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Étale categories, restriction semigroups, groupoid extensions, and their operator algebras

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Natã Machado[1]

Étale categories, restriction semigroups, groupoid extensions, and their operator algebras

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Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de doutor em Matemática.

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RESUMO

Definimos álgebras de operadores não auto-adjuntas associadas a categorias étale e a semigrupos de restrição, e mostramos também que existe uma versão reduzida para estas álgebras no caso de categorias cancelativas à esquerda e semigrupos de restrição amplos à esquerda. Além disso, definimos a álgebra produto semicruzado de uma ação étale de um semigrupo de restrição em uma C^*-algebra, e esta se torna o componente principal para conectar a álgebra de operadores de um semigrupo de restrição com a álgebra de sua categoria étale correspondente. Mostramos também que nos casos particulares de grupoides étale e semigrupos inversos nossas álgebras de operadores coincidem com as C^*-algebras destes objetos. Além disso, apresentamos uma teoria de classificação geométrica de extensões não Abelianas de groupoides e que generaliza a teoria de classificação de Westman no caso abeliano, bem como a teoria de classificação de extensões de grupo devida a Schreier, Elenberg e Mac-lane. Como aplicação de nossas técnicas, demonstramos que uma extensão de grupoides $\mathcal{N} \to \mathcal{E} \to \mathcal{G}$ dá origem a um produto cruzado por grupoide de $\mathcal{G}$ pelo anel de grupoide de $\mathcal{N}$, e este recupera o anel de grupoide de $\mathcal{E}$ a menos de isomorfismo.

Palavras-chave: Categorias étale. semigrupos de restrição. ações de semigrupos de restrição. álgebras de operadores não auto-adjuntas. extensões não Abelianas de grupoides. sistema de fatores, cohomologia de grupoides. produtos cruzados por grupoides. C^*-álgebra de grupoide

RESUMO EXPANDIDO

INTRODUÇÃO

Em 1926, Brandt [14] introduziu a noção de grupoide como a generalização de grupo. Já a noção de categorias topológicas, e consequentemente de grupoides topológicos, teve sua gênese no trabalho de Ehresmann [21]. Desde então, a teoria de grupoides desenvolveu-se de modo que aplicações são encontradas em diversas áreas como geometria diferencial, teoria de folheações, topologia diferencial, geometria algébrica, álgebras de operadores, teoria ergódica, [15].

O estudo de C^*-álgebras associadas a grupoides topológicos teve início em 1980 na tese de doutorado de Jean Renault [55]. Neste mesmo trabalho, Renault estudou os grupoides chamados r-discretos, que mais tarde seriam chamados grupoides *étale*. Em [50], Paterson mostrou que todo grupoide étale $\mathcal{G}$ é isomorfo a um grupoide de germes $S \rtimes_\theta X$ de uma ação θ de um semigrupo inverso S em um espaço topológico X. Além disso, sob hipóteses no grupoide $\mathcal{G}$ e no semigrupo inverso S, mostrou que $C^*(\mathcal{G})$ é isomorfa a C^*-álgebra produto cruzado $S \rtimes_\alpha \mathrm{C}_0(X)$ obtida a partir da ação induzida de θ, $\alpha : S \to \mathrm{C}_0(X)$. Em 2008, com o objetivo de estudar C^*-álgebras associadas a objetos combinatórios, Exel [24] aperfeiçoou os resultados de Paterson removendo hipóteses tanto no semigrupo quanto na ação.

O problema de classificar todas as extensões de um grupo G por um grupo N foi sistematicamente tratado em muitos trabalhos [9, 22, 23, 59]. O conceito fundamental subjacente a uma extensão de grupos de $N \to E \to G$ é o de *sistema de fatores*, que é um par (L, σ) consistindo de uma ação $L : G \to \mathrm{Aut}(N)$ de G em N torcida por um 2-cociclo $\sigma : G \times G \to N$. Exemplos de extensões de grupos podem ser encontrados em quase todas as disciplinas da matemática moderna. Por exemplo, extensões não Abelianas de grupos de Lie aparecem naturalmente no contexto de fibrados principais sobre variedades compactas. No contexto de extensões de grupoides, extensões de grupoides por fibrados de grupos Abelianos foram classificadas por Westman, veja por exemplo [55, 69]. Já o caso de extensões por fibrados não Abelianos foi tratado em partes por [12].

OBJETIVOS

Existe uma generalização natural para um grupoide étale que é o conceito de categoria étale. Já para semigrupos inversos, temos a generalização natural que é o conceito de semigrupo de restrição. Gudryavtseva e Lawson estabeleceram uma equivalência categórica entre a categoria das categorias étale e a categoria dos semigrupos de restrição, [36]. Nosso principal objetivo no estudo de categorias étale e semigrupos de restrição é associar algebras de operadores a estes objetos de tal forma que resultados previamente conhecidos no contexto de grupoides étale e semigrupos inversos sejam estendidos para estas novas classes.

No estudo de extensões de grupóide, temos o objetivo de classificar extensões de um grupóide $\mathcal{G}$ por um fibrado não Abeliano $\mathcal{N}$. Além disso, se L é uma família *exterior*, buscamos estabelecer um critério para a existência de sistemas de fatores do tipo $(L, \cdot)$. Também queremos investigar se nossos métodos de classificação se transferem para a classe de anéis chamados *produto cruzados por grupoide*, e verificar se a C^*-álgebra de uma extensão tem uma estrutura de produto cruzado por grupoide.

METODOLOGIA

Pesquisa bibliográfica em livros e periódicos. Apresentação de seminários e reuniões de discussão com pesquisadores especialistas no tema para apresentação e avaliação dos resultados obtidos.

RESULTADOS E DISCUSSÃO

No que tange ao estudo de categorias étale e semigrupos de restrição, dada uma categoria étale $\mathcal{C}$ e um semigrupo de restrição S, definimos as álgebras de operadores de $\mathcal{C}$ e de S, denotadas respectivamente por $\mathcal{A}(\mathcal{C})$ e $\mathcal{A}(S)$. Além disso, definimos uma série de novos conceitos e objetos como por exemplo ações de semigrupos de restrição em espaços topológicos e em C^*-álgebras, a categoria de germes de uma ação topológica de um semigrupo de restrição e a álgebra produto semicruzado associada a uma ação C^*-algébrica de semigrupo restrição. Além disso, mostramos que estes novos conceitos estendem resultados previamente conhecidos do contexto de grupoides étale e semigrupos de restrição como por exemplo:

Theorem 3.4. *Seja $\mathcal{D}$ uma categoria étale. Então $\mathcal{D}$ é isomorfa a categoria de germes $\mathcal{C}(\theta, \mathrm{Bis}(\mathcal{D}), \mathcal{D}^{(0)})$, em que $\theta : \mathrm{Bis}(\mathcal{D}) \to \mathcal{I}(\mathcal{D}^{(0)})$ é a ação do semigrupo de restrição das bisseções de $\mathcal{D}$ no espaço das unidades $\mathcal{D}^{(0)}$.*

Theorem 3.18. *Seja (S, E, λ, ρ) um semigrupo de restrição, e X um espaço localmente compacto Hausdorff e segundo-enumerável. Além disso, seja $\theta : S \to \mathcal{I}(X)$ uma ação étale, e considere α a ação induzida de θ. Então $\mathcal{A}(\mathcal{C}(\theta, S, X))$ é isomorfa ao produto semicruzado $\mathrm{C}_0(X) \rtimes_\alpha S$.*

Theorem 3.24. *Seja $\mathcal{G}$ um grupoide étale, e suponha que $\mathcal{G}^{(0)}$ é segundo-enumerável. Então a álgebra de operadores de $\mathcal{G}$, $\mathcal{A}(\mathcal{G})$, é isomorfa a $C^*(\mathcal{G})$.*

Com relação ao estudo de extensões não Abelianas de grupoides, mostramos que dados um grupoide $\mathcal{G}$ e um fibrado não Abeliano de grupos $\mathcal{N}$ sobre $\mathcal{G}^{(0)}$, uma família exterior L dá origem a um 3-cociclo $\chi(L) \in H^3(\mathcal{G}, Z(\mathcal{N}))_L$. Este, por sua vez, dá origem a um critério de existência de um sistema de fatores da forma (L, σ),

Corollary 4.29. *Seja L uma familia exterior. Então existe um* 2*-cociclo* σ *de modo que* (L, σ) *é um sistema de fatores se e somente se* $\chi(L)$ *é trivial em* $H^3(\mathcal{G}, Z(\mathcal{N}))_L$ *.*

Utilizamos também uma metodologia similar a empregada na construção do resultado acima para classificar anéis do tipo produto cruzado por grupóide (Section 4.4).

Além disso, dada uma extensão $\mathcal{N} \to \mathcal{E} \to \mathcal{G}$, a Proposição 4.39 nos dá uma decomposição da C^*-álgebra de $\mathcal{E}$ em termos do fibrado de $*$-álgebras $\mathcal{C}[\mathcal{N}]$ e do sistema de fatores (L, σ) associado à extensão $\mathcal{E}$.

CONSIDERAÇÕES FINAIS

Os resultados obtidos nesta tese respondem a uma questão publicada no artigo [36] que versa sobre a existência de álgebras não auto-adjuntas associadas a categorias étale e semigrupos de restrição. Podemos afirmar que nosso trabalho responde a questão feita pelos autores de [36] mas concomitantemente abre margem para uma infinidade de investigações futuras tanto do ponto de vista algébrico, quanto categórico. Na parte em que estudamos extensões não Abelianas de grupoides, provamos resultados de classificação que vão de encontro à teoria desenvolvida por Schreier e os estendemos para anéis do tipo produto cruzado.
Finalizando, elencamos abaixo possíveis linhas de investigação que seguem os resultados deste trabalho

a) Provar o Corolário 3.19 para as álgebras reduzidas.
b) Mostrar que os isomorfismos construídos nos Teoremas 2.33 e 3.18 são completamente isométricos e não apenas isométricos.
c) Encontrar critérios de simplicidade para as álgebras de operadores cheia e reduzida de uma categoria étale.
d) Dada uma categoria étale $\mathcal{C}$, considere B o C^*-envelope da álgebra de operadores $\mathcal{A}(C)$. Nos perguntamos se existe um grupoide étale $\mathcal{G}$ de forma que $B = C^*(\mathcal{G})$.
e) Generalizar álgebras de Steinberg de um grupóide com coeficientes num feixe de anéis (ver [27]) através de sistemas de fatores da Definição 4.5.

Palavras-chave: Categorias étale. semigrupos de restrição. ações de semigrupos de restrição. álgebras de operadores não auto-adjuntas. extensões não Abelianas

de grupoides. sistema de fatores, cohomologia de grupoides. produtos cruzados por grupoides. C^*-álgebra de grupoide

ABSTRACT

We define non-self-adjoint operator algebras associated with étale categories and restriction semigroups, and we also show that these algebras have a reduced version in the cases where the category is left cancellative and the restriction semigroup is left-ample. Moreover, we define the semicrossed product algebra of an étale action of a restriction semigroup on a C^*-algebra, which turns out to be the key point when connecting the operator algebra of a restriction semigroup to the operator algebra of its associated étale category. We also prove that in the particular cases of étale groupoids and inverse semigroups, our operator algebras coincide with the C^*-algebras of the referred objects. Furthermore, we present a geometrically oriented classification theory for non-Abelian extensions of groupoids generalizing the classification theory for Abelian extensions of groupoids by Westman as well as the familiar classification theory for non-Abelian extensions of groups by Schreier and Eilenberg-Mac Lane. As an application of our techniques we demonstrate that each extension of groupoids $\mathcal{N} \to \mathcal{E} \to \mathcal{G}$ gives rise to a groupoid crossed product of $\mathcal{G}$ by the groupoid ring of $\mathcal{N}$ which recovers the groupoid ring of $\mathcal{E}$ up to isomorphism.

Keywords: Étale categories. restriction semigroups. restriction semigroups actions. non-self-adjoint operator algebras. non-Abelian extension of groupoids. factor system. groupoid cohomology. groupoid crossed product. groupoid C^* algebra.

CONTENTS

INTRODUCTION

This thesis consists of two separate works. The first part was carried out in collaboration with my advisor Gilles Castro and is presented in chapters 1,2, and 3. In summary, it is an attempt to generalize the C^*- algebras of étale groupoids and inverse semigroups to the broader classes of étale categories and restriction semigroups. The second part was conducted in collaboration with Johan Öinert and Stefan Wagner during my stay at the Blekinge Institute of Technology and is presented in chapter 4. There, we investigate non-Abelian groupoid extensions, and their associated groupoid rings.

PART 1

The study of non-self-adjoint operator algebras began with the pioneering paper of Kadison and Singer [33] on triangular operators algebras, in 1960, and with Sarason's paper [58] on unstarred algebras. The works of Arveson and Hamana on the C^*-envelop program also were fundamental to the subject [4, 5, 7, 29, 30]. In 1990, Blecher, Ruan, and Sinclair [13] characterized abstractly an operator algebra up to completely isometric isomorphisms which gave to the community new insights and paths to follow. The connection between non-self-adjoint operator algebras and dynamical systems has its roots in Arveson's papers [6, 8], and since then many interesting examples and generalizations have appeared (see [19, 20, 34, 51] and the references therein).

The concept of restriction semigroup seems to have appeared for the first time in Schweizer and Sklar's series of papers about partial functions [60, 61, 62, 63]. Its motivation is traced back to the independent works of Wagner and Preston in inverse semigroups (see also [31] and the references therein). Restriction semigroups have many presentations and emerge in several different contexts, for more on this subject see Victoria Gould's survey [28] and also [36] for generalizations.

There is a well-known connection between étale groupoids, inverse semigroups, and C^*-algebras. Indeed, since Jean Renault's dissertation [55] was published, in 1980, the interplay between these objects has become very fruitful and has been explored by many authors (see for instance [2, 25, 67] and the references therein). Paterson [50] introduced the groupoid of germs of an inverse semigroup action, which is one of the keys when studying how the C^*-algebra of an inverse semigroup can be realized as an étale groupoid C^*-algebra. In fact, one starts with an inverse semigroup S, then considers the *canonical* action θ of S on its spectrum $\widehat{E(S)}$ and obtains the correspondent groupoid of germs $\mathcal{G}$. Using Sieben's C^*-crossed product algebra [65] of the action θ as an intermediary step, Paterson proved that $C^*(\mathcal{G})$ and $C^*(S)$ are isomorphic. Exel [24] improved his work by removing hypotheses in the inverse semigroup. There, Exel also provides a careful study of non-Hausdorff étale groupoids, and also introduces the tight C^*-algebra of an inverse semigroup.

Recently, Ganna Kudryavtseva and Mark Lawson published the paper [36] in which they use techniques of pointless topology to establish a duality between complete restriction monoids and étale topological categories (Theorem 7.22), which extends the duality presented in [40] between pseudogroups and étale groupoids. In their paper one can find the following comment:

> *"... there is the important question of how our work fits into the theory of operator algebras; the role of inverse semigroups and étale groupoids is of course well established but it is natural to ask if a theory of combinatorial non-self-adjoint operator algebras associated to étale categories could be developed that extended the theory developed in [24]."*

The main goal of this part is to give a positive answer to the above question and extend the works of Exel [24] and Paterson [50] to the context of restriction semigroups, étale topological categories, and non-self-adjoint operator algebras.

Here is an outline of our work. In chapter 1, we have defined the full and the reduced operator algebras of an étale category $\mathcal{C}$, and we have denoted them $\mathcal{A}(\mathcal{C})$, and $\mathcal{A}_r(\mathcal{C})$. In the case $\mathcal{G}$ is an étale groupoid we have shown that under suitable conditions $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}_r(\mathcal{G})$ are the very well-known full and reduced groupoid C^*-algebras of $\mathcal{G}$, which is somehow surprising. In chapter 2, we have also provided the notion of full and reduced operator algebras for restriction semigroups. And again we were able to show that this construction extends the C^*-algebra of an inverse semigroup. Furthermore, we have defined actions of restriction semigroups on both topological spaces and C^*-algebras and we have shown that these actions lead to a notion of semicrossed product algebra. Chapter 3 is in turn devoted to the study of the category of germs of an action of a restriction semigroups and its operator algebras. The main result of the chapter states that the category of germs of such an action is isomorphic to a semicrossed product, again extending the involutive case.

PART 2

The problem of classifying all extensions of a given group G by a group N is a core problem in group theory and may be found in many expositions. The first systematic treatments seem to originate in Schreier's PhD thesis from 1923 (see also [59]) and in the work of Baer [9] from the 1930s. Cohomological methods used to study group extensions first appeared in the seminal papers by Eilenberg and MacLane [22, 23]. Another curious reference is due to computer scientist Alan Turing [68]. The central concept underlying a group extension is that of a so-called *factor system*, which determines and is determined by the group extension. Examples and applications of group extensions can be found in almost all disciplines of modern mathematics. For instance, non-Abelian extensions of Lie groups occur quite naturally in the context of smooth principal bundles over compact manifolds, and as an application thereof in mathematical gauge theory (see, e. g., [45, 71] and the ref. therein). Furthermore, a counterexample to Kaplansky's famous unit conjecture for group rings has recently been given by Gardam [26] by means of Passman's fours group, which is a group extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}^3$.

In 1926, Brandt [14] introduced the notion of a groupoid as a generalization of a group. Since then, the theory of groupoids has flourished into an area of active research and applications of groupoids appear in areas such as

fibre bundle theory, differential geometry, foliation theory, differential topology, ergodic theory, functional analysis, homotopy theory, and algebraic geometry, [15].

The classification problem for Abelian groupoid extensions seems to originate in the work of Westman [69], in which he developed a cohomology theory for groupoids that extends the usual (Abelian) cohomology theory for groups. About a decade later Renault reproduced Westman's theory in his pioneering study of C*-algebras [55], thus spotlighting it for operator algebraists and functional analysts. Another two decades later, Blanco, Bullejos, and Faro [12] studied non-Abelian groupoid extensions from a 2-categorical point of view; their central result is the classification of non-Abelian groupoid extensions by means of a categorical cohomology theory for groupoids. Of particular interest is also the article [17], in which the authors study and classify fibrations of Lie groupoids. In recent years, there has been a renewed interest in groupoid extensions due to the fact that such extensions lead to many new and interesting algebraic structures (see, e. g., [3, 32, 37, 38, 55, 56] and the ref. therein).

To illustrate the latter circumstance, let us consider a, possibly non-Abelian, extension of groupoids $\mathcal{N} \to \mathcal{E} \to \mathcal{G}$. It is natural to ask whether the groupoid ring of $\mathcal{E}$ (resp. the groupoid C*-algebra of $\mathcal{E}$) can be described in terms of data associated with the building blocks $\mathcal{N}$ and $\mathcal{G}$. For groups this question has been studied by many authors (see, e. g., [47] and the ref. therein) and leads to the class of *group crossed products*, which is very well-understood and has numerous connections to geometry, operator algebras, and mathematical physics (see, e. g., [1, 49, 64] and the ref. therein). Furthermore, Renault [55, Prop. 1.22] proved that the groupoid C*-algebra of a *twist*, i. e., a groupoid extension by the trivial torus bundle, can be realized as a twisted groupoid C*-algebra. A treatment of the most general case of a proper non-Abelian groupoid extension has, however, to the best of our knowledge not been worked out yet.

Our investigations naturally leads to the class of groupoid crossed products, which is, in contrast to the class of group crossed products, relatively new (cf. [16, 48]) and thus provides fertile ground for further studies. More precisely, we establish that each, possibly non-Abelian, groupoid extension $\mathcal{N} \to \mathcal{E} \to \mathcal{G}$ gives rise to a groupoid crossed product of $\mathcal{G}$ by the groupoid ring of $\mathcal{N}$ which recovers the groupoid ring of $\mathcal{E}$ up to isomorphism. This also provides a natural class of examples of groupoid crossed products. In addition, it is our hope that this work will contribute to the development and understanding of groupoid C*-algebras and Steinberg algebras.

PART 1

1 ÉTALE CATEGORIES AND THEIR OPERATOR ALGEBRAS

In this chapter, we introduce (topological) étale categories and their operator algebras. The methods we are going to use are based on the works of [24] and [50] for étale groupoids, inverse semigroups, and their C^*-algebras, and we refer the reader to these references for a complete and systematic approach on the subject.

1.1 PRELIMINARIES

We begin by presenting some facts and definitions that will follow us along the way.

Definition 1.1. Let X be a set. We define

$$\ell_2(X) = \left\{ f : X \to \mathbb{C} \mid \sum_{x \in X} |f(x)|^2 < +\infty \right\}$$

to be the **Hilbert space associated with** X. The inner product on $\ell_2(X)$ is given by $\langle f,g\rangle = \sum_{x\in X} f(x)\overline{g(x)}$. Furthermore, denoting by δ_x the characteristic function of the singleton $\{x\}$ it is easily seen that $\{\delta_x\}_{x\in X}$ is a orthonormal basis for $\ell_2(X)$, and hence we can also write

$$\ell_2(X) = \left\{ \sum_{x\in X} a_x\delta_x \mid \sum_{x\in X} |a_x|^2 < +\infty \right\}.$$

Definition 1.2. Let H be a Hilbert space. A finite family of operators $\{T_i\}_{i=1}^n \subseteq B(H)$ is called **completely orthogonal** if $T_i^* T_j = 0$ and $T_i T_j^* = 0$, whenever $i \neq j$.

Suppose $\{T_i\}_{i=1}^n$ is a completely orthogonal family of operators on the Hilbert space H. Note that for $i \neq j$, we have

$$\left\langle T_j x, T_i y \right\rangle = \left\langle T_i^* T_j x, y \right\rangle = 0$$

for every $x,y \in H$. Hence, $\operatorname{ran}(T_i) \perp \operatorname{ran}(T_j)$, and similarly $\operatorname{ran}(T_i^*) \perp \operatorname{ran}(T_j^*)$. Moreover, since $\overline{\operatorname{ran}(T_i^*)} = \ker(T_i)^\perp$, we also have $\ker(T_i)^\perp \perp \ker(T_j)^\perp$, whenever $i \neq j$. Then H decomposes into the orthogonal direct sum $H = \bigoplus_{i=0}^{n} H_i$, where $H_i = \ker(T_i)^\perp$, for $i \geq 1$, and $H_0 = \bigcap_{i=1}^{n} \ker(T_i)$. Let x be in H, and write $x = \sum_{i=0}^n h_i$, where $h_i \in H_i$, for every $0 \leq i \leq n$. Then we have

$$\begin{aligned} \left\| \sum_{i=1}^n T_i(x) \right\|^2 &= \left\| \sum_{i=1}^n T_i(h_i) \right\|^2 = \sum_{i=1}^n \| T_i(h_i)\|^2 \\ &\leq \max_{i=1,\dots,n} \{\| T_i\|\}^2 \sum_{i=1}^n \|h_i\|^2 \leq \max_{i=1,\dots,n} \{\| T_i\|\}^2 \|x\|^2 \end{aligned}$$

and hence $\left\| \sum_{i=1}^{n} T_i \right\| \leq \max\limits_{i=1...n} \left\{ \| T_i \| \right\}$. Next, for $j \in \{1,...,n\}$ and $h \in H_j$, we have

$$\| T_j(h) \| = \left\| \sum_{i=1}^{n} T_i(h) \right\| \leq \left\| \sum_{i=1}^{n} T_i \right\| \|h\|.$$

Thus, $\| T_j \| \leq \left\| \sum_{i=1}^{n} T_i \right\|$, for every $j \in \{1,...,n\}$, and as a consequence we obtain

$$\left\| \sum_{i=1}^{n} T_i \right\| = \max_{i=1...n} \left\{ \| T_i \| \right\}. \tag{1.1}$$

Now, we recall the concept of generalized inverse of an operator.

Definition 1.3. Let H be a Hilbert space and $T \in B(H)$. An operator $S \in B(H)$ will be called a **generalized inverse** of T if $TST = T$ and $STS = S$.

Note that if T is a partial isometry then T^* is a generalized inverse of T. But, what can we say about the converse of this fact?

Theorem 1.4 (Theorem 3.1 of [43])**.** Let $T \in B(H)$ be a contraction, that is $\| T \| \leq 1$. Then T is a partial isometry if, and only if, T has a contractive generalized inverse.

Corollary 1.5 (Corollary 3.3 of [43])**.** Let $T \in B(H)$ be a contraction, and suppose that there exists a contractive generalized inverse S of T. Then $S = T^*$.

sketch. Let S be a contractive generalized inverse of T. At the end of the proof of Theorem 1.4, the author concludes that $S = T^*TS$. Then multiplying by T on both sides we obtain that $T = TST = TT^*TST = TT^*T$, and therefore T is partial isometry. Now, we focus on the equation

$$S = T^*TS.$$

Multiplying by T on the right side, we have $ST = T^*T$. Hence, since S^* is a contractive generalized inverse of the contraction T^*, we obtain $S^*T^* = TT^*$, and adjointing on both sides we obtain $TS = TT^*$. Then we conclude $S = T^*TS = T^*TT^* = T^*$. □

Let A be an algebra over $\mathbb{C}$, and let $\| \cdot \|_A : A \to \mathbb{R}_+$ be a seminorm. Recall that the standard procedure to obtain a normed algebra from the pair $(A, \| \cdot \|_A)$ is to take the quotient of A by the ideal $N = \{a \in A \mid \|a\|_A = 0\}$, and then induce on A/N the norm $\|[a]\| = \|a'\|_A$, where a' is any representative of $[a]$.

Definition 1.6. The normed algebra obtained by completing $(A/N, \| \cdot \|)$ is called the **Hausdorff completion** of $(A, \| \cdot \|_A)$, or simply A.

Last but not least, throughout the whole text we will use the concept of Boolean value to simplify the presentation of some equations or even definitions.

Definition 1.7. Let P be a statement. The **Boolean value** of P is the function denoted by

$$[\ P\ ] = \begin{cases} 1, & \text{if P is true} \\ 0, & \text{otherwise.} \end{cases}$$

We are now ready to begin with the main topic of this work. Let $\mathcal{C} = \left(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\right)$ be a small category, where $\mathcal{C}^{(0)}$ is the set of *objects* of $\mathcal{C}$ and $\mathcal{C}^{(1)}$ is the set of all *morphisms* between objects of $\mathcal{C}^{(0)}$. There are some special functions and sets to consider when dealing with a category: The maps *source* $\mathbf{d} : \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$, and *range* $\mathbf{r} : \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$, assigning to a morphism its domain and codomain, respectively. Moreover, we have the *unit* map $\mathbf{u} : \mathcal{C}^{(0)} \to \mathcal{C}^{(1)}$ assigning to an object u its identity id_u. Note that the set $\mathcal{C}^{(2)} = \{(x,y) \in \mathcal{C}^{(1)} \times \mathcal{C}^{(1)} : \mathbf{d}(x) = \mathbf{r}(y)\}$ is the set of all *composable morphisms*, and hence we denote the *composition* function by $\mathbf{m} : \mathcal{C}^{(2)} \to \mathcal{C}^{(1)}$ which assigns to every pair of composable morphisms (x,y) its composition xy. The maps $\mathbf{d}$, $\mathbf{r}$, $\mathbf{u}$ and $\mathbf{m}$ are usually called **structure maps** of $\mathcal{C}$.

Definition 1.8. For u and v in $\mathcal{C}^{(0)}$, we define $\mathcal{C}_u = \{x \in \mathcal{C}^{(1)} \mid \mathbf{d}(x) = u\}$, and $\mathcal{C}^u = \{x \in \mathcal{C}^{(1)} \mid \mathbf{r}(x) = u\}$. Moreover, we define $\mathcal{C}_u^v$ to denote the usual $\mathrm{Hom}_{\mathcal{C}}(u,v)$ set, which is precisely $\{x \in \mathcal{C}^{(1)} \mid \mathbf{d}(x) = u \text{ and } \mathbf{r}(x) = v\}$. Finally, for all $z \in \mathcal{C}^{(1)}$, we define M_z to be the set $\{(x,y) \in \mathcal{C}^{(2)} : xy = z\}$.

The maps $\mathbf{d}$, $\mathbf{r}$, $\mathbf{u}$ and $\mathbf{m}$ are usually called **structure maps**.

Definition 1.9. Let $\mathcal{C}$ be a small category. We call $\mathcal{C}$ a **topological category** if $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$ are topological spaces and the structure maps are continuous.

Before we define étale categories, recall that if X and Y are topological spaces, and $f : X \to Y$ is a map then we say that f is a topological embedding if the map $f : X \to f(X)$ is a homeomorphism, where $f(X)$ is equipped with the relative topology.

Definition 1.10. Let $\mathcal{C}$ be a topological category. We call $\mathcal{C}$ an **étale category** if the following conditions hold:

1. $\mathcal{C}^{(0)}$ is a locally compact Hausdorff space.
2. The maps $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms.
3. $\mathbf{u}$ is a topological embedding.

This definition of étale category is stronger than the one presented in [36] since they do not require $\mathcal{C}^{(0)}$ to be a locally compact Hausdorff space.

Definition 1.11. A category $\mathcal{C}$ is called left (resp. right) cancellative if $xy = xw$ (resp. $yx = wx$) implies $y = w$, for every triple of morphisms x,y, and w in $\mathcal{C}$. A category is called **cancellative** if it is both left and right cancellative.

Example 1.12 (Transformation category)**.** Let X be a compact Hausdorff topological space and $f : X \to X$ be a local homeomorphism. Define $\mathcal{C}^{(0)} = X$ and $\mathcal{C}^{(1)} = \{(y,n,x) : f^n(x) = y,\ n \in \mathbb{N}\}$. Here, $\mathcal{C}^{(0)}$ is equipped with the topology of X and $\mathcal{C}^{(1)}$ is equipped with the subspace topology of $X \times \mathbb{N} \times X$, where $\mathbb{N}$ is viewed as a discrete space. Moreover, we define the source and range maps to be $\mathbf{d}(y,n,x) = x$ and $\mathbf{r}(y,n,x) = y$. The identity of x is set to be $\mathbf{u}(x) = (x,0,x)$, and finally the composition of two triples is given by $(z,m,y)(y,n,x) = (z,m+n,x)$. It is routine to verify that $\mathcal{C} = \left(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\right)$ is a category with these structure maps. Furthermore, $\mathcal{C}$ is left (and right) cancellative because

$$(z,m,y)(y,n,x) = (z,m,y)(y,k,t) \Rightarrow (z,m+n,x) = (z,m+k,t) \Rightarrow k = n \text{ and } t = x.$$

Let $\gamma = (y,n,x) \in \mathcal{C}$ and $A \subseteq X$ be an open set such that $x \in A$ and f^n_A is a homeomorphism. The open set $U := (f^n(A) \times \{n\} \times A) \bigcap \mathcal{C}^{(2)} = \{(f^n(a),n,a) :\ a \in a\}$ is an open neighborhood of γ and we have that both $\mathbf{d}_U : U \to A$ and $\mathbf{r}_U : U \to f^n(A)$ are homeomorphisms, whose inverses are $a \mapsto (f^n(a),n,a)$ and $a \mapsto (a,n,(f^n)^{-1}(a))$, respectively. Moreover, if V is an open subset of X then $\mathbf{u}(V) = \{(v,0,v) :\ v \in V\} = (V \times \{0\} \times V) \bigcap \mathcal{C}^{(2)}$ which is open. Finally, the continuity of $\mathbf{u}$ and $\mathbf{m}$ can be observed noticing that both are compositions of continuous functions (projections, inclusions, the sum in $\mathbb{N}$).

From now on, let $\mathcal{C} = \left(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\right)$ be an étale category. Let us begin by showing that the range of $\mathbf{u}$ is an open subset of $\mathcal{C}^{(1)}$. To this end, we present the following lemma whose proof is an adaptation from [24, Proposition 3.2].

Lemma 1.13. Let X be a topological space and $Y \subseteq X$ a topological subspace of X. Suppose there exists a local homeomorphism $f : X \to Y$ such that $f(y) = y$ for every $y \in Y$. Then Y is open in X, and hence the map $f : X \to X$ is a local homeomorphism.

Proof. Take $y \in Y$, and let V be an open set of X such that $y \in V$ and $f_V : V \to f(V)$ is a homeomorphism. Define B to be the subset $V \cap f(V)$. Note that $y \in B$, and that B is open in $f(V)$. Hence, the subset $f_V^{-1}(B)$ is open in V. But note that B is equal to $f_V^{-1}(B)$ since $B \subseteq V$, and $f(B) = B$, and f_V is bijective. Therefore, B is open in V and, consequently, B is open in X. In conclusion, we have $y \in B \subseteq Y$. This gives that Y is open in X. □

Proposition 1.14. Let $\mathcal{C}$ be an étale category. Then $\mathbf{u}\left(\mathcal{C}^{(0)}\right) = \{\mathrm{id}_u \ :\ u \in \mathcal{C}^{(0)}\}$ is an open subset of $\mathcal{C}^{(1)}$, and hence $\mathbf{u}$ is an open map.

Proof. Note that $\mathbf{u} \circ \mathbf{r} : \mathcal{C}^{(1)} \to \{\mathrm{id}_u \mid u \in \mathcal{C}^{(0)}\}$ satisfies the hypothesis of Lemma 1.13, and hence $\{\mathrm{id}_u \mid u \in \mathcal{C}^{(0)}\}$ is open in $\mathcal{C}^{(1)}$. Moreover if $U \subseteq \mathcal{C}^{(0)}$ is an open set then $\mathbf{u}(U)$ is an open subset of $\mathbf{u}(\mathcal{C}^{(0)})$ which is, in turn, open in $\mathcal{C}^{(1)}$. Then $\mathbf{u}(U)$ is open in $\mathcal{C}^{(1)}$. □

For simplicity of the presentation, we will avoid working with two distinct sets $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$. In what follows we will identify an object $u \in \mathcal{C}^{(0)}$ with its associated identity morphism id_u, and hence we will see $\mathcal{C}^{(0)}$ as a subset of $\mathcal{C}^{(1)}$. The fact that $\mathbf{u}$ is an embedding ensures that there is no loss in this identification since $\mathbf{u}\left(\mathcal{C}^{(0)}\right)$ is homeomorphic to $\mathcal{C}^{(0)}$. Furthermore, we will just write $\mathcal{C}$ instead $\mathcal{C}^{(1)}$.

Definition 1.15. An open subset $U \subseteq \mathcal{C}$ is called a **bisection** if the restrictions $\mathbf{d}_U$ and $\mathbf{r}_U$ are injective. We will denote by Bis($\mathcal{C}$) the set of all bisections of $\mathcal{C}$.

Remark 1.16. Because local homeomorphisms are open maps, Bis($\mathcal{C}$) coincides with the family of open sets V such that $\mathbf{d}_V : V \to \mathbf{d}(V)$ and $\mathbf{r}_V : V \to \mathbf{r}(V)$ are homeomorphisms.

In [24], the author proves several topological facts about étale groupoids. It occurs that many of these facts are valid (with similar proofs) for the more general context of étale categories.

Proposition 1.17. 1. Bis($\mathcal{C}$) forms a basis for the topology of $\mathcal{C}$.

2. Every bisection is a locally compact Hausdorff subspace.
3. Every open subset of $\mathcal{C}$ is a locally compact subspace.
4. Every open Hausdorff subset of $\mathcal{C}$ is a locally compact Hausdorff subspace.
5. For all $u,v \in \mathcal{C}^{(0)}$, $\mathcal{C}_u$, $\mathcal{C}^v$ and $\mathcal{C}_u^v$ are closed subsets.
6. For all $u,v \in \mathcal{C}^{(0)}$, $\mathcal{C}_u$, $\mathcal{C}^v$ and $\mathcal{C}_u^v$ are discrete subspaces.
7. $\mathcal{C}^{(2)}$ is a closed subset of $\mathcal{C} \times \mathcal{C}$.
8. If $\mathcal{C}$ is Hausdorff then $\mathcal{C}^{(0)}$ is closed in $\mathcal{C}$.

Proof. 1. Fix $x \in \mathcal{C}$ and $W \subseteq \mathcal{C}$ an open subset such that $x \in W$. Let U and V be open subsets such that $x \in U \cap V$, $\mathbf{d}_U$ and $\mathbf{r}_V$ are homeomorphisms. Then $Z := U \cap V \cap W$ is a bisection and $x \in Z \subseteq W$.

2. Recall that open subsets of locally compact Hausdorff spaces are also locally compact Hausdorff spaces with the subspace topology. If U is a bisection, then it is homeomorphic to the open subset $\mathbf{d}(U)$ of $\mathcal{C}^{(0)}$ and, consequently, it is a locally compact Hausdorff subspace.
3. Let V be an open subset of $\mathcal{C}$ and $x \in V$. By 1 and 2, there is a bisection U and a compact K such that $x \in \mathring{K} \subseteq K \subseteq U \subseteq V$. Then x has a compact neighborhood in V and therefore V is locally compact.
4. Immediate.
5. $\mathcal{C}_u = \mathbf{d}^{-1}(\{u\})$; $\mathcal{C}^v = \mathbf{r}^{-1}(\{v\})$; $\mathcal{C}_u^v = \mathcal{C}_u \cap \mathcal{C}^v$.

6. Fix $u,v \in \mathcal{C}^{(0)}$. Take $x \in \mathcal{C}_u$ and U a bisection containing x. Then $U \cap \mathcal{C}_u$ is equal to $\{x\}$ and it is open in $\mathcal{C}_u$. Similarly, $\mathcal{C}^v$ and $\mathcal{C}_u^v$ are discrete.
7. The diagonal set $\Delta(\mathcal{C}^{(0)}) = \{(u,u) \mid u \in \mathcal{C}^{(0)}\}$ is closed because $\mathcal{C}^{(0)}$ is Hausdorff. Moreover, $f : \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{(0)} \times \mathcal{C}^{(0)}$ given by $f(x,y) = (\mathrm{d}(x), \mathrm{r}(y))$ is continuous. Thus, $\mathcal{C}^{(2)} = f^{-1}\left(\Delta\left(\mathcal{C}^{(0)}\right)\right)$ is a closed subset of $\mathcal{C} \times \mathcal{C}$.
8. Define $f : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ to be the map given by $f(x) = (x, \mathrm{r}(x))$. Note that f is continuous, and moreover $\mathcal{C}^{(0)} = f^{-1}(\Delta(\mathcal{C}))$.

□

The first item above tells us that the family $\{(U \times V) \cap \mathcal{C}^{(2)} : U,V \in \mathrm{Bis}(\mathcal{C})\}$ forms a basis for the topology of $\mathcal{C}^{(2)}$. Moreover, for U, $V \in \mathrm{Bis}(\mathcal{C})$ note that $(U \times V) \cap \mathcal{C}^{(2)} = (U_1 \times V_1) \cap \mathcal{C}^{(2)}$, where $U_1 = \mathrm{d}_U^{-1}(\mathrm{r}(V) \cap \mathrm{d}(U))$ and $V_1 = \mathrm{r}_V^{-1}(\mathrm{r}(V) \cap \mathrm{d}(U))$. Hence, $\mathrm{d}(U_1) = \mathrm{r}(V_1) = \mathrm{r}(V) \cap \mathrm{d}(U)$, and consequently another basis for the topology of $\mathcal{C}^{(2)}$ is the family $\{U \times V \cap \mathcal{C}^{(2)} : U,V \in \mathrm{Bis}(\mathcal{C}),\ \mathrm{d}(U) = \mathrm{r}(V)\}$. Furthermore, for any pair U,V of subsets of $\mathcal{C}$, we define UV to be the set $\{xy : (x,y) \in (U \times V) \cap \mathcal{C}^{(2)}\}$. Hence, because m is associative, it becomes clear that the product of subsets is also associative.

Proposition 1.18. The **composition** function $\mathrm{m} : \mathcal{C}^{(2)} \to \mathcal{C}$ is open.

Proof. We start by showing that if U,V is a pair of bisections such that $\mathrm{d}(U) = \mathrm{r}(V)$ and $UV \subseteq W$, for another bisection W, then UV is open. For $x \in U$, there exists $y \in V$ such that $\mathrm{d}(x) = \mathrm{r}(y)$. Therefore $\mathrm{r}(x) = \mathrm{r}(xy) \in \mathrm{r}(UV)$ and consequently $\mathrm{r}(U) = \mathrm{r}(UV)$. Then, $\mathrm{r}_W(UV)$ is equal to $\mathrm{r}(U)$, which is open. Hence, UV is open as r_W is a homeomorphism.

Next, we show that the above fact implies m is open. Let O be an open subset of $\mathcal{C}^{(2)}$, (x,y) be in O, W be a bisection such that $xy \in W$. Since m is continuous, there exists a pair U,V of bisections such that $\mathrm{d}(U) = \mathrm{r}(V)$, $(x,y) \in U \times V \cap \mathcal{C}^{(2)} \subseteq O$ and $UV \subseteq W$. By the previous case, UV is open and $xy \in UV \subseteq \mathrm{m}(O)$. Hence, $\mathrm{m}(O)$ is open. □

Remark 1.19. For bisections U and V, Proposition 1.18 says UV is open. Moreover, it is easy to verify that d_{UV} and r_{UV} are injective maps, and therefore $\mathrm{Bis}(\mathcal{C})$ forms a semigroup. In fact, it is a monoid where the identity is $\mathcal{C}^{(0)}$.

Lemma 1.20. If $\{U_1, \ldots, U_n\}$ is a finite family of bisections and $z \in U_1 \cdots U_n$ then, there exists is a unique n-tuple $(x_1, x_2, \ldots, x_n) \in U_1 \times \ldots \times U_n$ such that $x_1 \ldots x_n = z$.

Proof. The proof is by induction on n. The case $n = 1$ is trivial. Suppose the statement holds for $n = k - 1$, and let $U_1, U_2, \ldots, U_k$ be a family of bisections. For k-tuples $(x_1,\ldots,x_k), (y_1,\ldots,y_k) \in U_1 \times \ldots \times U_k$ such that $x_1 \ldots x_k = y_1 \ldots y_k$, applying r to both sides we get $x_1 = y_1$, since U_1 is a bisection. Therefore, $\mathrm{r}(x_2 \ldots x_k) = \mathrm{d}(x_1) = \mathrm{d}(y_1) = \mathrm{r}(y_2 \ldots y_k)$ which implies that $x_2 \ldots x_k = y_2 \ldots y_k$. Hence by the induction hypothesis $x_i = y_i$, for all $1 \leq i \leq k$. □

1.2 OPERATOR ALGEBRAS ASSOCIATED WITH ÉTALE CATEGORIES

Given an open subset $V \subseteq \mathcal{C}$ and $f : \mathcal{C} \to \mathbb{C}$, we will write $f \in \mathrm{C_c}(V)$ if $f_V \in \mathrm{C_c}(V)$ and $f_{\mathcal{C}\setminus V}$ is identically zero. As in [24], let $\mathcal{U}$ to be the family of open Hausdorff subsets of $\mathcal{C}$, and define the vector space

$$\mathcal{A}_0(\mathcal{C}) = \mathrm{span}\{f : \ f \in \mathrm{C_c}(U),\ U \in \mathcal{U}\}.$$

We write $\mathcal{A}_0(\mathcal{C})$ instead of $\mathrm{C_c}(\mathcal{C})$ because the elements of the former are not necessarily continuous. But they are not so far from each other, if $\mathcal{C}$ were Hausdorff then they coincide.

Proposition 1.21. If $\mathcal{F} \subseteq \mathrm{Bis}(\mathcal{C})$ covers $\mathcal{C}$, then $\mathcal{A}_0(\mathcal{C}) = \mathrm{span}\{f : \ f \in \mathrm{C_c}(U),\ U \in \mathcal{F}\}$. In particular, one has

$$\mathcal{A}_0(\mathcal{C}) = \mathrm{span}\{f : \ f \in \mathrm{C_c}(U),\ U \in \mathrm{Bis}(\mathcal{C})\}.$$

Proof. Let V be an open Hausdorff subset and $f \in \mathrm{C_c}(V)$. There is a finite family $\{U_i\}_{i=1}^n$ of bisections in $\mathcal{F}$ such that $\mathrm{supp}(f) \subseteq \cup_{i=1}^n U_i$. Using partitions of unity (see [57, Theorem 2.13]), there are functions $\eta_i \in \mathrm{C_c}(U_i \cap V)$ such that $\sum_{i=1}^n \eta_i(x) = 1$, for every $x \in \mathrm{supp}(f)$. Then, we have $f = \sum_{i=1}^n f\eta_i$ and $f\eta_i \in \mathrm{C_c}(U_i \cap V) \subseteq \mathrm{C_c}(U_i)$, for all $i \in \{1,...,n\}$. □

On $\mathcal{A}_0(\mathcal{C})$ we define the convolution product

$$f * g(z) = \sum_{(x,y)\in M_z} f(x)g(y). \tag{1.2}$$

We use the following lemma to show that the above product is well-defined.

Lemma 1.22. Assume U,V and W are bisections and $f \in \mathrm{C_c}(U)$, $g \in \mathrm{C_c}(V)$ and $h \in \mathrm{C_c}(W)$ then:

1. $|\,\mathrm{supp}(f) \cap \mathcal{C}_u| \leq 1$ and $|\,\mathrm{supp}(f) \cap \mathcal{C}^u| \leq 1$ for every $u \in \mathcal{C}^{(0)}$,
2. $f * g \in \mathrm{C_c}(UV)$.
3. If $U = \mathcal{C}^{(0)}$, then $f * g \in \mathrm{C_c}(V)$ and $(f * g)(z) = f(\mathbf{r}(z))g(z)$, for every $z \in V$.
4. If $V = \mathcal{C}^{(0)}$, then $f * g \in \mathrm{C_c}(U)$ and $(f * g)(z) = f(z)g(\mathbf{d}(z))$, for every $z \in U$.
5. If $U = V = \mathcal{C}^{(0)}$, then $f * g \in \mathrm{C_c}(\mathcal{C}^{(0)})$ and $(f * g)(z) = f(z)g(z)$, for every $z \in \mathcal{C}^{(0)}$.
6. $(f * g) * h = f * (g * h)$.

Proof. 1. Immediate.

2. By Lemma 1.20, given $z \in UV$, there is a unique pair (x,y) belonging to $M_z \cap (U \times V)$. Consequently, $f * g(z)$ is equal to $f(x)g(y)$. Now, writing x and y as functions of z, we obtain
$$f * g(z) = f \circ \mathbf{r}_U^{-1} \circ \mathbf{r}(z).g \circ \mathbf{d}_V^{-1} \circ \mathbf{d}(z).$$

Thus, the restriction of $f * g$ to UV is continuous. If $f * g(z) \neq 0$, there must be a pair $(x,y) \in M_z$ such that $f(x)g(y) \neq 0$. Then $x \in U$, $y \in V$ and, consequently, $z \in UV$. Moreover, $\{w \in \mathcal{C} \mid f * g(w) \neq 0\} \subseteq \operatorname{supp}(f)\operatorname{supp}(g) \subseteq UV$. Since UV is Hausdorff, $\operatorname{supp}(f)\operatorname{supp}(g)$ is compact and closed. Hence, $\operatorname{supp}(f * g)$ is compact.

3. Given $z \in V$, $(\mathrm{r}(z),z)$ is the unique element in $(U \times V) \cap M_z$, then $f * g(z) = f(\mathrm{r}(z))g(z)$.
4. Similar to 3.
5. Immediate from 3 and 4.
6. Note that $(f * g) * h$ and $f * (g * h)$ are both supported in UVW, by item 2. Moreover, for $z \in UVW$ there are unique pairs $(x,x_3) \in (UV \times W) \cap M_z$, and $(y_1,y) \in (U \times VW) \cap M_z$. Hence $(f * g) * h(z) = f * g(x)h(x_3)$ and $f * (g * h)(z) = f(y_1)g * h(y)$.

   Now, since there are unique pairs (x_1,x_2) and (y_2,y_3) belonging to $(U \times V) \cap M_x$ and $(V \times W) \cap M_y$, respectively, we obtain $(f * g) * h(z) = f(x_1)g(x_2)h(x_3)$ and $f * (g * h)(z) = f(y_1)g(y_2)h(y_3)$. Thus, by Lemma 1.20, we have $x_i = y_i$ for $1 \leq i \leq 3$, completing the proof.

□

Clearly, the product defined in Equation (1.2) is bilinear, and then to obtain that it is well-defined we just need to ensure that the sum appearing in Equation (1.2) is finite and that it is associative. Combining Proposition 1.21 with item 1 of Lemma 1.22 gives us the finiteness, and the associativity comes from combining the same proposition with item 6 of Lemma 1.22. Hence, $\mathcal{A}_0(\mathcal{C})$ is an associative algebra.

Remark 1.23. We can generalize items 3 and 4 of Lemma 1.22. In fact, for $f \in \mathcal{A}_0(\mathcal{C})$ and $g \in \mathrm{C_c}(\mathcal{C}^{(0)})$, we have $f * g(z) = f(z)g(\mathrm{d}(z))$ and $g * f(z) = g(\mathrm{r}(z))f(z)$, for every $z \in \mathcal{C}$. To see this just write f as a sum $\sum_{i=1}^{n} f_i$, where each f_i is supported on a bisection U_i, and apply the aforementioned items of Lemma 1.22.

Definition 1.24. A **representation** of $\mathcal{A}_0(\mathcal{C})$ on a Hilbert space H is an algebra homomorphism $\pi : \mathcal{A}_0(\mathcal{C}) \to B(H)$ such that:

1. $||\pi(f)|| \leq ||f||_\infty$, for every $f \in \mathrm{C_c}(U)$, and for every $U \in \operatorname{Bis}(\mathcal{C})$.
2. $\pi(\bar{f}) = \pi(f)^*$, for every $f \in \mathrm{C_c}(\mathcal{C}^{(0)})$.

The class of all representations of $\mathcal{A}_0(\mathcal{C})$ will be denoted by $\operatorname{Rep}(\mathcal{A}_0(\mathcal{C}))$.

Remark 1.25. A representation $\pi : \mathcal{A}_0(\mathcal{C}) \to B(H)$ gives rise to a C^*-algebra homomorphism $\pi_0 : \mathrm{C_0}(\mathcal{C}^{(0)}) \to B(H)$, where π_0 is obtained by extending the restriction $\pi_{|\,\mathrm{C_c}(\mathcal{C}^{(0)})}$.

Our goal is to build a normed algebra from $\mathcal{C}$ in which every representation of $\mathcal{A}_0(\mathcal{C})$ is contractive. Hence, we define

$$||f||_0 = \sup\left\{ \, ||\pi(f)|| \mid \pi \in \operatorname{Rep}(\mathcal{A}_0(\mathcal{C})) \right\}. \tag{1.3}$$

Note that $\{\, \|\pi(f)\| \mid \pi \in \mathrm{Rep}(\mathcal{A}_0(\mathcal{C}))\}$ is a set, regardless of $\mathrm{Rep}(\mathcal{A}_0(\mathcal{C}))$ being a set or a class, and therefore the above supremum is well-defined. In general $\|\cdot\|_0$ is just a semi-norm, and it is a norm if and only if there exists a faithful representation of $\mathcal{A}_0(\mathcal{C})$.

Definition 1.26. Let $\mathcal{C}$ be an étale category. The **operator algebra** of $\mathcal{C}$ is the Hausdorff completion $\mathcal{A}(\mathcal{C})$ of the normed algebra $(\mathcal{A}_0(\mathcal{C})\,, \|\cdot\|_0)$.

It is still not clear why $\mathcal{A}(\mathcal{C})$ is an operator algebra, that is, a closed subalgebra of the C^*-algebra of bounded linear operators on a Hilbert space H. At the end of the next subection we provide an argument justifying this nomenclature. Moreover, note that every representation of $\pi : \mathcal{A}_0(\mathcal{C}) \to B(H)$ extends to a contractive homomorphism to $\pi : \mathcal{A}(\mathcal{C}) \to B(H)$.

1.2.1 The reduced operator algebra of an étale category

We now proceed to show that if $\mathcal{C}$ is a left cancellative étale category then there exists a faithful representation of $\mathcal{A}_0(\mathcal{C})$, which ensures that $\mathcal{A}(\mathcal{C})$ is simply the completion of $(\mathcal{A}_0(\mathcal{C})\,, \|\cdot\|_0)$.

Let $\ell_2(\mathcal{C})$ be the Hilbert space associated with $\mathcal{C}$ (see Definition 1.1), and consider the map $\pi : \mathcal{A}_0(\mathcal{C}) \to B(\ell_2(\mathcal{C}))$, in which π_f is given by

$$\pi_f\left(\sum_{z\in\mathcal{C}} a_z\delta_z\right) = \sum_{z\in\mathcal{C}} a_z \sum_{x\in\mathcal{C}_{\mathrm{r}(z)}} f(x)\delta_{xz}. \tag{1.4}$$

Our strategy is to prove that for the bisections U and $V \in \mathrm{Bis}(\mathcal{C})$, and the maps $f \in \mathrm{C_c}(U)$ and $g \in \mathrm{C_c}(V)$ we have that $\|\pi_f\| \leq \|f\|_\infty$ and $\pi_{f*g} = \pi_f\pi_g$. Thus, the linearity of π will ensure that it is a well-defined representation of $\mathcal{A}_0(\mathcal{C})$.

Note that $\pi_f(\delta_z)$ is nonzero if and only if $\mathcal{C}_{\mathrm{r}(z)} \cap f^{-1}(\{0\})$ is non-empty. In such case $\pi_f(\delta_z) = f(x^z)\delta_{x^z z}$, where x^z is the unique element in $\mathcal{C}_{\mathrm{r}(z)} \cap f^{-1}(\{0\})$, by item (1) of Lemma 1.22. Now, if Γ is the auxiliary set $\left\{u \in \mathcal{C}^{(0)} \mid \mathcal{C}_u \cap f^{-1}(\{0\}) \neq \emptyset\right\}$ and $v = \sum_{z\in\mathcal{C}} a_z\delta_z$ we have

$$\pi_f(v) = \sum_{r(z)\in\Gamma} a_z f(x^z)\delta_{x^z z}.$$

Suppose that z and z' are such that $\mathrm{r}(z) \in \Gamma$, $\mathrm{r}(z') \in \Gamma$, and $x^z z = x^{z'} z'$. In this case, $x^z \in U$, $x^{z'} \in U$, and $\mathrm{r}(x^z) = \mathrm{r}\left(x^{z'}\right)$. Thus, $x^z = x^{z'}$, and because $\mathcal{C}$ is left cancellative $z = z'$. Which gives

$$\|\pi_f(v)\|^2 = \sum_{r(z)\in\Gamma} |a_z|^2|f(x^z)|^2 \leq \|f\|_\infty^2\|v\|^2.$$

To see that $\pi_{f*g} = \pi_f\pi_g$, we prove $\pi_{f*g}(\delta_z) = \pi_f\pi_g(\delta_z)$ for every $z \in \mathcal{C}$. Suppose that $\pi_f(\pi_g(\delta_z))$ is nonzero. Then $\pi_f\pi_g(\delta_z) = f(x)g(y)\delta_{xyz}$, for $x \in f^{-1}(\{0\}) \subseteq U$, and

$y \in g^{-1}(\{0\}) \subseteq V$. Note that, xy is the unique element in $\mathcal{C}_{\mathrm{r}(z)} \cap UV$, and hence

$$\pi_{f*g}(\delta_z) = f * g(xy)\delta_{xyz} = \pi_f(\pi_g(\delta_z)).$$

On the other hand, suppose $\pi_{f*g}(\delta_z)$ is nonzero. In this case $\pi_{f*g}(\delta_z) = f * g(w)\delta_{wz}$, for the unique element w in $\mathcal{C}_{\mathrm{r}(z)} \cap (f * g)^{-1}(\{0\})$. Moreover, there exists a unique pair $(x,y) \in U \times V$ such that $f * g(w) = f(x)g(y)$, by Lemma 1.20. Finally, $f(x)$ and $g(y)$ are nonzero, and hence

$$\pi_f(\pi_g(\delta_z)) = f(x)g(y)\delta_{xyz} = f * g(w)\delta_{wz} = \pi_{f*g}(\delta_z).$$

We have proven that $\pi_{f*g}(\delta_z)$ is nonzero if, and only if, $\pi_f(\pi_g(\delta_z))$ is nonzero and in such case they coincide. Therefore, $\pi_{f*g}(\delta_z) = \pi_f\pi_g(\delta_z)$ for every $z \in \mathcal{C}$.

The straightforward calculation below shows that $\pi_{\overline{f}} = \pi_f^*$. Indeed, for $f \in \mathrm{C_c}(\mathcal{C}^{(0)})$, and z, and z' be in $\mathcal{C}$ we have

$$\begin{aligned} \langle \pi_f(\delta_z),\delta_{z'}\rangle &= f(\mathrm{r}(z))\,\langle \delta_z,\delta_{z'}\rangle = [z = z']f(\mathrm{r}(z)) \\ &= [z = z']f(\mathrm{r}(z')) = f(\mathrm{r}(z'))\,\langle \delta_z,\delta_{z'}\rangle \\ &= \left\langle \delta_z,\overline{f(\mathrm{r}(z'))}\delta_{z'}\right\rangle = \left\langle \delta_z,\pi_{\overline{f}}(\delta_{z'})\right\rangle . \end{aligned}$$

We finish this discussion by proving that π is faithful. Note that if $f \in \mathcal{A}_0(\mathcal{C})$ is nonzero, there is $z \in \mathcal{C}$ such that $f(z)$ is nonzero, and then

$$\pi_f(\delta_{\mathrm{d}(z)}) = \sum_{x\in\mathcal{C}_{\mathrm{r}(\mathrm{d}(z))}} f(x)\delta_{x\,\mathrm{d}(z)} = \sum_{x\in\mathcal{C}_{\mathrm{d}(z)}} f(x)\delta_x = f(z)\delta_z + \sum_{\substack{x\in\mathcal{C}_{\mathrm{d}(z)}\\ x\neq z}} f(x)\delta_x \neq 0.$$

Definition 1.27. The above defined map π is called the **regular representation of** $\mathcal{C}$. The **reduced operator algebra of** $\mathcal{C}$ is the closure of $\pi(\mathcal{A}_0(\mathcal{C}))$ in $B(\ell_2(\mathcal{C}))$, and it is denoted by $\mathcal{A}_r(\mathcal{C})$.

Of course, we could also define the reduced operator algebra intrinsically as the completion of $\mathcal{A}_0(\mathcal{C})$ in the norm induced by the regular representation. Moreover, the regular representation is usually presented as a direct sum. If one desires to recover the definition in terms of a direct sum of representations, it suffices to note that for any $u \in \mathcal{C}^{(0)}$ the Hilbert (sub)space $\ell_2(\mathcal{C}_u)$ is an invariant subspace of π, and $\ell_2(\mathcal{C}) = \bigoplus_{u\in\mathcal{C}^{(0)}} \ell_2(\mathcal{C}_u)$. Therefore, denoting by π_u the restriction of π to $\ell_2(\mathcal{C}_u)$ we obtain

$$\pi = \bigoplus_{u\in\mathcal{C}^{(0)}} \pi_u. \tag{1.5}$$

Note that the cancellation assumption is crucial for the existence of the left regular representation. In fact, let $\mathcal{C} = \mathbb{N}$ be the category of natural numbers, where 1 is the unique object of $\mathcal{C}$ and the composition is given by multiplication. The topology in $\mathcal{C}$

is the discrete one and hence $\mathcal{A}_0(\mathcal{C})$ is simply the set of all functions whose support is finite. Let $f = \chi_0$ be the characteristic function of $\{0\}$. Note that $\pi_f(\delta_1 + \delta_2) = 2\delta_0$, which implies that $\|\pi_f\| \geq \sqrt{2}$. But, since f has support contained on a bisection we should have $\|\pi_f\| \leq \|f\|_\infty = 1$.

Proposition 1.28. Let $\mathcal{C}$ be a left cancellative étale category, and π be the regular representation. Then for any bisection U, and $f \in \mathrm{C_c}(U)$ it holds $||\pi(f)|| = ||f||_\infty$. In particular $\mathrm{C_0}(U)$ is a linear subspace of both $\mathcal{A}_r(\mathcal{C})$ and $\mathcal{A}(\mathcal{C})$. Furthermore, if $U = \mathcal{C}^{(0)}$ then $\mathrm{C_0}(\mathcal{C}^{(0)})$ is a subalgebra of $\mathcal{A}_r(\mathcal{C})$ and $\mathcal{A}(\mathcal{C})$.

Proof. Let U be a bisection, let $f \in \mathrm{C_c}(U)$, and let $x \in U$ be such that $|f(x)| = ||f||_\infty$. Note that $\pi_f(\delta_{\mathbf{d}(x)}) = f(x)\delta_x$, and consequently $||f||_\infty = |f(x)| \leq ||\pi(f)|| \leq ||f||_\infty$. This proves that f attains its supremum norm on $\mathcal{A}_r(\mathcal{C})$ and $\mathcal{A}(\mathcal{C})$ (cf. item 1 of Definition 1.24), and hence $\mathrm{C_c}(U)$ is an isometric linear subspace of both full and reduced algebras of $\mathcal{C}$. The claim then follows on from taking closure.

If $U = \mathcal{C}^{(0)}$, item 5 of Lemma 1.22 shows that the convolution product of functions supported on U reduces to the pointwise product. Therefore $\mathrm{C_c}(\mathcal{C}^{(0)})$ is a subalgebra of $\mathcal{A}_r(\mathcal{C})$ and $\mathcal{A}(\mathcal{C})$, as well as its closure $\mathrm{C_0}(\mathcal{C}^{(0)})$. □

Let $\mathcal{C}$ be a left cancellative étale category. We now argue why $\mathcal{A}(\mathcal{C})$ live up to the term *operator algebra*. Our strategy is to prove that the supremum in (1.3) can be taken over a certain set $\mathrm{Rep}'(\mathcal{A}_0(\mathcal{C}))$ of representations, and hence taking the direct sum of all representations in $\mathrm{Rep}'(\mathcal{A}_0(\mathcal{C}))$ we will obtain the universal representation of $\mathcal{A}_0(\mathcal{C})$.

Let A be a unital C^*-algebra, and $\rho : A \to B(H)$ be a representation of A. From [44, Theorem 5.1.3], we obtain that there exists a set X such that $\rho = \bigoplus_{x\in X} \rho_x$, where $\rho_x : A \to B(H_x)$ is a cyclic representation of A. Recall that $\|\rho(a)\| = \sup_{x\in X} \|\rho_x(a)\|$ and that $H_x = \overline{\rho_x(A)\xi_x}$ for some cyclic vector ξ_x. In particular, the spaces H_x have their dimensions bounded by $|A|^{\aleph_0}$.

Now, suppose that H is a Hilbert space and that B is nonzero a subalgebra of $B(H)$, which says, in particular, that B is uncountable. Let us find an upper bound for the cardinality of the C^*-subalgebra generated by B in $B(H)$, denoted by $C^*(B)$. Let $\mathcal{P}_{fin} = \{ p_1 \cdots p_n \mid p_i \in B \cup B^*, n \in \mathbb{N}^*\}$ denote the set of all finite products of elements in $B \cup B^*$, and let $B' = \mathrm{span}_{\mathbb{Q}+i\mathbb{Q}} \mathcal{P}_{fin}$ be the linear span of $\mathcal{P}_{fin}$ with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Note that $|\mathcal{P}_{fin}| \leq | \cup_{n\in\mathbb{N}} (B \cup B^*)^n| = \aleph_0|B| = |B|$. Moreover, note that $B' = \cup_{n\in\mathbb{N}^*} S_n$, where

$$S_n = \left\{ \sum_{i=1}^{n} a_i s_i \;\middle|\; a_i \in \mathbb{Q} + i\mathbb{Q},\, s_i \in \mathcal{P}_{fin} \right\}.$$

We claim that $|S_n|$ is bounded by $|B|$. In fact, since every element $\sum_{i=1}^{n} a_i s_i$ corresponds to a n-tuple $\xi = (a_1 s_1,...,a_n s_n) \in \big((\mathbb{Q} + i\mathbb{Q}) \times \mathcal{P}_{fin}\big)^n$ we have that

$$|S_n| \leq \left|\big((\mathbb{Q} + i\mathbb{Q}) \times \mathcal{P}_{fin}\big)^n\right| \leq \aleph_0|B| = |B|.$$

Thus $|B'| \leq \aleph_0 |B| = |B|$. Therefore $|C^*(B)| \leq |B'|^{\aleph_0} \leq |B|^{\aleph_0}$, since $C^*(B) = \overline{B'}$.

Finally, let $\pi : \mathcal{A}_0(\mathcal{C}) \to B(H)$ be a representation of $\mathcal{A}_0(\mathcal{C})$, and let B be the unital subalgebra generated by the range of π. Define $A = C^*(B)$, and consider the inclusion map $j : A \to B(H)$. Decompose j as a direct sum of cyclic representations $j = \oplus_{x \in X} j_x$, $j_x : A \to B(H_x)$, and note that $\pi = \oplus_{x \in X} \pi_x$, where $\pi_x = j_x \circ \pi$. Thus

$$\|\pi(f)\| = \sup_{x \in X} \|\pi_x(f)\|, \quad f \in \mathcal{A}_0(\mathcal{C})\,. \tag{1.6}$$

By the above discussion, the dimensions of the Hilbert spaces H_x are bounded by the cardinal number $\omega := \left(|\,\mathcal{A}_0(\mathcal{C})\,|^{\aleph_0}\right)^{\aleph_0}$. For each cardinal number $\lambda \leq \omega$, take H_λ to be a Hilbert space of dimension λ, and define $\Lambda = \{H_\lambda \mid \lambda \leq \omega\}$. Then consider the following set

$$\mathrm{Rep}'(\mathcal{A}_0(\mathcal{C})) = \{\pi : \mathcal{A}_0(\mathcal{C}) \to B(H) \mid \pi \in \mathrm{Rep}(\mathcal{A}_0(\mathcal{C})),\ H \in \Lambda\}.$$

By (1.6), we have $||f||_0 = \sup\left\{\, ||\pi(f)|| \mid \pi \in \mathrm{Rep}'(\mathcal{A}_0(\mathcal{C}))\right\}$ and hence defining

$$\Pi : \mathcal{A}_0(\mathcal{C}) \to B\left(\bigoplus_{\lambda \leq \omega} H_\lambda\right)$$

by

$$\Pi(f) := \bigoplus_{\pi \in \mathrm{Rep}'(\mathcal{A}_0(\mathcal{C}))} \pi(f)$$

we obtain a isometric representation of $\mathcal{A}_0(\mathcal{C})$, and hence $\mathcal{A}(\mathcal{C}) \cong \overline{\Pi(\mathcal{A}_0(\mathcal{C}))}$ which is in turn an operator algebra. The representation Π is commonly called the *universal* representation of $\mathcal{A}_0(\mathcal{C})$.

1.2.2 Graph algebras

Let $E = (E^0, E^1, d, r)$ be a directed graph. Define for $n \geq 2$ the set of paths of length n to be the set $E^n = \{a_1 ... a_n : \{a_i\}_{i=1}^n \subseteq E^1,\ d(a_i) = t(a_{i+1}), \forall\ 1 \leq i \leq n-1\}$. The *category associated with the graph E* is the pair $\mathcal{C}_E = \left(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\right)$ where $\mathcal{C}^{(0)} = E^0$, $\mathcal{C}^{(1)} = \bigcup_{n \in \mathbb{N}} E^n$ and the structure maps are as follows: we set $\mathbf{u}$ to be the inclusion map, moreover we define the *source* and *range* maps to be

$$\mathbf{d}(x) = \begin{cases} x, & \text{if } x \in E^0 \\ d(a_n), & \text{if } x = a_1 ... a_n,\ n \geq 1 \end{cases} \qquad \mathbf{r}(x) = \begin{cases} x, & \text{if } x \in E^0 \\ r(a_1), & \text{if } x = a_1 ... a_n,\ n \geq 1. \end{cases}$$

Finally, the composition is given by

$$\mathbf{m}(x,y) = \begin{cases} x, & \text{if } y \in E^0 \\ y, & \text{if } x \in E^0 \\ xy, & \text{otherwise.} \end{cases}$$

where xy means the usual concatenation of the paths. Hence, equipping $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$ with the discrete topology, it is easy to see that $\mathcal{C}_E$ is a cancellative étale category.

For a path $x \in \mathcal{C}_E$, define $L_x \in B(\ell_2(\mathcal{C}_E))$ to be

$$L_x(\delta_y) = \begin{cases} \delta_{xy}, & \text{if } (x,y) \in \mathcal{C}^{(2)} \\ 0, & \text{otherwise.} \end{cases}$$

In [34], one can find the following definition.

Definition 1.29. The **tensor algebra of** E is the closed algebra generated by the family $\{L_x \mid x \in \mathcal{C}_E\} \subseteq B(\ell_2(\mathcal{C}_E))$, and it is denoted by $\mathcal{T}^+(E)$.

Clearly, $\mathcal{T}^+(E)$ can also be viewed as the closed algebra generated by the small family $\{L_x \mid x \in E^0 \cup E^1\}$, since $L_x = L_{a_1}...L_{a_n}$ if $x = a_1...a_n$. We now show that the tensor algebra of E is isomorphic to the reduced operator algebra of $\mathcal{C}_E$.

Proposition 1.30. $\mathcal{T}^+(E) = \mathcal{A}_r(\mathcal{C}_E)$.

Proof. Let $\pi : \mathcal{A}_0(\mathcal{C}) \to B(\ell_2(\mathcal{C}))$ be the regular representation of $\mathcal{A}_0(\mathcal{C})$, let x be an element of $\mathcal{C}$, and let f be the characteristic function of $\{x\}$. Note that

$$\pi_f(\delta_z) = \sum_{y \in \mathcal{C}_{\mathrm{r}(z)}} f(y)\delta_{yz} = \left[(x,z) \in \mathcal{C}^{(2)}\right] \delta_{xz} = L_x(\delta_z).$$

Moreover, note that if U is a bisection and $g \in \mathrm{C_c}(U)$ then $g = \sum_{j=1}^{n} a_i \chi_{\{x_j\}}$ and hence $\pi_g = \sum_{j=1}^{n} a_j L_{x_i}$. The result then follows. □

Example 1.31. Let E the graph of a unique vertex e_0 and a unique loop e_1. Note that the unique path of length n is the path $e_n = \underbrace{e_1 e_1 ... e_1}_{n \text{ times}}$. Moreover, denote by δ_n the characteristic function of the singleton $\{e_n\}$.

Let $\mathcal{C}_E = \left(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\right)$ be the category of E. Note that $\mathcal{C}^{(0)} = \{e_0\}$, and $\mathcal{C}^{(1)} = \{e_0, e_1, e_2, ...\}$. Moreover, note that $\mathrm{Bis}(\mathcal{C}_E) = \{\{e_n\} \mid n \in \mathbb{N}\}$, and in particular it is isomorphic to $\mathbb{N}$, as a monoid, since $\{e_n\}\{e_m\} = \{e_{n+m}\}$.

Now, note that if $f \in \mathcal{A}_0(\mathcal{C}_E)$ has support contained in a bisection then $f = a\delta_n$, $a \in \mathbb{C}$ and hence

$$\mathcal{A}_0(\mathcal{C}_E) = \left\{ \sum_{i=1}^{n} a_i \delta_i \mid a_i \in \mathbb{C} \right\}.$$

It easily follows then that $\mathcal{A}_0(\mathcal{C}_E)$ is the polynomial algebra $\mathbb{C}[x]$, where $x = \delta_1$ and the constant polynomial 1 stands for the identity δ_0. Our goal is to describe the representations of $\mathbb{C}[x]$.

Proposition 1.32. The representations of $\mathbb{C}[x]$ are in correspondence with pairs (P, T) where $P \in B(H)$ is a projection, $T \in B(H)$ is a contraction and $PT = T = TP$.

Proof. Suppose $\pi : \mathbb{C}[x] \to B(H)$ is a representation in the sense of Definition 1.24. Then $\pi(x)$ is a contraction, and $\pi(1)$ is a projection since it is idempotent and self-adjoint. Moreover $\pi(\delta_n) = \pi(\delta_1)^n$, for every $n \geq 1$.

Conversely if one has in hands a contraction T and a projection P such that $PT = T = TP$ then mapping $1 \to P$ and $x \to T$ gives rise to a representation of $\mathbb{C}[x]$. □

Let H be a Hilbert space and (P, T) a pair like in the statement of Proposition 1.32. Moreover, let π be the representation generated by (P, T) and let π_1 be the representation generated by (id_H, T). For a polynomial $p = \sum_{i=1}^{n} a_i x^i$ note that

$$\begin{aligned} \|\pi(p)\| &= ||a_0 P + \sum_{i=1}^{n} a_i T^i|| = ||a_0 P + \sum_{i=1}^{n} a_i (PT)^i|| \\ &= ||P(a_0 \,\mathrm{id}_H + \sum_{i=1}^{n} a_i T^i)|| \leq ||a_0 \,\mathrm{id}_H + \sum_{i=1}^{n} a_i T^i|| \\ &= \|\pi_1(p)\|. \end{aligned} \tag{1.7}$$

Therefore, we can suppose without loss of generality that every representation of $\mathcal{A}_0(\mathcal{C})$ on H is like π, that is, $p \mapsto p(T)$, for a fixed contraction $T \in B(H)$.

Identifying e_n with n, we have that $\mathcal{C}$ is equal to the natural numbers $\mathbb{N}$ and hence $\ell_2(\mathcal{C}_E)$ is the canonical Hilbert space $\ell_2(\mathbb{N})$. Let $\pi : \mathbb{C}[x] \to B(\ell_2(\mathbb{N}))$ be the regular representation. We proved that $\pi(x) = \pi(\delta_1) = L_1$. Note that $L_1(\delta_n) = \delta_{n+1}$, then $\pi(x)$ is the (forward) shift operator $S \in \mathcal{B}(\ell_2(\mathbb{N}))$. Consequently, $\mathcal{A}_r(\mathcal{C}_E)$ is the closed subalgebra of $B(\ell_2(\mathbb{N}))$ generated by the identity and S, which is called the *disc algebra* and is a subalgebra of the Toeplitz C^*-algebra, the C^*-algebra generated by S.

We now show that $\mathcal{A}(\mathcal{C}_E)$ and $\mathcal{A}_r(\mathcal{C}_E)$ coincide. Let $\mathbb{D}$ be the closed unitary ball of $\mathbb{C}$. For a polynomial $p \in \mathbb{C}[x]$, define $\|p\| = \sup_{z \in \mathbb{D}} |p(z)|$. Von Neumann [46] proved that for every contraction T on a Hilbert space and every polynomial $p \in \mathbb{C}[x]$ it holds that $||p(T)|| \leq \|p\|$.

Conversely, for an operator T on a Hilbert space, let $\sigma(T)$ denote the spectrum of T and $r(T)$ denote the spectral radius of T. Recall that $r(T) \leq \|T\|$. Moreover, by the spectral mapping theorem, for a polynomial $p \in \mathbb{C}[x]$ we have

$$\sigma(p(S)) = p(\sigma(S)) = p(\mathbb{D}).$$

And hence $\|p\| \leq \|p(S)\|$, for every $p \in \mathbb{C}[x]$, where S is the shift operator aforementioned. Then, combining the facts listed above we get

$$\|p\|_0 = \sup\{\, \|p(T)\| \mid T \in B(H), ||T|| \leq 1\} \leq \|p\| \leq ||p(S)||.$$

Thus

$$\mathcal{A}(\mathcal{C}_E) = \mathcal{A}_r(\mathcal{C}_E) = \mathcal{T}^+(E).$$

Example 1.33. Now let E be the graph of one vertex and m loops $e_1,..., e_m$. It is easy to see that $\mathcal{A}_0(\mathcal{C}_E) = \mathbb{C}\{x_1,...x_m\}$ is the algebra of noncommutative polynomials in m variables and that a representation of $\mathcal{A}_0(\mathcal{C}_E)$ on H corresponds to a $(m+1)$–tuple $(P,T_1,...,T_m)$ where P is a projection, each T_i is a contraction, and $PT_i = T_i = T_iP$ for every $i \in \{1,...,m\}$. From a calculation similar to (1.7), we assume without loss of generality that $P = \mathrm{id}_H$.

In what follows we describe the regular representation of $\mathcal{A}_0(\mathcal{C}_E)$. First note that $\mathcal{C}_E = \bigcup_{n\in\mathbb{N}} E^n$ and hence

$$\ell_2(\mathcal{C}_E) = \bigoplus_{n\in\mathbb{N}} \ell_2(E^n).$$

Since $E^0 = \{e_0\}$, we have that $\ell_2(E^0) = \mathbb{C}$, with basis δ_0. Since $E^1 = \{e_1,...,e_m\}$, we have that $H := \ell_2(E^1) = \mathbb{C}^m$, and we denote by δ_i the basic element δ_{e_i}. We now look to $\ell_2(E^n)$ for $n \geq 2$. Note that a basic element of $\ell_2(E^n)$ is of the form $\delta_{e_{i_1}...e_{i_n}}$, and hence identifying $\delta_{e_{i_1}...e_{i_n}}$ with the tensor $\delta_{i_1} \otimes ... \otimes \delta_{i_n}$, we have $\ell_2(E^n) = H^{\otimes n}$. Therefore

$$\ell_2(\mathcal{C}_E) = \mathbb{C} \oplus \bigoplus_{n\geq 1} H^{\otimes n}.$$

This Hilbert space is known as the *full Fock space* of H. Moreover, through this identification the concatenation operators L_{e_j} become left creation operators L_j, where $L_j(\delta_0) = \delta_j$ and $L_j(\delta_{i_1} \otimes ... \otimes \delta_{i_n}) = \delta_j \otimes \delta_{i_1} \otimes ... \otimes \delta_{i_n}$, for every $n \geq 1$. Therefore $\mathcal{A}_r(\mathcal{C}_E)$ is the closed unital subalgebra of $B\big(\mathbb{C} \oplus \bigoplus_{n\geq 1} H^{\otimes n}\big)$ generated by the left creation operators $L_1,...,L_n$. This algebra is called the noncommutative disc algebra $\mathcal{A}_n$ and it was studied by Popescu in [52, 53].

Note that in this case we do not have $\mathcal{A}_r(\mathcal{C}_E) = \mathcal{A}(\mathcal{C}_E)$. For instance, consider the polynomial $p(x_1,...,x_n) = x_1 + ... + x_n$. In $\mathcal{A}_r(\mathcal{C}_E)$ the norm of p is

$$\|p\|_r = \|p(L_1,...,L_n)\| = \sqrt{n}$$

since $\{L_1,...,L_n\}$ is a family of n isometries with pairwise orthogonal ranges. On the other hand the norm of p in the full algebra $\mathcal{A}(\mathcal{C}_E)$ is n since we can take the representation sending each monomial x_i to id_H.

1.3 RELATIONAL COVERING MORPHISMS

We present below a class of morphisms between étale categories called relational covering morphisms (see [36, Section 7.2]). In this definition we have to put our convention aside and treat the set of objects and morphisms as two distinct sets.

Definition 1.34. Let $\mathcal{C} = \big(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\big)$ and $\mathcal{D} = \big(\mathcal{D}^{(0)},\mathcal{D}^{(1)}\big)$ be étale categories. A **relational covering morphism** from $\mathcal{C}$ to $\mathcal{D}$ is a pair (φ_0,φ_1) where $\varphi_0 : \mathcal{C}^{(0)} \to \mathcal{D}^{(0)}$ is a proper continuous function and $\varphi_1 : \mathcal{C}^{(1)} \to \mathcal{P}\big(\mathcal{D}^{(1)}\big)$ is a function such that the following conditions hold:

(M1) $\mathbf{u}(\varphi_0(v)) \in \varphi_1(\mathbf{u}(v))$, for every $v \in \mathcal{C}^{(0)}$.

(M2) If $b \in \varphi_1(a)$ then $\mathbf{d}(b) = \varphi_0(\mathbf{d}(a))$ and $\mathbf{r}(b) = \varphi_0(\mathbf{r}(a))$, for every $a \in \mathcal{C}^{(1)}$.

(M3) If $c \in \varphi_1(a)$ and $d \in \varphi_1(b)$ then $cd \in \varphi_1(ab)$, for every $(a,b) \in \mathcal{C}^{(2)}$.

(M4) If $\mathbf{d}(a) = \mathbf{d}(b)$ (resp. $\mathbf{r}(a) = \mathbf{r}(b)$) and $\varphi_1(a) \cap \varphi_1(b)$ is non-empty then $a = b$, for every a and b in $\mathcal{C}^{(1)}$.

(M5) If $\mathbf{d}(x) = \varphi_0(v)$ (resp. $\mathbf{r}(x) = \varphi_0(v)$) then there exists $a \in \mathcal{C}_v$ (resp. $a \in \mathcal{C}^v$) such that $x \in \varphi_1(a)$, for every $v \in \mathcal{C}^{(0)}$ and $x \in \mathcal{D}^{(1)}$.

(M6) If $A \in \mathrm{Bis}(\mathcal{D})$ then $\widehat{A} := \{z \in \mathcal{C}^{(1)} : \varphi_1(z) \cap A \neq \emptyset\} \in \mathrm{Bis}(\mathcal{C})$.

Throughout the following let $\mathcal{C}$ and $\mathcal{D}$ be étale categories and let $\varphi = (\varphi_0,\varphi_1) : \mathcal{C} \to \mathcal{D}$ be a relational covering morphism.

Proposition 1.35. Let $A \in \mathrm{Bis}(\mathcal{D})$ be a bisection, and let f be a map in $\mathrm{C_c}(A)$. Then the map $\widehat{f} : \mathcal{C} \to \mathbb{C}$, given by $\widehat{f}(z) = \sum_{x\in\varphi_1(z)} f(x)$, is in $\mathrm{C_c}(\widehat{A})$.

Proof. Suppose $\widehat{f}(z)$ is nonzero. Then there exists $x \in \varphi_1(z)$ such that $f(x) \neq 0$, which gives that $x \in A$, and consequently $x \in \varphi_1(z) \cap A$. Thus $z \in \widehat{A}$, and moreover $\widehat{f}_{\mathcal{C}\setminus\widehat{A}} \equiv 0$.

Now, let us check that $\widehat{f}_{\widehat{A}}$, the restriction of $\widehat{f}$ to $\widehat{A}$, is a continuous function with compact support. Note that if $z \in \widehat{A}$ then $\varphi_1(z) \cap A$ has a unique element since A is a bisection and since (M2) holds. Denote by x_z the unique element of $\varphi_1(z) \cap A$ and note that $\widehat{f}(z) = f(x_z)$. Thus, for an open subset B of $\mathbb{C}$, we have $z \in \widehat{f}_{\widehat{A}}^{-1}(B)$ if and only if $x_z \in f_A^{-1}(B)$, and moreover $x_z \in f_A^{-1}(B)$ if and only if $z \in \widehat{f_A^{-1}(B)}$, by (M6). Therefore, combining these two equivalences, we obtain $\widehat{f}_{\widehat{A}}^{-1}(B) = \widehat{f_A^{-1}(B)}$ which is open, by property (M6). This gives the continuity of $\widehat{f}$.

Next, we check that $\widehat{f}_{\widehat{A}}$ has compact support. Note that

$$\mathbf{r}_{\widehat{A}}\left(\{z : \widehat{f}(z) \neq 0\}\right) \subseteq \varphi_0^{-1}\Big(\mathbf{r}_A\left(\mathrm{supp}(f)\right)\Big).$$

Indeed, it easily follows from the fact that if $\widehat{f}(z) = f(x_z)$ and $\mathbf{r}(x_z) = \varphi_0(\mathbf{r}(z))$, by (M2). Finally, note that, $\varphi_0^{-1}\left(\mathbf{r}_A\left(\mathrm{supp}(f)\right)\right)$ is compact, since φ_0 is proper and $\mathbf{r}_{\widehat{A}}$ is a homeomorphism. Thus, we obtain

$$\mathbf{r}_{\widehat{A}}\left(\mathrm{supp}(\widehat{f})\right) = \overline{\mathbf{r}_{\widehat{A}}(\{z : \widehat{f}(z) \neq 0\})} \subseteq \varphi_0^{-1}(\mathbf{r}_A \mathrm{supp}(f)).$$

Which gives that $\mathbf{r}_{\widehat{A}}\left(\mathrm{supp}(\widehat{f})\right)$ is compact because it is closed in a compact set, and hence the subset supp($\widehat{f}$ is compact. □

Let $T_\varphi : \mathcal{A}_0(\mathcal{D}) \to \mathcal{A}_0(\mathcal{C})$ be the map $f \mapsto \widehat{f}$, where $\widehat{f}(z) = \sum_{x\in\varphi_1(z)} f(x)$. By Propositions 1.21 and 1.35, it is easy to see that T_φ is well-defined linear map. Indeed, we have even more.

Proposition 1.36. T_φ is an algebra homomorphism.

Proof. Let f and g be in $\mathcal{A}_0(\mathcal{D})$ and note that

$$\widehat{f*g}(z) = \sum_{x\in\varphi_1(z)} f*g(x) = \sum_{x\in\varphi_1(z)} \sum_{(c,d)\in M_x} f(c)g(d). \tag{1.8}$$

On the other hand

$$\widehat{f}*\widehat{g}(z) = \sum_{(a,b)\in M_z} \widehat{f}(a)\widehat{g}(b) = \sum_{(a,b)\in M_z} \sum_{\substack{\xi\in\varphi_1(a)\\ \eta\in\varphi_1(b)}} f(\xi)g(\eta). \tag{1.9}$$

By (M3), we obtain that every term of the sum (1.9) is also a term of the sum (1.8). Conversely, if a composition cd belongs to $\varphi_1(z)$ then $\mathbf{d}(d) = \mathbf{d}(cd) = \varphi_0(\mathbf{d}(z))$, by (M2). In this case, there is $b \in \mathcal{C}_{\mathbf{d}(z)}$ such that $d \in \varphi_1(b)$, by (M5). Moreover, $\mathbf{d}(c) = \mathbf{r}(d) = \varphi_0(\mathbf{r}(b))$ and hence there exists $a \in \mathcal{C}_{\mathbf{r}(b)}$ such that $c \in \varphi_1(a)$, again by (M5). Therefore, $cd \in \varphi_1(ab)$, by (M3). Moreover $\mathbf{d}(ab) = \mathbf{d}(b) = \mathbf{d}(z)$, and hence $ab = z$, by (M4). This gives that whenever a composition cd belongs to $\varphi_1(z)$, there exists a unique $(a,b) \in M_z$ such that $c \in \varphi_1(a)$ and $d \in \varphi_1(b)$. We conclude then that every term of the sum (1.8) also appears uniquely in the sum (1.9), and the result follows. □

The next goal is to show that T_φ extends to a morphism from $\mathcal{A}(\mathcal{D})$ to $\mathcal{A}(\mathcal{C})$. To this end, note that $\widehat{\mathbf{u}(\mathcal{D}^{(0)})} = \mathbf{u}(\mathcal{C}^{(0)})$. Indeed, the inclusion ($\supseteq$) follows easily from (M1). To prove the reverse inclusion, suppose $\mathbf{u}(v) \in \varphi_1(z)$, for $v \in \mathcal{D}^{(0)}$ and $z \in \mathcal{C}^{(1)}$. By (M2), we have that $v = \varphi_0(\mathbf{d}(z))$, and hence we have that $\mathbf{u}(v) = \mathbf{u}\left(\varphi_0(\mathbf{d}(z))\right) \in \varphi_1\left(\mathbf{u}(\mathbf{d}(z))\right)$, by (M1). Therefore $\mathbf{u}(v) \in \varphi_1\left(\mathbf{u}(\mathbf{d}(z))\right) \cap \varphi_1(z)$ and hence $z = \mathbf{u}\left(\mathbf{d}(z)\right)$, by (M4). With this result we can now prove the following proposition.

Proposition 1.37. Let $\pi : \mathcal{A}_0(\mathcal{C}) \to B(H)$ be a representation of $\mathcal{A}_0(\mathcal{C})$. Then $\pi \circ T_\varphi$ is a representation of $\mathcal{A}_0(\mathcal{D})$, and hence T_φ extends to a contractive homomorphism $\overline{T_\varphi} : \mathcal{A}(\mathcal{D}) \to \mathcal{A}(\mathcal{C})$.

Proof. We need to check conditions 1 and 2 of Definition 1.24. For condition 1, let A be a bisection and let $f \in \mathrm{C_c}(A)$ be a map. From Proposition 1.35 and its proof, we have that $\widehat{f} = T_\varphi(f)$ has support contained on the bisection $\widehat{A}$, and moreover $\|\widehat{f}\|_\infty \leq \|f\|_\infty$, since $\widehat{f}(z) = f(x_z)$. Hence $\|\pi \circ T_\varphi(f)\| \leq \|f\|_\infty$.

To prove condition 2, note that if $f \in \mathrm{C_c}\left(\mathbf{u}\left(\mathcal{D}^{(0)}\right)\right)$ then $T_\varphi(f) \in \mathrm{C_c}(\mathcal{C}^{(0)})$ and hence

$$\pi \circ T_\varphi(\overline{f}\,) = \pi\left(\overline{T_\varphi(f)}\right) = \left(\pi \circ T_\varphi(f)\right)^*.$$

□

Let $\mathcal{E}$ be an étale category and $\psi = (\psi_0,\psi_1) : \mathcal{D} \to \mathcal{E}$ be a relational covering morphism. Then, we define $\psi \circ \varphi = ((\psi \circ \varphi)_0,(\psi \circ \varphi)_1)$ to be the following relational covering morphism $(\psi\circ\varphi)_0 : \mathcal{C}^{(0)} \to \mathcal{E}^{(0)},\ (\psi\circ\varphi)_0(v) = \psi_0(\varphi_0(v))$, and

$$(\psi\circ\varphi)_1 : \mathcal{C}1 \to \mathcal{P}(\mathcal{E}^{(1)}), \qquad (\psi\circ\varphi)_1(z) = \bigcup_{x\in\varphi_1(z)} \psi_1(x),\ z\in\mathcal{C}^{(1)}.$$

It is straightforward to prove that (M1)-(M5) hold. Here, we just show that (M6) holds. Let $A \in \mathrm{Bis}(\mathcal{E})$ be a bisection. Denote by $\widehat{A}$ the subset $\{z\in\mathcal{C}^{(1)} :\ (\psi\circ\varphi)_1(z)\cap A \neq \emptyset\}$, and by $\widehat{A}_\psi$ the bisection $\{z\in\mathcal{D}^{(1)} :\ \psi_1(z)\cap A\neq\emptyset\}$. Moreover, for a bisection $B\in\mathrm{Bis}(\mathcal{D})$ let $\widehat{B}_\varphi$ denote the bisection $\{z\in\mathcal{C}^{(1)} :\ \varphi_1(z)\cap B\neq\emptyset\}$. We claim that

$$\widehat{A} = \widehat{(\widehat{A}_\psi)}_\varphi.$$

proof of the claim. If z be in $\widehat{A}$ then there exists $x\in\varphi_1(z)$ such that $\psi_1(x)\cap A$ is non-empty. Hence, $x\in\widehat{A}_\psi$, and consequently $z\in\widehat{(\widehat{A}_\psi)}_\varphi$. The converse is similar. □

Finally, note that for every $f\in\mathcal{A}_0(\mathcal{E})$, and $z\in\mathcal{C}$ we have

$$T_{\psi\circ\varphi}(f)(z) = \sum_{x\in(\psi\circ\varphi)_1(z)} f(x) = \sum_{y\in\varphi_1(z)}\ \sum_{x\in\psi_1(y)} f(x) = (T_\varphi\circ T_\psi)(f)(z).$$

This gives $\overline{T_{\psi\circ\varphi}} = \overline{T_\varphi}\circ\overline{T_\psi}$, and proves that we have a functor from the category of étale categories to the category of Banach algebras.

2 RESTRICTION SEMIGROUPS AND THEIR OPERATOR ALGEBRAS

In this chapter, we review the basic theory of restriction semigroups, developed in [36, chapter 2]. After that, we define $\mathcal{A}(S)$ and $\mathcal{A}_r(S)$ as the full and the reduced operator algebra of a restriction semigroup S, respectively. Then we devote the remainder of the chapter to prove that $\mathcal{A}(S)$ has a semicrossed product structure.

Below, we introduce the notion of restriction semigroup. A good picture for a restriction semigroup (S,E,λ,ρ) could be the one in which an element $s \in S$ is a function, and the maps λ and ρ assign the correspondent domain and codomain to s. The subset E, in turn, is a semilattice, which is a commutative semigroup consisting only of idempotents. In this case recall that $e \leq f \Leftrightarrow ef = e$ defines an order relation on E.

Definition 2.1. Let S be a semigroup, and $E(S)$ be the set of idempotents of S. Moreover, let $E \subseteq E(S)$ be a non-empty commutative subsemigroup of S, and let $\lambda : S \to E$ and $\rho : S \to E$ be functions satisfying:

(P1) $\lambda(f) = f$, for every $f \in E$.

(P2) $\rho(f) = f$, for every $f \in E$.

(P3) $s = s\lambda(s)$, for every $s \in S$.

(P4) $s = \rho(s)s$, for every $s \in S$.

(P5) $\lambda(st) = \lambda(\lambda(s)t)$, for every $s,t \in S$.

(P6) $\rho(st) = \rho(s\rho(t))$, for every $s,t \in S$.

(P7) $fs = s\lambda(fs)$, for every $f \in E$ and $s \in S$.

(P8) $sf = \rho(sf)s$, for every $f \in E$ and $s \in S$.

The quadruple (S,E,λ,ρ) is called a **restriction semigroup**. Further, the elements of the semilattice E are called projections, and the maps λ and ρ are called structure maps.

Remark 2.2. Under the conditions of Definition 2.1, a quadruple (S,E,λ,ρ) satisfying (P1)-(P6) is called an **Ehresmann semigroup**. In the literature, one can find examples in which the map ρ is not defined, and in this case, a triple (S,E,λ) satisfying (P1), (P3), (P5), and (P7) is called a **left restriction semigroup**.

Example 2.3. Let S be a monoid and e its unit. Defining $E = \{e\}$, note that there is just one possibility for λ and ρ. Thus, it can be easily seen that (S,E,λ,ρ) is a restriction semigroup.

Example 2.4 (cf. Proposition 3.12 of [36])**.** If $\mathcal{C}$ is an étale category then $(\mathrm{Bis}(\mathcal{C}),\ E, \lambda, \rho)$ is a restriction semigroup, where $E = \{U \in \mathrm{Bis}(\mathcal{C})\ |U \subseteq \mathcal{C}^{(0)}\}$, $\lambda(U) = \{\mathrm{d}(x)\ |x \in U\}$ and $\rho(U) = \{\mathrm{r}(x)\ |x \in U\}$.

Example 2.5 (cf. [39], [50])**.** If S is an inverse semigroup then $(S,E(S),\lambda,\rho)$ is a restriction semigroup, where $\lambda(s) = s^*s$ and $\rho(s) = ss^*$ are the usual *source* and *range* maps on S. This will be the canonical way of viewing an inverse semigroup as a restriction semigroup.

The following proposition shows that the structure maps are intrinsic to the pair (S,E).

Proposition 2.6. Let (S,E,λ,ρ) be a restriction semigroup, and let $s \in S$. Then $\lambda(s) = \min\{f \in E \mid sf = s\}$ and $\rho(s) = \min\{f \in E \mid fs = s\}$, where E is equipped with its semilattice order.

Proof. Note that the $\lambda(s) \in \{f \in E \mid sf = s\}$, by property (P3). Moreover, if $f \in E$ is such that $sf = s$ we have that $\lambda(s)f \in E$, and hence $\lambda(s)f = \lambda(\lambda(s)f) = \lambda(sf) = \lambda(s)$. Therefore $\lambda(s) \leq f$, proving that $\lambda(s) = \min\{f \in E \mid sf = s\}$. The result for ρ is similar. □

A restriction semigroup (S,E,λ,ρ) has an underlying category structure, where $\mathcal{C}_S^{(0)} := E$ is the set of objects, $\mathcal{C}_S^{(1)} := S$ is the set of all morphisms and the structure maps λ and ρ are the *source* and the *range*, respectively. In this case for elements s and t of S such that $\lambda(s) = \rho(t)$, we define the composition $s \cdot t$ to be the product st on S. The category $\mathcal{C}_S = \left(\mathcal{C}_S^{(0)}, \mathcal{C}_S^{(1)}\right)$ is called *the category of the restriction semigroup* (S,E,λ,ρ).

Like inverse semigroups, restriction semigroups have a natural partial order induced by the set of projections. In fact, the relation

$$s \leq t \iff \exists\, f \in E \text{ such that } s = tf,$$

is a partial order on S, which agrees with the semilattice order on E. Furthermore, the following equivalences hold.

1. $s \leq t$.
2. $\exists\, f \in E$ such that $s = ft$.
3. $s = \rho(s)t$.
4. $s = t\lambda(s)$.

The properties presented below are proved in chapter 2 of [36] and follow easily from the previous definitions. After that, we state some propositions whose proofs are very easy to understand if one thinks of the product as partial composition and λ and ρ as domain and codomain.

- (R1) $sf \leq s, \forall s \in S, f \in E$.
- (R2) $fs \leq s, \forall s \in S, f \in E$.
- (R3) λ and ρ are order-preserving.
- (R4) If $s \leq t$ and $\lambda(s) = \lambda(t)$ then $s = t$.
- (R5) If $s \leq t$ and $\rho(s) = \rho(t)$ then $s = t$.
- (R6) $\lambda(st) \leq \lambda(t), \forall s,t \in S$.
- (R7) $\rho(st) \leq \rho(s), \forall s,t \in S$.

Proposition 2.7. Let (S,E,λ,ρ) be a restriction semigroup, and let s,t be elements of S. Then $\rho(t) \leq \lambda(s)$ if and only if $\lambda(st) = \lambda(t)$.

Proof. If $\rho(t) \leq \lambda(s)$ then $\lambda(s)t = \lambda(s)\rho(t)t = \rho(t)t = t$. Applying λ to both sides, we get $\lambda(st) = \lambda(t)$. On the other hand, suppose $\lambda(st) = \lambda(t)$. By (R2), $\lambda(s)t \leq t$ and in addition $\lambda(\lambda(s)t) = \lambda(st) = \lambda(t)$. Then by (R4) $t = \lambda(s)t$. Thus, Proposition 2.6 ensures $\rho(t) \leq \lambda(s)$. □

Proposition 2.8. Let (S,E,λ,ρ) be a restriction semigroup, and let s, t, and y in S. Then $\rho(y) \leq \lambda(st)$ if and only if $\rho(y) \leq \lambda(t)$ and $\rho(ty) \leq \lambda(s)$.

Proof. Suppose $\rho(y) \leq \lambda(st)$. By (R6), $\rho(y) \leq \lambda(t)$, and moreover

$$\begin{aligned}\lambda(s)\rho(ty) &\overset{(P6)}{=} \lambda(s)\rho(t\rho(y)) \overset{(P2)}{=} \rho(\lambda(s)\rho(t\rho(y)))\\ &\overset{(P6)}{=} \rho(\lambda(s)t\rho(y)) \overset{(P7)}{=} \rho(t\lambda(\lambda(s)t)\rho(y))\\ &\overset{(P5)}{=} \rho(t\lambda(st)\rho(y)) \overset{hip.}{=} \rho(t\rho(y)) \overset{(P6)}{=} \rho(ty).\end{aligned}$$

On the other hand, suppose that $\rho(y) \leq \lambda(t)$ and that $\rho(ty) \leq \lambda(s)$. Then we have

$$\begin{aligned}\lambda(st)\rho(y) &\overset{(P1)}{=} \lambda(\lambda(st)\rho(y)) \overset{(P5)}{=} \lambda(st\rho(y)) \overset{(P5)}{=} \lambda(\lambda(s)t\rho(y)) \overset{(P8)}{=} \lambda(\lambda(s)\rho(t\rho(y))t)\\ &\overset{(P8)}{=} \lambda(\lambda(s)\rho(ty)t) \overset{hip.}{=} \lambda(\rho(ty)t) \overset{(P6)}{=} \lambda(\rho(t\rho(y))t) \overset{(P8)}{=} \lambda(t\rho(y))\\ &\overset{(P5)}{=} \lambda(\lambda(t)\rho(y)) \overset{(P1)}{=} \lambda(t)\rho(y) \overset{hip.}{=} \rho(y).\end{aligned}$$

□

We now show an example of a left restriction semigroup which is not a restriction semigroup.

Example 2.9 (Partial surjections on a set)**.** Let X be a non-empty set. Define

$$\mathcal{J}(X) = \{f : A \to B \mid A,B \subseteq X \text{ and } f \text{ is surjective}\}.$$

A straightforward calculation shows that $\mathcal{J}(X)$ is a semigroup equipped with the following product: for $f : A \to B$ and $g : C \to D$ in $\mathcal{J}(X)$ we define

$$\begin{aligned} fg :\ g^{-1}(A \cap D) &\longrightarrow f(A \cap D)\\ x &\longmapsto f(g(x)).\end{aligned} \tag{2.1}$$

Note that for $f : A \to B$ in $\mathcal{J}(X)$ and $C \subseteq X$, we have

$$\begin{aligned} f\,\mathrm{id}_C :\ A \cap C &\longrightarrow f(A \cap C)\\ x &\longmapsto f(x).\end{aligned} \tag{2.2}$$

And, on the other hand

$$\begin{aligned} \mathrm{id}_C\, f :\ f^{-1}(B \cap C) &\longrightarrow B \cap C\\ x &\longmapsto f(x).\end{aligned} \tag{2.3}$$

In particular, for subsets B and C of X we have $\mathrm{id}_C \mathrm{id}_B = \mathrm{id}_{B\cap C}$. Therefore, the set $E = \{\mathrm{id}_A \mid A \subseteq X\}$ is a subsemigroup of $\mathcal{J}(X)$ whose elements are idempotents, that is, E is semilattice. Moreover, for $f : A \to B \in \mathcal{J}(X)$, $C \subseteq A$ and $D \subseteq B$ we have

$$\begin{aligned} f\,\mathrm{id}_C : \quad C &\longrightarrow f(C) \\ x &\longmapsto f(x), \end{aligned} \tag{2.4}$$

and also

$$\begin{aligned} \mathrm{id}_D f : \quad f^{-1}(D) &\longrightarrow D \\ x &\longmapsto f(x). \end{aligned} \tag{2.5}$$

Finally, for $f : A \to B \in \mathcal{J}(X)$, define $\lambda(f) = \mathrm{id}_A$ and $\rho(f) = \mathrm{id}_B$. Let us see that $(\mathcal{J}(X), E, \lambda)$ is a left restriction semigroup and moreover let us see where the quadruple $(\mathcal{J}(X), E, \lambda, \rho)$ fails to be a restriction semigroup

(P1) $\lambda(\mathrm{id}_A) = \mathrm{id}_A$, for every $A \subseteq X$: Trivial.

(P2) $\rho(\mathrm{id}_A) = \mathrm{id}_A$, for every $A \subseteq X$: Trivial.

(P3) $f = f\lambda(f)$, for every $f \in \mathcal{J}(X)$: It follows easily from equation (2.4).

(P4) $f = \rho(f)f$, for every $f \in \mathcal{J}(X)$: It follows easily from equation (2.5).

(P5) $\lambda(fg) = \lambda(\lambda(f)g)$, for every $f,g \in \mathcal{J}(X)$:

Let $f : A \to B$ and $g : C \to D$ in $\mathcal{J}(X)$. By (2.1), we have that $\lambda(fg) = \mathrm{id}_{g^{-1}(A\cap D)}$. On the other hand $\lambda(f)g = \mathrm{id}_A g$ has domain $g^{-1}(A \cap D)$, by (2.3). Thus, $\lambda(\lambda(f)g) = \mathrm{id}_{g^{-1}(A\cap D)} = \lambda(fg)$.

(P6) $\rho(fg) = \rho(f\rho(g))$, for every $f,g \in \mathcal{J}(X)$:

Let $f : A \to B$ and $g : C \to D$ in $\mathcal{J}(X)$. By (2.1) we have that $\rho(fg) = \mathrm{id}_{f(A\cap D)}$. On the other hand $f\rho(g) = f\,\mathrm{id}_D$ has codomain $f(A \cap D)$, by (2.2). Thus, $\rho(f\rho(g)) = \mathrm{id}_{f(A\cap D)} = \rho(fg)$.

(P7) $\mathrm{id}_C f = f\lambda(\mathrm{id}_C f)$, for every $C \subseteq X$ and $f \in \mathcal{J}(X)$:

By (2.3), we have

$$\begin{aligned} \mathrm{id}_C f : \quad f^{-1}(B\cap C) &\longrightarrow B\cap C \\ x &\longmapsto f(x). \end{aligned}$$

On the other hand, since $\lambda(\mathrm{id}_C f) = \mathrm{id}_{f^{-1}(B\cap C)}$ and $f^{-1}(B \cap C) \subseteq A$, by (2.4) we have

$$\begin{aligned} f\lambda(\mathrm{id}_C f) : \quad f^{-1}(B\cap C) &\longrightarrow B\cap C \\ x &\longmapsto f(x). \end{aligned}$$

(P8) $f\,\mathrm{id}_C = \rho(f\,\mathrm{id}_C)f$, for every $C \subseteq X$ and $f \in \mathcal{J}(X)$:

So far, we have proved that $(\mathcal{J}(X), E, \lambda)$ is a left restriction semigroup. To obtain that the quadruple $(\mathcal{J}(X), E, \lambda, \rho)$ is a restriction semigroup we should be able to verify that property (P8) holds. But, unfortunately, it is not always true. Assume X

has at least two distinct elements $a,b \in X$ and let $f : \{a,b\} \to \{a\}$ be the constant map a. By (2.4), $f\,\mathrm{id}_{\{a\}} = \mathrm{id}_{\{a\}}$. However, by (2.5), $\rho(f\,\mathrm{id}_{\{a\}})f = \mathrm{id}_{\{a\}}\,f = f$. Thus $\mathcal{J}(X)$ is not a restriction semigroup. Moreover, we have shown $f\,\mathrm{id}_{\{a\}} \neq \mathrm{id}_{\{a\}}\,f$ and, in particular, the product of idempotents of $\mathcal{J}(X)$ does not commute.

Now, let us define morphisms between restriction semigroups.

Definition 2.10. Let S and T be left restriction semigroups. A **left restriction semigroup homomorphism** from S to T is a semigroup homomorphism $\varphi : S \to T$ such that $\varphi(\lambda(s)) = \lambda(\varphi(s))$, for every $s \in S$.

Definition 2.11. Let S and T be restriction semigroups. A **restriction semigroup homomorphism** from S to T is a semigroup homomorphism $\varphi : S \to T$ such that $\varphi(\lambda(s)) = \lambda(\varphi(s))$ and $\varphi(\rho(s)) = \rho(\varphi(s))$, for every $s \in S$.

Example 2.12 (cf. Example 2.41 of [36])**.** Let G be a group with the identity e and suppose $|G| \geq 2$. Let S be the group G with the adjoined zero 0. It is a restriction semigroup with respect to $E = \{e, 0\}$ with λ and ρ mapping $0 \mapsto 0$ and each nonzero element to e. Then $\varphi : S \to S$, which maps $x \mapsto e$, $x \in G$, and $0 \mapsto 0$, is a restriction semigroup homomorphism.

We say that a restriction semigroup S is left-ample (resp. right-ample) if $st = su$ (resp. $ts = us$) implies that $\lambda(s)t = \lambda(s)u$ (resp. $t\rho(s) = u\rho(s)$) for every triple $(s,t,u) \in S^3$. Moreover, S is called **ample** if it is both left and right ample. The following theorem is an analogous version of the Wagner-Preston theorem for left restriction semigroups. We will use the machinery developed in the proof in a subsequent section.

Theorem 2.13 (Theorem 14 of [60])**.** Let (S,E,λ) be a left restriction semigroup. For every $s \in S$, define $B_s = \{\lambda(s)t \mid t \in S\}$, $C_s = \{st \mid t \in B_s\}$ and define the map $\varphi_s : B_s \to C_s$, given by $\varphi_s(t) = st$. Then the map

$$\begin{aligned} \varphi :\ S &\longrightarrow \mathcal{J}(S) \\ s &\longmapsto \varphi_s \end{aligned}$$

is an injective left restriction semigroup homomorphism, and if in addition S is left-ample then each map φ_s is a bijection.

Proof. Clearly φ_s is surjective. Moreover note that B_s is equal to $\{t \in S \mid t = \lambda(s)t\}$ and C_s is equal to $\{st \mid t \in S\}$. In particular, if S is left-ample and $\varphi_s(t) = \varphi_s(t')$ then $t = \lambda(s)t = \lambda(s)t' = t'$, and therefore φ_s is an injective map for every $s \in S$. Furthermore, it is easy to see that $\varphi_e = \mathrm{id}_{B_e}$, for any $e \in E$. Note that for s and u in S

$$\begin{aligned} \varphi_s\varphi_u :\ \varphi_u^{-1}(B_s \cap C_u) &\longrightarrow \varphi_s(B_s \cap C_u) \\ t &\longmapsto sut. \end{aligned}$$

To show that $\varphi_s\varphi_u = \varphi_{su}$, it suffices to ensure that $\varphi_u^{-1}(B_s \cap C_u) = B_{su}$ and $\varphi_s(B_s \cap C_u) = C_{su}$, since both functions have the same output.

1. $\varphi_u^{-1}(B_s \cap C_u) = B_{su}$: For $t \in \varphi_u^{-1}(B_s \cap C_u)$, we have that $t = \lambda(u)t$ and $ut = \lambda(s)ut$. Hence,

$$\lambda(su)t \overset{(P7)}{=} t\lambda(\lambda(su)t) \overset{(P5)}{=} t\lambda(sut) \overset{(P5)}{=} t\lambda(\lambda(s)ut)$$
$$= t\lambda(ut) \overset{(P5)}{=} t\lambda(\lambda(u)t) \overset{(P7)}{=} \lambda(u)t = t.$$

   Hence, t belongs to B_{su}. On the other hand, if $x \in B_{su}$ then

$$\lambda(u)x = \lambda(u)\lambda(su)x = \lambda(su)\lambda(u)x = \lambda(\lambda(su)\lambda(u))x$$
$$= \lambda(su\lambda(u))x = \lambda(su)x = x.$$

   Therefore, $x \in B_u$. Moreover, $\lambda(s)ux = u\lambda(\lambda(s)u)x = u\lambda(su)x = ux$, and hence $\varphi_u(x) \in B_s$.

2. $\varphi_s(B_s \cap C_u) = C_{su}$: Since $C_{su} = \{sut \mid t \in S\}$, it is clear that $\varphi_s(B_s \cap C_u) \subseteq C_{su}$. On the other hand $sut = s(\lambda(s)ut) = \varphi_s(\lambda(s)ut)$, for every $t \in S$. Finally, note that $\lambda(s)ut \in B_s$, and that $\lambda(s)ut = u\lambda(su)t \in C_u$. Thus, $sut \in \varphi_s(B_s \cap C_u)$ for every $t \in S$.

In conclusion, φ is a semigroup homomorphism and moreover a restriction semigroup homomorphism since

$$\lambda(\varphi_s) = \mathrm{id}_{B_s} = \mathrm{id}_{B_{\lambda(s)}} = \varphi_{\lambda(s)}.$$

Now, suppose $\varphi_s = \varphi_u$. In this case, since $B_s = B_u$ we have that $\lambda(s) \in B_u$ and $\lambda(u) \in B_s$. Thus, $\lambda(s) = \lambda(s)\lambda(u) = \lambda(u)$ and, moreover, $s = \varphi_s(\lambda(s)) = \varphi_s(\lambda_u) = \varphi_u(\lambda_u) = u$, which proves the injectivity of φ. □

Let (S,E,λ,ρ) be a restriction semigroup. Our goal is to express the domain B_s and the codomain C_s of the maps φ_s Theorem 2.13 in terms of the structure maps of S. For instance, note that

$$B_s = \{t \in S \mid \rho(t) \leq \lambda(s)\}.$$

In fact, if $t = \lambda(s)t$ then $\rho(t) = \lambda(s)\rho(t) \leq \lambda(s)$ and, on the other hand, if $\rho(t) \leq \lambda(s)$ then $\lambda(s)t = \lambda(s)\rho(t)t = \rho(t)t = t$. Furthermore, this characterization of B_s gives $C_s = \{st \mid \rho(t) \leq \lambda(s)\}$. And, in this case note that C_s is contained in $\{t \in S \mid \rho(t) \leq \rho(s)\}$. In fact, if $\rho(t) \leq \lambda(s)$ then $\rho(st) = \rho(s\rho(t)) \leq \rho(s\lambda(s)) = \rho(s)$. The reader familiar with the Wagner-Preston theorem may wonder under what conditions C_s coincides with the set $\{t \in S \mid \rho(t) \leq \rho(s)\}$.

Proposition 2.14. Let S be a restriction semigroup. If the sets $C_s = \{st \in S \mid \rho(t) \leq \lambda(s)\}$ and $\{y \in S \mid \rho(y) \leq \rho(s)\}$ coincide for every $s \in S$, then S is a regular semigroup. Conversely, if S is regular and, in addition, if for every $s \in S$ there is an inverse s^* for s such that $ss^* \in E$ then $C_s = \{st \in S \mid \rho(t) \leq \lambda(s)\}$ coincides with the set $\{y \in S \mid \rho(y) \leq \rho(s)\}$, for every $s \in S$.

Proof. Let $s \in S$ be such that $C_s = \{y \in S \mid \rho(y) \leq \rho(s)\}$. Choosing $y = \rho(s)$, we have that

$$\rho(s) = y = ss^*, \text{ for } s^* \in S \text{ such that } \rho(s^*) \leq \lambda(s). \tag{2.6}$$

Hence, applying ρ to (2.6), we obtain that $\rho(s) = \rho(ss^*) = \rho(s\rho(s^*))$. Then, by (R1) $s\rho(s^*) \leq s$, and by (R5) $s = s\rho(s^*)$. Thus, by Proposition 2.6, $\lambda(s) = \rho(s^*)$. Applying λ to (2.6), we get $\rho(s) = \lambda(\lambda(s)s^*) = \lambda(\rho(s^*)s^*) = \lambda(s^*)$. Therefore, $ss^*s = \rho(s)s = s$ and $s^*ss^* = s^*\rho(s) = s^*\lambda(s^*) = s^*$.

Conversely, let $s \in S$ and an inverse s^* for s such that $ss^* \in E$. Note that if $t \in S$ is such that $\rho(t) \leq \rho(s)$ then $\rho(t) \leq ss^*$ and then $t = \rho(t)t = ss^*\rho(t)t = ss^*t$. □

2.1 OPERATOR ALGEBRAS ASSOCIATED WITH RESTRICTION SEMIGROUPS

Definition 2.15. Let (S,E,λ,ρ) be a restriction semigroup. A **representation** of S on a Hilbert space H is a semigroup homomorphism $\sigma : S \to B(H)$, $s \mapsto \sigma_s$, such that

1. $\|\sigma_s\| \leq 1$ for every $s \in S$.
2. $\sigma_e^* = \sigma_e^2 = \sigma_e$ for every $e \in E$.

The class of all representations of (S,E,λ,ρ) will be denoted by Rep(S).

For a restriction semigroup (S,E,λ,ρ), let $\mathbb{C}[S]$ denote the complex semigroup algebra of S. If $\sigma : S \to B(H)$ is a representation of S on H then it extends to a representation $\widetilde{\sigma} : \mathbb{C}[S] \to \mathrm{B(H)}$, $\widetilde{\sigma}\left(\sum_{s\in S} a_s\delta_s\right) = \sum_{s\in S} a_s\sigma_s$, which is contractive if $\mathbb{C}[S]$ is equipped with the ℓ_1 norm, $\left\|\sum_{s\in S} a_s\delta_s\right\| = \sum_{s\in S} |a_s|$. Thus, we have the following seminorm on $\mathbb{C}[S]$

$$\|\mathrm{x}\|_0 = \sup\left\{ \widetilde{\sigma}(\mathrm{x}) \mid \sigma \in \mathrm{Rep}(S)\right\}.$$

Definition 2.16. Let (S,E,λ,ρ) be a restriction semigroup. The *operator algebra* of S is the Hausdorff completion $\mathcal{A}(S)$ of $(\mathbb{C}[S], \|\cdot\|_0)$

2.1.1 The reduced operator algebra of a restriction semigroup

Let (S,E,λ,ρ) be a left-ample restriction semigroup. From Theorem 2.13 and its subsequent discussion, for each $s \in S$ there exists a bijective map φ_s from $B_s = \{t \in S \mid \rho(t) \leq \lambda(s)\}$ to $C_s = \{st \mid t \in B_s\}$, namely $\varphi_s(t) = st$. Let $\ell_2(S)$ be the Hilbert space associated with S (see Definition 1.1), and note that each map φ_s induces an operator

$$\varphi'_s : \ell_2(S) \to \ell_2(S) \text{ by } \varphi'_s(\delta_t) = [\rho(t) \leq \lambda(s)]\, \delta_{st}, \tag{2.7}$$

which is a partial isometry such that $\ker(\varphi'_s)^\perp = \overline{\mathrm{span}}\{\delta_t \mid t \in B_s\}$ and $\mathrm{ran}(\varphi'_s)_s = \overline{\mathrm{span}}\{\delta_t \mid t \in C_s\}$. Thus, the map $\varphi' : S \to B(\ell_2(S))$, $s \mapsto \varphi'_s$, is a representation of S called **the regular representation of** S. In fact, to see that $\varphi' \in \mathrm{Rep}(S)$ it suffices to note that Proposition 2.8 ensures

$$\varphi'_{st}(\delta_y) = \varphi'_s\varphi'_t(\delta_y),\ \forall y \in S.$$

Theorem 2.17. The extension $\widetilde{\varphi'} : \mathbb{C}[S] \to B(\ell_2(S))$ is faithful.

Proof. Replacing s^*s by $\lambda(s)$, ss^* by $\rho(s)$, and $E(S)$ by E, the proof of Theorem 2.17 is the verbatim copy of Wordingham's theorem proof (see [50, Theorem 2.1.1]). □

Definition 2.18. Let (S,E,λ,ρ) be a left-ample restriction semigroup. The **reduced operator algebra** of S is the closure $\mathcal{A}_r(S)$ of $\widetilde{\varphi'(\mathbb{C}[S]})$ in $B(\ell_2(S))$.

2.1.2 The inverse semigroup case

Let S be an inverse semigroup and let σ be a representation of S in the sense of Definition 2.15. For every $s \in s$, note that σ_{s^*} is a generalized inverse of σ_s (see Definition 1.3). Hence, by Corollary 1.5, we obtain that $\sigma_{s^*} = \sigma_s^*$, which implies that σ is a $*$-representation of S. This implies

$$\mathcal{A}(S) = C^*(S).$$

Moreover, by Proposition 2.14, the regular representation φ' presented on (2.7) is precisely the left regular representation from the Wagner-Preston theorem [50, Proposition 2.1.3]. And hence

$$\mathcal{A}_r(S) = C^*_{red}(S).$$

2.2 RESTRICTION SEMIGROUP ÉTALE ACTIONS

Now, we give the precise definition of what is an étale action of a restriction semigroup. We emphasize that there are different notions of restriction semigroups as well as restriction semigroup actions, and that is the reason why we have decided to call our actions *étale*. However, in the text either étale action or simply action will mean the same.

Let (S,E,λ,ρ) be a restriction semigroup, let X be a set and let $\mathcal{I}(X)$ be the inverse semigroup of partial bijections of X. Suppose $\theta : S \to \mathcal{I}(X)$ is a restriction semigroup homomorphism, and note that for every $s \in S$, there exist $D_s \subseteq X$, $R_s \subseteq X$ such that $\theta_s : D_s \to R_s$ is a bijection. For every idempotent $e \in E(S)$, recall that $D_e = R_e$, and moreover $\theta_e = \mathrm{id}_{D_e}$. Thus, for every $s \in S$

$$\mathrm{id}_{D_{\lambda(s)}} = \theta_{\lambda(s)} = \lambda(\theta_s) = \theta_s^{-1}\theta_s = \mathrm{id}_{D_s} ,$$

and

$$\mathrm{id}_{D_{\rho(s)}} = \theta_{\rho(s)} = \rho(\theta_s) = \theta_s\theta_s^{-1} = \mathrm{id}_{R_s} .$$

Therefore, $D_s = D_{\lambda(s)}$ and $R_s = D_{\rho(s)}$.

Definition 2.19. Let (S,E,λ,ρ) be a restriction semigroup and let X be a locally compact Hausdorff space. An **étale action** of S on X is a restriction semigroup homomorphism $\theta : S \to \mathcal{I}(X)$ satisfying:

1. For every $e \in E$, D_e is open and $X = \bigcup_{e \in E} D_e$.
2. For every $s \in S$, $\theta_s : D_{\lambda(s)} \to D_{\rho(s)}$ is a homeomorphism.

Definition 2.20. Let (S,E,λ,ρ) be a restriction semigroup and let A be a C^*-algebra. An **étale action** of S on A is a restriction semigroup homomorphism $\alpha : S \to \mathcal{I}(A)$, $\alpha_s : J_{\lambda(s)} \to J_{\rho(s)}$, satisfying:

1. For every $e \in E$, J_e is a closed ideal and $A = \overline{\text{span}} \bigcup_{e \in E} J_e$.
2. For every $s \in S$, $\alpha_s : J_{\lambda(s)} \to J_{\rho(s)}$ is a $*$-isomorphism.

Example 2.21. If $\mathcal{C}$ is an étale category, we have seen in Example 2.4 that $\text{Bis}(\mathcal{C})$ is a restriction semigroup. Furthermore, as in the groupoid case, a bisection U defines a homeomorphism from $\mathrm{d}(U)$ to $\mathrm{r}(U)$, namely $\theta_U = \mathrm{r}_U \circ \mathrm{d}_U^{-1}$. Easy calculations show that $\theta : \text{Bis}(\mathcal{C}) \to \mathcal{I}(\mathcal{C}^{(0)})$, $U \mapsto \theta_U$, defines an étale action of $\text{Bis}(\mathcal{C})$ on $\mathcal{C}^{(0)}$.

Remark 2.22. Every étale action of a restriction semigroup S on a locally comapct Hausdorff space X induces an étale action of S on $C_0(X)$. Indeed, define $J_e := C_0(D_e)$, and $\alpha_s : J_{\lambda(s)} \to J_{\rho(s)}$ by $\alpha_s(f) = f \circ \theta_s^{-1}$. It is straightforward to check that the map $\alpha : S \to \mathcal{I}(C_0(X))$, $\alpha(s) = \alpha_s$, is an étale action. In particular, the canonical action of S on $\widehat{E}$, presented below, induces an action on $C_0(\widehat{E})$, where $\alpha_s : C(D_{\lambda(s)}) \to C(D_{\rho(s)})$ is given by $\alpha_s(f) = f \circ \zeta_s$.

2.2.1 The canonical action of a restriction semigroup

Let E be a semilattice. Recall that the *spectrum* of E is the semicharacter space $\widehat{E} = \{\varphi : E \to \{0,1\} \mid \varphi(ef) = \varphi(e)\varphi(f) \text{ and } \varphi \neq 0\}$ endowed with the subspace topology of $\{0,1\}^E$, which agrees with the *pointwise convergence* topology. By Tychonoff's Theorem, $\{0,1\}^E$ is a compact Hausdorff space and hence $\widehat{E} \cup \{0\}$ is a compact Hausdorff subspace, because it is closed. Thus, $\widehat{E}$ itself is a locally compact Hausdorff space.

For every $e \in E$, let D_e denote the subset $\{\varphi \in \widehat{E} \mid \varphi(e) = 1\}$. Note that each subset D_e is clopen (open and closed) in the relative topology on $\widehat{E}$ because $D_e = P_e^{-1}(\{1\}) \cap \widehat{E}$, where $P_e : \{0,1\}^E \to \{0,1\}$ is the projection on coordinate e. Moreover via convergence of nets one easily sees that D_e is closed on $\{0,1\}^E$, and hence D_e is compact. Therefore, the family $\{D_e \mid e \in E\}$ forms a cover of compact open sets for $\widehat{E}$.

Denoting by 1_e the characteristic function of D_e, we have that $1_{ef} = 1_e 1_f$, since $D_e \cap D_f = D_{ef}$. In particular, $\text{span}\{1_e \mid e \in E\}$ is a subalgebra of $C_0(\widehat{E})$. Moreover, note that

$$1_e(\varphi) = \varphi(e), \forall\, \varphi \in \widehat{E}. \tag{2.8}$$

From (2.8), we see that the family of functions $\{1_e \mid e \in E\}$ separates the points of $\widehat{E}$, and that for every semicharacter $\varphi \in \widehat{E}$ there exists e such that $1_e(\varphi) \neq 0$. Then by the Stone-Weierstrass theorem, $\text{span}\{1_e \mid e \in E\}$ is a dense subalgebra of $C_0(\widehat{E})$. Thus, for $f \in E$ and $F \in C(D_f) \subseteq C_0(\widehat{E})$, we have that F is the limit of a sequence

$\{F_n\} \subseteq \text{span}\{1_e \mid e \in E\}$. In particular, since the support of F is contained in D_f, we have that

$$F = \lim_{n\to\infty} 1_f F_n.$$

But note that if $G \in \text{span}\{1_e \mid e \in E\}$, then we can write $G = \sum_{e\in E} a_e 1_e$ and in this case $1_f G = \sum_{e\in E} a_e 1_{ef}$. We have therefore proven that for every $f \in E$ the subalgebra $\text{span}\{1_e \mid e \in E \text{ and } e \leq f\}$ is dense in $\mathrm{C}(D_f)$.

Theorem 2.23 (Proposition 10.6 from [24])**.** Let $\sigma : E \to B(H)$ be a semigroup homomorphism whose range consists of (orthogonal) projections. Then there is a unique C^*-algebra homomorphism $\pi_\sigma : \mathrm{C}_0(\widehat{E}) \to B(H)$ sending $\pi_\sigma(1_e) = \sigma_e$.

Proof. Let A denote the C^*-subalgebra of $B(H)$ generated by the family of commuting projections $\{\sigma_e \mid e \in E\}$. Since $\sigma_e\sigma_f = \sigma_{ef}$ and $\sigma_e^* = \sigma_e$, the subspace span $\{\sigma_e \mid e \in E\}$ is already a self-adjoint subalgebra of $B(H)$. Therefore A is the commutative subalgebra $\overline{\text{span}}\ \{\sigma_e \mid e \in E\}$.

Recall that the spectrum of A is the locally compact Hausdorff space $\widehat{A}$ of all nonzero continuous C^*-algebra homomorphisms (also called characters) $\psi : A \to \mathbb{C}$. Furthermore, define $\imath_1 : \widehat{A} \cup \{0\} \to \widehat{E} \cup \{0\}$ by $\imath_1(\psi)(e) = \psi(\sigma_e)$. Via convergence of nets, one can easily see that $\imath_1$ is a continuous map. Moreover, if $\imath_1(\psi) = \imath_1(\psi')$ we have that ψ and ψ' coincide on the generator set $\{\sigma_e \mid e \in E\}$, which implies that $\psi = \psi'$ and hence $\imath_1$ is injective. Note that, $\imath_1$ is a continuous map between compact Hausdorff spaces and in particular it is proper. Since 0 is mapped to 0, $\imath_1$ restricts to a proper injective map $\imath : \widehat{A} \to \widehat{E}$. In this case we have a C^*-algebra homomorphism $\pi : \mathrm{C}_0(\widehat{E}) \to \mathrm{C}_0(\widehat{A})$, given by $\pi(f) = f \circ \imath$.

For $a \in A$, recall that $\widehat{a} \in \mathrm{C}_0(\widehat{A})$ denotes the function that evaluates a character τ on a, that is, $\widehat{a}(\tau) = \tau(a)$. Moreover, recall that if $T : \mathrm{C}_0(\widehat{A}) \to A \subseteq B(H)$ denotes the inverse of the Gelfand transform we have $T(\widehat{a}) = a$. Finally, for $e \in E$ and $\psi \in \widehat{A}$ we obtain

$$\pi(1_e)(\psi) = 1_e(\imath(\psi)) \overset{(2.8)}{=} \imath(\psi)(e) = \psi(\sigma_e) = \widehat{\sigma_e}(\psi).$$

Thus, defining $\pi_\sigma := T \circ \pi$ we have the desired C^*-algebra homomorphism from $\mathrm{C}_0(\widehat{E})$ to $B(H)$ satisfying $\pi_\sigma(1_e) = \sigma_e$. The uniqueness of π_σ follows from the fact that $\text{span}\{1_e \mid e \in E\}$ is a dense subalgebra of $\mathrm{C}_0(\widehat{E})$, and therefore two continuous linear maps coinciding on $\{1_e \mid e \in E\}$ are the same. □

Let (S,E,λ,ρ) be a restriction semigroup. It is well known that an inverse semigroup T acts on $\widehat{E(T)}$. We proceed now to show that a similar construction holds for S. Recall that $\widehat{E}$ is covered by the family of compact open sets $D_e = \{\varphi \in \widehat{E} \mid \varphi(e) = 1\}$, and define $\theta_s : D_{\lambda(s)} \to D_{\rho(s)}$ by $\theta_s(\varphi)(f) = \varphi(\lambda(fs))$. Note that for $\varphi \in D_{\lambda(s)}$ the following holds

$$\theta_s(\varphi)(\rho(s)) = \varphi(\lambda(\rho(s)s)) = \varphi(\lambda(s)) = 1.$$

Hence, $\theta_s(\varphi) \in D_{\rho(s)}$, and moreover it is a homomorphism since

$$\begin{aligned}\theta_s(\varphi)(ef) &= \varphi(\lambda(efs)) = \varphi(\lambda(es\lambda(fs)))\\ &= \varphi(\lambda(es)\lambda(fs))) = \varphi(\lambda(es))\varphi(\lambda(fs)))\\ &= \theta_s(\varphi)(e)\theta_s(\varphi)(f).\end{aligned}$$

Now, if $\{\varphi_j\}_{j\in J}$ is a net in $D_{\lambda(s)}$ converging to φ then $\varphi_j(\lambda(fs))$ converges to $\varphi(\lambda(fs))$, for every $f \in E$. Then $\theta_s(\varphi_j)$ converges to $\theta_s(\varphi)$ and θ_s is continuous. The inverse of θ_s is the map $\zeta_s : D_{\rho(s)} \to D_{\lambda(s)}$ given by $\zeta_s(\varphi)(f) = \varphi(\rho(sf))$. Similar calculations show that it is a well-defined continuous map. We now pass to show that ζ_s and θ_s are inverses. Note that for $\varphi \in D_{\lambda(s)}$ we have

$$\begin{aligned}(\zeta_s \circ \theta_s)(\varphi)(e) &= \theta_s(\varphi)(\rho(se)) = \varphi(\lambda(\rho(se)s))\\ &= \varphi(\lambda(se)) = \varphi(\lambda(\lambda(s)e))\\ &= \varphi(\lambda(s)e) = \varphi(\lambda(s))\varphi(e)\\ &= \varphi(e).\end{aligned}$$

Similarly, $\theta_s \circ \zeta_s = \mathrm{id}_{D_{\rho(s)}}$, and therefore θ_s is a homeomorphism.

The following calculations show that $\theta : S \to \mathcal{I}(\widehat{E})$ is an étale action.

1. $D_{\lambda(st)} = \theta_t^{-1}(D_{\rho(t)} \cap D_{\lambda(s)})$: Let $\varphi \in D_{\rho(t)} \cap D_{\lambda(s)}$, then $\theta_t^{-1}(\varphi) = \zeta_t(\varphi)$ and, since $t\lambda(st) = t\lambda(\lambda(s)t) = \lambda(s)t$, we have

$$\begin{aligned}\zeta_t(\varphi)(\lambda(st)) &= \varphi(\rho(t\lambda(st))) = \varphi(\rho(\lambda(s)t))\\ &= \varphi(\rho(\lambda(s)\rho(t))) = \varphi(\lambda(s)\rho(t))\\ &= \varphi(\lambda(s))\varphi(\rho(t)) = 1.\end{aligned}$$

   Thus, we have the inclusion ($\supseteq$). For ($\subseteq$), note that if $\varphi \in D_{\lambda(st)}$ then

$$\theta_t(\varphi)(\lambda(s)) = \varphi(\lambda(\lambda(s)t)) = \varphi(\lambda(st)) = 1.$$

2. $\theta_s\theta_t = \theta_{st}$:

$$\begin{aligned}\theta_s\theta_t(\varphi)(e) &= \theta_t(\varphi)(\lambda(es)) = \varphi(\lambda(\lambda(es)t))\\ &= \varphi(\lambda(est)) = \theta_{st}(\varphi)(e).\end{aligned}$$

   Hence, θ_{st} is equal to $\theta_s\theta_t$ since they are two bijections with the same domain and rule.

3. $\theta_{\lambda(s)} = \mathrm{id}_{D_{\lambda(s)}}$: For $\varphi \in D_{\lambda(s)}$, we have

$$\begin{aligned}\theta_{\lambda(s)}(\varphi)(e) &= \varphi(\lambda(e\lambda(s))) = \varphi(e\lambda(s))\\ &= \varphi(e)\varphi(\lambda(s)) = \varphi(e).\end{aligned}$$

4. $\theta_{\rho(s)} = \mathrm{id}_{D_{\rho(s)}}$: For $\varphi \in D_{\rho(s)}$, we have

$$\begin{aligned}\theta_{\rho(s)}(\varphi)(e) &= \varphi(\lambda(e\rho(s))) = \varphi(e\rho(s))\\ &= \varphi(e)\varphi(\rho(s)) = \varphi(e).\end{aligned}$$

2.2.1.1 The invariance of the tight spectrum

In this subsection, we use the notation present in [24, Section 11]. Suppose that (S,E,λ,ρ) is a restriction semigroup with 0, and moreover assume that 0 is in E.

Recall that $\widehat{E}_0 = \{\varphi \in \widehat{E} \mid \varphi(0) = 0\}$ is a locally compact Hausdorff space and its elements are called *characters*. Moreover, for $x,y \in E$ we say that x *intersects* y if, and only if, $xy \neq 0$ and, in this case, we write $x \Cap y$. Otherwise, we say that x and y are *orthogonal*, and we write $x \perp y$. Let us see that $\widehat{E}_0$ is invariant under the canonical action, that is, if $\varphi \in \widehat{E}_0 \cap D_{\lambda(s)}$ then $\theta_s(\varphi) \in \widehat{E}_0$. Note that

$$\theta_s(\varphi)(0) = \varphi(\lambda(0s)) = \varphi(\lambda(0)) = \varphi(0) = 0.$$

Definition 2.24. For $x \in E$ the **upper set** of x is the set $x^{\uparrow} = \{f \in E \mid x \leq f\}$ and the **down set** of x is the set $x^{\downarrow} = \{f \in E \mid f \leq x\}$.

Definition 2.25. Let F be a subset of E. A subset $Z \subseteq F$ is a **cover** for F if for every nonzero $x \in F$, there exists $z \in Z$ such that $z \Cap x$. For $y \in E$, we say that Z is a cover for y if Z is a cover for $y^{\downarrow}$.

In view of [24, Prop 11.8], we define a tight character as follows.

Definition 2.26. Let φ be in $\widehat{E}_0$. We call φ a **tight character** if for every $x \in E$ and for every finite cover Z for x one has that

$$\bigvee_{z \in Z} \varphi(z) \geq \varphi(x).$$

The **tight spectrum** of E is the set of all tight characters, and it is denoted by $\widehat{E}_{tight}$.

Lemma 2.27. Let $x \in E$, Z be a cover for x, and $s \in S$. Then $Z_s = \{\lambda(zs) \mid z \in Z\}$ is a cover for $\lambda(xs)$.

Proof. Since λ is order-preserving, $Z_s \subseteq \lambda(xs)^{\downarrow}$. Now, let $y \in \lambda(xs)^{\downarrow}$ be a nonzero element. Note that

$$x\rho(sy) = \rho(xsy) = \rho(s\lambda(xs)y) = \rho(sy).$$

Hence, $\rho(sy) \leq x$ and

$$y = \lambda(xs)y = \lambda(xsy) = \lambda(x\rho(sy)s) = \lambda(\rho(sy)s).$$

Thus, since y is nonzero and $y = \lambda(\rho(sy)s)$, $\rho(sy)$ is nonzero. Therefore, there exists $z \in Z$ such that $z\rho(sy) \neq 0$. Next, note that

$$z\rho(sy) = \rho(zsy) = \rho(s\lambda(zs)y).$$

In particular, $\lambda(zs)y \neq 0$, and Z_s covers $\lambda(xs)$. $\square$

Proposition 2.28. Le $\varphi \in \widehat{E}_0 \cap D_{\lambda(s)}$ be a tight character. Then $\theta_s(\varphi)$ is a tight character.

Proof. Let $x \in E$, and Z be a finite cover for x. By the previous Lemma, $Z_s = \{\lambda(zs) \mid z \in Z\}$ is a finite cover for $\lambda(xs)$. Therefore

$$\bigvee_{z\in Z} \theta_s(\varphi)(z) = \bigvee_{z\in Z} \varphi(\lambda(zs)) \geq \varphi(\lambda(xs)) = \theta_s(\varphi)(x).$$

□

2.3 THE SEMICROSSED PRODUCT ALGEBRA

Let (S,E,λ,ρ) be a restriction semigroup, let A be a C^*-algebra, and let $\alpha : S \to \mathcal{I}(A)$, $\alpha_s : J_{\lambda(s)} \to J_{\rho(s)}$, be an action of S on A. Consider the vector space L_α of all finite formal sums $\sum_{s\in S} a_s\delta_s$, where $a_s \in J_{\rho(s)}$. Furthermore, define the product of two monomials $a\delta_s$ and $b\delta_t$ to be

$$(a\delta_s)(b\delta_t) := \alpha_s(\alpha_s^{-1}(a)b)\delta_{st},$$

and hence extend this product to L_α. Following [65, Proposition 4.1], and replacing α_{s^*} by α_s^{-1}, we can easily convince ourselves that L_α is an associative algebra.

Definition 2.29. A **covariant pair** for (α,S,A), or simply α, on a Hilbert space H is a pair (π,σ) where σ is a representation of S, π is a representation of A and

1. $\pi(\alpha_s(a))\sigma_s = \sigma_s\pi(a)$, for every $a \in J_{\lambda(s)}$, $s \in S$.
2. $\overline{\text{span}}\, \pi(J_e)H = \sigma_e(H)$, for every $e \in E$.

The equality presented in item 1 is usually called covariance relation.

Every covariant pair (π,σ) on H for the action α integrates to an algebra homomorphism $\pi \times \sigma : L_\alpha \to B(H)$ given by $\pi \times \sigma(a\delta_s) = \pi(a)\sigma_s$. Moreover, $\pi \times \sigma$ is contractive if we endow L_α with the ℓ_1 norm. We will only show that $\pi \times \sigma$ separates the product:

$$\begin{aligned}\pi \times \sigma(a\delta_s b\delta_t) &= \pi(\alpha_s(\alpha_s^{-1}(a)b))\sigma_{st} = \pi(\alpha_s(\alpha_s^{-1}(a)b))\sigma_s\sigma_t \\ &= \sigma_s\pi(\alpha_s^{-1}(a)b)\sigma_t = \sigma_s\pi(\alpha_s^{-1}(a))\pi(b)\sigma_t \\ &= \pi(\alpha_s(\alpha_s^{-1}(a)))\sigma_s\pi(b)\sigma_t = \pi(a)\sigma_s\pi(b)\sigma_t \\ &= \pi \times \sigma(a\delta_s)\pi \times \sigma(b\delta_t).\end{aligned}$$

Definition 2.30. The **semicrossed product algebra** of A by S with respect to α, denoted $A \rtimes_\alpha S$, is the *Hausdorff completion* of L_α equipped with the seminorm induced by the class of all covariant pairs, which is

$$||\sum_{s\in S} a_s\delta_s||_0 = \sup_{(\pi,\sigma)\text{ cov. pair}} ||\pi \times \sigma(\sum_{s\in S} a_s\delta_s)||.$$

Would it be possible to find a “faithful" covariant pair in such a way the quotient is not necessary? In general, the answer is No: Note that if $a \in J_e$, for every covariant pair (π,σ) one has $\pi(a)\sigma_e = \pi(a)$. Indeed $\pi(a)\sigma_e = \pi(\alpha_e(a))\sigma_e = \sigma_e\pi(a) = \pi(a)$. Where the last equality comes form the fact that σ_e is the projection on the subspace $\overline{\pi(J_e)H}$. Then, if $s \leq t$ and $a \in J_{\rho(s)} \subseteq J_{\rho(t)}$ one has

$$\pi \times \sigma(a\delta_s) = \pi(a)\sigma_s = \pi(a)\sigma_{\rho(s)t} = \pi(a)\sigma_{\rho(s)}\sigma_t = \pi(a)\sigma_t = \pi \times \sigma(a\delta_t). \tag{2.9}$$

Remark 2.31. Let s be an element of S. Suppose that $\{a_n\} \subseteq J_{\rho(s)}$ converges to a. Then, for every covariant pair (π,σ), one has $||\pi\times\sigma(a_n\delta_s - a\delta_s)|| = ||\pi(a_n - a)\sigma_s|| \leq ||a_n - a||$. In particular, $a_n\delta_s$ converges to $a\delta_s$ on $A \rtimes_\alpha S$. Thus, if we suppose that $\{J'_e\}_{e\in E}$ is a family of subsets such that J'_e is dense on J_e then the subset $L'_\alpha = \{f \in C_c(S,A) \mid f(s) \in J'_{\rho(s)}\}$ is dense on $A \rtimes_\alpha S$. In particular, if for every $e \in E$ the subset J'_e is an ideal of J_e then L'_α is a dense subalgebra of $A \rtimes_\alpha S$.

In particular, when α arises from a topological action, as in Remark 2.22, we obtain that

$$C_0(X) \rtimes_\alpha S = \overline{\left\{ \sum_{s\in S} a_s\delta_s \mid a_s \in C_c(D_{\rho(s)}) \right\}}. \tag{2.10}$$

Remark 2.32. Let B a unital C^*-algebra and $\tau : B \to B(H)$ a C^*-algebra homomorphism. Note that $\tau(1)H$ is closed because $\tau(1)$ is a projection. Now, for every $b \in B$ and $h \in H$, one has $\tau(b)h = \tau(1)\tau(b)h \in \tau(1)H$. Hence $\overline{\text{span}}\,\tau(B)H = \tau(1)H$.

Theorem 2.33. Let (S,E,λ,ρ) be a restriction semigroup, let $\theta : S \to \mathcal{I}(\widehat{E})$ be the canonical action, and let $\alpha : S \to \mathcal{I}(C_0(\widehat{E}))$ be the induced action of θ. Then

$$\mathcal{A}(S) \cong C_0(\widehat{E}) \rtimes_\alpha S.$$

Proof. Define $\psi : \mathbb{C}[S] \to L_\alpha$ by $\psi(\sum_{s\in S} a_s\delta_s) = \sum_{s\in S} a_s 1_{\rho(s)}\delta_s$, where $1_e := 1_{D_e}$ is the characteristic function of D_e. Since ψ is obviously linear, we are reduced to prove that it preserves the product. For $s,t \in S$ one has

$$\begin{aligned}
\psi(\delta_s)\psi(\delta_t) &= 1_{\rho(s)}\delta_s 1_{\rho(t)}\delta_t \\
&= \alpha_s(\alpha_s^{-1}(1_{\rho(s)})1_{\rho(t)})\delta_{st} \\
&= \alpha_s((1_{\rho(s)} \circ \theta_s)1_{\rho(t)})\delta_{st} \\
&= \alpha_s(1_{\lambda(s)}1_{\rho(t)})\delta_{st} \\
&= 1_{D_{\lambda(s)}\cap D_{\rho(t)}} \circ \zeta_s\delta_{st} \\
&= 1_{\theta_s(D_{\lambda(s)}\cap D_{\rho(t)})}\delta_{st} \\
&= 1_{\rho(st)}\delta_{st} \\
&= \psi(\delta_{st}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\psi\left(\sum_{s\in S} a_s\delta_s \sum_{t\in S} b_t\delta_t\right) &= \psi\left(\sum_{s,t\in S} a_s b_t \delta_{st}\right)\\
&= \sum_{s,t\in S} a_s b_t \psi(\delta_{st})\\
&= \sum_{s,t\in S} a_s b_t \psi(\delta_s)\psi(\delta_t)\\
&= \sum_{s,t\in S} \psi(a_s\delta_s)\psi(b_t\delta_t)\\
&= \psi\left(\sum_{s\in S} a_s\delta_s\right)\psi\left(\sum_{t\in S} b_t\delta_t\right).
\end{aligned}$$

Next, we show that there is a bijection between representations of S and covariant pairs for α. For a representation $\sigma : S \to B(H)$ of S, we have by definition that σ restricts to E as a semigroup homomorphism whose range consists of (orthogonal) projections, and hence let $\pi_\sigma : C_0(\widehat{E}) \to B(H)$ denote the homomorphism from Theorem 2.23. We claim that (π_σ,σ) defines a covariant pair for α. In fact, for $s \in S$ let us check that the *covariance relation* holds. Since $\text{span}\{1_e \mid e \in E \text{ and } e \leq \lambda(s)\}$ is dense on $C(D_{\lambda(s)})$, we only need to check the equality for the family $\{1_e \mid e \in E \text{ and } e \leq \lambda(s)\}$. Take $e \leq \lambda(s)$ and note that

$$\alpha_s(1_e)(\varphi) = 1_e \circ \zeta_s(\varphi) = 1_e(\zeta_s(\varphi)) \overset{(2.8)}{=} \zeta_s(\varphi)(e) = \varphi(\rho(se)) = 1_{\rho(s.e)}(\varphi).$$

Therefore

$$\pi_\sigma(\alpha_s(1_e))\sigma_s = \pi_\sigma(1_{\rho(se)})\sigma_s = \sigma_{\rho(se)}\sigma_s = \sigma_{\rho(se)s} = \sigma_{se} = \sigma_s\sigma_e = \sigma_s\pi(1_e).$$

Now, note that $\sigma_e(H) = \pi_\sigma(1_e)H$ and $\pi_\sigma(1_e)H = \overline{\text{span}}\pi_\sigma(C(D_e))H$, where the latter equality comes from Remark 2.32. Then (π_σ,σ) indeed defines a covariant pair.

Let us see that the map $\sigma \to (\pi_\sigma,\sigma)$ defines an bijective correspondence from the set of representation of S to the set of covariant pairs for α. Clearly, it is injective. Moreover, if (π,σ) is a covariant pair then $\pi(1_e)$ is a projection and again by Remark 2.32 $\pi(1_e)H = \overline{\text{span}}\ \pi(C(D_e))H$, which is equal to $\sigma_e(H)$, by definition of covariant pair. Then $\pi(1_e)$ and σ_e are projections with the same range. Hence, they are equal, and $\pi = \pi_\sigma$.

Finally, take $x = \sum_{s\in S} a_s\delta_s \in \mathbb{C}[S]$ and σ a representation of S, and note that

$$\begin{aligned}
||\pi_\sigma \times \sigma(\psi(x))|| &= \left\|\pi_\sigma \times \sigma\Big(\sum_{s\in S} a_s 1_{\rho(s)}\delta_s\Big)\right\|\\
&= ||\sum_{s\in S} a_s\pi_\sigma(1_{\rho(s)})\sigma_s|| = ||\sum_{s\in S} a_s\sigma_{\rho(s)}\sigma_s||\\
&= ||\sum_{s\in S} a_s\sigma_s|| = ||\widetilde{\sigma}(\sum_{s\in S} a_s\delta_s)|| = ||\widetilde{\sigma}(x)||.
\end{aligned} \tag{2.11}$$

Hence, taking supremum over σ we obtain that ψ is isometric and extends to an isometric homomorphism $\psi_1 : \mathcal{A}(S) \to \mathrm{C}_0(\widehat{E}) \rtimes_\alpha S$.

We finish by showing that the range of ψ_1 contains a dense subset. If $s \in S$ and $e \le \rho(s)$, we have $\rho(es) = \rho(e\rho(s)) = e\rho(s) = e$. Thus, $\psi_1(\delta_{es}) = 1_{\rho(es)}\delta_{es} = 1_e\delta_{es} \overset{(2.9)}{=} 1_e\delta_s$. Then, since $\mathrm{span}\{1_e \mid e \in E \text{ and } e \le \rho(s)\}$ is dense on $\mathrm{C}(D_{\rho(s)})$, by Remark 2.31 we have that ψ_1 is an isomorphism $\square$

3 THE CATEGORY OF GERMS OF A RESTRICTION SEMIGROUP ACTION

Let (S,E,λ,ρ) be a restriction semigroup, let X a locally compact Hausdorff space and let $\theta : S \to \mathcal{I}(X)$ an action. Our goal is to construct an étale category associated with the action θ, similar to the groupoid of germs of an inverse semigroup action. To this end, let Ξ_0 denote the subset $\{(s,x) \mid x \in D_{\lambda(s)}\}$ of $S \times X$, and consider the following equivalence relation on Ξ_0: $(s,x) \sim (t,y)$ if, and only if, $x = y$ and there exists $f \in E$ such that $sf = tf$ an $x \in D_f$. For simplicity, we just show that $\sim$ is transitive. Suppose $(s,x) \sim (t,x)$ and $(t,x) \sim (w,x)$. Then, there exist $e \in E$ and $f \in E$ such that $x \in D_e \cap D_f$, $se = te$, and $tf = wf$. Thus, $sef = tef = tfe = wfe = wef$ and therefore $(s,x) \sim (w,x)$, since $D_{ef} = D_e \cap D_f$. Hence $\sim$ is, in fact, an equivalence relation. The equivalence class of a pair (s,x) will be called the *germ* of s at the point x. Moreover, note that if $(s,x) \sim (t,x)$ and $f \in E$ is a projection implementing the equivalence then

$$\theta_s(x) = \theta_s(\theta_f(x)) = \theta_{sf}(x) = \theta_{tf}(x) = \theta_t(\theta_f(x)) = \theta_t(x). \tag{3.1}$$

Proposition 3.1. Let (s,x) and (t,y) be elements of Ξ_0, with $x = \theta_t(y)$. In addition, suppose $(s',x) \sim (s,x)$ and $(t',y) \sim (t,y)$. Then $(s't',y) \sim (st,y)$.

Proof. Take $e,f \in E$ such that $s'f = sf$, $t'e = te$, $x \in D_f$ and $y \in D_e$. By property (P8) of restriction semigroups, one has $\rho(te)t = te$, and $\rho(te)t' = \rho(t'e)t' = t'e = te$. Thus, if $h = f\rho(te)$ then $ht = fte$, $ht' = ft'e$, and since $te = t'e$, $ht = ht'$. Hence,

$$\begin{aligned} st\lambda(ht) &= sht = (sf)(te) = (s'f)(t'e) \\ &= s'ht' = s't'\lambda(ht') = s't'\lambda(ht). \end{aligned}$$

Moreover, note that $\lambda(ht) = \lambda(fte) = \lambda(\lambda(ft)e) = \lambda(ft)e$. Hence, $D_{\lambda(ht)} = D_{\lambda(ft)} \cap D_e$, which gives that $y \in D_{\lambda(ht)}$ if and only if $y \in D_{\lambda(ft)}$, since y already belongs to D_e. But, recalling that $\theta_t(y) = x$, and $x \in D_f$ we have that $x = \theta_f(x) = \theta_f(\theta_t(y)) = \theta_{ft}(y)$, which means $y \in D_{\lambda(ft)}$, and therefore $y \in D_{\lambda(ht)}$. □

Remark 3.2. Let $x \in X$, $e \in E$, and $f \in E$ be such that $x \in D_e \cap D_f$. Note that $e(ef) = f(ef)$, and $x \in D_{ef}$, and therefore $(e,x) \sim (f,x)$. Moreover, suppose $(s,x) \sim (e,x)$ for $s \in S$ and $e \in E$. In this case, there exists $f \in E$ such that $sf = ef$ and $x \in D_f$, which gives $sef = ef$ and $x \in D_{ef}$. Thus, we easily conclude that two pairs (s,x) and (e,x) are equivalent if and only if there exists a projection h such that $sh = h$ and $x \in D_h$.

Having disposed of this preliminary steps, we can now define the categorical structure associated to the action θ. Define $\mathcal{C}^{(0)} = X$ and $\mathcal{C}^{(1)} = {}^{\Xi_0}\!/\!_{\sim}$. Moreover, define the source map to be $\mathbf{d} : \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$, $\mathbf{d}([s,x]) = x$, and the range map to be $\mathbf{r} : \mathcal{C}^{(1)} \to \mathcal{C}^{(0)}$, $\mathbf{r}([s,x]) = \theta_s(x)$. In this case, the set of composable pairs is $\mathcal{C}^{(2)} = \{([s,x],[t,y]) \in \mathcal{C}^{(1)} \times \mathcal{C}^{(1)} : x = \theta_t(y)\}$, and we define the composition map to be $\mathbf{m} : \mathcal{C}^{(2)} \to \mathcal{C}^{(1)}$ given by $\mathbf{m}([s,x],[t,y]) = [st,y]$. Note that the $\mathbf{m}$ is well-defined by

Proposition 3.1. Finally, we define the unit map to be $\mathbf{u} : \mathcal{C}^{(0)} \to \mathcal{C}^{(1)}$, $\mathbf{u}(x) = [e,x]$ for any $e \in E$ such $x \in D_e$, which is a well-defined map by Remark 3.2. It is a simple matter to check that $\mathcal{C}(\theta,S,X) = \left(\mathcal{C}^{(0)}, \mathcal{C}^{(1)}\right)$ with the above defined structure maps is, in fact, a category. Furthermore, for simplicity of notation whenever the action is clear in the context we will refer to $\mathcal{C}(\theta,S,X)$ just as $\mathcal{C}$.

We now proceed to equip $\mathcal{C}^{(1)}$ with a topology. To this end, for every $s \in S$ and every open set $U \subseteq D_{\lambda(s)}$, define

$$\Theta(s,U) := \{[s,x] \mid x \in U\}.$$

In the case, $U = D_{\lambda(s)}$, we denote $\Theta\left(s,D_{\lambda(s)}\right)$ by Θ_s. We claim that the subset family $\left\{\Theta(s,U) \mid s \in S,\ U \subseteq D_{\lambda(s)} \text{ open set}\right\}$ forms a basis for a topology on $\mathcal{C}^{(1)}$ [70, Theorem 5.3]. In fact, this family is a cover for $\mathcal{C}^{(1)}$. Morover, if $[s,u] = [t,v] \in \Theta(s,U) \cap \Theta(t,V)$, there exists $e \in E$ such that $se = te$ and $u = v \in U \cap V \cap D_e$. Hence, defining $W = U \cap V \cap D_e$, we have that $[s,u] \in \Theta(s,W) \subseteq \Theta(s,U) \cap \Theta(t,V)$.

Proposition 3.3. Equip $\mathcal{C}^{(1)}$ with the topology generated by the basis

$$\left\{\Theta(s,U) \mid s \in S,\ U \subseteq D_{\lambda(s)} \text{ open set}\right\}.$$

Then $\mathcal{C}(\theta,S,X)$ is étale.

Proof. Let $\Theta(s,U)$ be a basic open set of $\mathcal{C}^{(1)}$. By Remark 3.2, one has

$$\mathbf{u}^{-1}(\Theta(s,U)) = \bigcup_{\substack{h \in E \\ \text{s.t } sh=h}} D_h \cap U.$$

On the other hand, if $U \subseteq X$ is an open subset

$$\mathbf{u}(U) = \mathbf{u}(\bigcup_{e \in E} U \cap D_e) = \bigcup_{e \in E} \Theta(e, U \cap D_e).$$

Thus, $\mathbf{u}$ is a one-to-one continuous open map, since the right sides of the above equation are open subsets. In particular, $\mathbf{u}$ is an embedding. Furthermore, note that

$$\mathbf{r}^{-1}(U) = \bigcup_{s \in S} \Theta\left(s, \theta_s^{-1}\left(U \cap D_{\rho(s)}\right)\right) \text{ and } \mathbf{d}^{-1}(U) = \bigcup_{s \in S} \Theta\left(s, U \cap D_{\lambda(s)}\right).$$

The above equations show that both the range and source are continuous maps. Moreover, for every $s \in S$ it is easy to see that $\mathbf{d}_{\Theta_s} : \Theta_s \to D_{\lambda(s)}$ and $\mathbf{r}_{\Theta_s} : \Theta_s \to D_{\rho(s)}$ are bijective maps, and hence $\mathbf{d}(\Theta(s,U)) = U$ and $\mathbf{r}(\Theta(s,U)) = \theta_s(U)$, which gives $\mathbf{d}_{\Theta_s}$ and $\mathbf{r}_{\Theta_s}$ are open maps, since $\{\Theta(s,U) \mid U \subseteq D_{\lambda(s)},\ U \text{ open}\}$ is a basis for the topology of Θ_s. Therefore, $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms, since $\{\Theta_s\}_{s \in S}$ is a cover of $\mathcal{C}^{(1)}$.

We finish by showing the continuity of $\mathbf{m}$. Let $([s,\theta_t(x)],[t,x])$ a composable pair and let $\Theta(r,V)$ a basic open set containing $\mathbf{m}([s,\theta_t(x)],[t,x]) = [st,x]$. There exists $e \in E$

such that $x \in D_e \cap V \cap D_{\lambda(st)}$ and $ste = re$. Hence, defining $W := D_e \cap V \cap D_{\lambda(st)}$, we easily obtain that $\mathbf{m}\Big(\big(\,\Theta_s \times \Theta(t, W)\,\big) \cap \mathcal{C}^{(2)}\Big) \subseteq \Theta(r,V)$, with e implementing the equivalences. □

We would like to draw attention to the fact that we will proceed as before (see the comment after proof of Proposition 1.14) and identify an object x with $[e,x]$, where e is any projection such that $x \in D_e$. Hence, we can without lost of generality say that X is an open subset of $\mathcal{C}(\theta,S,X)$. Note that with this identification we have $\Theta_e = D_e$, for every $e \in E$. Moreover the following diagram commutes

$$\begin{array}{ccc} & \Theta_s & \\ {}^{\mathbf{d}}\swarrow & & \searrow^{\mathbf{r}} \\ D_{\lambda(s)} & \xrightarrow{\ \theta_s\ } & D_{\rho(s)} \end{array} \tag{3.2}$$

Suppose $\mathcal{D}$ is an étale category, and let $\theta : \mathrm{Bis}(\mathcal{D}) \to \mathcal{I}\big(\mathcal{D}^{(0)}\big)$ be the action of Example 2.21. Following the above construction, we have the category of germs $\mathcal{C}\big(\theta, \mathrm{Bis}(\mathcal{D}),\mathcal{D}^{(0)}\big)$ whose elements are equivalence classes of pairs (U,x), where $U \in \mathrm{Bis}(\mathcal{D})$ and $x \in \mathbf{d}(U)$.

Theorem 3.4 (cf. Proposition 5.4 of [24])**.** Let $\mathcal{D}$ be an étale category. Then $\mathcal{D}$ is isomorphic to the category of germs $\mathcal{C}\big(\theta, \mathrm{Bis}(\mathcal{D}),\mathcal{D}^{(0)}\big)$, where $\theta : \mathrm{Bis}(\mathcal{D}) \to \mathcal{I}\big(\mathcal{D}^{(0)}\big)$ is the action of the restriction semigroup of bisections of D on the space $\mathcal{D}^{(0)}$.

Proof. For simplicity, let $\mathcal{C}$ denote $\mathcal{C}\big(\theta, \mathrm{Bis}(\mathcal{D}),\mathcal{D}^{(0)}\big)$. We have to check that there exists a homeomorphism $\varphi : \mathcal{C} \to \mathcal{D}$ satisfying:

1. $\varphi\big(\mathcal{C}^{(0)}\big) = \mathcal{D}^{(0)}$
2. $\varphi([U,x][V,y]) = \varphi([U,x])\varphi([V,y])$, for every $([U,x],[V,y]) \in \mathcal{C}^{(2)}$.

To this end, define $\varphi : \mathcal{C} \to \mathcal{D}$, $\varphi([U,x]) = \mathbf{d}_U^{-1}(x)$. Note that if $[U,x] = [V,x]$, then there exists $F \subseteq \mathcal{D}^{(0)}$ such that $UF = VF$ and $x \in D_F = \mathbf{d}(F) = F$. In this case, $\mathbf{d}_U^{-1}(x) \in UF = VF$ and $\mathbf{d}_V^{-1}(x) \in VF = UF$. Therefore, $\mathbf{d}_U^{-1}(x) = \mathbf{d}_V^{-1}(x)$ since the source of these elements is x, and UF is a bisection. Therefore, φ is well-defined.

To see that φ is surjective, note that for every $z \in \mathcal{D}$, we have that $z = \varphi([Z, \mathbf{d}(z)])$, where $Z \in \mathrm{Bis}(\mathcal{D})$ is a bisection such that $z \in Z$. To see the injectivity of φ, note that if $\varphi([U,x]) = \varphi([V,y])$ then $\mathbf{d}_U^{-1}(x) = \mathbf{d}_V^{-1}(y) \in U \cap V$, and hence, defining $F = \mathbf{d}(U \cap V) \subseteq \mathcal{D}^{(0)}$, we have that $x = \mathbf{d}(\mathbf{d}_U^{-1}(x)) = \mathbf{d}(\mathbf{d}_V^{-1}(y)) = y \in F$. Moreover,

$$UF = U\,\mathbf{d}(U \cap V) = U \cap V = V\,\mathbf{d}(U \cap V) = VF.$$

Thus, F implements the equivalence $(U,x) \sim (V,y)$, and therefore $[U,x] = [V,y]$. Finally, since φ is bijective, it is easy to see that if $U \in \mathrm{Bis}(\mathcal{D})$ then $\varphi^{-1}(U) = \Theta_U =: \Theta(U, \mathbf{d}(U))$ and, on the other hand, $\varphi(\Theta(V,Z)) = \mathbf{d}_V^{-1}(Z)$, which gives that φ is continous and open, and hence a homeomorphism.

Once φ is a homeomorphism, it remains to show items 1 and 2 above. Consider the composable pair $([U,\theta_V(x)],[V,x])$. On the one hand, we have

$$\varphi([U,\theta_V(x)][V,x]) = \varphi([UV,x]) = \mathbf{d}_{UV}^{-1}(x).$$

On the other hand,

$$\begin{aligned}\mathbf{d}(\varphi([U,\theta_V(x)])) &= \mathbf{d}(\mathbf{d}_U^{-1}(\theta_V(x))) = \theta_V(x)\\ &= \mathbf{r}_V(\mathbf{d}_V^{-1}(x)) = \mathbf{r}(\varphi([V,x])).\end{aligned}$$

Which gives $(\varphi([U,\theta_V(x)]),\varphi([V,x])) \in \mathcal{D}^{(2)}$ is a composable pair. Note that the product $\varphi([U,\theta_V(x)])\varphi([V,x])$ belongs to UV, and moreover $\mathbf{d}\left(\varphi\left([U,\theta_V(x)]\right)\varphi\left([V,x]\right)\right) = \mathbf{d}\left(\varphi([V,x])\right) = x$. Hence, $\varphi\big([UV,x]\big)$ and $\varphi\big([U,\theta_V(x)]\big)\varphi\big([V,x]\big)$ belong to UV and have x as source, which gives the equality.

To finish, we show that φ preserves objects. An object of $\mathcal{C}^{(0)}$ is a class $[F,u]$, where F is a projection of *bis*($\mathcal{D}$), that is, F is an open subset of $\mathcal{D}^{(0)}$, and $u \in \mathbf{d}(F) = F$. Then $\varphi\big([F,u]\big) = \mathbf{d}_F^{-1}(u) = u \in \mathcal{D}^{(0)}$. □

3.1 THE CATEGORY OF GERMS OF THE CANONICAL ACTION

Let (S,E,λ,ρ) be a restriction semigroup, and let θ be the canonical action of S on $\widehat{E}$ (see Subsection 2.2.1). Moreover, let $\mathcal{C}$ denote the category of germs $\mathcal{C}(\theta,S,\widehat{E})$. Our purpose here is to study some properties of $\mathcal{C}$ and see which properties of S are reflected on $\mathcal{C}$. For instance, we prove that S is left-ample if and only if $\mathcal{C}$ is left-cancellative.

Recall that for every $e \in E$, the *upper set* of e is the set $e^{\uparrow} = \{f \in E \mid e \leq f\}$. Let ς_e be the semicharacter $1_{e^{\uparrow}} \in \widehat{E}$, which is simply the characteristic function of $e^{\uparrow}$. The family of semicharacters $\widetilde{E} = \{\varsigma_e \mid e \in E\}$ has a special importance when dealing with the Banach algebra $\ell_1(E)$ (e.g [50, Lemma 2.1.1]). Here, we will study these semicharacters in the context of the germs.

Proposition 3.5. If E is finite then $\widehat{E} = \widetilde{E}$. In general, $\widetilde{E}$ is dense on $\widehat{E}$. In particular, $\{\varsigma_e \mid e \in E, e \leq h\}$ is dense on D_h, for every $h \in E$.

Proof. Suppose that E is finite. For any $\varphi \in \widehat{E}$, the set $\varphi^{-1}(\{1\})$ is finite and, in particular, it has a minimum e, where

$$e = \prod_{f \in \varphi^{-1}(\{1\})} f.$$

We claim that $\varphi = \varsigma_e$. It suffices to show that $\varphi(f) = 1$ if and only if $\varsigma_e(f) = 1$, for every $f \in \mathcal{E}$. In fact suppose that $\varphi(f) = 1$. Then $e \leq f$, by definition of E, and hence $\varsigma_e(f) = 1$. Conversely, $\varsigma_e(f) = 1$ then $e \leq f$, and therefore $\varphi(f) = \varphi(f).1 = \varphi(f)\varphi(e) = \varphi(ef) = \varphi(e) = 1$.

For the general case, take $\varphi \in \widehat{E}$ and define the set $\Lambda = \{F \subseteq \varphi^{-1}(\{1\}) \mid |F| < +\infty\}$. Note that Λ is a directed set, ordered by inclusion. Hence, For any $\alpha \in \Lambda$, define

$$e_\alpha = \prod_{a \in \alpha} a. \tag{3.3}$$

Let us prove that ς_{e_α} converges to φ. For $f \in E$, we have either $\varphi(f) = 0$ or $\varphi(f) = 1$. Assuming $\varphi(f) = 0$, we have that for all $\alpha \in \Lambda$, $\varsigma_{e_\alpha}(f) = 0$. Otherwise, we would have $e_\alpha \leq f$ for some $\alpha \in \Lambda$ and, in this case, $1 = \varphi(e_\alpha) \leq \varphi(f)$. Hence, $\varsigma_{e_\alpha}(f)$ converges to $\varphi(f)$. On the other hand, suppose $\varphi(f) = 1$. In this case, $\alpha_0 := \{f\}$ belongs to Λ, and, $\varsigma_{e_\alpha}(f) = 1$, for all $\alpha \geq \alpha_0$. Thus, $\varsigma_{e_\alpha}(f)$ converges to $\varphi(f)$ again. In particular, if $\varphi \in D_h$, note that the subnet $\{\varsigma_{e_\alpha}\}_{\alpha \geq \{h\}}$ still converges to φ, and hence $\{\varsigma_e \mid e \in E, e \leq h\}$ is dense on D_h. $\square$

Note that if $\varsigma_e = \varsigma_f$ then $e \leq f$ and $f \leq e$, and thus $e = f$. Moreover, since ς_e belongs to D_e, we can consider the family of elements $\widetilde{S} = \left\{[s,\varsigma_{\lambda(s)}] \mid s \in S\right\} \subseteq \mathcal{C}$. Define the map

$$\Psi : S \to \widetilde{S}, \quad \Psi(s) = [s,\varsigma_{\lambda(s)}] \tag{3.4}$$

We claim that Ψ is a bijective map. To see that Ψ is injective, suppose that $[s,\varsigma_{\lambda(s)}] = [t,\varsigma_{\lambda(t)}]$. In this case $\lambda(s)$ is equal to $\lambda(t)$, and there exists a projection $e \in E$ such that $\varsigma_{\lambda(s)} \in D_e$, and $se = te$. In particular, this means that $\lambda(t) = \lambda(s) \leq e$, and hence $s = se = te = t$. We have then proved that Ψ is injective. The surjectivity is trivial.

For $t \in S$ and $e \in E$, we have $\lambda(t) \leq \lambda(et) \Leftrightarrow \lambda(t) = \lambda(et) \overset{Prop. 2.7}{\Leftrightarrow} \rho(t) \leq e$, and hence

$$\begin{aligned}
\theta_t\left(\varsigma_{\lambda(t)}\right)(e) = \varsigma_{\lambda(t)}(\lambda(et)) &= \begin{cases} 1, & \text{if } \lambda(t) \leq \lambda(et) \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{if } \rho(t) \leq e \\ 0, & \text{otherwise} \end{cases} \\
&= \varsigma_{\rho(t)}(e).
\end{aligned}$$

Therefore, $\theta_t\left(\varsigma_{\lambda(t)}\right) = \varsigma_{\rho(t)}$ and

$$\theta_t\left(\varsigma_{\lambda(t)}\right) \in D_{\lambda(s)} \Leftrightarrow \rho(t) \leq \lambda(s). \tag{3.5}$$

Lemma 3.6. If $\varsigma_e \in D_{\lambda(s)}$, then $[s,\varsigma_e]$ is equal to $[se,\varsigma_{\lambda(se)}]$. In particular, if E is finite then $\widetilde{S} = \mathcal{C}(\theta,S,\widehat{E})$, and, in general, $\widetilde{S}$ is dense on $\mathcal{C}(\theta,S,\widehat{E})$. Moreover, $\{[t,\varsigma_{\lambda(t)}] \mid t \leq s\}$ is dense in Θ_s.

Proof. Suppose that for $s \in S$ and $e \in E$ one has $\varsigma_e \in D_{\lambda(s)}$, then $e \leq \lambda(s)$ and $e = \lambda(s)e = \lambda(se)$. Hence, $[s,\varsigma_e] = [s,\varsigma_{\lambda(se)}] = [se,\varsigma_{\lambda(se)}]$, where clearly e implements the latter equivalence. By Proposition 3.5, if E is finite we have that any germ $[s,\varphi]$ is of the form $[s,\varsigma_e]$ which is equal to $[se,\varsigma_{\lambda(se)}] \in \widetilde{S}$.

In general, let $[s,\varphi]$ be a germ, and $\{\varsigma_{e_\alpha}\}_{\alpha\in\Lambda}$ be a net converging to φ, such that $\varsigma_{e_\alpha} \in D_{\lambda(s)}, \forall\, \alpha \in \Lambda$. Thus, since $\mathbf{d}_{\Theta_s}$ is a homeomorphism, we obtain that $[s,\varsigma_{e_\alpha}]$ converges to $[s,\varphi]$. But $[s,\varsigma_{e_\alpha}] = [se_\alpha,\varsigma_{\lambda(se_\alpha)}] \in \widetilde{S}$, which tells us that $\widetilde{S}$ in dense on $\mathcal{C}(\theta,S,\widehat{E})$. In particular, $\left\{[t,\varsigma_{\lambda(t)}] \mid t \leq s\right\}$ is dense in Θ_S. □

Proposition 3.7. $\widetilde{S}$ is closed by left composition. Moreover $\widetilde{S}$ is the subcategory of $\mathcal{C}(\theta,S,\widehat{E})$ obtained by restricting the objects to $\{\varsigma_e \mid e \in E\}$, that is, $\widetilde{S} = \mathbf{r}^{-1}(\widetilde{E}) \cap \mathbf{d}^{-1}(\widetilde{E})$. Furthermore, the category $\widetilde{S}$ is isomorphic to the restriction semigroup category $\mathcal{C}_S$ via Ψ.

Proof. We will show that whenever an element $[s,x]$ composes with $[t,\varsigma_{\lambda(t)}]$ the composition belongs to $\widetilde{S}$. So, assume the pair $([s,x],[t,\varsigma_{\lambda(t)}])$ is composable and note that $x = \theta_t(\varsigma_{\lambda(t)})$. By Equation (3.5), $\rho(t) \leq \lambda(s)$ which, by Prop 2.7, is equivalent to say $\lambda(t) = \lambda(st)$. Hence

$$[s,x][t,\varsigma_{\lambda(t)}] = [st,\varsigma_{\lambda(t)}] = [st,\varsigma_{\lambda(st)}]. \tag{3.6}$$

In particular, $\widetilde{S}$ is closed by multiplication. Furthermore, since $\mathbf{r}([t,\varsigma_{\lambda(t)}]) = \theta_t(\varsigma_{\lambda(t)}) = \varsigma_{\rho(t)}$, we have that $\widetilde{S} \subseteq \mathbf{r}^{-1}(\widetilde{E}) \cap \mathbf{d}^{-1}(\widetilde{E})$. The converse is given by the Lemma 3.6. In fact, if $[s,\varsigma_e]$ belongs to $\mathbf{r}^{-1}(\widetilde{E}) \cap \mathbf{d}^{-1}(\widetilde{E})$, then $[s,\varsigma_e] = [se,\varsigma_{\lambda(se)}] \in \widetilde{S}$. Next, note that a pair $([s,\varsigma_{\lambda(s)}],[t,\varsigma_{\lambda(t)}])$ is composable if, and only, if $\theta_t(\varsigma_{\lambda(t)}) = \varsigma_{\rho(t)} = \varsigma_{\lambda(s)}$ if, and only, if $\rho(t) = \lambda(s)$. In particular Ψ is a bijective functor from $\mathcal{C}_S$ to $\widetilde{S}$. □

Proposition 3.8. Let S be a restriction semigroup, X a locally compact Hausdorff space and $\theta : S \to \mathcal{I}(X)$ an action. Moreover, suppose that S is left-ample. Then for every triple $(s,t,v) \in S^3$ and $x \in X$ such that $[st,x] = [sv,x]$ we have $[t,x] = [v,x]$. In particular, $\mathcal{C}(\theta,S,X)$ is left cancellative.

Proof. Suppose $[st,x] = [sv,x]$. In this case, there exists $e \in E$ such that $ste = sve$, and $x \in D_e$. Therefore, $\lambda(s)te = \lambda(s)ve$, and equivalently, $t\lambda(st)e = v\lambda(sv)e$. Next, multiplying on both sides by $\lambda(st)\lambda(sv)$, we get $t\lambda(st)\lambda(sv)e = t\lambda(st)\lambda(sv)e$. Thus, since $D_{\lambda(st)\lambda(sv)e} = D_{\lambda(st)} \cap D_{\lambda(sv)} \cap D_e$, we have $[t,x] = [v,x]$.

Now, suppose that $[s,y][t,x] = [s,y][w,z]$. In this case, we have $[st,x] = [sw,z]$, and therefore x is equal to z. Then, by the previous calculation, $[t,x] = [w,z]$. □

The converse of the above Proposition is not valid. In fact, let S be a restriction semigroup with 0, in which 0 is a projection, and let $\theta : S \to \mathcal{I}(X)$ be the trivial action of S on a locally compact Hausdorff space X, that is, $\theta_S = \mathrm{id}_X, \forall\, s \in S$. The category of germs of θ consists only of objects since $(s,x) \sim (t,x)$ for all $s,t \in S$. In fact, it holds because $s0 = 0 = t0$ and $D_0 = X$. Therefore, $\mathcal{C}(\theta,S,X) = \mathcal{C}(\theta,S,X)^{(0)} \cong X$ and, in particular, it is (left) cancellative. However, it does not imply that S is left-ample. To see this, it suffices to present an example of a non left-ample restriction semigroup with 0 in which 0 is a projection.

Let X be a set with more than 2 elements. The set of all functions from X to X, $\mathcal{F}(X)$, is a monoid under composition. This monoid does not have a zero, i.e, it does not exist a function $\varsigma : X \to X$ such that $f\varsigma = \varsigma = \varsigma f, \forall f \in \mathcal{F}(X)$. Indeed, if $y_0,y_1 \in X$ and $y_0 \neq y_1$, considering the constant maps $f_0(x) = y_0$, $f_1(x) = y_1$ one have that $f_0\varsigma \neq f_1\varsigma$, for any function $\varsigma \in \mathcal{F}(X)$.

In order to get a restriction semigroup with 0, we add a formal 0 to $\mathcal{F}(X)$ and we define $E = \{0, \mathrm{id}_X\}$, $\lambda(s) = 0 \Leftrightarrow s = 0$, and $\rho(s) = 0 \Leftrightarrow s = 0$. This monoid has no zero divisors then it is straightforward to verify the requirements of Definition 2.1. Thus, $(\mathcal{F}(X) \cup \{0\}, \{0, \mathrm{id}_X\}, \lambda, \rho)$ is a restriction semigroup with 0 in which 0 is a projection. Of course $\mathcal{F}(X) \cup \{0\}$ is not left-ample since $f_0 f = f_0 g$, for any pair $f,g \in \mathcal{F}(X)$.

We have seen that it is possible to find a left cancellative category of germs without assuming the semigroup to be left-ample. Indeed, being cancellative has to do not only with the semigroup but also with the action. But, the next proposition shows us that being left-ample can be characterized in categorical terms

Proposition 3.9. *S is left-ample if, and only if, $\mathcal{C}(\theta, S, \widehat{E})$ is left cancellative.*

Proof. Proposition 3.8 gives us one side of the proof. So, assume $\mathcal{C}(\theta, S, \widehat{E})$ is left cancellative and suppose $st = sw$, for $s,t,w \in S$. Define $e = \lambda(st) = \lambda(sw)$ and note that

$$\theta_t(\varsigma_e)(\lambda(s)) = \varsigma_e(\lambda(\lambda(s)t)) = \varsigma_e(\lambda(st)) = \varsigma_e(e) = 1.$$

Hence, $\theta_t(\varsigma_e) \in D_{\lambda(s)}$, and similarly $\theta_w(\varsigma_e) \in D_{\lambda(s)}$. Note that

$$\begin{aligned}\mathbf{r}([s,\theta_t(\varsigma_e)]) &= \theta_s(\theta_t(\varsigma_e)) = \theta_{st}(\varsigma_e) = \theta_{sw}(\varsigma_e) \\ &= \theta_s(\theta_w(\varsigma_e))\, \mathbf{r}([s,\theta_w(\varsigma_e)]).\end{aligned}$$

Then $[s,\theta_t(\varsigma_e)]$ is equal to $[s,\theta_w(\varsigma_e)]$ because they have the same range and belong to the bisection Θ_s. Furthermore $[s,\theta_t(\varsigma_e)][t,\varsigma_e] = [st,\varsigma_e] = [sw,\varsigma_e] = [s,\theta_w(\varsigma_e)][w,\varsigma_e]$. Thus, by left cancellativity, we have

$$[t,\varsigma_e] = [w,\varsigma_e].$$

Then, there is a projection $f \in E$ such that $e \leq f$ and $tf = wf$. Multiplying both sides by e we get $te = we$, and therefore

$$\begin{aligned}\lambda(s)t &= t\lambda(\lambda(s)t) = t\lambda(st) = te = we \\ &= w\lambda(sw) = w\lambda(\lambda(s)w) = \lambda(s)w.\end{aligned}$$

In conclusion, S is left-ample. □

3.2 THE SEMICROSSED PRODUCT STRUCTURE OF THE OPERATOR ALGEBRA OF AN ÉTALE CATEGORY

Throughout this section, let (S,E,λ,ρ) be a restriction semigroup, and let $\theta : S \to \mathcal{I}(X)$ be an étale action of S on a second countable locally compact Hausdorff space X.

Moreover, let $\mathcal{C}$ be the category of germs $\mathcal{C}(\theta,S,X)$. We have defined so far two operator algebras associated with the action θ, which are $\mathcal{A}(\mathcal{C})$, the operator algebra of $\mathcal{C}$, and the semicrossed product algebra $\mathrm{C}_0(X) \rtimes_\alpha S$, where α is the induced action from θ (see Remark 2.22). The purpose of this section is to show that $\mathcal{A}(\mathcal{C})$ and $\mathrm{C}_0(X) \rtimes_\alpha S$ are isometrically isomorphic. The reader will note that the second countability assumption is crucial to obtaining this isomorphism. Indeed, we will strongly use that every open set of a locally compact Hausdorff space is itself a second countable locally compact Hausdorff topological with the relative topology, and hence it is σ-compact.

Proposition 3.10 (Borel measurable Functional Calculus)**.** Let Y be a locally compact Hausdorff space and $\pi : \mathrm{C}_0(Y) \to B(H)$ a C^*-homomorphism. Then π extends to a C^*-homomorphism $\widetilde{\pi} : B(Y) \to B(H)$, where $B(Y)$ is the set of bounded Borel measurable functions. Moreover, if g_n converges pointwise to g and $\sup \|g_n\|_\infty < +\infty$ then $\widetilde{\pi}(g_n)$ converges to $\widetilde{\pi}(g)$ in the weak operator topology.

Sketch of the proof. For every $\xi, \eta \in H$, we have the continuous linear functional $\tau_{\xi,\eta} : \mathrm{C}_0(Y) \to B(H)$ given by $f \mapsto \langle \pi(f)\xi,\eta\rangle$. Then, by Riesz representation theorem, there is a complex Borel regular measure $\nu_{\xi,\eta}$ such that

$$\tau_{\xi,\eta}(f) = \int f \, \mathrm{d}\nu_{\xi,\eta}, \forall\, f \in \mathrm{C}_0(Y).$$

Next, for every $g \in B(X)$, we have the bounded sesquilinear form $\sigma_g : H \times H \to \mathbb{C}$ given by

$$\sigma_g(\xi,\eta) = \int g \ \mathrm{d}\nu_{\xi,\eta}.$$

Again, by (another) Riesz representation theorem, there exists $\widetilde{\pi}(g) \in B(H)$ such that

$$\sigma_g(\xi,\eta) = \langle \widetilde{\pi}(g)\xi,\eta\rangle .$$

[10, Prop 3.14] shows $\widetilde{\pi} : B(X) \to B(H)$ is a C^*-homomorphism. Moreover, if $f \in \mathrm{C}_0(Y)$ we have that

$$\langle \widetilde{\pi}(f)\xi,\eta\rangle = \sigma_f(\xi,\eta) = \int f \ \mathrm{d}\nu_{\xi,\eta} = \tau_{\xi,\eta}(f) = \langle \pi(f)\xi,\eta\rangle , \forall\ \xi,\eta \in H.$$

Then $\widetilde{\pi}(f) = \pi(f)$, which shows that $\widetilde{\pi}$ extends π. To finish, if g_n converges pointwise to g and $\sup \|g_n\|_\infty < +\infty$, by Lebesgue's dominated converge theorem

$$\lim_{n\to\infty} \langle \widetilde{\pi}(g_n)\xi,\eta\rangle = \lim_{n\to\infty} \int g_n \ \mathrm{d}\nu_{\xi,\eta} = \int g \ \mathrm{d}\nu_{\xi,\eta} = \langle \widetilde{\pi}(g)\xi,\eta\rangle .$$

Hence, $\widetilde{\pi}(g_n)$ converges to $\widetilde{\pi}(g)$ in the weak operator topology. □

Corollary 3.11. Suppose we are in the same conditions of Proposition 3.10. Then, for a σ-compact open set $U \subseteq Y$ it holds that $\widetilde{\pi}(1_U)H = \overline{\mathrm{span}}\, \pi(\mathrm{C}_0(U))H$.

Proof. For every $f \in C_0(U)$ and $h \in H$, we have

$$\pi(f)h = \widetilde{\pi}(f)h = \widetilde{\pi}(1_U)\widetilde{\pi}(f)h \subseteq \widetilde{\pi}(1_U)H.$$

The function 1_U is a projection in the C^*-algebra $B(Y)$, since $\overline{1_U} = 1_U = 1_U 1_U$. Hence, the operator $\widetilde{\pi}(1_U)$ is an orthogonal projection, which implies its range is closed. Therefore,

$$\overline{\operatorname{span}}\, \pi(C_0(U))H \subseteq \widetilde{\pi}(1_U)H.$$

Conversely, let $\{K_n\}_{n\in\mathbb{N}}$ be an increasing sequence of compact sets such that $U = \cup_{n\in\mathbb{N}} K_n$. By Urysohn's Lemma, there exists $g_n \in C_c(U)$ such that $g_{n_{|K_n}} \equiv 1$ and $\|g_n\|_\infty = 1$. Note that g_n converges pointwise to 1_U and hence $\widetilde{\pi}(1_U)h$ is the weak limit of $\pi(g_n)h$, for every $h \in H$. The Hahn-Banach theorem then implies $\widetilde{\pi}(1_U)h \in \overline{\operatorname{span}}\pi(g_n)h \subseteq \overline{\operatorname{span}}\, \pi(C_0(U))H$, and hence $\widetilde{\pi}(1_U)H \subseteq \overline{\operatorname{span}}\pi(C_0(U))H$. □

Recall that each map $\theta_s : D_{\lambda(s)} \to D_{\rho(s)}$ is a homeomorphism. We proved that a basis for the topology of $\mathcal{C}$ is given by the sets $\Theta(s,U)$, and when $U = D_{\lambda(s)}$ the open $\Theta(s,U)$ is simply denoted by Θ_s. The family of open sets $\{\Theta_s \mid s \in S\}$ foms a cover for C, and hence, by Proposition 1.21,

$$\mathcal{A}_0(\mathcal{C}) = \operatorname{span}\{f :\ f \in C_c(\Theta_s),\ s \in S\}.$$

We have also proved that the maps $\mathbf{d}_{\Theta_s} : \Theta_s \to D_{\lambda(s)}$ and $\mathbf{r}_{\Theta_s} : \Theta_s \to D_{\rho(s)}$ are homeomorphisms. Therefore, the following aplications are isometric $*$-isomorphisms.

$$C_c(D_{\rho(s)}) \to C_c(\Theta_s), f \mapsto f \circ \mathbf{r}_{\Theta_s} \text{ and } C_c(D_{\lambda(s)}) \to C_c(\Theta_s), f \mapsto f \circ \mathbf{d}_{\Theta_s}.$$

For maps $f \in C_c\left(D_{\rho(s)}\right)$, and $g \in C_c\left(D_{\lambda(s)}\right)$ we will denote

$$f\delta_s := f \circ \mathbf{r}_{\Theta_s} \text{ and } \delta_s g := g \circ \mathbf{d}_{\Theta_s} \tag{3.7}$$

It then follows that

$$\mathcal{A}_0(\mathcal{C}) = \operatorname{span}\left\{f\delta_s :\ f \in C_c\left(D_{\rho(s)}\right),\ s \in S\right\}. \tag{3.8}$$

And

$$\mathcal{A}_0(\mathcal{C}) = \operatorname{span}\left\{\delta_s f :\ f \in C_c\left(D_{\lambda(s)}\right),\ s \in S\right\}.$$

Recall that the $*$-isomorphism $\alpha_s : C_0\left(D_{\lambda(s)}\right) \to C_0\left(D_{\rho(s)}\right)$ is given by $\alpha_s(f) = f \circ \theta_s^{-1}$. In this case, if $f \in C_c\left(D_{\lambda(s)}\right)$ we have that

$$\alpha_s(f)\delta_s = f \circ \theta_s^{-1} \circ \mathbf{r}_{\Theta_s} \overset{(3.2)}{=} f \circ \mathbf{d}_{\Theta_s} = \delta_s f. \tag{3.9}$$

Moreover, for $f \in \mathrm{C_c}\left(D_{\rho(s)}\right)$, and $g \in \mathrm{C_c}\left(D_{\rho(t)}\right)$, we have

$$\begin{aligned} f\delta_s * g\delta_t([st,x]) &= f\delta_s([s,\theta_t(x)])g\delta_t([t,x]) \\ &= f(\theta_{st}(x))g(\theta_t(x)) \\ &= \left(f\circ\theta_s\circ\theta_s^{-1}\right)\left(g\circ\theta_s^{-1}\right)(\theta_{st}(x)) \\ &= ((f\circ\theta_s)g)\circ\theta_s^{-1}(\theta_{st}(x)) \\ &= \alpha_s\left(\alpha_s^{-1}(f)g\right)(\theta_{st}(x)) \\ &= \alpha_s\left(\alpha_s^{-1}(f)g\right)(\mathrm{r}([st,x])) \\ &= \alpha_s\left(\alpha_s^{-1}(f)g\right)\delta_{st}([st,x]). \end{aligned} \tag{3.10}$$

Which gives

$$f\delta_s * g\delta_t = \alpha_s(\alpha_s^{-1}(f)g)\delta_{st}. \tag{3.11}$$

To the end of showing that there is a correspondence between covariant pairs for $(\alpha, S, \mathrm{C}_0(X))$ and representations of $\mathcal{A}_0(\mathcal{C})$, below we prove some preliminary results.

Definition 3.12. Let Z be a set, and let P and L be subsets of the power set $\mathcal{P}(Z)$. P will be called a π**-system** if it is closed under finite intersections, and L will be called a λ**-system** if satisfies:

1. $Z \in L$.
2. If $A \in L$ then $Z \setminus A \in L$, for every $A \in L$.
3. If $\{A_n\}_{n\in\mathbb{N}} \subseteq L$ then $\cup_{n\in\mathbb{N}} A_n \in L$, for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint open sets.

Theorem 3.13 (Dynkin's $\pi-\lambda$ theorem (cf. Thm 3.2 of [11]))**.** Let Z be a set, and let $P \subseteq \mathcal{P}(Z)$ a π-system and $L \subseteq \mathcal{P}(Z)$ a λ-system such that $P \subseteq L$. Then $\sigma(P) \subseteq L$, where $\sigma(P)$ denotes the σ-algebra generated by P.

Since the map $\theta_s : D_{\lambda(s)} \to D_{\rho(s)}$ is a homeomorphism, the following map is a C^*-isomorphism

$$\alpha_s : B(D_{\lambda(s)}) \longrightarrow B(D_{\rho(s)}),\ f \longmapsto f\circ\theta_s^{-1}.$$

Note that it extends the previously defined map $\alpha_s : \mathrm{C}_0\left(D_{\lambda(s)}\right) \to \mathrm{C}_0\left(D_{\rho(s)}\right)$, and for this reason we keep the same notation. Moreover, let (π,σ) be a covariant pair for α, and let $\widetilde{\pi}$ be the Borel extension of π. Recall that every subset D_e is σ-compact, thus by Corollary 3.11 and item 2 of Definition 2.29, we have that for every $e \in E$

$$\widetilde{\pi}(1_e) = \sigma_e,\ 1_e := 1_{D_e}. \tag{3.12}$$

Thus, for every $e \in E$, and $f \in B(X)$ we have that

$$\sigma_e\widetilde{\pi}(f) = \widetilde{\pi}(1_e f) = \widetilde{\pi}(f 1_e) = \widetilde{\pi}(f)\sigma_e. \tag{3.13}$$

Can the covariance relation be extended to the measurable context? Below, we give a positive answer to this question.

Proposition 3.14. Let (π,σ) be a covariant pair for $(\alpha, S, C_0(X)$ on a Hilbert space H and $\widetilde{\pi}$ the Borel extension of π. Then, for every $f \in B(D_{\lambda(s)})$ the following equation holds

$$\widetilde{\pi}(\alpha_s(f))\sigma_s = \sigma_s\widetilde{\pi}(f).$$

Proof. Let U be an open set of $D_{\lambda(s)}$. Note that U is σ-compact, and recall that there exists a sequence $\{g_n\} \subseteq C_c(U) \subseteq C_c(D_{\lambda(s)})$ such that g_n converges pointwise to 1_U and $\sup \|g_n\|_\infty = 1$ (see the proof of Corollary 3.11). Then, for every $\xi,\eta \in H$

$$\begin{aligned} \langle \widetilde{\pi}(\alpha_s(1_U))\sigma_s\xi,\eta\rangle &= \lim_{n\to\infty} \langle \widetilde{\pi}(\alpha_s(g_n))\sigma_s\xi,\eta\rangle \\ &= \lim_{n\to\infty} \langle \sigma_s\widetilde{\pi}(g_n)\xi,\eta\rangle \\ &= \langle \sigma_s\widetilde{\pi}(1_U)\xi,\eta\rangle . \end{aligned}$$

Thus, the covariance relation holds for characteristic functions of open sets. In what follows, we will check that the subset Λ, defined down below, is a λ-system.

$$\Lambda = \left\{ A \subseteq D_{\lambda(s)} \mid A \text{ is a Borel subset such that } \widetilde{\pi}(\alpha_s(1_A))\sigma_s = \sigma_s\widetilde{\pi}(1_A) \right\},$$

Note that $D_{\lambda(s)} \in \Lambda$, by the above calculation. Moreover, if $A \in \Lambda$, note that

$$\begin{aligned} \widetilde{\pi}(\alpha_s(1_A))\sigma_s + \widetilde{\pi}(\alpha_s(1_{D_{\lambda(s)}\setminus A}))\sigma_s &= \widetilde{\pi}(\alpha_s(1_{\lambda(s)}))\sigma_s \\ &= \sigma_s\widetilde{\pi}(1_{\lambda(s)}) \\ &= \sigma_s\widetilde{\pi}(1_A) + \sigma_s\widetilde{\pi}(1_{D_{\lambda(s)}\setminus A}) \\ &= \widetilde{\pi}(\alpha_s(1_A))\sigma_s + \sigma_s\widetilde{\pi}(1_{D_{\lambda(s)}\setminus A}). \end{aligned}$$

Hence, $D_{\lambda(s)} \setminus A \in \Lambda$. Finally, suppose $\{A_n\}_{n\in\mathbb{N}} \subseteq \Lambda$ is a sequence of pairwise disjoint open sets. Define $A = \cup_{n\in\mathbb{N}} A_n$ and note that 1_A is the pointwise limit of f_n, where $f_n = \sum_{i\le n} 1_{A_i}$. Then, for every $\xi, \eta \in H$ we have that

$$\begin{aligned} \langle \widetilde{\pi}(\alpha_s(1_A))\sigma_s\xi,\eta\rangle &= \lim_{n\to\infty} \langle \widetilde{\pi}(\alpha_s(f_n))\sigma_s\xi,\eta\rangle \\ &= \lim_{n\to\infty} \langle \sigma_s\widetilde{\pi}(f_n)\xi,\eta\rangle \quad . \\ &= \langle \sigma_s\widetilde{\pi}(1_A)\xi,\eta\rangle , \end{aligned}$$

Hence $A \in L$, and Λ is a λ-system.

Note that the topology of $D_{\lambda(s)}$ is a π-system contained in Λ and thus Λ is the Borel σ-algebra of $D_{\lambda(s)}$, by Theorem 3.13. In particular, the covariance relation holds for simple functions, and consequently for positive functions, since every positive function is a limit of simple functions (see [11, Theorem 13.5]) and $\widetilde{\pi}$ is weakly continuous. Finally, by linearity, we have that the covariance relation holds for the whole algebra $B(D_{\lambda(s)})$. □

The following Lemma generalizes [24, Lemma 8.4]. Furthermore, we provided a different proof that avoids the assumption that S is countable. Recall the fact presented in (3.8).

Lemma 3.15. Suppose $\sum_{s\in S} f_s\delta_s = 0$ in $\mathcal{A}_0(\mathcal{C})$. Then $\sum_{s\in S}\pi(f_s)\sigma_s = 0$ for every covariant pair (π,σ) for the induced action $(\alpha, S, \mathrm{C}_0(X))$.

Proof. Fix (π,σ) a covariant pair for the action α on a Hilbert space H. Recall that $\sigma : S \to B(H)$, and $\pi : \mathrm{C}_0(X) \to B(H)$ are representations of S and $\mathrm{C}_0(X)$ satisfying the conditions stated in Definition 2.29. By Proposition 3.10, we have the Borel extension $\widetilde{\pi}$ of π, and for every $\xi,\eta \in H$ we have the measures $\nu_{\xi,\eta}$ on X, satisfying

$$\nu_{\xi,\eta}(B) = \langle \widetilde{\pi}(1_B)\xi,\eta\rangle .$$

Let ξ and η be in H. For each $s \in S$, let ν_s be the Borel measure $\nu_{\xi,\sigma_s^*\eta}$ restricted to the Borel subspace $D_{\lambda(s)}$ of X. Moreover, let μ_s be the pushforward measure of ν_s by the homeomorphism $\mathrm{d}_{\Theta_s}^{-1} : D_{\lambda(s)} \to \Theta_s$, that is, μ_s is the Borel measure on Θ_s given by

$$\mu_s(B) = \nu_s(\mathrm{d}_{\Theta_s}(B)) = \left\langle \sigma_s\widetilde{\pi}(1_{\mathrm{d}_{\Theta_s}(B)})\xi,\eta \right\rangle , \tag{3.14}$$

for every Borel subset $B \subseteq \Theta_s$. Furthermore, note that $1_{\mathrm{d}_{\Theta_s}(B)} = 1_B \circ \mathrm{d}_{\Theta_s}^{-1}$ and hence

$$\int f \,\mathrm{d}\mu_s = \left\langle \sigma_s\widetilde{\pi}(f \circ \mathrm{d}_{\Theta_s}^{-1})\xi,\eta \right\rangle ,. \tag{3.15}$$

for every Borel function $f \in B(\Theta_s)$. In fact, equation (3.14) implies that (3.15) holds for simple functions and the extension to Borel functions comes from the fact that $\widetilde{\pi}$ is weakly continuous.

Now, we check that μ_s and μ_t coincide on $\Theta_s \cap \Theta_t$. Let $K \subseteq \Theta_s \cap \Theta_t$ be a compact subset. Note that for every $x \in \mathrm{d}(K)$, $[s,x] = [t,x]$. Hence, for every $x \in X$ there exists $e_x \in E$ such that $x \in D_{e_x}$ and $se_x = te_x$. Moreover, since $\mathrm{d}(K)$ is compact, there are $e_1,\ldots,e_n$ such that $\mathrm{d}(K) \subseteq \cup_{i=1}^n D_{e_i}$ and $se_i = te_i$. In this case, we write $\mathrm{d}(K)$ as a disjoint union

$$\mathrm{d}(K) = \cup_{i=1}^n A_i,$$

where $A_1 = \mathrm{d}(K) \cap D_{e_1}$ and $A_i = \mathrm{d}(K) \cap D_{e_i} \setminus (\cup_{j=1}^{i-1} \mathrm{d}(K) \cap D_{e_j})$, for $2 \le i \le n$. Note that $\{A_i\}_{i=1}^n$ is a family of pairwise disjoint and measurable subsets such that $A_i \subseteq D_{e_i}$. Hence, for every $i \in \{1,\ldots,n\}$

$$\begin{aligned}\sigma_s\widetilde{\pi}(1_{A_i}) &= \sigma_s\widetilde{\pi}(1_{e_i}1_{A_i}) = \sigma_s\widetilde{\pi}(1_{e_i})\widetilde{\pi}(1_{A_i})\\ &\overset{(3.12)}{=} \sigma_s\sigma_{e_i}\widetilde{\pi}(1_{A_i}) = \sigma_{se_i}\widetilde{\pi}(1_{A_i})\\ &= \sigma_{te_i}\widetilde{\pi}(1_{A_i}) = \sigma_t\widetilde{\pi}(1_{e_i})\widetilde{\pi}(1_{A_i})\\ &= \sigma_t\widetilde{\pi}(1_{A_i}).\end{aligned}$$

Thus,

$$\mu_s(K) \overset{(3.14)}{=} \sum_{i=1}^{n} \langle \sigma_s \widetilde{\pi}(1_{A_i})\xi,\eta \rangle = \sum_{i=1}^{n} \langle \sigma_t \widetilde{\pi}(1_{A_i})\xi,\eta \rangle = \mu_t(K).$$

Since $\Theta_s \cap \Theta_t$ isσ-compact, we obtain that every closed subset is a countable union of compact subsets and, therefore, μ_s and μ_t coincide in the family of closed subsets of $\Theta_s \cap \Theta_t$, which is a π-system. It is easy to see that the family of subsets where μ_s and μ_t coincide is a λ-system. Hence, by Theorem 3.13, μ_s and μ_t coincide on every Borel subset of $\Theta_s \cap \Theta_t$.

As a consequence of the fact that μ_s and μ_t coincide on $\Theta_s \cap \Theta_t$, we obtain that if $F \in B(\Theta_s \cap \Theta_t)$ then $\int F \,\mathrm{d}\mu_s = \int F \,\mathrm{d}\mu_t$. Therefore, by (3.15), we have

$$\left\langle \sigma_s \widetilde{\pi}(F \circ \mathrm{d}_{\Theta_s}^{-1})\xi,\eta \right\rangle = \left\langle \sigma_t \widetilde{\pi}(F \circ \mathrm{d}_{\Theta_t}^{-1})\xi,\eta \right\rangle .$$

Then, by Proposition 3.14, we have

$$\left\langle \widetilde{\pi}(F \circ \mathrm{r}_{\Theta_s}^{-1})\sigma_s\xi,\eta \right\rangle = \left\langle \widetilde{\pi}(F \circ \mathrm{r}_{\Theta_t}^{-1})\sigma_t\xi,\eta \right\rangle .$$

And hence, since ξ, η have been arbitrarily chosen, we obtain that for every Borel function $F \in B(\Theta_s \cap \Theta_t)$.

$$\widetilde{\pi}(F \circ \mathrm{r}_{\Theta_s}^{-1})\sigma_s = \widetilde{\pi}(F \circ \mathrm{r}_{\Theta_t}^{-1})\sigma_t. \tag{3.16}$$

Now, let $J = \{s \in S \mid f_s \neq 0\}$ and $M = \cup_{s\in J}\Theta_s$. The Borel σ-algebra of M coincides with the family $\{\cup_{s\in J}B_s \mid B_s$ is a Borel set of $\Theta_s, \forall\, s \in J\}$. Indeed, it easily comes from the fact: If V is an open subset of the topological space Z, then the Borel σ-algebra of V is the family $\{B \subseteq V \mid B$ is a Borel suset of $Z\}$. Furthermore, for a Borel subset $\cup_{s\in J}B_s$ of M, we can suppose that it is a disjoint union. Indeed, write $J = \{s_1,...,s_n\}$ and define $A_{s_1} = B_{s_1}$ and $A_{s_i} = B_{s_i} \setminus (\cup_{j=1}^{i-1} B_{s_j})$, for every $i \in \{2,...,n\}$. Then $\cup_{s\in J}B_s = \cup_{s\in J}A_s$, and $\{A_s\}_{s\in J}$ is a family of pairwise disjoint subsets such that $A_s \subseteq B_s \subseteq \Theta_s$. To complete the argument, we just need to check that A_s is a Borel subset of Θ_s, but note that it is the intersection of measurable subsets of M and, therefore measurable in M. As Borel subsets of Θ_s are precisely the Borel subsets of M contained in Θ_s, we obtain that A_s is a Borel subset of Θ_s.

Define on M the measure

$$\mu\Big(\bigsqcup_{s\in J} B_s\Big) = \sum_{s\in J} \mu_s(B_s).$$

Let us see that μ does not depend of choice. Assume $B = \sqcup_{s\in J}A_s = \sqcup_{t\in J}B_t$. Every A_s and B_t are respectively equal to $\sqcup_{t\in J}A_s \cap B_t$ and $\sqcup_{s\in J}A_s \cap B_t$. Hence

$$\sum_{s\in J}\mu_s(A_s) = \sum_{s\in J}\sum_{t\in J}\mu_s(A_s \cap B_t) = \sum_{s\in J}\sum_{t\in J}\mu_t(A_s \cap B_t) = \sum_{t\in J}\sum_{s\in J}\mu_t(A_s \cap B_t) = \sum_{t\in J}\mu_t(B_t).$$

Thus, there exists a Borel measure on M that agrees with μ_s on Θ_s, for every $s \in J$. In particular, for every Borel function $f \in \mathrm{C_c}(\Theta_s)$.

$$\int f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu_s.$$

Next, recall that $\sum_{s\in J} f_s\delta_s = 0$. Then, integrating with respect to μ on both sides and using the equation above, we have

$$\sum_{s\in J} \int f_s\delta_s \;\mathrm{d}\mu_s = 0.$$

And therefore

$$\begin{aligned} 0 &= \sum_{s\in J} \int f_s\delta_s \;\mathrm{d}\mu_s \overset{(3.15)}{=} \sum_{s\in J} \left\langle \sigma_s\pi((f_s\delta_s)\circ \mathbf{d}_{\Theta_s}^{-1})\xi,\eta\right\rangle \\ &= \sum_{s\in J} \left\langle \sigma_s\pi((f_s\circ \mathbf{r}_{\Theta_s})\circ \mathbf{d}_{\Theta_s}^{-1})\xi,\eta\right\rangle = \sum_{s\in J} \langle \sigma_s\pi(f_s\circ\theta_s)\xi,\eta\rangle \\ &= \sum_{s\in J} \left\langle \sigma_s\pi(\alpha_s^{-1}(f_s))\xi,\eta\right\rangle = \sum_{s\in J} \langle \pi(f_s)\sigma_s\xi,\eta\rangle \\ &= \left\langle \sum_{s\in J} \pi(f_s)\sigma_s\xi,\eta \right\rangle. \end{aligned}$$

which gives $\sum_{s\in J} \pi(f_s)\sigma_s = 0$, since $\xi,\eta \in H$ have been arbitrarily chosen. □

Next, we recall the concept of disjointification (see [18, Remark 2.4]). Loosely speaking, the disjointification of a family of subsets consists in collecting those small pieces generated in the Venn diagram. Let $\mathcal{F} = \{A_i\}_{i=1}^n$ be a family of sets and $I = \{1,\ldots,n\}$ be the index set. For a non-empty set $J \subseteq I$ define

$$P_J = \bigcap_{i\in J} A_i \setminus \bigcup_{i\in I\setminus J} A_i.$$

The *disjointification* of $\mathcal{F}$ is the family $\mathcal{D} = \{P_J \mid \emptyset \neq J \subseteq I\}$. Let us briefly recall some properties that the disjointification of the family $\mathcal{F}$ has. Note that for $i \in I$ and a non-empty subset $J \subseteq I$, either $i \in J$ or $i \in I\setminus J$, and respectively either $P_J \subseteq A_i$ or $P_J \cap A_i = \emptyset$. This, in particular, shows that the family $\{P_J \mid \emptyset \neq J \subseteq I\}$ is pairwise disjoint. Moreover, note that for every $i \in I$

$$A_i = \bigsqcup_{\{J\subseteq I \mid P_J\subseteq A_i\}} P_J.$$

In fact, if $i_0 \in I$ and $x \in A_{i_0}$, we have that $x \in P_J$, where $J = \{i \in I \mid x \in A_i\}$. And hence ($\subseteq$) holds. The other inclusion is easy.

Furthermore, if the family $\{A_i\}_{i=1}^n$ consists of open sets on a given topological space, it is easy to see that $\mathcal{D}$ forms a Borel partition for $\bigcup_{i\in I} A_i$.

Having disposed of the above machinery, we can now show that there is a correspondence between covariant pairs for $(\alpha, S, \mathrm{C}_0(X))$ and representations of $\mathcal{A}_0(\mathcal{C})$. Recall that a covariant pair (π,σ) for α on H gives rise to a homomorphism $\pi \times \sigma : L_\alpha \to B(H)$. Note that both L_α and $\mathcal{A}_0(\mathcal{C})$ have elements of the form $\sum_{s \in S} f_s \delta_s$, where in the former δ_s is a symbol while in the latter the meaning is explained in the discussion after Corollary 3.11. Throughout the following, on L_α we replace δ by $\widetilde{\delta}$ and hence the elements of L_α are finite sums of the form $\sum_{s \in S} f_s \widetilde{\delta}_s$.

Proposition 3.16. Let (π,σ) be a covariant pair for the system $(\alpha, S, \mathrm{C}_0(X))$. Then, (π,σ) gives rise to a representation $\overline{\pi \times \sigma}$ of $\mathcal{A}_0(\mathcal{C})$, where

$$\overline{\pi \times \sigma}\left(\sum_{s \in S} f_s \delta_s\right) = \sum_{s \in S} \pi(f_s)\sigma_s.$$

Proof. Define $\overline{\pi \times \sigma}\left(\sum_{s \in S} f_s \delta_s\right) = \sum_{s \in S} \pi(f_s)\sigma_s$. By Lemma 3.15, $\overline{\pi \times \sigma}$ is well-defined. Moreover, note that $\overline{\pi \times \sigma}$ is linear, and that for $f \in \mathrm{C_c}(D_{\rho(s)})$ and $g \in \mathrm{C_c}(D_{\rho(t)})$ we have

$$\begin{aligned}
\overline{\pi \times \sigma}(f\delta_s * g\delta_t) &\overset{(3.11)}{=} \overline{\pi \times \sigma}\left(\alpha_s(\alpha_s^{-1}(f)g)\delta_{st}\right) \\
&= \pi(\alpha_s\left(\alpha_s^{-1}(f)g\right)\sigma_{st} \\
&= \pi \times \sigma\left(\alpha_s(\alpha_s^{-1}(f)g)\widetilde{\delta}_{st}\right) \\
&= \pi \times \sigma\left(f\widetilde{\delta}_s g\widetilde{\delta}_t\right) \\
&= \pi \times \sigma(f\widetilde{\delta}_s)\pi \times \sigma(g\widetilde{\delta}_t) \\
&= \pi(f)\sigma_s\pi(g)\sigma_t \\
&= \overline{\pi \times \sigma}(f\delta_s)\overline{\pi \times \sigma}(g\delta_t).
\end{aligned}$$

Thus, $\overline{\pi \times \sigma}$ is an algebra homomorphism. Furthermore, note that $\mathcal{C}^{(0)} = \bigcup_{e \in E} \Theta_e$ and hence every $f \in \mathrm{C_c}\left(\mathcal{C}^{(0)}\right)$ is of the form $\sum_{e \in E} f_e \delta_e$ (see [57, Theorem 2.13]). In this case,

$$\begin{aligned}
\overline{\pi \times \sigma}\left(\overline{f}\right) = \overline{\pi \times \sigma}\left(\sum_{e \in E} \overline{f_e}\delta_e\right) &= \sum_{e \in E} \pi\left(\overline{f_e}\right)\sigma_e \\
\overset{(3.12)}{=} \sum_{e \in E} \sigma_e \pi\left(\overline{f_e}\right) &= \sum_{e \in E} \sigma_e^* \pi(f_e)^* \\
= \left(\sum_{e \in E} \pi(f_e)\sigma_e\right)^* &= \left(\overline{\pi \times \sigma}(f)\right)^*.
\end{aligned}$$

It remains to show that if $F \in \mathrm{C_c}(U)$, $U \in \mathrm{Bis}(\mathcal{C})$, then $\|\overline{\pi \times \sigma}(F)\| \leq \|F\|_\infty$. Write $F = \sum_{i=1}^n f_{s_i}\delta_{s_i}$ and let $\{P_j\}_{j=1}^m$ be the disjointification of $\{\Theta_i\}_{i=1}^n$, where $\Theta_i := \Theta_{s_i}$. We emphasize that $\{P_j\}_{j=1}^m$ is a Borel partition of $\bigcup_{i=1}^n \Theta_i$, and that for any pair (i,j), it holds that either $P_j \cap \Theta_i = \emptyset$ or $P_j \subseteq \Theta_i$. Moreover

$$\Theta_i = \bigsqcup_{\{j : P_j \subseteq \Theta_i\}} P_j$$

and, in consequence,

$$D_{\rho(s_i)} = \bigsqcup_{\{j : P_j \subseteq \Theta_i\}} \mathrm{r}_{\Theta_i}(P_j). \tag{3.17}$$

For every $j \in \{1, ..., m\}$, define $\mathfrak{L}_j = \{i \mid P_j \subseteq \Theta_i\}$, and $i(j) := \max \ \mathfrak{L}_j$. Moreover, for every $j \in \{1, ..., m\}$, define F_j to be the restriction of F to P_j, and the operator

$$T_j = \widetilde{\pi}(F_j \circ \mathrm{r}^{-1}_{\Theta_{i(j)}})\sigma_{s_{i(j)}}. \tag{3.18}$$

By (3.16), for every $i \in \mathfrak{L}_j$, and for every $G \in B(P_j)$ we have that

$$\widetilde{\pi}\left(G \circ \mathrm{r}^{-1}_{\Theta_i}\right)\sigma_{s_i} = \widetilde{\pi}\left(G \circ \mathrm{r}^{-1}_{\Theta_{i(j)}}\right)\sigma_{s_{i(j)}}. \tag{3.19}$$

Therefore,

$$\begin{aligned}
\overline{\pi \times \sigma}(F) &= \sum_{i=1}^{n} \pi(f_{s_i})\sigma_{s_i} \overset{(3.17)}{=} \sum_{i=1}^{n} \sum_{\{j : P_j \subseteq \Theta_i\}} \widetilde{\pi}\left(1_{\mathrm{r}_{\Theta_i}(P_j)} f_{s_i}\right)\sigma_{s_i} \\
&= \sum_{i=1}^{n} \sum_{\{j : P_j \subseteq \Theta_i\}} \widetilde{\pi}\left(1_{P_j} \circ \mathrm{r}^{-1}_{\Theta_i} f_{s_i}\right)\sigma_{s_i} \\
&= \sum_{i=1}^{n} \sum_{\{j : P_j \subseteq \Theta_i\}} \widetilde{\pi}\left((1_{P_j} f_{s_i} \circ \mathrm{r}_{\Theta_i}) \circ \mathrm{r}^{-1}_{\Theta_i}\right)\sigma_{s_i} \\
&= \sum_{j=1}^{m} \sum_{\{i : P_j \subseteq \Theta_i\}} \widetilde{\pi}\left((1_{P_j} f_{s_i}\delta_{s_i}) \circ \mathrm{r}^{-1}_{\Theta_i}\right)\sigma_{s_i} \\
&\overset{(3.19)}{=} \sum_{j=1}^{m} \sum_{\{i : P_j \subseteq \Theta_i\}} \widetilde{\pi}\left((1_{P_j} f_{s_i}\delta_{s_i}) \circ \mathrm{r}^{-1}_{\Theta_{i(j)}}\right)\sigma_{s_{i(j)}} \\
&= \sum_{j=1}^{m} \widetilde{\pi}\left(\left(1_{P_j} \sum_{\{i : P_j \subseteq \Theta_i\}} f_{s_i}\delta_{s_i}\right) \circ \mathrm{r}^{-1}_{\Theta_{i(j)}}\right)\sigma_{s_{i(j)}} \\
&= \sum_{j=1}^{m} \widetilde{\pi}(F_j \circ \mathrm{r}^{-1}_{\Theta_{i(j)}})\sigma_{s_{i(j)}} = \sum_{j=1}^{m} T_j.
\end{aligned}$$

Next, we show the operator family $\{T_j\}_{j=1}^m$ is completely orthogonal (see Definition 1.2). Let $k, l \in \{1, ..., m\}$ such that $k \neq l$. Suppose that $(\overline{F_k} \circ \mathrm{r}^{-1}_{\Theta_{i(k)}})(F_l \circ \mathrm{r}^{-1}_{\Theta_{i(l)}})(x) \neq 0$, for some $x \in X$. In this case, we obtain

$$\overline{F_k}([s_{i(k)}, \theta^{-1}_{s_{i(k)}}(x)])F_l([s_{i(l)}, \theta^{-1}_{s_{i(l)}}(x)]) \neq 0.$$

Then, since $\mathrm{r}\left([s_{i(k)}, \theta^{-1}_{s_{i(k)}}(x)]\right) = x$, $\mathrm{r}\left([s_{i(l)},\theta^{-1}_{s_{i(l)}}(x)]\right) = x$, and the support of F is contained in a bisection, we have that

$$[s_{i(k)},\theta^{-1}_{s_{i(k)}}(x)] = [s_{i(l)},\theta^{-1}_{s_{i(l)}}(x)].$$

Thus, $P_k \cap P_l \neq \emptyset$, which gives a contradiction. Therefore, $\left(\overline{F_k} \circ \mathrm{r}^{-1}_{\Theta_{i(k)}}\right)\left(F_l \circ \mathrm{r}^{-1}_{\Theta_{i(l)}}\right) \equiv 0$, and similarly $\left(\overline{F_k} \circ \mathrm{d}^{-1}_{\Theta_{i(k)}}\right)\left(F_l \circ \mathrm{d}^{-1}_{\Theta_{i(l)}}\right) \equiv 0$. A simple computation then gives

$$\begin{aligned} T_k^* T_l &= \sigma^*_{s_{i(k)}}\widetilde{\pi}(\overline{F_k} \circ \mathrm{r}^{-1}_{\Theta_{i(k)}})\widetilde{\pi}(F_l \circ \mathrm{r}^{-1}_{\Theta_{i(l)}})\sigma_{s_{i(l)}} \\ &= \sigma^*_{s_{i(k)}}\widetilde{\pi}(\overline{F_k} \circ \mathrm{r}^{-1}_{\Theta_{i(k)}}\; F_l \circ \mathrm{r}^{-1}_{\Theta_{i(l)}})\sigma_{s_{i(l)}} = 0. \end{aligned}$$

And

$$\begin{aligned} T_k T_l^* &= \widetilde{\pi}(F_k \circ \mathrm{r}^{-1}_{\Theta_{i(k)}})\sigma_{s_{i(k)}}(\widetilde{\pi}(F_l \circ \mathrm{r}^{-1}_{\Theta_{i(l)}})\sigma_{s_{i(l)}})^* \\ &\overset{3.14}{=} \sigma_{s_{i(k)}}\widetilde{\pi}(\alpha^{-1}_{s_{i(k)}}(F_k \circ \mathrm{r}^{-1}_{\Theta_{i(k)}}))(\sigma_{s_{i(l)}}\widetilde{\pi}(\alpha^{-1}_{s_{i(l)}}(F_l \circ \mathrm{r}^{-1}_{\Theta_{i(l)}})))^* \\ &= \sigma_{s_{i(k)}}\widetilde{\pi}(F_k \circ \mathrm{d}^{-1}_{\Theta_{i(k)}})\widetilde{\pi}(\overline{F_l} \circ \mathrm{d}^{-1}_{\Theta_{i(l)}})\sigma^*_{s_{i(l)}} \\ &= \sigma_{s_{i(k)}}\widetilde{\pi}(F_k \circ \mathrm{d}^{-1}_{\Theta_{i(k)}}\; \overline{F_l} \circ \mathrm{d}^{-1}_{\Theta_{i(l)}})\sigma^*_{s_{i(l)}} = 0. \end{aligned}$$

And, hence

$$\begin{aligned} \|\overline{\pi \times \sigma}(F)\| &= \left\|\sum_{j=1}^{m} T_j\right\| \overset{(1.1)}{=} \max_{j=1,\dots,m}\left\{\|T_j\|\right\} \\ &= \max_{j=1,\dots,m}\left\{\|\widetilde{\pi}\left(F_j \circ \mathrm{r}^{-1}_{\Theta_{i(j)}}\right)\sigma_{s_{i(j)}}\|\right\} \\ &\leq \max_{j=1,\dots,m}\{\|F_j\|_\infty\} = \|F\|_\infty \end{aligned}$$

□

Conversely, we have the following disintegration method

Proposition 3.17. Let $\Pi : \mathcal{A}_0(\mathcal{C}) \to B(H)$ be a representation of $\mathcal{A}_0(\mathcal{C})$. Then, there exists a covariant pair (π,σ) for α such that $\Pi = \overline{\pi \times \sigma}$.

Proof. We start constructing σ. Let $\Pi_s : B(\Theta_s) \to B(H)$ denote the Borel extension of the continuous linear map (see Proposition 3.10). Finally, define for every $s \in S$

$$\sigma_s = \Pi_s(1_{\Theta_s}).$$

We are going to use the notation of the proof of Proposition 3.10 to check that the map $\sigma : S \to B(H)$, $s \mapsto \sigma_s$, is a representation of S. Take $\xi,\eta \in H$, and note that

$$|\langle\sigma_s\xi,\eta\rangle| = |\langle\Pi_s(1_{\Theta_s})\xi,\eta\rangle| = |\nu_{\xi,\eta}(\Theta_s)| \leq \|\nu_{\xi,\eta}\| = \|\tau_{\xi,\eta}\| \leq \|\xi\|\|\eta\|.$$

Then, $\|\sigma_s\| \leq 1$. Now, since Θ_s is σ-compact, there is an increasing sequence $\{g^s_n\} \subseteq \mathrm{C_c}(\Theta_s)$ that converges pointwise to 1_{Θ_s}, for every $s \in S$. Let s and t be in S and

$g \in C_c(\Theta_s)$. Note that $\{g * g_n^t\} \subseteq C_c(\Theta_{st})$ is uniformly bounded and converges to $g \circ \mathbf{r}_{\Theta_s}^{-1} \circ \mathbf{r}_{\Theta_{st}} \subseteq C_c(\Theta_{st})$. Therefore, for $\xi,\eta \in H$ we have that

$$\langle \Pi(g)\sigma_t \xi,\eta\rangle = \lim_{n\to\infty} \left\langle \Pi(g)\Pi(g_n^t)\xi,\eta\right\rangle = \lim_{n\to\infty} \left\langle \Pi(g * g_n^t)\xi,\eta\right\rangle = \left\langle \Pi(g \circ \mathbf{r}_{\Theta_s}^{-1} \circ \mathbf{r}_{\Theta_{st}})\xi,\eta\right\rangle .$$

Which gives

$$\Pi(g)\sigma_t = \Pi(g \circ \mathbf{r}_{\Theta_s}^{-1} \circ \mathbf{r}_{\Theta_{st}}). \tag{3.20}$$

Similarly, if $h \in C_c(\Theta_t)$ we have that

$$\sigma_s \Pi(h) = \Pi(h \circ \mathbf{d}_{\Theta_t}^{-1} \circ \mathbf{d}_{\Theta_{st}}). \tag{3.21}$$

Hence

$$\begin{aligned}\langle \sigma_s\sigma_t\xi,\eta\rangle &= \lim_{n\to\infty} \left\langle \Pi(g_n^s)\sigma_t\xi,\eta\right\rangle \\ &= \lim_{n\to\infty} \left\langle \Pi(g_n^s \circ \mathbf{r}_{\Theta_s}^{-1} \circ \mathbf{r}_{\Theta_{st}})\xi,\eta\right\rangle \\ &= \left\langle \Pi(1_{\Theta_{st}})\xi,\eta\right\rangle = \langle \sigma_{st}\xi,\eta\rangle .\end{aligned}$$

Which gives $\sigma_{st} = \sigma_s\sigma_t$.

On the other hand, recall that $\mathcal{C}^{(0)} = X$ and hence define $\pi : C_0(X) \to B(H)$ to be the extension of the contractive $*$-homomorphism $\Pi_{|C_c(\mathcal{C}^{(0)})}$. Explicitly, since every $f \in C_c(\mathcal{C}^{(0)})$ is of the form $\sum_{e\in E} f_e\delta_e$, we have

$$\pi(f) = \sum_{e\in E} \Pi(f_e\delta_e).$$

We now show the covariance relation. Let s be in S, f be in $C_c\left(D_{\lambda(s)}\right)$. For $\xi,\eta \in H$, we have that

$$\begin{aligned}\langle \sigma_s\pi(f)\xi,\eta\rangle &= \left\langle \sigma_s\Pi\left(f\delta_{\lambda(s)}\right)\xi,\eta\right\rangle \overset{(3.21)}{=} \left\langle \Pi(f\delta_{\lambda_s} \circ \mathbf{d}_{\Theta_{\lambda(s)}}^{-1} \circ \mathbf{d}_{\Theta_{s\lambda(s)}})\xi,\eta\right\rangle \\ &= \left\langle \Pi(f \circ \mathbf{d}_{\Theta_{\lambda_s}} \circ \mathbf{d}_{\Theta_{\lambda(s)}}^{-1} \circ \mathbf{d}_{\Theta_s})\xi,\eta\right\rangle = \left\langle \Pi(f \circ \mathbf{d}_{\Theta_s})\xi,\eta\right\rangle \\ &= \langle \Pi(\delta_s f)\xi,\eta\rangle \overset{(3.9)}{=} \langle \Pi(\alpha_s(f)\delta_s)\xi,\eta\rangle = \left\langle \Pi(\alpha_s(f) \circ \mathbf{r}_{\Theta_s})\xi,\eta\right\rangle \\ &= \left\langle \Pi(\alpha_s(f) \circ \mathbf{r}_{\Theta_{\rho(s)}} \circ \mathbf{r}_{\Theta_{\rho(s)}}^{-1} \circ \mathbf{r}_{\Theta_{\rho(s)s}})\xi,\eta\right\rangle \overset{(3.20)}{=} \langle \pi(\alpha_s(f))\sigma_s\xi,\eta\rangle .\end{aligned}$$

The fact that $\overline{\text{span}}\, \pi(C_0(D_e))H = \sigma_e(H)$ follows from Corollary 3.11. To finish, we check that $\Pi = \overline{\pi \times \sigma}$. Let $f\delta_s$ be in $\mathcal{A}_0(\mathcal{C})$. Note that

$$\begin{aligned}\overline{\pi \times \sigma}(f\delta_s) &= \pi(f)\sigma_s = \Pi(f\delta_{\rho(s)})\sigma_s \\ &= \Pi(f \circ \mathbf{r}_{\Theta_{\rho(s)}} \circ \mathbf{r}_{\Theta_{\rho(s)}}^{-1} \circ \mathbf{r}_{\Theta_{\rho(s)s}}) \\ &= \Pi(f \circ \mathbf{r}_{\Theta_s}) = \Pi(f\delta_s).\end{aligned} \tag{3.22}$$

□

We are now ready to prove the main result of this chapter

Theorem 3.18. Let (S,E,λ,ρ) be a restriction semigroup, X a second countable locally compact Hausdorff space, and $\theta : S \to \mathcal{I}(X)$ an étale action. Moreover, let α be the induced étale action from θ. Then $\mathcal{A}(\mathcal{C}(\theta,S,X))$ and $\mathrm{C}_0(X) \rtimes_\alpha S$ are isomorphic.

Proof. By Remark 2.31, $L'_\alpha = \left\{\sum_{s\in S} f_s\widetilde{\delta}_s \mid f_s \in \mathrm{C_c}\left(D_{\rho(s)}\right)\right\}$ is a dense subalgebra of $\mathrm{C}_0(X) \rtimes_\alpha S$. Hence, define the following map

$$\psi : L'_\alpha \longrightarrow \mathcal{A}_0(\mathcal{C}), \qquad \psi\left(\sum f_s\widetilde{\delta}_s\right) \longmapsto \left(\sum f_s\delta_s\right).$$

From (3.8) and (3.10), we otain that ψ is a surjective algebra homomorphism. Moreover, for any covariant pair (π,σ) for α we have that

$$\left\|\overline{\pi \times \sigma}\left(\psi\left(\sum_{s\in S} f_s\delta_s\right)\right)\right\| = \left\|\sum_{s\in S}\pi(f_s)\sigma_s\right\| = \left\|\pi \times \sigma\left(\sum_{s\in S} f_s\widetilde{\delta}_s\right)\right\|. \tag{3.23}$$

Thus, By Propositions 3.16 and 3.17, we obtain that ψ is isometric (on the quotient) and then it extends to an isomorphism. □

Corollary 3.19. Let (S,E,λ,ρ) be a restriction semigroup, and suppose that $\widehat{E}$ is second countable. Then the map $\delta_s \mapsto 1_{\Theta_s}$ defines an isomorphism between $\mathcal{A}(S)$ and $\mathcal{A}(\mathcal{C}(\theta,S,\widehat{E}))$.

Proof. Combine the proof of Theorem 2.33 with the proof of Theorem 3.18. □

Corollary 3.20. Let $\mathcal{C}$ be an étale category, and suppose $\mathcal{C}^{(0)}$ is second countable. Then $\mathcal{A}(\mathcal{C})$ is isomorphic to a semicrossed product.

Proof. By Theorem 3.4, $\mathcal{C}$ is isomorphic to $\mathcal{C}\left(\theta, \mathrm{Bis}(\mathcal{C}), \mathcal{C}^{(0)}\right)$, and hence the algebras are isomorphic. □

3.2.1 The reduced case

Let (S,E,λ,ρ) be a left-ample restriction semigroup, and suppose that $\widehat{E}$ is second countable. Moreover, let θ be the canonical action of S on $\widehat{E}$, and let $\mathcal{C}$ be the category of germs $\mathcal{C}(\theta,S,\widehat{E})$. Equations (2.11) and (3.23) tell us that not only $\mathcal{A}(S)$ and $\mathcal{A}(\mathcal{C})$ are isomorphic but also for any representation σ of S, we have

$$\overline{\mathbb{C}[S]}^{\|\cdot\|_\sigma} \cong \overline{\mathcal{A}_0(\mathcal{C})}^{\|\cdot\|_{\overline{\pi_\sigma \times \sigma}}}. \tag{3.24}$$

Recall the Definition 1.27 of the regular representation of $\mathcal{C}$ $\Pi : \mathcal{A}_0(\mathcal{C}) \to B(\ell_2(\mathcal{C})$ given by

$$\Pi_f(\delta_z) = \sum_{x\in\mathcal{C}_{\mathbf{r}(\gamma)}} f(x)\delta_{xz},\ f \in \mathcal{A}_0(\mathcal{C})\,.$$

By Proposition 3.7, the subset $\widetilde{S} \subseteq \mathcal{C}$ is closed by left composition, and therefore the Hilbert subspace $\ell_2(\widetilde{S})$ of $\ell_2(\mathcal{C})$ is an invariant subspace of the regular representation

Π. Let (π,σ) be a covariant pair (π,σ) such that $\Pi = \overline{\pi \times \sigma}$. Since the subsets Θ_s are compact and open, we do not need to take the Borel extesion of Π to define σ (cf. Proposition 3.17), and hence $\sigma_s = \Pi(1_{\Theta_s})$. Note that for every $[t,\varphi] \in \mathcal{C}$ and $s \in S$ we have

$$\sigma_s(\delta_{[t,\varphi]}) = \left[\theta_t(\varphi) \in D_{\lambda(s)}\right] 1_{\Theta_s}\big([s,\theta_t(\varphi)]\big)\delta_{[st,\varphi]} = \left[\theta_t(\varphi) \in D_{\lambda(s)}\right]\delta_{[st,\varphi]}. \tag{3.25}$$

In particular, by Proposition 2.7 and Equation (3.5), if $[t,\varsigma_{\lambda(t)}] \in \widetilde{S}$ then it holds that

$$\sigma_s\big(\delta_{[t,\varsigma_{\lambda(t)}]}\big) = \left[\rho(t) \leq \lambda(s)\right] \delta_{[st,\varsigma_{\lambda(st)}]}.$$

Thus, by (2.7), we note that that $\sigma_{|\ell_2(\widetilde{S})} : S \to B(\ell_2(\widetilde{S}))$ is unitarily equivalent to the regular representation of S, $\varphi' : S \to B(\ell_2(s))$. In fact, the unitary operator implementing the equivalence is the one induced by $s \overset{\Psi}{\mapsto} [s,\varsigma_{\lambda(s)}]$, defined on (3.4). Moreover, since $\varphi'_e = \pi_{\varphi'}(1_e) =$ (cf. Theorem 2.23), $\sigma_e = \Pi(1_e)$, and $\operatorname{span}\{1_e \mid e \in E\}$ is a dense subalgebra of $C_0(\widehat{E})$ we have that $\pi_{|\ell_2(\widetilde{S})}$ and $\pi_{\varphi'}$ are unitarily equivalent via the same unitary operator. Thus, $\pi_{\varphi'} \times \varphi'$ is unitarily equivalent to $\Pi_{|\ell_2(\widetilde{S})}$.

By Lemma 3.6, If E is finite then $\widetilde{S} = \mathcal{C}$ and hence $\pi_{\varphi'} \times \varphi'$ and $\Pi_{|\ell_2(\widetilde{S})}$ induce the same norm on $\mathcal{A}_0(\mathcal{C})$.

If S is an inverse semigroup then we also have $\mathcal{A}_r(S) = \mathcal{A}_r(\mathcal{C})$, since $\mathcal{A}_r(S) = C^*(S)$ and $\mathcal{A}_r(\mathcal{C}) = C^*(\mathcal{C})$ (cf. subsection 3.2.2). The result then follows from [35, Theorem 3.5].

The case where E is countable remains open. We conclude this subsection by showing that σ is a representation by partial isometries. By Equation (3.25), we just need to show

Proposition 3.21. Let $[t,\varphi]$ and $[t',\varphi']$ be in $\mathcal{C}$. Suppose that $\theta_t(\varphi) \in D_{\lambda(s)}$, and $\theta'_t(\varphi') \in D_{\lambda(s)}$, and $[st,\varphi] = [st',\varphi']$. Then $[t,\varphi] = [t',\varphi']$.

Proof. Since $[st,\varphi] = [st',\varphi']$, we obtain that $\varphi = \varphi'$, and moreover that there exists $e \in E$ such that $\varphi \in D_e$, and $ste = st'e$. Recall that S is left-ample and hence

$$t\lambda(st)e = t'\lambda(st')e.$$

Then defining $h = \lambda(st)\lambda(st')e$, we have that $th = t'h$, and that $\varphi \in D_h$, since $\theta_t(\varphi) \in D_{\lambda(s)}$, and $\theta'_t(\varphi) \in D_{\lambda(s)}$. Therefore, $[t,\varphi] = [t',\varphi']$. □

3.2.2 The groupoid case

We now show how our work fits in the theory of groupoid C^*-algebras. Throughout this subsection let $\mathcal{G}$ be an étale groupoid. Recall that $\mathcal{A}_0(\mathcal{G})$ has an involution, given by $f^*(x) = \overline{f(x^{-1})}$, and that $C^*(\mathcal{G})$ is the completion of $\mathcal{A}_0(\mathcal{G})$ equipped with the norm induced by the class of all $*$-representations. Suppose U is a compact open bisection of $\mathcal{G}$. Then the C^*-algebra $\mathrm{C}(U)$ is unital, and hence is generated by unitary elements. Recall that a unitary element of $\mathrm{C}(U)$ is just a function f whose range is contained in $\mathbb{T}$, that is, $|f(x)| = 1$ for every $x \in U$.

Lemma 3.22. Let U be a compact open bisection, and let $f \in \mathrm{C}(U)$ be a unitary map. Then, f is a partial isometry on $\mathcal{A}_0(\mathcal{G})$. Moreover, if $\pi : \mathcal{A}_0(\mathcal{G}) \to B(H)$ is a representation in the sense of Definition 1.24, we have $\pi(f^*) = \pi(f)^*$.

Proof. Note that f^* belongs to $\mathrm{C}\left(U^{-1}\right)$ and hence $f^* * f \in \mathrm{C_c}\left(\mathcal{G}^{(0)}\right)$. By item 4 of Lemma 1.22, if $x \in U$ we have that

$$f * f^* * f(x) = f(x)(f^* f)(\mathrm{d}(x)) = f(x)f^*(x^{-1})f(x) = f(x)|f(x)|^2 = f(x).$$

Then, since $f * f^* * f \in \mathrm{C}(U)$, we have that $f * f^* * f = f$, and similarly $f^* * f * f^* = f^*$. To finish, note that the contraction $\pi(f^*)$ is a generalized inverse for $\pi(f)$, and therefore by Corollary 1.5 we obtain that $\pi(f^*) = \pi(f)^*$. □

We can now prove our first main result.

Theorem 3.23. Let $\mathcal{G}$ be an étale groupoid, and suppose that $\mathcal{G}$ has a cover of compact open bisections $\mathcal{F}$. Then the operator algebra of $\mathcal{G}$, $\mathcal{A}(\mathcal{G})$, coincides with the C^*-algebra of $\mathcal{G}$, $C^*(\mathcal{G})$.

Proof. By Proposition 1.21 and the above discussion, we have

$$\mathcal{A}_0(\mathcal{G}) = \mathrm{span}\{f : \ f \in \mathrm{C_c}(U),\ \mathrm{ran}(f) \subseteq \mathbb{T},\ U \in \mathcal{F}\}.$$

Then, it follows by Lemma 3.22 that every representation $\pi : \mathcal{A}_0(\mathcal{G}) \to B(H)$ is a $*$-homomorphism. This completes the proof. □

The next theorem proves that a similar fact holds without assuming that $\mathcal{G}$ is covered by compact bisection, but now we need an assumption regarding $\mathcal{G}^{(0)}$.

Theorem 3.24. Let $\mathcal{G}$ be an étale groupoid, and suppose that $\mathcal{G}^{(0)}$ is second countable. Then $\mathcal{A}(\mathcal{G}) = C^*(\mathcal{G})$.

Proof. We can without loss of generality suppose that $\mathcal{G}$ is the groupoid of germs of an action θ of an inverse semigroup S on a second countable locally compact Hausdorff space X, $\mathcal{G} = \mathcal{G}(\theta,S,X)$. Let $\Pi : \mathcal{G} \to B(H)$ be a representation of $\mathcal{A}_0(\mathcal{G})$. We now procceed to show that Π is a $*$-homomorphism. From Proposition 3.17, we have that $\Pi = \overline{\pi \times \sigma}$, for a covariant pair (π,σ) for $(\alpha, S, \mathrm{C}_0(X))$, where α is the induced by θ. Moreover, recall that σ is a $*$- representation of S (see Subsection 2.1.2). By (3.8), to show that $\Pi(f^*) = \Pi(f)^*$ for every f in $\mathcal{A}_0(\mathcal{G})$, it suffices to prove that $\Pi\big((f\delta_s)^*\big) = \Pi(f\delta_s)^*$

for every $f \in \mathrm{C_c}\left(D_{\rho(s)}\right)$, and for every $s \in S$. Furthermore, recall that the inverse of a germ $[s,x]$ is $[s^*,\theta_s(x)]$, and hence

$$(f\delta_s)^*\big([s^*,x]\big) = \overline{f\delta_s\big([s,\theta_{s^*}(x)]\big)} = \overline{f(x)} = (\delta_{s^*}\bar{f})\big([s^*,x]\big)$$
$$\overset{(3.9)}{=} \alpha_{s^*}\left(\,\bar{f}\,\right)\delta_{s^*}\big([s^*,x]\big),$$

for every $s \in S$, and $f \in \mathrm{C_c}\left(D_{\rho(s)}\right)$. Which gives $(f\delta_s)^* = \alpha_{s^*}(\bar{f})\delta_{s*}$, since these functions have their support contained in Θ_{s^*}. Therefore, for $\xi,\eta \in H$ we have that

$$\langle\Pi(f\delta_s)\xi,\eta\rangle = \langle\pi(f)\sigma_s\xi,\eta\rangle = \left\langle\xi,\sigma_{s^*}\pi\big(\bar{f}\big)\eta\right\rangle$$
$$= \left\langle\xi,\pi\left(\alpha_{s^*}\left(\,\bar{f}\,\right)\right)\sigma_{s^*}\eta\right\rangle = \left\langle\xi,\Pi\big((f\delta_s)^*\big)\right\rangle.$$

Which gives $\Pi(f\delta_s)^* = \Pi\big((f\delta_s)^*\big)$ and hence completes the proof. □

We finish this section with a result for the reduced version of these algebras.

Theorem 3.25. Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{A}_r(\mathcal{G}) = C^*_{red}(\mathcal{G})$.

Proof. The regular representation defined for étale categories on (1.4) is precisely the same for étale groupoids (see for instance [66, Section 9.3]) and hence the result follows. □

PART 2

4 NON-ABELIAN EXTENSIONS OF GROUPOIDS AND THEIR GROUPOID RINGS

Here is an outline of this chapter. In Section 4.1, we provide the necessary foundations on groupoids, Abelian groupoid cohomology, groupoid rings, and groupoid crossed products. In Section 4.2, we develop a geometrically oriented classification theory for non-Abelian extensions of groupoids by means of groupoid cohomology à la Westman (see, e. g., Corollary 4.22, Corollary 4.24, and Corollary 4.29). We wish to point out that our results, up to Theorem 4.23, are similar to the results obtained by Blanco, Bullejos, and Faro in [12], but presented in a more geometric and computational framework. Moreover, from Corollary 4.24 and on, our investigation goes further. In Section 4.3, we study groupoid crossed products associated with groupoid extenions (see Theorem 4.32) and show that the groupoid ring of a groupoid extension is isomorphic to a groupoid crossed product associated with the building blocks of the extension (see Corollary 4.33). We also extend our results to the realm of C*-algebras (see Proposition 4.39). In Section 4.4, we make use of the methods developed in Section 4.2 to provide a classification theory for groupoid crossed products (see, e. g., Proposition 4.42 and Theorem 4.44).

4.1 PRELIMINARIES

In this preliminary section we recall the most fundamental definitions and notation used throughout this article.

4.1.1 Groupoids

There are several ways to view groupoids. In this article we consider groupoids as objects with a geometric flavour. We refer the reader to [41, 55, 66] for equivalent definitions as well as for examples.

By a *groupoid* we mean a non-empty set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(0)}$, called the *unit space* of $\mathcal{G}$, together with structure maps $r,s : \mathcal{G} \to \mathcal{G}^{(0)}$, called respectively the *range* and the *source* maps, a partial multiplication $(x,y) \mapsto xy$ in $\mathcal{G}$ defined on the set $\mathcal{G}^{(2)} := \{(x,y) \in \mathcal{G} \times \mathcal{G} : s(x) = r(y)\}$ of *composable elements* of $\mathcal{G}$, and a map $\mathcal{G} \ni z \mapsto z^{-1} \in \mathcal{G}$, called *inversion*, satisfying the following properties for all $x,y,z \in \mathcal{G}$ and $u \in \mathcal{G}^{(0)}$:

(G1) $r(u) = u = s(u)$;

(G2) $r(z)z = z = zs(z)$;

(G3) $r(z^{-1}) = s(z)$ and $s(z^{-1}) = r(z)$;

(G4) $z^{-1}z = s(z)$ and $zz^{-1} = r(z)$;

(G5) $r(xy) = r(x)$ and $s(xy) = s(y)$ whenever $s(x) = r(y)$;

(G6) $(xy)z = x(yz)$ whenever $s(x) = r(y)$ and $s(y) = r(z)$.

To emphasize the unit space $\mathcal{G}^{(0)}$, we shall sometimes say that $\mathcal{G}$ is a groupoid over $\mathcal{G}^{(0)}$.

Given a groupoid $\mathcal{G}$, we write $\mathcal{G}^{(n)}$ for the set of all n-tuples of composable elements of $\mathcal{G}$, that is, $\mathcal{G}^{(n)} := \{(x_1,\dots,x_n) \in \mathcal{G}^n : s(x_i) = r(x_{i+1}),\ i = 1,\dots,n-1\}$. We also bring to mind that a *homomorphism* of groupoids $\mathcal{G}$ and $\mathcal{H}$ is a map $\varphi : \mathcal{G} \to \mathcal{H}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $(x,y) \in \mathcal{G}^{(2)}$ and $\varphi(z^{-1}) = \varphi(z)^{-1}$ for all $z \in \mathcal{G}$. Note that each homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ satisfies $\varphi(\mathcal{G}^{(0)}) \subseteq \mathcal{H}^{(0)}$, thus inducing a map $\varphi^0 : \mathcal{G}^{(0)} \to \mathcal{H}^{(0)}$. An *isomorphism* of groupoids is simply a bijective homomorphism.

4.1.2 Groupoid cohomology

We shall also be concerned with groupoid cohomology. For convenience of the reader we briefly recall the basics of this theory. For further reading we refer to [55, Sec. 1].

Let $\mathcal{C}$ be a category and let X be a non-empty set. A *$\mathcal{C}$-bundle* over X is a pair $(\mathcal{N},p)$, where $\mathcal{N}$ is a non-empty set and $p : \mathcal{N} \to X$ is a map with the property that each fiber $N_u := p^{-1}(u)$, $u \in X$, is an object of $\mathcal{C}$. If $\mathcal{C}$ is the category of groups (resp. rings), then $\mathcal{N}$ is called a *group (resp. ring) bundle*. In particular, we refer to $\mathcal{N}$ as *Abelian* if each fiber N_u is an Abelian group (resp. commutative ring). We use the symbol $\mathrm{Iso}_{\mathcal{C}}(\mathcal{N})$, or simply $\mathrm{Iso}(\mathcal{N})$, to denote the isomorphism groupoid of the $\mathcal{C}$-bundle $(\mathcal{N},p)$.

Each group bundle carries a natural groupoid structure. Indeed, let X be a set and let $(\mathcal{N},p)$ be a group bundle over X. For each $u \in X$ denote by 1_u the unit of the fiber N_u and put $\mathcal{N}^{(0)} := \{1_u : u \in X\}$. Define the source and the range of $n \in \mathcal{N}$ to be equal to $1_{p(n)}$. Consider the partial multiplication and the inversion defined by the respective operations on the fibers N_u, $u \in X$. This turns $\mathcal{N}$ into a groupoid over $\mathcal{N}^{(0)}$. Identifying $\mathcal{N}^{(0)}$ with X, in which case p becomes the source and the range map, yields the claim.

Let $\mathcal{G}$ be a groupoid. A *$\mathcal{G}$-module bundle* is a pair $((\mathcal{A},p),L)$, where $(\mathcal{A},p)$ is an Abelian group (or ring) bundle over $\mathcal{G}^{(0)}$ and L is a $\mathcal{G}$-module structure on $\mathcal{A}$, that is, L consists of a family $L_x : A_{s(x)} \to A_{r(x)}$, $x \in \mathcal{G}$, of group (or ring) isomorphisms such that $L_u = \mathrm{id}_{A_u}$ for all $u \in \mathcal{G}^{(0)}$ and $L_xL_y = L_{xy}$ whenever $(x,y) \in \mathcal{G}^{(2)}$.

Let $((\mathcal{A},p),L)$ be a $\mathcal{G}$-module bundle. For $n \in \mathbb{N}_0$ an *n-cochain* is a map $h : \mathcal{G}^{(n)} \to \mathcal{A}$ satisfying the following conditions:

1. $p\,(h(x_1,\dots,x_n)) = r(x_1)$ for every $(x_1,\dots,x_n) \in \mathcal{G}^{(n)}$.
2. If $n \geq 1$ and $x_i \in \mathcal{G}^{(0)}$ for some $i \in \{1,\dots,n\}$, then $h(x_1,\dots,x_n) \in \mathcal{G}^{(0)}$.

We denote by $C^n(\mathcal{G},\mathcal{A})$ the set of n-cochains and define $d^0_L : C^0(\mathcal{G},\mathcal{A}) \to C^1(\mathcal{G},\mathcal{A})$ by

$$d^0_L(h)(x) := L_x\,(h(s(x))) - h(r(x)).$$

For $n > 0$ we consider the map $d^n_L : C^n(\mathcal{G},\mathcal{A}) \to C^{n+1}(\mathcal{G},\mathcal{A})$ given by

$$d^n_L(h)(x_1,\dots,x_{n+1}) := L_{x_1}(h(x_2,\dots,x_{n+1})) + \sum_{i=1}^{n}(-1)^n h(x_1,\dots,x_ix_{i+1},\dots,x_{n+1})$$

$$+ (-1)^{n+1} h(x_1, \ldots, x_n).$$

This gives a chain complex $\left(C^n(\mathcal{G},\mathcal{A}), d_L^n\right)_{n\in\mathbb{N}_0}$. For $n \in \mathbb{N}_0$ we write $Z^n(\mathcal{G},\mathcal{A})_L$ for the *n-cocycles*, $B^n(\mathcal{G},\mathcal{A})_L$ for the *n-coboundaries*, and $H^n(\mathcal{G},\mathcal{A})_L := Z^n(\mathcal{G},\mathcal{A})_L/B^n(\mathcal{G},\mathcal{A})_L$ for the n-th *cohomology group*.

4.1.3 Groupoid rings

Let $\mathcal{G}$ be a groupoid and let R be a unital ring. We recall that the *groupoid ring* $R[\mathcal{G}]$ is the set of all finitely supported functions $f : \mathcal{G} \to R$ endowed with the addition given by taking the pointwise sum and the product given by

$$(fg)(z) := \sum_{xy=z} f(x)g(y).$$

For a finite subset $F \subseteq \mathcal{G}$, we let $\delta_F \in R[\mathcal{G}]$ stand for the corresponding characteristic function. In particular, for $F = \{x\}$ we simply write δ_x.

4.1.4 Groupoid crossed products

In what follows, we recall the foundations on groupoid crossed products (cf. [16, 48]).

Definition 4.1. Let $\mathcal{G}$ be a groupoid and let S be a ring. We say that S is $\mathcal{G}$*-graded* if there are additive subsets S_x of S, for $x \in \mathcal{G}$, such that $S = \oplus_{x\in\mathcal{G}} S_x$ and $S_x S_y \subseteq S_{xy}$ if $(x,y) \in \mathcal{G}^{(2)}$ and $S_x S_y = \{0\}$ otherwise.

Definition 4.2. A $\mathcal{G}$-graded ring S is *object unital* if for all $u \in \mathcal{G}^{(0)}$ the ring S_u is unital, and for all $x \in \mathcal{G}$ and all $r \in S_x$ the equalities $1_{S_{r(x)}} r = r 1_{S_{s(x)}} = r$ hold.

Definition 4.3 (cf. [16, Def. 10 and Def. 12])**.** Let $\mathcal{G}$ be a groupoid and let S be a $\mathcal{G}$-graded ring which is object unital.

1. We put $S_0 := \bigoplus_{u\in\mathcal{G}^{(0)}} S_u$ and consider S_0 as a $\mathcal{G}$-graded ring as follows: If $x \in \mathcal{G}$, then $(S_0)_x = S_x$, if $x \in \mathcal{G}^{(0)}$, and $(S_0)_x = \{0\}$, otherwise.
2. We say that a homogeneous element $r \in S_x$ is *object invertible* if there exists $s \in S_{x^{-1}}$ such that $sr = 1_{S_{s(x)}}$ and $rs = 1_{S_{r(x)}}$. We denote by $S^{\times}_{\text{gr}}$ the set of all object invertible elements of S.
3. We say that S is a $\mathcal{G}$*-crossed product* if for all $x \in \mathcal{G}$ the relation $S^{\times}_{\text{gr}} \bigcap S_x \neq \emptyset$ holds. By [16, Prop. 7(iv)], all object crossed products are strongly graded.

Definition 4.4. Let $\mathcal{G}$ be a groupoid, let $\mathcal{R}$ be a unital ring bundle over $\mathcal{G}^{(0)}$, and let R be the ring $\bigoplus_{u\in\mathcal{G}^{(0)}} R_u$.

1. We call a $\mathcal{G}$-crossed product S a $\mathcal{G}$*-crossed product over* R if $S_0 = R$.

2. Two $\mathcal{G}$-crossed products S and S' over R are called *equivalent* if there exists a graded isomorphism $\varphi : S \to S'$ such that $\varphi_{|R} = \mathrm{id}_R$.
3. We let $\mathrm{Ext}(\mathcal{G},\mathcal{R})$ stand for the set of all equivalence classes of $\mathcal{G}$-crossed products over R. Given a $\mathcal{G}$-crossed product S over R, we write $[S]$ for its class in $\mathrm{Ext}(\mathcal{G},\mathcal{R})$.

Definition 4.5 (cf. [16, Def. 13])**.** Let $\mathcal{G}$ be a groupoid, let $\mathcal{R}$ be a unital ring bundle over $\mathcal{G}^{(0)}$, and consider the induced group bundle $\mathcal{R}^\times$ over $\mathcal{G}^{(0)}$ given by $\mathcal{R}^\times := \bigcup_{u\in\mathcal{G}^{(0)}} R_u^\times$.

1. We define $C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{R}))$ as the set of all families of maps $\{M_x : R_{s(x)} \to R_{r(x)}\}_{x\in\mathcal{G}}$ of ring isomorphisms such that $M_u = \mathrm{id}_{R_u}$ for all $u \in \mathcal{G}^{(0)}$.
2. We write $C^2(\mathcal{G},\mathcal{R}^\times)$ for the set of all maps $\tau : \mathcal{G}^{(2)} \to \mathcal{R}^\times$ such that $\tau(x,y) \in R^\times_{r(x)}$ for all $(x,y) \in \mathcal{G}^{(2)}$ and $\tau(x,s(x)) = \tau(r(x),x) = r(x)$ for all $x \in \mathcal{G}$.
3. We call a pair $(M,\tau) \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{R})) \times C^2(\mathcal{G},\mathcal{R}^\times)$ a *factor system* for $(\mathcal{G},\mathcal{R})$ if the following conditions are satisfied:
   (C1) $M_xM_y(n) = \tau(x,y)M_{xy}(n)\tau(x,y)^{-1}$ for all $(x,y) \in \mathcal{G}^{(2)}$ and $n \in R_{s(y)}$,
   (C2) $\tau(x,y)\tau(xy,z) = M_x(\tau(y,z))\tau(x,yz)$ for all $(x,y,z) \in \mathcal{G}^{(3)}$.
4. We let $Z^2(\mathcal{G},\mathcal{R})$ stand for the set of all factor systems for $(\mathcal{G},\mathcal{R})$.

Proposition 4.6 (cf. [16, Def. 14 and Prop. 16])**.** Let $\mathcal{G}$ be a groupoid and let $\mathcal{R}$ be a unital ring bundle over $\mathcal{G}^{(0)}$. For a factor system (M,τ) for $(\mathcal{G},\mathcal{R})$ let $\mathcal{R} \times_{(M,\tau)} \mathcal{G}$ be the set of all functions $f : \mathcal{G} \to \mathcal{R}$ with finite support satisfying $p \circ f = r$. Then $\mathcal{R} \times_{(M,\tau)} \mathcal{G}$ becomes a ring when equipped with the pointwise sum and the product

$$\big(fg\big)(z) := \sum_{xy=z} f(x)M_x(g(y))\tau(x,y).$$

Moreover, $\mathcal{R}\times_{(M,\tau)}\mathcal{G}$ is a $\mathcal{G}$-graded ring which is a $\mathcal{G}$-crossed product over $\mathcal{R}$. Conversely, any $\mathcal{G}$-crossed product over $\mathcal{R}$ can be presented in this way.

Remark 4.7. Let $\mathcal{G}$ be a groupoid, let $\mathcal{R}$ be a unital ring bundle over $\mathcal{G}^{(0)}$, and let (M,τ) be a factor system for $(\mathcal{G},\mathcal{R})$. For all $x,y,z \in \mathcal{G}$ such that $xy = z$ the following identities hold:

$$\tau(x,x^{-1}) = M_x\left(\tau(x^{-1},x)\right), \tag{4.1}$$

$$\tau(z,y^{-1}) = \tau(x,y)^{-1}M_x\left(\tau\left(y,y^{-1}\right)\right), \tag{4.2}$$

$$\tau(z,y^{-1})\tau(x,x^{-1}) = M_z\left(\tau(y^{-1},x^{-1})\right)\tau(z,z^{-1}), \tag{4.3}$$

$$\tau(z,y^{-1})M_x(n) = M_z\left(M_{y^{-1}}(n)\right)\tau(z,y^{-1}), \qquad n \in R_{s(x)}. \tag{4.4}$$

4.2 NON-ABELIAN EXTENSIONS OF GROUPOIDS AND THEIR CLASSIFICATION

In this section we develop a geometrically oriented classification theory for non-Abelian extensions of groupoids in the spirit of Schreier, Baer, and Eilenberg-Mac Lane.

Throughout the following let $\mathcal{G}$ be a groupoid and let $(\mathcal{N},p)$ be a group bundle over $\mathcal{G}^{(0)}$, which we shall consider as a groupoid over $\mathcal{G}^{(0)}$ with respect to its natural groupoid structure described in Section 4.1.2.

Definition 4.8. A *groupoid extension* of $\mathcal{G}$ by $\mathcal{N}$ is a surjective homomorphism $j : \mathcal{E} \to \mathcal{G}$, where $\mathcal{E}$ is a groupoid over $\mathcal{G}^{(0)}$, j^0 is the identity map on $\mathcal{G}^{(0)}$ and $\mathcal{N} = \ker(j)$, i. e., the set of elements $e \in \mathcal{E}$ such that $j(e) \in \mathcal{G}^{(0)}$. Usually, we shall write

$$\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$$

to denote a groupoid extension of $\mathcal{G}$ by $\mathcal{N}$.

1. We call two groupoid extensions $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ and $\mathcal{N} \to \mathcal{E}' \xrightarrow{j'} \mathcal{G}$ of $\mathcal{G}$ by $\mathcal{N}$ *equivalent* if there exists a groupoid homomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{N} & \longrightarrow & \mathcal{E} & \xrightarrow{j} & \mathcal{G} \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{N}}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{G}}} \\ \mathcal{N} & \longrightarrow & \mathcal{E}' & \xrightarrow{j'} & \mathcal{G} \end{array}$$

   It is easily seen that any such φ is, in fact, an isomorphism of groupoids. We shall sometimes say that such a map φ is an *equivalence* of groupoid extensions.

2. We denote by $\mathrm{Ext}(\mathcal{G},\mathcal{N})$ the set of all equivalence classes of groupoid extensions of $\mathcal{G}$ by $\mathcal{N}$. Given an extension $\mathcal{E}$ of $\mathcal{G}$ by $\mathcal{N}$, we write $[\mathcal{E}]$ for its class in $\mathrm{Ext}(\mathcal{G},\mathcal{N})$.

Remark 4.9. Note that all groupoids involved in a groupoid extension necessarily have the same unit space.

Example 4.10. Let $\mathcal{G}$ be a groupoid. In [41, Chap. 1] the author introduces the notion of a *normal subgroupoid* of $\mathcal{G}$ and of the corresponding quotient groupoid $\mathcal{G}/\mathcal{N}$ of $\mathcal{G}$ by $\mathcal{N}$ with projection map $\mathrm{pr} : \mathcal{G} \to \mathcal{G}/\mathcal{N}$. In particular, each normal subgroupoid $\mathcal{N}$ of $\mathcal{G}$ yields a groupoid extension of the form $\mathcal{N} \to \mathcal{G} \xrightarrow{\mathrm{pr}} \mathcal{G}/\mathcal{N}$.

Example 4.11. Let $\mathcal{G}$ be a groupoid over $\mathcal{G}^{(0)}$. A *twist* of $\mathcal{G}$ is a groupoid extension of $\mathcal{G}$ by the trivial group bundle $\mathcal{G}^{(0)} \times \mathbb{T}$. Twists and their applications to operator algebras and related fields have recently regained major interest (see, e. g., [37, 54, 55]).

Example 4.12. A geometrically oriented example of a groupoid extension is given as follows: Let $q : P \to X$ be a locally trivial principal bundle with structure group G and consider the natural action of G on $P \times G$ given by $(p,g).h := (p.h,h^{-1}g)$ for $p \in P$ and

$g,h \in G$. The corresponding quotient $C_G(P) := (P \times G)/G$ is a group bundle over X, the so-called *conjugation bundle*, which is of particular interest in gauge theory, because its space of sections is isomorphic to the the gauge group of the principal bundle. Now, let $N \to E \xrightarrow{\pi} G$ be a short exact sequence of, possibly non-Abelian, Lie groups. Furthermore, suppose that there exists a locally trivial principal bundle $q' : P' \to X$ with structure group E such that $P'/N \cong P$. Then we obtain a short exact sequence of the corresponding conjugation bundles

$$C_N(P') \to C_E(P') \xrightarrow{j} C_G(P) \qquad \text{with} \qquad j([(p',e)]) := [([p'],\pi(e))],$$

and therefore an extension of groupoids. By passing over to the corresponding spaces of sections we get a short exact sequence of gauge groups. A particular simple example of the above situation is given in case of a trivial principal bundle $q_X : X \times G \to X$, $q_X(x,g) = x$. We may then look at $q'_X : X \times E \to X$, $q'_X(x,e) = x$, which in turn leads to the following extension of group bundles over X:

$$X \times N \to X \times E \xrightarrow{j} X \times G \qquad \text{with} \qquad j(x,e) := (x,\pi(e)).$$

We proceed to give a description of non-Abelian groupoid extensions in terms of factor systems in analogy with the classical theory of non-Abelian group extensions (see, e. g., [42, Chap. 4]).

Definition 4.13. 1. We define $C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$ to be the set of all families of group isomorphisms $\{L_x : N_{s(x)} \to N_{r(x)}\}_{x\in\mathcal{G}}$ such that $L_u = \mathrm{id}_{N_u}$ for all $u \in \mathcal{G}^{(0)}$.

2. We write $C^2(\mathcal{G},\mathcal{N})$ for the set of all maps $\sigma : \mathcal{G}^{(2)} \to \mathcal{N}$ such that $\sigma(x,y) \in N_{r(x)}$ for all $(x,y) \in \mathcal{G}^{(2)}$ and $\sigma(x,s(x)) = \sigma(r(x),x) = r(x)$ for all $x \in \mathcal{G}$.

3. We call a pair $(L,\sigma) \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N})) \times C^2(\mathcal{G},\mathcal{N})$ a *factor system* for $(\mathcal{G},\mathcal{N})$ if the following conditions are satisfied:

   (F1) $L_xL_y(n) = \sigma(x,y)L_{xy}(n)\sigma(x,y)^{-1}$ for all $(x,y) \in \mathcal{G}^{(2)}$ and $n \in N_{s(y)}$,

   (F2) $\sigma(x,y)\sigma(xy,z) = L_x(\sigma(y,z))\sigma(x,yz)$ for all $(x,y,z) \in \mathcal{G}^{(3)}$.

   We shall refer to Condition (F1) as the *twisted action condition* and to Condition (F2) as the *twisted cocycle condition*.

4. We let $Z^2(\mathcal{G},\mathcal{N})$ stand for the set of all factor systems for $(\mathcal{G},\mathcal{N})$.

Remark 4.14. For fixed $L \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$ we denote by $Z^2(\mathcal{G},\mathcal{N})_L$ the set of all elements $\sigma \in C^2(\mathcal{G},\mathcal{N})$ satisfying Condition (F1) and Condition (F2) in Definition 4.13. Note that we may then write $Z^2(\mathcal{G},\mathcal{N})$ as the disjoint union

$$Z^2(\mathcal{G},\mathcal{N}) = \bigcup_L Z^2(\mathcal{G},\mathcal{N})_L,$$

which explains the shift in notation from 2-cocycles σ as functions to pairs (L,σ). If $\mathcal{N}$ is a bundle of Abelian groups one can fix L and deal with each set $Z^2(\mathcal{G},\mathcal{N})_L$ separately, but that is not possible for bundles of non-Abelian groups.

The purpose of the following example is to show that every groupoid extension of $\mathcal{G}$ by $\mathcal{N}$ admits a factor system for $(\mathcal{G},\mathcal{N})$. There and subsequently, we use the notation

$$\mathcal{N} \times_{(p,r)} \mathcal{G} := \{(n,x) \in \mathcal{N} \times \mathcal{G} : p(n) = r(x)\}.$$

Example 4.15. Let $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ be a groupoid extension of $\mathcal{G}$ by $\mathcal{N}$. Furthermore, let $k : \mathcal{G} \to \mathcal{E}$ be a normalized section for j, i. e., $j \circ k = \mathrm{id}_{\mathcal{G}}$ and $k_{|\mathcal{G}^{(0)}} = \mathrm{id}_{\mathcal{G}^{(0)}}$. Then

$$\varphi : \mathcal{N} \times_{(p,r)} \mathcal{G} \to \mathcal{E}, \qquad (n,x) \mapsto nk(x) \tag{4.5}$$

is a bijection.

Proof of the claim. For each $x \in \mathcal{E}$ we have $x = xk(j(x))^{-1}k(j(x))$ and $xk(j(x))^{-1} \in \mathcal{N}$, the latter due to the section property. This shows that φ is surjective. To establish its injectivity, we assume that $nk(x) = mk(y)$ for some $n,m \in \mathcal{N}$ and $x,y \in \mathcal{G}$. Applying j then gives $x = y$, and further $n = m$ by cancellation. □

Now, each $x \in \mathcal{G}$ defines a group isomorphism

$$L_x : N_{s(x)} \to N_{r(x)}, \qquad n \mapsto k(x)nk(x)^{-1}.$$

Furthermore, the bijectivity of the map φ implies that $j^{-1}(x) = N_{r(x)}k(x)$ for all $x \in \mathcal{G}$. Since $j(k(x)k(y)) = xy$ for every $(x,y) \in \mathcal{G}^{(2)}$, we conclude that there exists a unique element $\sigma(x,y) \in N_{r(x)}$ such that

$$k(x)k(y) = \sigma(x,y)k(xy). \tag{4.6}$$

This gives a map $\sigma : \mathcal{G}^{(2)} \to \mathcal{N}$ with $\sigma(x,y) \in N_{r(x)}$ for all $(x,y) \in \mathcal{G}^{(2)}$. These maps are related as follows: for all $(x,y) \in \mathcal{G}^{(2)}$ and $n \in N_{s(y)}$ we have

$$\begin{aligned} L_xL_y(n) &= k(x)k(y)n(k(x)k(y))^{-1} \\ &= \sigma(x,y)k(xy)nk(xy)^{-1}\sigma(x,y)^{-1} \\ &= \sigma(x,y)L_{xy}(n)\sigma(x,y)^{-1}. \end{aligned}$$

Also, associativity entails that $(k(x)k(y))\,k(z) = k(x)\,(k(y)k(z))$ for all $(x,y,z) \in \mathcal{G}^{(3)}$. The left-hand side is equal to $\sigma(x,y)\sigma(xy,z)k(xyz)$, while the right-hand side yields

$$k(x)\,(k(y)k(z)) = k(x)\sigma(y,z)k(yz) = L_x(\sigma(y,z))\sigma(x,yz)k(xyz).$$

Consequently, $\sigma(x,y)\sigma(xy,z) = L_x(\sigma(y,z))\sigma(x,yz)$ for all $(x,y,z) \in \mathcal{G}^{(3)}$ by cancellation. Finally, the fact that the section k is normalized makes it obvious that $(L,\sigma) \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N})) \times C^2(\mathcal{G},\mathcal{N})$. Hence (L,σ) is a factor system for $(\mathcal{G},\mathcal{N})$.

Proposition 4.16. Let (L,σ) be a factor system for $(\mathcal{G},\mathcal{N})$. Then $\mathcal{N} \times_{(p,r)} \mathcal{G}$ becomes a groupoid over $\{(u,u) \mid u \in \mathcal{G}^{(0)}\} \cong \mathcal{G}^{(0)}$ equipped with the following structure maps:

1. The source and the range are given by $s(n,x) = s(x)$ and $r(n,x) = r(x)$, respectively. In particular, two elements (n,x) and (m,y) in $\mathcal{N} \times_{(p,r)} \mathcal{G}$ are composable if and only if x and y are.
2. For $s(n,x) = r(m,y)$ the product is given by $(n,x)(m,y) := (nL_x(m)\sigma(x,y),xy)$.
3. The inversion is given by $(n,x)^{-1} := \left(\sigma\left(x^{-1},x\right)^{-1} L_{x^{-1}}\left(n^{-1}\right),x^{-1}\right)$.

We write $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ for the set $\mathcal{N} \times_{(p,r)} \mathcal{G}$ endowed with the above groupoid structure.

Proof. Items (G1)-(G3) and (G5) in Section 4.1.1 are easily checked. Here, we just focus on (G4) and (G6). Applying the twisted cocycle condition to the triple (x, x^{-1},x) gives $\sigma(x,x^{-1}) = L_x(\sigma(x^{-1},x))$, and hence for $(n,x) \in \mathcal{N} \times_{(p,r)} \mathcal{G}$, we have

$$\begin{aligned}(n,x)(n,x)^{-1} &= \left(nL_x\left(\sigma(x^{-1},x)^{-1}\right) L_x\left(L_{x^{-1}}(n^{-1})\right)\sigma(x,x^{-1}),r(x)\right)\\ &= \left(nL_x\left(\sigma(x^{-1},x)^{-1}\right)\sigma(x,x^{-1})n^{-1},r(x)\right) = (nn^{-1},r(x)) = (r(x),r(x)).\end{aligned}$$

Similarly, we get $(n,x)^{-1}(n,x) = (s(x),s(x))$. Next, let $(n,x),(m,y), (l,z) \in \mathcal{N} \times_{(p,r)} \mathcal{G}$ be such that $(x,y,z) \in \mathcal{G}^{(3)}$. Then a straightforward computation yields

$$\begin{aligned}\big((n,x)(m,y)\big)(l,z) &= (nL_x(m)\sigma(x,y)L_{xy}(l)\sigma(x,y)^{-1}\sigma(x,y)\sigma(xy,z),xyz)\\ &= (nL_x(m)L_x(L_y(l))L_x(\sigma(y,z))\sigma(x,yz),xyz)\\ &= (nL_x\big(mL_y(l)\sigma(y,z)\big)\sigma(x,yz),xyz) = (n,x)\big((m,y)(l,z)\big).\end{aligned}$$ □

Summarizing, we get the following result:

Corollary 4.17. $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ is a groupoid extension of $\mathcal{G}$ by $\mathcal{N}$ for any factor system (L,σ) for $(\mathcal{G},\mathcal{N})$.

Proposition 4.18. Let $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ be a groupoid extension of $\mathcal{G}$ by $\mathcal{N}$. Furthermore, let $k : \mathcal{G} \to \mathcal{E}$ be a normalized section for j, i. e. $j \circ k = \mathrm{id}_{\mathcal{G}}$ and $k_{|\mathcal{G}^{(0)}} = \mathrm{id}_{\mathcal{G}^{(0)}}$, and let (L,σ) be the associated factor system. Then $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ and $\mathcal{E}$ are equivalent groupoid extensions via the map $\varphi : \mathcal{N} \times_{(L,\sigma)} \mathcal{G} \to \mathcal{E}$ given by $(n,x) \mapsto nk(x)$.

Proof. By Example 4.15, it suffices to verify the algebraic conditions. Indeed, we first note that $\varphi(n,s(n)) = nk(s(n)) = n$ for all $n \in \mathcal{N}$ and $j(\varphi(n,x)) = j(n)j(k(x)) = x$ for all $(n,x) \in \mathcal{N} \times_{(L,\sigma)} \mathcal{G}$. Now, let $(n,x),(m,y) \in \mathcal{N} \times_{(L,\sigma)} \mathcal{G}$. Then

$$\begin{aligned}\varphi((n,x)(m,y)) &= \varphi(nL_x(m)\sigma(x,y),xy) = nL_x(m)\sigma(x,y)k(xy)\\ &= (nk(x))(mk(y)) = \varphi(n,x))\varphi(m,y).\end{aligned}$$

Moreover, since $\varphi(u) = u$ for all $u \in \mathcal{G}^{(0)}$, we find $\varphi((n,x)^{-1}) = \varphi(n,x)^{-1}$. □

Definition 4.19. We denote by $C^1(\mathcal{G},\mathcal{N})$ the group of all maps $h : \mathcal{G} \to \mathcal{N}$ satisfying $h(x) \in N_{r(x)}$ for all $x \in \mathcal{G}$ and $h(u) = u$ for all $u \in \mathcal{G}^{(0)}$ with respect to the pointwise product. Note that this definition extends the definition of 1-cochains in Section 4.1.2 to the non-Abelian case.

Proposition 4.20. For $h \in C^1(\mathcal{G},\mathcal{N})$ and a factor system $(L,\sigma) \in Z^2(\mathcal{G},\mathcal{N})$ we define

$$(h.L)_x(n) := h(x)L_x(n)h(x)^{-1}, \qquad x \in \mathcal{G}, n \in N_{s(x)}, \tag{4.7}$$

$$(h.\sigma)(x,y) := h(x)L_x(h(y))\sigma(x,y)h(xy)^{-1}, \qquad (x,y) \in \mathcal{G}^{(2)}. \tag{4.8}$$

Then $h.(L,\sigma) := (h.L,h.\sigma)$ is a factor system for $(\mathcal{G},\mathcal{N})$ and the map

$$\alpha : C^1(\mathcal{G},\mathcal{N}) \times Z^2(\mathcal{G},\mathcal{N}) \to Z^2(\mathcal{G},\mathcal{N})$$

given by $\alpha_h(L,\sigma) := \alpha(h,(L,\sigma)) := h.(L,\sigma)$ defines an action of $C^1(\mathcal{G},\mathcal{N})$ on $Z^2(\mathcal{G},\mathcal{N})$.

Proof. We only show that $h.(L,\sigma)$ satisfies the twisted action condition (F1) and the twisted cocycle condition (F2). Let $(x,y) \in \mathcal{G}^{(2)}$ and $n \in N_{s(y)}$. Then

$$\begin{aligned}
&(h.\sigma)(x,y)(h.L)_{xy}(n)(h.\sigma)(x,y)^{-1}\\
&= h(x)L_x(h(y))\sigma(x,y)L_{xy}(n)\sigma(x,y)^{-1}L_x(h(y))^{-1}h(x)^{-1}\\
&= h(x)L_x(h(y))L_xL_y(n)L_x(h(y))^{-1}h(x)^{-1}\\
&= h(x)L_x\big(h(y)L_y(n)h(y)^{-1}\big)h(x)^{-1} = (h.L)_x\big((h.L)_y(n)\big),
\end{aligned}$$

which establishes the twisted action condition (F1). Now, let $(x,y,z) \in \mathcal{G}^{(3)}$. Then

$$\begin{aligned}
&(h.\sigma)(x,y)(h.\sigma)(xy,z)\\
&= h(x)L_x(h(y))L_xL_y(h(z)L_x(\sigma(y,z))\sigma(x,yz)h(xyz)^{-1}\\
&= h(x)L_x\big(h(y)L_y(h(z))\sigma(y,z)h(yz)^{-1}\big)L_x(h(yz))\sigma(x,yz)h(xyz)^{-1}\\
&= (h.L)_x\big((h.\sigma)(y,z)\big)(h.\sigma)(x,yz),
\end{aligned}$$

and the twisted cocycle condition (F2) is proved. Next, we show that $\alpha_{h'}\alpha_h = \alpha_{h'h}$ for all $h,h' \in C^1(\mathcal{G},\mathcal{N})$. For this let $h,h' \in C^1(\mathcal{G},\mathcal{N})$, let $(L,\sigma) \in Z^2(\mathcal{G},\mathcal{N})$, and let $(x,y) \in \mathcal{G}^{(2)}$. We see at once that $h'.(h.L) = (h'h).L$, and hence it remains to verify that $h'.(h.\sigma) = (h'h).\sigma$. Indeed,

$$\begin{aligned}
h'.(h.\sigma)(x,y) &= h'(x)(h.L)_x(h'(y))(h.\sigma)(x,y)h'(xy)^{-1}\\
&= h'(x)h(x)L_x(h'(y))L_x(h(y))\sigma(x,y)h(xy)^{-1}h'(xy)^{-1}\\
&= h'h(x)L_x\big((h'h)(y)\big)\sigma(x,y)(h'h)(xy)^{-1} = (h'h).\sigma(x,y)
\end{aligned}$$

□

By Proposition 4.20, we have an equivalence relation on the set $Z^2(\mathcal{G},\mathcal{N})$ of all factor systems given by

$$(L,\sigma) \sim (L',\sigma') \qquad \Longleftrightarrow \qquad \Big(\exists h \in C^1(\mathcal{G},\mathcal{N})\Big) \;\; (L',\sigma') = h.(L,\sigma).$$

That is, two factor systems are equivalent if they are in the same orbit under the action α. We denote the corresponding orbit space of α by $Z^2(\mathcal{G},\mathcal{N})/C^1(\mathcal{G},\mathcal{N})$.

Theorem 4.21. For two factor systems $(L,\sigma),(L',\sigma') \in Z^2(\mathcal{G},\mathcal{N})$ the following conditions are equivalent:

1. $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ and $\mathcal{N} \times_{(L',\sigma')} \mathcal{G}$ are equivalent groupoid extensions of $\mathcal{G}$ by $\mathcal{N}$.
2. $(L,\sigma) \sim (L',\sigma')$, i. e., there exists $h \in C^1(\mathcal{G},\mathcal{N})$ such that $(L',\sigma') = h.(L,\sigma)$.

If these conditions are satisfied, then the map

$$\psi : \mathcal{N} \times_{(L,\sigma)} \mathcal{G} \to \mathcal{N} \times_{(L',\sigma')} \mathcal{G}, \qquad (n,x) \mapsto (nh(x),x)$$

is an equivalence of groupoid extensions and, further, all equivalences of extensions $\mathcal{N} \times_{(L,\sigma)} \mathcal{G} \to \mathcal{N} \times_{(L',\sigma')} \mathcal{G}$ are of this form.

Proof. Let $\mathcal{N} \times_{(L',\sigma')} \mathcal{G}$ and $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ be equivalent groupoid extensions of $\mathcal{G}$ by $\mathcal{N}$ and let $\varphi : \mathcal{N} \times_{(L',\sigma')} \mathcal{G} \to \mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ be a homomorphism implementing the equivalence. Then there exists a map $\varphi_0 : \mathcal{N} \times_{(L',\sigma')} \mathcal{G} \to \mathcal{N}$ such that φ has the form $\varphi(n,x) = (\varphi_0(n,x),x)$. It is easily seen that $\varphi_0(n,x) \in N_{r(x)}$ for all $(n,x) \in \mathcal{N} \times_{(L',\sigma')} \mathcal{G}$ and $\varphi_0(n,s(n)) = 1_{p(n)}$ for all $n \in N$. Moreover, for each $(n,x) \in \mathcal{N} \times_{(L',\sigma')} \mathcal{G}$ we find

$$\varphi(n,x) = \varphi(n,r(x))\varphi(r(x),x) = (n,r(x))(\varphi_0(r(x),x),x) = (n\varphi_0(r(x),x),x).$$

Consequently, the map $h : \mathcal{G} \to \mathcal{N}$ given by $h(x) := \varphi_0(r(x),x)$ belongs to $C^1(\mathcal{G},\mathcal{N})$ and satisfies $\varphi(n,x) = (nh(x),x)$. To proceed, let $(n,x),(m,y) \in \mathcal{N} \times_{(L',\sigma')} \mathcal{G}$. Then $\varphi\big((n,x)(m,y)\big) = \varphi(n,x)\varphi(m,y)$, and hence

$$(nL'_x(m)\sigma'(x,y)h(xy),xy) = (nh(x)L_x(mh(y))\sigma(x,y),xy). \tag{4.9}$$

Considering $m \in \mathcal{G}^{(0)}$ and $y \in \mathcal{G}^{(0)}$, we thus get $(L',\sigma') = h.(L,\sigma)$. If, conversely, $(L',\sigma') = h.(L,\sigma)$ for some $h \in C^1(\mathcal{G},\mathcal{N})$, then we define

$$\varphi : \mathcal{N} \times_{(L',\sigma')} \mathcal{G} \to \mathcal{N} \times_{(L,\sigma)} \mathcal{G}, \qquad (n,x) \mapsto (nh(x),x)$$

and the considerations above show that φ implements an equivalence of groupoids. □

Corollary 4.22. The map $Z^2(\mathcal{G},\mathcal{N}) \to \operatorname{Ext}(\mathcal{G},\mathcal{N})$ sending (L,σ) to $[\mathcal{N} \times_{(L,\sigma)} \mathcal{G}]$ induces a bijection $H^2(\mathcal{G},\mathcal{N}) := Z^2(\mathcal{G},\mathcal{N})/C^1(\mathcal{G},\mathcal{N}) \to \operatorname{Ext}(\mathcal{G},\mathcal{N})$.

In what follows, we call an element $L \in C^1(\mathcal{G},\operatorname{Iso}(\mathcal{N}))$ *outer* if there exists $\sigma \in C^2(\mathcal{G},\mathcal{N})$ such that (L,σ) satisfies the twisted action condition (F1). We emphasize that

$$L \sim L' \qquad \Longleftrightarrow \qquad \Big(\exists h \in C^1(\mathcal{G},\mathcal{N})\Big) \;\; L' = h.L$$

defines an equivalence relation on the set of all outer elements. Given an outer element $L \in C^1(\mathcal{G},\operatorname{Iso}(\mathcal{N}))$, we denote by $[L]$ the equivalence class of L and call it a $\mathcal{G}$-*kernel* in accordance with the notion of kernels in the classical theory of non-Abelian extensions of groups (see, e. g., [42, Chap. 4]).

The preceding proposition shows in particular that if $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ and $\mathcal{N} \times_{(L',\sigma')} \mathcal{G}$ are equivalent extensions then $[L] = [L']$. We write $\mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]}$ for the set of equivalence classes of groupoid extensions of $\mathcal{G}$ by $\mathcal{N}$ corresponding to the $\mathcal{G}$-kernel $[L]$. Moreover, we put

$$Z(\mathcal{N}) := \bigcup_{u \in \mathcal{G}^{(0)}} Z(N_u)$$

and consider the induced $\mathcal{G}$-module bundle $(Z(\mathcal{N}),L)$ as well as its cohomology theory (cf. Section 4.1.2).

Theorem 4.23. Suppose that $L \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$ is outer with $\mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]} \neq \emptyset$. Then the following assertions hold:

1. Each class in $\mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]}$ can be represented by one of the form $N \times_{(L,\sigma)} G$.
2. Let (L,σ') and (L,σ) be factor systems for $(\mathcal{G},\mathcal{N})$. Then $\sigma^{-1} \cdot \sigma' \in Z^2(\mathcal{G},Z(\mathcal{N}))_L$, and moreover $(L,\sigma') \sim (L,\sigma)$ if and only if $\sigma^{-1} \cdot \sigma' \in B^2(\mathcal{G},Z(\mathcal{N}))_L$.

Proof. 1. From Proposition 4.18 we know that each groupoid extension of $\mathcal{G}$ by $\mathcal{N}$ is equivalent to one of the form $\mathcal{N} \times_{(L',\sigma')} \mathcal{G}$. If $[L'] = [L]$ and $h \in C^1(\mathcal{G},\mathcal{N})$ satisfies $L' = h.L$, then $h^{-1}.(L',\sigma') = (L,h^{-1}.\sigma')$ so that $\sigma'' := h^{-1}.\sigma'$ satisfies $[\mathcal{N} \times_{(L',\sigma')} \mathcal{G}] = [\mathcal{N} \times_{(L,\sigma'')} \mathcal{G}]$, which proves the first claim.

2. We first note that $\sigma(x,y)^{-1}\sigma'(x,y)$ is central for every $(x,y) \in \mathcal{G}^{(2)}$, because $\sigma(x,y)n\sigma(x,y)^{-1} = \sigma'(x,y)n\sigma'(x,y)^{-1}$ for all $(x,y) \in \mathcal{G}^{(2)}$ and $n \in N_{r(x)}$ by the twisted action condition. Now, we check that $\sigma^{-1} \cdot \sigma'$ is a 2-cocycle. For this let $(x,y,z) \in \mathcal{G}^{(3)}$. Then

$$\begin{aligned}
&\big(\sigma^{-1} \cdot \sigma'\big)(x,y)\big(\sigma^{-1} \cdot \sigma'\big)(xy,z) \\
&= \sigma^{-1}(xy,z)\sigma^{-1}(x,y)\sigma'(x,y)\sigma'(xy,z) \\
&= \sigma^{-1}(x,yz)L_x\big(\sigma(y,z)^{-1}\big)L_x\big(\sigma'(y,z)\big)\sigma'(x,yz) \\
&= L_x\big(\sigma(y,z)^{-1}\sigma'(y,z)\big)\sigma^{-1}(x,yz)\sigma'(x,yz) \\
&= L_x\big(\big(\sigma^{-1} \cdot \sigma'\big)(y,z)\big)\big(\sigma^{-1} \cdot \sigma'\big)(x,yz),
\end{aligned}$$

where we have used the twisted cocycle condition (F2) to get the third equation.

For the second part we first assume that $(L,\sigma') \sim (L,\sigma)$. Then there exists $h \in C^1(\mathcal{G},\mathcal{N})$ such that $(L,\sigma') = h.(L,\sigma)$. In particular $L_x(n) = h(x)L_x(n)h(x)^{-1}$ holds for all $x \in \mathcal{G}$ and $n \in N_{s(x)}$, and hence $h \in C^1(\mathcal{G},Z(\mathcal{N}))$. Since h is central and $\sigma' = h.\sigma$ we further obtain

$$\sigma(x,y)^{-1}\sigma'(x,y) = h(x)L_x(h(y))h(xy)^{-1} = d_L^1(h) \in B^2(\mathcal{G},Z(\mathcal{N}))_L.$$

If, conversely, $\sigma^{-1}\sigma' = d_L^1(h)$ for $h \in C^1(\mathcal{G},Z(\mathcal{N}))$, then $(L,\sigma') = h.(L,\sigma)$. $\square$

Corollary 4.24. For a $\mathcal{G}$-kernel $[L]$ with $\mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]} \neq \emptyset$ the following map is a well-defined simply transitive action:

$$H^2(\mathcal{G},Z(\mathcal{N}))_L \times \mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]} \to \mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]}, \quad \Big([\rho],[\mathcal{N} \times_{(L,\sigma)} \mathcal{G}]\Big) \mapsto [\mathcal{N} \times_{(L,\sigma\cdot\rho)} \mathcal{G}]$$

Remark 4.25. Suppose $\mathcal{A}$ is an Abelian group bundle. A factor system (L,σ) for $(\mathcal{G},\mathcal{A})$ consists of a $\mathcal{G}$-module structure L on $\mathcal{A}$, and an element $\sigma \in Z^2(\mathcal{G},\mathcal{A})_L$, and we write $\mathcal{A} \times_\sigma \mathcal{G}$ for the corresponding groupoid extension of $\mathcal{A} \times_{(L,\sigma)} \mathcal{G}$. Furthermore, we have $L \sim L'$ if and only if $L = L'$. Hence a $\mathcal{G}$-kernel $[L]$ is the same as a $\mathcal{G}$-module structure L on $\mathcal{A}$ and $\mathrm{Ext}(\mathcal{G},\mathcal{A})_L := \mathrm{Ext}(\mathcal{G},\mathcal{A})_{[L]}$ is the set of groupoid extensions of $\mathcal{G}$ by $\mathcal{A}$ for which the associated $\mathcal{G}$-module structure on $\mathcal{A}$ is L. According to Corollary 4.24, the equivalence classes of groupoid extensions correspond to cohomology classes of cocycles, so that the map

$$H^2(\mathcal{G},\mathcal{A})_L \to \mathrm{Ext}(\mathcal{G},\mathcal{A})_L, \qquad [\sigma] \mapsto [\mathcal{A} \times_\sigma \mathcal{G}]$$

is a well-defined bijection. In fact, by [55, Prop 1.14] it is not only a bijection but also a group isomorphism.

We conclude this section with a criterion for the nonemptyness of the set $\mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]}$. To the best of our knowledge, such a criterion has not been worked out yet.

Lemma 4.26. Suppose that $(L,\sigma) \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N})) \times C^2(\mathcal{G},\mathcal{N})$ satisfies the twisted action condition (F1). Then the map $\chi_{(L,\sigma)} : \mathcal{G}^{(3)} \to Z(\mathcal{N})$ given by

$$\chi_{(L,\sigma)}(x,y,z) := L_x(\sigma(y,z))\sigma(x,yz)\sigma(xy,z)^{-1}\sigma(x,y)^{-1}, \qquad (x,y,z) \in \mathcal{G}^{(3)}$$

defines a 3-cocycle, i. e., $\chi_{(L,\sigma)} \in Z^3(\mathcal{G},Z(\mathcal{N}))_L$.

Proof. For ease of notation we simply put $\chi := \chi_{(L,\sigma)}$. Let $(x,y,z) \in \mathcal{G}^{(3)}$, let $m \in N_{r(x)}$, and define $n = L_{xyz}^{-1}(m)$. Then

$$\sigma(x,y)\sigma(xy,z)m\sigma(xy,z)^{-1}\sigma(x,y)^{-1} = \sigma(x,y)L_{xy}L_z(n)\sigma(x,y)^{-1} = L_xL_yL_z(n)$$

and further

$$\begin{aligned} L_xL_yL_z(n) &= L_x\big(\sigma(y,z)L_{yz}(n)\sigma(y,z)^{-1}\big) \\ &= L_x\big(\sigma(y,z)\big)\big(L_xL_{yz}(n)\big)L_x\big(\sigma(y,z)\big)^{-1} \\ &= L_x\big(\sigma(y,z)\big)\sigma(x,yz)m\sigma(x,yz)^{-1}L_x\big(\sigma(y,z)\big)^{-1}. \end{aligned}$$

Therefore $L_x\big(\sigma(y,z)\big)\sigma(x,yz)$ and $\sigma(x,y)\sigma(xy,z)$ define the same inner automorphism of $N_{r(x)}$ and hence $\chi(x,y,z) = L_x\big(\sigma(y,z)\big)\sigma(x,yz)\sigma(xy,z)^{-1}\sigma(x,y)^{-1}$ is a central element. This shows that the map χ is well-defined. We proceed to show that χ lies in the kernel of the map $d_L^3 : C^3(\mathcal{G},Z(\mathcal{N}))_L \to C^4(\mathcal{G},Z(\mathcal{N}))_L$ given by

$$d_L^3(\chi)(x,y,z,w) := L_x(\chi(y,z,w))\chi(xy,z,w)^{-1}\chi(x,yz,w)\chi(x,y,zw)^{-1}\chi(x,y,z).$$

Below we explicitly write down all the factors that we have to multiply. We also emphasize that they can be multiplied in any order, because χ is central.

- $\chi(xy,z,w)^{-1} = \sigma(xy,z)\sigma(xyz,w)\sigma(xy,zw)^{-1}L_{xy}(\sigma(z,w)^{-1})$.
- $\chi(x,y,zw)^{-1} = \sigma(x,y)\sigma(xy,zw)\sigma(x,yzw)^{-1}L_x(\sigma(y,zw)^{-1})$.
- $\chi(x,yz,w) = L_x(\sigma(yz,w))\sigma(x,yzw)\sigma(xyz,w)^{-1}\sigma(x,yz)^{-1}$.
- $\chi(x,y,z) = L_x(\sigma(y,z))\sigma(x,yz)\sigma(xy,z)^{-1}\sigma(x,y)^{-1}$.
- $L_x(\chi(y,z,w)) = L_x\left(L_y(\sigma(z,w))\sigma(y,zw)\sigma(yz,w)^{-1}\sigma(y,z)^{-1}\right)$.

Moreover, for simplicity of the presentation we introduce the following auxiliary elements:

- $n_1 := L_{zw}^{-1}\left(\sigma(z,w)^{-1}L_{xy}^{-1}(\sigma(x,y))\right)$,
- $n_2 := L_{yzw}^{-1}\left(\sigma(y,zw)^{-1}\sigma(yz,w)\right)$,
- $n_3 := L_y\left(L_z\left(L_w(n_1 n_2)L_{yz}^{-1}(\sigma(y,z))\right)\right)$,
- $n_4 := L_y(\sigma(z,w))\sigma(y,zw)\sigma(yz,w)^{-1}$.

Using repeatedly the twisted action condition (F1), we obtain

$$\chi(xy,z,w)^{-1}\chi(x,y,zw)^{-1} = \sigma(xy,z)\sigma(xyz,w)L_{xyzw}(n_1)\sigma(x,yzw)^{-1}L_x(\sigma(y,zw))^{-1},$$

and further

$$\chi(xy,z,w)^{-1}\chi(x,y,zw)^{-1}\chi(x,yz,w) = \sigma(xy,z)L_{xyz}(L_w(n_1 n_2))\sigma(x,yz)^{-1}.$$

It follows that

$$\chi(xy,z,w)^{-1}\chi(x,y,zw)^{-1}\chi(x,yz,w)\chi(x,y,z) = \sigma(x,y)^{-1}L_x(n_3). \tag{4.10}$$

To proceed, we look more closely at n_3. Indeed, since $L_y L_{zw}(n_2) = \sigma(yz,w)\sigma(y,zw)^{-1}$, we conclude that

$$\begin{aligned}
n_3 &= L_y L_z\big(L_w(n_1 n_2)\big)L_y L_z\big(L_{yz}^{-1}(\sigma(y,z))\big)\\
&= L_y L_z\big(L_w(n_1 n_2)\big)\sigma(y,z)\\
&= L_y\left(\sigma(z,w)L_{zw}(n_1)L_{zw}(n_2)\sigma(z,w)^{-1}\right)\sigma(y,z)\\
&= L_y\left(L_{xy}^{-1}(\sigma(x,y))L_{zw}(n_2)\sigma(z,w)^{-1}\right)\sigma(y,z)\\
&= L_y\big(L_{xy}^{-1}(\sigma(x,y))\big)\sigma(yz,w)\sigma(y,zw)^{-1}L_y\big(\sigma(z,w)^{-1}\big)\sigma(y,z).
\end{aligned}$$

Combining the previous expression with Equation ((4.10)), we get

$$\begin{aligned}
&\chi(xy,z,w)^{-1}\chi(x,y,zw)^{-1}\chi(x,yz,w)\chi(x,y,z) = \sigma(x,y)^{-1}L_x(n_3)\\
&= L_x\left(\sigma(yz,w)\sigma(y,zw)^{-1}L_y\big(\sigma(z,w)^{-1}\big)\sigma(y,z)\right) = L_x\big(n_4^{-1}\sigma(y,z)\big),
\end{aligned}$$

and finally that

$$\begin{aligned} d_L^3(\chi)(x,y,z,w) &= L_x\big(n_4^{-1}\sigma(y,z)\big)L_x(\chi(y,z,w)) = L_x\big(n_4^{-1}\sigma(y,z)\chi(y,z,w)\big) \\ &= L_x\big(n_4^{-1}\chi(y,z,w)\sigma(y,z)\big) = L_x\big(n_4^{-1}n_4\sigma(y,z)^{-1}\sigma(y,z)\big) = 1_{N_{r(x)}}. \end{aligned}$$

□

Theorem 4.27. Suppose that $(L,\sigma), (L',\sigma') \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N})) \times C^2(\mathcal{G},\mathcal{N})$ satisfy the twisted action condition (F1) and that $L' \sim L$. Then $\chi := \chi_{(L,\sigma)}$ and $\chi' := \chi_{(L,'\sigma')}$ are cohomologous 3-cocycles in $Z^3(\mathcal{G},Z(\mathcal{N}))_L$.

Proof. To begin with, we note that $L' \sim L$ implies that $L' = L$ on the center $Z(\mathcal{N})$, and hence the cohomology groups $H^3(\mathcal{G},Z(\mathcal{N}))_L$ and $H^3(\mathcal{G},Z(\mathcal{N}))_{L'}$ are, in fact, identical. To show that χ and χ' are cohomologous, we first assume that $L' = L$ and recall that in this case $\sigma^{-1}\cdot\sigma'$ takes values in the center by item 2 of Theorem 4.23. Since we also have $\sigma^{-1}\cdot\sigma' = \sigma'\cdot\sigma^{-1}$, it follows that

$$\begin{aligned} &\chi'(x,y,z)\chi(x,y,z)^{-1} \\ &= L_x\big(\sigma'(y,z)\big)\sigma'(x,yz)\sigma'(xy,z)^{-1}\sigma'(x,y)^{-1}\sigma(x,y)\sigma(xy,z)\sigma(x,yz)^{-1}L_x\big(\sigma(y,z)^{-1}\big) \\ &= L_x\big(\sigma'(y,z)\big)\big(\sigma'\cdot\sigma^{-1}\big)(x,yz)L_x\big(\sigma(y,z)^{-1}\big)\big(\sigma'^{-1}\cdot\sigma\big)(xy,z)\big(\sigma'^{-1}\cdot\sigma\big)(x,y) \\ &= L_x\big(\big(\sigma'\cdot\sigma^{-1}\big)(y,z)\big)\big(\sigma'\cdot\sigma^{-1}\big)(x,yz)\big(\sigma'^{-1}\cdot\sigma\big)(xy,z)\big(\sigma'^{-1}\cdot\sigma\big)(x,y) \\ &= d_L^2\big(\sigma^{-1}\cdot\sigma'\big)(x,y,z). \end{aligned}$$

Now, if $L' = h.L$ for some $h \in C^1(\mathcal{G},\mathcal{N})$ and $\theta := h.\sigma$ is as in Equation (4.8), then Proposition 4.20 implies that (L',θ) satisfies the twisted action condition, and further

$$\begin{aligned} &L'_x(\theta(y,z))\theta(x,yz)h(xyz) \\ &= L'_x(\theta(y,z))h(x)L_x(h(yz))\sigma(x,yz) \\ &= L'_x\big(\theta(y,z)h(yz)\big)h(x)\sigma(x,yz) \\ &= L'_x\big(h(y)L_y(h(z))\sigma(y,z)\big)h(x)\sigma(x,yz) \\ &= L'_x\big(L'_y(h(z))h(y)\big)L'_x(\sigma(y,z))h(x)\sigma(x,yz) \\ &= L'_x\big(L'_y(h(z))h(y)\big)h(x)L_x(\sigma(y,z))\sigma(x,yz) \\ &= L'_x\big(L'_y(h(z))h(y)\big)h(x)\chi(x,y,z)\sigma(x,y)\sigma(xy,z) \\ &= \chi(x,y,z)L'_x\big(L'_y(h(z))\big)L'_x(h(y))h(x)\sigma(x,y)\sigma(xy,z) \\ &= \chi(x,y,z)L'_x\big(L'_y(h(z))\big)h(x)L_x(h(y))\sigma(x,y)\sigma(xy,z) \\ &= \chi(x,y,z)L'_x\big(L'_y(h(z))\big)\theta(x,y)h(xy)\sigma(xy,z) \\ &= \chi(x,y,z)\theta(x,y)L'_{xy}(h(z))h(xy)\sigma(xy,z) \\ &= \chi(x,y,z)\theta(x,y)h(xy)L_{xy}(h(z))h(xy)\sigma(xy,z) \\ &= \chi(x,y,z)\theta(x,y)\theta(xy,z)h(xyz). \end{aligned}$$

Hence $\chi = \chi_{(L',\theta)}$, and combining this with the first step completes the proof. □

Corollary 4.28. Suppose that $L \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$ is outer and choose $\sigma \in C^2(\mathcal{G},\mathcal{N})$ such that (L,σ) satisfies the twisted action condition (F1). Then the cohomology class $[\chi_{(L,\sigma)}] \in H^3(\mathcal{G},Z(\mathcal{N}))_L$ does not depend on the choice of σ and is constant on the equivalence class $[L]$.

On account of Corollary 4.28, each outer element $L \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$ gives rise to a characteristic class $\chi(L) \in H^3(\mathcal{G},Z(\mathcal{N}))_L$.

Corollary 4.29. For a $\mathcal{G}$-kernel $[L]$ we have $\mathrm{Ext}(\mathcal{G},\mathcal{N})_{[L]} \neq \emptyset$ if and only if the characteristic class $\chi(L) \in H^3(\mathcal{G},Z(\mathcal{N}))_L$ is trivial.

Proof. If there exists a groupoid extension $\mathcal{E}$ of $\mathcal{G}$ by $\mathcal{N}$ corresponding to $[L]$, then we may w. l. o. g. assume that it is of the form $\mathcal{N} \times_{(L,\sigma)} \mathcal{G}$ for some factor system (L,σ) for $(\mathcal{G},\mathcal{N})$. This in particular implies that $\chi_{(L,\sigma)} = 1$ and hence the characteristic class $\chi(L) \in H^3(\mathcal{G},Z(\mathcal{N}))_L$ is trivial. If, conversely, $L \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$ is outer and $\chi(L) \in H^3(\mathcal{G},Z(\mathcal{N}))_L$ is trivial, then there exists $\sigma \in C^2(\mathcal{G},\mathcal{N})$ and $\rho \in C^2(\mathcal{G},Z(\mathcal{N}))$ such that $\chi_{(L,\sigma)} = \chi_{(L,\rho^{-1})}$, so that $(L,\sigma \cdot \rho)$ is a factor system. It follows that $\mathcal{N} \times_{(L,\sigma\cdot\rho)} \mathcal{G}$ is a groupoid extension of $\mathcal{G}$ by $\mathcal{N}$ corresponding to $[L]$. This completes the proof. □

4.3 GROUPOID RINGS OF GROUPOID EXTENSIONS

Here and subsequently, let $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ be a groupoid extension. In this section we associate certain groupoid crossed products (cf. Section 4.1.4) with the groupoid extension, study their relationship, and establish, as an application, that the groupoid ring of $\mathcal{E}$ is isomorphic to a $\mathcal{G}$-crossed product over the groupoid ring of $\mathcal{N}$.

For a start we note that for $(u,v) \in \mathcal{G}^{(0)}$ there exists $x \in \mathcal{E}$ such that $u = s_{\mathcal{E}}(x)$ and $v = r_{\mathcal{E}}(x)$ if and only if there exists $y \in \mathcal{G}$ such that $u = s_{\mathcal{G}}(y)$ and $v = r_{\mathcal{G}}(y)$. Hence the equivalence relations $u \sim_{\mathcal{E}} v \iff$ there exists $x \in \mathcal{E}$ such that $s_{\mathcal{E}}(x) = u$ and $r_{\mathcal{E}}(x) = v$, and $u \sim_{\mathcal{G}} v \iff$ there exists $x \in \mathcal{G}$ such that $s_{\mathcal{G}}(x) = u$ and $r_{\mathcal{G}}(x) = v$ generate the same partition $\{S_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{G}^{(0)}$. To proceed, let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a family of unital rings. Below we present two constructions of factor systems in the sense of Definition 4.5:

1. For each $u \in \mathcal{G}^{(0)}$ we put $R_u := R_\lambda[N_u]$, where λ is the unique element in Λ such that $u \in S_\lambda$, and and consider the ring bundle $\mathcal{R} := \bigcup_{u\in\mathcal{G}^{(0)}} R_u$ over $\mathcal{G}^{(0)}$. Furthermore, let $k : \mathcal{G} \to \mathcal{E}$ be a normalized section for j and let (L,σ) be the associated factor system (cf. Example 4.15). Then a straightforward computation shows that the family of maps

$$M_x : R_{s(x)} \to R_{r(x)}, \qquad M_x(f) := f \circ L_x^{-1}, \qquad x \in \mathcal{G},$$

and

$$\tau : \mathcal{G}^{(2)} \to \mathcal{R}^\times, \qquad \tau(x,y) := \delta_{\sigma(x,y)}$$

yields a factor system (M,τ) for $(\mathcal{G},\mathcal{R})$.

2. For each $u \in \mathcal{G}^{(0)}$ we put $R'_u := R_\lambda$, where λ is the unique element in Λ such that $u \in S_\lambda$. Then $\mathcal{R}' := \bigcup_{u \in \mathcal{G}^{(0)}} R'_u$ is a ring bundle over $\mathcal{G}^{(0)}$ and the family of maps

$$M'_x = \mathrm{id} : R'_{s(x)} \to R'_{r(x)}, \qquad x \in \mathcal{E},$$

and

$$\tau' : \mathcal{E}^{(2)} \to \mathcal{R}^\times, \qquad \tau'(x,y) = 1_{R'_s(x)}$$

yields a factor system for $(\mathcal{E},\mathcal{R}')$.

For expedience we put all of this on record:

Proposition 4.30. Let $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ be a groupoid extension and let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a family of unital rings, where Λ indexes the partition of $\mathcal{G}^{(0)}$ w. r. t. the equivalence relation $\sim_\mathcal{G}$. Then the following assertions hold:

1. If $k : \mathcal{G} \to \mathcal{E}$ is a normalized section for j and (L,σ) is the associated factor system, then (M,τ) defined in item 1 above is a factor system for $(\mathcal{G},\mathcal{R})$.
2. (M',τ') defined in item 2 above is a factor system for $(\mathcal{E},\mathcal{R}')$.

Remark 4.31. 1. We shall refer to (M,τ) as *the factor system associated with* $\{R_\lambda\}_{\lambda\in\Lambda}$ *and* (L,σ). If $R_\lambda = R$ for all $\lambda \in \Lambda$, then we denote $\mathcal{R}$ by $R[\mathcal{N}]$.

2. We shall refer to (M',τ') as the *trivial factor system system associated with* $\mathcal{E}$ *and* $\{R_\lambda\}_{\lambda\in\Lambda}$. Note that if $R_\lambda = R$ for all $\lambda \in \Lambda$, then the associated groupoid crossed product is simply the groupoid ring $R[\mathcal{E}]$.

Having disposed of these preparatory steps, we are now ready to prove the following:

Theorem 4.32. Let $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ be a groupoid extension and let $\{R_\lambda\}_{\lambda\in\Lambda}$ be a family of unital rings, where Λ indexes the partition of $\mathcal{G}^{(0)}$ w. r. t. the equivalence relation $\sim_\mathcal{G}$. Furthermore, let $k : \mathcal{G} \to \mathcal{E}$ be a normalized section for j, let (L,σ) be the associated factor system, and let (M,τ) be the factor system associated with $\{R_\lambda\}_{\lambda\in\Lambda}$ and (L,σ). Finally, let (M',τ') be the trivial factor system associated with $\mathcal{E}$ and $\{R_\lambda\}_{\lambda\in\Lambda}$. Then the respective groupoid crossed products $\mathcal{R} \times_{(M,\tau)} \mathcal{G}$ and $\mathcal{R}' \times_{(M',\tau')} \mathcal{E}$ (cf. Proposition 4.6) are isomorphic.

Proof. Let us consider the maps

$$\Phi : \mathcal{R}' \times_{(M',\tau')} \mathcal{E} \to \mathcal{R} \times_{(M,\tau)} \mathcal{G}, \qquad \Phi(f)(x)(n) = f(nk(x))$$

and

$$\Psi : \mathcal{R} \times_{(M,\tau)} \mathcal{G} \to \mathcal{R}' \times_{(M',\tau')} \mathcal{E}, \qquad \Psi(f)(e) = f(x)(n),$$

where in the latter case $(n,x) \in \mathcal{N} \times \mathcal{G}$ is the unique element such that $e = nk(x)$ (cf. Example 4.15). We first establish that Φ and Ψ are mutually inverses. For this, let

$f \in \mathcal{R} \rtimes_{(M,\tau)} \mathcal{G}$, $x \in \mathcal{G}$, and $n \in N_{r(x)}$. Then $\Phi(\Psi(f))(x)(n) = \Psi(f)(nk(x)) = f(x)(n)$. Moreover, for $f \in \mathcal{R}' \times_{(M',\tau')} \mathcal{E}$ and $e = nk(x) \in \mathcal{E}$ we have $\Psi(\Phi(f))(e) = \Phi(f)(x)(n) = f(nk(x)) = f(e)$, which proves the assertion. Since Φ is clearly linear, it remains to show that Φ is multiplicative. To this end, let $z \in \mathcal{G}$ and let $n' \in N_{r(z)}$. Then

$$\{(s,t) \in \mathcal{E}^{(2)} : st = n'k(z)\} = \left\{ \left(nk(x), L_x^{-1}(m)k(y)\right) : xy = z,\ nm = n'\sigma(x,y)^{-1} \right\}.$$

Proof of the equality. Let $(s,t) \in \mathcal{E}^{(2)}$ be such that $st = n'k(z)$ and write $s = nk(x)$ and $t = m'k(y)$. Put $m := L_x(m')$ and note that $n'k(z) = nk(x)m'k(y) = nm\sigma(x,y)k(xy)$. By uniqueness, we may conclude that $xy = z$ and $nm = n'\sigma(x,y)^{-1}$, and therefore $(s,t) = \left(nk(x), L_x^{-1}(m)k(y)\right)$. The inverse inclusion follows from multiplication. □

Now, a standard calculation shows that

$$\begin{aligned}
\Phi(f)\Phi(g)(z)(n') &= \sum_{xy=z} \left(\Phi(f)(x)M_x(\Phi(g)(y))\tau(x,y)\right)(n') \\
&= \sum_{xy=z} \sum_{nm=n'\sigma(x,y)^{-1}} \Phi(f)(x)(n)M_x(\Phi(g)(y))(m) \\
&= \sum_{xy=z} \sum_{nm=n'\sigma(x,y)^{-1}} f(nk(x))g\left(L_x^{-1}(m)k(y)\right) \\
&= \sum_{st=n'k(z)} f(s)g(t) = \sum_{st=n'k(z)} f(s)M'_s(g(t))\tau'(s,t) = \Phi(fg)(z)(n'),
\end{aligned}$$

which in turn completes the proof. □

Corollary 4.33. Under the hypotheses of Theorem 4.32 with $R_\lambda = R$ for all $\lambda \in \Lambda$ we have that $R[\mathcal{N}] \times_{(M,\tau)} \mathcal{G}$ is isomorphic to the groupoid ring $R[\mathcal{E}]$ (cf. Remark 4.31).

In the remaining part of this section we extend Corollary 4.33 to the realm of C*-algebras. For this we first need to suitably adapt Definition 4.5:

Definition 4.34. Let $\mathcal{G}$ be a groupoid and let $\mathcal{R}$ be a bundle of normed unital *-algebras over $\mathcal{G}^{(0)}$. A **-factor system for* $(\mathcal{G},\mathcal{R})$ is a factor system (M,τ) in the sense of Definition 4.5 with the additional property that M is a family of isometric *-isomorphisms and that $\tau(x,y)^{-1} = \tau(x,y)^*$ for all $(x,y) \in \mathcal{G}^{(2)}$.

Example 4.35. Let $\mathcal{G}$ be a groupoid, let $\mathcal{N}$ be a group bundle over $\mathcal{G}^{(0)}$, and let (L,σ) be a factor system for $(\mathcal{G},\mathcal{N})$. Then the construction in item 1 above with $R_\lambda = \mathbb{C}$ for all $\lambda \in \Lambda$ yields, in fact, a *-factor system for $(\mathcal{G},\mathbb{C}[\mathcal{N}])$.

Proposition 4.36. Let $\mathcal{G}$ be a groupoid, let $\mathcal{R}$ be a bundle of normed unital *-algebras over $\mathcal{G}^{(0)}$, and let (M,τ) be a *-factor system for $(\mathcal{G},\mathcal{R})$. Then the ring $\mathcal{R} \times_{(M,\tau)} \mathcal{G}$ becomes a normed *-algebra when endowed with the norm $\|f\|_1 := \sum_{x\in\mathcal{G}} \|f(x)\|$ and the involution $f^*(x) := \tau(x,x^{-1})^{-1} M_x\left(f(x^{-1})\right)^*$, $x \in \mathcal{G}$.

Proof. Clearly, $\|\cdot\|_1$ is a norm and the involution is linear conjugate and isometric. For f, $g \in \mathcal{R} \times_{(M,\tau)} \mathcal{G}$ we now check that $\|fg\|_1 \leq \|f\|_1\|g\|_1$, $(f^*)^* = f$, and $(fg)^* = g^*f^*$. Indeed, we have

$$\|fg\|_1 \leq \sum_{z\in\mathcal{G}} \sum_{\substack{(x,y)\in\mathcal{G}^{(2)} \\ xy=z}} \|f(x)\|\|g(y)\| = \sum_{(x,y)\in\mathcal{G}^{(2)}} \|f(x)\|\|g(y)\|$$
$$\leq \left(\sum_{x\in\mathcal{G}} \|f(x)\|\right)\left(\sum_{y\in\mathcal{G}} \|g(y)\|\right) = \|f\|_1\|g\|_1.$$

Moreover, for each $x \in G$ we find

$$\begin{aligned}
(f^*)^*(x) &= \tau(x,x^{-1})^{-1} M_x \left(\tau(x^{-1},x)^{-1} M_{x^{-1}}(f(x))^*\right)^* \\
&= \tau(x,x^{-1})^{-1} M_x\Big(\left(M_{x^{-1}}(f(x))\tau(x^{-1},x)\right)^* \Big)^* \\
&= \tau(x,x^{-1})^{-1} M_x M_{x^{-1}} (f(x))\, M_x \left(\tau(x^{-1},x)\right) \\
&\overset{((4.1))}{=} \tau(x,x^{-1})^{-1} M_x M_{x^{-1}} (f(x))\, \tau(x,x^{-1}) = M_{r(x)}(f(x)) = f(x).
\end{aligned}$$

Finally, for $z \in G$ a straightforward computation gives

$$\begin{aligned}
g^*f^*(z) &= \sum_{xy=z} \tau(x,x^{-1})^{-1} M_x \left(g(x^{-1})\right)^* M_x \left(\tau(y,y^{-1})^{-1} M_y \left(f(y^{-1})\right)^*\right) \tau(x,y) \\
&= \sum_{xy=z} \tau(x,x^{-1})^{-1} M_x \left(g(x^{-1})\right)^* M_x \left(M_y \left(f(y^{-1})\right) \tau(y,y^{-1})\right)^* \tau(x,y) \\
&= \sum_{xy=z} \tau(x,x^{-1})^{-1} \left(\tau(x,y)^{-1} M_x \left(M_y \left(f(y^{-1})\right) \tau(y,y^{-1})\right) M_x \left(g(x^{-1})\right) \right)^* \\
&= \sum_{xy=z} \tau(x,x^{-1})^{-1} \left(M_z \left(f(y^{-1})\right) \tau(x,y)^{-1} M_x \left(\tau(y,y^{-1})\right) M_x \left(g(x^{-1})\right) \right)^* \\
&\overset{((4.2))}{=} \sum_{xy=z} \tau(x,x^{-1})^{-1} \left(M_z \left(f(y^{-1})\right) \tau(z,y^{-1}) M_x \left(g(x^{-1})\right) \right)^* \\
&\overset{((4.4))}{=} \sum_{xy=z} \tau(x,x^{-1})^{-1} \left(M_z \left(f(y^{-1}) M_{y^{-1}} \left(g(x^{-1})\right)\right) \tau(z,y^{-1})\right)^* \\
&= \sum_{xy=z} \tau(x,x^{-1})^{-1} \tau(z,y^{-1})^{-1} M_z\left(f(y^{-1}) M_{y^{-1}} \left(g(x^{-1})\right) \right)^* \\
&\overset{((4.3))}{=} \sum_{xy=z} \tau(z,z^{-1})^{-1} M_z\left(f(y^{-1}) M_{y^{-1}} \left(g(x^{-1})\right) \tau(y^{-1},x^{-1})\right)^* \\
&= \tau(z,z^{-1})^{-1} M_z\Big(\sum_{xy=z} f(y^{-1}) M_{y^{-1}} \left(g(x^{-1})\right) \tau(y^{-1},x^{-1})\Big)^* \\
&= \tau(z,z^{-1})^{-1} M_z \left(fg(z^{-1})\right) = (fg)^*(z).
\end{aligned}$$

□

Definition 4.37. Let $\mathcal{G}$ be a groupoid, let $\mathcal{R}$ be a bundle of normed unital *-algebras over $\mathcal{G}^{(0)}$, and let (M,τ) be a *-factor system for $(\mathcal{G},\mathcal{R})$. The *$C^*$-algebra for* $(\mathcal{G},\mathcal{R},M,\tau)$ is the universal enveloping C*-algebra of the normed *-algebra $(\mathcal{R} \times_{(M,\tau)} \mathcal{G}, \|\cdot\|_1, ^*)$ and will be denoted by $C^*(\mathcal{G},\mathcal{R},M,\tau)$.

Example 4.38. Let $\mathcal{N} \to \mathcal{E} \xrightarrow{j} \mathcal{G}$ be a groupoid extension and let (M',τ') be the trivial factor system associated with $\mathcal{E}$ and the family $R_\lambda = \mathbb{C}$, $\lambda \in \Lambda$ (cf. item 2). Then $C^*(\mathcal{E},\mathcal{R}', M',\tau')$ is the well-known groupoid C^*-algebra of $\mathcal{E}$, $C^*(\mathcal{E})$.

Proposition 4.39. Under the hypotheses of Theorem 4.32 with $R_\lambda = \mathbb{C}$ for all $\lambda \in \Lambda$ we have that the map $\Phi : \mathbb{C}[\mathcal{E}] \to \mathbb{C}[\mathcal{N}] \times_{(M,\tau)} \mathcal{G}$ given by $\Phi(f)(x)(n) = f(nk(x))$ is an isometric *-homomorphism, and therefore the C^*-algebras $C^*(\mathcal{E})$ and $C^*(\mathcal{G},\mathbb{C}[\mathcal{N}],M,\tau)$ are isomorphic.

Proof. We already know from Theorem 4.32 that Φ is a ring homomorphism. Since it is obviously $\mathbb{C}$-linear, we are reduced to proving that Φ is isometric and $\Phi(f^*) = \Phi(f)^*$. Indeed, a short computation shows that

$$\begin{aligned}\|\Phi(f)\|_1 &= \sum_{x\in\mathcal{G}} \|\Phi(f)(x)\| = \sum_{x\in\mathcal{G}} \sum_{n\in\mathcal{N}_{r(x)}} |\Phi(f)(x)(n)| \\ &= \sum_{x\in\mathcal{G}} \sum_{n\in\mathcal{N}_{r(x)}} |f(nk(x)| = \sum_{z\in\mathcal{E}} |f(z)| = \|f\|.\end{aligned}$$

Moreover, for $x \in \mathcal{G}$ and $n \in N_{r(x)}$ we have

$$\begin{aligned}\Phi(f)^*(x)(n) &= \left(\tau(x,x^{-1})^{-1} M_x \left(\Phi(f)(x^{-1})\right)^*\right)(n) \\ &= M_x \left(\Phi(f)(x^{-1})\right)^* \left(\sigma(x,x^{-1})n\right) \\ &= \left(\Phi(f)(x^{-1})\right)^* \left(L_x^{-1}\left(\sigma(x,x^{-1})n\right)\right) \\ &= \overline{\Phi(f)(x^{-1})\left(L_x^{-1}\left(n^{-1}\sigma(x,x^{-1})^{-1}\right)\right)} \\ &= \overline{f\left(L_x^{-1}\left(n^{-1}\sigma(x,x^{-1})^{-1}\right)k(x^{-1})\right)} \\ &= \overline{f\left(k(x)^{-1}n^{-1}\sigma(x,x^{-1})^{-1}k(x)k(x^{-1})\right)} \\ &= \overline{f\left(k(x)^{-1}n^{-1}\right)} = \Phi(f^*)(x)(n).\end{aligned}$$

□

4.4 GROUPOID CROSSED PRODUCTS AND THEIR CLASSIFICATION

In this section we provide a classification theory for groupoid crossed products by using the techniques developed in Section 4.2. The proofs of our statements may be handled in the exact same way as the proofs of the respective statements in Section 4.2 and are therefore omitted for the sake of a concise presentation.

Throughout the following let $\mathcal{G}$ be a groupoid. Furthermore, let $\mathcal{R}$ be a unital ring bundle over $\mathcal{G}^{(0)}$ and let $\mathcal{R}^\times$ be the induced group bundle over $\mathcal{G}^{(0)}$ (cf. Section 4.1.4). We start with the following definition:

Definition 4.40 (cf. Definition 4.19)**.** We let $C^1(\mathcal{G},\mathcal{R}^\times)$ stand for the group of all maps $h : \mathcal{G} \to \mathcal{R}^\times$ satisfying $h(x) \in R^\times_{r(x)}$ for all $x \in \mathcal{G}$ and $h(u) = 1_{R_{r(u)}}$ for all $u \in \mathcal{G}^{(0)}$ with respect to the pointwise product.

Proposition 4.41 (cf. Proposition 4.20)**.** For $h \in C^1(\mathcal{G},\mathcal{R}^\times)$ and a factor system $(M,\tau) \in Z^2(\mathcal{G},\mathcal{R})$ we define

$$(h.M)_x(n) := h(x)M_x(n)h(x)^{-1}, \qquad x \in \mathcal{G}, n \in R_{s(x)}, \tag{4.11}$$

$$(h.\tau)(x,y) := h(x)M_x(h(y))\tau(x,y)h(xy)^{-1}, \qquad (x,y) \in \mathcal{G}^{(2)}. \tag{4.12}$$

Then $h.(M,\tau) := (h.M,h.\tau)$ is a factor system for $(\mathcal{G},\mathcal{R})$ and the map

$$\beta : C^1(\mathcal{G},\mathcal{R}) \times Z^2(\mathcal{G},\mathcal{R}) \to Z^2(\mathcal{G},\mathcal{R})$$

defines an action of $C^1(\mathcal{G},\mathcal{R})$ on $Z^2(\mathcal{G},\mathcal{R})$.

We call two factor systems (M,τ) and (M',τ') for $(\mathcal{G},\mathcal{R})$ *equivalent*, written with symbols $(M,\tau) \sim (M',\tau')$, if they are in the same orbit under the action β. We denote the corresponding orbit space of β by $Z^2(\mathcal{G},\mathcal{R})/C^1(\mathcal{G},\mathcal{R})$.

Proposition 4.42 (cf. Proposition 4.21)**.** For two factor systems $(M,\tau), (M',\tau') \in Z^2(\mathcal{G},\mathcal{R})$ the following conditions are equivalent:

1. $\mathcal{R} \times_{(M,\tau)} \mathcal{G}$ and $\mathcal{R} \times_{(M',\tau')} \mathcal{G}$ are equivalent.
2. $(M,\tau) \sim (M',\tau')$, i. e., there exists $h \in C^1(\mathcal{G},\mathcal{R}^\times)$ such that $(M',\tau') = h.(M,\tau)$.

If these conditions are satisfied, then the map

$$\psi : \mathcal{R} \times_{(M,\tau)} \mathcal{G} \to \mathcal{R} \times_{(M',\tau')} \mathcal{G}, \qquad (n,x) \mapsto (nh(x),x)$$

is an equivalence of $\mathcal{G}$-crossed products over $\mathcal{R}$ and, further, all equivalences of $\mathcal{G}$-crossed products over $\mathcal{R}$, $\mathcal{R} \times_{(M,\tau)} \mathcal{G} \to \mathcal{R} \times_{(M',\tau')} \mathcal{G}$, are of this form.

Corollary 4.43 (cf. Corollary 4.22)**.** The map $Z^2(\mathcal{G},\mathcal{R}) \to \mathrm{Ext}(\mathcal{G},\mathcal{R})$ sending (M,τ) to $[\mathcal{R} \times_{(M,\tau)} \mathcal{G}]$ induces a bijection $H^2(\mathcal{G},\mathcal{R}) := Z^2(\mathcal{G},\mathcal{R})/C^1(\mathcal{G},\mathcal{R}^\times) \to \mathrm{Ext}(\mathcal{G},\mathcal{R})$.

In accordance with Section 3 we say that an element $M \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{R}))$ is *outer* if there exists $\tau \in C^2(\mathcal{G},\mathcal{R}^\times)$ such that (M,τ) satisfies the twisted action condition (C1) and note that

$$M \sim M' \qquad \Longleftrightarrow \qquad \left(\exists h \in C^1(\mathcal{G},\mathcal{R}^\times)\right) \ \ M' = h.M$$

defines an equivalence relation on the set of all outer elements. Given an outer element $M \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{N}))$, we write $[M]$ for the equivalence class of M and call it a $\mathcal{G}$*-kernel*. Proposition 4.42 entails that $\mathcal{R} \times_{(M,\tau)} \mathcal{G} \sim \mathcal{R} \times_{(M',\tau')} \mathcal{G}$ implies $[M] = [M']$, i. e., equivalent $\mathcal{G}$-crossed products over $\mathcal{R}$ correspond to the same $\mathcal{G}$-kernel $[M]$. We denote by

$\mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]}$ the set of equivalence classes of $\mathcal{G}$-crossed products over $\mathcal{R}$ corresponding to the $\mathcal{G}$-kernel $[M]$. Moreover, we put

$$Z(\mathcal{R})^\times := \bigcup_{u\in\mathcal{G}^{(0)}} Z(R_u)^\times.$$

and consider the induced $\mathcal{G}$-module bundle $(Z(\mathcal{R})^\times, M)$ as well as its cohomology theory (cf. Section 4.1.2).

Theorem 4.44 (cf. Theorem 4.23)**.** Let $M \in C^1(\mathcal{G}, \mathrm{Iso}(\mathcal{R}))$ with $\mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]} \neq \emptyset$. Then the following assertions hold:

1. Each class in $\mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]}$ can be represented by one of the form $\mathcal{R} \times_{(M,\tau)} G$.
2. Let (M,τ') and (L,τ) be factor systems for $(\mathcal{G},\mathcal{R})$. Then $\tau^{-1}\cdot\tau' \in Z^2(\mathcal{G},Z(\mathcal{R})^\times)_M$, and moreover $(M,\tau') \sim (M,\tau)$ if and only if $\tau^{-1}\cdot\tau' \in B^2(\mathcal{G},Z(\mathcal{R})^\times)_M$.

Corollary 4.45 (cf. Corollary 4.24)**.** For a $\mathcal{G}$-kernel $[M]$ with $\mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]} \neq \emptyset$ the following map is a well-defined simply transitive action:

$$H^2(\mathcal{G},Z(\mathcal{R})^\times)_M \times \mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]} \to \mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]}, \quad \Big([\rho],[\mathcal{R}\times_{(M,\tau)}\mathcal{G}]\Big) \mapsto [\mathcal{R}\times_{(M,\tau\cdot\rho)}\mathcal{G}].$$

Theorem 4.46 (cf. Lemma 4.26 and Theorem 4.27)**.** Suppose that $(M,\tau),(M',\tau') \in C^1(\mathcal{G},\mathrm{Iso}(\mathcal{R}))\times C^2(\mathcal{G},\mathcal{R}^\times)$ satisfy the twisted action condition (C1) and that $M' \sim M$. Then $\xi_{(M,\tau)}$ and $\xi_{(M',\tau')}$ are cohomologous 3-cocycles in $Z^3(\mathcal{G},Z(\mathcal{R})^\times)_M$, where

$$\xi_{(M,\tau)}(x,y,z) := M_x(\tau(y,z))\tau(x,yz)\tau(xy,z)^{-1}, \qquad (x,y,z) \in \mathcal{G}^{(3)}.$$

Corollary 4.47 (cf. Corollary 4.28)**.** Suppose that $M \in C^1(\mathcal{G},\mathrm{Iso}(\mathcal{R}))$ is outer and choose $\tau \in C^2(\mathcal{G},\mathcal{R}^\times)$ such that (M,τ) satisfies the twisted action condition (C1). Then the cohomology class $[\xi_{(M,\tau)}] \in H^3(\mathcal{G},Z(\mathcal{R})^\times)_M$ does not depend on the choice of τ and is constant on the equivalence class $[M]$.

On account of Corollary 4.28, each outer element $L \in C^1(\mathcal{G},\mathrm{Iso}(\mathcal{N}))$ gives rise to a characteristic class $\xi(M) \in H^3(\mathcal{G},Z(\mathcal{R})^\times)_M$.

Corollary 4.48 (cf. Corollary 4.29)**.** For a $\mathcal{G}$-kernel $[M]$ we have $\mathrm{Ext}(\mathcal{G},\mathcal{R})_{[M]} \neq \emptyset$ if and only if the characteristic class $\xi(M) \in H^3(\mathcal{G},Z(\mathcal{R})^\times)_M$ is trivial.

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