) = x_i \in X^\alpha]\!]
\end{aligned}$$

for every $g \in \mathcal{G}$, *which is not the unity of* $\mathcal{B}rt(\Bbbk\mathcal{G},A)$, *unless* $X^\alpha = \mathcal{G}^{(0)}$, *that is, unless* $\alpha_{(\varphi_1,\dots,\varphi_n)}$ *is a global biretraction. Therefore, we have that* $Gl\mathcal{B}rt(\Bbbk\mathcal{G},A)$ *is a group.*

Example 4.1.25 (The algebraic quantum torus) *Consider an algebra* T_q *over* $\mathbb{C}$, *generated by two invertible elements* U *and* V *satisfying* $UV = q\,VU$, *with* $q \in \mathbb{C}^\times$. *The algebra* T_q *has a structure of Hopf algebroid over the commutative* $\mathbb{C}$*-algebra* $A = \mathbb{C}[U]$:

- $s = s_l = t_l = s_r = t_r : A \to T_q$ *is the inclusion map;*
- $\Delta_l(U^nV^m) = U^nV^m \otimes_A V^m$ *and* $\varepsilon_l(U^nV^m) = U^n$;
- $\Delta_r(V^mU^n) = V^mU^n \otimes_A V^m$ *and* $\varepsilon_r(V^mU^n) = U^n$;
- $S(U^nV^m) = V^{-m}U^n$.

Observe that the only idempotent of A *is* 1. *Then we can only have global biretractions for* T_q. *By the Remark 4.1.22, a global biretraction for* T_q *can be described as a linear and multiplicative map* $\alpha : T_q \to A$ *such that* $\alpha|_A = Id_A$.

Moreover, since α *is multiplicative, we have that*

$$\alpha(V)\,\alpha(V^{-1}) = \alpha(V^{-1})\,\alpha(V) = \alpha(V^{-1}V) = \alpha(1_{\mathbb{C}}) = 1_{\mathbb{C}} \;\Rightarrow\; \alpha(V)^{-1} = \alpha(V^{-1}),$$

which implies that $\alpha(V)$ *is invertible in* A, *and consequently,*

$$U\alpha(V) = \alpha(UV) = q\,\alpha(VU) = q\,U\alpha(V)$$

$$\Rightarrow q = 1_{\mathbb{C}}.$$

So we only have global biretractions for the commutative torus T_1. *In this case, we have that a global biretraction for* T_1 *is a multiplicative and linear map* $\alpha : T_1 \to A$ *such that* $\alpha(U) = U$ *and* $\alpha(V) = q_\alpha U^{t_\alpha}$, *with* $q_\alpha \in \mathbb{C}$ *and* $t_\alpha \in \mathbb{Z}$.

Moreover, since the zero map is not a global biretraction, any global biretraction $\alpha : T_1 \to A$ *is in fact a morphism of algebras (since* $\alpha(1_{T_1}) = 1_A$*).* T_1 *and* A *are algebras of Laurent polynomials,* $T_1 = \mathbb{C}[U,U^{-1},V,V^{-1}]$ *and* $A = \mathbb{C}[U,U^{-1}]$. *General arguments from algebraic geometry show that algebra morphisms* $\alpha : T_1 \to A$ *correspond to maps* $f : \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$ *whose entries are Laurent polynomials in* z, *i.e.,* $f(z) = (p_1(z), p_2(z))$ *with* $p_i(U) \in A$. *Given such a map, the associated morphism of algebras is* $\alpha(U) = p_1(U), \alpha(V) = p_2(V)$.

In particular, given a real number θ *and an integer* n, *the biretraction* $\alpha : T_1 \to A$ *given by* $\alpha(V) = e^{2\pi i\theta}U^n$ *corresponds to the the map* $f : \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times, f(z) = (z, e^{2\pi i\theta}z^n)$. *The restriction of* f *to the unit circle* S^1 *yields the map*

$$g : S^1 \to S^1 \times S^1, \qquad e^{2\pi it} \mapsto (e^{2\pi it}, e^{2\pi i(\theta+tn)}).$$

Hence biretractions of the Hopf algebroid T_1 *include imersions of* T_1 *in* T^2. *Also, we can say that* α *rolls up the unit circle* S^1 *around the torus* T^2. *Indeed, observe that*

$$\begin{aligned} \alpha * \alpha(V) &= \alpha \circ \alpha(V)\,\alpha(V) \\ &= \alpha(e^{2\pi i\theta}U^n)\, e^{2\pi i\theta}U^n \\ &= e^{2\pi i\theta}\alpha(U)^n\, e^{2\pi i\theta}U^n \\ &= e^{2\pi i\, 2\theta}U^{2n} \end{aligned}$$

and, analogously,

$$\alpha^k := \underbrace{\alpha * \cdots * \alpha}_{k\ times}(V) = e^{2\pi i\, k\theta}U^{kn}$$

Consequently, we can associate α^k *with the restriction*

$$g_k : S^1 \to S^1 \times S^1, \qquad e^{2\pi it} \mapsto (e^{2\pi it}, e^{2\pi ik(\theta+tn)}).$$

We remark that g *is a closed curve that starts and ends at* $(1, e^{2\pi i\theta})$ *for* $t = 0$ *and for* $t = 1$, *and runs along the torus as shown in* Figure 1*.*

Observe that the curve g_2 *acts similarly to* g *but rolls up twice as fast (vertically) along the torus, starting and ending at* $(1, e^{4\pi i\theta})$. *In general, the map* g_k *rolls up the torus* k*–times faster than* g *vertically, starting and ending at* $(1, e^{2k\pi i\theta})$.

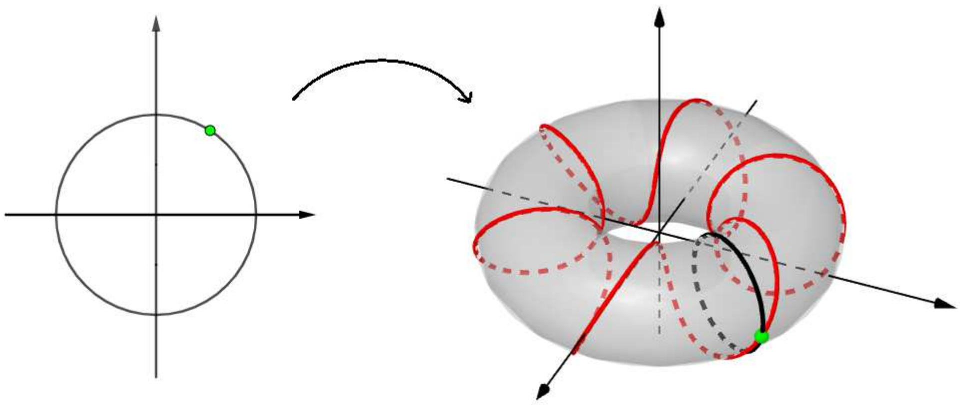

Figure 1 – Representation of the curve g

5 CONCLUSION

We started this work with a question: can we find a good generalization of inverse semigroups and Hopf algebras which can play the same role with Hopf algebroids as inverse semigroups do with groupoids?

Trying to find the answer, we first analysed the definition that relates inverse semigroups and groupoids: the bisections. We started working with known examples of commutative Hopf algebroids and tried to find the best definition of biretractions that would generalize the notion of bisections. Trying to dualize the definition of bisections for these Hopf algebroids, we started analysing maps from the Hopf algebroid to the base algebra defined in a natural way, and kept finding maps α that were at the same time multiplicative, right-module morphisms and that their composition with the target map were partially defined bijections.

Hence our first definition of a biretraction for a commutative Hopf algebroid $\mathcal{H}$ over a commutative algebra A was that $\alpha : \mathcal{H} \to A$ should be a multiplicative map satisfying

(BRT1) $\alpha \circ s(a) = a\alpha(1_{\mathcal{H}})$ for every $a \in A$.

(BRT2) The restriction

$$\alpha \circ t|_{A\alpha(1_{\mathcal{H}})} : A\alpha(1_{\mathcal{H}}) \longrightarrow A\alpha(1_{\mathcal{H}})$$

is a bijection.

With this definition, we were able to define a product and a pseudoinverse for the local biretractions in a way that the set $\mathcal{B}rt(\mathcal{H},A)$ is a regular monoid.

At the same time, we wanted to use the example of the Hopf algebroid of representative functions of a groupoid $\mathcal{G}$ to relate the set $\mathcal{B}(\mathcal{G})$ of all local bisections of groupoid with the set $\mathcal{B}rt(\mathcal{H},A)$ of all local biretractions of the Hopf algebroid of its representative functions. We had a map from $\mathcal{B}(\mathcal{G})$ to $\mathcal{B}rt(\mathcal{H},A)$ defined in a natural way, but this map was not necessarily a morphism os regular monoids. Analysing this map it was clear that we need to redefine the condition (BRT2). The problem is that the partial bijection $\alpha \circ t|_{A\alpha(1_{\mathcal{H}})}$ has equals domain and image. Thus inspired by the example of the representative functions and by the classic inverse semigroup of partially defined bijections of a set, which considers different domains and images, we thought that would be better to restrict the map $\alpha \circ t$ to the ideal Ae^{α}, where $\alpha \circ t(e^{\alpha}) = \alpha(1_{\mathcal{H}})$, such that the restriction $\alpha \circ t|_{Ae^{\alpha}} : Ae^{\alpha} \to A\alpha(1_{\mathcal{H}})$ is a bijection. And it turned out that the element e^{α} defined in this way is unique and idempotent.

On the other hand, we wanted the definition of quantum inverse semigroups to be a generalization of inverse semigroups in the same sense that Hopf algebras are a generalization of groups. With this is mind, it got created the Definition 3.1.1. Then we just needed to adjust the definitions of the product and the pseudoinverse in $\mathcal{B}rt(\mathcal{H},A)$ so the algebra $\mathcal{B}(\mathcal{H})$ generated by the local biretractions has a structure of quantum inverse semigroup, just like the bisections of a groupoid form an inverse semigroup.

So we were able to generalize the definition of bisections for Hopf algebroids and also created a structure that generalize inverse semigroups at least with this particular relation.

We also created a definition of biretractions for non necessarily commutative Hopf algebroids over a commutative algebra with $s_l = t_r$ and $t_l = s_r$. Observe that in a lot of instances from the proofs of the Theorems 4.1.19 and 4.1.21 we used the commutative property of the base algebra along with the properties $s_l = t_r$ and $s_r = t_l$ so we could maneuver between the left and right bialgebroid structures. Thus it seems like for the general case, we should define as biretraction a map that creates a relation between the two structures. Also, it is not expected, in the general case of noncommutative Hopf algebroids, for the biretractions to be multiplicative, but this definition should be an extension of the definition for the commutative case.

It remains for the future to find a good definition of biretractions for any Hopf algebroid. Being $\mathcal{H} = (\mathcal{H}_l, \mathcal{H}_r, S)$ a Hopf algebroid with bialgebroid structures $(\mathcal{H}_l, s_l, t_l, \Delta_l, \varepsilon_l)$ and $(\mathcal{H}_r, s_r, t_r, \Delta_r, \varepsilon_r)$, it is an initial thought to replace (BRT1) and the multiplicative property valid for the commutative case by the following conditions: a local biretraction of $\mathcal{H}$ is a pair of linear maps $\alpha : \mathcal{H} \to A$ and $\overline{\alpha} : \mathcal{H} \to \overline{A}$ satisfying for every $a \in A$, $b \in \overline{A}$ and $h,k \in \mathcal{H}$,

$$\alpha(t_l(a)\,h) = \alpha(h \triangleleft a) = \alpha(h)\,a \qquad \alpha(hk) = \alpha(h\,t_l \circ \alpha(k)) \tag{35}$$

$$\overline{\alpha}(h\,s_r(b)) = \overline{\alpha}(h \blacktriangleleft b) = \overline{\alpha}(h)\,b \qquad \overline{\alpha}(hk) = \overline{\alpha}(s_r \circ \overline{\alpha}(h)\,k). \tag{36}$$

that is, α should be a left A–module morphism and $\overline{\alpha}$ a $\overline{A}$–morphism, which extends (BRT1) and the second line is what replaces the multiplicative property. These conditions are inspired by (XIAO, 2021), where was proposed a definition of "bisections" for bialgebroids.

With these conditions, all maps $\alpha \circ s_l$, $\alpha \circ t_r$, $\overline{\alpha} \circ s_l$ and $\overline{\alpha} \circ t_r$ are multiplicative or antimultiplicative. Indeed, for every $a \in A$ and $h \in \mathcal{H}$,

$$\alpha(s_l(a)\,h) = \alpha(s_l(a)\,t_l \circ \alpha(h)) = \alpha(t_l \circ \alpha(h)\,s_l(a)) = \alpha \circ s_l(a)\,\alpha(h),$$

which implies that

$$\alpha \circ s_l(a_1\,a_2) = \alpha(s_l(a_1)\,s_l(a_2)) = \alpha \circ s_l(a_1)\,\alpha \circ s_l(a_2)$$

for all $a_1, a_2 \in A$. Consequently, $\alpha \circ s_l$ is multiplicative. Analogously, using the facts that s_l is multiplicative, t_r is antimultiplicative and that $s_l \circ \varepsilon_l \circ t_r = t_r$ and $t_r \circ \varepsilon_r \circ s_l = s_l$, we have that

$$\alpha(t_r(b)\,h) = \alpha \circ t_r(b)\,\alpha(h) \qquad \overline{\alpha}(h\,t_r(b)) = \overline{\alpha} \circ t_r(b)\,\overline{\alpha}(h) \qquad \overline{\alpha}(h\,s_l(a)) = \overline{\alpha} \circ s_l(a)\,\overline{\alpha}(h)$$

for every $a \in A$, $b \in \overline{A}$ and $h \in \mathcal{H}$. Hence $\overline{\alpha} \circ s_l$ and $\alpha \circ t_r$ are antimultiplicative and $\overline{\alpha} \circ t_r$ is multiplicative. And these are properties that will probably facilitate the work of defining a condition for the pair $(\alpha, \overline{\alpha})$ that extends (BRT2).

Moreover, is possible to define products for the maps satisfying the conditions (35) and (36): for any maps $\alpha, \beta : \mathcal{H} \to A$ satisfying (35) and $\overline{\alpha}, \overline{\beta} : \mathcal{H} \to \overline{A}$ satisfying (36), define for each $h \in \mathcal{H}$,

$$(\alpha * \beta)(h) = \beta(\alpha(h_{(1)}) \triangleright h_{(2)}) = \beta(s_l \circ \alpha(h_{(1)})\,h_{(2)}) = \beta \circ s_l \circ \alpha(h_{(1)})\,\beta(h_{(2)})$$

$$(\overline{\alpha} * \overline{\beta})(h) = \overline{\beta}(\overline{\alpha}(h^{(1)}) \blacktriangleright h^{(2)}) = \overline{\beta}(h^{(2)}\,t_r \circ \overline{\alpha}(h^{(1)})) = \overline{\beta} \circ t_r \circ \overline{\alpha}(h^{(1)})\,\overline{\beta}(h^{(2)}).$$

These products are associative and well defined, because for every $a \in A$ and $h, k \in \mathcal{H}$,

$$\begin{aligned}
(\alpha * \beta)(t_l(a)\, h) &= \beta \circ s_l \circ \alpha(h_{(1)})\beta(t_l(a)\, h_{(2)}) = \beta \circ s_l \circ \alpha(h_{(1)})\beta(h_{(2)})\, a = (\alpha * \beta)(h)\, a \\
(\alpha * \beta)(h\, t_l \circ (\alpha * \beta)(k)) &= \beta \circ s_l \circ \alpha(h_{(1)})\beta(h_{(2)}\, t_l \circ (\alpha * \beta)(k)) \\
&= \beta \circ s_l \circ \alpha(h_{(1)})\beta(h_{(2)}\, t_l\left(\beta \circ s_l \circ \alpha(k_{(1)})\beta(k_{(2)})\right)) \\
&= \beta \circ s_l \circ \alpha(h_{(1)})\beta(h_{(2)}\, t_l \circ \beta(k_{(2)})\, t_l \circ \beta \circ s_l \circ \alpha(k_{(1)})) \\
&= \beta \circ s_l \circ \alpha(h_{(1)})\beta(h_{(2)}\, t_l \circ \beta(k_{(2)})\, s_l \circ \alpha(k_{(1)})) \\
&= \beta \circ s_l \circ \alpha(h_{(1)})\beta(h_{(2)}\, s_l \circ \alpha(k_{(1)})\, t_l \circ \beta(k_{(2)})) \\
&\overset{(*)}{=} \beta \circ s_l \circ \alpha(h_{(1)}\, t_l \circ \alpha(k_{(1)}))\beta(h_{(2)}\, t_l \circ \beta(k_{(2)})) \\
&= \beta \circ s_l \circ \alpha(h_{(1)}\, k_{(1)})\beta(h_{(2)}\, k_{(2)}) \\
&= (\alpha * \beta)(hk),
\end{aligned}$$

where we used the property of the Takeuchi product on $(*)$. Hence $\alpha * \beta$ satisfies (35) and, analogously, $\overline{\alpha} * \overline{\beta}$ satisfies (36).

We can call the maps $\alpha : \mathcal{H} \to A$ satisfying (35) left-retractions of $\mathcal{H}$ and the maps $\overline{\alpha} : \mathcal{H} \to \overline{A}$ right-retractions of $\mathcal{H}$. This way, a local biretractions of $\mathcal{H}$ is a pair of a left-retraction and a right-retraction. Also, it is expected for a "pseudoinverse" of a left-retraction to be a right-retraction and for a "pseudoinverse" α^* of a right-retraction α to be a left-retraction, because of the action of S. Therefore, for this definition to be an extension of the definition for commutative Hopf algebroids, we can redefine a local biretraction of $\mathcal{H}$ as a pair (α,α^*) of a left-retraction and its "pseudoinverse". So it remains to create a condition that extends (BRT2) and a product that enable us to define a pseudoinverse $(\alpha,\alpha^*)^*$.

Finally, we want, in future works, to answer the questions:

- Can we extend the definition of local biretractions for any Hopf algebroid? Will we also be able to create a quantum inverse semigroup generated by these biretractions?
- Can we use the quantum inverse semigroups and Hopf algebroids' structures to extend other relations between inverse semigroups and groupoids?

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