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An equivalence of exact $\mathcal{C}$-module categories coming from internal Homs

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Orientadora: Profa. Virgínia Silva Rodrigues, Dra.

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O presente trabalho em nível de doutorado foi avaliado e aprovado, em 26 de julho de 2022, por banca examinadora composta pelos seguintes membros:

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Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de Doutor em Matemática com Área de Concentração em Álgebra.

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Este trabalho é dedicado a todos estudantes de
matemática.

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*“The mathematician’s patterns, like the
painter’s or the poet’s must be beautiful;
the ideas like the colours or the words, must fit
together in a harmonious way. Beauty is the first test:
there is no permanent place in the world for ugly mathematics.”
(G. H. HARDY, A Mathematician’s Apology, 1967)*

RESUMO

Neste trabalho desenvolvemos uma condição necessária e suficiente à definição de equivalência de categorias módulo quando estas são módulo exatas indecomponíveis sobre uma categoria tensorial finita $\mathcal{C}$. A existência de determinado isomorfismo natural e de equivalências de funtores de $\mathcal{C}$-módulos derivados de propriedades que envolvem Hom interno com outros resultados auxiliares são utilizados na demonstração. Um estudo detalhado das ferramentas usadas é dado.

Palavras-chave: Categoria Módulo. Adjunção. Funtores Representáveis. Lema de Yoneda. Hom Interno. Equivalência.

RESUMO EXPANDIDO

Introdução

A definição de uma categoria monoidal foi introduzida pela primeira vez no livro Natural Associativity and Commutativity de Saunders MacLane em 1963 (veja [12]), e pode ser pensada como uma “categorificação” (para uma melhor idéia do que significa categorificação com exemplos, recomendamos o excelente trabalho An Invitation to Categorification de Aaron Lauda e Joshua Sussan de 2022, veja [9]) da noção de um monóide, este que é um conjunto X equipado com uma operação binária associativa $(x, y) \mapsto x.y$, um elemento de identidade 1 satisfazendo $1.1 = 1$, e bijeções $1.x \mapsto x$ e $x.1 \mapsto x$ de X a X. Com esta mesma ideia, a noção de categoria tensorial pode ser pensada como uma categorificação da noção de anel, e a ideia de uma categoria módulo sobre uma categoria tensorial pode ser pensada como uma categorificação do conceito de módulo sobre um anel com unidade.

O conceito mais próximo da ideia de igualdade entre categorias é o conceito de *equivalência*. Por exemplo, duas categorias módulo $\mathcal{M}$ e $\mathcal{N}$ sobre $\mathcal{C}$ são ditas *equivalentes* se existirem dois funtores de $\mathcal{C}$-modulos $F : \mathcal{M} \to \mathcal{N}$ e $G : \mathcal{N} \to \mathcal{M}$, e um par de isomorfismos naturais de $\mathcal{C}$-modulos $G \circ F \to Id_{\mathcal{M}}$ e $Id_{\mathcal{N}} \to F \circ G$. Como podemos ver, pode ser uma tarefa bastante difícil verificar se duas categorias módulo sobre $\mathcal{C}$ são equivalentes. Nosso principal objetivo neste trabalho é reduzir esses requisitos sob certas condições e utilizando o conceito de adjunção.

O conceito de adjunção foi introduzido pela primeira vez por Daniel M. Kan em 1958 (ver [7]), e consiste em um par de funtores opostos que satisfazem uma relação. Um funtor $G : \mathcal{N} \to \mathcal{M}$ é *adjunto à esquerda* de um funtor $F : \mathcal{M} \to \mathcal{N}$ se existirem transformações naturais $e : G \circ F \to Id_{\mathcal{M}}$ e $c : Id_{\mathcal{N}} \to F \circ G$ satisfazendo duas condições. Neste caso, temos uma *adjunção* de $\mathcal{N}$ a $\mathcal{M}$.

No livro Category Theory in Context de Emily Riehl em 2016 (veja [20]), ela demonstrou que sempre que dois funtores formam uma equivalência de categorias, eles são adjuntos à esquerda e à direita um do outro. Observe que a definição de uma adjunção é, de certa forma, mais fraca do que a definição de equivalência de categoria.

Objetivos

Neste trabalho desenvolvemos uma condição necessária e suficiente à definição de equivalência de categorias módulo quando estas são módulo exatas indecomponíveis sobre uma categoria tensorial finita $\mathcal{C}$. Nosso objetivo aqui é fornecer um método alternativo (e com menos requerimentos) para verificar se duas categorias módulo exatas indecomponíveis $\mathcal{M}$ e $\mathcal{N}$ sobre uma categoria tensorial finita $\mathcal{C}$ são equivalentes usando a existência de um certo isomorfismo. Isto é, $\mathcal{M}$ e $\mathcal{N}$ são equivalentes como categorias módulo sobre $\mathcal{C}$ se, e somente se, existir um funtor de $\mathcal{C}$-módulos $F : \mathcal{M} \to \mathcal{N}$ admitindo um adjunto $G : \mathcal{N} \to \mathcal{M}$ e um objeto diferente de zero $M \in \mathcal{M}$ (ou $N \in \mathcal{N}$) tal que $e_M : G(F(M)) \to M$ (ou $c_N : N \to F(G(N))$) seja um isomorfismo. Para conseguirmos chegar em tal resultado, usamos fortemente o conceito de Hom interno, assim como certos isomorfismos naturais envolvendo estes funtores. Como aplicação, apresentamos um exemplo que usa o teorema principal deste trabalho considerando as categorias e condições presentes no Theorem 3.8 em [19].

Metodologia
Pesquisa bibliográfica, artigos publicados em jornais conceituados, discussões frequentes sobre os objetivos e resultados já obtidos com a orientadora, bem como os problemas a serem resolvidos e dificuldades encontradas. Foi feito um estudo detalhado das ferramentas utilizadas neste trabalho, em especial o conceito de Hom interno.

Resultados e Discussão
Achamos interessante fornecer as demonstrações de vários resultados aqui. As principais razões são que obtivemos uma demonstração ligeiramente diferente ou apenas o fato da demonstração ser bastante difícil de ser encontrada. Houveram também demonstrações que não encontramos na literatura (mas pode estar presente em algum lugar) e outros alguns resultados menores que usamos aqui e que acreditamos que não tenham sido enunciados antes.

Considerações Finais
Os resultados desejados foram obtidos. À medida que generalizamos alguns conceitos, claramente perdemos alguns resultados, mas conseguimos mesmo assim alcançar todos nossos objetivos sem ter de usar a hipótese de semissimplicidade nas categorias envolvidas. Nas nossas hipóteses do teorema principal, basta que as categorias módulos sejam exatas indecomponíveis sobre uma categoria tensorial finita $\mathcal{C}$, ou seja, não precisamos impor que as categorias módulo sejam semissimples, ou que $\mathcal{C}$ seja semissimples (ou de fusão) que são conceitos mais restritos que o conceito de exatas para as categorias módulo e tensorial finita para $\mathcal{C}$. De fato, toda categoria módulo que é semissimples é exata, mas o contrário nem sempre é verdade. Com nosso teorema principal, facilitamos o processo de se verificar quando duas categorias módulo exatas indecomponíveis sobre uma categoria tensorial finita são ou não equivalentes. Este trabalho pode ser visto também como a criação dessa ferramenta, abrindo muitas portas para aplicações como, por exemplo, auxiliando na classificação de categorias módulo exatas e, também, na classificação de categorias módulo semissimples.

Palavras-chave: Categoria Módulo. Adjunção. Funtores Representáveis. Lema de Yoneda. Hom Interno. Equivalência.

ABSTRACT

In this work we provide a necessary and sufficient condition for the definition of module category equivalence when these are exact indecomposable module categories over a finite tensor category $\mathcal{C}$. The existence of a certain natural isomorphism and $\mathcal{C}$-module functor equivalences derived from properties coming from internal Homs with other auxiliary results are used in its proof. A detailed study of the tools used to achieve this is given.

Keywords: Module Category. Adjunction. Representable Functor. Yoneda Lemma. Internal Hom. Equivalence.

CONTENTS

INTRODUCTION

The definition of a monoidal category was first introduced in the book Natural Associativity and Commutativity by Saunders MacLane in 1963 (see [12]), and it can be thought as a "categorification"[1] of the notion of a monoid which is a set X equipped with an associative binary operation $(x, y) \mapsto x.y$, an identity element 1 satisfying $1.1 = 1$, and bijections $1.x \mapsto x$ and $x.1 \mapsto x$ from X to X. With this same idea, the notion of a tensor category may be thought as a categorification of the notion of a ring, and the idea of a module category over a tensor category can be thought as a categorification of the concept of a module over a ring with unity.

"Category theory takes a bird's eye view of mathematics.
From high in the sky, details become invisible, but we can spot
patterns that were impossible to detect from ground level."
(Tom Leinster in the page 1 of [10], 2014)

The closest one can reach in terms of the idea of equality between categories is through the concept of *equivalence*. For instance, two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ are *equivalent* if there are two $\mathcal{C}$-module functors $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{M}$, and a pair of natural isomorphisms of $\mathcal{C}$-module functors $G \circ F \to Id_{\mathcal{M}}$ and $Id_{\mathcal{N}} \to F \circ G$. As we can see, it may be quite a hard task to check whether two $\mathcal{C}$-module categories are equivalent. Our main objective is to reduce these requirements under certain conditions.

The concept of an adjunction was firstly introduced by Daniel M. Kan in 1958 (see [7]), and it consists of a pair of opposing functors satisfying a relation. A functor $G : \mathcal{N} \to \mathcal{M}$ is *left adjoint* to a functor $F : \mathcal{M} \to \mathcal{N}$ if there are natural transformations $e : G \circ F \to Id_{\mathcal{M}}$ and $c : Id_{\mathcal{N}} \to F \circ G$ satisfying two conditions. In this case, we have an *adjunction* from $\mathcal{N}$ to $\mathcal{M}$.

In the book Category Theory in Context by Emily Riehl in 2016 (see [20]) she proved that whenever two functors form an equivalence of categories, they are left and right adjoint to each other. Notice that the definition of an adjunction is, in a certain way, weaker than the definition of a category equivalence.

Our objective here is to provide a different method for checking whether two exact indecomposable module categories $\mathcal{M}$ and $\mathcal{N}$ over a finite tensor category $\mathcal{C}$ are equivalent by using the existence of a certain isomorphism. Namely, $\mathcal{M}$ and $\mathcal{N}$ are equivalent as $\mathcal{C}$-module categories if, and only if, there exists a $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{N}$ admitting a left adjoint $G : \mathcal{N} \to \mathcal{M}$ and a nonzero object $M \in \mathcal{M}$ (or $N \in \mathcal{N}$) such that $e_M : G(F(M)) \to M$ (or $c_N : N \to F(G(N))$) is an isomorphism.

We then divided this work in six chapters as follows. In the first chapter we introduce basic definitions and properties regarding abelian categories, natural trans-

[1] For a better idea of what categorification means with examples, we recommend the excellent work An Invitation to Categorification by Aaron Lauda and Joshua Sussan in 2022, see [9].

formations, exact sequences, equivalences, adjunctions and exact functors. We tried to place these definitions close to each other to make it easier to find while reading the following chapters.

In Chapter 2 we present the concepts of monoidal, rigid, multitensor, tensor and fusion categories, also letting these definitions close together to make it easier to find them later. Some adjunctions involving the tensor functor and module product are also provided, as well as functors and natural transformations in the context of module categories. We show that the left and right adjoint of a $\mathcal{C}$-module functor admits a $\mathcal{C}$-module functor structure under certain hypothesis. At last, the notion of an exact module category over multitensor categories that will be used in Chapter 4 is given.

In the third chapter the notions of representable functors, the Yoneda Lemma (for the contravariant case) and universal elements, as well as a condition for a functor to be representable are introduced. We see that there is a certain one-to-one correspondence between representable functors and universal elements which is strongly used in the chapters to follow.

The chapter four contains the study of the internal Hom (bi)functor $\underline{Hom}(\ \_\ ,\ \_\ )$ which is largely used in the next chapter and in results to follow. We begin by defining the internal Hom object which is an object that represents a certain contravariant functor, and with this object we then define the internal Hom functor. This functor admits a $\mathcal{C}$-module functor structure and, if the category $\mathcal{M}$ is exact, it is an exact functor.

The chapter 5 begins with the notions of algebra and module over an algebra in the category context (for a monoidal category $\mathcal{C}$), and the category of the right A-modules over an algebra A (denoted by $\mathcal{C}_A$) which has a structure of left $\mathcal{C}$-module category. Later we give an algebra structure to the internal Hom object $\underline{Hom}(M, M)$ (for all $M \in \mathcal{M}$) and then construct a $\mathcal{C}$-module functor F from $\mathcal{M}$ to $\mathcal{C}_{\underline{Hom}(M,M)}$ which inherits many properties of the functor $\underline{Hom}(M,\ \_\ )$. At last, we present a result stating that F is an equivalence of $\mathcal{C}$-module categories under certain conditions.

In the last chapter we present the main result of this work. It uses a natural isomorphism present in Lemma 2 of [5] together with the equivalence F defined in the previous chapter to give an alternative (and arguably easier) way to verify whether two exact indecomposable $\mathcal{C}$-module categories are equivalent. Finally, an application of this theorem using the Theorem 3.8 in [19] is given.

We found it interesting to provide the proofs of a good few results here, and the main reasons are that we got a slightly different proof or just the proof was fairly difficult to find. There were proofs that we couldn't find in the literature (but it may be present in somewhere) and other minor results we used here which we believe were not stated before. For instance, in order of appearance,

a) every pair of functors that make an equivalence of categories are left and right adjoint to each other. This is Proposition 4.4.5 in [20] and Proposition 1.3.9 here;

b) given an adjunction between additive categories, the isomorphisms that define this adjunction are group isomorphisms. This is item (ii) of Proposition 1.3.10;

c) an additive functor between abelian categories takes a splitting short exact sequences into a splitting short exact sequence. This is Proposition 1.4.4;

d) a right (left) adjoint to any functor between abelian categories is left (right) exact. This is Proposition 1.4.5;

e) an equivalence between two definitions of equivalence of $\mathcal{C}$-module categories in Proposition 2.2.12;

f) the adjoint of a $\mathcal{C}$-module functor admits a $\mathcal{C}$-module functor structure. This is Theorem 2.3.2;

g) there is a one-to-one correspondence between representation of functors and universal elements, Proposition 3.4;

h) $\underline{Hom}(\_\,,\,\_) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is a bifunctor is Proposition 4.1.2;

i) the bifunctor $\underline{Hom}(\_\,,\,\_) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is left biexact is Proposition 4.2.3;

j) some properties of the internal Hom object and functor for a locally finite and exact indecomposable module category over a finite tensor category $\mathcal{C}$ in Lemma 4.4.2, and that the morphism in a universal element is an epimorphism in Lemma 4.4.3;

k) Propositions 5.1.9 and 5.1.10 regarding Morita equivalent algebras in a category $\mathcal{C}$, and objects in a category $\mathcal{C}_A$, respectively;

l) an universal element of the representable functor $Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, X \overline{\otimes} N)$ with Lemma 5.3.1;

m) a $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{C}_{\underline{Hom}(M,M)}$ in Proposition 5.3.2;

n) an equivalence between $\mathcal{M}$ and $\mathcal{C}_{\underline{Hom}(M,M)}$. This is Theorem 7.10.1 in [4] and Theorem 5.4.1 here;

o) a certain natural isomorphism present in [5] as Lemma 2. This is Lemma 6.2 here.

The result in a) is present in not so many references[2], and o) we included the proof since we use an explicit description of certain isomorphisms in our main theorem. A slight different proof is given for the result (with minor changes in the hypothesis) in n). The result in c) is (implicitly) used within the proof of Proposition 7.6.9 in [4], so we made a Proposition for it. The results in d), f), g), h), i) and m) are known results but we

2 Indeed, we've only found it in [20].

could not find the proof for them or they are hard to find. The results in b), e), j), k) and l) were created in order to reach our goal and we believe they are not present in the literature.

Throughout this work, $\Bbbk$ is a field and for a category $\mathcal{C}$ the notation $X \in \mathcal{C}$ will mean that X is an object of $\mathcal{C}$. Every category considered here are locally small, that is, for any objects X and $Y \in \mathcal{C}$, the collection of morphisms from X to Y, denoted by $Hom_{\mathcal{C}}(X, Y)$, is a set. A morphism $f \in Hom_{\mathcal{C}}(X, Y)$ can be denoted either by $f : X \to Y$ or $X \xrightarrow{f} Y$. The notation id_X is used to denote the identity morphism of an object $X \in \mathcal{C}$ and $Id_{\mathcal{C}}$ to denote the identity functor from $\mathcal{C}$ to $\mathcal{C}$.

1 BASIC NOTIONS

In this chapter we will briefly recall some basic definitions and results involving category theory. These can be found in [2], [4], [6], [15], [17] and [18] for example. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories.

1.1 KERNELS, COKERNELS AND ABELIAN CATEGORIES

Here we remember some notions regarding basic category theory and then present to the reader the definitions of abelian, $\Bbbk$-linear, locally finite and finite categories (among others) that will be used in the entire work. We found it easier to put these definitions together in one place to help finding them whenever necessary.

Definition 1.1.1. *Let X, Y and Z be objects in $\mathcal{C}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be*

(i) *a monomorphism if for any pair of morphisms $g, h : Z \to X$ in $\mathcal{C}$ such that $f \circ g = f \circ h$ implies $g = h$;*

(ii) *an epimorphism if for any pair of morphisms $g, h : Y \to Z$ in $\mathcal{C}$ such that $g \circ f = h \circ f$ implies $g = h$;*

(iii) *an isomorphism if there exists a morphism $g : Y \to X$ in $\mathcal{C}$ satisfying $f \circ g = id_Y$ and $g \circ f = id_X$. In this case, the object X is said to be isomorphic to Y and it's denoted by $X \cong Y$.*

Definition 1.1.2. *Let Y be an object in $\mathcal{C}$. Then*

(i) *a subobject of Y is a pair (X, ι) where X is an object in $\mathcal{C}$ and $\iota : X \to Y$ is a monomorphism in $\mathcal{C}$, and it's denoted by $X \subseteq Y$. A quotient object of Y is a pair (Z, π) where $Z \in \mathcal{C}$ and $\pi : Y \to Z$ is an epimorphism in $\mathcal{C}$;*

(ii) *two subobjects (X_1, ι_1) and (X_2, ι_2) of Y are said to be equal as subobjects of Y if there is an isomorphism $u : X_1 \to X_2$ in $\mathcal{C}$ satisfying $\iota_2 \circ u = \iota_1$, where $\iota_j : X_j \to Y$ ($j = 1, 2$) are said to be equivalent monomorphisms;*

(iii) *two quotient objects (Z_1, π_1) and (Z_2, π_2) of Y are said to be equal as quotient objects of Y if there is an isomorphism $v : Z_1 \to Z_2$ in $\mathcal{C}$ satisfying $v \circ \pi_1 = \pi_2$, where $\pi_j : Y \to Z_j$ ($j = 1, 2$) are said to be equivalent epimorphisms.*

Sometimes we just say that X is a subobject of Y and Z is a quotient object of Y, omitting the morphisms of their respective pairs. Moreover, we say that X_1 and X_2 are equal as subobjects of Y, and Z_1 and Z_2 are equal as quotient objects of Y also omitting the morphisms of their respective pairs.

For a subobject $X \subseteq Y$, the quotient object $Z = Y/X$ is defined to be the cokernel of the monomorphism $X \to Y$. This notation is used when we introduce the notion of filtration and the Jordan-Hölder composition series of an object.

It's possible to define a mathematical object zero in the context of category theory which will be unique up to isomorphism.

Definition 1.1.3. *An object $Z \in \mathcal{C}$ is called a zero object if for any $X \in \mathcal{C}$, there are unique morphisms $\phi_X : X \to Z$ and $\psi_X : Z \to X$, i.e., $Hom_{\mathcal{C}}(X, Z) = \{\phi_X\}$ and $Hom_{\mathcal{C}}(Z, X) = \{\psi_X\}$.*

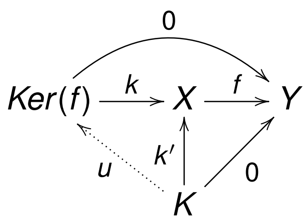
Lemma 1.1.4. *If $\mathcal{C}$ admits a zero object then it is unique up to isomorphism, and it's denoted by $0_{\mathcal{C}}$ or simply by 0.*

Moreover, there is also the notion of zero morphism in a category that admits a zero object. Suppose that our category $\mathcal{C}$ has a zero object and let X and Y be objects of $\mathcal{C}$. The *zero morphism from X to Y*, which is denoted by 0^X_Y (or simply by 0 when the source and target are implicit in the context) is defined by $0^X_Y := \psi_Y \circ \phi_X$, and it does not depend on the zero object of the category.

Definition 1.1.5. *A nonzero object $X \in \mathcal{C}$ is called simple if 0 and X are its only subobjects.*

Definition 1.1.6. *Let $\mathcal{C}$ be a category with a zero object and $f \in Hom_{\mathcal{C}}(X, Y)$.*

The kernel of f (if it exists) is a pair $(Ker(f), k)$ where $Ker(f)$ is an object of $\mathcal{C}$ and $k : Ker(f) \to X$ is a morphism in $\mathcal{C}$ such that $f \circ k = 0$, and if K is an object of $\mathcal{C}$ and $k' : K \to X$ is a morphism in $\mathcal{C}$ satisfying $f \circ k' = 0$ then there exists an unique morphism $u : K \to Ker(f)$ in $\mathcal{C}$ such that $k \circ u = k'$, that is, the diagram



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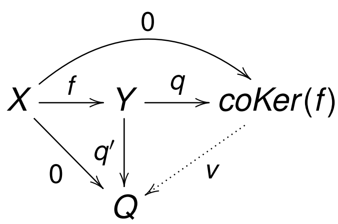
Dually, the cokernel of f (if it exists) is a pair $(coKer(f), q)$ where $coKer(f)$ is an object of $\mathcal{C}$ and $q : Y \to coKer(f)$ is a morphism in $\mathcal{C}$ such that $q \circ f = 0$, and if Q is an object of $\mathcal{C}$ and $q' : Y \to Q$ is a morphism in $\mathcal{C}$ satisfying $q' \circ f = 0$ then there exists an unique morphism $v : coKer(f) \to Q$ in $\mathcal{C}$ such that $v \circ q = q'$, that is, the diagram



commutes.

As we can see, the kernel and cokernel of a morphism, when they exist, are pairs containing an object and a morphism in the category. Sometimes we write just the morphism to denote the pair since in the case of kernel, its source (domain) is the object of the pair, and analogously, in the case of cokernel its target (codomain) is the object of the pair.

It's well known that every kernel is a monomorphism and every cokernel is an epimorphism.

We can see by this next proposition that the kernel (cokernel) of a morphism, when it exists, is unique up to monomorphism (epimorphism) equivalence.

Proposition 1.1.7. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$ with kernel $(Ker(f), k)$ and cokernel $(coKer(f), q)$. A subobject (K', k') of X is a kernel of f if, and only if, $(Ker(f), k) = (K', k')$ as subobjects of X. A quotient object (Q', q') of Y is a cokernel of f if, and only if, $(coKer(f), q) = (Q', q')$ as quotient objects of Y.*

Proof. The "if" implications are done in [18] as Proposição 1.1.9. For the converse we'll show that (K', k') is a kernel of f. Since $(K', k') = (Ker(f), k)$ as subobjects of X, it follows that there exists an isomorphism $u : K' \to Ker(f)$ in $\mathcal{C}$ such that $k \circ u \overset{(*)}{=} k'$. In fact, k' is a morphism satisfying $f \circ k' = f \circ k \circ u = 0$.

Now, let (K'', k'') where K'' is an object in $\mathcal{C}$ and $k'' : K'' \to X$ satisfying $f \circ k'' = 0$. We're going to verify that there is an unique morphism $v : K'' \to K'$ in $\mathcal{C}$ such that $k' \circ v = k''$. Again, considering that k is a kernel of f, there is an unique morphism $u'' : K'' \to Ker(f)$ in $\mathcal{C}$ satisfying $k \circ u'' \overset{(**)}{=} k''$.

The morphism $v : K'' \to K'$ defined as $v := u^{-1} \circ u''$ satisfies $k' \circ v = k' \circ u^{-1} \circ u'' \overset{(*)}{=} k \circ u'' \overset{(**)}{=} k''$. Lastly, we're going to check its uniqueness. For this, let $v' : K'' \to K'$ be a morphism in $\mathcal{C}$ satisfying $k' \circ v' = k''$. Then $k' \circ v = k' \circ v'$ which implies $v = v'$ (since k' is a monomorphism) as wanted. Hence, (K', k') is a kernel of f.

In a similar way it's possible to prove the converse for the other case involving cokernels. ■

For this reason, from now on we just say that a pair is *the* kernel (*the* cokernel) of a morphism, when it exists, even knowing it's just unique up to monomorphism (epimorphism) equivalence.

Remark 1.1.8 ([15], Ejercicio 2.7.9)**.** *If Z is an object in $\mathcal{C}$ and $id_Z = 0$ then $Z = 0$. This follows from the fact that both $Hom_{\mathcal{C}}(Z, X)$ and $Hom_{\mathcal{C}}(X, Z)$ are unitary for all $X \in \mathcal{C}$. Indeed, if $f \in Hom_{\mathcal{C}}(Z, X)$ then $f = f \circ id_Z = f \circ 0 = 0$, i.e., $Hom_{\mathcal{C}}(Z, X) = \{0\}$. Similarly one can prove that $Hom_{\mathcal{C}}(X, Z) = \{0\}$. Thus Z is the zero object of $\mathcal{C}$.*

This next lemma is used in the proof of quite a few results in this work. It's not so difficult to find authors using some of them without making any mention.

Lemma 1.1.9. *Let f be a morphism in $Hom_{\mathcal{C}}(X, Y)$ with kernel $k : Ker(f) \to X$ and cokernel $q : Y \to coKer(f)$. If*

(i) *$g \in Hom_{\mathcal{C}}(Y, Z)$ is a monomorphism then $k : Ker(f) \to X$ is also the kernel of $g \circ f$;*

(ii) *$z \in Hom_{\mathcal{C}}(W, X)$ is an isomorphism then $z^{-1} \circ k : Ker(f) \to W$ is the kernel of $f \circ z$;*

(iii) *$h \in Hom_{\mathcal{C}}(Z', X)$ is an epimorphism then $q : Y \to coKer(f)$ is also the cokernel of $f \circ h$;*

(iv) *$w \in Hom_{\mathcal{C}}(Y, W')$ is an isomorphism then $q \circ w^{-1} : W' \to coKer(f)$ is the cokernel of $w \circ f$;*

(v) *$z' \in Hom_{\mathcal{C}}(Z'', Ker(f))$ is an isomorphism then $k \circ z' : Z'' \to X$ is the kernel of f;*

(vi) *$w' \in Hom_{\mathcal{C}}(coKer(f), W'')$ is an isomorphism then $w' \circ q : Y \to W''$ is the cokernel of f.*

Proof. From the definition of kernel of f we know that if $k' : K' \to X$ is a morphism in $\mathcal{C}$ satisfying $f \circ k' = 0$ then there is an unique $u \in Hom_{\mathcal{C}}(K', Ker(f))$ satisfying $k \circ u = k'$.

(i) Firstly, we can notice that $(g \circ f) \circ k = g \circ (f \circ k) = g \circ 0 = 0$. Now let $k'' \in Hom_{\mathcal{C}}(K'', X)$ be a morphism such that $(g \circ f) \circ k'' = 0$. Since g is a monomorphism it follows that $f \circ k'' = 0$ and this implies that there is an unique $u' \in Hom_{\mathcal{C}}(K'', Ker(f))$ satisfying $k \circ u' = k''$. Hence $k : Ker(f) \to X$ is the kernel of $g \circ f$.

(ii) We can easily notice that $(f \circ z) \circ (z^{-1} \circ k) = f \circ k = 0$. Now, let $k'' \in Hom_{\mathcal{C}}(K'', W)$ be a morphism satisfying $(f \circ z) \circ k'' = 0$. It remains to show that there is an unique $u' \in Hom_{\mathcal{C}}(K'', Ker(f))$ such that $(z^{-1} \circ k) \circ u' = k''$. This can be done by considering $k' = z \circ k''$ in the definition of kernel of f. In fact, it follows that there exists an unique $u' \in Hom_{\mathcal{C}}(K'', Ker(f))$ such that $k \circ u' = z \circ k''$, as wanted.

The items (iii) and (iv) can be shown in a similar manner. The item (v) follows immediately from the fact that the pairs $(Ker(f), k)$ and $(Z'', k \circ z')$ are equal as subobjects of X. Therefore, $(Z'', k \circ z')$ is the kernel of f (by Proposition 1.1.7). The item (vi) can be checked analogously. ∎

One thing we can notice from this lemma is that the kernel and cokernel objects and morphisms of their respective pairs are still the same in the items (i) and (iii). However, in the items (ii) and (iv) the objects of their respective pairs remains the same, while the morphisms of the pairs change slightly.

Lemma 1.1.10 ([18], Lema 1.1.10)**.** *Let $f \in Hom_{\mathcal{C}}(X, Y)$ be a morphism, $k : Ker(f) \to X$ its kernel and $q : Y \to coKer(f)$ its cokernel. The following are equivalent:*

(i) *$(Ker(f), k) = (X, id_X)$ as subobjects of X;*

(ii) *k is an isomorphism;*

(iii) *$f = 0$;*

(iv) *$(coKer(f), q) = (Y, id_Y)$ as quotient objects of Y;*

(v) *q is an isomorphism.*

We're now going to introduce some important definitions for the work ahead. These definitions were gathered together to make it easier to find them.

Definition 1.1.11 ([4], Definition 1.5.3)**.** *The object X is said to have finite length if there exists a filtration*

$$0 = X_0 \subseteq X_1 \subseteq \cdots X_{n-1} \subseteq X_n = X$$

such that X_{i+1}/X_i is simple for all i. Such a filtration is called a Jordan-Hölder series of X. We say that this Jordan-Hölder series contain a simple object Y with multiplicity m if the number of values of i for which $X_{i+1}/X_i \cong Y$ is m, and it is denoted by $[X : Y] = m$.

The simple objects X_{i+1}/X_i of the definition above are called composition factors of X.

Theorem 1.1.12 ([4], Theorem 1.5.4 - Jordan-Hölder Theorem)**.** *Suppose that X has finite length. Then any filtration of X can be extended to a Jordan-Hölder series, and any two Jordan-Hölder series of X contain any simple object with the same multiplicity, so in particular have the same length.*

Definition 1.1.13 ([4], Definition 1.5.5)**.** *The length of an object X in $\mathcal{C}$ is the length of its Jordan-Hölder series (if it exists).*

Definition 1.1.14 ([15], Definición 2.7.48)**.** *An object $P \in \mathcal{C}$ is said to be projective if for any epimorphism $\pi : X \to Y$ in $\mathcal{C}$ and for all morphism $f : P \to Y$ in $\mathcal{C}$, there exists a morphism $g : P \to X$ in $\mathcal{C}$ satisfying $\pi \circ g = f$.*

Definition 1.1.15 ([4], Definition 1.6.6)**.** *Let $X \in \mathcal{C}$. A projective cover of X is a projective object $P(X) \in \mathcal{C}$ with an epimorphism $p : P(X) \to X$ such that if $g : P \to X$ is an epimorphism from a projective object P to X, then there exists an epimorphism $h : P \to P(X)$ satisfying $p \circ h = g$.*

Definition 1.1.16. *We say that a category $\mathcal{C}$ is*

a) *pre-additive, if it has a zero object, the collection $Hom_{\mathcal{C}}(X, Y)$ has a structure of abelian group for any $X, Y \in \mathcal{C}$ and the morphism composition in $\mathcal{C}$ is bilinear, i.e., $g \circ (f + f') = g \circ f + g \circ f'$ and $(g + g') \circ f = g \circ f + g' \circ f$, whenever these compositions are possible;*

b) *additive, if it is pre-additive and every pair of objects* $X_1, X_2 \in \mathcal{C}$ *has a direct sum, that is, a collection* $(Z, \pi_1, \pi_2, \iota_1, \iota_2)$ *such that* $\pi_1 : Z \to X_1$, $\pi_2 : Z \to X_2$, $\iota_1 : X_1 \to Z$ *and* $\iota_2 : X_2 \to Z$ *are morphisms in* $\mathcal{C}$ *satisfying* $\pi_1 \circ \iota_1 = id_{X_1}$, $\pi_2 \circ \iota_2 = id_{X_2}$ *and* $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = id_Z$.

   Considering that the direct sum of two objects is unique up to isomorphism, we may write $Z = X_1 \oplus X_2$. *It's easy to see that* π_1 *and* π_2 *are epimorphisms, while* ι_1 *and* ι_2 *are monomorphisms, and* $\pi_i \circ \iota_j = 0$ *whenever* $i \neq j$*;*

c) *abelian, if it is additive, every morphism has kernel and cokernel, every monomorphism is a kernel and every epimorphism is a cokernel.*

d) *semisimple, if it is abelian and every object* X *in* $\mathcal{C}$ *is semisimple, that is,* X *is a direct sum of simple objects;*

e) $\Bbbk$*-linear, if it is additive and the abelian group* $Hom_{\mathcal{C}}(X, Y)$ *admits a* $\Bbbk$*-linear vector space structure for any* $X, Y \in \mathcal{C}$ *such that the composition of morphisms is* $\Bbbk$*-bilinear, i.e., it's bilinear and* $k(f \circ g) = kf \circ g = f \circ kg$ *for all* $k \in \Bbbk$ *and whenever this composition is possible;*

f) *locally finite, if it is abelian,* $\Bbbk$*-linear, any object of* $\mathcal{C}$ *has finite length, and* $Hom_{\mathcal{C}}(X, Y)$ *is a vector space with* $dim_{\Bbbk}(Hom_{\mathcal{C}}(X, Y)) < \infty$ *for every* $X, Y \in \mathcal{C}$*;*

g) *finite, if it is locally finite,* $\mathcal{C}$ *has enough projectives, i.e., every simple object of* $\mathcal{C}$ *has a projective cover, and the number of isomorphism classes of simple objects is finite.*

One can easily notice that in any category an isomorphism is both a monomorphism and an epimorphism. An important result regarding abelian categories is that the converse of this statement is valid when the category is abelian.

Proposition 1.1.17 ([15], Corolario 2.8.8)**.** *A morphism in an abelian category* $\mathcal{C}$ *is a monomorphism and an epimorphism if, and only if, it is an isomorphism.*

Remark 1.1.18. *Every morphism in an abelian category admits a decomposition. In fact, let* $f : X \to Y$ *be a morphism in* $\mathcal{C}$*,* $(coKer(f), q)$ *be its cokernel and consider* $(Ker(q), k')$ *be the kernel of* q*. Since* k' *is the kernel of* q *and* $q \circ f = 0$*, it follows that there exists an unique morphism* $u : X \to Ker(q)$ *in* $\mathcal{C}$ *such that* $f = k' \circ u$*.*

Similarly, one can check that if $(Ker(f), k)$ *is the kernel of* f *and* $(coKer(k), q')$ *is the cokernel of* k *then there exists an unique morphism* $v : coKer(k) \to Y$ *in* $\mathcal{C}$ *satisfying* $f = v \circ q'$*.*

Furthermore, u *is an epimorphism and* v *is a monomorphism* (see [1], Corolário 2.3.10).

Definition 1.1.19. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $X, Y \in \mathcal{C}$.*

(i) *If $\mathcal{C}$ and $\mathcal{D}$ are additive categories then the functor F is called additive if*

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of abelian groups, that is, $F(f + g) = F(f) + F(g)$ for every pair of morphisms f and g in $Hom_{\mathcal{C}}(X, Y)$;

(ii) *If $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{k}$-linear categories then F is said to be $\mathbb{k}$-linear if*

$$F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$

is a $\mathbb{k}$-linear transformation of vector spaces, that is, for all $f, g \in Hom_{\mathcal{C}}(X, Y)$ and $k \in \mathbb{k}$ we have $F(f + kg) = F(f) + kF(g)$.

As an observation, in order to define additive functors we do not necessarily need the categories involved being additive. In fact, it's enough for them just being pre-additives.

Remark 1.1.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. Any additive functor $F : \mathcal{C} \to \mathcal{D}$ satisfies $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$. In fact, $F : \mathcal{C} \to \mathcal{D}$ being additive implies that it preserves products* (via [21], Proposição 3.19)*. Therefore, $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$* (see [21], Lema 3.17)*.*

1.2 EXACT SEQUENCES IN ABELIAN CATEGORIES AND COMPOSITION SERIES

In this section we'll be studying some well-known results about exact sequences in an abelian category $\mathcal{C}$. We start by introducing some definitions and results that are important for our objective.

Definition 1.2.1. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$ with cokernel $(coker(f), q)$, and $(Ker(q), k')$ be the kernel of q. The image of f, which is denoted by $Im(f)$, is the kernel of q, that is, the subobject $(Ker(q), k')$ of Y.*

As we could see in this definition, the image of a morphism is the kernel of its cokernel.

Definition 1.2.2. *Let $f \in Hom_{\mathcal{C}}(X, Y)$ and $g \in Hom_{\mathcal{C}}(Y, Z)$ be morphisms in $\mathcal{C}$. The sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact in Y if $Im(f) = Ker(g)$ as subobjects of Y.

A sequence

$$0 \longrightarrow X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow 0$$

is said to be exact if it is exact in X_i *for all* $i \in \{1, 2, \cdots, n\}$.

An exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is called a short exact sequence.

We say that a short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *splits if there is a morphism* $\iota : Z \to Y$ *in* $\mathcal{C}$ *such that* $g \circ \iota = id_Z$.

This following remark is part of the Proposición 2.7.49 in [15].

Remark 1.2.3. *It follows directly from the definition of split sequence that, if* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} P \longrightarrow 0$ *is a short exact sequence in* $\mathcal{C}$ *with the object* P *being projective then this sequence splits. In fact, by considering the morphism* $id_P : P \to P$ *in the definition of projective object, it follows that there is a morphism* $h : P \to Y$ *in* $\mathcal{C}$ *satisfying* $g \circ h = id_P$, *that is, the short exact sequence* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} P \longrightarrow 0$ *splits.*

Now we remember some other useful results regarding exact sequences.

Proposition 1.2.4 ([15], Ejercicio 2.8.13)**.** *Let* $\mathcal{C}$ *be an abelian category and*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

a short exact sequence in $\mathcal{C}$. *The following are equivalent:*

(i) *this short exact sequence splits;*

(ii) *there is a morphism* $\pi : Y \to X$ *in* $\mathcal{C}$ *such that* $\pi \circ f = id_X$ *(or, equivalently, a morphism* $\iota : Z \to Y$ *such that* $g \circ \iota = id_Z$*);*

(iii) *there are morphisms* $\pi : Y \to X$ *and* $\iota : Z \to Y$ *in* $\mathcal{C}$ *satisfying* $id_Y = f \circ \pi + \iota \circ g$.

Proposition 1.2.5 ([18], Lema 1.1.16)**.** *Let* $f \in Hom_{\mathcal{C}}(X, Y)$, *and* $(Ker(f), k)$ *be its kernel. Then the affirmations are equivalent:*

(i) f *is a monomorphism;*

(ii) $(0, 0)$ *is the kernel of* f;

(iii) $0 \longrightarrow X \xrightarrow{f} Y$ *is an exact sequence.*

And for epimorphisms we have the following result.

Proposition 1.2.6 ([18], Lema 1.1.17)**.** *Let* $f \in Hom_{\mathcal{C}}(X, Y)$, *and* $(coKer(f), q)$ *be its cokernel. The affirmations are equivalent:*

(i) *f is an epimorphism;*

(ii) $(0,0)$ *is the cokernel of f;*

(iii) $X \xrightarrow{f} Y \longrightarrow 0$ *is an exact sequence.*

This following lemma asserts that, in an abelian category, any monomorphism is the kernel of its cokernel and any epimorphism is the cokernel of its kernel.

Lemma 1.2.7 ([15], Lema 2.8.4)**.** *Let $f : X \to Y$ be a monomorphism in $\mathcal{C}$ with cokernel $(coKer(f), q)$, and $g : X \to Y$ be an epimorphism in $\mathcal{C}$ with kernel $(Ker(g), k)$. Then*

(i) (X, f) *is the kernel of q (i.e., the image of f);*

(ii) (Y, g) *is the cokernel of k.*

Proof. We prove the item (ii), and the item (i) can be checked similarly. The morphism g is an epimorphism and our category $\mathcal{C}$ is abelian, thus there exists a morphism $h : Z \to X$ in $\mathcal{C}$ such that $(Y, g : X \to Y)$ is the cokernel of h. Particularly, this implies that $g \circ h = 0$ and since $(Ker(g), k)$ is the kernel of g, there exists an unique morphism $u : Z \to Ker(g)$ in $\mathcal{C}$ satisfying

$$k \circ u = h. \tag{1}$$

In order to show that g is the cokernel of k, let $j : X \to W$ be a morphism in $\mathcal{C}$ such that $j \circ k = 0$. Given that g is the cokernel of h and $j \circ h \overset{(1)}{=} j \circ k \circ u = 0$, we can conclude that there exists an unique morphism $v : Y \to W$ in $\mathcal{C}$ satisfying $v \circ g = j$, as wanted. Therefore, g is the cokernel of k. ∎

As we can see by item (i), the image of a monomorphism is the monomorphism itself.

Remark 1.2.8. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$ with kernel $(Ker(f), k)$ and cokernel $(coKer(f), q)$. The sequences*

$$0 \longrightarrow Ker(f) \xrightarrow{k} X \xrightarrow{f} Y$$

and

$$X \xrightarrow{f} Y \xrightarrow{q} coKer(f) \longrightarrow 0$$

are exact.

Indeed, it follows that the first sequence is exact by defining the cokernel of k as $(coKer(k), q')$ and then by item (i) of the lemma above, $(Ker(f), k)$ is going to be the kernel of q', i.e., $Ker(f) = Ker(q')$ as subobjects of X. Moreover, by definition we have $Im(k) = Ker(q')$ which implies $Im(k) = Ker(f)$ as subobjects of X.

For the second sequence there is not much to do since $Im(f) = Ker(q)$ as subobjects of Y by definition.

This next proposition can be very useful when checking whether a short sequence is exact.

Proposition 1.2.9 ([18], Proposição 1.1.20)**.** *Let $X, Y, Z \in \mathcal{C}$. Consider the sequence*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

The next sentences are equivalent:

(i) *the above sequence is a short exact sequence;*

(ii) *(X, f) is the kernel of g and (Z, g) is the cokernel of f.*

Corollary 1.2.10. *Let $S \in \mathcal{C}$ be a simple objects and X a nonzero object in $\mathcal{C}$. Then any nonzero morphism*

(i) *$f : S \to X$ in $\mathcal{C}$ is a monomorphism;*

(ii) *$g : X \to S$ in $\mathcal{C}$ is an epimorphism.*

Proof. For the item (i), let $(Ker(f), k)$ be the kernel of f. Because $(Ker(f), k)$ is a subobject of the simple object S then $Ker(f) = 0$ or $Ker(f) = S$ as subobjects of S. However, if $Ker(f) = S$ as subobjects of S, then by the Lemma 1.1.10 we can conclude that $f = 0$ which is a contradiction.

If $Ker(f) = 0$ as subobjects of S then f is a monomorphism via the Proposition 1.2.5. The item (ii) can be proven in a similar way. ■

1.3 NATURAL TRANSFORMATIONS, EQUIVALENCES AND ADJUNCTIONS

The study of any mathematical object necessarily requires consideration of the "maps" of such objects. Functors are the closest as morphisms (or maps) between categories which preserves the appropriate structure, and natural transformations are the closest as morphisms between functors. A functor may describe an equivalence of categories, in which case the objects in one can be translated into and reconstructed from the objects of another. For example, the notions of monomorphism, epimorphism and isomorphism are invariant under certain classes of functors including, in particular, functors that form an equivalence.

"The multiple examples, here and elsewhere, of adjoint functors tend to show that adjoints occur almost everywhere in many branches of Mathematics.", (Mac Lane in [11]).

Definition 1.3.1. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\alpha : F \to G$ is a family of morphisms*

$$\alpha = \{\alpha_X : F(X) \to G(X)\}_{X \in \mathcal{C}}$$

in $\mathcal{D}$ such that

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\alpha_Y]{} & G(Y) \end{array}$$

commutes, i.e., $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$, for all $X, Y \in \mathcal{C}$ and $f \in Hom_{\mathcal{C}}(X, Y)$.

We can also say that α is a natural transformation in $\mathcal{C}$ (or even in $X \in \mathcal{C}$) since the family α is indexed in the objects of the category $\mathcal{C}$ (the domain of the functors F and G is $\mathcal{C}$). It's always good to observe that even though the family α is indexed in $\mathcal{C}$, each α_X is a morphism in $\mathcal{D}$ (the codomain of the functors F and G).

If $\alpha_X : F(X) \to G(X)$ is an isomorphism for all $X \in \mathcal{C}$, then α is said to be a *natural isomorphism*. In this case we say that the functor F is equivalent to G (or F and G are equivalent functors), and this fact is denoted by $F \overset{\alpha}{\sim} G$ or simply by $F \sim G$.

Definition 1.3.2. *The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \sim Id_{\mathcal{D}}$ and $G \circ F \sim Id_{\mathcal{C}}$.*

When this happens we say that the functor F (the same can be said about the functor G) is an *equivalence of categories*, and this equivalence is denoted by $\mathcal{C} \simeq \mathcal{D}$.

This following definition can be found in the reference [2].

Definition 1.3.3. *A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be*

(i) *faithful if $F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ is injective, for all $X, Y \in \mathcal{C}$;*

(ii) *full if $F : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ is surjective, for all $X, Y \in \mathcal{C}$;*

(iii) *dense if for all $Z \in \mathcal{D}$ there exists $X \in \mathcal{C}$ such that $F(X) \cong Z$.*

If a functor satisfies these three items then it is an equivalence (and vice versa) as we can see in this next result.

Theorem 1.3.4 ([21], Teorema 2.20)**.** *Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if, and only if, there exists a faithful, full and dense functor $F : \mathcal{C} \to \mathcal{D}$.*

Any time that there are two categories, let us say $\mathcal{C}$ and $\mathcal{D}$, it's possible to define the product of them, which is called the *product category* between $\mathcal{C}$ and $\mathcal{D}$ (see [2], Definition 1.6.5). This product category is denoted by $\mathcal{C} \times \mathcal{D}$, and its objects are the pairs (X, Y) such that $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, and if (W, Z) is another object in $\mathcal{C} \times \mathcal{D}$, then

$$Hom_{\mathcal{C}\times\mathcal{D}}((X, Y), (W, Z)) = Hom_{\mathcal{C}}(X, W) \times Hom_{\mathcal{D}}(Y, Z).$$

Let $f = (f', f'') \in Hom_{\mathcal{C}\times\mathcal{D}}((U, V), (W, Z))$ and $g = (g', g'') \in Hom_{\mathcal{C}\times\mathcal{D}}((X, Y), (U, V))$. The morphism composition $f \circ g$ is defined as

$$\begin{aligned} \circ : Hom_{\mathcal{C}\times\mathcal{D}}((U, V), (W, Z)) \times Hom_{\mathcal{C}\times\mathcal{D}}((X, Y), (U, V)) &\to Hom_{\mathcal{C}\times\mathcal{D}}((X, Y), (W, Z)) \\ ((f', f''), (g', g'')) &\mapsto (f', f'') \circ (g', g'') := (f' \circ g', f'' \circ g'') \end{aligned}$$

and, in particular, $id_{(X,Y)} = (id_X, id_Y)$.

For this definition, see page 23 of [2].

Definition 1.3.5. *A bifunctor is a functor defined on the product of two categories.*

A functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is also called a bifunctor from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$. One famous example of bifunctor is $Hom_{\mathcal{C}}(\ \_\ ,\ \_\ ) : \mathcal{C}^{op} \times \mathcal{C} \to Set$.

For the next definition, let's remember that given a category $\mathcal{C}$, we denote by $\mathcal{C}^{op}$ the category whose objects are the same as in $\mathcal{C}$, and $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$. If $f \in Hom_{\mathcal{C}^{op}}(X, Y)$ and $g \in Hom_{\mathcal{C}^{op}}(Y, Z)$ then its composition is given by $g \circ^{op} f = f \circ g$. The category $\mathcal{C}^{op}$ is called the *opposite category* of $\mathcal{C}$. It is easy to see that a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is a covariant (contravariant) functor if, and only if, $F : \mathcal{C} \to \mathcal{D}$ is contravariant (covariant).

We can see that an adjunction consists of an opposing pair of functors that enjoy a special relationship to one another.

Definition 1.3.6. *An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, G, ϕ) such that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors and $\phi = \{\phi_{X,Y} : Hom_{\mathcal{D}}(F(X), Y) \to Hom_{\mathcal{C}}(X, G(Y))\}_{(X,Y) \in \mathcal{C} \times \mathcal{D}}$ is a natural isomorphism in $\mathcal{C}^{op} \times \mathcal{D}$. The functor F is said to be a left adjoint to G, and G is a right adjoint to F.*

We'll see with Proposition 1.4.7 that a left (or right) adjoint to a functor, when it exists, is unique up to a functor equivalence.

About the natural isomorphism ϕ,

$$\phi : Hom_{\mathcal{D}}(\ \_\ ,\ \_\ ) \circ (F \times Id_{\mathcal{D}}) \to Hom_{\mathcal{C}}(\ \_\ ,\ \_\ ) \circ (Id_{\mathcal{C}^{op}} \times G)$$

in which both functors are defined from the product category $\mathcal{C}^{op} \times \mathcal{D}$ to Set, and for every morphism

$$(f', f'') \in Hom_{\mathcal{C}^{op} \times \mathcal{D}}((X, Y), (W, Z)) = Hom_{\mathcal{C}}(W, X) \times Hom_{\mathcal{D}}(Y, Z)$$

we have

$$Hom_{\mathcal{D}}(F(f'), f'')(\alpha) = f'' \circ \alpha \circ F(f'), \quad \text{and}$$
$$Hom_{\mathcal{C}}(f', G(f''))(\beta) = G(f'') \circ \beta \circ f'$$

for all $\alpha \in Hom_{\mathcal{D}}(F(X), Y)$ and $\beta \in Hom_{\mathcal{C}}(X, G(Y))$. The naturality of ϕ can be expressed as the commutativity of the diagram

$$\begin{array}{ccc}
Hom_{\mathcal{D}}(F(X), Y) & \xrightarrow{\phi_{X,Y}} & Hom_{\mathcal{C}}(X, G(Y)) \\
{\scriptstyle Hom_{\mathcal{D}}(F(f'),f'')}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(f',G(f''))} \\
Hom_{\mathcal{D}}(F(W), Z) & \xrightarrow[\phi_{W,Z}]{} & Hom_{\mathcal{C}}(W, G(Z)).
\end{array}$$

Remark 1.3.7. *We can notice that whenever we have a natural isomorphism in a product category, (for example $\mathcal{C}^{op} \times \mathcal{D}$ as in the definition above) we can fix one entry and this natural isomorphism is still going to be natural on the other entry. For this reason, one can say that the natural isomorphism α above is natural in $\mathcal{C}^{op}$ (and also $\mathcal{C}$) and $\mathcal{D}$. We are going to use this fact in many situations without further mention. We can observe this fact by the commutativity of the diagram*

$$\begin{array}{ccc} Hom_{\mathcal{D}}(F(X), Y) & \xrightarrow{\phi_{X,Y}} & Hom_{\mathcal{C}}(X, G(Y)) \\ {\scriptstyle Hom_{\mathcal{D}}(F(id_X),g)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(id_X,G(g))} \\ Hom_{\mathcal{D}}(F(X), Z) & \xrightarrow[\phi_{X,Z}]{} & Hom_{\mathcal{C}}(X, G(Z)) \end{array}$$

for every morphism $g : Y \to Z$ in $\mathcal{C}$.

This next well-known result gives an equivalent definition of adjunction.

Proposition 1.3.8 ([21], Teorema 2.28)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. The following affirmations are equivalent:*

(i) *(F, G, ϕ) is an adjunction;*

(ii) *there are natural transformations $e : F \circ G \to Id_{\mathcal{D}}$ and $c : Id_{\mathcal{C}} \to G \circ F$ such that for any $Y \in \mathcal{D}$ and $X \in \mathcal{C}$, the equalities*

$$id_{G(Y)} = G(e_Y) \circ c_{G(Y)} \quad \text{and} \quad id_{F(X)} = e_{F(X)} \circ F(c_X)$$

hold.

The natural transformations e and c are called *counit* and *unit* of the adjunction[1], respectively. Furthermore, the counit and unit are defined as $e_Y := \phi^{-1}_{G(Y),Y}(id_{G(Y)})$ and $c_X := \phi_{X,F(X)}(id_{F(X)})$, for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Additionally, in its proof it can be seen that, for any $f \in Hom_{\mathcal{D}}(F(X), Y)$ and $g \in Hom_{\mathcal{C}}(X, G(Y))$, the equalities

$$\phi_{X,Y}(f) = G(f) \circ c_X \tag{2}$$

and

$$\phi^{-1}_{X,Y}(g) = e_Y \circ F(g) \tag{3}$$

hold. These two equations will be used later.

This next proposition affirms that whenever two functors make an equivalence, they are adjoints. It can be found in [20] as Proposition 4.4.5.

Proposition 1.3.9 ([20], Proposition 4.4.5)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories, and $G : \mathcal{D} \to \mathcal{C}$ be a functor such that $F \circ G \sim Id_{\mathcal{D}}$ and $G \circ F \sim Id_{\mathcal{C}}$. Then F is left and right adjoint to G.*

[1] Sometimes we denote the counit and unit by α and β, respectively.

Proof. We start by showing that F is left adjoint to G. By the definition of category equivalence, there are natural isomorphisms $e' : F \circ G \to Id_{\mathcal{D}}$ and $c : Id_{\mathcal{C}} \to G \circ F$. This c will be the unit of the adjunction. By setting

$$\gamma_Y \coloneqq G(e'_Y) \circ c_{G(Y)} : G(Y) \to (G \circ F \circ G)(Y) = G((F \circ G)(Y)) \to G(Y),$$

it's possible to define the counit e of the adjunction as

$$e_Y \coloneqq e'_Y \circ F(\gamma_Y^{-1}) : F(G(Y)) \to F(G(Y)) \to Y$$

for all $Y \in \mathcal{D}$, and show that $e : F \circ G \to Id_{\mathcal{D}}$ is a natural transformation which satisfies $id_{G(Y)} = G(e_Y) \circ c_{G(Y)}$ and $id_{F(X)} = e_{F(X)} \circ F(c_X)$. This will imply that F is left adjoint to G by Proposition 1.3.8.

It is easy to see that γ_Y is an isomorphism for each $Y \in \mathcal{D}$ with inverse given by $c_{G(Y)}^{-1} \circ G(e'^{-1}_Y)$. The natural property of $\gamma : G \to G$ comes directly from the commutativity of the following diagram, for every $f \in Hom_{\mathcal{D}}(Y, Y')$:

$$\begin{array}{ccccc}
 & & \overset{\gamma_Y}{\frown} & & \\
G(Y) & \xrightarrow{c_{G(Y)}} & (G \circ F \circ G)(Y) & \xrightarrow{G(e'_Y)} & G(Y) \\
{\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle (G\circ F\circ G)(f)} & & \downarrow{\scriptstyle G(f)} \\
G(Y') & \xrightarrow[c_{G(Y')}]{} & (G \circ F \circ G)(Y') & \xrightarrow[G(e'_{Y'})]{} & G(Y'). \\
 & & \underset{\gamma_{Y'}}{\smile} & &
\end{array}$$

The commutativity of the first square follows immediately from the naturality of c, and the second from the naturality of e' and the fact that G is a functor.

We can also affirm that $e_Y = e'_Y \circ F(\gamma_Y^{-1})$ is an isomorphism for every $Y \in \mathcal{D}$ by noticing that $F(\gamma_Y) \circ e'^{-1}_Y$ is its inverse. Analogously as we did before, the fact that γ^{-1} is natural (because γ is a natural isomorphism) and e' being natural implies that $e : F \circ G \to Id_{\mathcal{D}}$ is also natural, and the diagram of this naturality is given by

$$\begin{array}{ccccc}
 & & \overset{e_Y}{\frown} & & \\
F(G(Y)) & \xrightarrow{F(\gamma_Y^{-1})} & F(G(Y)) & \xrightarrow{e'_Y} & Y \\
{\scriptstyle (F\circ G)(f)}\downarrow & & \downarrow{\scriptstyle (F\circ G)(f)} & & \downarrow{\scriptstyle f} \\
F(G(Y')) & \xrightarrow[F(\gamma_{Y'}^{-1})]{} & F(G(Y')) & \xrightarrow[e'_{Y'}]{} & Y'. \\
 & & \underset{e_{Y'}}{\smile} & &
\end{array}$$

Now, to check the equality $id_{G(Y)} = G(e_Y) \circ c_{G(Y)}$ notice that

$$\begin{aligned}
G(e_Y) \circ c_{G(Y)} &= G(e'_Y \circ F(\gamma_Y^{-1})) \circ c_{G(Y)} = G(e'_Y) \circ G(F(\gamma_Y^{-1})) \circ c_{G(Y)} \\
&\overset{(*)}{=} G(e'_Y) \circ c_{G(Y)} \circ \gamma_Y^{-1} = \gamma_Y \circ \gamma_Y^{-1} = id_{G(Y)}
\end{aligned}$$

where $(*)$ holds via the naturality of c with the morphism γ_Y^{-1}, that is, through the commutativity of the diagram

$$\begin{array}{ccc} G(Y) & \xrightarrow{c_{G(Y)}} & (G \circ F \circ G)(Y) \\ {\scriptstyle \gamma_Y^{-1}}\downarrow & & \downarrow{\scriptstyle (G\circ F)(\gamma_Y^{-1})} \\ G(Y) & \xrightarrow[c_{G(Y)}]{} & (G \circ F \circ G)(Y). \end{array}$$

Finally,

$$\begin{aligned}(e_{F(X)} \circ F(c_X)) \circ (e_{F(X)} \circ F(c_X)) &= e_{F(X)} \circ (F(c_X) \circ e_{F(X)}) \circ F(c_X) \\ &\overset{(a)}{=} e_{F(X)} \circ e_{F((G\circ F)(X))} \circ (F \circ G)(F(c_X)) \circ F(c_X) \\ &= e_{F(X)} \circ e_{F((G\circ F)(X))} \circ F((G \circ F)(c_X) \circ c_X) \\ &\overset{(b)}{=} e_{F(X)} \circ e_{(F\circ G)(F(X))} \circ F(c_{(G\circ F)(X)} \circ c_X) \\ &\overset{(c)}{=} e_{F(X)} \circ (F \circ G)(e_{F(X)}) \circ F(c_{(G\circ F)(X)}) \circ F(c_X) \\ &= e_{F(X)} \circ F(G(e_{F(X)}) \circ c_{G(F(X))}) \circ F(c_X) \\ &\overset{(d)}{=} e_{F(X)} \circ F(id_{G(F(X))}) \circ F(c_X) \\ &= e_{F(X)} \circ F(c_X)\end{aligned}$$

in which the equalities (a) and (c) hold from the fact that e is natural. In fact, this can be seen with the commutativity of the diagrams

$$\begin{array}{ccc} (F \circ G)(F(X)) & \xrightarrow{e_{F(X)}} & F(X) \\ {\scriptstyle (F\circ G)(F(c_X))}\downarrow & & \downarrow{\scriptstyle F(c_X)} \\ (F \circ G)(F((G \circ F)(X))) & \xrightarrow[e_{F((G\circ F)(X))}]{} & F((G \circ F)(X)) \end{array}$$

and

$$\begin{array}{ccc} (F \circ G)((F \circ G)(F(X))) & \xrightarrow{e_{(F\circ G)(F(X))}} & (F \circ G)(F(X)) \\ {\scriptstyle (F\circ G)(e_{F(X)})}\downarrow & & \downarrow{\scriptstyle e_{F(X)}} \\ (F \circ G)(F(X)) & \xrightarrow[e_{F(X)}]{} & F(X), \end{array}$$

respectively. The equality (b) is due to the naturality of c, i.e., the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{c_X} & (G \circ F)(X) \\ {\scriptstyle c_X}\downarrow & & \downarrow{\scriptstyle (G\circ F)(c_X)} \\ (G \circ F)(X) & \xrightarrow[c_{(G\circ F)(X)}]{} & (G \circ F)((G \circ F)(X)), \end{array}$$

and the equality (d) comes directly from $G(e_Y) \circ c_{G(Y)} = id_{G(Y)}$.

From the fact that $e_{F(X)} \circ F(c_X)$ is an isomorphism for all $X \in \mathcal{C}$, we obtain $e_{F(X)} \circ F(c_X) = id_{F(X)}$. Hence F is left adjoint to G.

In order to show that G is left adjoint to F (or, equivalently, F is right adjoint to G) we don't need to do much. Indeed, we can define the unit and counit of this adjunction as $\overline{c} \coloneqq e^{-1} : Id_{\mathcal{D}} \to F \circ G$ and $\overline{e} \coloneqq c^{-1} : G \circ F \to Id_{\mathcal{C}}$, respectively. By definition we have that $\overline{e}$ and $\overline{c}$ are natural isomorphisms, and the equalities $id_{F(X)} = F(\overline{e}_X) \circ \overline{c}_{F(X)}$ and $id_{G(Y)} = \overline{e}_{G(Y)} \circ G(\overline{c}_Y)$ follows directly from the definition of $\overline{c}$ and $\overline{e}$ which have just been verified. In fact, it's only necessary to notice that

$$F(\overline{e}_X) \circ \overline{c}_{F(X)} = F(c_X^{-1}) \circ e_{F(X)}^{-1} = (e_{F(X)} \circ F(c_X))^{-1} = (id_{F(X)})^{-1} = id_{F(X)}, \text{ and}$$
$$\overline{e}_{G(Y)} \circ G(\overline{c}_Y) = c_{G(Y)}^{-1} \circ G(e_Y^{-1}) = (G(e_Y) \circ c_{G(Y)})^{-1} = (id_{G(Y)})^{-1} = id_{G(Y)}$$

as wanted. ■

This next proposition states that whenever there is an adjunction between additive categories, the functors involved are going to be additive, and each isomorphism involved in the collection is going to be a group isomorphism. The first item can be found in [21] as Teorema 3.20, and the second we couldn't find in the literature.

Proposition 1.3.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors, and (F, G, ϕ) an adjunction. Then*

(i) *the functors F and G are additive;*

(ii) *for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, each $\phi_{X,Y} : Hom_{\mathcal{D}}(F(X), Y) \to Hom_{\mathcal{C}}(X, G(Y))$ is a group isomorphism.*

Proof. For the item (ii), it'll be shown that each $\phi_{X,Y}^{-1}$ is a group homomorphism. Let f and g be morphisms in $Hom_{\mathcal{C}}(X, G(Y))$ and consider the following commutative diagram

$$\begin{array}{ccc} Hom_{\mathcal{C}}(G(Y), G(Y)) & \xrightarrow{\phi_{G(Y),Y}^{-1}} & Hom_{\mathcal{D}}(F(G(Y)), Y) \\ {\scriptstyle Hom_{\mathcal{C}}(f+g, G(id_Y))}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{D}}(F(f+g), id_Y)} \\ Hom_{\mathcal{C}}(X, G(Y)) & \xrightarrow[\phi_{X,Y}^{-1}]{} & Hom_{\mathcal{D}}(F(X), Y). \end{array}$$

Its commutativity implies $\phi_{X,Y}^{-1}(f + g) = \phi_{X,Y}^{-1}(f) + \phi_{X,Y}^{-1}(g)$ as we can see below

$$\begin{aligned} \phi_{X,Y}^{-1}(f + g) &= \phi_{X,Y}^{-1}(G(id_Y) \circ id_{G(Y)} \circ (f + g)) \\ &= \phi_{X,Y}^{-1}(Hom_{\mathcal{C}}(f + g, G(id_Y))(id_{G(Y)})) \\ &= (\phi_{X,Y}^{-1} \circ Hom_{\mathcal{C}}(f + g, G(id_Y)))(id_{G(Y)}) \\ &= (Hom_{\mathcal{D}}(F(f + g), id_Y) \circ \phi_{G(Y),Y}^{-1})(id_{G(Y)}) \\ &= Hom_{\mathcal{D}}(F(f + g), id_Y)(\phi_{G(Y),Y}^{-1}(id_{G(Y)})) \\ &= id_Y \circ \phi_{G(Y),Y}^{-1}(id_{G(Y)}) \circ F(f + g) \\ &\overset{(a)}{=} \phi_{G(Y),Y}^{-1}(id_{G(Y)}) \circ (F(f) + F(g)) \end{aligned}$$

$$\overset{(b)}{=} \phi^{-1}_{G(Y),Y}(id_{G(Y)}) \circ F(f) + \phi^{-1}_{G(Y),Y}(id_{G(Y)}) \circ F(g)$$
$$\overset{(c)}{=} \phi^{-1}_{X,Y}(f) + \phi^{-1}_{X,Y}(g).$$

The equality (a) is valid for the reason that F is additive (via the item (i) of this proposition), the equality (b) holds due to the fact that $\mathcal{D}$ is an additive category and therefore the morphism composition is bilinear. Lastly, (c) comes from the commutativity of the diagram above when using the morphisms f and g separately instead of $f + g$.

Therefore, each morphism $\phi^{-1}_{X,Y}$ in the collection ϕ^{-1} is a group isomorphism. Hence, $\phi_{X,Y}$ is a group isomorphism for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. ■

As we can see, the functors F and G considered above are covariant, but one can easily check that the same outcome holds when these functors are contravariant.

Suppose we have a function $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ such that, for all $X \in \mathcal{C}$ and $Z \in \mathcal{D}$, both

$$\begin{aligned} F^1_X : \mathcal{D} &\to \mathcal{E} \\ Y &\mapsto F^1_X(Y) = F(X, Y) \\ g : Y \to Z &\mapsto F^1_X(g) = F(X, g) = F(id_X, g) \end{aligned}$$

and

$$\begin{aligned} F^2_Z : \mathcal{C} &\to \mathcal{E} \\ W &\mapsto F^2_Z(W) = F(W, Z) \\ f : X \to W &\mapsto F^2_Z(f) = F(f, Z) = F(f, id_Z) \end{aligned}$$

are functors. One natural question that may arise is whether F has or not a structure of bifunctor. In other words, can we say that F is a bifunctor if F restricted to the first and second variables (or entries) are functors? The answer is no, but we can speculate about a condition.

Lemma 1.3.11. *Let $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be an application with $F^1_X : \mathcal{D} \to \mathcal{E}$ and $F^2_Z : \mathcal{C} \to \mathcal{E}$ being functors as above. If the equalities*

$$F(f, g) = F((f, id_Z) \circ (id_X, g)) = F(f, id_Z) \circ F(id_X, g) \tag{4}$$

and

$$F(f, g) = F((id_W, g) \circ (f, id_Y)) = F(id_W, g) \circ F(f, id_Y) \tag{5}$$

hold for any $f \in Hom_{\mathcal{C}}(X, W)$ and $g \in Hom_{\mathcal{D}}(Y, Z)$ then F is a bifunctor, and vice versa.

Proof. We know that $F(X, Y) \in \mathcal{E}$ since $F(X, Y) = F^1_X(Y)$ and F^1_X is a functor for all $X \in \mathcal{C}$. Moreover, $F(id_{(X,Y)}) = F(id_X, id_Y) = F^1_X(id_Y) = id_{F^1_X(Y)} = id_{F(X,Y)}$. Lastly, let

$h = (h', h'') : (X, Y) \to (U, V)$ and $j = (j', j'') : (U, V) \to (W, Z)$ be morphisms in $\mathcal{C} \times \mathcal{D}$. Therefore,

$$\begin{aligned}
F(j) \circ F(h) &= F(j', j'') \circ F(h', h'') \\
&= F((j', id_Z) \circ (id_U, j'')) \circ F((h', id_V) \circ (id_X, h'')) \\
&\overset{(4)}{=} F(j', id_Z) \circ F(id_U, j'') \circ F(h', id_V) \circ F(id_X, h'') \\
&\overset{(5)}{=} F(j', id_Z) \circ F(h', j'') \circ F(id_X, h'') \\
&\overset{(4)}{=} F(j', id_Z) \circ F(h', id_Z) \circ F(id_X, j'') \circ F(id_X, h'') \\
&= F^2_Z(j') \circ F^2_Z(h') \circ F^1_X(j'') \circ F^1_X(h'') \\
&= F^2_Z(j' \circ h') \circ F^1_X(j'' \circ h'') \\
&= F(j' \circ h', id_Z) \circ F(id_X, j'' \circ h'') \\
&\overset{(4)}{=} F(j' \circ h', j'' \circ h'') \\
&= F(j \circ h)
\end{aligned}$$

as wanted. Then $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a functor.

On the other hand, the converse clearly is valid seeing that, if F is a bifunctor, then both equalities (4) and (5) hold. ■

This previous lemma is handy when showing that the internal Hom functor (that is going to be studied later) is a bifunctor.

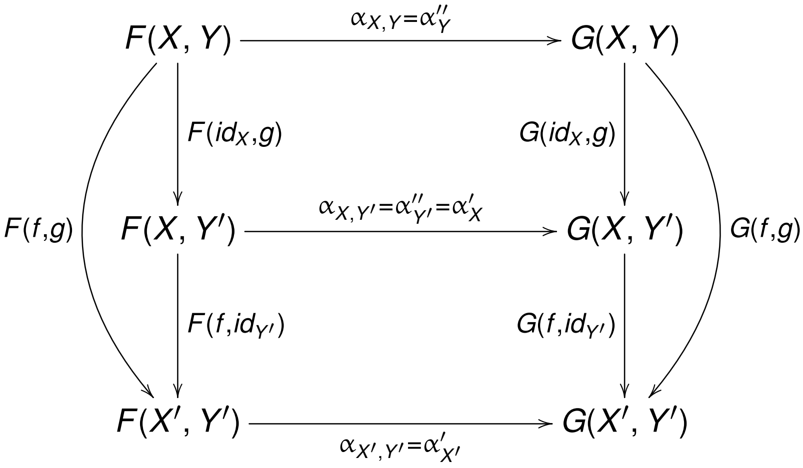
Remark 1.3.12. *Suppose we have two bifunctors F and G from a product category $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and we want to show that a family $\alpha = \{\alpha_{X,Y} : F(X, Y) \to G(X, Y)\}_{(X,Y) \in \mathcal{C} \times \mathcal{D}}$ is a natural transformation between the bifunctors F and G in $\mathcal{C} \times \mathcal{D}$. One way to check this is by fixing each entry and checking that α is natural on the other. Indeed, assume that $\alpha' = \{\alpha'_X = \alpha_{X,Y'}\}_{X \in \mathcal{C}}$ and $\alpha'' = \{\alpha''_Y = \alpha_{X,Y}\}_{Y \in \mathcal{D}}$ are natural transformation in $\mathcal{C}$ and $\mathcal{D}$, respectively*[2]*.*

Let $(f, g) \in Hom_{\mathcal{C} \times \mathcal{D}}((X, Y), (X', Y')) = Hom_{\mathcal{C}}(X, X') \times Hom_{\mathcal{D}}(Y, Y')$. The diagram on the Definition 1.3.1

$$\begin{array}{ccc}
F(X, Y) & \xrightarrow{\alpha_{X,Y}} & G(X, Y) \\
{\scriptstyle F(f,g)} \downarrow & & \downarrow {\scriptstyle G(f,g)} \\
F(X', Y') & \xrightarrow[\alpha_{X',Y'}]{} & G(X', Y')
\end{array}$$

2 In the first we're fixing an object $Y' \in \mathcal{D}$, while in the second $X \in \mathcal{C}$ is the one being fixed.

commutes since we can write it as



and then notice that

$$\begin{aligned}
G(f,g)\circ\alpha_{X,Y} &= G((f,id_{Y'})\circ(id_X,g))\circ\alpha''_Y\\
&\overset{(4)}{=} G(f,id_{Y'})\circ G(id_X,g)\circ\alpha''_Y\\
&\overset{(a)}{=} G(f,id_{Y'})\circ\alpha''_{Y'}\circ F(id_X,g)\\
&= G(f,id_{Y'})\circ\alpha'_X\circ F(id_X,g)\\
&\overset{(b)}{=} \alpha'_{X'}\circ F(f,id_{Y'})\circ F(id_X,g)\\
&\overset{(4)}{=} \alpha'_{X'}\circ F((f,id_{Y'})\circ(id_X,g))\\
&= \alpha_{X',Y'}\circ F(f,g)
\end{aligned}$$

where the equalities labeled with (a) *and* (b) *are valid due the naturalities of* α'' *and* α'*, respectively. Hence,* α *is a natural transformation in* $\mathcal{C}\times\mathcal{D}$*.*

1.4 EXACT FUNCTORS AND NATURAL ISOMORPHISMS

Here we introduce the notion of exact functors and the construction of some useful natural isomorphisms. In this section, all the categories involved are abelian unless stated otherwise.

Definition 1.4.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be abelian categories, and*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an arbitrary short exact sequence $\mathcal{C}$*. An additive covariant functor* $F:\mathcal{C}\to\mathcal{D}$ *is called left exact if the sequence*

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact in $\mathcal{D}$*.*

The functor F is called right exact if the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact in $\mathcal{D}$.

Similarly, an additive contravariant functor $G : \mathcal{C} \to \mathcal{D}$ is called left exact if the sequence

$$0 \longrightarrow G(Z) \xrightarrow{G(g)} G(Y) \xrightarrow{G(f)} F(X)$$

is exact in $\mathcal{D}$.

Moreover, G is called right exact if the sequence

$$G(Z) \xrightarrow{G(g)} G(Y) \xrightarrow{G(f)} G(X) \longrightarrow 0$$

is exact in $\mathcal{D}$.

Finally, a functor is said to be exact if it is both left and right exact.

It's a fact that the bifunctor $Hom_{\mathcal{C}}(\_ , \_) : \mathcal{C}^{op} \times \mathcal{C} \to Ab$ is left exact, which means it is left exact in each entry. This bifunctor being left exact in the first entry implies that the functor $Hom_{\mathcal{C}}(\_ , L) : \mathcal{C}^{op} \to Ab$ is left exact for all $L \in \mathcal{C}$, that is, for any short exact sequence $0 \longrightarrow Z \xrightarrow{g} Y \xrightarrow{f} X \longrightarrow 0$ in $\mathcal{C}^{op}$, the sequence

$$0 \longrightarrow Hom_{\mathcal{C}}(Z, L) \xrightarrow{Hom_{\mathcal{C}}(g,L)} Hom_{\mathcal{C}}(Y, L) \xrightarrow{Hom_{\mathcal{C}}(f,L)} Hom_{\mathcal{C}}(X, L)$$

is short exact in Ab, where

$$Hom_{\mathcal{C}}(g, L)(h) = h \circ g, \text{ for every } h \in Hom_{\mathcal{C}}(Z, L) \text{ and}$$
$$Hom_{\mathcal{C}}(f, L)(h') = h' \circ f, \text{ for every } h' \in Hom_{\mathcal{C}}(Y, L).$$

Similarly, one can easily see what it means to say that the bifunctor $Hom_{\mathcal{C}}(\_ , \_)$ is left exact in the second entry.

This next corollary is often used by many authors and it has quite a simple proof.

Corollary 1.4.2 ([15], Ejercicio 2.7.46). *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, f a morphism in $\mathcal{C}$, $F : \mathcal{C} \to \mathcal{D}$ a covariant functor and $G : \mathcal{C} \to \mathcal{D}$ a contravariant functor. If the functor*

(i) *F is left exact and f is a monomorphism then $F(f)$ is a monomorphism in $\mathcal{D}$. If F is right exact and f is an epimorphism then $F(f)$ is an epimorphism in $\mathcal{D}$;*

(ii) *G is left exact and f is an epimorphism then $G(f)$ is a monomorphism in $\mathcal{D}$. If G is right exact and f is a monomorphism then $G(f)$ is an epimorphism in $\mathcal{D}$.*

Proof. For the item (i), suppose $F : \mathcal{C} \to \mathcal{D}$ is left exact and $f : X \to Y$ is a monomorphism in $\mathcal{C}$. Let $(coKer(f), q)$ be the cokernel of f and consider the short sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{q} coKer(f) \longrightarrow 0$$

which is exact by Remark 1.2.8 and Proposition 1.2.5. Because F is left exact, it follows that

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(q)} F(coKer(f))$$

is exact which implies that $F(f)$ is monomorphism.

In a similar way one can prove the other assertion and the item (ii). ■

This next result gives a characterization of left and right exact functors that are going to be often used, and it can be also found as the definition of left and right exact functors, e.g., Definición 2.7.35 in [15].

Proposition 1.4.3 ([15], Definición 2.7.35)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories, $F : \mathcal{C} \to \mathcal{D}$ be an additive functor and $f : X \to Y$ be a morphism in $\mathcal{C}$ with kernel $k : Ker(f) \to X$ and cokernel $q : Y \to coKer(f)$. If the functor F is left exact, then $F(k) : F(Ker(f)) \to F(X)$ is the kernel of $F(f)$. Additionally, if F is right exact then $F(q) : F(Y) \to F(coKer(f))$ is the cokernel of $F(f)$.*

Proof. Suppose the functor F is left exact and consider the sequence

$$0 \longrightarrow Ker(f) \xrightarrow{k} X \xrightarrow{f} Y$$

which is exact directly from Remark 1.2.8. Now, let $(Ker(q), k')$ be the kernel of q. Using Remark 1.1.18 we can obtain a decomposition for f, i.e., there exists an unique morphism (which is an epimorphism) $u \in Hom_{\mathcal{C}}(X, Ker(q))$ satisfying $f = k' \circ u$.

The sequence $0 \longrightarrow Ker(f) \xrightarrow{k} X \xrightarrow{u} Ker(q) \longrightarrow 0$ is short exact since k a monomorphism, u is an epimorphism and the image of k is (via Lemma 1.2.7)

$$(Ker(f), k) = (Ker(k' \circ u), k) = (Ker(u), k)$$

as subobjects of X, where the last equality comes from item (i) of Lemma 1.1.9.

For the reason that the functor F is left exact we have that the sequence

$$0 \longrightarrow F(Ker(f)) \xrightarrow{F(k)} F(X) \xrightarrow{F(u)} F(Ker(q))$$

is exact, that is, $F(k)$ is a monomorphism and the image of $F(k)$ is the kernel of $F(u)$.

From the fact that $F(k)$ is a monomorphism, $(F(Ker(f)), F(k))$ is the image of $F(k)$ (by Lemma 1.2.7). And using that this sequence is exact in $F(X)$ we can conclude that the kernel of $F(u)$ is the image of $F(k)$, that is, $(F(Ker(f)), F(k))$ is the kernel of $F(u)$.

Therefore, by the item (i) of Lemma 1.1.9 we can affirm that $(F(Ker(f)), F(k))$ is the kernel of $F(k') \circ F(u) = F(k' \circ u) = F(f)$ (since $F(k')$ is a monomorphism by Corollary 1.4.2) as wanted.

Analogously, it's possible to prove that if F is right exact then $F(q)$ is the cokernel of $F(f)$. ■

This proposition we've just seen can also be found as the definition of left and right exact functors, that is, an additive functor between abelian categories $F : \mathcal{C} \to \mathcal{D}$ is left (or right) exact if, for any morphism $f : X \to Y$ in $\mathcal{C}$ with kernel $k : Ker(f) \to X$ (cokernel $q : Y \to coKer(f)$), the kernel (cokernel) of $F(f)$ is $F(k) : F(Ker(f)) \to F(X)$ $(F(q) : F(Y) \to F(coKer(f)))$.

Proposition 1.4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, $F : \mathcal{C} \to \mathcal{D}$ an additive functor and $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ a short exact sequence in $\mathcal{C}$ that splits. Then the short sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ in $\mathcal{D}$ is exact and splits.*

Proof. Firstly, we begin by showing that the short sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact in $\mathcal{D}$, i.e., $F(f)$ is a monomorphism, $F(g)$ is an epimorphism and the image of $F(f)$ is the kernel of $F(g)$. The short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ splits, so there are morphisms $\pi : Y \to X$ and $\iota : Z \to Y$ in $\mathcal{C}$ such that $\pi \circ f = id_X$, $g \circ \iota = id_Z$ and $f \circ \pi + \iota \circ g = id_Y$ (see Proposition 1.2.4). Using that F is an additive functor, the equalities

$$\begin{aligned} F(\pi) \circ F(f) &\overset{(a)}{=} id_{F(X)} \\ F(g) \circ F(\iota) &= id_{F(Z)}, \quad \text{and} \\ F(f) \circ F(\pi) + F(\iota) \circ F(g) &\overset{(b)}{=} id_{F(Y)} \end{aligned}$$

hold.

To show that $F(f)$ is a monomorphism, let a and b morphisms in $Hom_{\mathcal{D}}(W, F(X))$ satisfying $F(f) \circ a = F(f) \circ b$. This implies that $F(\pi) \circ F(f) \circ a = F(\pi) \circ F(f) \circ b$, that is, $a = b$ by using the equality (a). Therefore, $F(f)$ is a monomorphism in $\mathcal{D}$. Analogously, one can prove that $F(g)$ is an epimorphism in $\mathcal{D}$.

To check that the image of the monomorphism $F(f)$ (which is $(F(X), F(f))$ via Lemma 1.2.7) is the kernel of $F(g)$, observe that $F(g) \circ F(f) = F(g \circ f) = 0$ and consider a morphism $k' : K' \to F(Y)$ in $\mathcal{D}$ such that $F(g) \circ k' \overset{(*)}{=} 0$. We'll show that there is an unique $u \in Hom_{\mathcal{D}}(K', F(X))$ satisfying $F(f) \circ u = k'$.

By defining $u := F(\pi) \circ k'$ it follows that

$$\begin{aligned} F(f) \circ u &= F(f) \circ F(\pi) \circ k' \\ &\overset{(*)}{=} F(f) \circ F(\pi) \circ k' + F(\iota) \circ F(g) \circ k' \\ &= (F(f) \circ F(\pi) + F(\iota) \circ F(g)) \circ k' \\ &\overset{(b)}{=} id_{F(Y)} \circ k' = k' \end{aligned}$$

as wanted. For the uniqueness, let $u' : K' \to F(X)$ be a morphism in $\mathcal{D}$ satisfying $F(f) \circ u' = k'$. Thus, $F(f) \circ u = F(f) \circ u'$ and because $F(f)$ is a monomorphism, it follows that $u = u'$. Hence the short sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ is exact in $\mathcal{D}$.

It follows directly from the item (iii) of Proposition 1.2.4 and the equality (b) that the short exact sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ splits. ■

The items (i) and (ii) of the next proposition can be found as Exercise 1.6.4 in [4].

Proposition 1.4.5. *We have the following:*

(i) *the right adjoint to any functor between abelian categories is left exact;*

(ii) *the left adjoint to any functor between abelian categories is right exact;*

(iii) *every functor equivalence between abelian categories is exact.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between abelian categories and (F, G, ϕ) an adjunction.

(i) Considering an arbitrary short exact sequence

$$0 \longrightarrow Y \xrightarrow{\ f\ } Y' \xrightarrow{\ g\ } Y'' \longrightarrow 0$$

in $\mathcal{D}$, we need to show that the sequence

$$0 \longrightarrow G(Y) \xrightarrow{G(f)} G(Y') \xrightarrow{G(g)} G(Y'')$$

is exact in $\mathcal{C}$, that is, $G(f)$ is a monomorphism and the image of $G(f)$ is the kernel of $G(g)$.

To check that $G(f)$ is a monomorphism, let $a, b \in Hom_{\mathcal{C}}(Z, G(Y))$ such that $G(f) \circ a = G(f) \circ b$ and consider the commutative diagram

$$\begin{array}{ccc} Hom_{\mathcal{C}}(id_Z, G(Y)) & \xrightarrow{\phi^{-1}_{Z,Y}} & Hom_{\mathcal{D}}(F(id_Z), Y) \\ \downarrow{\scriptstyle Hom_{\mathcal{C}}(id_Z, G(f))} & & \downarrow{\scriptstyle Hom_{\mathcal{D}}(F(id_Z), f)} \\ Hom_{\mathcal{C}}(id_Z, G(Y')) & \xrightarrow[\phi^{-1}_{Z,Y'}]{} & Hom_{\mathcal{D}}(F(id_Z), Y'). \end{array}$$

Thus,

$$\begin{aligned} (f \circ \phi^{-1}_{Z,Y})(a) &= (f \circ \phi^{-1}_{Z,Y}(a) \circ F(id_Z)) \\ &= Hom_{\mathcal{D}}(F(id_Z), f)(\phi^{-1}_{Z,Y}(a)) \\ &= (Hom_{\mathcal{D}}(F(id_Z), f) \circ \phi^{-1}_{Z,Y})(a) \\ &= (\phi^{-1}_{Z,Y'} \circ Hom_{\mathcal{C}}(id_Z, G(f)))(a) \\ &= \phi^{-1}_{Z,Y'}(Hom_{\mathcal{C}}(id_Z, G(f))(a)) \\ &= \phi^{-1}_{Z,Y'}(G(f) \circ a \circ id_Z) \\ &= \phi^{-1}_{Z,Y'}(G(f) \circ a), \end{aligned}$$

i.e., $f \circ \phi^{-1}_{Z,Y}(a) = \phi^{-1}_{Z,Y'}(G(f) \circ a)$. Using that $\phi_{Z,Y'}$ is an isomorphism, it follows that $G(f) \circ a = \phi_{Z,Y'}(f \circ \phi^{-1}_{Z,Y}(a))$ and, similarly, $G(f) \circ b = \phi_{Z,Y'}(f \circ \phi^{-1}_{Z,Y}(b))$.

Given that $G(f) \circ a = G(f) \circ b$ we obtain

$$\begin{aligned} \phi_{Z,Y'}(f \circ \phi^{-1}_{Z,Y}(a)) = \phi_{Z,Y'}(f \circ \phi^{-1}_{Z,Y}(b)) &\implies f \circ \phi^{-1}_{Z,Y}(a) = f \circ \phi^{-1}_{Z,Y}(b) \\ &\implies \phi^{-1}_{Z,Y}(a) = \phi^{-1}_{Z,Y}(b) \\ &\implies a = b \end{aligned}$$

where the second implication is due to the morphism f being a monomorphism. Hence $G(f)$ is a monomorphism in $\mathcal{C}$.

Lastly, we show that the image of $G(f)$ is the kernel of $G(g)$. Given that $G(f)$ is a monomorphism, $(G(Y), G(f))$ is the kernel of its cokernel (see Lemma 1.2.7), i.e., the image of $G(f)$. Notice that $G(g) \circ G(f) = G(g \circ f) = 0$ since $g \circ f = 0$ (via Proposition 1.2.9), and consider $k' : K' \to G(Y')$ a morphism in $\mathcal{C}$ satisfying $G(g) \circ k' = 0$. We need to check that there exists an unique morphism $u \in Hom_{\mathcal{C}}(K', G(Y))$ such that $G(f) \circ u = k'$.

Affirmation 1: $g \circ \phi^{-1}_{K',Y'}(k') = 0$.

In fact, by the commutativity of

$$\begin{array}{ccc} Hom_{\mathcal{C}}(id_{K'}, G(Y')) & \xrightarrow{\phi^{-1}_{K',Y'}} & Hom_{\mathcal{D}}(F(id_{K'}), Y') \\ {\scriptstyle Hom_{\mathcal{C}}(id_{K'},G(g))}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{D}}(F(id_{K'}),g)} \\ Hom_{\mathcal{C}}(id_{K'}, G(Y'')) & \xrightarrow[\phi^{-1}_{K',Y''}]{} & Hom_{\mathcal{D}}(F(id_{K'}), Y''), \end{array}$$

it follows that

$$\begin{aligned} (g \circ \phi^{-1}_{K',Y'})(k') &= Hom_{\mathcal{D}}(F(id_{K'}), g)(\phi^{-1}_{K',Y'}(k')) \\ &= (Hom_{\mathcal{D}}(F(id_{K'}), g) \circ \phi^{-1}_{K',Y'})(k') \\ &= (\phi^{-1}_{K',Y''} \circ Hom_{\mathcal{C}}(id_{K'}, G(g)))(k') \\ &= \phi^{-1}_{K',Y''}(Hom_{\mathcal{C}}(id_{K'}, G(g))(k')) \\ &= \phi^{-1}_{K',Y''}(G(g) \circ k') \\ &= \phi^{-1}_{K',Y''}(0) = 0 \end{aligned}$$

where the last equality comes from the fact that each morphism $\phi^{-1}_{K',Y''}$ in the collection ϕ^{-1} is a group morphism (via the item (ii) of Proposition 1.3.10).

Using the hypothesis that (Y, f) is the kernel of g, there exists an unique $v \in Hom_{\mathcal{D}}(F(K'), Y)$ such that

$$f \circ v = \phi^{-1}_{K',Y'}(k') \iff \phi_{K',Y'}(f \circ v) = k'. \tag{6}$$

Affirmation 2: $u := \phi_{K',Y}(v)$ is the only morphism in $\mathcal{C}$ which satisfies $G(f) \circ u = k'$.

By considering the commutativity of

$$\begin{CD}
Hom_{\mathcal{D}}(F(K'), Y) @>{\phi_{K',Y}}>> Hom_{\mathcal{C}}(K', G(Y)) \\
@V{Hom_{\mathcal{D}}(F(id_{K'}),f)}VV @VV{Hom_{\mathcal{C}}(id_{K'},G(f))}V \\
Hom_{\mathcal{D}}(F(K'), Y') @>>{\phi_{K',Y'}}> Hom_{\mathcal{C}}(K', G(Y'))
\end{CD}$$

we obtain

$$\begin{aligned}
G(f) \circ u &= G(f) \circ \phi_{K',Y}(v) \\
&= Hom_{\mathcal{C}}(id_{K'}, G(f))(\phi_{K',Y}(v)) \\
&= (Hom_{\mathcal{C}}(id_{K'}, G(f)) \circ \phi_{K',Y})(v) \\
&= (\phi_{K',Y'} \circ Hom_{\mathcal{D}}(F(id_{K'}), f))(v) \\
&= \phi_{K',Y'}(Hom_{\mathcal{D}}(F(id_{K'}), f)(v)) \\
&= \phi_{K',Y'}(f \circ v) \overset{(6)}{=} k'.
\end{aligned}$$

For the uniqueness, consider $u' \in Hom_{\mathcal{C}}(K', G(Y))$ such that $G(f) \circ u' = k'$. This implies that $G(f) \circ u' = G(f) \circ u$ and since $G(f)$ is a monomorphism, $u = u'$. Thus, $G(f)$ is the kernel of $G(g)$ and, therefore, the functor $G : \mathcal{D} \to \mathcal{C}$ is left exact.

(ii) It's analogous to item (i).

(iii) Every functor equivalence is left and right adjoint to its inverse (as seen in Proposition 1.3.9), so it's left and right exact by items (i) and (ii). ■

It follows directly from the items (i) and (ii) that if a functor admits a left and a right adjoint then it is exact.

From this point to the end of this section the categories involved need not to be abelian. Let $\mathcal{C}$ be a category.

Let us consider the bifunctor $Hom_{\mathcal{C}}(\_\,,\,\_) : \mathcal{C}^{op} \times \mathcal{C} \to Set$. For all $X \in \mathcal{C}$, we can define a covariant functor $L_X \coloneqq Hom_{\mathcal{C}}(X, \_\,) : \mathcal{C} \to Set$ and a contravariant functor $R_X \coloneqq Hom_{\mathcal{C}}(\_\,, X) : \mathcal{C} \to Set$ as $R_X(Y) = Hom_{\mathcal{C}}(Y, X)$ on the objects and, for every morphism $f : Y \to Z$ in $\mathcal{C}$, $R_X(f) = Hom_{\mathcal{C}}(f, X)$ is defined as $Hom_{\mathcal{C}}(f, X)(g) = g \circ f$, for any $g \in Hom_{\mathcal{C}}(Z, X)$. These functors will be used in some results of this work, mainly because these next two propositions.

Proposition 1.4.6. *Let $X, Y \in \mathcal{C}$. Then $R_X \overset{\varphi}{\sim} R_Y$ (and $L_X \sim L_Y$) if, and only if, $X \cong Y$.*

Proof. Let us begin by showing that $X \cong Y$ when $R_X \overset{\varphi}{\sim} R_Y$. This will be done by verifying that the morphism $\varphi_X(id_X) : X \to Y$ admits an inverse given by $\varphi_Y^{-1}(id_Y) : Y \to X$. Given that φ is a natural isomorphism, the diagrams

$$\begin{CD}
Hom_{\mathcal{C}}(X, X) @>{\varphi_X}>> Hom_{\mathcal{C}}(X, Y) \\
@V{Hom_{\mathcal{C}}(\varphi_Y^{-1}(id_Y),X)}VV @VV{Hom_{\mathcal{C}}(\varphi_Y^{-1}(id_Y),Y)}V \\
Hom_{\mathcal{C}}(Y, X) @>>{\varphi_Y}> Hom_{\mathcal{C}}(Y, Y)
\end{CD}
\quad \text{and} \quad
\begin{CD}
Hom_{\mathcal{C}}(Y, Y) @>{\varphi_Y^{-1}}>> Hom_{\mathcal{C}}(Y, X) \\
@V{Hom_{\mathcal{C}}(\varphi_X(id_X),Y)}VV @VV{Hom_{\mathcal{C}}(\varphi_X(id_X),X)}V \\
Hom_{\mathcal{C}}(X, Y) @>>{\varphi_X^{-1}}> Hom_{\mathcal{C}}(X, X)
\end{CD}$$

commute. So by taking $id_X \in Hom_{\mathcal{C}}(X,X)$ in the first diagram it follows that

$$\begin{aligned}
\varphi_X(id_X) \circ \varphi_Y^{-1}(id_Y) &= Hom_{\mathcal{C}}(\varphi_Y^{-1}(id_Y), Y)(\varphi_X(id_X)) \\
&= (Hom_{\mathcal{C}}(\varphi_Y^{-1}(id_Y), Y) \circ \varphi_X)(id_X) \\
&= (\varphi_Y \circ Hom_{\mathcal{C}}(\varphi_Y^{-1}(id_Y), X))(id_X) \\
&= \varphi_Y(Hom_{\mathcal{C}}(\varphi_Y^{-1}(id_Y), X)(id_X)) \\
&= \varphi_Y(\varphi_Y^{-1}(id_Y)) = id_Y.
\end{aligned}$$

Doing the same in the second diagram with the morphism $id_Y \in Hom_{\mathcal{C}}(Y,Y)$, it follows the equality $\varphi_Y^{-1}(id_Y) \circ \varphi_X(id_X) = id_X$ and thus $\varphi_X(id_X)$ is an isomorphism. Hence $X \cong Y$.

On the other hand, let $f : X \to Y$ be an isomorphism in $\mathcal{C}$ and define

$$\begin{aligned}
\varphi_Z : R_X(Z) = Hom_{\mathcal{C}}(Z,X) &\longrightarrow R_Y(Z) = Hom_{\mathcal{C}}(Z,Y) \\
g : Z \to X &\longmapsto \varphi_Z(g) := f \circ g
\end{aligned}$$

which is an isomorphism in *Set* for all $Z \in \mathcal{C}$ with inverse $h : Z \to Y \longmapsto f^{-1} \circ h$. Finally, for the natural structure of $\varphi := \{\varphi_Z : R_X(Z) \longrightarrow R_Y(Z)\}_{Z \in \mathcal{C}}$, consider an arbitrary $j \in Hom_{\mathcal{C}}(W,Z)$ and the following diagram

$$\begin{array}{ccc}
Hom_{\mathcal{C}}(Z,X) & \xrightarrow{\varphi_Z} & Hom_{\mathcal{C}}(Z,Y) \\
{\scriptstyle Hom_{\mathcal{C}}(j,X)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(j,Y)} \\
Hom_{\mathcal{C}}(W,X) & \xrightarrow{\varphi_W} & Hom_{\mathcal{C}}(W,Y).
\end{array}$$

which commutes. Indeed, let g be an arbitrary morphism in $Hom_{\mathcal{C}}(Z,X)$ and notice that

$$\begin{aligned}
(Hom_{\mathcal{C}}(j,Y) \circ \varphi_Z)(g) &= Hom_{\mathcal{C}}(j,Y)((\varphi_Z)(g)) \\
&= Hom_{\mathcal{C}}(j,Y)(f \circ g) \\
&= f \circ g \circ j \\
&= \varphi_W(g \circ j) \\
&= \varphi_W(Hom_{\mathcal{C}}(j,X)(g)) \\
&= (\varphi_W \circ Hom_{\mathcal{C}}(j,X))(g)
\end{aligned}$$

This implies that $Hom_{\mathcal{C}}(j,Y) \circ \varphi_Z = \varphi_W \circ Hom_{\mathcal{C}}(j,X)$ which is equivalent to say that φ is a natural isomorphism between the contravariant functors R_X and R_Y.

Analogous as we just did for the contravariant functor R_X, it's possible do to with covariant functor L_X, i.e., $L_X \sim L_Y$ if, and only if, $X \cong Y$ (see [21], Proposição 2.24). ■

This next result is about the uniqueness of left and right adjoints of a functor up to a functor equivalence, and it can be found in the reference [21] as Proposição 2.32.

Proposition 1.4.7 ([21], Proposição 2.32)**.** *The left (or right) adjoint of a functor, when it exists, is unique up to a functor equivalence.*

If a functor F admits a left adjoint, we may denote such functor as $F^{l.a.}$. Analogously, if F has a right adjoint we may denote this functor as $F^{r.a.}$. As we have just seen, these functors are unique up to a functor equivalence.

2 MONOIDAL AND MODULE CATEGORIES

Here we present the concepts of monoidal, rigid, multitensor, tensor and fusion categories, and then we introduce the notions module categories over monoidal categories.

2.1 MONOIDAL CATEGORIES

As we've just seen, the notion of abelian categories are a categorification of abelian groups. In this section we're going to study the categorification of the notion of a monoid (which is a set with associative multiplication and an identity) by replacing the equalities in its definition by isomorphisms satisfying some properties.

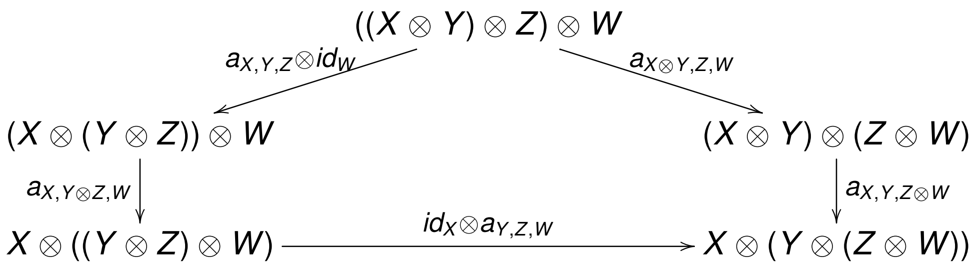
In this section we give some definitions and results about monoidal categories which will be useful later.

Definition 2.1.1. *A monoidal category is a sextuple* $(\mathcal{C}, \otimes, a, l, r, \mathbf{1})$ *such that* $\mathcal{C}$ *is a category,* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *is a bifunctor called tensor functor,* $\mathbf{1}$ *is an object in* $\mathcal{C}$ *called unit, and*

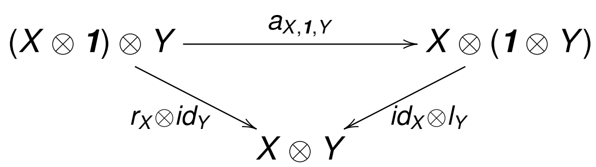
$$a : \otimes \circ (\otimes \times Id_{\mathcal{C}}) \to \otimes \circ (Id_{\mathcal{C}} \times \otimes),$$

$$l : \mathbf{1} \otimes \_ \to Id_{\mathcal{C}} \quad \text{and} \quad r : \_ \otimes \mathbf{1} \to Id_{\mathcal{C}}$$

are natural isomorphisms, such that for any objects X, Y, Z, W *in* $\mathcal{C}$*, the diagrams*



and



commute. The commutativity of these two diagrams can be written as

$$a_{X,Y,Z\otimes W} \circ a_{X\otimes Y,Z,W} = (id_X \otimes a_{Y,Z,W}) \circ a_{X,Y\otimes Z,W} \circ (a_{X,Y,Z} \otimes id_W)$$

and

$$r_X \otimes id_Y = (id_X \otimes l_Y) \circ a_{X,\mathbf{1},Y}.$$

The first diagram is called the pentagon axiom and the second is called the triangle axiom. The natural isomorphism a is called associativity isomorphism of the monoidal category $(\mathcal{C}, \otimes, a, l, r, \mathbf{1})$. We often omit the sextuple and denote a monoidal category $(\mathcal{C}, \otimes, a, l, r, \mathbf{1})$ simply by $\mathcal{C}$. From now on $\mathcal{C}$ is a monoidal category unless stated otherwise.

Let $f \in Hom_{\mathcal{C}}(X, Y)$ and $g \in Hom_{\mathcal{C}}(W, Z)$. We have that

$$\otimes(f, g) = f \otimes g \in Hom_{\mathcal{C}}(X \otimes W, Y \otimes Z)$$

and since $(id_X, id_Y) = id_{(X,Y)}$ is the identity morphism of the object (X, Y) in $\mathcal{C} \times \mathcal{C}$, then

$$id_X \otimes id_Y = \otimes(id_X, id_Y) = \otimes(id_{(X,Y)}) = id_{\otimes(X,Y)} = id_{X \otimes Y}.$$

Example 2.1.2. Let $\mathbb{k}$ be a field. The category $Vect_{\mathbb{k}}$ ($vect_{\mathbb{k}}$) of arbitrary dimensional (finite dimensional) vector spaces over $\mathbb{k}$ are monoidal with the bifunctor $\otimes$ being the tensor product over the field $\mathbb{k}$, i.e., $\otimes = \otimes_{\mathbb{k}}$. The unit object is the field $\mathbb{k}$ and the natural isomorphisms a, l and r are canonical.

In these following definitions, the same symbol $\otimes$ is used to denote the tensor functor of any monoidal categories $\mathcal{C}$ and $\mathcal{D}$. The same can be said about the object $\mathbf{1}$ which is going to be used to denote both $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$.

Definition 2.1.3. *A monoidal functor between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a triple (F, ξ, ϕ), such that $F : \mathcal{C} \to \mathcal{D}$ is a functor, $\xi : \otimes \circ (F \times F) \to F \circ \otimes$ is a natural isomorphism in $\mathcal{C} \times \mathcal{C}$ and $\phi : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ is an isomorphism in $\mathcal{D}$ such that the equalities*

(i) $\xi_{X,Y\otimes Z} \circ (id_{F(X)} \otimes \xi_{Y,Z}) \circ a_{F(X),F(Y),F(Z)} = F(a_{X,Y,Z}) \circ \xi_{X\otimes Y,Z} \circ (\xi_{X,Y} \otimes id_{F(Z)})$;

(ii) $l_{F(X)} = F(l_X) \circ \xi_{\mathbf{1},X} \circ (\phi \otimes id_{F(X)})$;

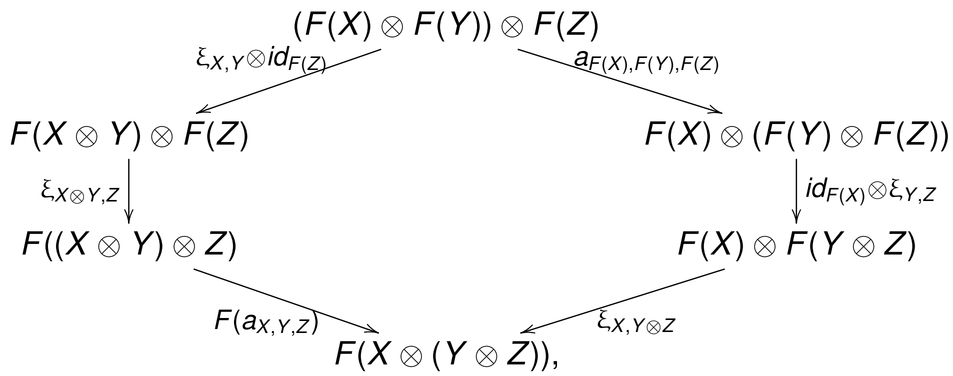
(iii) $r_{F(X)} = F(r_X) \circ \xi_{X,\mathbf{1}} \circ (id_{F(X)} \otimes \phi)$,

hold, for every $X, Y, Z \in \mathcal{C}$.

The natural isomorphism ξ is the collection

$$\xi = \{\xi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)\}_{(X,Y) \in \mathcal{C}\times\mathcal{C}}$$

of isomorphisms in $\mathcal{D}$, and the equalities (i), (ii) and (iii) can be expressed through the commutativity of the diagrams



$$\begin{array}{ccc} \mathbf{1}\otimes F(X) & \xrightarrow{l_{F(X)}} & F(X) \\ {\scriptstyle \phi\otimes id_{F(X)}}\downarrow & & \uparrow{\scriptstyle F(l_X)} \\ F(\mathbf{1})\otimes F(X) & \xrightarrow{\xi_{\mathbf{1},X}} & F(\mathbf{1}\otimes X) \end{array} \quad \text{and} \quad \begin{array}{ccc} F(X)\otimes \mathbf{1} & \xrightarrow{r_{F(X)}} & F(X) \\ {\scriptstyle id_{F(X)}\otimes\phi}\downarrow & & \uparrow{\scriptstyle F(r_X)} \\ F(X)\otimes F(\mathbf{1}) & \xrightarrow{\xi_{X,\mathbf{1}}} & F(X\otimes \mathbf{1}), \end{array}$$

respectively.

We often omit the triple (F, ξ, ϕ) and denote a monoidal functor simply by F.

Proposition 2.1.4 ([14], Proposição 2.10)**.** *The composition (when possible) of monoidal functors is a monoidal functor.*

Definition 2.1.5. *Let (F, ξ, ϕ) and (F', ξ', ϕ') be monoidal functors between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$. A monoidal natural transformation*

$$\theta : (F, \xi, \phi) \to (F', \xi', \phi')$$

is a natural transformation $\theta : F \to F'$ such that, for any $X, Y \in \mathcal{C}$, the equalities

(i) $\theta_{\mathbf{1}} \circ \phi = \phi'$ *and*

(ii) $\theta_{X\otimes Y} \circ \xi_{X,Y} = \xi'_{X,Y} \circ (\theta_X \otimes \theta_Y)$

hold.

If θ is a monoidal natural transformation and θ_X is an isomorphism for all $X \in \mathcal{C}$, then θ is called *monoidal natural isomorphism*.

A *natural equivalence* between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor $(F, \xi, \phi) : \mathcal{C} \to \mathcal{D}$ such that there exists another monoidal functor $(F', \xi', \phi') : \mathcal{D} \to \mathcal{C}$ and monoidal natural isomorphisms $\theta_1 : F \circ F' \to Id_{\mathcal{D}}$ and $\theta_2 : F' \circ F \to Id_{\mathcal{C}}$.

We'll now see the notion of dual object in a monoidal category. A dual object is an analogue of a dual vector space from linear algebra for objects in arbitrary monoidal categories. It is only a partial generalization, based upon the categorical properties of duality for finite-dimensional vector spaces.

Definition 2.1.6. *Let X an object in a monoidal category $\mathcal{C}$. A right dual of X is a triple $(X^*, ev_X, coev_X)$ such that X^* is an object in $\mathcal{C}$, and $ev_X : X^* \otimes X \to \mathbf{1}$ and $coev_X : \mathbf{1} \to X \otimes X^*$ are morphisms in $\mathcal{C}$ in which the following compositions*

$$X \xrightarrow{l_X^{-1}} \mathbf{1}\otimes X \xrightarrow{coev_X\otimes id_X} (X\otimes X^*)\otimes X \xrightarrow{a_{X,X^*,X}} X\otimes(X^*\otimes X) \xrightarrow{id_X\otimes ev_X} X\otimes\mathbf{1} \xrightarrow{r_X} X,$$

and

$$X^* \xrightarrow{r_{X^*}^{-1}} X^*\otimes\mathbf{1} \xrightarrow{id_{X^*}\otimes coev_X} X^*\otimes(X\otimes X^*) \xrightarrow{a_{X^*,X,X^*}^{-1}} (X^*\otimes X)\otimes X^* \xrightarrow{ev_X\otimes id_{X^*}} \mathbf{1}\otimes X^* \xrightarrow{l_{X^*}} X^*$$

are id_X and id_{X^*}, respectively.

Dually, a left dual of X is a triple $({}^*X, ev'_X, coev'_X)$ such that ${}^*X$ is an object in $\mathcal{C}$, and $ev'_X : X \otimes {}^*X \to \mathbf{1}$ and $coev'_X : \mathbf{1} \to {}^*X \otimes X$ are morphisms in $\mathcal{C}$ in which the compositions

$$X \xrightarrow{r_X^{-1}} X \otimes \mathbf{1} \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} \mathbf{1} \otimes X \xrightarrow{l_X} X,$$

and

$${}^*X \xrightarrow{l^{-1}_{{}^*X}} \mathbf{1} \otimes {}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X \otimes \mathbf{1} \xrightarrow{r_{{}^*X}} {}^*X$$

are id_X and $id_{{}^*X}$, respectively.

Some useful adjunctions can be seen in this proposition below.

Proposition 2.1.7 ([4], Proposition 2.10.8)**.** *Let $\mathcal{C}$ be a monoidal category and $X \in \mathcal{C}$. If X has a right dual X^* then*

(i) *the functor $\_ \otimes X : \mathcal{C} \to \mathcal{C}$ is left adjoint to $\_ \otimes X^* : \mathcal{C} \to \mathcal{C}$;*

(ii) *the functor $X^* \otimes \_ : \mathcal{C} \to \mathcal{C}$ is left adjoint to $X \otimes \_ : \mathcal{C} \to \mathcal{C}$.*

Moreover, if X has a left dual ${}^*X$ then

(iii) *the functor $\_ \otimes {}^*X : \mathcal{C} \to \mathcal{C}$ is left adjoint to $\_ \otimes X : \mathcal{C} \to \mathcal{C}$;*

(iv) *the functor $X \otimes \_ : \mathcal{C} \to \mathcal{C}$ is left adjoint to ${}^*X \otimes \_ : \mathcal{C} \to \mathcal{C}$.*

Now we introduce some definitions that will often be used in this work. All items presented here can be found in the book [4].

Definition 2.1.8. *A category $\mathcal{C}$ is said to be*

a) *rigid, if it is monoidal and every object has left and right duals;*

b) *multitensor, if it is locally finite, rigid, and the tensor bifunctor $\otimes$ is $\Bbbk$-bilinear on morphisms, i.e., $X \otimes \_$ and $\_ \otimes X$ are $\Bbbk$-linear functors*[1] *for every $X \in \mathcal{C}$;*

c) *tensor, if it is multitensor and $Hom_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) \cong \Bbbk$,*[2]

d) *fusion, if it is finite, tensor and semisimple.*

---

1 From $\mathcal{C}$ to $\mathcal{C}$.

2 The object **1** is a simple object in a tensor category. Indeed, in any tensor category the tensor functor $\otimes$ is biexact (see Remark 2.1.9) and hence the unit object $\mathbf{1} \in \mathcal{C}$ is simple (via Theorem 4.3.8 in [4] while noticing that every tensor category is a ring category).

Remark 2.1.9 ([4], Proposition 4.2.1). *We can observe that in any rigid category $\mathcal{C}$ the tensor functor $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is biexact. In fact, for all $X \in \mathcal{C}$ the functors $X \otimes \_$ and $\_ \otimes X$ have left and right adjoints (by Proposition 2.1.7). Using the item (iii) of Proposition 1.4.5, it follows that these functors are exact, i.e., the tensor functor $\otimes$ is biexact.*

Lemma 2.1.10. *If $\mathcal{C}$ is a tensor category and $0 \neq X \in \mathcal{C}$ then $coev_X$ ($coev'_X$) is a monomorphism and ev_X (ev'_X) is an epimorphism in $\mathcal{C}$.*

Proof. To begin, we want to show that $coev_X$ and ev_X are nonzero morphisms in $\mathcal{C}$. So, suppose that $coev_X$ is a zero morphism in $\mathcal{C}$. Since $X \neq 0$ we get $id_X \neq 0$ (see Remark 1.1.8), i.e.,

$$0 \neq id_X = r_X \circ (id_X \otimes ev_X) \circ a_{X,X^*,X} \circ (coev_X \otimes id_X) \circ l_X^{-1}.$$

The functor $\_ \otimes X$ is additive[3] and therefore a homomorphism of abelian groups (see Definition 1.1.19). Hence, $coev_X \otimes id_X = (\_ \otimes X)(coev_X) = 0$ implying $id_X = 0$, which is a contradiction. Analogously, one can check that $ev_X \neq 0$.

Since $coev_X$ and ev_X are nonzero morphisms in $\mathcal{C}$ and the object $\mathbf{1} \in \mathcal{C}$ is simple, it follows from Corollary 1.2.10 that $coev_X$ is a monomorphism and ev_X is an epimorphism. In a similar way, one can prove that $coev'_X$ is a monomorphism and ev'_X is an epimorphism. ∎

2.2 MODULE CATEGORIES OVER MONOIDAL CATEGORIES

We've seen that the notion of a monoidal category categorifies the notion of a monoid. From this point of view it is natural to define a module category over a monoidal category as a categorification of the notion of the module over a monoid.

"This theory is interesting by itself, but is also crucial for understanding the structure of tensor categories, similarly to how the study of modules is important in understanding the structure of rings." ([4], page 131).

Firstly, we begin by defining what is a module category over a monoidal category and later we define functors and natural transformations in this module context.

Let $\mathcal{C} = (\mathcal{C}, \otimes, a, l, r, \mathbf{1})$ be a monoidal category.

Definition 2.2.1. *A left $\mathcal{C}$-module category is a collection $(\mathcal{M}, \overline{\otimes}, m, l)$ such that*

(i) *$\mathcal{M}$ is a category;*

(ii) *$\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is an action (or module product) bifunctor;*

[3] It is exact (see Remark 2.1.9) and thus additive.

(iii) $m : \overline{\otimes} \circ (\otimes \times Id_{\mathcal{M}}) \to \overline{\otimes} \circ (Id_{\mathcal{C}} \times \overline{\otimes})$ *(called module associativity constraint) and* $l : \mathbf{1}\,\overline{\otimes}\,\_ \to Id_{\mathcal{M}}$ *(called unit constraint) are natural isomorphisms with* $m = \{m_{X,Y,M} : (X \otimes Y)\overline{\otimes}M \to X\overline{\otimes}(Y\overline{\otimes}M)\}_{X,Y\in\mathcal{C},M\in\mathcal{M}}$ *and* $l = \{l_M : \mathbf{1}\overline{\otimes}M \to M\}_{M\in\mathcal{M}}$ *satisfying*

$$m_{X,Y,Z\overline{\otimes}M} \circ m_{X\otimes Y,Z,M} = (id_X\overline{\otimes}m_{Y,Z,M}) \circ m_{X,Y\otimes Z,M} \circ (a_{X,Y,Z}\overline{\otimes}id_M),$$

and

$$(id_X\overline{\otimes}l_M) \circ m_{X,\mathbf{1},M} = r_X\overline{\otimes}id_M.$$

These equalities can also be express by the commutativity of the diagrams

$$\begin{array}{ccccc} & & ((X\otimes Y)\otimes Z)\overline{\otimes}M & & \\ & \swarrow^{a_{X,Y,Z}\overline{\otimes}id_M} & & \searrow^{m_{X\otimes Y,Z,M}} & \\ (X\otimes(Y\otimes Z))\overline{\otimes}M & & & & (X\otimes Y)\overline{\otimes}(Z\overline{\otimes}M) \\ \downarrow{\scriptstyle m_{X,Y\otimes Z,M}} & & & & \downarrow{\scriptstyle m_{X,Y,Z\overline{\otimes}M}} \\ X\overline{\otimes}((Y\otimes Z)\overline{\otimes}M) & & \xrightarrow{id_X\overline{\otimes}m_{Y,Z,M}} & & X\overline{\otimes}(Y\overline{\otimes}(Z\overline{\otimes}M)) \end{array}$$

and

$$\begin{array}{ccc} (X\otimes \mathbf{1})\overline{\otimes}M & \xrightarrow{m_{X,\mathbf{1},M}} & X\overline{\otimes}(\mathbf{1}\overline{\otimes}M) \\ & {\scriptstyle r_X\overline{\otimes}id_M}\searrow \quad \swarrow{\scriptstyle id_X\overline{\otimes}l_M} & \\ & X\overline{\otimes}M, & \end{array}$$

called the pentagon and triangle diagram, respectively. We could also define a right $\mathcal{C}$-module category by making some small changes on the definition above. For simplicity we omit the collection and denote a $\mathcal{C}$-module category $(\mathcal{M}, \overline{\otimes}, m, l)$ simply by $\mathcal{M}$. Analogously as we've said for monoidal categories, for module categories we have $id_X\overline{\otimes}id_M = id_{X\overline{\otimes}M}$, for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

Example 2.2.2. Any monoidal category $\mathcal{C}$ is a left and right module category over itself with $\overline{\otimes} := \otimes$ and $m := a$.

This next result will be used in many instances from now on when commuting diagrams, and it's possible to get the idea of this proof on the book [8] with Lemma XI.2.2 for the case in which the category is tensor. With some tweaks we adapted this for module categories in the following proposition.

Proposition 2.2.3. *If* $\mathcal{M}$ *is a* $\mathcal{C}$*-module category,* $X \in \mathcal{C}$ *and* $M \in \mathcal{M}$*. The diagram*

$$\begin{array}{ccc} (\mathbf{1}\otimes X)\overline{\otimes}M & \xrightarrow{m_{\mathbf{1},X,M}} & \mathbf{1}\overline{\otimes}(X\overline{\otimes}M) \\ & {\scriptstyle l_X\overline{\otimes}id_M}\searrow \quad \swarrow{\scriptstyle l_{X\overline{\otimes}M}} & \\ & X\overline{\otimes}M & \end{array}$$

also commutes, that is, $l_{X\overline{\otimes}M} \circ m_{\mathbf{1},X,M} = l_X\overline{\otimes}id_M$.

Proof. To begin, let $Y \in \mathcal{C}$ and consider the diagram

$$\begin{array}{ccc}
Y\overline{\otimes}((\mathbf{1} \otimes X)\overline{\otimes}M) & \xrightarrow{id_Y\overline{\otimes}m_{\mathbf{1},X,M}} & Y\overline{\otimes}(\mathbf{1}\overline{\otimes}(X\overline{\otimes}M)) \\
& {\scriptstyle id_Y\overline{\otimes}(l_X\overline{\otimes}id_M)} \searrow \quad \swarrow {\scriptstyle id_Y\overline{\otimes}l_{X\overline{\otimes}M}} & \\
& Y\overline{\otimes}(X\overline{\otimes}M). &
\end{array}$$

This diagram commutes because

$$\begin{aligned}
(id_Y\overline{\otimes}l_{X\overline{\otimes}M}) \circ (id_Y\overline{\otimes}m_{\mathbf{1},X,M}) &\overset{(a)}{=} (id_Y\overline{\otimes}l_{X\overline{\otimes}M}) \circ m_{Y,\mathbf{1},X\overline{\otimes}M} \circ m_{Y\otimes\mathbf{1},X,M} \circ (a^{-1}_{Y,\mathbf{1},X}\overline{\otimes}id_M)\circ \\
&\quad\ m^{-1}_{Y,\mathbf{1}\otimes X,M} \\
&\overset{(b)}{=} (r_Y\overline{\otimes}id_{X\overline{\otimes}M}) \circ m_{Y\otimes\mathbf{1},X,M} \circ (a^{-1}_{Y,\mathbf{1},X}\overline{\otimes}id_M) \circ m^{-1}_{Y,\mathbf{1}\otimes X,M} \\
&= (r_Y\overline{\otimes}(id_X\overline{\otimes}id_M)) \circ m_{Y\otimes\mathbf{1},X,M} \circ (a^{-1}_{Y,\mathbf{1},X}\overline{\otimes}id_M) \circ m^{-1}_{Y,\mathbf{1}\otimes X,M} \\
&\overset{(c)}{=} m_{Y,X,M} \circ ((r_Y \otimes id_X)\overline{\otimes}id_M) \circ (a^{-1}_{Y,\mathbf{1},X}\overline{\otimes}id_M) \circ m^{-1}_{Y,\mathbf{1}\otimes X,M} \\
&= m_{Y,X,M} \circ (((r_Y \otimes id_X) \circ a^{-1}_{Y,\mathbf{1},X})\overline{\otimes}id_M) \circ m^{-1}_{Y,\mathbf{1}\otimes X,M} \\
&\overset{(d)}{=} m_{Y,X,M} \circ ((id_Y \otimes l_X)\overline{\otimes}id_M) \circ m^{-1}_{Y,\mathbf{1}\otimes X,M} \\
&\overset{(c)}{=} (id_Y\overline{\otimes}(l_X\overline{\otimes}id_M))
\end{aligned}$$

where the equalities (a) and (b) hold by the pentagon and triangle diagram of the $\mathcal{C}$-module category $\mathcal{M}$, respectively. The equalities labeled with (c) are valid due to the naturality of m and (d) is via the triangle axiom of the monoidal category $\mathcal{C}$.

Next, by the naturality of the unit constraint l, the diagram

$$\begin{array}{ccc}
\mathbf{1}\overline{\otimes}((\mathbf{1} \otimes X)\overline{\otimes}M) & \xrightarrow{l_{(\mathbf{1}\otimes X)\overline{\otimes}M}} & (\mathbf{1} \otimes X)\overline{\otimes}M \\
{\scriptstyle id_{\mathbf{1}}\overline{\otimes}(l_{X\overline{\otimes}M}\circ m_{\mathbf{1},X,M})}\downarrow & & \downarrow{\scriptstyle l_{X\overline{\otimes}M}\circ m_{\mathbf{1},X,M}} \\
\mathbf{1}\overline{\otimes}(X\overline{\otimes}M) & \xrightarrow[l_{X\overline{\otimes}M}]{} & X\overline{\otimes}M
\end{array}$$

commutes, that is,

$$l_{X\overline{\otimes}M} \circ m_{\mathbf{1},X,M} \circ l_{(\mathbf{1}\otimes X)\overline{\otimes}M} = l_{X\overline{\otimes}M} \circ (id_{\mathbf{1}}\overline{\otimes}(l_{X\overline{\otimes}M} \circ m_{\mathbf{1},X,M})). \tag{7}$$

Notice that we are using the morphism $l_{X\overline{\otimes}M}\circ m_{\mathbf{1},X,M} \in Hom_{\mathcal{M}}((\mathbf{1}\otimes X)\overline{\otimes}M, X\overline{\otimes}M)$ in this diagram. We could do the same with the morphism $l_X\overline{\otimes}id_M \in Hom_{\mathcal{M}}((\mathbf{1}\otimes X)\overline{\otimes}M, X\overline{\otimes}M)$ and thus obtain

$$(l_X\overline{\otimes}id_M) \circ l_{(\mathbf{1}\otimes X)\overline{\otimes}M} = l_{X\overline{\otimes}M} \circ (id_{\mathbf{1}}\overline{\otimes}(l_X\overline{\otimes}id_M)). \tag{8}$$

At last, notice that

$$
\begin{aligned}
l_{X\overline{\otimes}M} \circ m_{\mathbf{1},X,M} &\overset{(7)}{=} l_{X\overline{\otimes}M} \circ (id_{\mathbf{1}}\overline{\otimes}(l_{X\overline{\otimes}M} \circ m_{\mathbf{1},X,M})) \circ l^{-1}_{(\mathbf{1}\otimes X)\overline{\otimes}M} \\
&= l_{X\overline{\otimes}M} \circ (id_{\mathbf{1}}\overline{\otimes} l_{X\overline{\otimes}M}) \circ (id_{\mathbf{1}}\overline{\otimes} m_{\mathbf{1},X,M}) \circ l^{-1}_{(\mathbf{1}\otimes X)\overline{\otimes}M} \\
&\overset{(*)}{=} l_{X\overline{\otimes}M} \circ (id_{\mathbf{1}}\overline{\otimes}(l_X\overline{\otimes} id_M)) \circ l^{-1}_{(\mathbf{1}\otimes X)\overline{\otimes}M} \\
&\overset{(8)}{=} (l_X\overline{\otimes} id_M)
\end{aligned}
$$

in which the equality $(*)$ is valid due to the commutativity of the triangle diagram on the beginning of this proof for $Y = \mathbf{1}$. Hence, $l_{X\overline{\otimes}M} \circ m_{\mathbf{1},X,M} = l_X\overline{\otimes} id_M$ for all objects $X \in \mathcal{C}$ and $M \in \mathcal{M}$. ■

Remark 2.2.4. *We'll see later that if the category $\mathcal{C}$ ir rigid, the module product bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is exact in the second variable. Moreover, if $\mathcal{C}$ is finite multitensor and $\mathcal{M}$ is locally finite then $\overline{\otimes}$ is also right exact in the first variable. For some upcoming results we'll need $\overline{\otimes}$ being $\mathbb{k}$-linear and left exact in the first variable*[4].

Proposition 2.2.5 ([4], Proposition 7.1.6)**.** *Let $\mathcal{C}$ be a rigid category, $\mathcal{M}$ a $\mathcal{C}$-module category and $X \in \mathcal{C}$. Then the functor*

(i) *$X^*\overline{\otimes}\,\_\ : \mathcal{M} \to \mathcal{M}$ is left adjoint to $X\overline{\otimes}\,\_\ : \mathcal{M} \to \mathcal{M}$;*

(ii) *$X\overline{\otimes}\,\_\ : \mathcal{M} \to \mathcal{M}$ is left adjoint to $^*X\overline{\otimes}\,\_\ : \mathcal{M} \to \mathcal{M}$.*

In particular, the functor $X\overline{\otimes}\,\_$ is exact.

Its proof is very similar as the one found in Proposition 2.1.7 (see Proposition 2.10.8 in [4]). For example, let $(X^*\overline{\otimes}\,\_\,, X\overline{\otimes}\,\_\,, \theta)$ be the adjunction of item (i) with $\theta = \{\theta_{M,M'} : Hom_{\mathcal{M}}(X^*\overline{\otimes}M, M') \to Hom_{\mathcal{M}}(M, X\overline{\otimes}M')\}_{M,M'\in\mathcal{M}}$, $f \in Hom_{\mathcal{M}}(X^*\overline{\otimes}M, M')$ and $g \in Hom_{\mathcal{M}}(M, X\overline{\otimes}M')$. The morphism $\theta_{M,M'}(f)$ in $\mathcal{M}$ is defined as

$$\theta_{M,M'}(f) \coloneqq (id_X\overline{\otimes}f) \circ m_{X,X^*,M} \circ (coev_X\overline{\otimes}id_M) \circ l^{-1}_M \tag{9}$$

with inverse

$$\theta^{-1}_{M,M'}(g) \coloneqq l_{M'} \circ (ev_X\overline{\otimes}id_{M'}) \circ m^{-1}_{X^*,X,M'} \circ (id_{X^*}\overline{\otimes}g). \tag{10}$$

Indeed, for all $g \in Hom_{\mathcal{M}}(M, X\overline{\otimes}M')$ we have

$$
\begin{aligned}
(\theta_{M,M'} \circ \theta^{-1}_{M,M'})(g) &= \theta_{M,M'}(\theta^{-1}_{M,M'}(g)) \\
&= (id_X\overline{\otimes}\theta^{-1}_{M,M'}(g)) \circ m_{X,X^*,M} \circ (coev_X\overline{\otimes}id_M) \circ l^{-1}_M \\
&= (id_X\overline{\otimes}(l_{M'} \circ (ev_X\overline{\otimes}id_{M'}) \circ m^{-1}_{X^*,X,M'} \circ (id_{X^*}\overline{\otimes}g))) \circ m_{X,X^*,M}\circ \\
&\quad (coev_X\overline{\otimes}id_M) \circ l^{-1}_M
\end{aligned}
$$

[4] That is, the functor $\_\,\overline{\otimes}M : \mathcal{C} \to \mathcal{M}$ being $\mathbb{k}$-linear and exact, for every $M \in \mathcal{M}$

$$= (id_X \overline{\otimes} l_{M'}) \circ (id_X \overline{\otimes} (ev_X \overline{\otimes} id_{M'})) \circ (id_X \overline{\otimes} m^{-1}_{X^*,X,M'}) \circ (id_X \overline{\otimes} (id_{X^*} \overline{\otimes} g)) \circ m_{X,X^*,M} \circ (coev_X \overline{\otimes} id_M) \circ l^{-1}_M$$

$$\overset{(a)}{=} (id_X \overline{\otimes} l_{M'}) \circ (id_X \overline{\otimes} (ev_X \overline{\otimes} id_{M'})) \circ (id_X \overline{\otimes} m^{-1}_{X^*,X,M'}) \circ m_{X,X^*,X\overline{\otimes} M'} \circ ((id_X \otimes id_{X^*}) \overline{\otimes} g) \circ (coev_X \overline{\otimes} id_M) \circ l^{-1}_M$$

$$\overset{(b)}{=} (id_X \overline{\otimes} l_{M'}) \circ (id_X \overline{\otimes} (ev_X \overline{\otimes} id_{M'})) \circ m_{X,X^*\otimes X,M'} \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ m^{-1}_{X\otimes X^*,X,M'} \circ ((id_X \otimes id_{X^*}) \overline{\otimes} g) \circ (coev_X \overline{\otimes} id_M) \circ l^{-1}_M$$

$$\overset{(a)}{=} (id_X \overline{\otimes} l_{M'}) \circ m_{X,\mathbf{1},M'} \circ ((id_X \otimes ev_X) \overline{\otimes} id_{M'}) \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ m^{-1}_{X\otimes X^*,X,M'} \circ ((id_X \otimes id_{X^*}) \overline{\otimes} g) \circ (coev_X \overline{\otimes} id_M) \circ l^{-1}_M$$

$$\overset{(c)}{=} (r_X \overline{\otimes} id_{M'}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{M'}) \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ m^{-1}_{X\otimes X^*,X,M'} \circ (id_{X\otimes X^*} \overline{\otimes} g) \circ (coev_X \overline{\otimes} id_M) \circ l^{-1}_M$$

$$= (r_X \overline{\otimes} id_{M'}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{M'}) \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ m^{-1}_{X\otimes X^*,X,M'} \circ (coev_X \overline{\otimes} id_{X\overline{\otimes} M'}) \circ (id_{\mathbf{1}} \overline{\otimes} g) \circ l^{-1}_M$$

$$\overset{(a)}{=} (r_X \overline{\otimes} id_{M'}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{M'}) \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ ((coev_X \otimes id_X) \overline{\otimes} id_{M'}) \circ m^{-1}_{\mathbf{1},X,M'} \circ (id_{\mathbf{1}} \overline{\otimes} g) \circ l^{-1}_M$$

$$\overset{(d)}{=} (r_X \overline{\otimes} id_{M'}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{M'}) \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ ((coev_X \otimes id_X) \overline{\otimes} id_{M'}) \circ m^{-1}_{\mathbf{1},X,M'} \circ l^{-1}_{X\overline{\otimes} M'} \circ g$$

$$\overset{(e)}{=} (r_X \overline{\otimes} id_{M'}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{M'}) \circ (a_{X,X^*,X} \overline{\otimes} id_{M'}) \circ ((coev_X \otimes id_X) \overline{\otimes} id_{M'}) \circ (l_X \overline{\otimes} id_{M'}) \circ g$$

$$= ((r_X \circ (id_X \otimes ev_X) \circ a_{X,X^*,X} \circ (coev_X \otimes id_X) \circ l_X) \overline{\otimes} id_{M'}) \circ g$$

$$= (id_X \overline{\otimes} id_{M'}) \circ g$$

$$= id_{X\overline{\otimes} M'} \circ g = g$$

in which the equalities labeled with (a) hold via the naturality of m. We are using the pentagon and triangle diagrams of the $\mathcal{C}$-module category $\mathcal{M}$ in the equalities (b) and (c), respectively. Finally, (d) is due to the naturality of l and (e) by using the Proposition 2.2.3. Therefore, $\theta_{M,M'} \circ \theta^{-1}_{M,M'} = id_{Hom_{\mathcal{M}}(M, X\overline{\otimes} M')}$, and the equality $\theta^{-1}_{M,M'} \circ \theta_{M,M'} = id_{Hom_{\mathcal{M}}(X^*\overline{\otimes} M, M')}$ can be verified analogously.

Similarly as we've done in Remark 2.1.9, by using Proposition 1.4.5 it's possible to conclude that the functor $X\overline{\otimes}\ \_$ is exact.

We now define the notions of functor between $\mathcal{C}$-module categories and natural transformation between $\mathcal{C}$-module functors. All definitions and results we will work with are presented for the case of left $\mathcal{C}$-modules, but the case of right $\mathcal{C}$-modules is entirely analogous. For simplicity we will just write $\mathcal{C}$-module category instead of left $\mathcal{C}$-module category.

Definition 2.2.6. *Let $\mathcal{M} = (\mathcal{M}, \overline{\otimes}, m, l)$ and $\mathcal{N} = (\mathcal{N}, \overline{\otimes}, m, l)$ be two $\mathcal{C}$-module categories. A $\mathcal{C}$-module functor between $\mathcal{M}$ and $\mathcal{N}$ is a pair (F, c) such that $F : \mathcal{M} \to \mathcal{N}$ is a functor and $c : F \circ \overline{\otimes} \to \overline{\otimes} \circ (Id_{\mathcal{C}} \times F)$ is a natural isomorphism in $\mathcal{C} \times \mathcal{M}$ with $c = \{c_{X,M} : F(X \overline{\otimes} M) \to X \overline{\otimes} F(M)\}_{(X,M) \in \mathcal{C} \times \mathcal{M}}$ satisfying*

$$(id_X \overline{\otimes} c_{Y,M}) \circ c_{X, Y \overline{\otimes} M} \circ F(m_{X,Y,M}) = m_{X,Y,F(M)} \circ c_{X \otimes Y, M},$$

and

$$l_{F(M)} \circ c_{\mathbf{1},M} = F(l_M).$$

These equalities can also be expressed via the commutativity of the diagrams

$$\begin{array}{ccccc} & & F((X \otimes Y) \overline{\otimes} M) & & \\ & \swarrow^{c_{X \otimes Y, M}} & & \searrow^{F(m_{X,Y,M})} & \\ (X \otimes Y) \overline{\otimes} F(M) & & & & F(X \overline{\otimes} (Y \overline{\otimes} M)) \\ \downarrow{\scriptstyle m_{X,Y,F(M)}} & & & & \downarrow{\scriptstyle c_{X, Y \overline{\otimes} M}} \\ X \overline{\otimes} (Y \overline{\otimes} F(M)) & & \xleftarrow{\ id_X \overline{\otimes} c_{Y,M}\ } & & X \overline{\otimes} F(Y \overline{\otimes} M) \end{array}$$

and

$$\begin{array}{ccc} F(\mathbf{1} \overline{\otimes} M) & \xrightarrow{\ c_{\mathbf{1},M}\ } & \mathbf{1} \overline{\otimes} F(M) \\ & {\scriptstyle F(l_M)} \searrow \quad \swarrow {\scriptstyle l_{F(M)}} & \\ & F(M), & \end{array}$$

for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

Sometimes we omit the pair of a $\mathcal{C}$-module functor (F, c) and say simply that F is a $\mathcal{C}$-module functor.

Example 2.2.7. The identity functor $Id_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}$-module functor and its $\mathcal{C}$-module functor structure is given by the natural isomorphism identity $ID : Id_{\mathcal{M}} \to Id_{\mathcal{M}}$ defined by $ID = \{ID_{X,M} := id_{X \overline{\otimes} M} : Id_{\mathcal{M}}(X \overline{\otimes} M) \to X \overline{\otimes} ID_{\mathcal{M}}(M)\}_{(X,M) \in \mathcal{C} \times \mathcal{M}}$.

The natural isomorphism ID that gives $Id_{\mathcal{M}}$ a $\mathcal{C}$-module functor structure is often omitted.

Example 2.2.8. The functor $(\_\ \overline{\otimes} M : \mathcal{C} \to \mathcal{M}, c)$ is a $\mathcal{C}$-module functor with $c = \{c_{X,Y} := m_{X,Y,M} : (X \otimes Y) \overline{\otimes} M \to X \overline{\otimes} (Y \overline{\otimes} M)\}_{X,Y \in \mathcal{C}}$.

Definition 2.2.9. *Let $(F, c), (G, d) : \mathcal{M} \to \mathcal{N}$ be two $\mathcal{C}$-module functors. A natural transformation of $\mathcal{C}$-module functors is a natural transformation $\theta : F \to G$ such that the diagram*

$$\begin{array}{ccc} F(X \overline{\otimes} M) & \xrightarrow{\ \theta_{X \overline{\otimes} M}\ } & G(X \overline{\otimes} M) \\ \downarrow{\scriptstyle c_{X,M}} & & \downarrow{\scriptstyle d_{X,M}} \\ X \overline{\otimes} F(M) & \xrightarrow[\ id_X \overline{\otimes} \theta_M\ ]{} & X \overline{\otimes} G(M) \end{array}$$

commutes, i.e., the equality $d_{X,M} \circ \theta_{X\overline{\otimes}M} = (id_X \overline{\otimes} \theta_M) \circ c_{X,M}$ *holds for all* $X \in \mathcal{C}$ *and* $M \in \mathcal{M}$.

If θ_M *is an isomorphism in* $\mathcal{N}$ *for every* $M \in \mathcal{M}$ *then* $\theta = \{\theta_M\}_{M\in\mathcal{M}}$ *is called natural isomorphism of* $\mathcal{C}$*-modules. In this case, the functor* (F, c) *is said to be equivalent to* (G, d) *as* $\mathcal{C}$*-module functors, and it is denoted by* $(F, c) \overset{\theta}{\sim} (G, d)$ *or simply by* $(F, c) \sim (G, d)$.

This following well known result states that the composition of two $\mathcal{C}$-module functors (when the composition is possible) is also a $\mathcal{C}$-module functor.

Proposition 2.2.10 ([14], Proposição 3.6)**.** *Let* $(F, c) : \mathcal{M}_1 \to \mathcal{M}_2$ *and* $(G, d) : \mathcal{M}_2 \to \mathcal{M}_3$ *two* $\mathcal{C}$*-module functors. Then* $(G \circ F, b) : \mathcal{M}_1 \to \mathcal{M}_3$ *is a* $\mathcal{C}$*-module functor with* $b_{X,M} = d_{X,F(M)} \circ G(c_{X,M})$*, for all* $X \in \mathcal{C}$ *and* $M \in \mathcal{M}_1$.

Definition 2.2.11 ([4], Definition 7.2.1)**.** *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be module categories over a monoidal category* $\mathcal{C}$*. The categories* $\mathcal{M}$ *and* $\mathcal{N}$ *are said to be equivalent as* $\mathcal{C}$*-module categories if there exists a functor* $F : \mathcal{M} \to \mathcal{N}$ *that is an equivalence of categories and it admits a structure of* $\mathcal{C}$*-module functor.*

In [15] the author defines that two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ are equivalent (as $\mathcal{C}$-modules) if there are $\mathcal{C}$-module functors $(F, c) : \mathcal{M} \to \mathcal{N}$ and $(G, d) : \mathcal{N} \to \mathcal{M}$ such that $(F, c) \circ (G, d) \sim (Id_{\mathcal{N}}, ID)$ and $(G, d) \circ (F, c) \sim (Id_{\mathcal{M}}, ID)$. These two definitions used in [4] and [15] are equivalent and we could not find this proof in the literature, so the following proposition was created.

Proposition 2.2.12. *Two* $\mathcal{C}$*-module categories* $\mathcal{M}$ *and* $\mathcal{N}$ *are equivalent (as* $\mathcal{C}$*-module categories) if, and only if, there are two* $\mathcal{C}$*-module functors* $(F, c) : \mathcal{M} \to \mathcal{N}$ *and* $(G, d) : \mathcal{N} \to \mathcal{M}$ *such that* $(F, c) \circ (G, d) \sim (Id_{\mathcal{N}}, ID)$ *and* $(G, d) \circ (F, c) \sim (Id_{\mathcal{M}}, ID)$.

Proof. The converse clearly holds since $(F, c) : \mathcal{M} \to \mathcal{N}$ being a $\mathcal{C}$-module functor equivalence implies that F is an equivalence of categories.

On the other hand, let $F : \mathcal{M} \to \mathcal{N}$ be an equivalence of categories such that F has a structure of $\mathcal{C}$-module functor $(F, c) : \mathcal{M} \to \mathcal{N}$. Since F is an equivalence, there exists a functor $G : \mathcal{N} \to \mathcal{M}$ and natural isomorphisms $\alpha' : F \circ G \to Id_{\mathcal{N}}$ and $\beta : Id_{\mathcal{M}} \to G \circ F$. Via Proposition 1.3.9, the functor F is left adjoint to G, β is the unit of this adjunction and the counit $\alpha : F \circ G \to Id_{\mathcal{N}}$ is given by

$$\alpha = \{\alpha_N \coloneqq \alpha'_N \circ F(\beta^{-1}_{G(N)}) \circ F(G({\alpha'_N}^{-1})) : F(G(N)) \to N\}_{N\in\mathcal{N}}.$$

Furthermore, β is a natural isomorphism by definition and α is a natural isomorphism by Proposition 1.3.9. Moreover, they satisfy $id_{G(N)} = G(\alpha_N) \circ \beta_{G(N)}$ and $id_{F(M)} = \alpha_{F(M)} \circ F(\beta_M)$.

The $\mathcal{C}$-module functor structure of G can be defined as the composition

$$G(X\overline{\otimes}N) \xrightarrow{G(id_X\overline{\otimes}\alpha_N^{-1})} G(X\overline{\otimes}F(G(N))) \xrightarrow{G(c^{-1}_{X,G(N)})} G(F(X\overline{\otimes}G(N))) \xrightarrow{\beta^{-1}_{X\overline{\otimes}G(N)}} X\overline{\otimes}G(N)$$

for all $X \in \mathcal{C}$ and $N \in \mathcal{N}$, that is, (G, d) has a structure of $\mathcal{C}$-module functor with

$$d = \{d_{X,N} := \beta^{-1}_{X\overline{\otimes}G(N)} \circ G(c^{-1}_{X,G(N)}) \circ G(id_X\overline{\otimes}\alpha_N^{-1})\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}.$$

In fact, let see that d is a natural isomorphism in $\mathcal{C}\times\mathcal{N}$, i.e., for every

$$(f, g) \in Hom_{\mathcal{C}\times\mathcal{M}}((X, N), (X', N')) = Hom_{\mathcal{C}}(X, X') \times Hom_{\mathcal{M}}(N, N')$$

the diagram

$$\begin{array}{ccc} G(X\overline{\otimes}N) & \xrightarrow{d_{X,N}} & X\overline{\otimes}G(N) \\ {\scriptstyle G(f\overline{\otimes}g)}\downarrow & & \downarrow{\scriptstyle f\overline{\otimes}G(g)} \\ G(X'\overline{\otimes}N') & \xrightarrow[d_{X',N'}]{} & X'\overline{\otimes}G(N') \end{array}$$

commutes. We have

$$\begin{aligned}
(f\overline{\otimes}G(g)) \circ d_{X,N} &= (f\overline{\otimes}G(g)) \circ \beta^{-1}_{X\overline{\otimes}G(N)} \circ G(c^{-1}_{X,G(N)}) \circ G(id_X\overline{\otimes}\alpha_N^{-1}) \\
&\overset{(a)}{=} \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(F(f\overline{\otimes}G(g))) \circ G(c^{-1}_{X,G(N)}) \circ G(id_X\overline{\otimes}\alpha_N^{-1}) \\
&= \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(F(f\overline{\otimes}G(g)) \circ c^{-1}_{X,G(N)} \circ (id_X\overline{\otimes}\alpha_N^{-1})) \\
&\overset{(b)}{=} \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(c^{-1}_{X',G(N')} \circ (f\overline{\otimes}F(G(g))) \circ (id_X\overline{\otimes}\alpha_N^{-1})) \\
&= \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(c^{-1}_{X',G(N')} \circ (f\overline{\otimes}(F(G(g)) \circ \alpha_N^{-1}))) \\
&\overset{(c)}{=} \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(c^{-1}_{X',G(N')} \circ (f\overline{\otimes}(\alpha_{N'}^{-1} \circ g))) \\
&= \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(c^{-1}_{X',G(N')} \circ ((id_{X'} \circ f)\overline{\otimes}(\alpha_{N'}^{-1} \circ g))) \\
&= \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(c^{-1}_{X',G(N')} \circ (id_{X'}\overline{\otimes}\alpha_{N'}^{-1}) \circ (f\overline{\otimes}g)) \\
&= \beta^{-1}_{X'\overline{\otimes}G(N')} \circ G(c^{-1}_{X',G(N')}) \circ G(id_{X'}\overline{\otimes}\alpha_{N'}^{-1}) \circ G(f\overline{\otimes}g) \\
&= d_{X',N'} \circ G(f\overline{\otimes}g)
\end{aligned}$$

in which the equality (a) is due to the naturality of β^{-1}, (b) is by the naturality of c^{-1} and (c) via the naturality of α^{-1}.

The inverse[5] of d is

$$d^{-1} = \{d^{-1}_{X,N} = G(id_X\overline{\otimes}\alpha_N) \circ G(c_{X,G(N)}) \circ \beta_{X\overline{\otimes}G(N)} : X\overline{\otimes}G(N) \to G(X\overline{\otimes}N)\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}. \tag{11}$$

[5] This inverse of d will only be used on a result later on this work which states that the right adjoint of a $\mathcal{C}$-module functor, when exists, is a $\mathcal{C}$-module functor.

Furthermore, the pentagon and triangle diagrams of the functor G commute. Indeed,

$$
\begin{aligned}
&(id_X\overline{\otimes}d_{Y,N})\circ d_{X,Y\overline{\otimes}N}\circ G(m_{X,Y,N})\\
&= (id_X\overline{\otimes}(\beta^{-1}_{Y\otimes G(N)}\circ G(c^{-1}_{Y,G(N)})\circ G(id_Y\overline{\otimes}\alpha^{-1}_N)))\circ \beta^{-1}_{X\otimes G(Y\overline{\otimes}N)}\circ G(c^{-1}_{X,G(Y\overline{\otimes}N)})\circ\\
&\quad G(id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}N})\circ G(m_{X,Y,N})\\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ(id_X\overline{\otimes}G(id_Y\overline{\otimes}\alpha^{-1}_N))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}N)}\circ\\
&\quad G(c^{-1}_{X,G(Y\overline{\otimes}N)})\circ G(id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}N})\circ G(m_{X,Y,N})\\
&\overset{(a)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(F(id_X\overline{\otimes}G(id_Y\overline{\otimes}\alpha^{-1}_N)))\circ\\
&\quad G(c^{-1}_{X,G(Y\overline{\otimes}N)})\circ G(id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}N})\circ G(m_{X,Y,N})\\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(F(id_X\overline{\otimes}G(id_Y\overline{\otimes}\alpha^{-1}_N))\circ\\
&\quad c^{-1}_{X,G(Y\overline{\otimes}N)}\circ(id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}N})\circ m_{X,Y,N})\\
&\overset{(b)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}F(G(id_Y\overline{\otimes}\alpha^{-1}_N)))\circ(id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}N})\circ m_{X,Y,N})\\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}(F(G(id_Y\overline{\otimes}\alpha^{-1}_N))\circ\alpha^{-1}_{Y\overline{\otimes}N}))\circ m_{X,Y,N})\\
&\overset{(c)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}(\alpha^{-1}_{Y\overline{\otimes}F(G(N))}\circ(id_Y\overline{\otimes}\alpha^{-1}_N)))\circ m_{X,Y,N})\\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}F(G(N))})\circ(id_X\overline{\otimes}(id_Y\overline{\otimes}\alpha^{-1}_N))\circ m_{X,Y,N})\\
&\overset{(d)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ\beta^{-1}_{X\otimes G(Y\overline{\otimes}F(G(N)))}\circ G(c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}F(G(N))})\circ m_{X,Y,F(G(N))}\circ(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N))\\
&\overset{(a)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ\beta^{-1}_{X\otimes G(F(Y\otimes G(N)))}\circ G(F(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)})))\circ G(c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}F(G(N))})\circ m_{X,Y,F(G(N))}\circ(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N))\\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ\beta^{-1}_{X\otimes G(F(Y\otimes G(N)))}\circ G(F(id_X\overline{\otimes}G(c^{-1}_{Y,G(N)}))\circ c^{-1}_{X,G(Y\overline{\otimes}F(G(N)))}\circ\\
&\quad (id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}F(G(N))})\circ m_{X,Y,F(G(N))}\circ(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N))\\
&\overset{(b)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ\beta^{-1}_{X\otimes G(F(Y\otimes G(N)))}\circ G(c^{-1}_{X,G(F(Y\otimes G(N)))}\circ(id_X\overline{\otimes}F(G(c^{-1}_{Y,G(N)})))\circ\\
&\quad (id_X\overline{\otimes}\alpha^{-1}_{Y\overline{\otimes}F(G(N))})\circ m_{X,Y,F(G(N))}\circ(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N))\\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ\beta^{-1}_{X\otimes G(F(Y\otimes G(N)))}\circ G(c^{-1}_{X,G(F(Y\otimes G(N)))}\circ(id_X\overline{\otimes}(F(G(c^{-1}_{Y,G(N)}))\circ\\
&\quad \alpha^{-1}_{Y\overline{\otimes}F(G(N))}))\circ m_{X,Y,F(G(N))}\circ(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N))\\
&\overset{(c)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\otimes G(N)})\circ\beta^{-1}_{X\otimes G(F(Y\otimes G(N)))}\circ G(c^{-1}_{X,G(F(Y\otimes G(N)))}\circ(id_X\overline{\otimes}(\alpha^{-1}_{F(Y\otimes G(N))}\circ\\
&\quad c^{-1}_{Y,G(N)}))\circ m_{X,Y,F(G(N))}\circ(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N))
\end{aligned}
$$

$$
\begin{aligned}
&= (id_X\overline{\otimes}\beta^{-1}_{Y\overline{\otimes}G(N)}) \circ \beta^{-1}_{X\overline{\otimes}G(F(Y\overline{\otimes}G(N)))} \circ G(c^{-1}_{X,G(F(Y\overline{\otimes}G(N)))} \circ (id_X\overline{\otimes}\alpha^{-1}_{F(Y\overline{\otimes}G(N))})\circ \\
&\quad (id_X\overline{\otimes}c^{-1}_{Y,G(N)}) \circ m_{X,Y,F(G(N))} \circ (id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N)) \\
&\stackrel{(e)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\overline{\otimes}G(N)}) \circ \beta^{-1}_{X\overline{\otimes}G(F(Y\overline{\otimes}G(N)))} \circ G(c^{-1}_{X,G(F(Y\overline{\otimes}G(N)))} \circ (id_X\overline{\otimes}\alpha^{-1}_{F(Y\overline{\otimes}G(N))})\circ \\
&\quad c_{X,Y\overline{\otimes}G(N)} \circ F(m_{X,Y,G(N)}) \circ c^{-1}_{X\otimes Y,G(N)} \circ (id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N)) \\
&\stackrel{(f)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\overline{\otimes}G(N)}) \circ \beta^{-1}_{X\overline{\otimes}G(F(Y\overline{\otimes}G(N)))} \circ G(c^{-1}_{X,G(F(Y\overline{\otimes}G(N)))} \circ (id_X\overline{\otimes}F(\beta_{Y\overline{\otimes}G(N)}))\circ \\
&\quad c_{X,Y\overline{\otimes}G(N)} \circ F(m_{X,Y,G(N)}) \circ c^{-1}_{X\otimes Y,G(N)} \circ (id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N)) \\
&\stackrel{(g)}{=} (id_X\overline{\otimes}\beta^{-1}_{Y\overline{\otimes}G(N)}) \circ \beta^{-1}_{X\overline{\otimes}G(F(Y\overline{\otimes}G(N)))} \circ G(F(id_X\overline{\otimes}\beta_{Y\overline{\otimes}G(N)}) \circ F(m_{X,Y,G(N)})\circ \\
&\quad c^{-1}_{X\otimes Y,G(N)} \circ (id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N)) \\
&= (id_X\overline{\otimes}\beta^{-1}_{Y\overline{\otimes}G(N)}) \circ \beta^{-1}_{X\overline{\otimes}G(F(Y\overline{\otimes}G(N)))} \circ G(F(id_X\overline{\otimes}\beta_{Y\overline{\otimes}G(N)})) \circ G(F(m_{X,Y,G(N)}))\circ \\
&\quad G(c^{-1}_{X\otimes Y,G(N)}) \circ G(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N) \\
&\stackrel{(a)}{=} \beta^{-1}_{X\overline{\otimes}(Y\overline{\otimes}G(N))} \circ G(F(m_{X,Y,G(N)})) \circ G(c^{-1}_{X\otimes Y,G(N)}) \circ G(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N) \\
&\stackrel{(a)}{=} m_{X,Y,G(N)} \circ \beta^{-1}_{(X\otimes Y)\overline{\otimes}G(N)} \circ G(c^{-1}_{X\otimes Y,G(N)}) \circ G(id_{X\otimes Y}\overline{\otimes}\alpha^{-1}_N) \\
&= m_{X,Y,G(N)} \circ d_{X\otimes Y,N}
\end{aligned}
$$

in which the equalities labeled with (a) are valid due to the naturality of β^{-1}, those labeled with (b) are due to the naturality of c^{-1} and the ones with (c) hold for the reason that α^{-1} is a natural isomorphism. Moreover, the naturality of m implies the equality (d), while the commutativity of the pentagon diagram of the $\mathcal{C}$-module functor (F, c) implies (e). Finally, (f) and (g) comes from the equality $id_{F(Y\overline{\otimes}G(N))} = \alpha_{F(Y\overline{\otimes}G(N))} \circ F(\beta_{Y\overline{\otimes}G(N)})$ seen before and the naturality of c, respectively.

Moreover, we have that

$$
\begin{aligned}
l_{G(N)} \circ d_{\mathbf{1},N} &= l_{G(N)} \circ \beta^{-1}_{\mathbf{1}\overline{\otimes}G(N)} \circ G(c^{-1}_{\mathbf{1},G(N)}) \circ G(id_{\mathbf{1}}\overline{\otimes}\alpha^{-1}_N) \\
&\stackrel{(a)}{=} l_{G(N)} \circ \beta^{-1}_{\mathbf{1}\overline{\otimes}G(N)} \circ G(F(l^{-1}_{G(N)}) \circ l_{F(G(N))}) \circ G(id_{\mathbf{1}}\overline{\otimes}\alpha^{-1}_N) \\
&= l_{G(N)} \circ \beta^{-1}_{\mathbf{1}\overline{\otimes}G(N)} \circ G(F(l^{-1}_{G(N)})) \circ G(l_{F(G(N))}) \circ G(id_{\mathbf{1}}\overline{\otimes}\alpha^{-1}_N) \\
&= l_{G(N)} \circ \beta^{-1}_{\mathbf{1}\overline{\otimes}G(N)} \circ G(F(l^{-1}_{G(N)})) \circ G(l_{F(G(N))} \circ (id_{\mathbf{1}}\overline{\otimes}\alpha^{-1}_N)) \\
&\stackrel{(b)}{=} l_{G(N)} \circ \beta^{-1}_{\mathbf{1}\overline{\otimes}G(N)} \circ G(F(l^{-1}_{G(N)})) \circ G(\alpha^{-1}_N \circ l_N) \\
&\stackrel{(c)}{=} l_{G(N)} \circ l^{-1}_{G(N)} \circ \beta^{-1}_{G(N)} \circ G(\alpha^{-1}_N \circ l_N) \\
&= \beta^{-1}_{G(N)} \circ G(\alpha^{-1}_N \circ l_N) \\
&= \beta^{-1}_{G(N)} \circ G(\alpha^{-1}_N) \circ G(l_N) \\
&\stackrel{(d)}{=} id_{G(N)} \circ G(l_N) \\
&= G(l_N),
\end{aligned}
$$

i.e., the commutativity of the triangle diagram of the functor G. The equality (a) is valid due to the triangle diagram of the $\mathcal{C}$-module functor (F, c). The equalities (b) and (c) are due to the naturality of l and β, respectively. Lastly, (d) comes from the equality $id_{G(N)} = G(\alpha_N) \circ \beta_{G(N)}$. Therefore, $(G, d) : \mathcal{N} \to \mathcal{M}$ is a $\mathcal{C}$-module functor.

By the Proposition 2.2.10, the $\mathcal{C}$-module functor composition between (F, c) and (G, d) is defined as $(F, c) \circ (G, d) \coloneqq (F \circ G, b)$ with $b_{X,N} = c_{X,G(N)} \circ F(d_{X,N})$. Similarly, the composition $(G, d) \circ (F, c) \coloneqq (G \circ F, b')$ with $b'_{X,M} = d_{X,F(M)} \circ G(c_{X,M})$. Let us now see that α is a natural isomorphism of $\mathcal{C}$-module functors between $(F, c) \circ (G, d) = (F \circ G, b)$ and $Id_{\mathcal{N}}$. This can be described as the commutativity of the diagram

$$\begin{array}{ccc} F(G(X\overline{\otimes}N)) & \xrightarrow{\alpha_{X\overline{\otimes}N}} & X\overline{\otimes}N \\ {\scriptstyle b_{X,N}}\downarrow & & \downarrow{\scriptstyle ID_{X,N}=id_{X\overline{\otimes}N}} \\ X\overline{\otimes}F(G(N)) & \xrightarrow[id_X\overline{\otimes}\alpha_N]{} & X\overline{\otimes}N \end{array}$$

for all $X \in \mathcal{C}$ and $N \in \mathcal{N}$ given by

$$\begin{aligned} (id_X\overline{\otimes}\alpha_N) \circ b_{X,N} &= (id_X\overline{\otimes}\alpha_N) \circ c_{X,G(N)} \circ F(d_{X,N}) \\ &= (id_X\overline{\otimes}\alpha_N) \circ c_{X,G(N)} \circ F(\beta^{-1}_{X\overline{\otimes}G(N)} \circ G(c^{-1}_{X,G(N)}) \circ G(id_X\overline{\otimes}\alpha^{-1}_N)) \\ &= (id_X\overline{\otimes}\alpha_N) \circ c_{X,G(N)} \circ F(\beta^{-1}_{X\overline{\otimes}G(N)}) \circ F(G(c^{-1}_{X,G(N)})) \circ F(G(id_X\overline{\otimes}\alpha^{-1}_N)) \\ &\overset{(a)}{=} (id_X\overline{\otimes}\alpha_N) \circ c_{X,G(N)} \circ \alpha_{F(X\overline{\otimes}G(N))} \circ F(G(c^{-1}_{X,G(N)})) \circ F(G(id_X\overline{\otimes}\alpha^{-1}_N)) \\ &\overset{(b)}{=} (id_X\overline{\otimes}\alpha_N) \circ \alpha_{X\overline{\otimes}F(G(N))} \circ F(G(id_X\overline{\otimes}\alpha^{-1}_N)) \\ &\overset{(b)}{=} \alpha_{X\overline{\otimes}N} \end{aligned}$$

where (a) comes from the equality $id_{F(X\overline{\otimes}G(N))} = \alpha_{F(X\overline{\otimes}G(N))} \circ F(\beta_{X\overline{\otimes}G(N)})$, and the equalities labeled with (b) are valid due to the naturality of α. Therefore, α is a natural isomorphism of $\mathcal{C}$-module functors.

The last checking we must do is the one to verify that β is a natural isomorphism of $\mathcal{C}$-module functors between $Id_{\mathcal{M}}$ and $(G, d) \circ (F, c) = (G \circ F, b')$, i.e., the commutativity of

$$\begin{array}{ccc} X\overline{\otimes}M & \xrightarrow{\beta_{X\overline{\otimes}M}} & G(F(X\overline{\otimes}M)) \\ {\scriptstyle ID_{X,M}=id_{X\overline{\otimes}M}}\downarrow & & \downarrow{\scriptstyle b'_{X,M}} \\ X\overline{\otimes}M & \xrightarrow[id_X\overline{\otimes}\beta_M]{} & X\overline{\otimes}G(F(M)) \end{array}$$

for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$. In fact,

$$\begin{aligned} b'_{X,M} \circ \beta_{X\overline{\otimes}M} &= d_{X,F(M)} \circ G(c_{X,M}) \circ \beta_{X\overline{\otimes}M} \\ &= \beta^{-1}_{X\overline{\otimes}G(F(M))} \circ G(c^{-1}_{X,G(F(M))}) \circ G(id_X\overline{\otimes}\alpha^{-1}_{F(M)}) \circ G(c_{X,M}) \circ \beta_{X\overline{\otimes}M} \\ &= \beta^{-1}_{X\overline{\otimes}G(F(M))} \circ G(c^{-1}_{X,G(F(M))} \circ (id_X\overline{\otimes}\alpha^{-1}_{F(M)}) \circ c_{X,M}) \circ \beta_{X\overline{\otimes}M} \end{aligned}$$

$$\overset{(a)}{=} \beta^{-1}_{X\overline{\otimes}G(F(M))} \circ G(c^{-1}_{X,G(F(M))} \circ (id_X \overline{\otimes} F(\beta_M)) \circ c_{X,M}) \circ \beta_{X\overline{\otimes}M}$$
$$\overset{(b)}{=} \beta^{-1}_{X\overline{\otimes}G(F(M))} \circ G(F(id_X \overline{\otimes} \beta_M)) \circ \beta_{X\overline{\otimes}M}$$
$$\overset{(c)}{=} id_X \overline{\otimes} \beta_M$$

with the equality (a) being valid from the identity $id_{F(M)} = \alpha_{F(M)} \circ F(\beta_M)$, and the naturalities of c and β implying in the equalities labeled with (b) and (c), respectively. Thus β is a natural isomorphism of $\mathcal{C}$-module functors.

Therefore, these two definitions of $\mathcal{C}$-module category equivalence are equivalent, as wanted. ■

2.3 ADJOINT OF A $\mathcal{C}$-MODULE FUNCTOR

Let $\mathcal{M}$ and $\mathcal{N}$ be abelian and module categories over a rigid category $\mathcal{C}$ and $(F, c) : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{C}$-module functor.

Our goal with this section is to show that if (F, c) admits a left $F^{l.a.} : \mathcal{N} \to \mathcal{M}$ or right $F^{r.a.} : \mathcal{N} \to \mathcal{M}$ adjoint, then these adjoint functors admit a $\mathcal{C}$-module functor structure. This result and an idea of its proof is written in the beginning of Section 7.12 on the book [4] (for the case in which the categories involved have more properties), and we couldn't find its proof in any reference. We'll focus on the case that F admits a right adjoint $F^{r.a.}$ for the similarity of the other case.

For a better understanding, we divided this in some steps.

Step 1: For all $X \in \mathcal{C}$ and $N \in \mathcal{N}$, consider the contravariant functors given to the reader in the end of Section 1.4

$$R_{F^{r.a.}(X\overline{\otimes}N)} = Hom_{\mathcal{M}}(\_\_, F^{r.a.}(X\overline{\otimes}N)) : \mathcal{M} \to Set, \quad \text{and}$$
$$R_{X\overline{\otimes}F^{r.a.}(N)} = Hom_{\mathcal{M}}(\_\_, X\overline{\otimes}F^{r.a.}(N)) : \mathcal{M} \to Set.$$

We are going to show that there is a natural isomorphism

$$\varphi = \{\varphi^{X,N}_M : R_{F^{r.a.}(X\overline{\otimes}N)}(M) \to R_{X\overline{\otimes}F^{r.a.}(N)}(M)\}_{M\in\mathcal{M}}$$

where

$$\varphi^{X,N}_M : Hom_{\mathcal{M}}(M, F^{r.a.}(X\overline{\otimes}N)) \to Hom_{\mathcal{M}}(M, X\overline{\otimes}F^{r.a.}(N))$$

is an isomorphism in *Set*. This step will be further developed in Lemma 2.3.1.

Step 2: Next, by using the Proposition 1.4.6 we'll obtain that

$$\varphi^{X,N}_{F^{r.a.}(X\overline{\otimes}N)}(id_{F^{r.a.}(X\overline{\otimes}N)}) : F^{r.a.}(X\overline{\otimes}N) \to X\overline{\otimes}F^{r.a.}(N)$$

is an isomorphism in $\mathcal{M}$. This is going to be our candidate for the natural isomorphism[6] $d : F^{r.a.} \circ \overline{\otimes} \to \overline{\otimes} \circ (Id_{\mathcal{C}} \times F^{r.a.})$ with $d = \{d_{X,N} = \varphi^{X,N}_{F^{r.a.}(X\overline{\otimes}N)}(id_{F^{r.a.}(X\overline{\otimes}N)})\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}$

[6] With both functors from $\mathcal{C} \times \mathcal{N}$ to $\mathcal{M}$

and which gives the functor $F^{r.a.}$ a $\mathcal{C}$-module functor structure $(F^{r.a.}, d) : \mathcal{N} \to \mathcal{M}$. We'll explicit each morphism $d_{X,N}$ in the family d right after Lemma 2.3.1, and use it in the proof of the Theorem 2.3.2.

Step 3: We'll construct a natural transformation

$$d' = \{d'_{X,N} : X\overline{\otimes}F^{r.a.}(N) \to F^{r.a.}(X\overline{\otimes}N)\}_{(X,N)\in\mathcal{C}\times\mathcal{N}},$$

proof that this is the inverse of d and also that d' satisfy the pentagon and triangle diagrams which gives the functor $F^{r.a.}$ a $\mathcal{C}$-module functor structure. This step can be found as Theorem 2.3.2.

For the following results, suppose that $(F, c) : \mathcal{M} \to \mathcal{N}$ is a $\mathcal{C}$-module functor admitting a right adjoint $F^{r.a.} : \mathcal{N} \to \mathcal{M}$.

Lemma 2.3.1. *The collection φ defined in Step 1 is a natural isomorphism in $\mathcal{M}$.*

Proof. Since F has a right adjoint $F^{r.a.} : \mathcal{N} \to \mathcal{M}$, there is a natural isomorphism

$$\phi = \{\phi_{M,N} : Hom_{\mathcal{N}}(F(M), N) \to Hom_{\mathcal{M}}(M, F^{r.a.}(N))\}_{(M,N)\in\mathcal{M}^{op}\times\mathcal{N}}$$

in $\mathcal{M}$ and $\mathcal{N}$. As we've seen in Remark 1.3.7, we are going to fix the second entry with the object $X\overline{\otimes}N \in \mathcal{N}$. So we have

$$\phi^{-1}_{M,X\overline{\otimes}N} : Hom_{\mathcal{M}}(M, F^{r.a.}(X\overline{\otimes}N)) \to Hom_{\mathcal{N}}(F(M), X\overline{\otimes}N). \tag{12}$$

Next, we define the family of morphisms in *Set*

$$\overline{\phi} = \{\overline{\phi}_M : Hom_{\mathcal{N}}(F(X^*\overline{\otimes}M), N) \to Hom_{\mathcal{M}}(X^*\overline{\otimes}M, F^{r.a.}(N))\}$$

as $\overline{\phi}_M \coloneqq \phi_{X^*\overline{\otimes}M,N}$. The naturality of $\overline{\phi}$ in $\mathcal{M}$ can be obtained from the naturality of ϕ, and it's an isomorphism by construction.

We'll use the morphism

$$\overline{\phi}_M = \phi_{X^*\overline{\otimes}M,N} : Hom_{\mathcal{N}}(F(X^*\overline{\otimes}M), N) \to Hom_{\mathcal{M}}(X^*\overline{\otimes}M, F^{r.a.}(N)). \tag{13}$$

Now, using that the functor $X^*\overline{\otimes}\,\_ : \mathcal{N} \to \mathcal{N}$ is left adjoint to $X\overline{\otimes}\,\_ : \mathcal{N} \to \mathcal{N}$ (see Proposition 2.2.5), there exists a natural isomorphism $\theta' = \{\theta'_{N,N'} : Hom_{\mathcal{N}}(X^*\overline{\otimes}N, N') \to Hom_{\mathcal{N}}(N, X\overline{\otimes}N')\}_{N,N'\in\mathcal{N}}$. Similar as we've just done, let us define the family

$$\overline{\theta'} = \{\overline{\theta'}_M : Hom_{\mathcal{N}}(F(M), X\overline{\otimes}N) \to Hom_{\mathcal{N}}(X^*\overline{\otimes}F(M), N)\}_{M\in\mathcal{M}}$$

of morphisms in *Set* as $\overline{\theta'}_M \coloneqq \theta'^{-1}_{F(M),N}$. This is a natural isomorphism in $M \in \mathcal{M}$ via the adjunction θ'.

We're going to consider the isomorphism

$$\overline{\theta'^{-1}}_M = \theta'^{-1}_{F(M),N} : Hom_{\mathcal{N}}(F(M), X\overline{\otimes}N) \to Hom_{\mathcal{N}}(X^*\overline{\otimes}F(M), N) \tag{14}$$

in the collection $\overline{\theta'^{-1}}$.

Using the $\mathcal{C}$-module structure of F, that is, the natural isomorphism

$$c = \{c_{X,M} : F(X\overline{\otimes}M) \to X\overline{\otimes}F(M)\}_{(X,M)\in\mathcal{C}\times\mathcal{M}}$$

we can define

$$\varepsilon = \{\varepsilon_M \coloneqq Hom_{\mathcal{N}}(c_{X^*,M}, N) : Hom_{\mathcal{N}}(X^*\overline{\otimes}F(M), N) \to Hom_{\mathcal{N}}(F(X^*\overline{\otimes}M), N)\}_{M\in\mathcal{M}}, \tag{15}$$

which is a natural isomorphism[7] with inverse $\varepsilon^{-1} = \{\varepsilon_M^{-1} = Hom_{\mathcal{N}}(c_{X^*,M}^{-1}, N)\}_{M\in\mathcal{M}}$.

Lastly, consider the adjunction $(X^*\overline{\otimes}\,\_ : \mathcal{M} \to \mathcal{M}, X\overline{\otimes}\,\_ : \mathcal{M} \to \mathcal{M}, \theta)$ from the Proposition 2.2.5. We'll fix the second entry with $F^{r.a.}(N) \in \mathcal{M}$ and consider the isomorphism

$$\theta_{M,F^{r.a.}(N)} : Hom_{\mathcal{M}}(X^*\overline{\otimes}M, F^{r.a.}(N)) \to Hom_{\mathcal{M}}(M, X\overline{\otimes}F^{r.a.}(M)) \tag{16}$$

in the collection θ.

We define the composition of the isomorphisms in *Set* (16), (13), (15), (14) and (12)

$$R_{F^{r.a.}(X\overline{\otimes}N)}(M) = Hom_{\mathcal{M}}(M, F^{r.a.}(X\overline{\otimes}N)) \xrightarrow{\phi^{-1}_{M,X\overline{\otimes}N}} Hom_{\mathcal{N}}(F(M), X\overline{\otimes}N)$$

$$\xrightarrow{\overline{\theta'^{-1}}_M} Hom_{\mathcal{N}}(X^*\overline{\otimes}F(M), N) \xrightarrow{\varepsilon_M} Hom_{\mathcal{N}}(F(X^*\overline{\otimes}M), N)$$

$$\xrightarrow{\overline{\phi}_M} Hom_{\mathcal{M}}(X^*\overline{\otimes}M, F^{r.a.}(N)) \xrightarrow{\theta_{M,F^{r.a.}(N)}} Hom_{\mathcal{M}}(M, X\overline{\otimes}F^{r.a.}(N)) = R_{X\overline{\otimes}F^{r.a.}(N)}(M)$$

to be $\varphi_M^{X,N}$, that is,

$$\varphi_M^{X,N} \coloneqq \theta_{M,F^{r.a.}(N)} \circ \overline{\phi}_M \circ \varepsilon_M \circ \overline{\theta'^{-1}}_M \circ \phi^{-1}_{M,X\overline{\otimes}N}.$$

Furthermore, it's easy to see that the collection $\varphi = \{\varphi_M^{X,N}\}_{M\in\mathcal{M}}$ is natural in $\mathcal{M}$ because it is a composition of natural isomorphisms in $\mathcal{M}$. ∎

Via Proposition 1.4.6, $\varphi^{X,N}_{F^{r.a.}(X\overline{\otimes}N)}(id_{F^{r.a.}(X\overline{\otimes}N)}) : F^{r.a.}(X\overline{\otimes}N) \to X\overline{\otimes}F^{r.a.}(N)$ is an isomorphism in $\mathcal{M}$. This is our candidate $d = \{d_{X,N} \coloneqq \varphi^{X,N}_{F^{r.a.}(X\overline{\otimes}N)}(id_{F^{r.a.}(X\overline{\otimes}N)})\}_{X\in\mathcal{C},N\in\mathcal{N}}$ for the natural isomorphism that gives our functor $F^{r.a.}$ a $\mathcal{C}$-module functor structure.

Before analyzing what morphism is going to be the inverse of $d_{X,N}$, let us define the counit $\alpha = \{\alpha_N \overset{(*)}{\coloneqq} \phi^{-1}_{F^{r.a.}(N),N}(id_{F^{r.a.}(N)})\}_{N\in\mathcal{N}}$ and unit $\beta = \{\beta_M \coloneqq \phi_{M,F(M)}(id_{F(M)})\}_{M\in\mathcal{M}}$

[7] This follows from the natural isomorphism c.

of the adjunction $(F, F^{r.a.}, \phi)$ as in the Proposition 1.3.8. They satisfy the relations

$$id_{F^{r.a.}(N)} = F^{r.a.}(\alpha_N) \circ \beta_{F^{r.a.}(N)}, \quad \text{and} \tag{17}$$

$$id_{F(M)} = \alpha_{F(M)} \circ F(\beta_M) \tag{18}$$

for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$. Moreover, it's possible to explicitly write each morphism in the family d as

$$\begin{aligned}
d_{X,N} &= \varphi^{X,N}_{F^{r.a.}(X\overline{\otimes}N)}(id_{F^{r.a.}(X\overline{\otimes}N)}) \\
&= (\theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)} \circ \overline{\phi}_{F^{r.a.}(X\overline{\otimes}N)} \circ \varepsilon_{F^{r.a.}(X\overline{\otimes}N)} \circ \overline{\theta'^{-1}}_{F^{r.a.}(X\overline{\otimes}N)} \circ \\
&\quad \phi^{-1}_{F^{r.a.}(X\overline{\otimes}N),X\overline{\otimes}N})(id_{F^{r.a.}(X\overline{\otimes}N)}) \\
&= (\theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)} \circ \phi_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N),N} \circ \varepsilon_{F^{r.a.}(X\overline{\otimes}N)} \circ \theta'^{-1}_{F(F^{r.a.}(X\overline{\otimes}N)),N} \circ \\
&\quad \phi^{-1}_{F^{r.a.}(X\overline{\otimes}N),X\overline{\otimes}N})(id_{F^{r.a.}(X\overline{\otimes}N)}) \\
&= \theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)}(\phi_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N),N}(Hom_{\mathcal{N}}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}, N)(\theta'^{-1}_{F(F^{r.a.}(X\overline{\otimes}N)),N} \\
&\quad (\phi^{-1}_{F^{r.a.}(X\overline{\otimes}N),X\overline{\otimes}N}(id_{F^{r.a.}(X\overline{\otimes}N)}))))) \\
&\overset{(*)}{=} \theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)}(\phi_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N),N}(Hom_{\mathcal{N}}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}, N) \\
&\quad (\theta'^{-1}_{F(F^{r.a.}(X\overline{\otimes}N)),N}(\alpha_{X\overline{\otimes}N})))) \\
&\overset{(10)}{=} \theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)}(\phi_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N),N}(Hom_{\mathcal{N}}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}, N)(l_N \circ (ev_X\overline{\otimes}id_N) \circ \\
&\quad m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})))) \\
&= \theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)}(\phi_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N),N}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ \\
&\quad c_{X^*,F^{r.a.}(X\overline{\otimes}N)})) \\
&\overset{(2)}{=} \theta_{F^{r.a.}(X\overline{\otimes}N),F^{r.a.}(N)}(F^{r.a.}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ \\
&\quad c_{X^*,F^{r.a.}(X\overline{\otimes}N)}) \circ \beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}) \\
&\overset{(9)}{=} (id_X\overline{\otimes}(F^{r.a.}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}) \circ \\
&\quad \beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})) \circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)} \circ (coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a}(X\overline{\otimes}N)} \\
&= (id_X\overline{\otimes}(F^{r.a.}(l_N) \circ F^{r.a.}(ev_X\overline{\otimes}id_N) \circ F^{r.a.}(m^{-1}_{X^*,X,N}) \circ F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ \\
&\quad F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}) \circ \beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})) \circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)} \circ (coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ \\
&\quad l^{-1}_{F^{r.a.}(X\overline{\otimes}N)} \\
&= (id_X\overline{\otimes}F^{r.a.}(l_N)) \circ (id_X\overline{\otimes}F^{r.a.}(ev_X\overline{\otimes}id_N)) \circ (id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N})) \circ \\
&\quad (id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})) \circ (id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)})) \circ (id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}) \circ \\
&\quad m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)} \circ (coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)},
\end{aligned}$$

and this can visually be seen as the composition

$$
\begin{array}{l}
F^{r.a.}(X\overline{\otimes}N) \xrightarrow{l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}} \mathbf{1}\overline{\otimes}F^{r.a.}(X\overline{\otimes}N) \\
\xrightarrow{coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}} (X\otimes X^*)\overline{\otimes}F^{r.a.}(X\overline{\otimes}N) \xrightarrow{m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}} X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)) \\
\xrightarrow{id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}} X\overline{\otimes}F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))) \xrightarrow{id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)})} X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N))) \\
\xrightarrow{id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})} X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N)) \xrightarrow{id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N})} X\overline{\otimes}F^{r.a.}((X^*\otimes X)\overline{\otimes}N) \\
\xrightarrow{id_X\overline{\otimes}F^{r.a.}(ev_X\overline{\otimes}id_N)} X\overline{\otimes}F^{r.a.}(\mathbf{1}\overline{\otimes}N) \xrightarrow{id_X\overline{\otimes}F^{r.a.}(l_N)} X\overline{\otimes}F^{r.a.}(N).
\end{array}
$$

Theorem 2.3.2. *Let $(F, c) : \mathcal{M} \to \mathcal{N}$ a $\mathcal{C}$-module functor. If the functor F admits a left $F^{l.a.} : \mathcal{N} \to \mathcal{M}$ or right $F^{r.a.} : \mathcal{N} \to \mathcal{M}$ adjoint, then this functor has a structure of $\mathcal{C}$-module functor.*

Proof. Suppose that the functor F admits a right $F^{r.a.} : \mathcal{N} \to \mathcal{M}$ adjoint and consider the natural isomorphism $d = \{d_{X,N}\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}$ to be the one defined above. Firstly, the natural transformation[8]

$$d' = \{d'_{X,N} : X\overline{\otimes}F^{r.a.}(N) \to F^{r.a.}(X\overline{\otimes}N)\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}$$

in which $d'_{X,N}$ is the inverse of $d_{X,N}$ for all $(X, N) \in \mathcal{C} \times \mathcal{N}$ is exactly d^{-1} defined in Proposition 2.2.12 as equation (11), i.e.,

$$d' = \{d'_{X,N} := F^{r.a.}(id_X\overline{\otimes}\alpha_N) \circ F^{r.a.}(c_{X,F^{r.a.}(N)}) \circ \beta_{X\overline{\otimes}F^{r.a.}(N)} : X\overline{\otimes}F^{r.a.}(N) \to F^{r.a.}(X\overline{\otimes}N)\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}.$$

Notice that we can only guarantee that d' is a natural transformation, and not necessarily a natural isomorphism since in our hypothesis α and β are just natural transformations instead of natural isomorphisms (as in Proposition 2.2.12).

[8] Between the functors $\overline{\otimes} \circ (Id_{\mathcal{C}} \times F^{r.a.})$ and $F^{r.a.} \circ \overline{\otimes}$.

Each $d_{X,N}$ is the inverse of $d'_{X,N}$ for all $X \in \mathcal{C}$ and $N \in \mathcal{N}$ as we can see by

$$\begin{aligned}
d'_{X,N} \circ d_{X,N} &= F^{r.a.}(id_X \overline{\otimes} \alpha_N) \circ F^{r.a.}(c_{X,F^{r.a.}(N)}) \circ \beta_{X \overline{\otimes} F^{r.a.}(N)} \circ (id_X \overline{\otimes} F^{r.a.}(l_N)) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)} \\
&= F^{r.a.}((id_X \overline{\otimes} \alpha_N) \circ c_{X,F^{r.a.}(N)}) \circ \beta_{X \overline{\otimes} F^{r.a.}(N)} \circ (id_X \overline{\otimes} F^{r.a.}(l_N)) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)} \\
&\overset{(a)}{=} F^{r.a.}((id_X \overline{\otimes} \alpha_N) \circ c_{X,F^{r.a.}(N)}) \circ F^{r.a.}(F(id_X \overline{\otimes} F^{r.a.}(l_N))) \circ \beta_{X \overline{\otimes} F^{r.a.}(\mathbf{1} \overline{\otimes} N)} \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)} \\
&= F^{r.a.}((id_X \overline{\otimes} \alpha_N) \circ c_{X,F^{r.a.}(N)} \circ F(id_X \overline{\otimes} F^{r.a.}(l_N))) \circ \beta_{X \overline{\otimes} F^{r.a.}(\mathbf{1} \overline{\otimes} N)} \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)} \\
&\overset{(b)}{=} F^{r.a.}((id_X \overline{\otimes} \alpha_N) \circ (id_X \overline{\otimes} F(F^{r.a.}(l_N))) \circ c_{X,F^{r.a.}(\mathbf{1} \overline{\otimes} N)}) \circ \beta_{X \overline{\otimes} F^{r.a.}(\mathbf{1} \overline{\otimes} N)} \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)} \\
&= F^{r.a.}((id_X \overline{\otimes} (\alpha_N \circ F(F^{r.a.}(l_N)))) \circ c_{X,F^{r.a.}(\mathbf{1} \overline{\otimes} N)}) \circ \beta_{X \overline{\otimes} F^{r.a.}(\mathbf{1} \overline{\otimes} N)} \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)} \\
&\overset{(c)}{=} F^{r.a.}((id_X \overline{\otimes} (l_N \circ \alpha_{\mathbf{1} \overline{\otimes} \mathcal{N}})) \circ c_{X,F^{r.a.}(\mathbf{1} \overline{\otimes} N)}) \circ \beta_{X \overline{\otimes} F^{r.a.}(\mathbf{1} \overline{\otimes} N)} \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(ev_X \overline{\otimes} id_N)) \circ (id_X \overline{\otimes} F^{r.a.}(m^{-1}_{X^*,X,N})) \circ (id_X \overline{\otimes} F^{r.a.}(id_{X^*} \overline{\otimes} \alpha_{X \overline{\otimes} N})) \circ \\
&\quad (id_X \overline{\otimes} F^{r.a.}(c_{X^*,F^{r.a.}(X \overline{\otimes} N)})) \circ (id_X \overline{\otimes} \beta_{X^* \overline{\otimes} F^{r.a.}(X \overline{\otimes} N)}) \circ m_{X,X^*,F^{r.a.}(X \overline{\otimes} N)} \circ \\
&\quad (coev_X \overline{\otimes} id_{F^{r.a.}(X \overline{\otimes} N)}) \circ l^{-1}_{F^{r.a.}(X \overline{\otimes} N)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ\alpha_{\mathbf{1}\overline{\otimes}\mathcal{N}}))\circ c_{X,F^{r.a.}(\mathbf{1}\overline{\otimes}N)})\circ F^{r.a.}(F(id_X\overline{\otimes}F^{r.a.}(ev_X\overline{\otimes}id_N)))\circ\\
&\quad\beta_{X\overline{\otimes}F^{r.a.}((X^*\otimes X)\overline{\otimes}N)}\circ(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N}))\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ\alpha_{\mathbf{1}\overline{\otimes}\mathcal{N}}))\circ c_{X,F^{r.a.}(\mathbf{1}\overline{\otimes}N)}\circ F(id_X\overline{\otimes}F^{r.a.}(ev_X\overline{\otimes}id_N)))\circ\\
&\quad\beta_{X\overline{\otimes}F^{r.a.}((X^*\otimes X)\overline{\otimes}N)}\circ(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N}))\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(b)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ\alpha_{\mathbf{1}\overline{\otimes}\mathcal{N}}))\circ(id_X\overline{\otimes}F(F^{r.a.}(ev_X\overline{\otimes}id_N)))\circ c_{X,F^{r.a.}((X^*\otimes X)\overline{\otimes}N)})\circ\\
&\quad\beta_{X\overline{\otimes}F^{r.a.}((X^*\otimes X)\overline{\otimes}N)}\circ(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N}))\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ\alpha_{\mathbf{1}\overline{\otimes}\mathcal{N}}\circ F(F^{r.a.}(ev_X\overline{\otimes}id_N))))\circ c_{X,F^{r.a.}((X^*\otimes X)\overline{\otimes}N)})\circ\\
&\quad\beta_{X\overline{\otimes}F^{r.a.}((X^*\otimes X)\overline{\otimes}N)}\circ(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N}))\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(c)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ\alpha_{(X^*\otimes X)\overline{\otimes}\mathcal{N}}))\circ c_{X,F^{r.a.}((X^*\otimes X)\overline{\otimes}N)})\circ\\
&\quad\beta_{X\overline{\otimes}F^{r.a.}((X^*\otimes X)\overline{\otimes}N)}\circ(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N}))\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ\alpha_{(X^*\otimes X)\overline{\otimes}\mathcal{N}}))\circ c_{X,F^{r.a.}((X^*\otimes X)\overline{\otimes}N)})\circ\\
&\quad F^{r.a.}(F(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N})))\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))}\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ\alpha_{(X^*\otimes X)\overline{\otimes}\mathcal{N}}))\circ c_{X,F^{r.a.}((X^*\otimes X)\overline{\otimes}N)}\circ\\
&\quad F(id_X\overline{\otimes}F^{r.a.}(m^{-1}_{X^*,X,N})))\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))}\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ\\
&\quad(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}
\end{aligned}$$

$$\overset{(b)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ\alpha_{(X^*\otimes X)\overline{\otimes}\mathcal{N}}))\circ(id_X\overline{\otimes}F(F^{r.a.}(m^{-1}_{X^*,X,N})))\circ$$
$$c_{X,F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))})\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))}\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ$$
$$(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ\alpha_{(X^*\otimes X)\overline{\otimes}\mathcal{N}}\circ F(F^{r.a.}(m^{-1}_{X^*,X,N}))))\circ$$
$$c_{X,F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))})\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))}\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ$$
$$(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(c)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ\alpha_{X^*\overline{\otimes}(X\overline{\otimes}\mathcal{N})}))\circ c_{X,F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))})\circ$$
$$\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))}\circ(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))\circ(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ\alpha_{X^*\overline{\otimes}(X\overline{\otimes}\mathcal{N})}))\circ c_{X,F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))})\circ$$
$$F^{r.a.}(F(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})))\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))}\circ$$
$$(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ\alpha_{X^*\overline{\otimes}(X\overline{\otimes}\mathcal{N})}))\circ c_{X,F^{r.a.}(X^*\overline{\otimes}(X\overline{\otimes}N))}\circ$$
$$F(id_X\overline{\otimes}F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})))\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))}\circ$$
$$(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(b)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ\alpha_{X^*\overline{\otimes}(X\overline{\otimes}\mathcal{N})}))\circ(id_X\overline{\otimes}F(F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})))\circ$$
$$c_{X,F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))})\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))}\circ(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ\alpha_{X^*\overline{\otimes}(X\overline{\otimes}\mathcal{N})}\circ F(F^{r.a.}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))))\circ$$
$$c_{X,F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))})\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))}\circ(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(c)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ\alpha_{X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ$$
$$c_{X,F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))})\circ\beta_{X\overline{\otimes}F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))}\circ(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$
\begin{aligned}
&\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ\alpha_{X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ\\
&\quad c_{X,F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))})\circ F^{r.a.}(F(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)})))\circ\\
&\quad \beta_{X\overline{\otimes}F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))}\circ(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad (coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ\alpha_{X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ\\
&\quad c_{X,F^{r.a.}(X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}N)))}\circ F(id_X\overline{\otimes}F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)})))\circ\beta_{X\overline{\otimes}F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))}\circ\\
&\quad (id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(b)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ\alpha_{X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ\\
&\quad (id_X\overline{\otimes}F(F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)})))\circ c_{X,F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))})\circ\beta_{X\overline{\otimes}F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))}\circ\\
&\quad (id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ\alpha_{X^*\overline{\otimes}F(F^{r.a.}(X\overline{\otimes}\mathcal{N}))}\circ\\
&\quad F(F^{r.a.}(c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))))\circ c_{X,F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))})\circ\beta_{X\overline{\otimes}F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))}\circ\\
&\quad (id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(c)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad \alpha_{F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ c_{X,F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))})\circ\beta_{X\overline{\otimes}F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))}\circ\\
&\quad (id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad \alpha_{F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ c_{X,F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))})\circ F^{r.a.}(F(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}))\circ\\
&\quad \beta_{X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))}\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad \alpha_{F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ c_{X,F^{r.a.}(F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)))}\circ F(id_X\overline{\otimes}\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}))\circ\\
&\quad \beta_{X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))}\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(b)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad \alpha_{F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ(id_X\overline{\otimes}F(\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}))\circ c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ\\
&\quad \beta_{X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))}\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&= F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad \alpha_{F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}\mathcal{N}))}\circ F(\beta_{X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})))\circ c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ\\
&\quad \beta_{X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))}\circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}\\
&\overset{(18)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N\circ(ev_X\overline{\otimes}id_N)\circ m^{-1}_{X^*,X,N}\circ(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}\circ\\
&\quad id_{F(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}\mathcal{N}))}))\circ c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)})\circ\beta_{X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))}\circ\\
&\quad m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})\circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}
\end{aligned}
$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}) \circ \beta_{X\overline{\otimes}(X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N))} \circ m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}) \circ F^{r.a.}(F(m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{(X\otimes X^*)\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}) \circ c_{X^*,F^{r.a.}(X\overline{\otimes}N)}))\circ$$
$$c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)} \circ F(m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{(X\otimes X^*)\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ (id_X\overline{\otimes}c_{X^*,F^{r.a.}(X\overline{\otimes}N)})\circ$$
$$c_{X,X^*\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)} \circ F(m_{X,X^*,F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{(X\otimes X^*)\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)}\circ$$
$$(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(d)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$c_{X\otimes X^*,F^{r.a.}(X\overline{\otimes}N)}) \circ \beta_{(X\otimes X^*)\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)} \circ (coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)}) \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$c_{X\otimes X^*,F^{r.a.}(X\overline{\otimes}N)}) \circ F^{r.a.}(F(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{\mathbf{1}\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)} \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$c_{X\otimes X^*,F^{r.a.}(X\overline{\otimes}N)} \circ F(coev_X\overline{\otimes}id_{F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{\mathbf{1}\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)} \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(b)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$(coev_X\overline{\otimes}F(id_{F^{r.a.}(X\overline{\otimes}N)})) \circ c_{\mathbf{1},F^{r.a.}(X\overline{\otimes}N)}) \circ \beta_{\mathbf{1}\overline{\otimes}F^{r.a.}(X\overline{\otimes}N)} \circ l^{-1}_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(a)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$(coev_X\overline{\otimes}id_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ c_{\mathbf{1},F^{r.a.}(X\overline{\otimes}N)}) \circ F^{r.a.}(F(l^{-1}_{F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$(coev_X\overline{\otimes}id_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ c_{\mathbf{1},F^{r.a.}(X\overline{\otimes}N)} \circ F(l^{-1}_{F^{r.a.}(X\overline{\otimes}N)})) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\overset{(e)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N} \circ (id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N}))) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$(coev_X\overline{\otimes}id_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ l^{-1}_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ (id_X\overline{\otimes}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$(coev_X\overline{\otimes}id_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ l^{-1}_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)}$$

$$= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ (id_X\overline{\otimes}(id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})) \circ m_{X,X^*,F(F^{r.a.}(X\overline{\otimes}N))}\circ$$
$$(coev_X\overline{\otimes}id_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ l^{-1}_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)}$$

$$\begin{aligned}
&\stackrel{(f)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ m_{X,X^*,X\overline{\otimes}N} \circ (id_{X\otimes X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ \\
&\quad (coev_X\overline{\otimes}id_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ l^{-1}_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)} \\
&= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ m_{X,X^*,X\overline{\otimes}N} \circ (coev_X\overline{\otimes}id_{X\overline{\otimes}N}) \circ (id_{\mathbf{1}}\overline{\otimes}\alpha_{X\overline{\otimes}N})\circ \\
&\quad l^{-1}_{F(F^{r.a.}(X\overline{\otimes}N))}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)} \\
&\stackrel{(g)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ m_{X,X^*,X\overline{\otimes}N} \circ (coev_X\overline{\otimes}id_{X\overline{\otimes}N}) \circ l^{-1}_{X\overline{\otimes}N}\circ \\
&\quad \alpha_{X\overline{\otimes}N}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)} \\
&= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ m_{X,X^*,X\overline{\otimes}N} \circ (coev_X\overline{\otimes}id_{X\overline{\otimes}N}) \circ l^{-1}_{X\overline{\otimes}N})\circ \\
&\quad F^{r.a.}(\alpha_{X\overline{\otimes}N}) \circ \beta_{F^{r.a.}(X\overline{\otimes}N)} \\
&\stackrel{(17)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N) \circ m^{-1}_{X^*,X,N})) \circ m_{X,X^*,X\overline{\otimes}N} \circ (coev_X\overline{\otimes}id_{X\overline{\otimes}N}) \circ l^{-1}_{X\overline{\otimes}N})\circ \\
&\quad id_{F^{r.a.}(X\overline{\otimes}N)} \\
&= F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N))) \circ (id_X\overline{\otimes}m^{-1}_{X^*,X,N}) \circ m_{X,X^*,X\overline{\otimes}N} \circ (coev_X\overline{\otimes}id_{X\overline{\otimes}N}) \circ l^{-1}_{X\overline{\otimes}N}) \\
&\stackrel{(h)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N))) \circ m_{X,X^*\otimes X,N} \circ (a_{X,X^*,X}\overline{\otimes}id_N) \circ m^{-1}_{X\otimes X^*,X,N}\circ \\
&\quad (coev_X\overline{\otimes}id_{X\overline{\otimes}N}) \circ l^{-1}_{X\overline{\otimes}N}) \\
&\stackrel{(f)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N))) \circ m_{X,X^*\otimes X,N} \circ (a_{X,X^*,X}\overline{\otimes}id_N) \circ ((coev_X \otimes id_X)\overline{\otimes}id_N)\circ \\
&\quad m^{-1}_{\mathbf{1},X,N} \circ l^{-1}_{X\overline{\otimes}N}) \\
&\stackrel{(i)}{=} F^{r.a.}((id_X\overline{\otimes}(l_N \circ (ev_X\overline{\otimes}id_N))) \circ m_{X,X^*\otimes X,N} \circ (a_{X,X^*,X}\overline{\otimes}id_N) \circ ((coev_X \otimes id_X)\overline{\otimes}id_N)\circ \\
&\quad (l_X^{-1}\overline{\otimes}id_N)) \\
&= F^{r.a.}((id_X\overline{\otimes}l_N) \circ (id_X\overline{\otimes}(ev_X\overline{\otimes}id_N)) \circ m_{X,X^*\otimes X,N} \circ (a_{X,X^*,X}\overline{\otimes}id_N) \circ ((coev_X \otimes id_X)\overline{\otimes}id_N)\circ \\
&\quad (l_X^{-1}\overline{\otimes}id_N)) \\
&\stackrel{(f)}{=} F^{r.a.}((id_X\overline{\otimes}l_N) \circ m_{X,\mathbf{1},N} \circ ((id_X \otimes ev_X)\overline{\otimes}id_N) \circ (a_{X,X^*,X}\overline{\otimes}id_N) \circ ((coev_X \otimes id_X)\overline{\otimes}id_N)\circ \\
&\quad (l_X^{-1}\overline{\otimes}id_N)) \\
&\stackrel{(j)}{=} F^{r.a.}((r_X\overline{\otimes}id_N) \circ ((id_X \otimes ev_X)\overline{\otimes}id_N) \circ (a_{X,X^*,X}\overline{\otimes}id_N) \circ ((coev_X \otimes id_X)\overline{\otimes}id_N) \circ (l_X^{-1}\overline{\otimes}id_N)) \\
&= F^{r.a.}((r_X \circ (id_X \otimes ev_X) \circ a_{X,X^*,X} \circ (coev_X \otimes id_X) \circ l_X^{-1})\overline{\otimes}id_N) \\
&\stackrel{(k)}{=} F^{r.a.}(id_X\overline{\otimes}id_N) \\
&= F^{r.a.}(id_{X\overline{\otimes}N}) \\
&= id_{F^{r.a.}(X\overline{\otimes}N)},
\end{aligned}$$

where the equalities labeled with (a) hold via the naturality of β, those labeled with (b) are valid due to the naturality of c, and the ones labeled with (c) come from the naturality of α. The commutativity of the pentagon and triangle diagrams of the $\mathcal{C}$-module functor (F, c) imply in the equalities (d) and (e), respectively, while the natural property of m and l are used in the equalities (f) and (g), respectively. The equality labeled with (h) is valid due to the pentagon diagram of the $\mathcal{C}$-module category $\mathcal{N}$, while the triangle diagram is used in the one labeled with (j). The Proposition 2.2.3 is used in the equality (i), and an identity of Definition 2.1.6 (the right dual X^* of the object $X \in \mathcal{C}$) for the one labeled with (k).

Using the fact that $d_{X,N}$ is an isomorphism for all $X \in \mathcal{C}$ and $N \in \mathcal{N}$, we can conclude that $d^{-1}_{X,N} \circ d'_{X,N} = id_{X \overline{\otimes} F^{r.a.}(N)}$. This implies that $d'_{X,N}$ is an isomorphism (it is the inverse of $d_{X,N}$) for all $X \in \mathcal{C}$ and $N \in \mathcal{N}$. Furthermore, for all $X, Y \in \mathcal{C}$ and $N \in \mathcal{N}$, the commutativity of the pentagon and triangle diagrams

$$\begin{array}{ccccc} & & F^{r.a.}((X \otimes Y)\overline{\otimes}N) & & \\ & \nearrow^{d'_{X\otimes Y,N}} & & \searrow^{F^{r.a.}(m_{X,Y,N})} & \\ (X \otimes Y)\overline{\otimes}F^{r.a.}(N) & & & & F^{r.a.}(X\overline{\otimes}(Y\overline{\otimes}N)) \\ \downarrow^{m_{X,Y,F^{r.a.}(N)}} & & & & \uparrow^{d'_{X,Y\overline{\otimes}N}} \\ X\overline{\otimes}(Y\overline{\otimes}F^{r.a.}(N)) & & \xrightarrow{id_X\overline{\otimes}d'_{Y,N}} & & X\overline{\otimes}F^{r.a.}(Y\overline{\otimes}N), \end{array}$$

and

$$\begin{array}{ccc} F^{r.a.}(\mathbf{1}\overline{\otimes}N) & \xleftarrow{d'_{\mathbf{1},N}} & \mathbf{1}\overline{\otimes}F^{r.a.}(N) \\ & {}_{F^{r.a.}(l_N)}\searrow \quad \swarrow_{l_{F^{r.a.}(N)}} & \\ & F^{r.a.}(N) & \end{array}$$

can be checked similarly as how they were done in the Proposition 2.2.12 (with the natural isomorphism d).

Hence, $d' = d^{-1} = \{d'_{X,N} = d^{-1}_{X,N}\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}$ is a natural isomorphism in $\mathcal{C}\times\mathcal{N}$ satisfying the pentagon and triangle diagram for the functor $F^{r.a.}$, which clearly implies that $d = \{d_{X,N}\}_{(X,N)\in\mathcal{C}\times\mathcal{N}}$ also satisfies the pentagon and the triangle diagrams of the functor $F^{r.a.}$. Then, $(F^{r.a.}, d) : \mathcal{N} \to \mathcal{M}$ is a $\mathcal{C}$-module functor.

Analogously, one can check that if the $\mathcal{C}$-module functor (F, c) admits a left adjoint $F^{l.a.} : \mathcal{N} \to \mathcal{M}$, then $F^{l.a.}$ has a structure of $\mathcal{C}$-module functor. We now explicit the $\mathcal{C}$-module structure d of the functor $F^{l.a.}$ since we'll use it later.

Let $\alpha : Id_{\mathcal{N}} \to F \circ F^{l.a.}$ and $\beta : F^{l.a.} \circ F \to Id_{\mathcal{M}}$ be the counit and unit of this adjunction, respectively. For all $X \in \mathcal{C}$ and $N \in \mathcal{N}$, the morphism $d_{X,N}$ is defined as the composition

$$F^{l.a.}(X\overline{\otimes}N) \xrightarrow{F^{l.a.}(id_X\overline{\otimes}\beta_N)} F^{l.a.}(X\overline{\otimes}F(F^{l.a.}(N))) \xrightarrow{F^{l.a.}(c^{-1}_{X,F^{l.a.}(N)})} F^{l.a.}(F(X\overline{\otimes}F^{l.a.}(N))) \xrightarrow{\alpha_{X\overline{\otimes}F^{l.a.}(N)}} X\overline{\otimes}F^{l.a.}(N),$$

so we may define the natural isomorphism d that gives $F^{l.a.}$ a $\mathcal{C}$-module functor structure by

$$d = \{d_{X,N} \coloneqq \alpha_{X\overline{\otimes}F^{l.a.}(N)} \circ F^{l.a.}(c^{-1}_{X,F^{l.a.}(N)}) \circ F^{l.a.}(id_X\overline{\otimes}\beta_N)\}_{(X,N)\in\mathcal{C}^{op}\times\mathcal{N}}. \tag{19}$$

The checking that $(F^{l.a.}, d)$ is indeed a $\mathcal{C}$-module functor can be done similarly as the case present in the first part of this proof. ■

2.4 EXACT MODULE CATEGORIES OVER MULTITENSOR CATEGORIES

We'll be interested in module categories with some more properties. From now on our category $\mathcal{C}$ is at least multitensor over $\Bbbk$, and $\mathcal{M}$ is a locally finite and module category over $\mathcal{C}$, unless stated otherwise.

Proposition 2.4.1 ([4], Proposition 7.3.4)**.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two module categories over $\mathcal{C}$. Then the category $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with module product*

$$\overline{\otimes} : \mathcal{C} \times \mathcal{M}_1 \oplus \mathcal{M}_2 \to \mathcal{M}_1 \oplus \mathcal{M}_2$$
$$X\overline{\otimes}(M, N) \mapsto (X\overline{\otimes}M, X\overline{\otimes}N),$$

associativity and unit being sums of those of $\mathcal{M}_1$ and $\mathcal{M}_2$ is a module category over $\mathcal{C}$.

The module category $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is called the *direct sum* of the module categories $\mathcal{M}_1$ and $\mathcal{M}_2$.

Definition 2.4.2. *Let $\mathcal{C}$ be a multitensor category and $\mathcal{M}$ a locally finite and module category over $\mathcal{C}$. The $\mathcal{C}$-module category $\mathcal{M}$ is said to be*

a) *indecomposable, if it is not equivalent to a nontrivial direct sum of nonzero module categories;*

b) *exact, if $\mathcal{C}$ has also enough projective objects, and for any projective object $P \in \mathcal{C}$ and any object $M \in \mathcal{M}$ the object $P\overline{\otimes}M$ is projective in $\mathcal{M}$.*

We can see that the notion of an exact module category may be regarded as the categorical analog of the notion of a projective module in ring theory.

Example 2.4.3 ([4], Example 7.5.5)**.** Any multitensor category with enough projective objects $\mathcal{C}$ considered as a module category over itself is exact.

Indeed, let P be a projective object in $\mathcal{C}$, and $X \in \mathcal{C}$. Since P is projective, the functor $Hom_{\mathcal{C}}(P, \_\ ) : \mathcal{C} \to vect_{\Bbbk}$ is exact (see Proposición 2.7.49 in [15]). Moreover, the functor $\_ \otimes X^* : \mathcal{C} \to \mathcal{C}$ is exact since the tensor functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is biexact (see Remark 2.1.9). Hence the functor composition

$$Hom_{\mathcal{C}}(P, \_\ ) \circ (\_ \otimes X^*) = Hom_{\mathcal{C}}(P, \_ \otimes X^*) : \mathcal{C} \to Ab$$

is exact because the composition of exact functors is exact. The functor $\_ \otimes X$ is left adjoint to $\_ \otimes X^*$ by Proposition 2.1.7, i.e., there is a natural isomorphism

$$\phi = \{\phi_{Y,Z} : Hom_{\mathcal{C}}(Y \otimes X, Z) \to Hom_{\mathcal{C}}(Y, Z \otimes X^*)\}_{Y,Z\in\mathcal{C}}$$

in $\mathcal{C}^{op} \times \mathcal{C}$. By fixing $Y = P$ and defining

$$\phi' = \{\phi'_Z \coloneqq \phi_{P,Z} : Hom_{\mathcal{C}}(P \otimes X, Z) \to Hom_{\mathcal{C}}(P, Z \otimes X^*)\}_{Z\in\mathcal{C}},$$

it follows that ϕ' is a natural isomorphism between the functors $Hom_{\mathcal{C}}(P \otimes X, \_\ )$ and $Hom_{\mathcal{C}}(P, \_ \otimes X^*)$, in other words, $Hom_{\mathcal{C}}(P \otimes X, \_\ )$ and $Hom_{\mathcal{C}}(P, \_ \otimes X^*)$ are equivalent functors. This implies that the functor $Hom_{\mathcal{C}}(P \otimes X, \_\ )$ is also exact, that is, the object $P \otimes X$ of $\mathcal{C}$ is projective (via Proposición 2.7.49 in [15]), as wanted.

Example 2.4.4 ([15], Ejercicio 5.1.8)**.** If $\mathcal{C}$ is a tensor category, it is an indecomposable module category over itself.

In fact, suppose the $\mathcal{C}$-module category $\mathcal{C}$ isn't indecomposable. This implies we can write the $\mathcal{C}$-module category $\mathcal{C}$ as a direct sum of two nonzero $\mathcal{C}$-module categories, i.e., $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$.

From this, it's possible to write $\mathbf{1}_{\mathcal{C}} = A \oplus B$ with $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$. From the definition of direct sum of two objects it follows that there are monomorphisms $\iota_1 : A \to \mathbf{1}_{\mathcal{C}}$ and $\iota_2 : B \to \mathbf{1}_{\mathcal{C}}$ (and epimorphisms $\pi_1 : \mathbf{1}_{\mathcal{C}} \to A$ and $\pi_2 : \mathbf{1}_{\mathcal{C}} \to B$ satisfying some conditions). This implies that A and B are subobjects of the simple object $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ (the unit object $\mathbf{1}_{\mathcal{C}}$ is simple whenever $\mathcal{C}$ is a tensor category), and hence A and B are either the zero object or the unit $\mathbf{1}_{\mathcal{C}}$.

Notice that we cannot have $A = B = \mathbf{1}_{\mathcal{C}}$ because we would get $\iota_1 = \iota_2$ and $\pi_1 = \pi_2$ which contradicts the fact that $\pi_i \circ \iota_j = 0$ whenever $i \neq j$.

So, suppose that $A = \mathbf{1}_{\mathcal{C}}$ and $B = 0$ which implies $\mathbf{1}_{\mathcal{C}} = \mathbf{1}_{\mathcal{C}} \oplus 0 = (\mathbf{1}_{\mathcal{C}}, 0) \in \mathcal{C}_1 \oplus \mathcal{C}_2$. Given that $\mathcal{C}_1$ is a $\mathcal{C}$-module category and $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}_1$, it follows that $X \cong X \otimes \mathbf{1}_{\mathcal{C}} \in \mathcal{C}_1$ for all $X \in \mathcal{C}$, that is, the category $\mathcal{C}_1$ is exactly $\mathcal{C}$. This same kind of contradiction we'd get by considering $A = 0$ and $B = \mathbf{1}_{\mathcal{C}}$. Hence, $\mathcal{C}$ is an indecomposable module category over itself.

3 REPRESENTABLE FUNCTORS AND THE YONEDA LEMMA

A representable functor is a functor from a locally small category into the category of sets which satisfies a certain property. Such functors give representations of an abstract category in terms of known structures (e.g., sets and functions) allowing one to utilize, as much as possible, knowledge about the category of sets in other settings.

"The Yoneda Lemma is arguably the most important result in category theory, although it takes some time to explore the depths of the consequences of this simple statement", (Emily Riehl in [20]).

The definitions of representable functors and the Yoneda Lemma can be found in both their covariant and contravariant versions. Our approach is on the contravariant case, but the definition and the proof we will see below can easily be adjusted for the covariant case (which is the standard version one can find in numerous books).

Definition 3.1. *A contravariant functor $F : \mathcal{C} \to Set$ (covariant functor $G : \mathcal{C} \to Set$) is said to be representable if there is a natural isomorphism $\phi : Hom_{\mathcal{C}}(\_\,, X) \to F$ ($\phi' : Hom_{\mathcal{C}}(X, \_\,) \to G$) in $\mathcal{C}$, for some object $X \in \mathcal{C}$. In this case, the pair (X, ϕ) (the pair (X, ϕ')) is called a representation of the functor F (of the functor G).*

The functors $Hom_{\mathcal{C}}(\_\,, X)$ and $Hom_{\mathcal{C}}(X, \_\,)$ were defined in the end of Section 1.4 and they are denoted by R_X and L_X, respectively. We can also say that the object $X \in \mathcal{C}$ represents the functors F (the functor G), or even that F (G) is represented by $X \in \mathcal{C}$. Moreover, one can easily verify that the object representing a functor is unique up to isomorphism by using the Proposition 1.4.6.

It appears that the representation of a functor is only possible if its target (codomain) is the category *Set*, but as we can see now this is not exactly the only case. If $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{D}$ is a category whose objects are sets (for example $Vect_k$ and the most categories we studied in this work) there always exists the forgetful functor $Forget : \mathcal{D} \to Set$, and the functor $F : \mathcal{C} \to \mathcal{D}$ is said to be representable if the functor composition $Forget \circ F : \mathcal{C} \to Set$ is. Moreover, we often omit the forgetful functor $Forget : \mathcal{D} \to Set$ and simply say that the functor $F : \mathcal{C} \to Set$ is representable.

We now present a very important and useful result in category theory, namely, the Yoneda Lemma[1] (for the contravariant case).

Lemma 3.2 ([6], Corollary 1.8 - Yoneda Lemma)**.** *Let $F : \mathcal{C} \to Set$ be a contravariant functor and $X \in \mathcal{C}$. Then there is a bijection (in Set) between the set[2] of natural*

[1] For a historical context of its origin, see the interesting story present in [13].

[2] Since the category $\mathcal{C}$ is only locally small, the collection of natural transformations $Nat(Hom_{\mathcal{C}}(\_\,, X), F)$ might be large for some $X \in \mathcal{C}$. However, the bijection in the Yoneda Lemma guarantees that this particular collection of natural transformations indeed forms a set.

transformations from $Hom_{\mathcal{C}}(\_\,, X)$ to F and the set $F(X)$ via

$$\begin{aligned}\Psi : Nat(Hom_{\mathcal{C}}(\_\,, X), F) &\longrightarrow F(X)\\ \tau \qquad &\longmapsto \Psi(\tau) \coloneqq \tau_X(id_X).\end{aligned}$$

Proof. Let us begin by defining the inverse of Ψ as

$$\begin{aligned}\Psi' : F(X) &\longrightarrow Nat(Hom_{\mathcal{C}}(\_\,, X), F)\\ x &\longmapsto \Psi'(x) : Hom_{\mathcal{C}}(\_\,, X) \to F\end{aligned}$$

where

$$\Psi'(x) = \{\Psi'(x)_Y : Hom_{\mathcal{C}}(Y, X) \longrightarrow F(Y)\}_{Y \in \mathcal{C}}$$

is defined by $\Psi'(x)_Y(f) \coloneqq F(f)(x)$, for every $f \in Hom_{\mathcal{C}}(Y, X)$. The natural property of $\Psi'(x)$ can be translated into the commutativity of the diagram

$$\begin{array}{ccc} Hom_{\mathcal{C}}(Y, X) & \xrightarrow{\Psi'(x)_Y} & F(Y) \\ {\scriptstyle Hom_{\mathcal{C}}(g,X)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ Hom_{\mathcal{C}}(Z, X) & \xrightarrow{\Psi'(x)_Z} & F(Z), \end{array}$$

for all $g \in Hom_{\mathcal{C}}(Z, Y)$. In fact, let f be any morphism in $Hom_{\mathcal{C}}(Y, X)$ and notice that

$$\begin{aligned}(F(g) \circ \Psi'(x)_Y)(f) &= F(g)(\Psi'(x)_Y(f))\\ &= F(g)(F(f)(x))\\ &= (F(g) \circ F(f))(x)\\ &= F(f \circ g)(x)\\ &= \Psi'(x)_Z(f \circ g)\\ &= \Psi'(x)_Z(Hom_{\mathcal{C}}(g, X)(f))\\ &= (\Psi'(x)_Z \circ Hom_{\mathcal{C}}(g, X))(f).\end{aligned}$$

Therefore, $F(g) \circ \Psi'(x)_Y = \Psi'(x)_Z \circ Hom_{\mathcal{C}}(g, X)$ which implies that $\Psi'(x)$ is a natural transformation.

For its inverse, let x be any element in the set $F(X)$ and notice that

$$(\Psi \circ \Psi')(x) = \Psi(\Psi'(x)) = \Psi'(x)_X(id_X) = F(id_X)(x) = id_{F(X)}(x)$$

i.e., $\Psi \circ \Psi' = id_{F(X)}$. To show that $\Psi' \circ \Psi = id_{Nat(Hom_{\mathcal{C}}(\_\,, X), F)}$ consider $\tau \in Nat(Hom_{\mathcal{C}}(\_\,, X), F)$ and the commutativity of

$$\begin{array}{ccc} Hom_{\mathcal{C}}(X, X) & \xrightarrow{\tau_X} & F(X) \\ {\scriptstyle Hom_{\mathcal{C}}(f,X)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ Hom_{\mathcal{C}}(Y, X) & \xrightarrow{\tau_Y} & F(Y) \end{array}$$

for any $f \in Hom_{\mathcal{C}}(Y,X)$. Using this commutativity to the morphism $id_X \in Hom_{\mathcal{C}}(X,X)$ we have

$$\begin{aligned}\tau_Y(f) &= \tau_Y(Hom_{\mathcal{C}}(f,X)(id_X))\\ &= (\tau_Y \circ Hom_{\mathcal{C}}(f,X))(id_X)\\ &= (F(f)\circ\tau_X)(id_X)\\ &= F(f)(\tau_X(id_X))\\ &= F(f)(\Psi(\tau)),\end{aligned}$$

that is, $F(f)(\Psi(\tau)) \overset{(*)}{=} \tau_Y(f)$. Finally,

$$((\Psi'\circ\Psi)(\tau))_Y(f) = (\Psi'(\Psi(\tau)))_Y(f) = F(f)(\Psi(\tau)) \overset{(*)}{=} \tau_Y(f)$$

which implies $((\Psi'\circ\Psi)(\tau))_Y = \tau_Y$, for all $Y \in \mathcal{C}$. Then $(\Psi'\circ\Psi)(\tau) = \tau$ and from the fact that $\tau \in Nat(Hom_{\mathcal{C}}(\_\_ ,X),F)$ is arbitrary, $\Psi'\circ\Psi = id_{Nat(Hom_{\mathcal{C}}(\_\_ ,X),F)}$. Hence, Ψ is an isomorphism. ∎

It's good to remember that this is the contravariant version of the Yoneda Lemma. The covariant version can be easily adapted from this one as well as its proof.

Before defining the internal Hom functors, let us see some definitions and results that are going to be very useful from now on. A mathematical object that will be quite used is the universal element of a representable functor which is introduced in this following definition.

Definition 3.3. *A universal element of a contravariant functor $F : \mathcal{C} \to Set$ is a pair (X,x) such that X is an object of $\mathcal{C}$ and $x \in F(X)$ satisfying the following condition: for any pair (Y,y) with $Y \in \mathcal{C}$ and $y \in F(Y)$ there is an unique morphism $g : Y \to X$ in $Hom_{\mathcal{C}}(Y,X)$ satisfying $F(g)(x) = y$.*

This next result shows that there is a certain one-to-one correspondence between representation of functors and universal elements.

Proposition 3.4. *Let $F : \mathcal{C} \to Set$ be a contravariant functor.*

(i) *If F is representable and (X,ϕ) is a representation of F, that is, $X \in \mathcal{C}$ and $\phi : Hom_{\mathcal{C}}(\_\_ ,X) \to F$ is a natural isomorphism in $\mathcal{C}$ then $(X, x = \phi_X(id_X))$ is an universal element of F and $\Psi'(x) = \phi$ where Ψ' is the bijection of the Yoneda Lemma.*

(ii) *If (X,x) is an universal element of F then $(X,\phi = \Psi'(x))$ is a representation of F.*

Proof. (i) Let $Y \in \mathcal{C}$ and $y \in F(Y)$. The unique morphism $t \in Hom_{\mathcal{C}}(Y,X)$ satisfying $F(t)(x) = y$ comes from the isomorphism $\phi_Y : Hom_{\mathcal{C}}(Y,X) \to F(Y)$. Indeed, there is an unique morphism $t \in Hom_{\mathcal{C}}(Y, X)$ satisfying $\phi_Y(t) = y$. By the naturality of ϕ, the diagram

$$\begin{array}{ccc} Hom_{\mathcal{C}}(X,X) & \xrightarrow{\phi_X} & F(X) \\ {\scriptstyle Hom_{\mathcal{C}}(t,X)}\downarrow & & \downarrow{\scriptstyle F(t)} \\ Hom_{\mathcal{C}}(Y,X) & \xrightarrow{\phi_Y} & F(Y) \end{array}$$

commutes, i.e.,

$$\begin{aligned} F(t)(x) &= F(t)(\phi_X(id_X)) \\ &= (F(t) \circ \phi_X)(id_X) \\ &= (\phi_Y \circ Hom_{\mathcal{C}}(t, X))(id_X) \\ &= \phi_Y(Hom_{\mathcal{C}}(t, X)(id_X)) \\ &= \phi_Y(id_X \circ t) \\ &= \phi_Y(t) = y. \end{aligned}$$

This implies that $(X, x = \phi_X(id_X))$ is an universal element of the functor F. For the equality $\Psi'(x) = \phi$ notice that $\Psi(\phi) = \phi_X(id_X) = x$, so

$$\Psi'(x) = \Psi'(\Psi(\phi)) = (\Psi' \circ \Psi)(\phi) = (id_{Nat(Hom_{\mathcal{C}}(\_\_ ,X),F)})(\phi) = \phi$$

as wanted.

(ii) Let us show that $(X, \phi = \Psi'(x))$ is a representation of F, that is, $\Psi'(x) : Hom_{\mathcal{C}}(\_\_ , X) \to F$ is a natural isomorphism in $\mathcal{C}$.

The morphism $\Psi'(x)_Y$ is an isomorphism in the category *Set* for all $Y \in \mathcal{C}$. Indeed, it is injective since for any morphisms g and h in $Hom_{\mathcal{C}}(Y, X)$ satisfying $\Psi'(x)_Y(g) = \Psi'(x)_Y(h)$ we get $F(g)(x) = F(h)(x)$, and by using that $F(h)(x) \in F(Y)$ and (X, x) is an universal element, there is an unique morphism $t \in Hom_{\mathcal{C}}(Y, X)$ satisfying $F(t)(x) = F(h)(x)$. Because $F(t)(x) = F(h)(x) = F(g)(x)$ it follows that $t = h = g$ as wanted.

To check that $\Psi'(x)_Y$ is surjective, consider $y \in F(Y)$. By noticing that (X, x) is an universal element of F, there is an unique $t \in Hom_{\mathcal{C}}(Y, X)$ satisfying $F(t)(x) = y$. Since $F(t)(x) = \Psi'(x)_Y(t)$ it follows that $\Psi'(x)_Y(t) = y$, i.e., $\Psi'(x)_Y$ is surjective and therefore, an isomorphism in *Set*.

Since $\Psi'(x)$ is a already natural transformation between the functors $Hom_{\mathcal{C}}(\_\_ , X)$ and F, it follows that $\Psi'(x)$ is a natural isomorphism between the functors $Hom_{\mathcal{C}}(\_\_ , X)$ and F from $\mathcal{C}$ to *Set*. ■

This following proposition is important since it guarantees the existence of the mathematical object *internal Hom* which is in the core of our main result.

Proposition 3.5 ([4], Corollary 1.8.11)**.** *Let* $\mathcal{C}$ *be a finite category over* $\Bbbk$ *and* $F : \mathcal{C} \to vect_{\Bbbk}$ *a* $\Bbbk$*-linear left exact contravariant functor. Then the functor* F *is representable*[3]*, i.e., for some* $X \in \mathcal{C}$ *there is a natural isomorphism*[4] $\phi : Hom_{\mathcal{C}}(\_\_, X) \to F$.

[3] Notice that we are omitting a functor composition in here. To be precise, the functor $Forget \circ F : \mathcal{C} \to vect_{\Bbbk} \to Set$ is the one that is representable.

[4] Again, to be precise, $\phi : Hom_{\mathcal{C}}(\_\_, X) \to Forget \circ F$.

4 INTERNAL HOM

In this chapter we'll introduce an important technical tool in the study of module categories which are going to be frequently used in this work. They're called the internal Hom object and functor and have a strong relation with a particular representable functor. The theory presented here can be found in [4] and [15], for example.

4.1 INTERNAL HOM OBJECT AND FUNCTOR

The internal Hom functor arises from the definition of the internal Hom object, which is an object that represent a certain functor. Let $\mathcal{C}$ be a finite multitensor category (the finiteness condition is not strictly necessary in this chapter but simplifies the exposition)[1] and $\mathcal{M}$ be a locally finite and module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable[2]. For every pair of objects $M, N \in \mathcal{M}$ the contravariant functor

$$F := Hom_{\mathcal{M}}(\_\,, N) \circ (\_\,\overline{\otimes} M) = Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, N) : \mathcal{C} \to vect_{\Bbbk}$$

is $\Bbbk$-linear and left exact since it is the composition of $\Bbbk$-linear and left exact functors[3]. Using the Proposition 3.5, it follows that F is representable, i.e., the functor $Forget \circ F : \mathcal{C} \to vect_{\Bbbk} \to Set$ is representable. Before going the next definition it is prudent to remember that a functor representation is a pair containing an object (in $\mathcal{C}$ in our case) and a natural isomorphism. Let us begin by the definition of the internal Hom object.

Definition 4.1.1. *The object in $\mathcal{C}$ which represents the contravariant functor $F = Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, N) : \mathcal{C} \to vect_{\Bbbk}$ is called the internal Hom object from M to N, and it's denoted by $\underline{Hom}_{\mathcal{M}}(M, N)$ or simply by $\underline{Hom}(M, N)$.*

Therefore, there is a natural isomorphism

$$\phi : Hom_{\mathcal{C}}(\_\,, \underline{Hom}(M, N)) \to Forget \circ Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, N)$$

in $\mathcal{C}$, and thus the pair $(\underline{Hom}(M, N), \phi)$ is a representation of the functor $Forget \circ F$. About the family ϕ,

$$\phi = \{\phi_X : Hom_{\mathcal{C}}(X, \underline{Hom}(M, N)) \to Hom_{\mathcal{M}}(X \overline{\otimes} M, N)\}_{X \in \mathcal{C}}.$$

Considering that for all $M, N \in \mathcal{M}$, the object $\underline{Hom}(M, N) \in \mathcal{C}$ represents the functor $Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, N)$, we may think about the applications $\underline{Hom}(M, \_\,) : \mathcal{M} \to \mathcal{C}$ and $\underline{Hom}(\_\,, M) : \mathcal{M}^{op} \to \mathcal{C}$. The following result asserts that these applications are in fact functors, and also that $\underline{Hom}(\_\,, \_\,) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is a bifunctor. This result is

[1] See Remark 7.9.1 in [4].
[2] That is, $\_\,\overline{\otimes} M : \mathcal{C} \to \mathcal{M}$ is a $\Bbbk$-linear and left exact functor.
[3] The functor $Hom_{\mathcal{M}}(\_\,, N)$ is $\Bbbk$-linear by the fact that $\mathcal{M}$ is a $\Bbbk$-linear category ($\mathcal{M}$ is multitensor).

considered to be valid by many authors, but we could not find a proof anywhere. Many of them simply state that $\underline{Hom}(\,\_\,,\,\_\,)$ is a bifunctor right from the Yoneda Lemma, but no description on how they act on morphisms is provided in most cases.

Proposition 4.1.2. *The application $\mathcal{F} := \underline{Hom}(\,\_\,,\,\_\,) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is a bifunctor.*

Proof. To begin, we show that both $\underline{Hom}(M,\ \_\,) : \mathcal{M} \to \mathcal{C}$ and $\underline{Hom}(\,\_\,, M) : \mathcal{M}^{op} \to \mathcal{C}$ are functors. By using these facts and Lemma 1.3.11 we'll be able to conclude that $\underline{Hom}(\,\_\,,\,\_\,) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is a bifunctor.

Firstly, let $M, N, P, Q \in \mathcal{M}$ and consider the representable functor $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, N) : \mathcal{C} \to vect_{\Bbbk}$. Since the object $\underline{Hom}(M, N)$ represents this functor, there is a natural isomorphism

$$\phi^1 : Hom_{\mathcal{C}}(\,\_\,, \underline{Hom}(M, N)) \to Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, N)$$

in $\mathcal{C}$.

From the item (i) in Proposition 3.4 we can affirm that

$$(\underline{Hom}(M, N), z = \phi^1_{\underline{Hom}(M,N)}(id_{\underline{Hom}(M,N)}))$$

is an universal element of the functor $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, N)$, and $\Psi^N(z) = \phi^1$ where Ψ^N is the bijection of Yoneda Lemma. Notice that $z \in Hom_{\mathcal{M}}(\underline{Hom}(M, N)\overline{\otimes} M, N)$ and

$$\begin{aligned} \phi^1_Y = \Psi^N(z)_Y : Hom_{\mathcal{C}}(Y, \underline{Hom}(M, N)) &\longrightarrow Hom_{\mathcal{M}}(Y\overline{\otimes} M, N) \\ j &\longmapsto \Psi^N(z)_Y(m) = Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, N)(j)(z), \end{aligned}$$

that is,

$$\Psi^N(z)_Y(j) = Hom_{\mathcal{M}}(j\overline{\otimes} M, N)(z) = z \circ (j\overline{\otimes} id_M).$$

Consider the representable functors $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, P)$ and $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, Q)$, and its representations $(\underline{Hom}(M, P), \phi^2)$ and $(\underline{Hom}(M, Q), \phi^3)$, respectively. Analogously as we did before, let us define the elements

$$\begin{aligned} y &:= \phi^2_{\underline{Hom}(M,P)}(id_{\underline{Hom}(M,P)}) \in Hom_{\mathcal{M}}(\underline{Hom}(M, P)\overline{\otimes} M, P), \quad \text{and} \\ x &:= \phi^3_{\underline{Hom}(M,Q)}(id_{\underline{Hom}(M,Q)}) \in Hom_{\mathcal{M}}(\underline{Hom}(M, Q)\overline{\otimes} M, Q) \end{aligned}$$

in *Set*. So we have that the pairs $(\underline{Hom}(M, N), z)$, $(\underline{Hom}(M, P), y)$ and $(\underline{Hom}(M, Q), x)$ are universal elements of the functors $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, N)$, $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, P)$ and $Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, Q)$, respectively. Moreover, we have the natural isomorphisms

$$\begin{aligned} \phi^2 &:= \Psi^P(y) : Hom_{\mathcal{C}}(\,\_\,, \underline{Hom}(M, P)) \to Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, P), \quad \text{and} \\ \phi^3 &:= \Psi^Q(x) : Hom_{\mathcal{C}}(\,\_\,, \underline{Hom}(M, Q)) \to Hom_{\mathcal{M}}(\,\_\,\overline{\otimes} M, Q). \end{aligned}$$

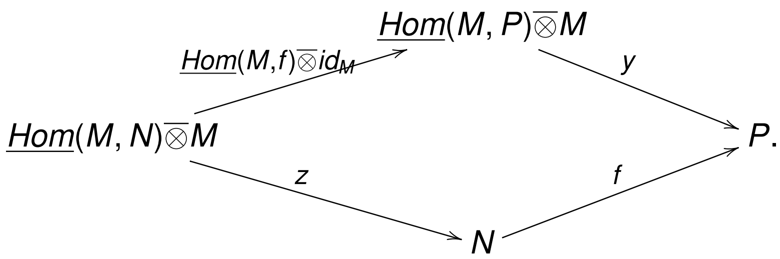
We now see how it is possible to define $\underline{Hom}(M,\ \_\,)$ on morphisms of $\mathcal{M}$. So, consider $f : N \to P$ an arbitrary morphism in $\mathcal{M}$ and let's determine what the morphism $\underline{Hom}(M, f) : \underline{Hom}(M, N) \to \underline{Hom}(M, P)$ is. Notice that the composition $f \circ z : \underline{Hom}(M, N)\overline{\otimes}M \to N \to P$ is a morphism in $\mathcal{M}$ and because

$$\Psi^P(y)_{\underline{Hom}(M,N)} : Hom_{\mathcal{C}}(\underline{Hom}(M, N), \underline{Hom}(M, P)) \to Hom_{\mathcal{M}}(\underline{Hom}(M, N)\overline{\otimes}M, P)$$

is an isomorphism in *Set*, there exists an unique $g \in Hom_{\mathcal{C}}(\underline{Hom}(M, N), \underline{Hom}(M, P))$ satisfying

$$f \circ z = \Psi^P(y)_{\underline{Hom}(M,N)}(g) = Hom_{\mathcal{M}}(\_\overline{\otimes}M, P)(g)(y) = y \circ (g\overline{\otimes}id_M).$$

Therefore, we define $\underline{Hom}(M, f) \coloneqq g$. This can also be seen by the commutativity of



Let $f' : P \to Q$ be a morphism in $\mathcal{M}$. Similarly as we've just done, we may define $\underline{Hom}(M, f') \coloneqq g' \in Hom_{\mathcal{C}}(\underline{Hom}(M, P), \underline{Hom}(M, Q))$ such that $f' \circ y = x \circ (g'\overline{\otimes}id_M)$, and $\underline{Hom}(M, f' \circ f) \coloneqq h \in Hom_{\mathcal{C}}(\underline{Hom}(M, N), \underline{Hom}(M, Q))$ satisfying $f' \circ f \circ z = x \circ (h\overline{\otimes}id_M)$. Notice that

$$\begin{aligned}
\Psi^Q(x)_{\underline{Hom}(M,N)}(h) &= x \circ (h\overline{\otimes}id_M) \\
&= f' \circ f \circ z \\
&= f' \circ y \circ (g\overline{\otimes}id_M) \\
&= x \circ (g'\overline{\otimes}id_M) \circ (g\overline{\otimes}id_M) \\
&= x \circ ((g' \circ g)\overline{\otimes}id_M) \\
&= \Psi^Q(x)_{\underline{Hom}(M,N)}(g' \circ g),
\end{aligned}$$

which implies

$$\underline{Hom}(M, f' \circ f) = h = g' \circ g = \underline{Hom}(M, f') \circ \underline{Hom}(M, f)$$

since $\Psi^Q(x)_{\underline{Hom}(M,N)}$ is an isomorphism by item (ii) of Proposition 3.4.

Finally, by considering $\underline{Hom}(M, id_N) \coloneqq j$ we have

$$\begin{aligned}
\Psi^N(z)_{\underline{Hom}(M,N)}(id_{\underline{Hom}(M,N)}) &= Hom_{\mathcal{M}}(_\overline{\otimes}M, N)(id_{\underline{Hom}(M,N)})(z) \\
&= z \circ (id_{\underline{Hom}(M,N)}\overline{\otimes}id_M) \\
&= z \circ id_{\underline{Hom}(M,N)\overline{\otimes}M} \\
&= z \\
&= id_N \circ z \\
&= z \circ (j\overline{\otimes}id_M) \\
&= \Psi^N(z)_{\underline{Hom}(M,N)}(j)
\end{aligned}$$

implying the equality $id_{\underline{Hom}(M,N)} = j$ since $\Psi^N(z)_{\underline{Hom}(M,N)}$ is an isomorphism. It follows that $id_{\underline{Hom}(M,\ \_\ )(N)} = \underline{Hom}(M,\ \_\ )(id_N)$ and therefore, $\underline{Hom}(M,\ \_\ ) : \mathcal{M} \to \mathcal{C}$ is a (covariant) functor.

Next, we will define the contravariant functor $\underline{Hom}(\ \_\ , M) : \mathcal{M} \to \mathcal{C}$ (or equivalently, the covariant functor $\underline{Hom}(\ \_\ , M) : \mathcal{M}^{op} \to \mathcal{C}$).

Let $(\underline{Hom}(N, M), \sigma^1)$, $(\underline{Hom}(P, M), \sigma^2)$ and $(\underline{Hom}(Q, M), \sigma^3)$ be representations of the functors $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} N, M)$, $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} P, M)$ and $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} Q, M)$, respectively. By defining $x := \sigma^1_{\underline{Hom}(N,M)}(id_{\underline{Hom}(N,M)})$, $y := \sigma^2_{\underline{Hom}(P,M)}(id_{\underline{Hom}(P,M)})$ and $z := \sigma^3_{\underline{Hom}(Q,M)}(id_{\underline{Hom}(Q,M)})$, we get

$$\begin{aligned}
\sigma^1 &= \Psi^N(x) : Hom_{\mathcal{C}}(\ _\ , \underline{Hom}(N, M)) \to Hom_{\mathcal{M}}(\ _\ \overline{\otimes} N, M),\\
\sigma^2 &= \Psi^P(y) : Hom_{\mathcal{C}}(\ _\ , \underline{Hom}(P, M)) \to Hom_{\mathcal{M}}(\ _\ \overline{\otimes} P, M) \ \ \textit{and}\\
\sigma^3 &= \Psi^Q(z) : Hom_{\mathcal{C}}(\ _\ , \underline{Hom}(Q, M)) \to Hom_{\mathcal{M}}(\ _\ \overline{\otimes} Q, M).
\end{aligned}$$

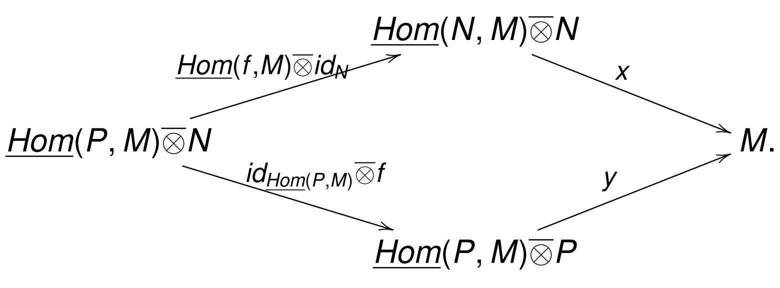
Consider an arbitrary morphism $f : N \to P$ in $\mathcal{M}$. Using that $y \circ (id_{\underline{Hom}(P,M)} \overline{\otimes} f) \in Hom_{\mathcal{M}}(\underline{Hom}(P, M) \overline{\otimes} N, M)$ and

$$\Psi^N(x)_{\underline{Hom}(P,M)} : Hom_{\mathcal{C}}(\underline{Hom}(P, M), \underline{Hom}(N, M)) \to Hom_{\mathcal{M}}(\underline{Hom}(P, M) \overline{\otimes} N, M)$$

is an isomorphism in *Set*, it follows that there is an unique $g \in Hom_{\mathcal{C}}(\underline{Hom}(P, M), \underline{Hom}(N, M))$ such that

$$y \circ (id_{\underline{Hom}(P,M)} \overline{\otimes} f) = \Psi^N(x)_{\underline{Hom}(P,M)}(g) = Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} N, M)(g)(x) = x \circ (g \overline{\otimes} id_N).$$

So we define $\underline{Hom}(f, M) := g$ which satisfies[4]



Doing the same for the morphism $f' : P \to Q$ in $\mathcal{M}$, there exists an unique $g' \in Hom_{\mathcal{C}}(\underline{Hom}(Q, M), \underline{Hom}(P, M))$ such that

$$z \circ (id_{\underline{Hom}(Q,M)} \overline{\otimes} f') = \Psi^P(y)_{\underline{Hom}(Q,M)}(g') = Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} P, M)(g')(y) = y \circ (g' \overline{\otimes} id_P).$$

The uniqueness comes from the fact that $\Psi^P(y)_{\underline{Hom}(Q,M)}$ is an isomorphism in *Set*.

Analogously, it's possible to define $\underline{Hom}(f' \circ f, M) := h$ satisfying

$$z \circ (id_{\underline{Hom}(Q,M)} \overline{\otimes} (f' \circ f)) = \Psi^N(x)_{\underline{Hom}(Q,M)}(h) = x \circ (h \overline{\otimes} id_N).$$

[4] It's always a good idea to have this diagram observation while checking how the functors $\underline{Hom}(M,\ \_\ )$ and $\underline{Hom}(\ \_\ , M)$ act on morphisms of $\mathcal{M}$.

Thus we have

$$\begin{aligned}\Psi^N(x)_{\underline{Hom}(Q,M)}(h) &= x \circ (h\overline{\otimes} id_N)\\ &= z \circ (id_{\underline{Hom}(Q,M)}\overline{\otimes} f') \circ (id_{\underline{Hom}(Q,M)}\overline{\otimes} f)\\ &= y \circ (g'\overline{\otimes} id_P) \circ (id_{\underline{Hom}(Q,M)}\overline{\otimes} f)\\ &= y \circ (id_{\underline{Hom}(P,M)}\overline{\otimes} f) \circ (g'\overline{\otimes} id_N)\\ &= x \circ (g\overline{\otimes} id_N) \circ (g'\overline{\otimes} id_N)\\ &= x \circ ((g \circ g')\overline{\otimes} id_N)\\ &= \Psi^N(x)_{\underline{Hom}(Q,M)}(g \circ g').\end{aligned}$$

Given that $\Psi^N(x)_{\underline{Hom}(Q,M)}$ is an isomorphism, the equality $h = g \circ g'$ holds and, equivalently,

$$\underline{Hom}(f' \circ f, M) = \underline{Hom}(f, M) \circ \underline{Hom}(f', M).$$

By defining $\underline{Hom}(id_N, M) := j$ in which

$$\Psi^N(x)_{\underline{Hom}(N,M)}(j) = x \circ (id_{\underline{Hom}(N,M)}\overline{\otimes} id_N) = x$$

we may notice that

$$\Psi^N(x)_{\underline{Hom}(N,M)}(id_{\underline{Hom}(N,M)}) = x \circ (id_{\underline{Hom}(N,M)}\overline{\otimes} id_N) = x = \Psi^N(x)_{\underline{Hom}(N,M)}(j).$$

Since $\Psi^N(x)_{\underline{Hom}(N,M)}$ is an isomorphism we have the following

$$id_{\underline{Hom}(\_\,,M)(N)} = id_{\underline{Hom}(N,M)} = j = \underline{Hom}(id_N, M) = \underline{Hom}(\_\,, M)(id_N).$$

Therefore, $\underline{Hom}(\_\,, M) : \mathcal{M} \to \mathcal{C}$ is a contravariant functor.

At last, we want to use Lemma 1.3.11 to get that $\mathcal{F} = \underline{Hom}(\_\,,\_\,) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is a bifunctor. Let $f : M \to N$ and $g : P \to Q$ be morphisms in $\mathcal{M}$. We have just shown that both $\mathcal{F}^1_L = \underline{Hom}(L, \_\,) : \mathcal{M} \to \mathcal{C}$ and $\mathcal{F}^2_L = \underline{Hom}(\_\,, L) : \mathcal{M}^{op} \to \mathcal{C}$ are functors, for all $L \in \mathcal{M}$. Notice we can define how the application $\underline{Hom}(\_\,,\_\,)$ acts on morphisms, as long as it satisfies $\mathcal{F}^1_L(g) = \mathcal{F}(L, g) = \underline{Hom}(L, g)$ and $\mathcal{F}^2_L(f) = \mathcal{F}(f, L) = \underline{Hom}(f, L)$. One canonical way to do this is by setting

$$\mathcal{F}(f, g) := \underline{Hom}(f, Q) \circ \underline{Hom}(N, g) = \mathcal{F}^2_Q(f) \circ \mathcal{F}^1_N(g)$$

which clearly satisfies these two conditions.

To use Lemma 1.3.11 it only remains to check the equality

$$\mathcal{F}(f, g) = \underline{Hom}(M, g) \circ \underline{Hom}(f, P)$$

since

$$\mathcal{F}(f, g) = \underline{Hom}(f, Q) \circ \underline{Hom}(N, g)$$

holds by definition.

For this purpose, let the pairs $(\underline{Hom}(M,P),x)$, $(\underline{Hom}(M,Q),y)$, $(\underline{Hom}(N,P),t)$ and $(\underline{Hom}(N,Q),z)$ be universal elements of the functors $Hom_{\mathcal{M}}(\_\,\overline{\otimes}M,P)$, $Hom_{\mathcal{M}}(\_\,\overline{\otimes}M,Q)$, $Hom_{\mathcal{M}}(\_\,\overline{\otimes}N,P)$ and $Hom_{\mathcal{M}}(\_\,\overline{\otimes}N,Q)$, respectively. Furthermore, from the definition of the morphisms $\underline{Hom}(M,g)$, $\underline{Hom}(f,Q)$, $\underline{Hom}(N,g)$ and $\underline{Hom}(f,P)$ in $\mathcal{C}$, we obtain the following equalities

$$g\circ x = y\circ(\underline{Hom}(M,g)\overline{\otimes}id_M), \tag{20}$$

$$z\circ(id_{\underline{Hom}(N,Q)}\overline{\otimes}f) = y\circ(\underline{Hom}(f,Q)\overline{\otimes}id_M), \tag{21}$$

$$g\circ t = z\circ(\underline{Hom}(N,g)\overline{\otimes}id_N), \quad \text{and} \tag{22}$$

$$t\circ(id_{\underline{Hom}(N,P)}\overline{\otimes}f) = x\circ(\underline{Hom}(f,P)\overline{\otimes}id_M). \tag{23}$$

Considering that $\Psi^Q(y) : Hom_{\mathcal{C}}(\_\,,\underline{Hom}(M,Q)) \to Hom_{\mathcal{M}}(\_\,\overline{\otimes}M,Q)$ is a natural isomorphism,

$$\Psi^Q(y)_{\underline{Hom}(N,P)} : Hom_{\mathcal{C}}(\underline{Hom}(N,P),\underline{Hom}(M,Q)) \to Hom_{\mathcal{M}}(\underline{Hom}(N,P)\overline{\otimes}M,Q)$$

is an isomorphism in *Set*.

Noticing that

$$\begin{aligned}
\Psi^Q(y)_{\underline{Hom}(N,P)}(\mathcal{F}(f,g)) &= \Psi^Q(y)_{\underline{Hom}(N,P)}(\underline{Hom}(f,Q)\circ\underline{Hom}(N,g))\\
&= y\circ((\underline{Hom}(f,Q)\circ\underline{Hom}(N,g))\overline{\otimes}id_M)\\
&= y\circ(\underline{Hom}(f,Q)\overline{\otimes}id_M)\circ(\underline{Hom}(N,g)\overline{\otimes}id_M)\\
&\overset{(21)}{=} z\circ(id_{\underline{Hom}(N,Q)}\overline{\otimes}f)\circ(\underline{Hom}(N,g)\overline{\otimes}id_M)\\
&= z\circ(\underline{Hom}(N,g)\overline{\otimes}id_N)\circ(id_{\underline{Hom}(N,P)}\overline{\otimes}f)\\
&\overset{(22)}{=} g\circ t\circ(id_{\underline{Hom}(N,P)}\overline{\otimes}f)\\
&\overset{(23)}{=} g\circ x\circ(\underline{Hom}(f,P)\overline{\otimes}id_M)\\
&\overset{(20)}{=} y\circ(\underline{Hom}(M,g)\overline{\otimes}id_M)\circ(\underline{Hom}(f,P)\overline{\otimes}id_M)\\
&= y\circ((\underline{Hom}(M,g)\circ\underline{Hom}(f,P))\overline{\otimes}id_M)\\
&= \Psi^Q(y)_{\underline{Hom}(N,P)}(\underline{Hom}(M,g)\circ\underline{Hom}(f,P)),
\end{aligned}$$

we obtain

$$\mathcal{F}(f,g) = \underline{Hom}(M,g)\circ\underline{Hom}(f,P)$$

because $\Psi^Q(y)_{\underline{Hom}(N,P)}$ is an isomorphism. Therefore, $\underline{Hom}(\_\,,\_\,) : \mathcal{M}^{op}\times\mathcal{M}\to\mathcal{C}$ is a bifunctor. ■

4.2 EXACTNESS OF THE INTERNAL HOM BIFUNCTOR

Here we discuss about the exactness of the internal Hom bifunctor $\underline{Hom}_{\mathcal{M}}(\_\,,\_\,) : \mathcal{M}^{op}\times\mathcal{M}\to\mathcal{C}$ defined in the previous section.

Proposition 4.2.1. *The $\mathcal{C}$-module functor $\_\,\overline{\otimes}M : \mathcal{C} \to \mathcal{M}$ is left adjoint to $\underline{Hom}(M, \_\,) : \mathcal{M} \to \mathcal{C}$.*

Proof. It suffices to show that there is a natural isomorphism

$$\phi = \{\phi_{X,N} : Hom_{\mathcal{M}}(X\overline{\otimes}M, N) \to Hom_{\mathcal{C}}(X, \underline{Hom}(M,N))\}_{(X,N)\in\mathcal{C}^{op}\times\mathcal{M}}$$

in $\mathcal{C}^{op} \times \mathcal{M}$.

Let $N, P \in \mathcal{M}$ and $\underline{Hom}(M,N), \underline{Hom}(M,P) \in \mathcal{C}$, and consider the universal elements $(\underline{Hom}(M,N), z)$ and $(\underline{Hom}(M,P), y)$. From Proposition 3.4 there are natural isomorphisms

$$\begin{aligned} &\Psi^N(z) : Hom_{\mathcal{C}}(\_\,, \underline{Hom}(M,N)) \to Hom_{\mathcal{M}}(\_\,\overline{\otimes}M, N), \quad \text{and} \\ &\Psi^P(y) : Hom_{\mathcal{C}}(\_\,, \underline{Hom}(M,P)) \to Hom_{\mathcal{M}}(\_\,\overline{\otimes}M, P) \end{aligned}$$

in $\mathcal{C}$ given by $\Psi^N(z)_X(g) = z \circ (g\overline{\otimes}id_M)$ and $\Psi^P(y)_Y(h) = y \circ (h\overline{\otimes}id_M)$, respectively.

Define

$$\phi = \{\phi_{X,N} := (\Psi^N(z))_X^{-1} : Hom_{\mathcal{M}}(X\overline{\otimes}M, N) \to Hom_{\mathcal{C}}(X, \underline{Hom}(M,N))\}_{(X,N)\in\mathcal{C}^{op}\times\mathcal{M}}.$$

We have to keep in mind that for each $N \in \mathcal{M}$ there exists a morphism[5] $z \in Hom_{\mathcal{M}}(\underline{Hom}(M,N)\overline{\otimes}M,$ and hence a natural isomorphism $\Psi^N(z)$. It follows directly from this definition that $\phi_{X,N}$ is an isomorphism for all $X \in \mathcal{C}$ and $N \in \mathcal{M}$. Furthermore, ϕ is already natural in $\mathcal{C}$ because $\Psi^N(z)$ is natural in $\mathcal{C}$. It only remains to show that ϕ is natural in $\mathcal{M}$ (see Remark 1.3.12), that is, the diagram

$$\begin{array}{ccc} Hom_{\mathcal{M}}(X\overline{\otimes}M, N) & \xrightarrow{\phi_{X,N}} & Hom_{\mathcal{C}}(X, \underline{Hom}(M,N)) \\ {\scriptstyle Hom_{\mathcal{M}}(X\overline{\otimes}M,f)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(X,\underline{Hom}(M,f))} \\ Hom_{\mathcal{M}}(X\overline{\otimes}M, P) & \xrightarrow{\phi_{X,P}} & Hom_{\mathcal{C}}(X, \underline{Hom}(M,P)) \end{array}$$

commutes for all $f \in Hom_{\mathcal{M}}(N,P)$. Let u be an arbitrary morphism in $Hom_{\mathcal{C}}(X, \underline{Hom}(M,N))$ and notice that

$$\begin{aligned} (\phi_{X,P}^{-1} \circ Hom_{\mathcal{C}}(X, \underline{Hom}(M,f)))(u) &= (\Psi^P(y)_X \circ Hom_{\mathcal{C}}(X, \underline{Hom}(M,f)))(u) \\ &= \Psi^P(y)_X(Hom_{\mathcal{C}}(X, \underline{Hom}(M,f))(u)) \\ &= \Psi^P(y)_X(\underline{Hom}(M,f) \circ u) \\ &= y \circ ((\underline{Hom}(M,f) \circ u)\overline{\otimes}id_M) \\ &= y \circ (\underline{Hom}(M,f)\overline{\otimes}id_M) \circ (u\overline{\otimes}id_M) \\ &\overset{(*)}{=} f \circ z \circ (u\overline{\otimes}id_M) \\ &= Hom_{\mathcal{M}}(X\overline{\otimes}M, f)(z \circ (u\overline{\otimes}id_M)) \\ &= Hom_{\mathcal{M}}(X\overline{\otimes}M, f)(\Psi^N(z)_X(u)) \end{aligned}$$

5 Later, we'll denote this morphism by $ev_{M,N}$ (which will be called evaluation) since it depends on a pair M and N of objects in $\mathcal{M}$.

$$= (Hom_{\mathcal{M}}(X\overline{\otimes}M, f) \circ \Psi^N(z)_X)(u)$$
$$= (Hom_{\mathcal{M}}(X\overline{\otimes}M, f) \circ \phi^{-1}_{X,N})(u)$$

where the equality $(*)$ comes from the definition of the morphism $\underline{Hom}(M, f)$. This implies that $\phi^{-1}_{X,P} \circ Hom_{\mathcal{C}}(X, \underline{Hom}(M, f)) = Hom_{\mathcal{M}}(X\overline{\otimes}M, f) \circ \phi^{-1}_{X,N}$, i.e., ϕ is also natural in $\mathcal{M}$. Therefore, $(\_\,\overline{\otimes}M, \underline{Hom}(M, \_\,), \phi)$ is an adjunction. ■

Particularly, this implies that the left exact functor $\_\,\overline{\otimes}M$ is also right exact (see Proposition 1.4.5) and, therefore, exact.

Corollary 4.2.2. *The functor $\underline{Hom}(M, \_\,) : \mathcal{M} \to \mathcal{C}$ admits a $\mathcal{C}$-module functor structure.*

The functor $(\_\,\overline{\otimes}M : \mathcal{C} \to \mathcal{M}, c)$ is a $\mathcal{C}$-module functor with $c = \{c_{X,Y} := m_{X,Y,M}\}_{X,Y\in\mathcal{C}}$ (see Example 2.2.8) and it's left adjoint to $\underline{Hom}(M, \_\,) : \mathcal{M} \to \mathcal{C}$. We've seen in Theorem 2.3.2 that the right (and left) adjoint of a $\mathcal{C}$-module functor admits a natural structure of $\mathcal{C}$-module functor. Let us denote by d this $\mathcal{C}$-module functor structure of $\underline{Hom}(M, \_\,)$, i.e., $(\underline{Hom}(M, \_\,), d)$ is a $\mathcal{C}$-module functor.

For every $(X, N) \in \mathcal{C} \times \mathcal{M}$, the element $d_{X,N}$ in the family d can be seen as the composition (see Section 2.3)

$$\underline{Hom}(M, X\overline{\otimes}N) \xrightarrow{l^{-1}_{\underline{Hom}(M,X\overline{\otimes}N)}} \mathbf{1} \otimes \underline{Hom}(M, X\overline{\otimes}N)$$
$$\xrightarrow{coev_X \otimes id_{\underline{Hom}(M,X\overline{\otimes}N)}} (X \otimes X^*) \otimes \underline{Hom}(M, X\overline{\otimes}N) \xrightarrow{a_{X,X^*,\underline{Hom}(M,X\overline{\otimes}N)}} X \otimes (X^* \otimes \underline{Hom}(M, X\overline{\otimes}N))$$
$$\xrightarrow{id_X \otimes \beta_{X^*\otimes\underline{Hom}(M,X\overline{\otimes}N)}} X \otimes \underline{Hom}(M, (X^* \otimes \underline{Hom}(M, X\overline{\otimes}N))\overline{\otimes}M) \xrightarrow{id_X \otimes \underline{Hom}(M, c_{X^*,\underline{Hom}(M,X\overline{\otimes}N)})} X \otimes \underline{Hom}(M, X^*\overline{\otimes}(\underline{Hom}(M, X\overline{\otimes}N)\overline{\otimes}M))$$
$$\xrightarrow{id_X \otimes \underline{Hom}(M, id_{X^*}\overline{\otimes}\alpha_{X\overline{\otimes}N})} X \otimes \underline{Hom}(M, X^*\overline{\otimes}(X\overline{\otimes}N)) \xrightarrow{id_X \otimes \underline{Hom}(M, m^{-1}_{X^*,X,N})} X \otimes \underline{Hom}(M, (X^* \otimes X)\overline{\otimes}N)$$
$$\xrightarrow{id_X \otimes \underline{Hom}(M, ev_X\overline{\otimes}id_N)} X \otimes \underline{Hom}(M, \mathbf{1}\overline{\otimes}N) \xrightarrow{id_X \otimes \underline{Hom}(M, l_N)} X \otimes \underline{Hom}(M, N),$$

with inverse

$$d^{-1}_{X,N} = \underline{Hom}(M, id_X\overline{\otimes}\alpha_N) \circ \underline{Hom}(M, m_{X,\underline{Hom}(M,N),M}) \circ \beta_{X\otimes\underline{Hom}(M,N)} \tag{24}$$

which also can be seen as the composition

$$\begin{array}{ccc} X \otimes \underline{Hom}(M,N) & \xrightarrow{\beta_{X\otimes \underline{Hom}(M,N)}} & \underline{Hom}(M,(X \otimes \underline{Hom}(M,N))\overline{\otimes} M) \\ & \swarrow_{\underline{Hom}(M, m_{X,\underline{Hom}(M,N),M})} & \\ \underline{Hom}(M, X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes} M)) & \xrightarrow{\underline{Hom}(M, id_X \overline{\otimes} \alpha_N)} & \underline{Hom}(M, X\overline{\otimes} N). \end{array}$$

Proposition 4.2.3. *The bifunctor* $\underline{Hom} : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ *is left biexact, that is, left exact in both entries.*

Proof. The functor $\underline{Hom}(M, \_\,)$ admits a left adjoint (which is the functor $\_\,\overline{\otimes} M$), so it follows that $\underline{Hom}(M, \_\,)$ is additive via Proposition 1.3.10 and left exact from item (i) of Proposition 1.4.5.

We could also check this fact by the definition of left exact functor as we are going to do now with the contravariant functor $\underline{Hom}(\_\,, M) : \mathcal{M} \to \mathcal{C}$. We begin by showing that this functor is additive. Let $a, b \in Hom_{\mathcal{M}}(R, S)$, and consider the universal elements $(\underline{Hom}(R, M), y)$ and $(\underline{Hom}(S, M), x)$ of their respective functors. The morphisms $\underline{Hom}(a+b, M) \coloneqq h$, $\underline{Hom}(a, M) \coloneqq g$ and $\underline{Hom}(b, M) \coloneqq g'$ satisfy

$$\Psi^R(y)_{\underline{Hom}(S,M)}(h) = y \circ (h \overline{\otimes} id_R) = x \circ (id_{\underline{Hom}(S,M)} \overline{\otimes} (a+b)), \tag{25}$$
$$\Psi^R(y)_{\underline{Hom}(S,M)}(g) = y \circ (g \overline{\otimes} id_R) = x \circ (id_{\underline{Hom}(S,M)} \overline{\otimes} a) \quad \text{and} \tag{26}$$
$$\Psi^R(y)_{\underline{Hom}(S,M)}(g') = y \circ (g' \overline{\otimes} id_R) = x \circ (id_{\underline{Hom}(S,M)} \overline{\otimes} b) \tag{27}$$

by definition. We then get

$$\begin{aligned} \Psi^R(y)_{\underline{Hom}(S,M)}(h) &\overset{(25)}{=} y \circ (h \overline{\otimes} id_R) = x \circ (id_{\underline{Hom}(S,M)} \overline{\otimes} (a+b)) \\ &= x \circ ((id_{\underline{Hom}(S,M)} \overline{\otimes} a) + (id_{\underline{Hom}(S,M)} \overline{\otimes} b)) \\ &= x \circ (id_{\underline{Hom}(S,M)} \overline{\otimes} a) + x \circ (id_{\underline{Hom}(S,M)} \overline{\otimes} b) \\ &\overset{(26),(27)}{=} y \circ (g \overline{\otimes} id_R) + y \circ (g' \overline{\otimes} id_R) \\ &= y \circ ((g \overline{\otimes} id_R) + (g' \overline{\otimes} id_R)) \\ &= y \circ ((g + g') \overline{\otimes} id_R) \\ &= \Psi^R(y)_{\underline{Hom}(S,M)}(g + g'), \end{aligned}$$

by using the additivity of $\overline{\otimes}$ (it is exact in both entries and thus additive) and the fact that morphism composition in $\mathcal{M}$ is bilinear. Considering that $\Psi^R(y)_{\underline{Hom}(S,M)}$ is an isomorphism, it follows that $h = g + g'$ or, equivalently,

$$\underline{Hom}(a+b, M) = \underline{Hom}(a, M) + \underline{Hom}(b, M).$$

Hence, the functor $\underline{Hom}(\_\,, M) : \mathcal{M} \to \mathcal{C}$ is additive.

Now, let $0 \longrightarrow N \xrightarrow{f} P \xrightarrow{f'} Q \longrightarrow 0$ be a short exact sequence in $\mathcal{M}$. We now show that

$$0 \longrightarrow \underline{Hom}(Q,M) \xrightarrow{\underline{Hom}(f',M)} \underline{Hom}(P,M) \xrightarrow{\underline{Hom}(f,M)} \underline{Hom}(N,M)$$

is exact, i.e., $\underline{Hom}(f',M)$ is a monomorphism and $Ker(\underline{Hom}(f,M)) = Im(\underline{Hom}(f',M))$ as subobjects of $\underline{Hom}(P,M)$.

For these two following affirmations, let $(\underline{Hom}(N,M),w)$, $(\underline{Hom}(P,M),t)$ and $(\underline{Hom}(Q,M),z)$ be universal elements of their respective functors. Moreover, by definition we have $\underline{Hom}(f',M) \coloneqq g'$ satisfying

$$t \circ (g' \overline{\otimes} id_P) = z \circ (id_{\underline{Hom}(Q,M)} \overline{\otimes} f'). \tag{28}$$

Affirmation 1: $\underline{Hom}(f',M)$ is a monomorphism.

Let $c,d : X \to \underline{Hom}(Q,M)$ be morphisms in $\mathcal{C}$ such that

$$\underline{Hom}(f',M) \circ c = \underline{Hom}(f',M) \circ d.$$

Then

$$\begin{aligned} g' \circ c = g' \circ d &\implies (g' \circ c)\overline{\otimes} id_P = (g' \circ d)\overline{\otimes} id_P \\ &\implies (g'\overline{\otimes} id_P) \circ (c \overline{\otimes} id_P) = (g'\overline{\otimes} id_P) \circ (d \overline{\otimes} id_P) \\ &\implies t \circ (g'\overline{\otimes} id_P) \circ (c \overline{\otimes} id_P) = t \circ (g'\overline{\otimes} id_P) \circ (d \overline{\otimes} id_P) \\ &\overset{(28)}{\implies} z \circ (id_{\underline{Hom}(Q,M)} \overline{\otimes} f') \circ (c \overline{\otimes} id_P) = z \circ (id_{\underline{Hom}(Q,M)} \overline{\otimes} f') \circ (d \overline{\otimes} id_P) \\ &\implies z \circ (c \overline{\otimes} id_Q) \circ (id_X \overline{\otimes} f') = z \circ (d \overline{\otimes} id_Q) \circ (id_X \overline{\otimes} f'), \end{aligned}$$

and since the functor $X \overline{\otimes}\ \_\ : \mathcal{M} \to \mathcal{M}$ is right exact and f' is an epimorphism, it follows that $id_X \overline{\otimes} f'$ is an epimorphism. Hence, $z \circ (c \overline{\otimes} id_Q) = z \circ (d \overline{\otimes} id_Q)$.

Considering that

$$\Psi^Q(z)_X : Hom_{\mathcal{C}}(X, \underline{Hom}(Q,M)) \to Hom_{\mathcal{M}}(X \overline{\otimes} Q, M)$$

is an isomorphism in the family $\Psi^Q(z) = \{\Psi^Q(z)_X\}_{X \in \mathcal{C}}$ and

$$\Psi^Q(z)_X(c) = z \circ (c \overline{\otimes} id_Q) = z \circ (d \overline{\otimes} id_Q) = \Psi^Q(z)_X(d)$$

we finally get $c = d$. Therefore, $\underline{Hom}(f',M)$ is a monomorphism.

Affirmation 2: $Ker(\underline{Hom}(f,M)) = Im(\underline{Hom}(f',M))$ as subobjects of $\underline{Hom}(P,M)$.

We already know that every monomorphism is the kernel of its cokernel (see Lemma 1.2.7) and, because $\underline{Hom}(f',M)$ is a monomorphism, it suffices to show that $(\underline{Hom}(Q,M), \underline{Hom}(f',M))$ is the kernel of $\underline{Hom}(f,M)$.

Notice that $\underline{Hom}(f,M) \circ \underline{Hom}(f',M) = \underline{Hom}(f' \circ f, M) = \underline{Hom}(0,M) = 0$, where the last equality holds since the functor $\underline{Hom}(\ \_\ ,M)$ is additive (see Remark 1.1.20).

Let $k \in Hom_{\mathcal{C}}(K, \underline{Hom}(P, M))$ be a morphism such that $\underline{Hom}(f, M) \circ k = 0$. We'll show that there exists an unique morphism $u : K \to \underline{Hom}(Q, M)$ in $\mathcal{C}$ which satisfies $k = \underline{Hom}(f', M) \circ u$.

The functor $K\overline{\otimes}\_ : \mathcal{M} \to \mathcal{M}$ is exact, so it follows that the sequence

$$0 \longrightarrow K\overline{\otimes}N \xrightarrow{id_K\overline{\otimes}f} K\overline{\otimes}P \xrightarrow{id_K\overline{\otimes}f'} K\overline{\otimes}Q \longrightarrow 0$$

is short exact in $\mathcal{M}$ and thus $(K\overline{\otimes}Q, id_K\overline{\otimes}f')$ is the cokernel of $id_K\overline{\otimes}f$ (via Proposition 1.2.9). By definition, if $q' \in Hom_{\mathcal{M}}(K\overline{\otimes}P, U')$ is a morphism satisfying $q' \circ (id_K\overline{\otimes}f) = 0$ then there is an unique $u' \in Hom_{\mathcal{M}}(K\overline{\otimes}Q, U')$ such that $q' = u' \circ (id_K\overline{\otimes}f')$.

Let us check that $t\circ(k\overline{\otimes}id_P)$ is a morphism in $\mathcal{M}$ satisfying $t\circ(k\overline{\otimes}id_P)\circ(id_K\overline{\otimes}f) = 0$. In fact,

$$\begin{aligned} t \circ (k\overline{\otimes}id_P) \circ (id_K\overline{\otimes}f) &= t \circ (id_{\underline{Hom}(P,M)}\overline{\otimes}f) \circ (k\overline{\otimes}id_N) \\ &= w \circ (g\overline{\otimes}id_N) \circ (k\overline{\otimes}id_N) \\ &= w \circ ((g \circ k)\overline{\otimes}id_N) \\ &= w \circ (0\overline{\otimes}id_N) = 0 \end{aligned}$$

where the second equality holds by the definition of the morphism $\underline{Hom}(f, M)$, and the last by the additivity of the functor $\_\overline{\otimes}N$. So, by the definition of the cokernel of $id_K\overline{\otimes}f$, there exists an unique morphism $u' : K\overline{\otimes}Q \to M$ in $\mathcal{M}$ satisfying

$$t \circ (k\overline{\otimes}id_P) = u' \circ (id_K\overline{\otimes}f'). \tag{29}$$

Using the isomorphism $\Psi^Q(z)_K : Hom_{\mathcal{C}}(K, \underline{Hom}(Q, M)) \to Hom_{\mathcal{M}}(K\overline{\otimes}Q, M)$ we can define $u : K \to \underline{Hom}(Q, M)$ to be the unique morphism in $\mathcal{C}$ such that

$$u' = \Psi^Q(z)_K(u) = z \circ (u\overline{\otimes}id_Q). \tag{30}$$

Finally, let us verify that $k = g' \circ u$ by showing the equality

$$\Psi^P(t)_K(k) = \Psi^P(t)_K(g' \circ u).$$

Noticing that

$$\begin{aligned} \Psi^P(t)_K(g' \circ u) &= t \circ ((g' \circ u)\overline{\otimes}id_P) \\ &= t \circ (g'\overline{\otimes}id_P) \circ (u\overline{\otimes}id_P) \\ &\overset{(28)}{=} z \circ (id_{\underline{Hom}(Q,M)}\overline{\otimes}f') \circ (u\overline{\otimes}id_P) \\ &= z \circ (u\overline{\otimes}id_Q) \circ (id_K\overline{\otimes}f') \\ &\overset{(30)}{=} u' \circ (id_K\overline{\otimes}f') \\ &\overset{(29)}{=} t \circ (k\overline{\otimes}id_P) \\ &= \Psi^P(t)_K(k), \end{aligned}$$

it is possible to conclude that $k = g' \circ u = \underline{Hom}(f', M) \circ u$ as wanted. Thus $(\underline{Hom}(Q, M), \underline{Hom}(f', M))$ is the kernel of $\underline{Hom}(f, M)$ and therefore, the contravariant functor $\underline{Hom}(\ \_\ , M) : \mathcal{M} \to \mathcal{C}$ is left exact. ■

4.3 THE RIGHT BIEXACTNESS OF THE BIFUNCTOR $\underline{Hom}(\ \_\ ,\ \_\ )$ WHEN $\mathcal{M}$ IS EXACT

In this section, let $\mathcal{C}$ be a tensor category over $\mathbb{k}$, $\mathcal{M}$ an abelian and module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\mathbb{k}$-linear and left exact in the first variable, unless stated otherwise. Our main objective here is to verify a result stating that the bifunctor $\underline{Hom}(\ \_\ ,\ \_\ ) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ is also right biexact (i.e., right exact in both entries) when the category $\mathcal{C}$ is finite tensor and $\mathcal{M}$ is module exact over $\mathcal{C}$. In its proof we'll use the auxiliary propositions and theorem below.

Proposition 4.3.1 ([15], Lema 5.1.10)**.** *Let $0 \neq M \in \mathcal{M}$, $0 \neq X \in \mathcal{C}$, $0 \neq f \in Hom_{\mathcal{C}}(X, Y)$ and $0 \neq g \in Hom_{\mathcal{M}}(M, N)$. Then*

(i) *$X\overline{\otimes}M$ is a nonzero object of $\mathcal{M}$;*

(ii) *$id_X\overline{\otimes}g$ is a nonzero morphism in $\mathcal{M}$;*

(iii) *$f\overline{\otimes}id_M$ is a nonzero morphism in $\mathcal{M}$.*

Proof. (i) Let us suppose that $X\overline{\otimes}M = 0$ and consider the composition

$$M \xrightarrow{l_M^{-1}} 1\overline{\otimes}M \xrightarrow{coev_X\overline{\otimes}id_M} ({}^*X \otimes X)\overline{\otimes}M \xrightarrow{m_{{}^*X,X,M}} {}^*X\overline{\otimes}(X\overline{\otimes}M) = 0\ .$$

The morphism $coev_X\overline{\otimes}id_M$ is a monomorphism in $\mathcal{M}$ since $coev_X$ is a monomorphism (see Lemma 2.1.10) and $\_\ \overline{\otimes}M$ is left exact. This implies that the composition $h := m_{{}^*X,X,M} \circ (coev_X\overline{\otimes}id_M) \circ l_M^{-1} : M \to 0$ is a monomorphism. One way to get a contradiction is by noticing that $(Ker(h), k) = (M, id_M)$ as subobjects of M since $h = 0$ (see Lemma 1.1.10), and also $(0, 0)$ is the kernel of h since h is a monomorphism (see Proposition 1.2.5). Then $(M, id_M) = (0, 0)$ as subjects of M, that is, $M \cong 0$.

(ii) Suppose that $id_X\overline{\otimes}g = 0$ and let $k : Ker(g) \to M$ be the kernel of g. We are going to show that k is an epimorphism, which will imply that k is an isomorphism for the reason that any kernel is already a monomorphism. We know that the functor $X\overline{\otimes}\ \_\ : \mathcal{M} \to \mathcal{M}$ is exact[6] (and, particularly, left exact). From Proposition 1.4.3, $id_X\overline{\otimes}k : X\overline{\otimes}Ker(g) \to X\overline{\otimes}M$ is the kernel of $id_X\overline{\otimes}g$, and using that $id_X\overline{\otimes}g = 0$ we get by Lemma 1.1.10 that $id_X\overline{\otimes}k$ is an isomorphism. Given that $id_X\overline{\otimes}k$ is an epimorphism (it is an isomorphism) we can conclude by Proposition 1.2.6 that $(0, 0)$ is the cokernel of $id_X\overline{\otimes}k$.

[6] See Proposition 2.2.5.

On the other hand, let $(coKer(k), q)$ be the cokernel of k. Since the functor $X\overline{\otimes}\,\_ : \mathcal{M} \to \mathcal{M}$ is right exact and Proposition 1.4.3, the pair $(X\overline{\otimes}coKer(k), id_X\overline{\otimes}q)$ is the cokernel of $id_X\overline{\otimes}k$. This implies that $(0,0) = (X\overline{\otimes}coKer(k), id_X\overline{\otimes}q)$ as quotient objects of $X\overline{\otimes}M$ and, particularly, $X\overline{\otimes}coKer(k) \cong 0$. By the item (i) we get $coKer(k) \cong 0$ and by Proposition 1.2.6, k is an epimorphism. Because k is already a monomorphism (it's the kernel of g) then k is an isomorphism and via Lemma 1.1.10, $g = 0$ which is a contradiction.

(iii) It's similar to the proof of item (ii). ∎

Proposition 4.3.2 ([15], Lema 5.1.12)**.** *Let X be a nonzero object in $\mathcal{C}$. If the sequence*

$$0 \longrightarrow X\overline{\otimes}M \xrightarrow{id_X\overline{\otimes}f} X\overline{\otimes}N \xrightarrow{id_X\overline{\otimes}g} X\overline{\otimes}U \longrightarrow 0$$

is short exact in $\mathcal{M}$ then the sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} U \longrightarrow 0$ is also short exact in $\mathcal{M}$.

Proof. Firstly, we begin by showing that (M, f) is the kernel of g. The kernel of $id_X\overline{\otimes}g$ is the pair $(X\overline{\otimes}M, id_X\overline{\otimes}f)$ via the Proposition 1.2.9. Furthermore, let $(Ker(g), k)$ be the kernel of g and notice that, by using the left exactness of the functor $X\overline{\otimes}\,\_ : \mathcal{M} \to \mathcal{M}$ (it is exact[7]), the kernel of $id_X\overline{\otimes}g$ is the pair $(X\overline{\otimes}Ker(g), id_X\overline{\otimes}k)$ (via Proposition 1.4.3). On the other hand, since the first sequence is short exact, it follows that $(X\overline{\otimes}M, id_X\overline{\otimes}f)$ is the kernel of $id_X\overline{\otimes}g$. This implies that these two pairs are equal as subobjects of $X\overline{\otimes}N$ (see Proposition 1.1.7), that is, there exists an isomorphism $u : X\overline{\otimes}M \to X\overline{\otimes}Ker(g)$ in $\mathcal{M}$ such that $(id_X\overline{\otimes}k) \circ u \overset{(*)}{=} id_X\overline{\otimes}f$.

Secondly, the morphism f satisfies $g \circ f = 0$. In fact, since $(id_X\overline{\otimes}g) \circ (id_X\overline{\otimes}f) = 0$ we get $id_X\overline{\otimes}(g \circ f) = 0$, implying $g \circ f = 0$ by (ii) of Proposition 4.3.1. Considering that $(Ker(g), k)$ is the kernel of g and $g \circ f = 0$, there exists an unique morphism $u' : M \to Ker(g)$ in $\mathcal{M}$ such that $k \circ u' = f$. By applying the functor $X\overline{\otimes}\,\_ : \mathcal{M} \to \mathcal{M}$ in this last equality of morphisms we obtain $(id_X\overline{\otimes}k) \circ (id_X\overline{\otimes}u') = id_X\overline{\otimes}f$. Moreover, by this last equality and $(*)$ it follows that

$$(id_X\overline{\otimes}k) \circ (id_X\overline{\otimes}u') = (id_X\overline{\otimes}k) \circ u.$$

Because k is a monomorphism (it's the kernel of g) and the functor $X\overline{\otimes}\,\_ : \mathcal{M} \to \mathcal{M}$ is left exact we can conclude that $id_X\overline{\otimes}u' = u$ and, hence, $id_X\overline{\otimes}u'$ is an isomorphism.

Finally, we prove that $u' : M \to Ker(g)$ is an isomorphism (i.e., it is a monomorphism and an epimorphism since our category $\mathcal{M}$ is abelian) which will imply that k and f are equivalent as subobjects of N. Let h and h' be morphisms in $Hom_{\mathcal{M}}(Q, M)$ satisfying $u' \circ h = u' \circ h'$. By applying the functor $X\overline{\otimes}\,\_$ in this last morphism equality we get $(id_X\overline{\otimes}u') \circ (id_X\overline{\otimes}h) = (id_X\overline{\otimes}u') \circ (id_X\overline{\otimes}h')$, and using that $id_X\overline{\otimes}u'$ is a monomorphism

[7] Proposition 2.2.5.

(it is exactly the isomorphism u), we obtain $id_X\overline{\otimes}h = id_X\overline{\otimes}h'$, i.e., $id_X\overline{\otimes}(h-h') = 0$. Via Proposition 4.3.1 it follows the equality $h = h'$, i.e., u' is a monomorphism. Analogously, one can also verify that u' is an epimorphism and hence, an isomorphism satisfying $k \circ u' = f$. This implies that the monomorphisms f and k are equivalent, that is (M, f) and $(Ker(g), k)$ are equal as subobjects of N. Therefore, (M, f) is the kernel of g (see Proposition 1.1.7).

The fact that (U, g) is the cokernel of f can be checked analogously. Hence, the sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} U \longrightarrow 0$ is exact in $\mathcal{M}$ (via Proposition 1.2.9). ■

Theorem 4.3.3 ([4], Proposition 7.6.9)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two abelian and module categories over $\mathcal{C}$ with both module products $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and $\overline{\otimes} : \mathcal{C} \times \mathcal{N} \to \mathcal{N}$ being $\Bbbk$-linear and left exact in the first variable, and assume that $\mathcal{M}$ is locally finite and exact. Then any additive $\mathcal{C}$-module functor $(F, c) : \mathcal{M} \to \mathcal{N}$ is exact.*

Proof. Suppose the $\mathcal{C}$-module functor $(F, c) : \mathcal{M} \to \mathcal{N}$ is not exact. This implies that there is a short exact sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} U \longrightarrow 0$ in $\mathcal{M}$ such that

$$0 \longrightarrow F(M) \xrightarrow{F(f)} F(N) \xrightarrow{F(g)} F(U) \longrightarrow 0$$

is not exact in $\mathcal{N}$. Using the contrapositive of Proposition 4.3.2, we can say that for any nonzero object $X \in \mathcal{C}$ the sequence

$$0 \longrightarrow X\overline{\otimes}F(M) \xrightarrow{id_X\overline{\otimes}F(f)} X\overline{\otimes}F(N) \xrightarrow{id_X\overline{\otimes}F(g)} X\overline{\otimes}F(U) \longrightarrow 0$$

is not exact in $\mathcal{N}$.

Without loss of generality, suppose that $X \in \mathcal{C}$ is projective. The category $\mathcal{M}$ being exact particularly implies that the object $X\overline{\otimes}U \in \mathcal{M}$ is projective.

On the other hand, given that the sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} U \longrightarrow 0$ is short exact in $\mathcal{M}$ and the functor $X\overline{\otimes}\ \_ : \mathcal{M} \to \mathcal{M}$ is exact, it follows that the sequence

$$0 \longrightarrow X\overline{\otimes}M \xrightarrow{id_X\overline{\otimes}f} X\overline{\otimes}N \xrightarrow{id_X\overline{\otimes}g} X\overline{\otimes}U \longrightarrow 0$$

is short exact in $\mathcal{M}$ and it splits since $X\overline{\otimes}U \in \mathcal{M}$ is projective (see Remark 1.2.3).

Using the Proposition 1.4.4, we can conclude that the sequence

$$0 \longrightarrow F(X\overline{\otimes}M) \xrightarrow{F(id_X\overline{\otimes}f)} F(X\overline{\otimes}N) \xrightarrow{F(id_X\overline{\otimes}g)} F(X\overline{\otimes}U) \longrightarrow 0$$

is short exact in $\mathcal{N}$ (and splits).

Notice that the first sequence of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F(X\overline{\otimes}M) & \xrightarrow{F(id_X\overline{\otimes}f)} & F(X\overline{\otimes}N) & \xrightarrow{F(id_X\overline{\otimes}g)} & F(X\overline{\otimes}U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_{X,M}} & & \downarrow{\scriptstyle c_{X,N}} & & \downarrow{\scriptstyle c_{X,U}} & & \\ 0 & \longrightarrow & X\overline{\otimes}F(M) & \xrightarrow{id_X\overline{\otimes}F(f)} & X\overline{\otimes}F(N) & \xrightarrow{id_X\overline{\otimes}F(g)} & X\overline{\otimes}F(U) & \longrightarrow & 0 \end{array}$$

is short exact in $\mathcal{N}$ and the two square diagrams commute because $c = \{c_{X,M}\}_{(X,M)\in\mathcal{C}\times\mathcal{M}}$ is a natural isomorphism in $\mathcal{C} \times \mathcal{M}$ which implies that the second sequence

$$0 \longrightarrow X\overline{\otimes}F(M) \xrightarrow{id_X\overline{\otimes}F(f)} X\overline{\otimes}F(N) \xrightarrow{id_X\overline{\otimes}F(g)} X\overline{\otimes}F(U) \longrightarrow 0$$

is also short exact in $\mathcal{N}$. In fact, it is enough to show that $(X\overline{\otimes}F(M), id_X\overline{\otimes}F(f))$ is the kernel of $id_X\overline{\otimes}F(g)$ and $(X\overline{\otimes}F(U), id_X\overline{\otimes}F(g))$ is the cokernel of $id_X\overline{\otimes}F(f)$.

We already know that $F(id_X\overline{\otimes}f)$ is the kernel of $F(id_X\overline{\otimes}g)$. Using that $c^{-1}_{X,M}$ is an isomorphism in $\mathcal{M}$, it follows that $F(id_X\overline{\otimes}f)\circ c^{-1}_{X,M}$ is also the kernel of $F(id_X\overline{\otimes}g)$ (Lemma 1.1.9, (v)). Using (i) of this Lemma 1.1.9 it's possible to conclude that $F(id_X\overline{\otimes}f)\circ c^{-1}_{X,M}$ is the kernel of $c_{X,U} \circ F(id_X\overline{\otimes}g)$ since $c_{X,U}$ is a monomorphism in $\mathcal{M}$. Finally, using that $c^{-1}_{X,N}$ is an isomorphism in $\mathcal{M}$ and item (ii) of this same lemma, $c_{X,N}\circ F(id_X\overline{\otimes}f)\circ c^{-1}_{X,M}$ is the kernel of $c_{X,U} \circ F(id_X\overline{\otimes}g) \circ c^{-1}_{X,N}$. Via the commutativity of the two square diagrams above,

$$c_{X,N} \circ F(id_X\overline{\otimes}f) \circ c^{-1}_{X,M} = id_X\overline{\otimes}F(f)$$

and

$$c_{X,U} \circ F(id_X\overline{\otimes}g) \circ c^{-1}_{X,N} = id_X\overline{\otimes}F(g),$$

that is, $id_X\overline{\otimes}F(f)$ is the kernel of $id_X\overline{\otimes}F(g)$ as wanted.

Similarly, one can prove that $id_X\overline{\otimes}F(g)$ is the cokernel of $id_X\overline{\otimes}F(f)$ and consequently the sequence

$$0 \longrightarrow X\overline{\otimes}F(M) \xrightarrow{id_X\overline{\otimes}F(f)} X\overline{\otimes}F(N) \xrightarrow{id_X\overline{\otimes}F(g)} X\overline{\otimes}F(U) \longrightarrow 0.$$

is short exact in $\mathcal{N}$, which is a contradiction. Therefore, the additive $\mathcal{C}$-module functor $(F, c) : \mathcal{M} \to \mathcal{N}$ is exact.

■

With these two results, it follows immediately that the bifunctor $\underline{Hom}(\_\,,\,\_) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is exact when $\mathcal{C}$ is a finite tensor category (remembering that the finiteness is required to define the internal Hom object and functor) and $\mathcal{M}$ is a locally finite and exact module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C}\times\mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable.

Corollary 4.3.4. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ be a locally finite and exact module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable. The $\mathcal{C}$-module bifunctor $\underline{Hom}(\_\,,\,\_) : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{C}$ is also right biexact.*

Proof. It follows immediately from the last theorem by noticing that $\mathcal{M}$ is exact and $\underline{Hom}(M, \_\,) : \mathcal{M} \to \mathcal{C}$ and $\underline{Hom}(\_\,, M) : \mathcal{M}^{op} \to \mathcal{C}$ are additive functors for every $M \in \mathcal{M}$. ■

Namely, the bifunctor $\underline{Hom}(\_\,,\,\_) : \mathcal{M}^{op}\times\mathcal{M} \to \mathcal{C}$ is biexact, i.e., exact in both entries.

4.4 OTHER PROPERTIES OF INTERNAL HOM OBJECTS AND FUNCTOR

In this section let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a locally finite and exact indecomposable $\mathcal{C}$-module category with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable.

Proposition 4.4.1 ([15], Corolario 5.3.14)**.** *Let $\mathcal{N}$ be an abelian and module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{N} \to \mathcal{N}$ being $\Bbbk$-linear and left exact in the first variable and $F : \mathcal{M} \to \mathcal{N}$ a nonzero additive $\mathcal{C}$-module functor* [8]. *If M is an object in $\mathcal{M}$ satisfying $F(M) \cong 0_{\mathcal{N}}$ then $M \cong 0_{\mathcal{M}}$.*

These two following results will be used when proving an important theorem regarding an equivalence of $\mathcal{C}$-module categories in the end part of this work. We couldn't find any mention of these results in the literature. This first one is inspired on the Proposition 4.3.1.

Lemma 4.4.2. *Let $0 \neq M \in \mathcal{M}$, $0 \neq N \in \mathcal{M}$ and $0 \neq f \in Hom_{\mathcal{M}}(R, S)$. Then*

(i) *$\underline{Hom}(M, N)$ is a nonzero object of $\mathcal{C}$;*

(ii) *$\underline{Hom}(M, f) : \underline{Hom}(M, R) \to \underline{Hom}(M, S)$ is a nonzero morphism in $\mathcal{C}$;*

(iii) *$\underline{Hom}(f, M) : \underline{Hom}(S, M) \to \underline{Hom}(R, M)$ is a nonzero morphism in $\mathcal{C}$.*

Proof. Let $(\underline{Hom}(M, N), y)$ be an universal element of the functor $Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, N)$.

(i) We want to use the Proposition 4.4.1 in this item. It's known that the $\mathcal{C}$-module functor $\underline{Hom}(M, \_\,) : \mathcal{M} \to \mathcal{C}$ is additive (it is exact). It is also nonzero since the object $\underline{Hom}(M, \_\,)(M) = \underline{Hom}(M, M)$ represents the functor $Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, M)$, implying that there is a natural isomorphism

$$\phi : Hom_{\mathcal{C}}(\_\,, \underline{Hom}(M, M)) \to Hom_{\mathcal{M}}(\_\,\overline{\otimes} M, M)$$

in $\mathcal{C}$ and, particularly, a group isomorphism (via Proposition 1.3.10)

$$\phi_{\mathbf{1}} : Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}(M, M)) \to Hom_{\mathcal{M}}(\mathbf{1} \overline{\otimes} M, M).$$

Moreover, let us define $\theta : Hom_{\mathcal{M}}(\mathbf{1} \overline{\otimes} M, M) \to Hom_{\mathcal{M}}(M, M)$ by $\theta(f) := f \circ l_M^{-1}$. It's easy to see that θ is a group isomorphism with inverse $g \mapsto g \circ l_M$. Hence, the composition

$$\theta \circ \phi_{\mathbf{1}} : Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}(M, M)) \to Hom_{\mathcal{M}}(\mathbf{1} \overline{\otimes} M, M) \to Hom_{\mathcal{M}}(M, M)$$

is a group isomorphism and then

$$(\phi_{\mathbf{1}}^{-1} \circ \theta^{-1})(id_M) \in Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}(M, M)) \tag{31}$$

[8] And thus exact by the Theorem 4.3.3.

is not the zero morphism because $id_M \neq 0$ (see Remark 1.1.8). This implies $\underline{Hom}(M, M) \neq 0_{\mathcal{C}}$, i.e., the $\mathcal{C}$-module functor $\underline{Hom}(M, \_\ )$ is nonzero. Using that $N \neq 0_{\mathcal{M}}$ we can conclude that $\underline{Hom}(M, \_\ )(N) = \underline{Hom}(M, N) \neq 0_{\mathcal{C}}$ by Proposition 4.4.1.

(ii) Suppose $\underline{Hom}(M, f) = 0$ and let $k : Ker(f) \to R$ be the kernel of f. We now show that k is an epimorphism which will imply that k is an isomorphism (leading us to a contradiction) given that any kernel is already a monomorphism. The functor $\underline{Hom}(M, \_\ ) : \mathcal{M} \to \mathcal{C}$ is exact, so it follows that $\underline{Hom}(M, k)$ is the kernel of $\underline{Hom}(M, f)$ (see Proposition 1.4.3). Using the equality $\underline{Hom}(M, f) = 0$, it follows by Lemma 1.1.10 that $\underline{Hom}(M, k)$ is an isomorphism.

Now, let $(coKer(k), q)$ be the cokernel of $k : Ker(f) \to R$. The sequence

$$0 \longrightarrow Ker(f) \xrightarrow{k} R \xrightarrow{q} coKer(k) \longrightarrow 0$$

is short exact since k is a monomorphism, q is an epimorphism and the image of k is the kernel of q. From the fact that the functor $\underline{Hom}(M, \_\ ) : \mathcal{M} \to \mathcal{C}$ is exact, the sequence

$$0 \longrightarrow \underline{Hom}(M, Ker(f)) \xrightarrow{\underline{Hom}(M,k)} \underline{Hom}(M, R) \xrightarrow{\underline{Hom}(M,q)} \underline{Hom}(M, coKer(k)) \longrightarrow 0$$

is also short exact and therefore, $(\underline{Hom}(M, coKer(k)), \underline{Hom}(M, q))$ is the cokernel of $\underline{Hom}(M, k)$ (see Proposition 1.2.9).

On the other hand, because $\underline{Hom}(M, k)$ is an epimorphism (it is an isomorphism) we can conclude by Proposition 1.2.6 that $(0, 0)$ is the cokernel of $\underline{Hom}(M, k)$, that is, $(\underline{Hom}(M, coKer(k)), \underline{Hom}(M, q)) = (0, 0)$ as quotient objects of $\underline{Hom}(M, R)$ (see Proposition 1.1.7). This implies $\underline{Hom}(M, coKer(k)) \cong 0$ and since $M \in \mathcal{M}$ is a nonzero object, the object $coKer(k) \in \mathcal{M}$ has to be 0 by the item (i) of this lemma.

Thus, $(0, 0)$ is the cokernel of k (which is equal to $(coKer(k), q : R \to coKer(k))$ as quotient objects of R) and, via Proposition 1.2.6, k is an epimorphism. Since k is already a monomorphism (it's the kernel of f) it follows that k is an isomorphism and by Lemma 1.1.10 we get $f = 0$, i.e., a contradiction. Hence, if f is a nonzero morphism in $\mathcal{M}$ then $\underline{Hom}(M, f) \neq 0$.

(iii) It's similar to the proof of item (ii). ∎

Lemma 4.4.3. *If N and $0 \neq M$ are two objects in $\mathcal{M}$ and $(\underline{Hom}(M, N), y)$ is an universal element of the functor $Hom_{\mathcal{M}}(\_\ \overline{\otimes} M, N)$ then the morphism $y : \underline{Hom}(M, N)\overline{\otimes} M \to N$ is an epimorphism in $\mathcal{M}$.*

Proof. Let $(\underline{Hom}(M, P), z)$ be an universal element of its respective functor and

$$\Psi^P(z) = \{\Psi^P(z)_X : Hom_{\mathcal{C}}(X, \underline{Hom}(M, P)) \to Hom_{\mathcal{M}}(X\overline{\otimes} M, P)\}_{X \in \mathcal{C}}$$

be the natural isomorphism given by this universal element (see Proposition 3.4).

Let g and h be morphisms in $Hom_{\mathcal{M}}(N, P)$ satisfying $g \circ y = h \circ y$. From the definition of the morphisms $\underline{Hom}(M, g)$ and $\underline{Hom}(M, h)$ in $Hom_{\mathcal{C}}(\underline{Hom}(M, N), \underline{Hom}(M, P))$, we have

$$g \circ y = z \circ (\underline{Hom}(M, g)\overline{\otimes} id_M) = \Psi^P(z)_{\underline{Hom}(M,N)}(\underline{Hom}(M, g))$$

and

$$h \circ y = z \circ (\underline{Hom}(M, h)\overline{\otimes} id_M) = \Psi^P(z)_{\underline{Hom}(M,N)}(\underline{Hom}(M, h)).$$

Using that $\Psi^P_{\underline{Hom}(M,N)}$ is an isomorphism we obtain $\underline{Hom}(M, g) = \underline{Hom}(M, h)$, that is, $\underline{Hom}(M, a - b) = 0$. Hence, by the item (ii) of Lemma 4.4.2, $a - b = 0$ as wanted.

Therefore, y is an epimorphism in $\mathcal{M}$. ■

5 THE CATEGORY $\mathcal{C}_A$

It is possible to define the algebra notion in a categorical context. An algebra in a monoidal category $\mathcal{C}$ is an object $A \in \mathcal{C}$ with multiplication and unit morphisms satisfying some compatibilities through the commutativity of certain diagrams. These commutativities express, in a certain way, the properties of an algebra over a field that we are used to work with. This construction can be seen as the categorification of the concept of algebra over a field.

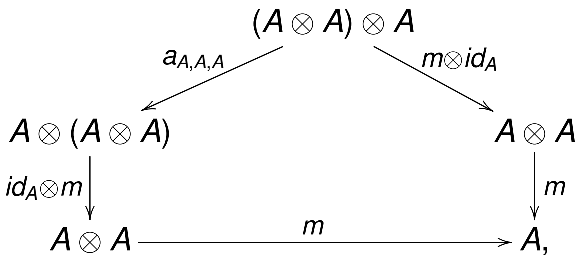
As a consequence, one can define the notion of right (and left) A-module in $\mathcal{C}$ in a similar way, which forms a category denoted by $\mathcal{C}_A$ (${}_A\mathcal{C}$). This category admits a structure of left (right) $\mathcal{C}$-module category. In this chapter we'll be studying these categories for a particular algebra given by an internal Hom object, namely $\underline{Hom}(M, M) \in \mathcal{C}$ for any $M \in \mathcal{M}$.

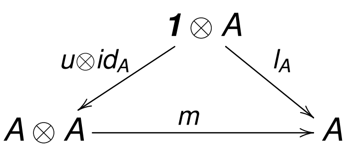
The main references for this chapter are [4] and [16]. Let $\mathcal{C}$ be a monoidal category unless stated otherwise.

5.1 ALGEBRA IN A MONOIDAL CATEGORY AND THE MODULE CATEGORY $\mathcal{C}_A$ OVER $\mathcal{C}$

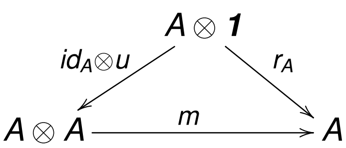
We begin by introducing some definitions and basic examples involving algebra and module over an algebra in a monoidal category context.

Definition 5.1.1. (i) *An algebra in $\mathcal{C}$ is a triple (A, m, u) where $A \in \mathcal{C}$, $m : A \otimes A \to A$ and $u : \mathbf{1} \to A$ are morphisms in $\mathcal{C}$ called multiplication and unit, respectively, and the diagrams*





and



commute. The triple of an algebra (A, m, u) is often omitted, and it's denoted simply by the object A.

(ii) *A morphism between two algebras $A = (A, m, u)$ and $B = (B, m', u')$ is a morphism*

$f \in Hom_{\mathcal{C}}(A, B)$ such that the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle u}\nwarrow \quad \nearrow{\scriptstyle u'} & \\ & \mathbf{1} & \end{array}$$

commute.

(iii) *A right module over an algebra A (or right A-module) in $\mathcal{C}$ is an object $M \in \mathcal{C}$ together with an action morphism $\rho_M : M \otimes A \to M$ in $\mathcal{C}$, that is, a pair (M, ρ_M) such that the diagrams*

$$\begin{array}{ccccc} (M \otimes A) \otimes A & & \xrightarrow{a_{M,A,A}} & & M \otimes (A \otimes A) \\ {\scriptstyle \rho_M \otimes id_A}\downarrow & & & & \downarrow{\scriptstyle id_M \otimes m} \\ M \otimes A & & & & M \otimes A \\ & {\scriptstyle \rho_M}\searrow & & \swarrow{\scriptstyle \rho_M} & \\ & & M & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M \otimes \mathbf{1} & \xrightarrow{r_M} & M \\ & {\scriptstyle id_M \otimes u}\searrow \quad \nearrow{\scriptstyle \rho_M} & \\ & M \otimes A & \end{array}$$

commute. In a similar way we can define a left module over an algebra A.

(iv) *A morphism between two right modules (M, ρ_M) and (N, ρ_N) over an algebra A (or an A-module morphism) is a morphism $f \in Hom_{\mathcal{C}}(M, N)$ such that the diagram*

$$\begin{array}{ccc} M \otimes A & \xrightarrow{f \otimes id_A} & N \otimes A \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

commutes.

We focused on the definition of right module over an algebra because this is the case we have the most interest in working with here.

Example 5.1.2. The object $\mathbf{1} \in \mathcal{C}$ is an algebra with multiplication $m = l_{\mathbf{1}} = r_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ and unit $u = id_{\mathbf{1}} : \mathbf{1} \to \mathbf{1}$.

Example 5.1.3. An algebra $A = (A, m, u)$ has a structure of (left and right) A-module with the action being its multiplication, that is, $\rho_A = m$.

It is well known that the right A-modules in $\mathcal{C}$ with the A-module morphisms form a category which is often denoted by $\mathcal{C}_A$ or $Mod_{\mathcal{C}}(A)$.

This following exercise will be helpful when showing that the category $\mathcal{C}_A$ has a structure of left $\mathcal{C}$-module category.

Proposition 5.1.4 ([4], Exercise 7.8.8)**.** *Let* $(M, \rho_M) \in \mathcal{C}_A$ *and* $X \in \mathcal{C}$*. Then the object* $X \otimes M \in \mathcal{C}$ *has a structure of a right A-module with action given by the composition*

$$(X \otimes M) \otimes A \xrightarrow{a_{X,M,A}} X \otimes (M \otimes A) \xrightarrow{id_X \otimes \rho_M} X \otimes M,$$

that is, $\rho_{X \otimes M} = (id_X \otimes \rho_M) \circ a_{X,M,A}$*.*

The application that is going to be the action bifunctor $\overline{\otimes}$ of the $\mathcal{C}$-module category $\mathcal{C}_A$ can be defined in the objects as

$$\begin{aligned} \overline{\otimes} : \mathcal{C} \times \mathcal{C}_A &\to \mathcal{C}_A \\ (X, (M, \rho_M)) &\mapsto X \overline{\otimes} (M, \rho_M) \coloneqq (X \otimes M, \rho_{X \otimes M}), \end{aligned}$$

where $\rho_{X \otimes M} = (id_X \otimes \rho_M) \circ a_{X,M,A}$. In the morphisms it is defined as $\overline{\otimes}(f, g) = f \overline{\otimes} g \coloneqq f \otimes g$, for every morphism (f, g) in $\mathcal{C} \times \mathcal{C}_A$.

Let $X, Y \in \mathcal{C}$ and $(M, \rho_M) \in \mathcal{C}_A$. We now define the associativity m' and unit l' constraint for the $\mathcal{C}$-module category $\mathcal{C}_A$. Before doing so, notice that

$$\begin{aligned} (X \otimes Y) \overline{\otimes} (M, \rho_M) &= ((X \otimes Y) \otimes M, \rho_{(X \otimes Y) \otimes M}), \quad \text{and} \\ X \overline{\otimes} (Y \overline{\otimes} (M, \rho_M)) &= X \overline{\otimes} (Y \otimes M, \rho_{Y \otimes M}) = (X \otimes (Y \otimes M), \rho_{X \otimes (Y \otimes M)}), \end{aligned}$$

so we may define

$$m'_{X,Y,(M,\rho_M)} : ((X \otimes Y) \otimes M, \rho_{(X \otimes Y) \otimes M}) \to (X \otimes (Y \otimes M), \rho_{X \otimes (Y \otimes M)})$$

as $m'_{X,Y,(M,\rho_M)} \coloneqq a_{X,Y,M}$. It remains to show that $a_{X,Y,M}$ is a morphism in $\mathcal{C}_A$, i.e., a morphism of right A-modules. In fact, the diagram

$$\begin{array}{ccc} ((X \otimes Y) \otimes M) \otimes A & \xrightarrow{a_{X,Y,M} \otimes id_A} & (X \otimes (Y \otimes M)) \otimes A \\ \Big\downarrow{\scriptstyle \rho_{(X \otimes Y) \otimes M}} & & \Big\downarrow{\scriptstyle \rho_{X \otimes (Y \otimes M)}} \\ (X \otimes Y) \otimes M & \xrightarrow[a_{X,Y,M}]{} & X \otimes (Y \otimes M) \end{array}$$

commutes since

$$\begin{aligned} \rho_{X \otimes (Y \otimes M)} \circ (a_{X,Y,M} \otimes id_A) &= (id_X \otimes \rho_{Y \otimes M}) \circ a_{X,Y \otimes M,A} \circ (a_{X,Y,M} \otimes id_A) \\ &= (id_X \otimes ((id_Y \otimes \rho_M) \circ a_{Y,M,A})) \circ a_{X,Y \otimes M,A} \circ (a_{X,Y,M} \otimes id_A) \\ &= (id_X \otimes (id_Y \otimes \rho_M)) \circ (id_X \otimes a_{Y,M,A}) \circ a_{X,Y \otimes M,A} \circ \\ &\quad\ (a_{X,Y,M} \otimes id_A) \\ &\overset{(a)}{=} (id_X \otimes (id_Y \otimes \rho_M)) \circ a_{X,Y,M \otimes A} \circ a_{X \otimes Y,M,A} \\ &\overset{(b)}{=} a_{X,Y,M} \circ (id_{X \otimes Y} \otimes \rho_M) \circ a_{X \otimes Y,M,A} \\ &= a_{X,Y,M} \circ \rho_{(X \otimes Y) \otimes M} \end{aligned}$$

where the equality (a) comes from the pentagon axiom of the monoidal category $\mathcal{C}$, and (b) via the naturality of a. One can easily see that $m' = \{m'_{X,Y,(M,\rho_M)} = a_{X,Y,M}\}_{X,Y\in\mathcal{C},(M,\rho_M)\in\mathcal{C}_A}$ is a natural isomorphism right from the fact that a is a natural isomorphism.

Furthermore, $\mathbf{1}\overline{\otimes}(M,\rho_M) = (\mathbf{1}\otimes M, \rho_{\mathbf{1}\otimes M})$ and thus we can define

$$l'_{(M,\rho_M)} : (\mathbf{1}\otimes M, \rho_{\mathbf{1}\otimes M}) \to (M, \rho_M)$$

simply as $l'_{(M,\rho_M)} := l_M$. One can easily check that l_M is an A-module morphism and $l' = \{l'_{(M,\rho_M)} := l_M\}_{(M,\rho_M)\in\mathcal{C}_A}$ is a natural isomorphism in $\mathcal{C}_A$ in a similar manner as we've done above.

The following result is present in [4] as Proposition 7.8.10.

Proposition 5.1.5 ([4], Proposition 7.8.10)**.** *The category $\mathcal{C}_A$ together with the action bifunctor, associativity and unit constraints defined above is a left $\mathcal{C}$-module category.*

For the next lemma, we've included an idea of the proof because we will need later an explicit description of a certain isomorphism. Let A be an algebra in $\mathcal{C}$ and define two functors

$$\begin{aligned} G : \mathcal{C} &\to \mathcal{C}_A \\ X &\mapsto (X\otimes A, \rho_{X\otimes A} = (id_X\otimes m)\circ a_{X,A,A}) \end{aligned}$$

and

$$\begin{aligned} Forg : \mathcal{C}_A &\to \mathcal{C} \\ (M,\rho_M) &\mapsto M. \end{aligned}$$

We now show that G is left adjoint to the forgetful functor $Forg$.

Lemma 5.1.6 ([4], Lemma 7.8.12)**.** *The functor G is left adjoint to Forg, i.e., there is a natural isomorphism*

$$\phi = \{\phi_{X,(M,\rho_M)} : Hom_{\mathcal{C}_A}((X\otimes A, \rho_{X\otimes A}), (M,\rho_M)) \to Hom_{\mathcal{C}}(X, M)\}_{(X,(M,\rho_M))\in\mathcal{C}^{op}\times\mathcal{C}_A}$$

in $\mathcal{C}^{op}\times\mathcal{C}_A$.

Let $X\in\mathcal{C}$, $(M,\rho_M)\in\mathcal{C}_A$, $f\in Hom_{\mathcal{C}_A}((X\otimes A, \rho_{X\otimes A}), (M,\rho_M))$ and $g\in Hom_{\mathcal{C}}(X,M)$. In its proof, the natural isomorphism ϕ is defined by $\phi_{X,(M,\rho_M)}(f) := f\circ(id_X\otimes u)\circ r_X^{-1}$ with inverse $\phi^{-1}_{X,(M,\rho_M)}(g) := \rho_M\circ(g\otimes id_A)$.

Proposition 5.1.7 ([4], Exercise 7.8.14)**.** *For every $(M,\rho_M)\in\mathcal{C}_A$, the action morphism ρ_M is an epimorphism in $\mathcal{C}_A$.*

Proof. Considering the diagram

$$\begin{array}{ccc} (M \otimes A) \otimes A & \xrightarrow{\rho_M \otimes id_A} & M \otimes A \\ \rho_{M\otimes A} \downarrow & & \downarrow \rho_M \\ M \otimes A & \xrightarrow[\rho_M]{} & M \end{array}$$

we may notice that

$$\rho_M \circ \rho_{M\otimes A} = \rho_M \circ (id_M \otimes \rho_A) \circ a_{M,A,A} = \rho_M \circ (id_M \otimes m) \circ a_{M,A,A} \overset{(*)}{=} \rho_M \circ (\rho_M \otimes id_A),$$

where the equality $(*)$ holds via the pentagon diagram of the object (M, ρ_M) in $\mathcal{C}_A$ (see Definition 5.1.1).

Finally, let g and h be morphisms in $Hom_{\mathcal{C}}(M, N)$ satisfying $g \circ \rho_M = h \circ \rho_M$. Since $\rho_M \circ (id_M \otimes u) = r_M$ (by definition) we have

$$g \circ \rho_M \circ (id_M \otimes u) = h \circ \rho_M \circ (id_M \otimes u) \implies g \circ r_M = h \circ r_M \implies g = h$$

and therefore, $\rho_M : (M \otimes A, \rho_{M\otimes A}) \to (M, \rho_M)$ is an epimorphism in $\mathcal{C}_A$. ■

Definition 5.1.8 ([4], Definition 7.8.17)**.** *Two algebras A and B in $\mathcal{C}$ are Morita equivalent if the categories $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent as $\mathcal{C}$-module categories.*

These next two propositions involving algebras and modules over an algebra will be used in some instances. We couldn't find them in the literature, but we believe they are given as a fact by many authors.

Proposition 5.1.9. *Let $A = (A, m, u)$ be an algebra in $\mathcal{C}$, $B \in \mathcal{C}$ and $f : A \to B$ be an isomorphism in $\mathcal{C}$. Then B is an algebra in $\mathcal{C}$, f is an algebra isomorphism, and the algebras A and B are Morita equivalent.*

Proof. We begin by defining the multiplication $m' : B \otimes B \to B$ of $B \in \mathcal{C}$ as the composition

$$m' : B \otimes B \xrightarrow{f^{-1}\otimes f^{-1}} A \otimes A \xrightarrow{m} A \xrightarrow{f} B,$$

and the unit $u' : 1 \to B$ as

$$u' : \mathbf{1} \xrightarrow{u} A \xrightarrow{f} B\,.$$

The pentagon diagram commutes since

$$m' \circ (id_B \otimes m') \circ a_{B,B,B} = f \circ m \circ (f^{-1} \otimes f^{-1}) \circ (id_B \otimes f) \circ (id_B \otimes m) \circ (id_B \otimes (f^{-1} \otimes f^{-1}))\circ$$
$$a_{B,B,B}$$
$$\overset{(a)}{=} f \circ m \circ (f^{-1} \otimes id_A) \circ (id_B \otimes m) \circ a_{B,A,A} \circ ((id_B \otimes f^{-1}) \otimes f^{-1})$$

$$
\begin{aligned}
&= f \circ m \circ (id_A \otimes m) \circ (f^{-1} \otimes id_{A\otimes A}) \circ a_{B,A,A} \circ ((id_B \otimes f^{-1}) \otimes f^{-1}) \\
&\overset{(a)}{=} f \circ m \circ (id_A \otimes m) \circ a_{A,A,A} \circ ((f^{-1} \otimes id_A) \otimes id_A) \circ ((id_B \otimes f^{-1}) \otimes f^{-1}) \\
&\overset{(b)}{=} f \circ m \circ (m \otimes id_A) \circ ((f^{-1} \otimes f^{-1}) \otimes f^{-1}) \\
&= f \circ m \circ (m \otimes id_A) \circ (id_{A\otimes A} \otimes f^{-1}) \circ ((f^{-1} \otimes f^{-1}) \otimes id_B) \\
&= f \circ m \circ (id_A \otimes f^{-1}) \circ (m \otimes id_B) \circ ((f^{-1} \otimes f^{-1}) \otimes id_B) \\
&= f \circ m \circ (f^{-1} \otimes f^{-1}) \circ (f \otimes id_B) \circ (m \otimes id_B) \circ ((f^{-1} \otimes f^{-1}) \otimes id_B) \\
&= m' \circ (m' \otimes id_B)
\end{aligned}
$$

where the equalities labeled with (a) come from the naturality of a, and (b) is due to the pentagon diagram of the algebra A. The first triangle diagram of the definition is also commutative. In fact,

$$
\begin{aligned}
m' \circ (u' \otimes id_B) &= f \circ m \circ (f^{-1} \otimes f^{-1}) \circ (f \otimes id_B) \circ (u \otimes id_B) \\
&= f \circ m \circ (id_A \otimes f^{-1}) \circ (u \otimes id_B) \\
&= f \circ m \circ (u \otimes id_A) \circ (id_{\mathbf{1}} \otimes f^{-1}) \\
&\overset{(c)}{=} f \circ l_A \circ (id_{\mathbf{1}} \otimes f^{-1}) \\
&\overset{(d)}{=} f \circ f^{-1} \circ l_B \\
&= l_B
\end{aligned}
$$

in which the equality (c) is valid via the first triangle diagram of the algebra A, and (d) is due to the naturality of l. The other triangle diagram can be verified in a similar way as this one above. Hence $B = (B, m', u')$ is an algebra in $\mathcal{C}$.

Moreover, the isomorphism $f : A \to B$ is an algebra isomorphism considering that

$$m' \circ (f \otimes f) = f \circ m \circ (f^{-1} \otimes f^{-1}) \circ (f \otimes f) = f \circ m \quad \text{and} \quad f \circ u = u'.$$

Finally, let us verify that the algebras A and B are Morita equivalent, that is, $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent as $\mathcal{C}$-module categories.

Affirmation 1: If (M, ρ_M) is an object in $\mathcal{C}_A$ then

$$(M, \rho'_M := \rho_M \circ (id_M \otimes f^{-1}) : M \otimes B \to M \otimes A \to M)$$

is an object in $\mathcal{C}_B$.

Indeed, the pentagon diagram commutes given that

$$
\begin{aligned}
\rho'_M \circ (id_M \otimes m') \circ a_{M,B,B} &= \rho_M \circ (id_M \circ f^{-1}) \circ (id_M \otimes (f \circ m \circ (f^{-1} \otimes f^{-1})))\circ \\
&\quad a_{M,B,B} \\
&= \rho_M \circ (id_M \circ f^{-1}) \circ (id_M \otimes f) \circ (id_M \otimes m) \circ (id_M \otimes (f^{-1} \otimes f^{-1}))\circ \\
&\quad a_{M,B,B}
\end{aligned}
$$

$$
\begin{aligned}
&= \rho_M \circ (id_M \otimes m) \circ (id_M \otimes (f^{-1} \otimes f^{-1})) \circ a_{M,B,B} \\
&\overset{(a)}{=} \rho_M \circ (id_M \otimes m) \circ a_{M,A,A} \circ ((id_M \otimes f^{-1}) \otimes f^{-1}) \\
&\overset{(b)}{=} \rho_M \circ (\rho_M \otimes id_A) \circ ((id_M \otimes f^{-1}) \otimes f^{-1}) \\
&= \rho_M \circ (\rho_M \otimes id_A) \circ (id_{M\otimes A} \otimes f^{-1}) \circ ((id_M \otimes f^{-1}) \otimes id_B) \\
&= \rho_M \circ (id_M \otimes f^{-1}) \circ (\rho_M \otimes id_B) \circ ((id_M \otimes f^{-1}) \otimes id_B) \\
&= \rho_M \circ (id_M \otimes f^{-1}) \circ ((\rho_M \circ (id_M \otimes f^{-1})) \otimes id_B) \\
&= \rho'_M \circ (\rho'_M \otimes id_B)
\end{aligned}
$$

in which the equality (a) comes from the naturality of a, and (b) is due to the pentagon diagram of the right A-module (M, ρ_M).

Lastly, notice that

$$
\begin{aligned}
\rho'_M \circ (id_M \otimes u') &= \rho_M \circ (id_M \otimes f^{-1}) \circ (id_M \otimes (f \circ u)) \\
&= \rho_M \circ (id_M \otimes f^{-1}) \circ (id_M \otimes f) \circ (id_M \otimes u) \\
&= \rho_M \circ (id_M \otimes u) \\
&= r_M
\end{aligned}
$$

by using the triangle diagram of the object $(M, \rho_M) \in \mathcal{C}_A$ in the last equality. It follows that $(M, \rho'_M = \rho_M \circ (id_M \otimes f^{-1}))$ is a right B-module.

Affirmation 2: If $g \in Hom_{\mathcal{C}_A}((M, \rho_M), (N, \rho_N))$ then $g \in Hom_{\mathcal{C}_B}((M, \rho'_M), (N, \rho'_N))$.

This is done by checking that $\rho'_N \circ (g \otimes id_B) = g \circ \rho'_M$. We have

$$
\begin{aligned}
\rho'_N \circ (g \otimes id_B) &= \rho_N \circ (id_N \otimes f^{-1}) \circ (g \otimes id_B) \\
&= \rho_N \circ (g \otimes id_A) \circ (id_M \otimes f^{-1}) \\
&= g \circ \rho_M \circ (id_M \otimes f^{-1}) \\
&= g \circ \rho'_M
\end{aligned}
$$

where in the third equality we are using the hypothesis of g being a morphism in $\mathcal{C}_A$. Hence $g : (M, \rho'_M) \to (N, \rho'_N)$ is a morphism in $\mathcal{C}_B$.

Affirmation 3: The application

$$
\begin{aligned}
G : \mathcal{C}_A &\longrightarrow \mathcal{C}_B \\
(M, \rho_M) &\longmapsto G(M, \rho_M) = (M, \rho'_M) \\
g : (M, \rho_M) \to (N, \rho_N) &\longmapsto G(g) = g : (M, \rho'_M) \to (N, \rho'_N)
\end{aligned}
$$

defines a functor equivalence.

This application is well defined via Affirmations 1 and 2. Moreover, G is clearly a functor since $G(id_{(M,\rho_M)}) = id_{(M,\rho'_M)} = id_{G(M,\rho_M)}$ and, if h is a morphism in $Hom_{\mathcal{C}_A}((N, \rho_N), (P, \rho_P))$ then $G(h \circ g) = h \circ g = G(h) \circ G(g)$.

Moreover, we can see that G is an equivalence by defining an application

$$\begin{aligned} H : \mathcal{C}_B &\longrightarrow \mathcal{C}_A \\ (Q, \rho_Q) &\longmapsto H(Q, \rho_Q) = (Q, \bar{\rho}_Q \coloneqq \rho_Q \circ (id_M \otimes f)) \\ t : (Q, \rho_Q) \to (R, \rho_R) &\longmapsto H(t) = t : (Q, \bar{\rho}_Q) \to (R, \bar{\rho}_R). \end{aligned}$$

which is going to be the inverse of G.

Similarly as we did for G, one can easily check that H is well defined and also a functor. This functor will be an inverse of G. In fact,

$$\begin{aligned} (H \circ G)(M, \rho_M) &= H(G(M, \rho_M)) \\ &= H(M, \rho'_M) \\ &= (M, \bar{\rho'}_M) \\ &= (M, \rho'_M \circ (id_M \otimes f)) \\ &= (M, \rho_M \circ (id_M \otimes f^{-1}) \circ (id_M \otimes f)) \\ &= (M, \rho_M) \end{aligned}$$

and

$$\begin{aligned} (G \circ H)(Q, \rho_Q) &= G(Q, \bar{\rho}_Q) \\ &= (Q, \bar{\rho}'_Q) \\ &= (Q, \bar{\rho}_Q \circ (id_Q \otimes f^{-1})) \\ &= (Q, \rho_Q \circ (id_Q \otimes f) \circ (id_Q \otimes f^{-1})) \\ &= (Q, \rho_Q) \end{aligned}$$

for all $(M, \rho_M) \in \mathcal{C}_A$ and $(Q, \rho_Q) \in \mathcal{C}_B$. This implies that $G \circ H = Id_{\mathcal{C}_B}$ and $H \circ G = Id_{\mathcal{C}_A}$[1] and hence, the functor G is an equivalence of categories.

Affirmation 4: G is an equivalence of $\mathcal{C}$-module categories.

The $\mathcal{C}$-module structure of $\mathcal{C}_A$ (and $\mathcal{C}_B$) is defined as in Proposition 5.1.5 via

$$\begin{aligned} \overline{\otimes} : \mathcal{C} \times \mathcal{C}_A &\longrightarrow \mathcal{C}_A \\ (X, (M, \rho_M)) &\longmapsto X \overline{\otimes} (M, \rho_M) = (X \otimes M, \rho_{X \otimes M} = (id_X \otimes \rho_M) \circ a_{X,M,A}) \\ (f, g) &\longmapsto f \overline{\otimes} g, \end{aligned}$$

$m' = \{m'_{X,Y,(M,\rho_M)} = a_{X,Y,M}\}_{X,Y \in \mathcal{C},(M,\rho_M) \in \mathcal{C}_A}$ and $l' = \{l'_{(M,\rho_M)} = l_M\}_{(M,\rho_M) \in \mathcal{C}_A}$. In a similar way one can define the $\mathcal{C}$-module structure of $\mathcal{C}_B$.

Let us now see what natural isomorphism c gives the functor G a structure of $\mathcal{C}$-module functor. Notice that

$$G(X \overline{\otimes} (M, \rho_M)) = G(X \otimes M, \rho_{X \otimes M}) = (X \otimes M, \rho'_{X \otimes M})$$

[1] The natural transformation between these functors is the identity ID.

where

$$\rho'_{X\otimes M} = \rho_{X\otimes M} \circ (id_M \otimes f^{-1}) = (id_X \otimes \rho_M) \circ a_{X,M,A} \circ (id_M \otimes f^{-1}) \tag{32}$$

and, on the other hand,

$$X\overline{\otimes}G(M, \rho_M) = X\overline{\otimes}(M, \rho'_M) = (X \otimes M, (id_X \otimes \rho'_M) \circ a_{X,M,B})$$

with

$$\begin{aligned}(id_X \otimes \rho'_M) \circ a_{X,M,B} &= (id_X \otimes (\rho_M \circ (id_M \otimes f^{-1}))) \circ a_{X,M,B}\\ &= (id_X \otimes \rho_M) \circ (id_X \otimes (id_M \otimes f^{-1})) \circ a_{X,M,B}\\ &= (id_X \otimes \rho_M) \circ a_{X,M,A} \circ (id_M \otimes f^{-1})\\ &\overset{(32)}{=} \rho'_{X\otimes M}.\end{aligned}$$

The naturality of a is used in the third equality. This implies

$$G(X\overline{\otimes}(M, \rho_M)) = X\overline{\otimes}G(M, \rho_M) = (X\otimes M, \rho'_{X\otimes M}) = (X\otimes M, (id_X\otimes\rho_M)\circ a_{X,M,A}\circ(id_M\otimes f^{-1}))$$

as objects of $\mathcal{C}_B$ and thus it's possible to define

$$c = \{c_{X,(M,\rho_M)} \coloneqq id_{(X\otimes M,\rho'_{X\otimes M})} = id_{X\otimes M}\}_{X\in\mathcal{C},(M,\rho_M)\in\mathcal{C}_A},$$

that is, $c = ID$ [2]. Lastly, it remains to show that two diagrams commute, but since $c_{X,(M,\rho_M)} = id_{X\otimes M}$ these diagrams become much simpler. The pentagon becomes simply $G(m'_{X,Y,(M,\rho_M)}) = m'_{X,Y,G(M,\rho_M)}$ and the triangle $G(l'_{(M,\rho_M)}) = l'_{G(M,\rho_M)}$. Thus

$$G(m_{X,Y,(M,\rho_M)}) = G(a_{X,Y,M}) = a_{X,Y,M} = m_{X,Y,G(M,\rho_M)}$$

and

$$G(l'_{(M,\rho_M)}) = G(l_M) = l_M = l'_{(M,\rho'_M)} = l'_{G(M,\rho_M)}$$

as wanted. Since G is an equivalence of categories and a $\mathcal{C}$-module functor, it implies that $(G, c) : \mathcal{C}_A \to \mathcal{C}_B$ is an equivalence of $\mathcal{C}$-module categories by definition. Hence the algebras A and B are Morita equivalent. ∎

Proposition 5.1.10. *Let $A = (A, m, u)$ be an algebra in $\mathcal{C}$, (M, ρ_M) an object in $\mathcal{C}_A$ and $f \in Hom_{\mathcal{C}}(M, N)$ an isomorphism. Then $(N, f \circ \rho_M \circ (f^{-1} \otimes id_A))$ is an object in $\mathcal{C}_A$ and $f : M \to N$ is an isomorphism in $\mathcal{C}_A$.*

[2] In this case, ID is the identity natural isomorphism between the functors $G \circ \overline{\otimes}$ and $\overline{\otimes} \circ (Id_{\mathcal{C}} \times G)$.

Proof. Let us begin by showing that $(N, f \circ \rho_M \circ (f^{-1} \otimes id_A))$ is an object of $\mathcal{C}_A$, i.e., the commutativity of two diagrams. The pentagon diagram is commutative because

$$\begin{aligned} f \circ \rho_M \circ (f^{-1} \otimes id_A) \circ (id_N \otimes m) \circ a_{N,A,A} &= f \circ \rho_M \circ (id_M \otimes m) \circ (f^{-1} \otimes id_{A\otimes A}) \circ a_{N,A,A} \\ &\overset{(a)}{=} f \circ \rho_M \circ (id_M \otimes m) \circ a_{M,A,A} \circ ((f^{-1} \otimes id_A) \otimes id_A) \\ &\overset{(b)}{=} f \circ \rho_M \circ (\rho_M \otimes id_A) \circ ((f^{-1} \otimes id_A) \otimes id_A) \\ &= f \circ \rho_M \circ (f^{-1} \otimes id_A) \circ (f \otimes id_A) \circ (\rho_M \otimes id_A) \circ \\ &\quad ((f^{-1} \otimes id_A) \otimes id_A) \\ &= f \circ \rho_M \circ (f^{-1} \otimes id_A) \circ ((f \circ \rho_M \circ (f^{-1} \otimes id_A)) \otimes id_A) \end{aligned}$$

in which the equality (a) comes from the naturality of a, and (b) is due to the pentagon diagram of the object $(M, \rho_M) \in \mathcal{C}_A$.

The triangle diagram also commutes because

$$\begin{aligned} f \circ \rho_M \circ (f^{-1} \otimes id_A) \circ (id_N \otimes u) &= f \circ \rho_M \circ (id_M \otimes u) \circ (f^{-1} \otimes id_{\mathbf{1}}) \\ &= f \circ r_M \circ (f^{-1} \otimes id_{\mathbf{1}}) \\ &= f \circ f^{-1} \circ r_N \\ &= r_N \end{aligned}$$

where the second equality is valid via the triangle diagram of the object $(M, \rho_M) \in \mathcal{C}_A$ and the third holds by the naturality of r.

Lastly, the morphism $f : M \to N$ is an isomorphism in $\mathcal{C}_A$ since it is an isomorphism in $\mathcal{C}$ and

$$f \circ \rho_M \circ (f^{-1} \otimes id_A) \circ (f \otimes id_A) = f \circ \rho_M$$

Therefore, the objects (M, ρ_M) and $(N, f \circ \rho_M \circ (f^{-1} \otimes id_A))$ are isomorphic as objects in the category $\mathcal{C}_A$. ■

Lemma 5.1.11. *Let A be an algebra in a multitensor category $\mathcal{C}$. Then*

(i) *the category $\mathcal{C}_A$ $({}_A\mathcal{C})$ is abelian;*

(ii) *the action bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{C}_A \to \mathcal{C}_A$ is $\Bbbk$-linear and left exact in the first variable.*

Proof. The item (i) can be found in [16] as Lemma 3. For the item (ii), we have to check that for all $(M, \rho_M) \in \mathcal{C}_A$, the functor $\_\,\overline{\otimes}(M, \rho_M) : \mathcal{C} \to \mathcal{C}_A$ is $\Bbbk$-linear and left exact. This is done by using the $\Bbbk$-linearity of the functor $\_ \otimes M$. Indeed, let $k \in \Bbbk$ and $f \in Hom_{\mathcal{C}}(X, Y)$ and notice that

$$(\_\,\overline{\otimes}(M, \rho_M))(kf) = kf\overline{\otimes}(M, \rho_M) = kf \otimes id_M = k(f \otimes id_M) = k(f\overline{\otimes}(M, \rho_M)) = k(\_\,\overline{\otimes}(M, \rho_M))(f).$$

The additivity property can be shown analogously.

The left exactness of $\_\ \overline{\otimes}(M, \rho_M) : \mathcal{C} \to \mathcal{C}_A$ follows from the fact that $\_\ \otimes M$ is a left exact functor[3]. Indeed, consider a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\mathcal{C}$. We now show that the sequence

$$0 \longrightarrow X\overline{\otimes}(M, \rho_M) \xrightarrow{f\overline{\otimes} id_{(M,\rho_M)}} Y\overline{\otimes}(M, \rho_M) \xrightarrow{g\overline{\otimes} id_{(M,\rho_M)}} Z\overline{\otimes}(M, \rho_M)$$

is exact in $\mathcal{C}_A$. By using the definition of the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{C}_A \to \mathcal{C}_A$ we can write this sequence as

$$0 \longrightarrow (X \otimes M, \rho_{X\otimes M}) \xrightarrow{f\otimes id_M} (Y \otimes M, \rho_{Y\otimes M}) \xrightarrow{g\otimes id_M} (Z \otimes M, \rho_{Z\otimes M}) .$$

Let a and b be morphisms in $Hom_{\mathcal{C}_A}((U, \rho_U), (X \otimes M, \rho_{X\otimes M}))$ satisfying

$$(f \otimes id_M) \circ a = (f \otimes id_M) \circ b.$$

We can see this morphism equality in $\mathcal{C}_A$ as a morphism equality in $\mathcal{C}$ considering that every morphism in $\mathcal{C}_A$ is a morphism in $\mathcal{C}$. Notice that $f \otimes id_M$ is a monomorphism in $\mathcal{C}$ (f is a monomorphism in $\mathcal{C}$ and $\_\ \otimes M$ is an exact functor) and therefore, $a = b$ as morphisms in $\mathcal{C}$ (and, consequently, $a = b$ as morphisms in $\mathcal{C}_A$).

The image of the morphism $f \otimes id_M$ is the morphism itself (see Lemma 1.2.7) so it only suffices to show that $f \otimes id_M$ is the kernel of $g \otimes id_M$. For this, consider the short exact sequence[4]

$$0 \longrightarrow X \otimes M \xrightarrow{f\otimes id_M} Y \otimes M \xrightarrow{g\otimes id_M} Z \otimes M \longrightarrow 0$$

in $\mathcal{C}$. Notice that $(g\otimes id_M)\circ(f\otimes id_M) = (g\circ f)\otimes id_M = 0$ and let $h : (K, \rho_K) \to (Y\otimes M, \rho_{Y\otimes M})$ be a morphism in $\mathcal{C}_A$ satisfying $(g\otimes id_M)\circ h = 0$. We may see this equality $(g\otimes id_M)\circ h = 0$ in $\mathcal{C}$ while considering the short exact sequence above. So there is an unique morphism $u : K \to X \otimes M$ in $\mathcal{C}$ such that $(f \otimes id_M) \circ u = h$. Let us now verify that this morphism u is, in fact, a morphism in $\mathcal{C}_A$ (from (K, ρ_K) to $(X \otimes M, \rho_{X\otimes M})$), i.e., $\rho_{X\otimes M} \circ (u \otimes id_A) = u \circ \rho_K$. We have

$$\begin{aligned}
(f \otimes id_M) \circ \rho_{X\otimes M} \circ (u \otimes id_A) &= (f \otimes id_M) \circ (id_X \otimes \rho_M) \circ a_{X,M,A} \circ (u \otimes id_A) \\
&= (id_Y \otimes \rho_M) \circ (f \otimes id_{M\otimes A}) \circ a_{X,M,A} \circ (u \otimes id_A) \\
&= (id_Y \otimes \rho_M) \circ a_{Y,M,A} \circ ((f \otimes id_M) \otimes id_A) \circ (u \otimes id_A) \\
&= \rho_{Y\otimes M} \circ (((f \otimes id_M) \circ u) \otimes id_A) \\
&= \rho_{Y\otimes M} \circ (h \otimes id_A) \\
&\overset{(*)}{=} h \circ \rho_K \\
&= (f \otimes id_M) \circ u \circ \rho_K
\end{aligned}$$

[3] The category $\mathcal{C}$ is rigid, and by the Remark 2.1.9, the functor $\_\ \otimes M$ is exact for every $M \in \mathcal{C}$.

[4] Here we are using the exactness of the functor $\_\ \otimes M : \mathcal{C} \to \mathcal{C}$.

in which the third equality holds by the naturality of a, and $(*)$ comes from the fact that the morphism $h : (K, \rho_K) \to (Y \otimes M, \rho_{Y\otimes M})$ is in $\mathcal{C}_A$. Hence, $\rho_{X\otimes M} \circ (u \otimes id_A) = u \circ \rho_K$ (since $f \otimes id_M$ is a monomorphism), that is, $u : (K, \rho_K) \to (X \otimes M, \rho_{X\otimes M})$ is a morphism in $\mathcal{C}_A$.

For the uniqueness, let $u' : (K, \rho_K) \to (X \otimes M, \rho_{X\otimes M})$ be a morphism in $\mathcal{C}_A$ satisfying $(f \otimes id_M) \circ u' = h$. Then $(f \otimes id_M) \circ u' = (f \otimes id_M) \circ u'$ which implies $u = u'$.

Therefore, the sequence

$$0 \longrightarrow X\overline{\otimes}(M, \rho_M) \xrightarrow{f\overline{\otimes} id_{(M,\rho_M)}} Y\overline{\otimes}(M, \rho_M) \xrightarrow{g\overline{\otimes} id_{(M,\rho_M)}} Z\overline{\otimes}(M, \rho_M)$$

is exact in $\mathcal{C}_A$, i.e., the functor $\_\ \overline{\otimes}(M, \rho_M) : \mathcal{C} \to \mathcal{C}_A$ is left exact. ■

5.2 THE ALGEBRA $\underline{Hom}(M, M)$ AND THE $\mathcal{C}$-MODULE CATEGORY $\mathcal{C}_{\underline{Hom}(M,M)}$

Let $\mathcal{C}$ be a finite multitensor category and $\mathcal{M}$ a locally finite and module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable. Our objective here is to show that the object $\underline{Hom}(M, M) \in \mathcal{C}$ has a structure of algebra in $\mathcal{C}$ for every object $M \in \mathcal{M}$, but before doing so let us remember some definitions and morphisms that will be useful in this section.

Let M_1, M_2 and M_3 be objects in $\mathcal{M}$. From what we've seen in the last chapter, the object $\underline{Hom}(M_1, M_2) \in \mathcal{C}$ represents the functor $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} M_1, M_2) : \mathcal{C} \to vect_{\Bbbk}$ and therefore there exists a natural isomorphism

$$\phi : Hom_{\mathcal{C}}(\ \_\ , \underline{Hom}(M_1, M_2)) \to Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} M_1, M_2)$$

in $\mathcal{C}$. We can define a canonical morphism in $\mathcal{M}$ by

$$ev_{M_1,M_2} := \phi_{\underline{Hom}(M_1,M_2)}(id_{\underline{Hom}(M_1,M_2)}) : \underline{Hom}(M_1, M_2)\overline{\otimes} M_1 \to M_2$$

called *evaluation*. We can see that this is exactly the morphism in the universal element[5] of the functor $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} M_1, M_2)$ that we've just seen in Proposition 3.4.

To define a multiplication and a unit for $\underline{Hom}(M, M) \in \mathcal{C}$ notice that the functor $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} M_1, M_3)$ is representable, so there exists a natural isomorphism

$$\varphi : Hom_{\mathcal{C}}(\ \_\ , \underline{Hom}(M_1, M_3)) \to Hom_{\mathcal{M}}(\ \_\ \overline{\otimes} M_1, M_3)$$

in $\mathcal{C}$. Consider the composition

$$(\underline{Hom}(M_2, M_3) \otimes \underline{Hom}(M_1, M_2))\overline{\otimes} M_1 \xrightarrow{m_{\underline{Hom}(M_2,M_3),\underline{Hom}(M_1,M_2),M_1}} \underline{Hom}(M_2, M_3)\overline{\otimes}(\underline{Hom}(M_1, M_2)\overline{\otimes} M_1)$$

$$\xrightarrow{id_{\underline{Hom}(M_2,M_3)}\overline{\otimes} ev_{M_1,M_2}} \underline{Hom}(M_2, M_3)\overline{\otimes} M_2 \xrightarrow{ev_{M_2,M_3}} M_3$$

[5] We mainly use this notation in the last chapter.

of morphism in $\mathcal{M}$, and then define

$$\mu_{M_1,M_2,M_3} := \varphi^{-1}_{\underline{Hom}(M_2,M_3)\otimes\underline{Hom}(M_1,M_2)}(ev_{M_2,M_3} \circ (id_{\underline{Hom}(M_2,M_3)}\overline{\otimes}ev_{M_1,M_2}) \circ m_{\underline{Hom}(M_2,M_3),\underline{Hom}(M_1,M_2),M_1})$$

which is a morphism in $\mathcal{C}$ from the object $\underline{Hom}(M_2, M_3)\otimes\underline{Hom}(M_1, M_2)$ to $\underline{Hom}(M_1, M_3)$.

By taking $M = M_1 = M_2 = M_3$, the multiplication μ for the algebra $\underline{Hom}(M, M)$ is defined as

$$\mu := \mu_{M,M,M} : \underline{Hom}(M, M) \otimes \underline{Hom}(M, M) \to \underline{Hom}(M, M).$$

Notice that we're considering the representable functor $Hom_{\mathcal{M}}(\_\ \overline{\otimes}M, M)$ and a representation (see Proposition 3.4)

$$\Psi^M(ev_{M,M}) : Hom_{\mathcal{C}}(\_\ , \underline{Hom}(M, M)) \to Hom_{\mathcal{M}}(\_\ \overline{\otimes}M, M)$$

. We can then write

$$\mu = (\Psi^M(ev_{M,M}))^{-1}_{\underline{Hom}(M,M)\otimes\underline{Hom}(M,M)}(ev_{M,M} \circ (id_{\underline{Hom}(M,M)}\overline{\otimes}ev_{M,M}) \circ m_{\underline{Hom}(M,M),\underline{Hom}(M,M),M}).$$

For the unit, let us consider the same composition given by equation (31) (with $\phi = \Psi^M(ev_{M,M})$ in this case)

$$((\Psi^M(ev_{M,M}))^{-1}_{\mathbf{1}}\circ\theta^{-1})(id_M) = (\Psi^M(ev_{M,M}))^{-1}_{\mathbf{1}}(\theta^{-1}(id_M)) = (\Psi^M(ev_{M,M}))^{-1}_{\mathbf{1}}(l_M) \in Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}(M, M))$$

and define the unit u of $\underline{Hom}(M, M)$ as $u := (\Psi^M(ev_{M,M}))^{-1}_{\mathbf{1}}(l_M) : \mathbf{1} \to \underline{Hom}(M, M)$.

This following proposition asserts that the internal Hom from an object to itself has an algebra structure in $\mathcal{C}$ with these multiplication and unit morphisms just defined.

Proposition 5.2.1 ([15], Lema 5.4.7 and [4], p. 149)**.** *Let M and N be objects in $\mathcal{M}$. Then the object*

(i) *$\underline{Hom}(M, M)$ in $\mathcal{C}$ together with the multiplication μ and unit u maps defined above is an algebra in $\mathcal{C}$;*

(ii) *$\underline{Hom}(M, N)$ in $\mathcal{C}$ is a right $\underline{Hom}(M, M)$-module with action defined as $\rho_{\underline{Hom}(M,N)} := \mu_{M,M,N} : \underline{Hom}(M, N)\otimes\underline{Hom}(M, M) \to \underline{Hom}(M, N)$, that is, $(\underline{Hom}(M, N), \rho_{\underline{Hom}(M,N)})$ is an object in the category $\mathcal{C}_{\underline{Hom}(M,M)}$.*

It follows directly from the Proposition 5.1.5 that the category $\mathcal{C}_{\underline{Hom}(M,M)}$ admits a left $\mathcal{C}$-module category structure.

5.3 THE $\mathcal{C}$-MODULE FUNCTOR $F : \mathcal{M} \to \mathcal{C}_{\underline{Hom}(M,M)}$

Suppose we are still within the hypothesis of the last section. Our objective here is to define a functor from the category $\mathcal{M}$ to $\mathcal{C}_{\underline{Hom}(M,M)}$ and show it has a $\mathcal{C}$-module functor structure.

Set $A \coloneqq \underline{Hom}(M, M)$. Let $(\underline{Hom}(M, M), x)$ and $(\underline{Hom}(M, N), y)$ be universal elements of their respective functors, and consider the $\mathcal{C}$-module functor $(\underline{Hom}(M, \_\ ), d) : \mathcal{M} \to \mathcal{C}$ defined in Corollary 4.2.2. We can then define an application

$$\begin{aligned} F : \mathcal{M} &\longrightarrow \mathcal{C}_{\underline{Hom}(M,M)} \\ N &\longmapsto F(N) \coloneqq (\underline{Hom}(M, N), \rho_{\underline{Hom}(M,N)}) \\ f &\longmapsto F(f) \coloneqq \underline{Hom}(M, f) \end{aligned}$$

with $\rho_{\underline{Hom}(M,N)} = \mu_{M,M,N} = (\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}(y \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M})$ in which

$$\Psi^N(y) = \{\Psi^N(y)_X : Hom_{\mathcal{C}}(X, \underline{Hom}(M, N)) \to Hom_{\mathcal{M}}(X\overline{\otimes}M, N)\}_{X\in\mathcal{C}}$$

is the natural isomorphism given by the universal element $(\underline{Hom}(M, N), y)$ (see Proposition 3.4). Before showing that the application F is well defined and also a $\mathcal{C}$-module functor, we'll check that

$$(\underline{Hom}(M, X\overline{\otimes}N), (id_X\overline{\otimes}y) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N}\overline{\otimes}id_M))$$

is an universal element of the functor $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes}M, X\overline{\otimes}N)$ with the following lemma. We couldn't find any mention of this result in the literature.

Lemma 5.3.1. *The pair $(\underline{Hom}(M, X\overline{\otimes}N), (id_X\overline{\otimes}y) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N}\overline{\otimes}id_M))$ is an universal element of the representable functor $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes}M, X\overline{\otimes}N)$.*

Proof. This will be done by defining a natural isomorphism[6]

$$\omega : Hom_{\mathcal{C}}(\ \_\ , \underline{Hom}(M, X\overline{\otimes}N)) \to Hom_{\mathcal{M}}(\ \_\ \overline{\otimes}M, X\overline{\otimes}N)$$

satisfying $\omega_{\underline{Hom}(M,X\overline{\otimes}N)}(id_{\underline{Hom}(M,X\overline{\otimes}N)}) = (id_X\overline{\otimes}y) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N}\overline{\otimes}id_M)$ (see Proposition 3.4).

We begin by defining some natural isomorphisms in $\mathcal{C}$ whose composition are going to be ω.

For the first, consider

$$\begin{aligned} \alpha_Z : Hom_{\mathcal{C}}(Z, \underline{Hom}(M, X\overline{\otimes}N)) &\longrightarrow Hom_{\mathcal{C}}(Z, X \otimes \underline{Hom}(M, N)) \\ h &\longmapsto \alpha_Z(h) \coloneqq d_{X,N} \circ h. \end{aligned}$$

It's not difficult to check that $\alpha = \{\alpha_Z\}_{Z\in\mathcal{C}}$ is a natural isomorphism[7] in $\mathcal{C}$.

The second comes from the fact that the functor $X^* \otimes \ \_\ : \mathcal{C} \to \mathcal{C}$ is left adjoint to $X\otimes \ \_\ : \mathcal{C} \to \mathcal{C}$ (see Proposition 2.1.7). This implies that there is a natural isomorphism

6 Which is going to be the natural isomorphism for the representable functor $Hom_{\mathcal{M}}(\ \_\ \overline{\otimes}M, X\overline{\otimes}N)$.

7 Between the functors $Hom_{\mathcal{C}}(\ \_\ , \underline{Hom}(M, X\overline{\otimes}N))$ and $Hom_{\mathcal{C}}(\ \_\ , X \otimes \underline{Hom}(M, N))$.

$\phi = \{\phi_{Z,W}\}_{Z,W\in\mathcal{C}}$[8] with inverse given by

$$\phi^{-1}_{Z,W} : Hom_{\mathcal{C}}(Z, X \otimes W) \longrightarrow Hom_{\mathcal{C}}(X^* \otimes Z, W)$$
$$h \longmapsto l_W \circ (ev_X \otimes id_W) \circ a^{-1}_{X^*,X,W} \circ (id_{X^*} \otimes h),$$

that is, $\phi^{-1} = \{\phi^{-1}_{Z,W}\}_{Z,W\in\mathcal{C}}$. In our case we'll fix the second entry $W \in \mathcal{C}$ of this natural isomorphism with the object $\underline{Hom}(M, N) \in \mathcal{C}$.

For the third, we'll use the natural isomorphism

$$\Psi^N(y) : Hom_{\mathcal{C}}(\_\_ , \underline{Hom}(M, N)) \longrightarrow Hom_{\mathcal{M}}(\_\_ \overline{\otimes} M, N)$$

in $\mathcal{C}$. by defining

$$\nu_Z := \Psi^N(y)_{X^*\otimes Z} : Hom_{\mathcal{C}}(X^* \otimes Z, \underline{Hom}(M, N)) \longrightarrow Hom_{\mathcal{M}}((X^* \otimes Z)\overline{\otimes} M, N)$$

for all $Z \in \mathcal{C}$. The naturality[9] of $\nu = \{\nu_Z = \Psi^N(y)_{X^*\otimes Z}\}_{Z\in\mathcal{C}}$ comes directly from the naturality of $\Psi^N(y)$, and by definition $\nu_Z(h) = \Psi^N(y)_{X^*\otimes Z}(h) = y \circ (h \otimes id_M)$ for every morphism $h \in Hom_{\mathcal{C}}(X^* \otimes Z, \underline{Hom}(M, N))$.

Next, for all $Z \in \mathcal{C}$ let us consider the morphism

$$\beta_Z : Hom_{\mathcal{M}}((X^* \otimes Z)\overline{\otimes} M, N) \longrightarrow Hom_{\mathcal{M}}(X^*\overline{\otimes}(Z\overline{\otimes} M), N)$$
$$h \longmapsto h \circ m^{-1}_{X^*,Z,M}$$

in *Set*. The naturality of m is used to verify the naturality of $\beta = \{\beta_Z\}_{Z\in\mathcal{C}}$.

For the fifth and last, we use that the functor $X^*\overline{\otimes} \_\_ : \mathcal{M} \to \mathcal{M}$ is left adjoint to $X\overline{\otimes} \_\_ : \mathcal{M} \to \mathcal{M}$ (see Proposition 2.2.5), so there is a natural isomorphism $\theta = \{\theta_{M,M'} : Hom_{\mathcal{M}}(X^*\overline{\otimes} M, M') \to Hom_{\mathcal{M}}(X\overline{\otimes} M, M')\}_{M,M'\in\mathcal{M}}$. We define the family

$$\theta' = \{\theta'_Z := \theta_{Z\overline{\otimes} M,N} : Hom_{\mathcal{M}}(X^*\overline{\otimes}(Z\overline{\otimes} M), N) \to Hom_{\mathcal{M}}(Z\overline{\otimes} M, X\overline{\otimes} N)\}_{Z\in\mathcal{C}}$$

which is a natural isomorphism[10] by the naturality of θ. By Proposition 2.2.5 it follows that $\theta'_Z(h) = (id_X\overline{\otimes} h)\circ m_{X,X^*,Z\overline{\otimes} M}\circ(coev_X\overline{\otimes} id_{Z\overline{\otimes} M})\circ l^{-1}_{Z\overline{\otimes} M}$ for all $h \in Hom_{\mathcal{M}}(X^*\overline{\otimes}(Z\overline{\otimes} M), N)$.

Therefore, $\omega = \{\omega_Z := \theta'_Z \circ \beta_Z \circ \nu_Z \circ \phi^{-1}_{Z,\underline{Hom}(M,N)} \circ \alpha_Z\}_{Z\in\mathcal{C}}$ is a natural isomorphism in $\mathcal{C}$ from the functor $Hom_{\mathcal{C}}(\_\_ , \underline{Hom}(M, X\overline{\otimes} N))$ to $Hom_{\mathcal{M}}(\_\_ \overline{\otimes} M, X\overline{\otimes} N)$. This implies that $(\underline{Hom}(M, X\overline{\otimes} N), \omega)$ is a representation of the functor $Hom_{\mathcal{M}}(\_\_ \overline{\otimes} M, X\overline{\otimes} N)$ and, by Proposition 3.4,

$$(\underline{Hom}(M, X\overline{\otimes} N), t := \omega_{\underline{Hom}(M,X\overline{\otimes} N)}(id_{\underline{Hom}(M,X\overline{\otimes} N)}))$$

is an universal element of this functor.

[8] And hence, a natural isomorphism $\phi^{-1} = \{\phi^{-1}_{Z,W}\}_{Z,W\in\mathcal{C}}$.
[9] From the functor $Hom_{\mathcal{C}}(\_\_ , \underline{Hom}(M, N)) \circ (X^* \otimes \_\_)$ to $Hom_{\mathcal{M}}(\_\_ \overline{\otimes} M, N) \circ (X^* \otimes \_\_)$.
[10] Between the functors $Hom_{\mathcal{M}}(X^*\overline{\otimes} \_\_ , N) \circ (\_\_ \overline{\otimes} M)$ and $Hom_{\mathcal{M}}(\_\_ , X\overline{\otimes} N) \circ (\_\_ \overline{\otimes} M)$.

For the equality $t = (id_X\overline{\otimes}y)\circ m_{X,\underline{Hom}(M,N),M}\circ(d_{X,N}\overline{\otimes}id_M)$ notice that

$$
\begin{aligned}
t &= \omega_{\underline{Hom}(M,X\otimes N)}(id_{\underline{Hom}(M,X\otimes N)}) = (\theta'_{\underline{Hom}(M,X\otimes N)}\circ\beta_{\underline{Hom}(M,X\otimes N)}\circ\nu_{\underline{Hom}(M,X\otimes N)}\circ\\
&\quad \phi^{-1}_{\underline{Hom}(M,X\otimes N),\underline{Hom}(M,N)}\circ\alpha_{\underline{Hom}(M,X\otimes N)})(id_{\underline{Hom}(M,X\otimes N)})\\
&= \theta'_{\underline{Hom}(M,X\otimes N)}(\beta_{\underline{Hom}(M,X\otimes N)}(\nu_{\underline{Hom}(M,X\otimes N)}(\phi^{-1}_{\underline{Hom}(M,X\otimes N),\underline{Hom}(M,N)}\\
&\quad (\alpha_{\underline{Hom}(M,X\otimes N)}(id_{\underline{Hom}(M,X\otimes N)})))))\\
&= \theta'_{\underline{Hom}(M,X\otimes N)}(\beta_{\underline{Hom}(M,X\otimes N)}(\nu_{\underline{Hom}(M,X\otimes N)}(\phi^{-1}_{\underline{Hom}(M,X\otimes N),\underline{Hom}(M,N)}(d_{X,N}))))\\
&= \theta'_{\underline{Hom}(M,X\otimes N)}(\beta_{\underline{Hom}(M,X\otimes N)}(\nu_{\underline{Hom}(M,X\otimes N)}(l_{\underline{Hom}(M,N)}\circ(ev_X\otimes id_{\underline{Hom}(M,N)})\circ\\
&\quad a^{-1}_{X^*,X,\underline{Hom}(M,N)}\circ(id_{X^*}\otimes d_{X,N}))))\\
&= \theta'_{\underline{Hom}(M,X\otimes N)}(\beta_{\underline{Hom}(M,X\otimes N)}(y\circ(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M)\circ((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M)\circ\\
&\quad (a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M)\circ((id_{X^*}\otimes d_{X,N})\overline{\otimes}id_M)))\\
&= \theta'_{\underline{Hom}(M,X\otimes N)}(y\circ(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M)\circ((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M)\circ(a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M)\circ\\
&\quad ((id_{X^*}\otimes d_{X,N})\overline{\otimes}id_M)\circ m^{-1}_{X^*,\underline{Hom}(M,X\otimes N),M})\\
&= (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M))\circ\\
&\quad (id_X\overline{\otimes}(a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((id_{X^*}\otimes d_{X,N})\overline{\otimes}id_M))\circ(id_X\overline{\otimes}m^{-1}_{X^*,\underline{Hom}(M,X\otimes N),M})\circ\\
&\quad m_{X,X^*,\underline{Hom}(M,X\otimes N)\overline{\otimes}M}\circ(coev_X\overline{\otimes}id_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M})\circ l^{-1}_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M}\\
&\overset{(a)}{=} (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M))\circ\\
&\quad (id_X\overline{\otimes}(a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}m^{-1}_{X^*,X\otimes\underline{Hom}(M,N),M})\circ(id_X\overline{\otimes}(id_{X^*}\overline{\otimes}(d_{X,N}\overline{\otimes}id_M)))\circ\\
&\quad m_{X,X^*,\underline{Hom}(M,X\otimes N)\overline{\otimes}M}\circ(coev_X\overline{\otimes}id_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M})\circ l^{-1}_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M}\\
&\overset{(a)}{=} (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M))\circ\\
&\quad (id_X\overline{\otimes}(a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}m^{-1}_{X^*,X\otimes\underline{Hom}(M,N),M})\circ m_{X,X^*,(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ\\
&\quad (id_{X\otimes X^*}\overline{\otimes}(d_{X,N}\overline{\otimes}id_M))\circ(coev_X\overline{\otimes}id_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M})\circ l^{-1}_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M}\\
&= (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M))\circ\\
&\quad (id_X\overline{\otimes}(a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}m^{-1}_{X^*,X\otimes\underline{Hom}(M,N),M})\circ m_{X,X^*,(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ\\
&\quad (coev_X\overline{\otimes}id_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M})\circ(id_{\mathbf{1}}\overline{\otimes}(d_{X,N}\overline{\otimes}id_M))\circ l^{-1}_{\underline{Hom}(M,X\otimes N)\overline{\otimes}M}\\
&\overset{(b)}{=} (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M))\circ\\
&\quad (id_X\overline{\otimes}(a^{-1}_{X^*,X,\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}m^{-1}_{X^*,X\otimes\underline{Hom}(M,N),M})\circ m_{X,X^*,(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ\\
&\quad (coev_X\overline{\otimes}id_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M})\circ l^{-1}_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ(d_{X,N}\overline{\otimes}id_M)\\
&\overset{(c)}{=} (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(l_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ(id_X\overline{\otimes}((ev_X\otimes id_{\underline{Hom}(M,N)})\overline{\otimes}id_M))\circ\\
&\quad (id_X\overline{\otimes}m^{-1}_{X^*\otimes X,\underline{Hom}(M,N),M})\circ(id_X\overline{\otimes}m^{-1}_{X^*,X,\underline{Hom}(M,N)\overline{\otimes}M})\circ(id_X\overline{\otimes}(id_{X^*}\overline{\otimes}m_{X,\underline{Hom}(M,N),M}))\circ\\
&\quad m_{X,X^*,(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ(coev_X\overline{\otimes}id_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M})\circ l^{-1}_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ(d_{X,N}\overline{\otimes}id_M)
\end{aligned}
$$

$$= (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes}(l_{\underline{Hom}(M,N)} \overline{\otimes} id_M)) \circ (id_X \overline{\otimes} m^{-1}_{\mathbf{1},\underline{Hom}(M,N),M}) \circ (id_X \overline{\otimes}(ev_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}))\circ$$

$$(id_X \overline{\otimes} m^{-1}_{X^*,X,\underline{Hom}(M,N)\overline{\otimes}M}) \circ (id_X \overline{\otimes}(id_{X^*} \overline{\otimes} m_{X,\underline{Hom}(M,N),M})) \circ m_{X,X^*,(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}\circ$$

$$(coev_X \overline{\otimes} id_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}) \circ l^{-1}_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(a)}{=} (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes}(l_{\underline{Hom}(M,N)} \overline{\otimes} id_M)) \circ (id_X \overline{\otimes} m^{-1}_{\mathbf{1},\underline{Hom}(M,N),M}) \circ (id_X \overline{\otimes}(ev_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}))\circ$$

$$(id_X \overline{\otimes} m^{-1}_{X^*,X,\underline{Hom}(M,N)\overline{\otimes}M}) \circ m_{X,X^*,X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ (id_{X\otimes X^*} \overline{\otimes} m_{X,\underline{Hom}(M,N),M})\circ$$

$$(coev_X \overline{\otimes} id_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}) \circ l^{-1}_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(d)}{=} (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes} l_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ (id_X \overline{\otimes}(ev_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}))\circ$$

$$(id_X \overline{\otimes} m^{-1}_{X^*,X,\underline{Hom}(M,N)\overline{\otimes}M}) \circ m_{X,X^*,X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ (id_{X\otimes X^*} \overline{\otimes} m_{X,\underline{Hom}(M,N),M})\circ$$

$$(coev_X \overline{\otimes} id_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M}) \circ l^{-1}_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$= (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes} l_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ (id_X \overline{\otimes}(ev_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}))\circ$$

$$(id_X \overline{\otimes} m^{-1}_{X^*,X,\underline{Hom}(M,N)\overline{\otimes}M}) \circ m_{X,X^*,X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ (coev_X \overline{\otimes} id_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)})\circ$$

$$(id_{\mathbf{1}} \overline{\otimes} m_{X,\underline{Hom}(M,N),M}) \circ l^{-1}_{(X\otimes\underline{Hom}(M,N))\overline{\otimes}M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(b)}{=} (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes} l_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ (id_X \overline{\otimes}(ev_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}))\circ$$

$$(id_X \overline{\otimes} m^{-1}_{X^*,X,\underline{Hom}(M,N)\overline{\otimes}M}) \circ m_{X,X^*,X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ (coev_X \overline{\otimes} id_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)})\circ$$

$$l^{-1}_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(c)}{=} (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes} l_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ (id_X \overline{\otimes}(ev_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M})) \circ m_{X,X^*\otimes X,\underline{Hom}(M,N)\overline{\otimes}M}\circ$$

$$(a_{X,X^*,X} \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ m^{-1}_{X\otimes X^*,X,\underline{Hom}(M,N)\overline{\otimes}M} \circ (coev_X \overline{\otimes} id_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)})\circ$$

$$l^{-1}_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(a)}{=} (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes} l_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ m_{X,\mathbf{1},\underline{Hom}(M,N)\overline{\otimes}M} \circ ((id_X \otimes ev_X) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M})\circ$$

$$(a_{X,X^*,X} \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ ((coev_X \otimes id_X) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ m^{-1}_{\mathbf{1},X,\underline{Hom}(M,N)\overline{\otimes}M}\circ$$

$$l^{-1}_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(e)}{=} (id_X \overline{\otimes} y) \circ (r_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M})\circ$$

$$(a_{X,X^*,X} \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ ((coev_X \otimes id_X) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ m^{-1}_{\mathbf{1},X,\underline{Hom}(M,N)\overline{\otimes}M}\circ$$

$$l^{-1}_{X\overline{\otimes}(\underline{Hom}(M,N)\overline{\otimes}M)} \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\stackrel{(d)}{=} (id_X \overline{\otimes} y) \circ (r_X \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ ((id_X \otimes ev_X) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M})\circ$$

$$(a_{X,X^*,X} \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ ((coev_X \otimes id_X) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M}) \circ (l_X^{-1} \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M})\circ$$

$$m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$= (id_X \overline{\otimes} y) \circ ((r_X \circ (id_X \otimes ev_X) \circ a_{X,X^*,X} \circ (coev_X \otimes id_X) \circ l_X^{-1}) \overline{\otimes} id_{\underline{Hom}(M,N)\overline{\otimes}M})\circ$$

$$m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

$$\overset{(f)}{=} (id_X \overline{\otimes} y) \circ (id_X \overline{\otimes} id_{\underline{Hom}(M,N) \overline{\otimes} M}) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$
$$= (id_X \overline{\otimes} y) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M)$$

where the equalities labeled with (a) and (b) are valid due to the naturality of m and l, respectively. The equalities with (c) hold due to the pentagon diagram of the $\mathcal{C}$-module category $\mathcal{M}$, and in (d) we used Proposition 2.2.3. The equality (e) holds via the triangle diagram of the $\mathcal{C}$-module category $\mathcal{M}$ and, finally, (f) holds by an identity in the Definition 2.1.6 (X^* is a right dual of the object $X \in \mathcal{C}$).

Hence, $(\underline{Hom}(M, X \overline{\otimes} N), t = (id_X \overline{\otimes} y) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M))$ is an universal element of the representable functor $hom_{\mathcal{M}}(\ \_\ , \overline{\otimes} M, X \overline{\otimes} N)$. ■

In the beginning of this section we've defined an application

$$\begin{aligned} F : \mathcal{M} &\longrightarrow \mathcal{C}_{\underline{Hom}(M,M)} \\ N &\longmapsto F(N) := (\underline{Hom}(M,N), \rho_{\underline{Hom}(M,N)}) \\ f &\longmapsto F(f) := \underline{Hom}(M,f) \end{aligned}$$

with $\rho_{\underline{Hom}(M,N)} = \mu_{M,M,N} = (\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}(y \circ (id_{\underline{Hom}(M,N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M})$. This following proposition shows that this application is a $\mathcal{C}$-module functor.

Proposition 5.3.2. *The application F is a $\mathcal{C}$-module functor.*

Proof. Set $A := \underline{Hom}(M,M)$ and let $(\underline{Hom}(M,M), x)$, $(\underline{Hom}(M,N), y)$ and $(\underline{Hom}(M,P), z)$ be universal elements of their respective functors. For all $N \in \mathcal{M}$ the object $F(N) = (\underline{Hom}(M,N), \rho_{\underline{Hom}(M,N)})$ is in $\mathcal{C}_{\underline{Hom}(M,M)}$ by the Proposition 5.2.1.

Affirmation 1: For all $f \in Hom_{\mathcal{M}}(N,P)$, the morphism
$F(f) = \underline{Hom}(M,f) : \underline{Hom}(M,N) \to \underline{Hom}(M,P)$ is in $\mathcal{C}_A$, that is, the diagram

$$\begin{array}{ccc} \underline{Hom}(M,N) \otimes A & \xrightarrow{\underline{Hom}(M,f) \otimes id_A} & \underline{Hom}(M,P) \otimes A \\ \downarrow \rho_{\underline{Hom}(M,N)} & & \downarrow \rho_{\underline{Hom}(M,P)} \\ \underline{Hom}(M,N) & \xrightarrow[\underline{Hom}(M,f)]{} & \underline{Hom}(M,P) \end{array}$$

commutes.

The actions $\rho_{\underline{Hom}(M,N)}$ and $\rho_{\underline{Hom}(M,P)}$ are defined (via Proposition 5.2.1) as

$$\rho_{\underline{Hom}(M,N)} := \mu_{M,M,N} = (\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}(y \circ (id_{\underline{Hom}(M,N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M}),$$

and

$$\rho_{\underline{Hom}(M,P)} := \mu_{M,M,P} = (\Psi^P(z))^{-1}_{\underline{Hom}(M,P)\otimes A}(z \circ (id_{\underline{Hom}(M,P)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,P),A,M}),$$

respectively. Then

$$
\begin{aligned}
&\underline{Hom}(M,f) \circ \rho_{\underline{Hom}(M,N)} \\
&= Hom_{\mathcal{C}}((\underline{Hom}(M,N) \otimes A), \underline{Hom}(M,f))(\rho_{\underline{Hom}(M,N)}) \\
&= Hom_{\mathcal{C}}((\underline{Hom}(M,N) \otimes A), \underline{Hom}(M,f))((\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}(y \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,N),A,M})) \\
&= (Hom_{\mathcal{C}}((\underline{Hom}(M,N) \otimes A), \underline{Hom}(M,f)) \circ (\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A})(y \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,N),A,M}) \\
&\overset{(a)}{=} ((\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A} \circ Hom_{\mathcal{M}}((\underline{Hom}(M,N) \otimes A)\overline{\otimes} M, f))(y \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,N),A,M}) \\
&= (\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}(Hom_{\mathcal{M}}((\underline{Hom}(M,N) \otimes A)\overline{\otimes} M, f)(y \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,N),A,M})) \\
&= (\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}(f \circ y \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M}) \\
&\overset{(b)}{=} (\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}(z \circ (\underline{Hom}(M,f)\overline{\otimes} id_M) \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M}) \\
&= (\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}(z \circ (id_{\underline{Hom}(M,P)}\overline{\otimes} x) \circ (\underline{Hom}(M,f)\overline{\otimes} id_{\underline{Hom}(M,M)\overline{\otimes} M})\circ \\
&\quad m_{\underline{Hom}(M,N),A,M}) \\
&\overset{(c)}{=} (\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}(z \circ (id_{\underline{Hom}(M,P)}\overline{\otimes} x) \circ m_{\underline{Hom}(M,P),A,M}\circ \\
&\quad ((\underline{Hom}(M,f) \otimes id_A)\overline{\otimes} id_M)) \\
&= (\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}(Hom_{\mathcal{M}}((\underline{Hom}(M,f) \otimes id_A)\overline{\otimes} id_M, P)(z \circ (id_{\underline{Hom}(M,P)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,P),A,M})) \\
&= ((\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A} \circ Hom_{\mathcal{M}}((\underline{Hom}(M,f) \otimes id_A)\overline{\otimes} id_M, P))(z \circ (id_{\underline{Hom}(M,P)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,P),A,M}) \\
&\overset{(d)}{=} (Hom_{\mathcal{C}}(\underline{Hom}(M,f) \otimes id_A, \underline{Hom}(M,P)) \circ (\Psi^P(z))^{-1}_{\underline{Hom}(M,P)\otimes A})(z \circ (id_{\underline{Hom}(M,P)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,P),A,M}) \\
&= Hom_{\mathcal{C}}(\underline{Hom}(M,f) \otimes id_A, \underline{Hom}(M,P))((\Psi^P(z))^{-1}_{\underline{Hom}(M,P)\otimes A}(z \circ (id_{\underline{Hom}(M,P)}\overline{\otimes} x)\circ \\
&\quad m_{\underline{Hom}(M,P),A,M})) \\
&= Hom_{\mathcal{C}}(\underline{Hom}(M,f) \otimes id_A, \underline{Hom}(M,P))(\rho_{\underline{Hom}(M,P)}) \\
&= \rho_{\underline{Hom}(M,P)} \circ (\underline{Hom}(M,f) \otimes id_A).
\end{aligned}
$$

For the equality (a) consider the adjunction ($\_\ \overline{\otimes} M, \underline{Hom}(M,\ \_\ ), \phi$) (see Proposition 4.2.1), with the natural isomorphism

$$\phi = \{\phi_{X,N} = (\Psi^N(y))^{-1}_X : Hom_{\mathcal{M}}(X\overline{\otimes} M, N) \to Hom_{\mathcal{C}}(X, \underline{Hom}(M,N))\}_{X\in\mathcal{C}, N\in\mathcal{M}}.$$

By fixing the first entry with $X = \underline{Hom}(M,N) \otimes A \in \mathcal{C}$, the diagram

$$\begin{array}{ccc}
Hom_{\mathcal{M}}((\underline{Hom}(M,N)\otimes A)\overline{\otimes}M, N) & \xrightarrow{(\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}} & Hom_{\mathcal{C}}(\underline{Hom}(M,N)\otimes A, \underline{Hom}(M,N)) \\
\Big\downarrow {\scriptstyle Hom_{\mathcal{M}}((\underline{Hom}(M,N)\otimes A)\overline{\otimes}M, f)} & & \Big\downarrow {\scriptstyle Hom_{\mathcal{C}}(\underline{Hom}(M,N)\otimes A, \underline{Hom}(M,f))} \\
Hom_{\mathcal{M}}((\underline{Hom}(M,N)\otimes A)\overline{\otimes}M, P) & \xrightarrow[(\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}]{} & Hom_{\mathcal{C}}(\underline{Hom}(M,N)\otimes A, \underline{Hom}(M,P))
\end{array}$$

commutes by the naturality in $\mathcal{M}$, implying that (a) holds. The equality (b) is valid via the definition of the morphism $\underline{Hom}(M,f)$, (c) is due to the naturality of m, and (d) via the naturality of $(\Psi^P(z))^{-1}$, i.e., the commutativity of

$$\begin{array}{ccc}
Hom_{\mathcal{M}}((\underline{Hom}(M,P)\otimes A)\overline{\otimes}M, P) & \xrightarrow{(\Psi^P(z))^{-1}_{\underline{Hom}(M,P)\otimes A}} & Hom_{\mathcal{C}}(\underline{Hom}(M,P)\otimes A, \underline{Hom}(M,P)) \\
\Big\downarrow {\scriptstyle Hom_{\mathcal{M}}((\underline{Hom}(M,f)\otimes id_A)\overline{\otimes}id_M, P)} & & \Big\downarrow {\scriptstyle Hom_{\mathcal{C}}(\underline{Hom}(M,f)\otimes id_A, \underline{Hom}(M,P))} \\
Hom_{\mathcal{M}}((\underline{Hom}(M,N)\otimes A)\overline{\otimes}M, P) & \xrightarrow[(\Psi^P(z))^{-1}_{\underline{Hom}(M,N)\otimes A}]{} & Hom_{\mathcal{C}}(\underline{Hom}(M,N)\otimes A, \underline{Hom}(M,P)).
\end{array}$$

Hence, the application F is well defined.

Affirmation 2: F is a functor.

In fact, let $g \in Hom_{\mathcal{M}}(P,Q)$ and notice that

$$F(g\circ f) = \underline{Hom}(M, g\circ f) = \underline{Hom}(M,g)\circ \underline{Hom}(M,f) = F(g)\circ F(f), \quad \text{and}$$
$$F(id_N) = \underline{Hom}(M, id_N) = id_{\underline{Hom}(M,\ \_\ )(N)} = id_{\underline{Hom}(M,N)} = id_{(\underline{Hom}(M,N),\rho_{\underline{Hom}(M,N)})} = id_{F(N)}$$

as wanted.

Affirmation 3: (F,d) is a $\mathcal{C}$-module functor with the natural isomorphism d being the same as the one present in the $\mathcal{C}$-module functor $(\underline{Hom}(M,\ \_\ ), d)$ (see Corollary 4.2.2).

Before checking this, let us have a thought about the objects $X\overline{\otimes}F(N)$ and $F(X\overline{\otimes}N)$. We have

$$X\overline{\otimes}F(N) = X\overline{\otimes}(\underline{Hom}(M,N), \rho_{\underline{Hom}(M,N)}) = (X\otimes \underline{Hom}(M,N), \rho_{X\otimes\underline{Hom}(M,N)})$$

with $\rho_{X\otimes\underline{Hom}(M,N)} = (id_X \otimes \rho_{\underline{Hom}(M,N)})\circ a_{X,\underline{Hom}(M,N),A}$ (via Proposition 5.1.5), and

$$F(X\overline{\otimes}N) = (\underline{Hom}(M, X\overline{\otimes}N), \rho_{\underline{Hom}(M,X\overline{\otimes}N)}) \xrightarrow{d_{X,N}} (X\otimes\underline{Hom}(M,N), \rho'_{X\otimes\underline{Hom}(M,N)}).$$

Using the Proposition 5.1.10 with the isomorphism $d_{X,N}$ in $\mathcal{C}$, it follows that the object $(X\otimes\underline{Hom}(M,N), \rho'_{X\otimes\underline{Hom}(M,N)})$ is in $\mathcal{C}_A$ with action $\rho'_{X\otimes\underline{Hom}(M,N)}$ defined as

$$\rho'_{X\otimes\underline{Hom}(M,N)} = d_{X,N}\circ\rho_{\underline{Hom}(M,X\overline{\otimes}N)}\circ(d^{-1}_{X,N}\otimes id_A),$$

and $d_{X,N}$ is an isomorphism in $\mathcal{C}_A$, for all $X \in \mathcal{C}$ and $N \in \mathcal{M}$.

We'd like to have these two actions $\rho_{X\otimes\underline{Hom}(M,N)} = (id_X \otimes \rho_{\underline{Hom}(M,N)}) \circ a_{X,\underline{Hom}(M,N),A}$ and $\rho'_{X\otimes\underline{Hom}(M,N)} = d_{X,N} \circ \rho_{\underline{Hom}(M,X\overline{\otimes}N)} \circ (d^{-1}_{X,N} \otimes id_A)$ being the same, or equivalently,

$$\rho_{\underline{Hom}(M,X\overline{\otimes}N)} \circ (d^{-1}_{X,N} \otimes id_A) \circ a^{-1}_{X,\underline{Hom}(M,N),A} = d^{-1}_{X,N} \circ (id_X \otimes \rho_{\underline{Hom}(M,N)}). \qquad (33)$$

In order to verify this, we'll use that

$$(\underline{Hom}(M, X\overline{\otimes}N), t = (id_X \overline{\otimes} y) \circ m_{X,\underline{Hom}(M,N),M} \circ (d_{X,N} \overline{\otimes} id_M))$$

is an universal element of the functor $Hom_{\mathcal{M}}(\_\ \overline{\otimes} M, X\overline{\otimes}N)$ which we've shown in the Lemma 5.3.1. Moreover, the object $(\underline{Hom}(M, X\overline{\otimes}N), \rho_{\underline{Hom}(M,X\overline{\otimes}N)})$ is in $\mathcal{C}_A$ with action

$$\rho_{\underline{Hom}(M,X\overline{\otimes}N)} := \mu_{M,M,X\overline{\otimes}N} = (\Psi^{X\overline{\otimes}N}(t))^{-1}_{\underline{Hom}(M,X\overline{\otimes}N)\otimes A}(t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M}).$$

Notice that

$$\begin{aligned}
&\rho_{\underline{Hom}(M,X\overline{\otimes}N)} \circ (d^{-1}_{X,N} \otimes id_A) \circ a^{-1}_{X,\underline{Hom}(M,N),A}\\
&= Hom_{\mathcal{C}}(a^{-1}_{X,\underline{Hom}(M,N),A}, \underline{Hom}(M, X\overline{\otimes}N))(\rho_{\underline{Hom}(M,X\overline{\otimes}N)} \circ (d^{-1}_{X,N} \otimes id_A))\\
&= Hom_{\mathcal{C}}(a^{-1}_{X,\underline{Hom}(M,N),A}, \underline{Hom}(M, X\overline{\otimes}N))(Hom_{\mathcal{C}}(d^{-1}_{X,N} \otimes id_A, \underline{Hom}(M, X\overline{\otimes}N))(\rho_{\underline{Hom}(M,X\overline{\otimes}N)}))\\
&= Hom_{\mathcal{C}}(a^{-1}_{X,\underline{Hom}(M,N),A}, \underline{Hom}(M, X\overline{\otimes}N))(Hom_{\mathcal{C}}(d^{-1}_{X,N} \otimes id_A, \underline{Hom}(M, X\overline{\otimes}N))\\
&\quad ((\Psi^{X\overline{\otimes}N}(t))^{-1}_{\underline{Hom}(M,X\overline{\otimes}N)\otimes A}(t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M})))\\
&= (Hom_{\mathcal{C}}(a^{-1}_{X,\underline{Hom}(M,N),A}, \underline{Hom}(M, X\overline{\otimes}N)) \circ Hom_{\mathcal{C}}(d^{-1}_{X,N} \otimes id_A, \underline{Hom}(M, X\overline{\otimes}N))\circ\\
&\quad (\Psi^{X\overline{\otimes}N}(t))^{-1}_{\underline{Hom}(M,X\overline{\otimes}N)\otimes A})(t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M})\\
&= (Hom_{\mathcal{C}}((d^{-1}_{X,N} \otimes id_A) \circ a^{-1}_{X,\underline{Hom}(M,N),A}, \underline{Hom}(M, X\overline{\otimes}N)) \circ (\Psi^{X\overline{\otimes}N}(t))^{-1}_{\underline{Hom}(M,X\overline{\otimes}N)\otimes A})\\
&\quad (t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M})\\
&\overset{(a)}{=} ((\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)} \circ Hom_{\mathcal{M}}(((d^{-1}_{X,N} \otimes id_A) \circ a^{-1}_{X,\underline{Hom}(M,N),A})\overline{\otimes} id_M, X\overline{\otimes}N))\\
&\quad (t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M})\\
&= ((\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)} \circ Hom_{\mathcal{M}}(a^{-1}_{X,\underline{Hom}(M,N),A} \overline{\otimes} id_M, X\overline{\otimes}N)\circ\\
&\quad Hom_{\mathcal{M}}((d^{-1}_{X,N} \otimes id_A)\overline{\otimes} id_M, X\overline{\otimes}N))(t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M})\\
&= (\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)}(Hom_{\mathcal{M}}(a^{-1}_{X,\underline{Hom}(M,N),A} \overline{\otimes} id_M, X\overline{\otimes}N)\\
&\quad (Hom_{\mathcal{M}}((d^{-1}_{X,N} \otimes id_A)\overline{\otimes} id_M, X\overline{\otimes}N)(t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M})))\\
&= (\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)}(Hom_{\mathcal{M}}(a^{-1}_{X,\underline{Hom}(M,N),A} \overline{\otimes} id_M, X\overline{\otimes}N)\\
&\quad (t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M} \circ ((d^{-1}_{X,N} \otimes id_A)\overline{\otimes} id_M)))\\
&= (\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)}(t \circ (id_{\underline{Hom}(M,X\overline{\otimes}N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M}\circ\\
&\quad ((d^{-1}_{X,N} \otimes id_A)\overline{\otimes} id_M) \circ (a^{-1}_{X,\underline{Hom}(M,N),A} \overline{\otimes} id_M))
\end{aligned}$$

$$\overset{(b)}{=} (\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)}(t\circ((d^{-1}_{X,N}\circ(id_X\otimes\rho_{\underline{Hom}(M,N)}))\overline{\otimes}id_M))$$
$$= (\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)}(\Psi^{X\overline{\otimes}N}(t)_{X\otimes(\underline{Hom}(M,N)\otimes A)}(d^{-1}_{X,N}\circ(id_X\otimes\rho_{\underline{Hom}(M,N)})))$$
$$= d^{-1}_{X,N}\circ(id_X\otimes\rho_{\underline{Hom}(M,N)}).$$

For the equality (a), the commutativity of a diagram given by the natural isomorphism $\Psi^{X\overline{\otimes}N}(t)$ in $\mathcal{C}$ is used, i.e., the commutativity of

$$\begin{array}{ccc}
Hom_{\mathcal{M}}((\underline{Hom}(M,X\overline{\otimes}N)\otimes A)\overline{\otimes}M, X\overline{\otimes}N) & \xrightarrow{(\Psi^{X\overline{\otimes}N}(t))^{-1}_{\underline{Hom}(M,X\overline{\otimes}N)\otimes A}} & Hom_{\mathcal{C}}(\underline{Hom}(M,X\overline{\otimes}N)\otimes A, \underline{Hom}(M,X\overline{\otimes}N)) \\
\downarrow{\scriptstyle Hom_{\mathcal{M}}(((d^{-1}_{X,N}\otimes id_A)\circ a^{-1}_{X,\underline{Hom}(M,N),A})\overline{\otimes}id_M, X\overline{\otimes}N)} & & \downarrow{\scriptstyle Hom_{\mathcal{C}}((d^{-1}_{X,N}\otimes id_A)\circ a^{-1}_{X,\underline{Hom}(M,N),A}, \underline{Hom}(M,X\overline{\otimes}N))} \\
Hom_{\mathcal{M}}((X\otimes(\underline{Hom}(M,N)\otimes A))\overline{\otimes}M, X\overline{\otimes}N) & \xrightarrow{(\Psi^{X\overline{\otimes}N}(t))^{-1}_{X\otimes(\underline{Hom}(M,N)\otimes A)}} & Hom_{\mathcal{C}}(X\otimes(\underline{Hom}(M,N)\otimes A), \underline{Hom}(M,X\overline{\otimes}N))
\end{array}$$

evaluated on the morphism $t\circ(id_{\underline{Hom}(M,X\overline{\otimes}N)}\overline{\otimes}x)\circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M}$ in $Hom_{\mathcal{M}}((\underline{Hom}(M,X\overline{\otimes}N)\otimes A)\overline{\otimes}M, X\overline{\otimes}N)$.

The equality (b) holds because

$$t\circ(id_{\underline{Hom}(M,X\overline{\otimes}N)}\overline{\otimes}x)\circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M}\circ((d^{-1}_{X,N}\otimes id_A)\overline{\otimes}id_M)\circ(a^{-1}_{X,\underline{Hom}(M,N),A}\overline{\otimes}id_M)$$
$$\overset{(c)}{=} (id_X\overline{\otimes}y)\circ m_{X,\underline{Hom}(M,N),M}\circ(d_{X,N}\overline{\otimes}id_M)\circ(id_{\underline{Hom}(M,X\overline{\otimes}N)}\overline{\otimes}x)\circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M}\circ$$
$$((d^{-1}_{X,N}\otimes id_A)\overline{\otimes}id_M)\circ(a^{-1}_{X,\underline{Hom}(M,N),A}\overline{\otimes}id_M)$$
$$= (id_X\overline{\otimes}y)\circ m_{X,\underline{Hom}(M,N),M}\circ(id_{X\otimes\underline{Hom}(M,N)}\overline{\otimes}x)\circ(d_{X,N}\overline{\otimes}id_{A\overline{\otimes}M})\circ m_{\underline{Hom}(M,X\overline{\otimes}N),A,M}\circ$$
$$((d^{-1}_{X,N}\otimes id_A)\overline{\otimes}id_M)\circ(a^{-1}_{X,\underline{Hom}(M,N),A}\overline{\otimes}id_M)$$
$$\overset{(d)}{=} (id_X\overline{\otimes}y)\circ m_{X,\underline{Hom}(M,N),M}\circ(id_{X\otimes\underline{Hom}(M,N)}\overline{\otimes}x)\circ m_{X\otimes\underline{Hom}(M,N),A,M}\circ$$
$$((d_{X,N}\otimes id_A)\overline{\otimes}id_M)\circ((d^{-1}_{X,N}\otimes id_A)\overline{\otimes}id_M)\circ(a^{-1}_{X,\underline{Hom}(M,N),A}\overline{\otimes}id_M)$$
$$\overset{(d)}{=} (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(id_{\underline{Hom}(M,N)}\overline{\otimes}x))\circ m_{X,\underline{Hom}(M,N),A\overline{\otimes}M}\circ m_{X\otimes\underline{Hom}(M,N),A,M}\circ$$
$$(a^{-1}_{X,\underline{Hom}(M,N),A}\overline{\otimes}id_M)$$
$$\overset{(e)}{=} (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(id_{\underline{Hom}(M,N)}\overline{\otimes}x))\circ(id_X\overline{\otimes}m_{\underline{Hom}(M,N),A,M})\circ m_{X,\underline{Hom}(M,N)\otimes A,M}$$
$$= (id_X\overline{\otimes}(y\circ(id_{\underline{Hom}(M,N)}\overline{\otimes}x)\circ m_{\underline{Hom}(M,N),A,M}))\circ m_{X,\underline{Hom}(M,N)\otimes A,M}$$
$$\overset{(f)}{=} (id_X\overline{\otimes}(y\circ(\rho_{\underline{Hom}(M,N)}\overline{\otimes}id_M)))\circ m_{X,\underline{Hom}(M,N)\otimes A,M}$$
$$= (id_X\overline{\otimes}y)\circ(id_X\overline{\otimes}(\rho_{\underline{Hom}(M,N)}\overline{\otimes}id_M))\circ m_{X,\underline{Hom}(M,N)\otimes A,M}$$
$$\overset{(d)}{=} (id_X\overline{\otimes}y)\circ m_{X,\underline{Hom}(M,N),M}\circ((id_X\otimes\rho_{\underline{Hom}(M,N)})\overline{\otimes}id_M)$$
$$= (id_X\overline{\otimes}y)\circ m_{X,\underline{Hom}(M,N),M}\circ(d_{X,N}\overline{\otimes}id_M)\circ(d^{-1}_{X,N}\overline{\otimes}id_M)\circ((id_X\otimes\rho_{\underline{Hom}(M,N)})\overline{\otimes}id_M)$$
$$\overset{(c)}{=} t\circ(d^{-1}_{X,N}\overline{\otimes}id_M)\circ((id_X\otimes\rho_{\underline{Hom}(M,N)})\overline{\otimes}id_M)$$
$$= t\circ((d^{-1}_{X,N}\circ(id_X\otimes\rho_{\underline{Hom}(M,N)}))\overline{\otimes}id_M)$$

where we use the definition of t in in the equalities labeled with (c), and the naturality of m is used in those labeled with (d). The equality (e) is valid due to the pentagon diagram of the $\mathcal{C}$-module category $\mathcal{M}$. Lastly, for (f) we can observe that by applying $\Psi^N(y)_{\underline{Hom}(M,N)\otimes A}$ in both sides of the equality

$$(\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}(y \circ (id_{\underline{Hom}(M,N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M}) = \rho_{\underline{Hom}(M,N)}$$

we get

$$\begin{aligned}
& y \circ (id_{\underline{Hom}(M,N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M} \\
&= \Psi^N(y)_{\underline{Hom}(M,N)\otimes A}((\Psi^N(y))^{-1}_{\underline{Hom}(M,N)\otimes A}(y \circ (id_{\underline{Hom}(M,N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M})) \\
&= \Psi^N(y)_{\underline{Hom}(M,N)\otimes A}(\rho_{\underline{Hom}(M,N)}) \\
&= y \circ (\rho_{\underline{Hom}(M,N)} \overline{\otimes} id_M),
\end{aligned}$$

that is,

$$y \circ (id_{\underline{Hom}(M,N)} \overline{\otimes} x) \circ m_{\underline{Hom}(M,N),A,M} = y \circ (\rho_{\underline{Hom}(M,N)} \overline{\otimes} id_M). \tag{34}$$

This implies that the equation (33) holds and, therefore, $\rho_{X\otimes\underline{Hom}(M,N)} = \rho'_{X\otimes\underline{Hom}(M,N)}$ as wanted.

At last, it remains to show the commutativity of two diagrams (see Definition 2.2.6) in order to (F, d) be a $\mathcal{C}$-module functor. These diagrams commute directly from the fact that $(\underline{Hom}(M, \_\ ), d)$ is a $\mathcal{C}$-module functor as we'll see. Namely, the diagrams we would like to commute are

$$\begin{array}{ccccc}
 & & (\underline{Hom}(M,(X\otimes Y)\overline{\otimes}N), \rho_{\underline{Hom}(M,(X\otimes Y)\overline{\otimes}N)}) & & \\
 & \swarrow^{d_{X\otimes Y,N}} & & \searrow^{\underline{Hom}(M,m_{X,Y,N})} & \\
((X\otimes Y)\otimes\underline{Hom}(M,N), \rho_{(X\otimes Y)\otimes\underline{Hom}(M,N)}) & & & & (\underline{Hom}(M,X\overline{\otimes}(Y\overline{\otimes}N)), \rho_{\underline{Hom}(M,X\overline{\otimes}(Y\overline{\otimes}N))}) \\
\downarrow{\scriptstyle a_{X,Y,\underline{Hom}(M,N)}} & & & & \downarrow{\scriptstyle d_{X,Y\overline{\otimes}N}} \\
(X\otimes(Y\otimes\underline{Hom}(M,N)), \rho_{X\otimes(Y\otimes\underline{Hom}(M,N))}) & & \xleftarrow{\quad id_X\overline{\otimes}d_{Y,N}\quad} & & (X\otimes\underline{Hom}(M,Y\overline{\otimes}N), \rho_{X\otimes\underline{Hom}(M,Y\overline{\otimes}N)})
\end{array}$$

and

$$\begin{array}{ccc}
(\underline{Hom}(M,\mathbf{1}\overline{\otimes}N), \rho_{\underline{Hom}(M,\mathbf{1}\overline{\otimes}N)}) & \xrightarrow{\quad d_{\mathbf{1},N}\quad} & (\mathbf{1}\otimes\underline{Hom}(M,N), \rho_{\mathbf{1}\otimes\underline{Hom}(M,N)}) \\
{\scriptstyle \underline{Hom}(M,l_N)}\searrow & & \swarrow{\scriptstyle l_{\underline{Hom}(M,N)}} \\
 & (\underline{Hom}(M,N), \rho_{\underline{Hom}(M,N)}), &
\end{array}$$

for any $X, Y \in \mathcal{C}$ and $N \in \mathcal{M}$. But since $(\underline{Hom}(M, \_\ ), d)$ is a $\mathcal{C}$-module functor, the diagrams

$$\begin{array}{ccccc}
 & & \underline{Hom}(M,(X\otimes Y)\overline{\otimes}N) & & \\
 & \swarrow^{d_{X\otimes Y,N}} & & \searrow^{\underline{Hom}(M,m_{X,Y,N})} & \\
(X\otimes Y)\otimes\underline{Hom}(M,N) & & & & \underline{Hom}(M,X\overline{\otimes}(Y\overline{\otimes}N)) \\
\downarrow{\scriptstyle a_{X,Y,\underline{Hom}(M,N)}} & & & & \downarrow{\scriptstyle d_{X,Y\overline{\otimes}N}} \\
X\otimes(Y\otimes\underline{Hom}(M,N)) & & \xleftarrow{\quad id_X\overline{\otimes}d_{Y,N}\quad} & & X\otimes\underline{Hom}(M,Y\overline{\otimes}N)
\end{array}$$

and

$$\begin{array}{ccc} \underline{Hom}(M, \mathbf{1}\overline{\otimes}N) & \xrightarrow{d_{\mathbf{1},N}} & \mathbf{1} \otimes \underline{Hom}(M,N) \\ & \searrow^{\underline{Hom}(M,l_N)} \quad \swarrow_{l_{\underline{Hom}(M,N)}} & \\ & \underline{Hom}(M,N), & \end{array}$$

commute, for all $X, Y \in \mathcal{C}$ and $N \in \mathcal{M}$. Therefore, (F, d) is a $\mathcal{C}$-module functor between the $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{C}_{\underline{Hom}(M,M)}$. ■

Remark 5.3.3. *It's always good to remember that $d_{X,N}$ is an isomorphism in $\mathcal{C}_A$ for all $X \in \mathcal{C}$ and $N \in \mathcal{M}$ as we have seen with the Affirmation 3 of the previous proposition. This fact will be used in some upcoming results.*

5.4 THE EQUIVALENCE F

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a locally finite and exact indecomposable $\mathcal{C}$-module category with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable. One can easily see that the $\mathcal{C}$-module functor[11]

$$\begin{aligned} F : \mathcal{M} &\longrightarrow \mathcal{C}_{\underline{Hom}(M,M)} \\ N &\longmapsto F(N) := (\underline{Hom}(M,N), \rho_{\underline{Hom}(M,N)}) \\ f &\longmapsto F(f) := \underline{Hom}(M,f) \end{aligned}$$

is additive directly from the fact that $\underline{Hom}(M, \_\ ) : \mathcal{M} \to \mathcal{C}$ is additive. Therefore, it is exact by Theorem 4.3.3[12]. This section is devoted to the proof that this functor is an equivalence under our hypothesis, which is largely used in our main theorem present in the next chapter. This result and its proof can be found in [4] and [15] with some details left to the reader.

Theorem 5.4.1 ([4], Theorem 7.10.1)**.** *The $\mathcal{C}$-module functor $(F : \mathcal{M} \to \mathcal{C}_{\underline{Hom}(M,M)}, d)$ is an equivalence of $\mathcal{C}$-module categories for every nonzero object $M \in \mathcal{M}$.*

Proof. Let M be a nonzero object in $\mathcal{M}$ and set $A := \underline{Hom}(M,M)$. Firstly, we are going to show that the functor F is faithful and full, that is, for any $N_1, N_2 \in \mathcal{M}$ the application[13]

$$F : Hom_{\mathcal{M}}(N_1, N_2) \to Hom_{\mathcal{C}_A}((\underline{Hom}(M,N_1), \rho_{\underline{Hom}(M,N_1)}), (\underline{Hom}(M,N_2), \rho_{\underline{Hom}(M,N_2)}))$$

is injective and surjective, i.e., an isomorphism[14].

[11] See Proposition 5.3.2.

[12] While noticing that $\mathcal{C}_{\underline{Hom}(M,M)}$ is abelian with the module product over $\mathcal{C}$ being $\Bbbk$-linear and left exact in the first entry via Lemma 5.1.11.

[13] As we already know from the definition of a functor, there exists an application F for all pairs of objects N_1 and N_2 in $\mathcal{M}$. We could be more precise and denote this application by F_{N_1,N_2}, but for simplicity we (and most authors) denote just by F as we can always find the objects in the context.

[14] Of abelian groups for the reason that F is already a group homomorphism (it is additive).

Before doing this we'll show a particular case in which this application F is an isomorphism whenever N_1 has the form $X\overline{\otimes}M$, for some $X \in \mathcal{C}$. To begin, we're going to consider an isomorphism composition γ (in *Set*) from $Hom_{\mathcal{M}}(X\overline{\otimes}M, N_2)$ to $Hom_{\mathcal{C}_A}((\underline{Hom}(M, X\overline{\otimes}M), \rho_{\underline{Hom}(M,X\overline{\otimes}M)}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)}))$ and check that F is equal to γ on the morphisms of $\mathcal{M}$. This being done, we'll be able to affirm that the application F is a group isomorphism (i.e., injective and surjective) whenever $N_1 = X\overline{\otimes}M$, for some $X \in \mathcal{C}$.

For the first isomorphism let $(\underline{Hom}(M, N_2), y_2)$ be an universal element of the representable functor $Hom_{\mathcal{M}}(\_\,\overline{\otimes}M, N_2)$ and consider the natural isomorphism $\Psi^{N_2}(y_2) : Hom_{\mathcal{C}}(\_\,, \underline{Hom}(M, N_2)) \to Hom_{\mathcal{M}}(\_\,\overline{\otimes}M, N_2)$ in $\mathcal{C}$ (see Proposition 3.4). We're going to use the inverse of $\Psi^{N_2}(y_2)_X$.

The second isomorphism is the inverse of the isomorphism $\phi_{X,(\underline{Hom}(M,N_2),\rho_M)}$ from the adjunction $(G, Forg, \phi)$ of Lemma 5.1.6. The third and last morphism is α from

$$Hom_{\mathcal{C}_A}((X \otimes \underline{Hom}(M, M), \rho_{X\otimes\underline{Hom}(M,M)}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)})), \text{ to}$$

$$Hom_{\mathcal{C}_A}((\underline{Hom}(M, X\overline{\otimes}M), \rho_{\underline{Hom}(M,X\overline{\otimes}M)}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)}))$$

defined by $\alpha(h) = h \circ d_{X,M}$. It's known that each $d_{X,N}$ is an isomorphism in $\mathcal{C}_A$ from the Remark 5.3.3 and, hence, α is well defined. It's also easy to see that α is an isomorphism (with inverse $f \mapsto f \circ d^{-1}_{X,M}$).

We have the following composition of isomorphisms in *Set*

$$\begin{array}{l}
Hom_{\mathcal{M}}(X\overline{\otimes}M, N_2) \\
\quad \Big\downarrow (\Psi^{N_2}(y_2))^{-1}_X \\
Hom_{\mathcal{C}}(X, \underline{Hom}(M, N_2)) \\
\quad \Big\downarrow \phi^{-1}_{X,(\underline{Hom}(M,N_2),\rho_{\underline{Hom}(M,N_2)})} \\
Hom_{\mathcal{C}_A}((X \otimes A, \rho_{X\otimes A}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)})) \\
\quad \| \\
Hom_{\mathcal{C}_A}((X \otimes \underline{Hom}(M, M), \rho_{X\otimes\underline{Hom}(M,M)}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)})) \\
\quad \Big\downarrow \alpha \\
Hom_{\mathcal{C}_A}((\underline{Hom}(M, X\overline{\otimes}M), \rho_{\underline{Hom}(M,X\overline{\otimes}M)}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)}))
\end{array}$$

which we define as being γ, i.e., $\gamma := \alpha \circ \phi^{-1}_{X,(\underline{Hom}(M,N_2),\rho_{\underline{Hom}(M,N_2)})} \circ (\Psi^{N_2}(y_2))^{-1}_X$.

It remains to show that γ is indeed the application F (on the morphisms of $\mathcal{M}$)[15].

[15] This will also imply that γ is a group homomorphism.

For this purpose, consider an arbitrary $h \in Hom_{\mathcal{M}}(X\overline{\otimes}M, N_2)$ and notice that

$$\begin{aligned}\gamma(h) &= (\alpha \circ \phi^{-1}_{X,(\underline{Hom}(M,N_2),\rho_{\underline{Hom}(M,N_2)})} \circ (\Psi^{N_2}(y_2))^{-1}_X)(h)\\ &= \alpha(\phi^{-1}_{X,(\underline{Hom}(M,N_2),\rho_{\underline{Hom}(M,N_2)})}((\Psi^{N_2}(y_2))^{-1}_X(h)))\\ &= \alpha(\rho_{\underline{Hom}(M,N_2)} \circ (\Psi^{N_2}(y_2)^{-1}_X(h) \otimes id_A))\\ &= \rho_{\underline{Hom}(M,N_2)} \circ (\Psi^{N_2}(y_2)^{-1}_X(h) \otimes id_A) \circ d_{X,M}.\end{aligned}$$

The equality $\gamma(h) = F(h)$ will be verified by showing that

$$\Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}(\gamma(h)) = \Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}(F(h)).$$

Before we begin, let us consider an universal element $(\underline{Hom}(M, X\overline{\otimes}M), t)$ of the functor $Hom_{\mathcal{M}}(\_\,\overline{\otimes}M, X\overline{\otimes}M)$, whose morphism t is defined as $t = (id_X\overline{\otimes}x) \circ m_{X,\underline{Hom}(M,M),M} \circ (d_{X,M}\overline{\otimes}id_M)$ (see Lemma 5.3.1). Thus, we have

$$\begin{aligned}&\Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}(\gamma(h))\\ &= \Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}(\rho_{\underline{Hom}(M,N_2)} \circ ((\Psi^{N_2}(y_2))^{-1}_X(h) \otimes id_A) \circ d_{X,M})\\ &= y_2 \circ ((\rho_{\underline{Hom}(M,N_2)} \circ ((\Psi^{N_2}(y_2))^{-1}_X(h) \otimes id_A) \circ d_{X,M})\overline{\otimes}id_M)\\ &= y_2 \circ (\rho_{\underline{Hom}(M,N_2)}\overline{\otimes}id_M) \circ (((\Psi^{N_2}(y_2))^{-1}_X(h) \otimes id_A)\overline{\otimes}id_M) \circ (d_{X,M}\overline{\otimes}id_M)\\ &\overset{(a)}{=} y_2 \circ (id_{\underline{Hom}(M,N_2)}\overline{\otimes}x) \circ m_{\underline{Hom}(M,N_2),A,M} \circ (((\Psi^{N_2}(y_2))^{-1}_X(h) \otimes id_A)\overline{\otimes}id_M) \circ (d_{X,M}\overline{\otimes}id_M)\\ &\overset{(b)}{=} y_2 \circ (id_{\underline{Hom}(M,N_2)}\overline{\otimes}x) \circ ((\Psi^{N_2}(y_2))^{-1}_X(h)\overline{\otimes}(id_A\overline{\otimes}id_M)) \circ m_{X,A,M} \circ (d_{X,M}\overline{\otimes}id_M)\\ &= y_2 \circ (id_{\underline{Hom}(M,N_2)}\overline{\otimes}x) \circ ((\Psi^{N_2}(y_2))^{-1}_X(h)\overline{\otimes}(id_{A\overline{\otimes}M})) \circ m_{X,A,M} \circ (d_{X,M}\overline{\otimes}id_M)\\ &= y_2 \circ ((\Psi^{N_2}(y_2))^{-1}_X(h)\overline{\otimes}id_M) \circ (id_X\overline{\otimes}x) \circ m_{X,A,M} \circ (d_{X,M}\overline{\otimes}id_M)\\ &= \Psi^{N_2}(y_2)_X((\Psi^{N_2}(y_2))^{-1}_X(h)) \circ (id_X\overline{\otimes}x) \circ m_{X,A,M} \circ (d_{X,M}\overline{\otimes}id_M)\\ &= h \circ (id_X\overline{\otimes}x) \circ m_{X,A,M} \circ (d_{X,M}\overline{\otimes}id_M)\\ &\overset{(c)}{=} h \circ t\\ &\overset{(d)}{=} y_2 \circ (\underline{Hom}(M,h)\overline{\otimes}id_M)\\ &= \Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}(\underline{Hom}(M,h))\\ &= \Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}(F(h))\end{aligned}$$

in which the equality (a) holds by the equation (34). We used the naturality of m in (b) and the definition of the morphism t in (c). For the equality (d), it's used the definition of the morphism $\underline{Hom}(M,h) : \underline{Hom}(M, X\overline{\otimes}M) \to \underline{Hom}(M, N_2)$ in $\mathcal{C}$.

Since $\Psi^{N_2}(y_2)_{\underline{Hom}(M,X\overline{\otimes}M)}$ is an isomorphism, it follows that $\gamma(h) = F(h)$ for every $h \in Hom_{\mathcal{M}}(X\overline{\otimes}M, N_2)$, i.e., the application

$$F = \gamma = \alpha \circ \phi^{-1}_{X,(\underline{Hom}(M,N_2),\rho_{\underline{Hom}(M,N_2)})} \circ (\Psi^{N_2}(y_2))^{-1}_X$$

is a group isomorphism whenever N_1 has the form $X\overline{\otimes}M$, for some $X \in \mathcal{C}$. We're now going to use this for the general case, the case where $N_1 \in \mathcal{M}$ is arbitrary.

Let $(\underline{Hom}(M, N_1), y_1)$ be an universal element of the representable functor $Hom_{\mathcal{M}}(\_\ \overline{\otimes}M, N_1)$. The morphism $y_1 : \underline{Hom}(M, N_1)\overline{\otimes}M \to N_1$ is an epimorphism in $\mathcal{M}$ by Lemma 4.4.3. Using that the category $\mathcal{M}$ is abelian, it follows that there exists a morphism $g \in Hom_{\mathcal{M}}(P, \underline{Hom}(M, N_1)\overline{\otimes}M)$ such that (N_1, y_1) is the cokernel[16] of g. Therefore, the sequence[17]

$$P \xrightarrow{\ g\ } \underline{Hom}(M, N_1)\overline{\otimes}M \xrightarrow{\ y_1\ } N_1 \longrightarrow 0$$

is exact in $\mathcal{M}$.

Let $(\underline{Hom}(M, P), z)$ be an universal element of the representable functor $Hom_{\mathcal{M}}(\_\ \overline{\otimes}M, P)$. Since $z : \underline{Hom}(M, P)\overline{\otimes}M \to P$ in an epimorphism in $\mathcal{M}$ (via Lemma 4.4.3), it follows by the item (iii) of Lemma 1.1.9 that (N_1, y_1) is also the cokernel of $g \circ z$, and hence the sequence

$$\underline{Hom}(M, P)\overline{\otimes}M \xrightarrow{\ g\circ z\ } \underline{Hom}(M, N_1)\overline{\otimes}M \xrightarrow{\ y_1\ } N_1 \longrightarrow 0.$$

is exact in $\mathcal{M}$.

The additive contravariant functors $Hom_{\mathcal{M}}(\_\ , N_2)$ and $Hom_{\mathcal{C}_A}(\_\ , F(N_2)) \circ F$ from $\mathcal{M}$ to Ab are left exact. When applying them to the exact sequence above we obtain the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow Hom_{\mathcal{M}}(N_1, N_2) & \xrightarrow{Hom_{\mathcal{M}}(y_1, N_2)} & Hom_{\mathcal{M}}(\underline{Hom}(M, N_1)\overline{\otimes}M, N_2) & \xrightarrow{Hom_{\mathcal{M}}(g\circ z, N_2)} & Hom_{\mathcal{M}}(\underline{Hom}(M, P)\overline{\otimes}M, N_2) \\
\downarrow F & & \downarrow F & & \downarrow F \\
0 \longrightarrow Hom_{\mathcal{C}_A}(F(N_1), F(N_2)) & \xrightarrow[Hom_{\mathcal{C}_A}(F(y_1), F(N_2))]{} & Hom_{\mathcal{C}_A}(F(\underline{Hom}(M, N_1)\overline{\otimes}M), F(N_2)) & \xrightarrow[Hom_{\mathcal{C}_A}(F(g\circ z), F(N_2))]{} & Hom_{\mathcal{C}_A}(F(\underline{Hom}(M, P)\overline{\otimes}M), F(N_2))
\end{array}$$

whose lines are exact (by the left exactness of both functors). In addition, the two rectangles commutes. In fact, let h be a morphism in $Hom_{\mathcal{M}}(N_1, N_2)$ and notice that

$$\begin{aligned}
(Hom_{\mathcal{C}_A}(F(y_1), F(N_2)) \circ F)(h) &= Hom_{\mathcal{C}_A}(F(y_1), F(N_2))(F(h)) \\
&= F(h) \circ F(y_1) \\
&= \underline{Hom}(M, h) \circ \underline{Hom}(M, y_1) \\
&= \underline{Hom}(M, h \circ y_1) \\
&= F(h \circ y_1) \\
&= F(Hom_{\mathcal{M}}(y_1, N_2)(h)) \\
&= (F \circ Hom_{\mathcal{M}}(y_1, N_2))(h)
\end{aligned}$$

[16] In an abelian category any epimorphism is the cokernel of some morphism.

[17] See Remark 1.2.8.

which implies the commutativity of the first rectangle, i.e., $F \circ Hom_{\mathcal{M}}(y_1, N_2) = F(y_1) \circ F$.

The commutativity of the second rectangle can be verified in a similar way. Hence, the whole diagram is commutative with the second and third columns applications being isomorphisms[18]. All of these arrows are group homomorphisms[19], so we may look to this diagram as one in the theory of modules over a ring, for example[20] and then conclude that the first column application is also an isomorphism, that is,

$$F : Hom_{\mathcal{M}}(N_1, N_2) \to Hom_{\mathcal{C}_A}((\underline{Hom}(M, N_1), \rho_{\underline{Hom}(M,N_1)}), (\underline{Hom}(M, N_2), \rho_{\underline{Hom}(M,N_2)}))$$

is a group isomorphism[21], for all N_1 and N_2 in $\mathcal{M}$. Therefore, the functor F is faithful and full.

Next, let us check that this functor is dense, i.e., surjective in the isomorphism classes of objects in $\mathcal{C}_A$. For this consider an object $(L, \rho_L) \in \mathcal{C}_A$. We'll show that there exists an object $N \in \mathcal{M}$ such that $F(N) \cong (L, \rho_L)$.

Since $\rho_L : (L \otimes A, \rho_{L\otimes A}) \to (L, \rho_L)$ is an epimorphism in $\mathcal{C}_A$ (via Proposition 5.1.7) and $\mathcal{C}_A$ is abelian, it follows that $((L, \rho_L), \rho_L)$ is the cokernel of some morphism $g' : (Q, \rho_Q) \to (L \otimes A, \rho_{L\otimes A})$ in $\mathcal{C}_A$.

Using the fact that ρ_Q is an epimorphism and also an argument we've already used in this proof, the pair (L, ρ_L) is the cokernel of $g' \circ \rho_Q$. Thus the sequence

$$(Q \otimes A, \rho_{Q\otimes A}) \xrightarrow{g'\circ\rho_Q} (L \otimes A, \rho_{L\otimes A}) \xrightarrow{\rho_L} (L, \rho_L) \longrightarrow 0$$

is exact in $\mathcal{C}_A$. By noticing that

$$\begin{aligned} F(Q\overline{\otimes}M) &= (\underline{Hom}(M, Q\overline{\otimes}M), \rho_{\underline{Hom}(M,Q\overline{\otimes}M)}) \overset{d_{Q,M}}{\cong} (Q \otimes \underline{Hom}(M, M), \rho_{Q\otimes\underline{Hom}(M,M)}) \\ &= (Q \otimes A, \rho_{Q\otimes A}) \end{aligned}$$

and

$$F(L\overline{\otimes}M) \overset{d_{L,M}}{\cong} (L \otimes A, \rho_{L\otimes A})$$

we have the isomorphism given by the composition

$$Hom_{\mathcal{C}_A}((Q\otimes A, \rho_{Q\otimes A}), (L\otimes A, \rho_{L\otimes A})) \overset{\beta}{\cong} Hom_{\mathcal{C}_A}(F(Q\overline{\otimes}M), F(L\overline{\otimes}M)) \overset{F^{-1}}{\cong} Hom_{\mathcal{M}}(Q\overline{\otimes}M, L\overline{\otimes}M),$$

where $\beta(h) = d_{L,M}^{-1} \circ h \circ d_{Q,M}$, for every $h \in Hom_{\mathcal{C}_A}((Q \otimes A, \rho_{Q\otimes A}), (L \otimes A, \rho_{L\otimes A}))$. Consider the morphism $f := F^{-1}(\beta(g' \circ \rho_Q)) \in Hom_{\mathcal{M}}(Q\overline{\otimes}M, L\overline{\otimes}M)$ and let $(coKer(f), q : L\overline{\otimes}M \to coKer(f))$ be the cokernel of f.

[18] Via the particular case (when $N_1 = X\overline{\otimes}M$ for some $X \in \mathcal{C}$) we showed in the first part of this proof.

[19] The horizontal lines are group homomorphisms by the additivity of the functors $Hom_{\mathcal{M}}(\_\_ , N_2)$ and $Hom_{\mathcal{C}_A}(\_\_ , F(N_2))$.

[20] Every abelian group is a module over the integers.

[21] Or equivalently, an isomorphism in Ab.

Finally we're going to show that there is an isomorphism $F(coKer(f)) \cong (L, \rho_L)$ in $\mathcal{M}$. By the (right) exactness of the functor F we know, by Proposition 1.4.3, that $F(q) : F(L\overline{\otimes}M) \to F(coKer(f))$ is the cokernel of

$$F(f) = F(F^{-1}(\beta(g' \circ \rho_Q))) = \beta(g' \circ \rho_Q) = d^{-1}_{L,M} \circ g' \circ \rho_Q \circ d_{Q,M},$$

that is, $F(coKer(f)) = coKer(d^{-1}_{L,M} \circ g' \circ \rho_Q \circ d_{Q,M})$ as quotient objects of $F(L\overline{\otimes}M)$. Furthermore,

$$\begin{aligned} coKer(d^{-1}_{L,M} \circ g' \circ \rho_Q \circ d_{Q,M}) &= coKer(d^{-1}_{L,M} \circ g' \circ \rho_Q) \\ &= coKer(g' \circ \rho_Q) \\ &= (L, \rho_L) \end{aligned}$$

as quotient objects of $F(L\overline{\otimes}M)$, where the first equality is valid by the item (iii) of Lemma 1.1.9, and the second equality via the item (iv) of this same lemma.

Thus the functor $F : \mathcal{M} \to \mathcal{C}_A$ is also dense and therefore, it is an equivalence of categories[22]. Since this functor is already a $\mathcal{C}$-module functor, it follows that (F, d) is an equivalence of $\mathcal{C}$-module categories (by Definition 2.2.11). ■

[22] See Theorem 1.3.4.

6 FINAL RESULTS

This final chapter is devoted to the proof of our main result, and we begin by showing two important lemmas that are going to be used. The first is about a natural isomorphism involving the inverse of the Yoneda Lemma, and the second is a certain naturality involving internal Homs and can be found in [5] as Lemma 2, and in in [3] as Lemma 2.6. At last, we present an example giving an application for our theorem based on Theorem 3.8 in [19].

Let $\mathcal{C}$ be a finite multitensor category and $\mathcal{M}$ a locally finite and module category over $\mathcal{C}$ with the module product $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ being $\Bbbk$-linear and left exact in the first variable.

Consider $X \in \mathcal{C}$ and an universal element[1] $(\underline{Hom}(M, N), ev_{M,N})$ of the representable functor $Hom_{\mathcal{M}}(\_ \overline{\otimes} M, N)$. Our first objective here is to define a family of isomorphisms

$$\gamma = \{\gamma_M \coloneqq (\Psi^N(ev_{M,N}))_X^{-1} : Hom_{\mathcal{M}}(X\overline{\otimes}M, N) \to Hom_{\mathcal{C}}(X, \underline{Hom}(M, N))\}_{M \in \mathcal{M}^{op}}$$

between the functors $Hom_{\mathcal{M}}(\_ , N) \circ (X\overline{\otimes} \_)$ and $Hom_{\mathcal{C}}(X, \_) \circ \underline{Hom}(\_ , N)$, and check this is natural in $\mathcal{M}^{op}$.

Lemma 6.1. *Let $(\underline{Hom}(M, N), ev_{M,N})$ be an universal element of its respective functor and $X \in \mathcal{C}$. Then*

$$\gamma = \{\gamma_M = (\Psi^N(ev_{M,N}))_X^{-1} : Hom_{\mathcal{M}}(X\overline{\otimes}M, N) \to Hom_{\mathcal{C}}(X, \underline{Hom}(M, N))\}_{M \in \mathcal{M}^{op}}$$

is a natural isomorphism in $\mathcal{M}^{op}$.

Proof. Let f be a morphism in $Hom_{\mathcal{M}^{op}}(M, M') = Hom_{\mathcal{M}}(M', M)$, and $(\underline{Hom}(M', N), ev_{M',N})$ be an universal element of its respective functor. The naturality of γ can be translated into the commutativity of the diagram

$$\begin{array}{ccc} Hom_{\mathcal{M}}(X\overline{\otimes}M, N) & \xrightarrow{\gamma_M} & Hom_{\mathcal{C}}(X, \underline{Hom}(M, N)) \\ {\scriptstyle Hom_{\mathcal{M}}(X\overline{\otimes}f,N)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathcal{C}}(X,\underline{Hom}(f,N))} \\ Hom_{\mathcal{M}}(X\overline{\otimes}M', N) & \xrightarrow{\gamma_{M'}} & Hom_{\mathcal{C}}(X, \underline{Hom}(M', N)). \end{array}$$

By noticing that $\Psi^N(ev_{M',N})_X$ is an isomorphism (in *Set*) we'll verify the equality

$$\Psi^N(ev_{M',N})_X((Hom_{\mathcal{C}}(X, \underline{Hom}(f, N))\circ\gamma_M)(h)) = \Psi^N(ev_{M',N})_X((\gamma_{M'}\circ Hom_{\mathcal{M}}(id_X\overline{\otimes}f, N))(h))$$

for every $h \in Hom_{\mathcal{M}}(X\overline{\otimes}M, N)$. We have

$$\begin{aligned} \Psi^N(ev_{M',N})_X((Hom_{\mathcal{C}}(X, \underline{Hom}(f, N)) \circ \gamma_M)(h)) &= \Psi^N(ev_{M',N})_X(Hom_{\mathcal{C}}(X, \underline{Hom}(f, N))(\gamma_M(h))) \\ &= \Psi^N(ev_{M',N})_X(\underline{Hom}(f, N) \circ \gamma_M(h)) \end{aligned}$$

[1] Notice that here we're using the notation introduced in Section 5.2.

$$= \Psi^N(ev_{M',N})_X(\underline{Hom}(f,N) \circ (\Psi^N(ev_{M,N}))_X^{-1}(h))$$
$$= ev_{M',N} \circ ((\underline{Hom}(f,N) \circ (\Psi^N(ev_{M,N}))_X^{-1}(h))\overline{\otimes} id_{M'})$$
$$= ev_{M',N} \circ (\underline{Hom}(f,N)\overline{\otimes} id_{M'}) \circ ((\Psi^N(ev_{M,N}))_X^{-1}(h)\overline{\otimes} id_{M'})$$
$$\overset{(*)}{=} ev_{M,N} \circ (id_{\underline{Hom}(M,N)}\overline{\otimes} f) \circ ((\Psi^N(ev_{M,N}))_X^{-1}(h)\overline{\otimes} id_{M'})$$
$$= ev_{M,N} \circ ((\Psi^N(ev_{M,N}))_X^{-1}(h)\overline{\otimes} id_{M}) \circ (id_X\overline{\otimes} f)$$
$$= \Psi^N(ev_{M,N})_X((\Psi^N(ev_{M,N}))_X^{-1}(h)) \circ (id_X\overline{\otimes} f)$$
$$= h \circ (id_X\overline{\otimes} f)$$
$$= \Psi^N(ev_{M',N})_X((\Psi^N(ev_{M',N}))_X^{-1}(h \circ (id_X\overline{\otimes} f)))$$
$$= \Psi^N(ev_{M',N})_X(\gamma_{M'}(h \circ (id_X\overline{\otimes} f)))$$
$$= \Psi^N(ev_{M',N})_X(\gamma_{M'}(Hom_{\mathcal{M}}(id_X\overline{\otimes} f, N)(h)))$$
$$= \Psi^N(ev_{M',N})_X((\gamma_{M'} \circ Hom_{\mathcal{M}}(X\overline{\otimes} f, N))(h))$$

where the equality $(*)$ holds via the definition of the morphism $\underline{Hom}(f,N) : \underline{Hom}(M,N) \to \underline{Hom}(M',N)$ in $\mathcal{C}$. This implies the equality

$$(Hom_{\mathcal{C}}(X, \underline{Hom}(f,N)) \circ \gamma_M)(h) = (\gamma_{M'} \circ Hom_{\mathcal{M}}(id_X\overline{\otimes} f, N))(h)$$

holds and hence, $Hom_{\mathcal{C}}(X, \underline{Hom}(f,N)) \circ \gamma_M = \gamma_{M'} \circ Hom_{\mathcal{M}}(id_X\overline{\otimes} f, N)$ as wanted. ∎

This following lemma asserts that whenever there is an adjunction between two $\mathcal{C}$-module categories, their internal Hom bifunctors are related in a certain way. Indeed, the internal Hom bifunctor behaves as the Hom bifunctor when dealing with adjoint functors.

Lemma 6.2 ([5], Lemma 2)**.** *Let $F : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{C}$-module functor with left adjoint $F^{l.a.} : \mathcal{N} \to \mathcal{M}$. Then there is a family ξ of isomorphisms in $\mathcal{C}$*

$$\xi_{N,M} : \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M) \to \underline{Hom}_{\mathcal{N}}(N, F(M)),$$

for all $N \in \mathcal{N}$ and $M \in \mathcal{M}$ such that $\xi = \{\xi_{N,M}\}_{(N,M)\in\mathcal{N}^{op}\times\mathcal{M}}$ is natural[2] in $\mathcal{N}^{op} \times \mathcal{M}$.

Proof. We know that $F^{l.a.}$ admits a $\mathcal{C}$-module functor structure by Theorem 2.3.2 which we denote by $(F^{l.a.}, d)$. Firstly, we construct a natural isomorphism from

$$R_{\underline{Hom}_{\mathcal{N}}(N,F(M))} = Hom_{\mathcal{C}}(\_\_, \underline{Hom}_{\mathcal{N}}(N, F(M))) \text{ to } R_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)} = Hom_{\mathcal{C}}(\_\_, \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M))$$

and use Proposition 1.4.6 to obtain an isomorphism $\underline{Hom}_{\mathcal{N}}(N, F(M)) \to \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M)$ in $\mathcal{C}$.

Let $(\underline{Hom}_{\mathcal{N}}(N, F(M)), ev_{N,F(M)})$ and $(\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M), ev_{F^{l.a.}(N),M})$ be universal elements of their respective functors.

[2] Between the functors $\underline{Hom}_{\mathcal{M}}(\_\_,\_\_) \circ (F^{l.a.} \times Id_{\mathcal{M}})$ and $\underline{Hom}_{\mathcal{N}}(\_\_,\_\_) \circ (Id_{\mathcal{N}} \times F)$ from $\mathcal{N}^{op} \times \mathcal{M}$ to $\mathcal{C}$.

The functor $\_\,\overline{\otimes}N : \mathcal{C} \to \mathcal{N}$ is left adjoint to $\underline{Hom}_{\mathcal{N}}(N, \_\,) : \mathcal{N} \to \mathcal{C}$ (see Proposition 4.2.1), so there is a natural isomorphism

$$\theta = \{\theta_{X,N'} : Hom_{\mathcal{N}}(X\overline{\otimes}N, N') \to Hom_{\mathcal{C}}(X, \underline{Hom}(N, N'))\}_{(X,N')\in\mathcal{C}^{op}\times\mathcal{N}}$$

in $\mathcal{C}^{op} \times \mathcal{N}$. Notice that $\theta_{X,F(M)} = (\Psi^{F(M)}(ev_{N,F(M)}))_X^{-1}$.

Given that $F^{l.a.}$ is left adjoint to F, there is a natural isomorphism

$$\phi = \{\phi_{N,M} : Hom_{\mathcal{M}}(F^{l.a.}(N), M) \to Hom_{\mathcal{N}}(N, F(M))\}_{(N,M)\in\mathcal{N}^{op}\times\mathcal{M}}$$

in $\mathcal{N}^{op} \times \mathcal{M}$. We'll use that $\phi' = \{\phi'_X \coloneqq \phi_{X\overline{\otimes}N,M}\}_{X\in\mathcal{C}}$ is a natural isomorphism in $\mathcal{C}$.

The next isomorphism is similar to the equation (15) in Lemma 2.3.1 which is

$$\varepsilon = \{\varepsilon_X \coloneqq Hom_{\mathcal{M}}(d_{X,N}, M) : Hom_{\mathcal{M}}(X\overline{\otimes}F^{l.a.}(N), M) \to Hom_{\mathcal{M}}(F^{l.a.}(X\overline{\otimes}N), M)\}_{X\in\mathcal{C}^{op}}.$$

Finally, we consider the natural isomorphism given by the adjunction
$(\_\,\overline{\otimes}F^{l.a.}(N), \underline{Hom}(F^{l.a.}(N), \_\,), \varphi)$, namely

$$\varphi = \{\varphi_{X,M} = (\Psi^M(ev_{F^{l.a.}(N),M}))_X^{-1} : Hom_{\mathcal{M}}(X\overline{\otimes}F^{l.a.}(N), M) \to Hom_{\mathcal{C}}(X, \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M))\}_{(X,M)\in\mathcal{C}^{op}\times\mathcal{M}}.$$

We then define the following composition of isomorphisms in *Set*

$$R_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(X) = Hom_{\mathcal{C}}(X, \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M)) \xrightarrow{\varphi^{-1}_{X,M}} Hom_{\mathcal{M}}(X\overline{\otimes}F^{l.a.}(N), M)$$
$$\xrightarrow{\varepsilon_X} Hom_{\mathcal{M}}(F^{l.a.}(X\overline{\otimes}N), M) \xrightarrow{\phi'_X} Hom_{\mathcal{N}}(X\overline{\otimes}N, F(M))$$
$$\xrightarrow{\theta_{X,F(M)}} R_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(X) = Hom_{\mathcal{C}}(X, \underline{Hom}_{\mathcal{N}}(N, F(M))).$$

The family $\{\theta_{X,F(M)} \circ \phi'_X \circ \varepsilon_X \circ \varphi^{-1}_{X,M}\}_{X\in\mathcal{C}}$ is natural in $\mathcal{C}$ since it is the composition of natural isomorphisms in $\mathcal{C}$. Via Proposition 1.4.6 it follows that

$$\xi_{N,M} \coloneqq (\theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)} \circ \phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)} \circ \varepsilon_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)} \circ \varphi^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),M})(id_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)})$$

is an isomorphism in $Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M), \underline{Hom}_{\mathcal{N}}(N, F(M)))$ with inverse

$$\xi^{-1}_{N,M} \coloneqq (\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M} \circ \varepsilon^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))} \circ \phi'^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))} \circ \theta^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),F(M)})(id_{\underline{Hom}_{\mathcal{N}}(N,F(M))}).$$

We can explicit $\xi_{N,M}$ as

$$\begin{aligned}
\xi_{N,M} &= (\theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)} \circ \phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)} \circ \varepsilon_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)} \circ \\
&\qquad \varphi^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),M})(id_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}) \\
&= \theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)}(\phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(\varepsilon_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(\varphi^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),M} \\
&\qquad (id_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}))))
\end{aligned}$$

$$= \theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)}(\phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(\varepsilon_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}$$
$$(\Psi^{M}(ev_{F^{l.a.}(N),M})_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(id_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}))))$$
$$= \theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)}(\phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(\varepsilon_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(ev_{F^{l.a.}(N),M})))$$
$$= \theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)}(\phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(Hom_{\mathcal{M}}(d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N}, M)(ev_{F^{l.a.}(N),M})))$$
$$= \theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)}(\phi'_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(ev_{F^{l.a.}(N),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N}))$$
$$= \theta_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),F(M)}(\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)\overline{\otimes}N,M}(ev_{F^{l.a.}(N),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N}))$$
$$= (\Psi^{F(M)}(ev_{N,F(M)}))^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)\overline{\otimes}N,M}(ev_{F^{l.a.}(N),M}\circ$$
$$d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N})).$$

Notice that this particularly gives us a certain relation[3] between $ev_{N,F(M)}$ and $ev_{F^{l.a.}(N),M}$ given by

$$ev_{N,F(M)} \circ (\xi_{N,M}\overline{\otimes}id_N) = \phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)\overline{\otimes}N,M}(ev_{F^{l.a.}(N),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N}). \tag{35}$$

In fact,

$$ev_{N,F(M)} \circ (\xi_{N,M}\overline{\otimes}id_N) = \Psi^{F(M)}(ev_{N,F(M)})_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}(\xi_{N,M})$$
$$= \Psi^{F(M)}(ev_{N,F(M)})_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}((\Psi^{F(M)}(ev_{N,F(M)}))^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)}$$
$$(\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)\overline{\otimes}N,M}(ev_{F^{l.a.}(N),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N})))$$
$$= \phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M)\overline{\otimes}N,M}(ev_{F^{l.a.}(N),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M),N}).$$

Moreover, $\xi^{-1}_{N,M}$ may be written as

$$\xi^{-1}_{N,M} = (\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M} \circ \varepsilon^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))} \circ \phi'^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))} \circ \theta^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),F(M)})$$
$$(id_{\underline{Hom}_{\mathcal{N}}(N,F(M))})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M}(\varepsilon^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(\phi'^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(\theta^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),F(M)}$$
$$(id_{\underline{Hom}_{\mathcal{N}}(N,F(M))}))))$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M}(\varepsilon^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(\phi'^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(\Psi^{F(M)}(ev_{N,F(M)})_{\underline{Hom}_{\mathcal{N}}(N,F(M))}$$
$$(id_{\underline{Hom}_{\mathcal{N}}(N,F(M))}))))$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M}(Hom_{\mathcal{M}}(d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N}, M)(\phi'^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(ev_{N,F(M)})))$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M}(\phi'^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= (\Psi^{M}(ev_{F^{l.a.}(N),M}))^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N}).$$

[3] This relation is only used in the proofs of our next theorem and corollary.

We'll show that $\xi^{-1} = \{\xi^{-1}_{N,M} : \underline{Hom}_{\mathcal{N}}(N, F(M)) \to \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M)\}_{(N,M)\in\mathcal{N}^{op}\times\mathcal{M}}$ is a natural isomorphism in $\mathcal{N}^{op} \times \mathcal{M}$, i.e., the commutativity of

$$\begin{array}{ccc} \underline{Hom}_{\mathcal{N}}(N, F(M)) & \xrightarrow{\xi^{-1}_{N,M}} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M) \\ {\scriptstyle \underline{Hom}_{\mathcal{N}}(f,F(g))}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{Hom}_{\mathcal{M}}(F^{l.a.}(f),g)} \\ \underline{Hom}_{\mathcal{N}}(N', F(M')) & \xrightarrow[\xi^{-1}_{N',M'}]{} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N'), M') \end{array}$$

for all $f \in Hom_{\mathcal{N}^{op}}(N, N')$ and $g \in Hom_{\mathcal{M}}(M, M')$.

The Remark 1.3.12 will be used in this verification, i.e., the commutativity of

$$\begin{array}{ccccc} & \underline{Hom}_{\mathcal{N}}(N, F(M)) & \xrightarrow{\xi^{-1}_{N,M}} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M) & \\ & {\scriptstyle \underline{Hom}_{\mathcal{N}}(N,F(g))}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),g)} & \\ {\scriptstyle \underline{Hom}_{\mathcal{N}}(f,F(g))}\Bigg\downarrow & \underline{Hom}_{\mathcal{N}}(N, F(M')) & \xrightarrow{\xi^{-1}_{N,M'}} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M') & \Bigg\downarrow{\scriptstyle \underline{Hom}_{\mathcal{M}}(F^{l.a.}(f),g)} \\ & {\scriptstyle \underline{Hom}_{\mathcal{N}}(f,F(M'))}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{Hom}_{\mathcal{M}}(F^{l.a.}(f),M')} & \\ & \underline{Hom}_{\mathcal{N}}(N', F(M')) & \xrightarrow[\xi^{-1}_{N',M'}]{} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N'), M'). & \end{array}$$

For the first rectangle, let $(\underline{Hom}_{\mathcal{N}}(N, F(M')), ev_{N,F(M')})$ be an universal element of its respective functor. Then

$$\begin{aligned}
&\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), g) \circ \xi^{-1}_{N,M} \\
&= \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), g) \circ (\Psi^{M}(ev_{F^{l.a.}(N),M}))^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ \\
&\quad d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N}) \\
&= \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), g) \circ \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ \\
&\quad d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N}) \\
&= (Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{N}}(N, F(M)), \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), g)) \circ \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M}) \\
&\quad (\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N}) \\
&\overset{(a)}{=} (\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'} \circ Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{N}}(N, F(M))\overline{\otimes}F^{l.a.}(N), g)) \\
&\quad (\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N}) \\
&= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{N}}(N, F(M))\overline{\otimes}F^{l.a.}(N), g) \\
&\quad (\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})) \\
&= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(g \circ \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})
\end{aligned}$$

$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(Hom_{\mathcal{M}}(F^{l.a.}(\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N),g)$$
$$(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M}(ev_{N,F(M)})) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}((Hom_{\mathcal{M}}(F^{l.a.}(\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N),g)\circ$$
$$\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M})(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$\overset{(b)}{=} \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}((\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M'}\circ$$
$$Hom_{\mathcal{N}}(\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,F(g)))(ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M'}(Hom_{\mathcal{N}}(\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,F(g))(ev_{N,F(M)}))\circ$$
$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M'}(F(g)\circ ev_{N,F(M)}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$\overset{(c)}{=} \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M'}(ev_{N,F(M')} \circ (\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_N))\circ$$
$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M'}$$
$$(Hom_{\mathcal{N}}(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_N,F(M'))(ev_{N,F(M')})) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}((\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M))\overline{\otimes}N,M'}\circ$$
$$Hom_{\mathcal{N}}(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_N,F(M')))(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$\overset{(b)}{=} \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}((Hom_{\mathcal{M}}(F^{l.a.}(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_N),M')\circ$$
$$\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'})(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(Hom_{\mathcal{M}}(F^{l.a.}(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_N),M')$$
$$(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')})) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ F^{l.a.}(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_N)\circ$$
$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M)),N})$$
$$\overset{(d)}{=} \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N}\circ$$
$$(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_{F^{l.a.}(N)}))$$
$$= (\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M)),M'} \circ Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{N}}(N,F(g))\overline{\otimes}id_{F^{l.a.}(N)},M'))$$
$$(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N})$$
$$\overset{(a)}{=} (Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{N}}(N,F(g)),\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M')) \circ \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M')),M'})$$
$$(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N})$$
$$= Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{N}}(N,F(g)),\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N),M'))(\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M')),M'}$$
$$(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N}))$$

$$= \varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M')),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N})\circ$$
$$\underline{Hom}_{\mathcal{N}}(N, F(g))$$
$$= (\Psi^{M'}(ev_{F^{l.a.}(N),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N})\circ$$
$$\underline{Hom}_{\mathcal{N}}(N, F(g))$$
$$= \xi^{-1}_{N,M'} \circ \underline{Hom}_{\mathcal{N}}(N, F(g))$$

where the equalities labeled with (a) hold by the naturality of φ, the ones with (b) are due to the naturality of ϕ^{-1} and (c) is via the definition of the morphism $\underline{Hom}_{\mathcal{N}}(N, F(g)) \in Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{N}}(N, F(M)), \underline{Hom}_{\mathcal{N}}(N, F(M')))$, i.e.,

$$F(g) \circ ev_{N,F(M)} = ev_{N,F(M')} \circ (\underline{Hom}_{\mathcal{N}}(N, F(g))\overline{\otimes}id_N).$$

At last, the naturality of d^{-1} is used in (d).

For the second retangle, let

$$(\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N), M'), ev_{F^{l.a.}(N),M'})$$
$$(\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N'), M'), ev_{F^{l.a.}(N'),M'}), \text{ and}$$
$$(\underline{Hom}_{\mathcal{N}}(N', F^{l.a.}(M')), ev_{N',F(M')})$$

be universal elements of their respective functors. We now check the equality

$$\Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M')\circ\xi^{-1}_{N,M'}) = \Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\xi^{-1}_{N',M'}\circ\underline{Hom}_{\mathcal{N}}(f, F(M')))$$

to get $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M') \circ \xi^{-1}_{N,M'} = \xi^{-1}_{N',M'} \circ \underline{Hom}_{\mathcal{N}}(f, F(M'))$[4]. In fact,

$$\Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M') \circ \xi^{-1}_{N,M'})$$
$$= ev_{F^{l.a.}(N'),M'} \circ ((\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M') \circ \xi^{-1}_{N,M'})\overline{\otimes}id_{F^{l.a.}(N')})$$
$$= ev_{F^{l.a.}(N'),M'} \circ (\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M')\overline{\otimes}id_{F^{l.a.}(N')}) \circ (\xi^{-1}_{N,M'}\overline{\otimes}id_{F^{l.a.}(N')})$$
$$\overset{(e)}{=} ev_{F^{l.a.}(N),M'} \circ (id_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(N'),M')}\overline{\otimes}F^{l.a.}(f)) \circ (\xi^{-1}_{N,M'}\overline{\otimes}id_{F^{l.a.}(N')})$$
$$= ev_{F^{l.a.}(N),M'} \circ (\xi^{-1}_{N,M'}\overline{\otimes}id_{F^{l.a.}(N)}) \circ (id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}F^{l.a.}(f))$$
$$= \Psi^{M'}(ev_{F^{l.a.}(N),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\xi^{-1}_{N,M'}) \circ (id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}F^{l.a.}(f))$$
$$= \Psi^{M'}(ev_{F^{l.a.}(N),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M')),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')})\circ$$
$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N})) \circ (id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}F^{l.a.}(f))$$
$$= \varphi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),M'}(\varphi_{\underline{Hom}_{\mathcal{N}}(N,F(M')),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')})\circ$$
$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N})) \circ (id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}F^{l.a.}(f))$$
$$= \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N} \circ (id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}F^{l.a.}(f))$$
$$\overset{(d)}{=} \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}) \circ F^{l.a.}(id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}f) \circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$$

4 $\Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))} : Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{N}}(N, F(M')), \underline{Hom}_{\mathcal{M}}(F^{l.a.}(N'), M')) \to Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{N}}(N, F(M'))\overline{\otimes}F^{l.a.}(N'), M')$ is an isomorphism (in *Set*).

$= Hom_{\mathcal{M}}(F^{l.a.}(id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}f), M')(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'}(ev_{N,F(M')}))\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= (Hom_{\mathcal{M}}(F^{l.a.}(id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}f), M')\circ \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N,M'})(ev_{N,F(M')})\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$\overset{(b)}{=} (\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N',M'}\circ Hom_{\mathcal{N}}(id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}f, F(M')))(ev_{N,F(M')})\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N',M'}(Hom_{\mathcal{N}}(id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}f, F(M'))(ev_{N,F(M')}))\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N',M'}(ev_{N,F(M')}\circ(id_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}\overline{\otimes}f))\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$\overset{(f)}{=} \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N',M'}(ev_{N',F(M')}\circ(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{N'}))\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N',M'}(Hom_{\mathcal{N}}(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{N'}, F(M'))(ev_{N',F(M')}))\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= (\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M'))\overline{\otimes}N',M'}\circ Hom_{\mathcal{N}}(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{N'}, F(M')))(ev_{N',F(M')})\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$\overset{(b)}{=} (Hom_{\mathcal{M}}(F^{l.a.}(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{N'}), M')\circ \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))\overline{\otimes}N',M'})(ev_{N',F(M')})\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= Hom_{\mathcal{M}}(F^{l.a.}(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{N'}), M')(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))\overline{\otimes}N',M'}(ev_{N',F(M')}))\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$= \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))\overline{\otimes}N',M'}(ev_{N',F(M')})\circ F^{l.a.}(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{N'})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N,F(M')),N'}$
$\overset{(d)}{=} \phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))\overline{\otimes}N',M'}(ev_{N',F(M')})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M')),N'}\circ(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{F^{l.a.}(N')})$
$= \Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N',F(M'))}((\Psi^{M'}(ev_{F^{l.a.}(N'),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))}$
$(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))\overline{\otimes}N',M'}(ev_{N',F(M')})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M')),N'}))\circ(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{F^{l.a.}(N')})$
$= \Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N',F(M'))}(\varphi_{\underline{Hom}_{\mathcal{N}}(N',F(M')),M'}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M'))\overline{\otimes}N',M'}(ev_{N',F(M')})\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(N',F(M')),N'}))\circ(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{F^{l.a.}(N')})$
$= \Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N',F(M'))}(\xi_{N',M'})\circ(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{F^{l.a.}(N')})$
$= ev_{F^{l.a.}(N'),M'}\circ(\xi_{N',M'}\overline{\otimes}id_{F^{l.a.}(N')})\circ(\underline{Hom}_{\mathcal{N}}(f,F(M'))\overline{\otimes}id_{F^{l.a.}(N')})$
$= ev_{F^{l.a.}(N'),M'}\circ((\xi_{N',M'}\circ\underline{Hom}_{\mathcal{N}}(f,F(M')))\overline{\otimes}id_{F^{l.a.}(N')})$
$= \Psi^{M'}(ev_{F^{l.a.}(N'),M'})_{\underline{Hom}_{\mathcal{N}}(N,F(M'))}(\xi_{N',M'}\circ\underline{Hom}_{\mathcal{N}}(f,F(M')))$

in which the equalities (e) and (f) come from the definition of the morphisms $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M')$ and $\underline{Hom}_{\mathcal{N}}(f, F(M'))$, respectively.

Therefore, $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(f), M')\circ\xi^{-1}_{N,M'} = \xi^{-1}_{N',M'}\circ\underline{Hom}_{\mathcal{N}}(f, F(M'))$ implying that

ξ^{-1} (and hence ξ) is a natural isomorphism in $\mathcal{N}^{op} \times \mathcal{M}$. ■

The following theorem is our main result. It uses the Theorem 5.4.1, Lemma 6.2 and some minor equivalences to give another method to check whether two exact indecomposable $\mathcal{C}$-module categories are equivalent. To be under the hypothesis of the Theorem 5.4.1, let $\mathcal{C}$ be a finite tensor category, and $\mathcal{M}$ and $\mathcal{N}$ be locally finite and exact indecomposable $\mathcal{C}$-module categories with their respective module product $\overline{\otimes}$ being $\mathbb{k}$-linear and left exact in the first variable.

Theorem 6.3. *Two categories $\mathcal{M}$ and $\mathcal{N}$ are equivalent as $\mathcal{C}$-module categories if, and only if, there is an object $0 \neq M \in \mathcal{M}$ (respectively, $0 \neq N \in \mathcal{N}$) and a $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{N}$ admitting an adjunction $(F^{l.a.}, F, \phi)$ such that*

$$\phi^{-1}_{F(M),M}(id_{F(M)}) : (F^{l.a.} \circ F)(M) \to M$$

(respectively, $\phi_{N,F^{l.a.}(N)}(id_{F^{l.a.}(N)}) : N \to (F \circ F^{l.a.})(N)$) is an isomorphism in $\mathcal{M}$ (in $\mathcal{N}$).

Proof. Suppose that $G : \mathcal{M} \to \mathcal{N}$ is an equivalence of $\mathcal{C}$-module categories. We know that there exists a $\mathcal{C}$-module functor $H : \mathcal{N} \to \mathcal{M}$ satisfying $G \circ H \sim Id_{\mathcal{N}}$ and $H \circ G \sim Id_{\mathcal{M}}$, and also H is left (and right) adjoint to G (see Proposition 1.3.9). Moreover, the counit and unit of this adjunction are natural isomorphisms[5] implying that e_M and c_N are isomorphisms, for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

For the converse, let $(F, b) : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{C}$-module functor, $(F^{l.a.}, F, \phi)$ an adjunction with counit (see Proposition 1.3.8)

$$e = \{e_M = \phi^{-1}_{F(M),M}(id_{F(M)}) : F^{l.a.}(F(M)) \to M\}_{M \in \mathcal{M}}$$

and unit

$$c = \{c_N = \phi_{N,F^{l.a.}(N)}(id_{F^{l.a.}(N)}) : N \to F(F^{l.a.(N)})\}_{N \in \mathcal{N}},$$

$0 \neq M$ an object in $\mathcal{M}$ and $e_M = \phi^{-1}_{F(M),M}(id_{F(M)}) : (F^{l.a.} \circ F)(M) \to M$ an isomorphism in $\mathcal{M}$.

Via Theorem 5.4.1 there is an equivalence

$$\begin{aligned} F_1 : \mathcal{M} &\longrightarrow \mathcal{C}_{\underline{Hom}_{\mathcal{M}}(M,M)} \\ M' &\longmapsto F_1(M') = (\underline{Hom}_{\mathcal{M}}(M, M'), \rho_{\underline{Hom}_{\mathcal{M}}(M,M')}) \\ h &\longmapsto F_1(h) = \underline{Hom}_{\mathcal{M}}(M, h) \end{aligned}$$

of $\mathcal{C}$-module categories where the algebra $\underline{Hom}_{\mathcal{M}}(M, M) = (\underline{Hom}_{\mathcal{M}}(M, M), \mu, u)$ was defined in the Section 5.2 with multiplication

$$\mu = \mu_{M,M,M} = (\Psi^M(ev_{M,M}))^{-1}_{\underline{Hom}_{\mathcal{M}}(M,M) \otimes \underline{Hom}_{\mathcal{M}}(M,M)}(ev_{M,M} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M)} \overline{\otimes} ev_{M,M}) \circ m_{\underline{Hom}_{\mathcal{M}}(M,M),\underline{Hom}_{\mathcal{M}}(M,M),M}), \quad \text{and unit}$$

$$u = (\Psi^M(ev_{M,M}))^{-1}_{\mathbf{1}}(l_M).$$

[5] This follows (by construction) from Proposition 1.3.9.

For every $M' \in \mathcal{M}$, let us define $\beta_{M'} \coloneqq \xi_{F(M),M'} \circ \underline{Hom}_{\mathcal{M}}(e_M, M')$ which can be seen as the composition[6]

$$\underline{Hom}_{\mathcal{M}}(M, M') \xrightarrow{\underline{Hom}_{\mathcal{M}}(e_M,M')} \underline{Hom}_{\mathcal{M}}((F^{l.a.} \circ F)(M), M') \xrightarrow{\xi_{F(M),M'}} \underline{Hom}_{\mathcal{N}}(F(M), F(M'))$$

of isomorphisms in $\mathcal{C}$. By Proposition 5.1.9, $\underline{Hom}_{\mathcal{N}}(F(M), F(M))$ is an algebra with multiplication $\beta_M \circ \mu \circ (\beta_M^{-1} \otimes \beta_M^{-1})$ and unit $\beta_M \circ u$, the morphism β_M is an algebra isomorphism and the algebras $\underline{Hom}_{\mathcal{M}}(M, M)$ and $\underline{Hom}_{\mathcal{N}}(F(M), F(M))$ are Morita equivalent, i.e., there exists an equivalence

$$\begin{aligned} G : \mathcal{C}_{\underline{Hom}_{\mathcal{M}}(M,M)} &\longrightarrow \mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))} \\ (X, \rho_X) &\longmapsto G(X, \rho_X) = (X, \rho'_X) = (X, \rho_X \circ (id_X \otimes \beta_M^{-1})) \\ g &\longmapsto G(g) = g \end{aligned}$$

of $\mathcal{C}$-module categories[7].

Affirmation: The algebra structure (multiplication and unit) of $\underline{Hom}_{\mathcal{N}}(F(M), F(M))$ defined as in the Section 5.2 is the same as the one given in terms of the algebra $\underline{Hom}_{\mathcal{M}}(M, M)$, that is,

$$\beta_M \circ \mu \circ (\beta_M^{-1} \otimes \beta_M^{-1}) = \mu_{F(M),F(M),F(M)}$$

and

$$\beta_M \circ u = (\Psi^{F(M)}(ev_{F(M),F(M)}))_{\mathbf{1}}^{-1}(l_{F(M)}).$$

In fact, by denoting the objects $A \coloneqq \underline{Hom}_{\mathcal{M}}(M, M)$ and $B \coloneqq \underline{Hom}_{\mathcal{N}}(F(M), F(M))$ we get

$$\begin{aligned}
\beta_M \circ u &= \xi_{F(M),M} \circ \underline{Hom}_{\mathcal{M}}(e_M, M) \circ (\Psi^M(ev_{M,M}))_{\mathbf{1}}^{-1}(l_M) \\
&= \xi_{F(M),M} \circ Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}_{\mathcal{M}}(e_M, M))((\Psi^M(ev_{M,M}))_{\mathbf{1}}^{-1}(l_M)) \\
&= \xi_{F(M),M} \circ (Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}_{\mathcal{M}}(e_M, M)) \circ (\Psi^M(ev_{M,M}))_{\mathbf{1}}^{-1})(l_M) \\
&\overset{(a)}{=} \xi_{F(M),M} \circ (Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}_{\mathcal{M}}(e_M, M)) \circ \gamma_M)(l_M) \\
&\overset{(b)}{=} \xi_{F(M),M} \circ (\gamma_{F^{l.a.}(F(M))} \circ Hom_{\mathcal{M}}(\mathbf{1} \overline{\otimes} e_M, M))(l_M) \\
&= \xi_{F(M),M} \circ \gamma_{F^{l.a.}(F(M))}(Hom_{\mathcal{M}}(\mathbf{1} \overline{\otimes} e_M, M)(l_M)) \\
&= \xi_{F(M),M} \circ \gamma_{F^{l.a.}(F(M))}(l_M \circ (id_{\mathbf{1}} \overline{\otimes} e_M)) \\
&\overset{(c)}{=} \xi_{F(M),M} \circ \gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))}) \\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)}^{-1}(\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M) \overline{\otimes} F(M),M} \\
&\quad (ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)})) \circ \gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))}) \\
&= (Hom_{\mathcal{C}}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))}), B) \circ (\Psi^{F(M)}(ev_{F(M),F(M)}))_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)}^{-1}) \\
&\quad (\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M) \overline{\otimes} F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)}))
\end{aligned}$$

[6] We'll proof in the following result that the family $\beta = \{\beta_{M'}\}_{M' \in \mathcal{M}}$ is a natural isomorphism.

[7] The $\mathcal{C}$-module functor structure of G is the natural isomorphism identity ID.

$$
\begin{aligned}
&\overset{(d)}{=} ((\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}} \circ Hom_{\mathcal{N}}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}F(M), F(M)))\\
&\quad (\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)}))\\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(Hom_{\mathcal{N}}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}F(M), F(M))\\
&\quad (\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)})))\\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}((Hom_{\mathcal{N}}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}id_{F(M)}, F(M))\circ\\
&\quad \phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)\overline{\otimes}F(M),M})(ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)}))\\
&\overset{(e)}{=} (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}((\phi_{\mathbf{1}\overline{\otimes}F(M),M}\circ\\
&\quad Hom_{\mathcal{M}}(F^{l.a.}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}id_{F(M)}), M))(ev_{F^{l.a.}(F(M)),M}\circ\\
&\quad d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)}))\\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}\\
&\quad (Hom_{\mathcal{M}}(F^{l.a.}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}id_{F(M)}), M)(ev_{F^{l.a.}(F(M)),M}\circ\\
&\quad d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)})))\\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)}\circ\\
&\quad F^{l.a.}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}id_{F(M)})))\\
&\overset{(f)}{=} (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M}\circ\\
&\quad (\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}F^{l.a.}(id_{F(M)})) \circ d_{\mathbf{1},F(M)}))\\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M}\circ\\
&\quad (\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}id_{F^{l.a.}(F(M))}) \circ d_{\mathbf{1},F(M)}))\\
&\overset{(g)}{=} (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}(e_M \circ l_{F^{l.a.}(F(M))} \circ d_{\mathbf{1},F(M)}))\\
&\overset{(h)}{=} (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}(e_M \circ F^{l.a.}(l_{F(M)})))\\
&\overset{(3)}{=} (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(\phi_{\mathbf{1}\overline{\otimes}F(M),M}(\phi^{-1}_{\mathbf{1}\overline{\otimes}F(M),M}(l_{F(M)})))\\
&= (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(l_{F(M)})
\end{aligned}
$$

where the equality (a) holds via the definition of the natural isomorphism

$$\gamma = \{\gamma_M \coloneqq (\Psi^M(ev_{M,M}))^{-1}_{\mathbf{1}} : Hom_{\mathcal{M}}(\mathbf{1}\overline{\otimes}M, M) \to Hom_{\mathcal{C}}(\mathbf{1}, \underline{Hom}_{\mathcal{M}}(M, M))\}_{M\in\mathcal{M}^{op}}$$

(see Lemma 6.1), and (b) uses its naturality. The naturalities of l, $\Psi^{F(M)}(ev_{F(M),F(M)})$, ϕ and d are used in the equalities (c), (d), (e) and (f), respectively. The equality (g) holds by noticing that

$$
\begin{aligned}
e_M \circ l_{F^{l.a.}(F(M))} &= \gamma^{-1}_{F^{l.a.}(F(M))}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))}))\\
&= \Psi^M(ev_{F^{l.a.}(F(M)),M})_{\mathbf{1}}(\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))}))\\
&= ev_{F^{l.a.}(F(M)),M} \circ (\gamma_{F^{l.a.}(F(M))}(e_M \circ l_{F^{l.a.}(F(M))})\overline{\otimes}id_{F^{l.a.}(F(M))}),
\end{aligned}
$$

and (h) via the triangle diagram of the $\mathcal{C}$-module functor $(F^{l.a.}, d)$ (see Definition 2.2.6).

A slight different (and more general) version of the equality

$$\mu_{F(M),F(M),F(M)} = \beta_M \circ \mu \circ (\beta_M^{-1} \otimes \beta_M^{-1})$$

has to be verified on the next corollary (Corollary 6.4). So to not solve it twice, we now check that a more general equality holds which is

$$\begin{aligned}
\mu_{F(M),F(M),F(M')} &= \beta_{M'} \circ \mu_{M,M,M'} \circ (\beta_{M'}^{-1} \otimes \beta_M^{-1}) \qquad (36)\\
&= \xi_{F(M),M'} \circ \underline{Hom}_{\mathcal{M}}(e_M, M') \circ \mu_{M,M,M'} \circ \\
&\quad ((\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \circ \xi^{-1}_{F(M),M'}) \otimes (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M) \circ \xi^{-1}_{F(M),M}))\\
&= \xi_{F(M),M'} \circ \underline{Hom}_{\mathcal{M}}(e_M, M') \circ \mu_{M,M,M'} \circ \\
&\quad (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)) \circ (\xi^{-1}_{F(M),M'} \otimes \xi^{-1}_{F(M),M})
\end{aligned}$$

for all $M' \in \mathcal{M}$. This equality can described as the commutativity of the diagram

$$\begin{array}{ccccc}
\underline{Hom}_{\mathcal{N}}(F(M), F(M')) \otimes B & \xrightarrow{\xi^{-1}_{F(M),M'} \otimes \xi^{-1}_{F(M),M}} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') \otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M) & \xrightarrow{\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)} & \underline{Hom}_{\mathcal{M}}(M, M') \otimes A \\
\Big\downarrow {\scriptstyle \mu_{F(M),F(M),F(M')}} & & & & \Big\downarrow {\scriptstyle \mu_{M,M,M'}} \\
\underline{Hom}_{\mathcal{N}}(F(M), F(M')) & \xleftarrow[\xi_{F(M),M'}]{} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') & \xleftarrow[\underline{Hom}_{\mathcal{M}}(e_M, M')]{} & \underline{Hom}_{\mathcal{M}}(M, M').
\end{array}$$

To make this easier, let us consider the morphism

$$\begin{aligned}
\nu = (\Psi^{M'}(ev_{F^{l.a}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M') \otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)}(ev_{M,M'} \circ \\
(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \overline{\otimes} id_M) \circ (id_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M')} \overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ \\
m_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M'),\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F^{l.a.}(F(M))})
\end{aligned}$$

in $\mathcal{C}$ from $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') \otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M)$ to $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M')$ and check the commutativity of the diagram

$$\begin{array}{ccccc}
\underline{Hom}_{\mathcal{N}}(F(M), F(M')) \otimes B & \xrightarrow{\xi^{-1}_{F(M),M'} \otimes \xi^{-1}_{F(M),M}} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') \otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M) & \xrightarrow{\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)} & \underline{Hom}_{\mathcal{M}}(M, M') \otimes A \\
\Big\downarrow {\scriptstyle \mu_{F(M),F(M),F(M')}} & & \Big\downarrow {\scriptstyle \nu} & & \Big\downarrow {\scriptstyle \mu_{M,M,M'}} \\
\underline{Hom}_{\mathcal{N}}(F(M), F(M')) & \xleftarrow[\xi_{F(M),M'}]{} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') & \xleftarrow[\underline{Hom}_{\mathcal{M}}(e_M, M')]{} & \underline{Hom}_{\mathcal{M}}(M, M').
\end{array}$$

Let us define $C \coloneqq \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M)$ and $C' \coloneqq \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M')$, and see that the first rectangle is commutative. In fact,

$$\begin{aligned}
&\xi^{-1}_{F(M),M'} \circ \mu_{F(M),F(M),F(M')} \circ (\xi_{F(M),M'} \otimes \xi_{F(M),M})\\
&= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')) \otimes B}(ev_{F(M),F(M')} \circ \\
&\quad (id_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))} \overline{\otimes} ev_{F(M),F(M)}) \circ m_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),B,F(M)}) \circ (\xi_{F(M),M'} \otimes \xi_{F(M),M})
\end{aligned}$$

$$= \xi^{-1}_{F(M),M'} \circ (Hom_{\mathcal{C}}(\xi_{F(M),M'} \otimes \xi_{F(M),M}, \underline{Hom}_{\mathcal{N}}(F(M), F(M')))\circ$$
$$(\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))\otimes B})(ev_{F(M),F(M')}\circ$$
$$(id_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}\overline{\otimes} ev_{F(M),F(M)}) \circ m_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),B,F(M)})$$
$$\stackrel{(i)}{=} \xi^{-1}_{F(M),M'} \circ ((\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}\circ$$
$$Hom_{\mathcal{N}}((\xi_{F(M),M'} \otimes \xi_{F(M),M})\overline{\otimes} F(M), F(M')))(ev_{F(M),F(M')}\circ$$
$$(id_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}\overline{\otimes} ev_{F(M),F(M)}) \circ m_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),B,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(Hom_{\mathcal{N}}((\xi_{F(M),M'} \otimes \xi_{F(M),M})\overline{\otimes} id_{F(M)}, F(M'))$$
$$(ev_{F(M),F(M')} \circ (id_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}\overline{\otimes} ev_{F(M),F(M)}) \circ m_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),B,F(M)}))$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(ev_{F(M),F(M')} \circ (id_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}\overline{\otimes} ev_{F(M),F(M)})\circ$$
$$m_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),B,F(M)} \circ ((\xi_{F(M),M'} \otimes \xi_{F(M),M})\overline{\otimes} id_{F(M)}))$$
$$\stackrel{(j)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(ev_{F(M),F(M')}\circ$$
$$(id_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}\overline{\otimes} ev_{F(M),F(M)}) \circ (\xi_{F(M),M'}\overline{\otimes}(\xi_{F(M),M}\overline{\otimes} id_{F(M)})) \circ m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(ev_{F(M),F(M')}\circ$$
$$(\xi_{F(M),M'}\overline{\otimes} ev_{F(M),F(M)}) \circ (id_{C'}\overline{\otimes}(\xi_{F(M),M}\overline{\otimes} id_{F(M)})) \circ m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(ev_{F(M),F(M')}\circ$$
$$(\xi_{F(M),M'}\overline{\otimes} id_{F(M)}) \circ (id_{C'}\overline{\otimes}(ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)}))) \circ m_{C',C,F(M)})$$
$$\stackrel{(k)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes} F(M),M'}(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)})\circ$$
$$(id_{C'}\overline{\otimes}(ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)}))) \circ m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}((Hom_{\mathcal{N}}(id_{C'}\overline{\otimes}(ev_{F(M),F(M)}\circ$$
$$(\xi_{F(M),M}\overline{\otimes} id_{F(M)})), F(M')) \circ \phi_{C'\overline{\otimes} F(M),M'})(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)}) \circ m_{C',C,F(M)})$$
$$\stackrel{(e)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}((\phi_{C'\overline{\otimes}(C\overline{\otimes} F(M)),M'}\circ$$
$$Hom_{\mathcal{M}}(F^{l.a.}(id_{C'}\overline{\otimes}(ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)}))), M'))(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)})\circ$$
$$m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes} F(M)),M'}$$
$$(Hom_{\mathcal{M}}(F^{l.a.}(id_{C'}\overline{\otimes}(ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)}))), M')(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)}))\circ$$
$$m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes} F(M)),M'}(ev_{F^{l.a.}(F(M)),M'}\circ$$
$$d_{C',F(M)} \circ F^{l.a.}(id_{C'}\overline{\otimes}(ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)})))) \circ m_{C',C,F(M)})$$
$$\stackrel{(f)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes} F(M)),M'}(ev_{F^{l.a.}(F(M)),M'}\circ$$
$$(id_{C'}\overline{\otimes} F^{l.a.}(ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)}))) \circ d_{C',C\overline{\otimes} F(M)}) \circ m_{C',C,F(M)})$$

$$\overset{(k)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes}F(M)),M'}(ev_{F^{l.a.}(F(M)),M'}\circ$$
$$(id_{C'}\overline{\otimes}F^{l.a.}(\phi_{C\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{C,F(M)}))) \circ d_{C',C\overline{\otimes}F(M)}) \circ m_{C',C,F(M)})$$
$$\overset{(l)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes}F(M)),M'}(ev_{F^{l.a.}(F(M)),M'}\circ$$
$$(id_{C'}\overline{\otimes}(e^{-1}_M \circ \phi^{-1}_{C\overline{\otimes}F(M),M}(\phi_{C\overline{\otimes}F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{C,F(M)})))) \circ d_{C',C\overline{\otimes}F(M)})\circ$$
$$m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes}F(M)),M'}(ev_{F^{l.a.}(F(M)),M'}\circ$$
$$(id_{C'}\overline{\otimes}(e^{-1}_M \circ ev_{F^{l.a.}(F(M)),M} \circ d_{C,F(M)})) \circ d_{C',C\overline{\otimes}F(M)}) \circ m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes}F(M)),M'}(ev_{F^{l.a.}(F(M)),M'}\circ$$
$$(id_{C'}\overline{\otimes}e^{-1}_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}d_{C,F(M)}) \circ d_{C',C\overline{\otimes}F(M)}) \circ m_{C',C,F(M)})$$
$$\overset{(m)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{C'\overline{\otimes}(C\overline{\otimes}F(M)),M'}(ev_{M,M'}\circ$$
$$(\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}d_{C,F(M)}) \circ d_{C',C\overline{\otimes}F(M)})\circ$$
$$m_{C',C,F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}((Hom_{\mathcal{N}}(m_{C',C,F(M)}, F(M')) \circ \phi_{C'\overline{\otimes}(C\overline{\otimes}F(M)),M'})$$
$$(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}d_{C,F(M)}) \circ d_{C',C\overline{\otimes}F(M)}))$$
$$\overset{(e)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}((\phi_{(C'\otimes C)\overline{\otimes}F(M),M'} \circ Hom_{\mathcal{M}}(F^{l.a.}(m_{C',C,F(M)}), M'))$$
$$(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}d_{C,F(M)}) \circ d_{C',C\overline{\otimes}F(M)}))$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{(C'\otimes C)\overline{\otimes}F(M),M'}(Hom_{\mathcal{M}}(F^{l.a.}(m_{C',C,F(M)}), M')$$
$$(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}d_{C,F(M)}) \circ d_{C',C\overline{\otimes}F(M)})))$$
$$= \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{(C'\otimes C)\overline{\otimes}F(M),M'}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M)\circ$$
$$(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}d_{C,F(M)}) \circ d_{C',C\overline{\otimes}F(M)} \circ F^{l.a.}(m_{C',C,F(M)})))$$
$$\overset{(n)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(\phi_{(C'\otimes C)\overline{\otimes}F(M),M'}(ev_{M,M'}\circ$$
$$(\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}))$$
$$\overset{(2)}{=} \xi^{-1}_{F(M),M'} \circ (\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}(F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M)\circ$$
$$(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes}F(M)})$$
$$= \xi^{-1}_{F(M),M'} \circ ((\Psi^{F(M')}(ev_{F(M),F(M')}))^{-1}_{C'\otimes C}\circ$$
$$Hom_{\mathcal{N}}((C' \otimes C)\overline{\otimes}F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M)\circ$$
$$(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes}F(M)}))(id_{(C'\otimes C)\overline{\otimes}F(M)})$$
$$\overset{(o)}{=} \xi^{-1}_{F(M),M'} \circ (Hom_{\mathcal{C}}(C' \otimes C, \underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_M, M')\overline{\otimes}id_M)\circ$$
$$(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes}F(M)}))\circ$$
$$(\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C})(id_{(C'\otimes C)\overline{\otimes}F(M)})$$

$$= \xi^{-1}_{F(M),M'} \circ Hom_{\mathcal{C}}(C' \otimes C, \underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$

$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)}))$$

$$((\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)}))$$

$$= \xi^{-1}_{F(M),M'} \circ \underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M})\circ$$

$$m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)})\circ$$

$$(\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))\overline{\otimes} F(M),M'}(ev_{F(M),F(M')})\circ$$

$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}) \circ \underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$

$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)})\circ$$

$$(\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$$= (Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M})\circ$$

$$m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)}, C'))\circ$$

$$(\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))\overline{\otimes} F(M),M'}(ev_{F(M),F(M')})\circ$$

$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}) \circ (\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$$\overset{(p)}{=} ((\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),(C'\otimes C)\overline{\otimes} F(M))} \circ Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ$$

$$(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)})\circ$$

$$c_{(C'\otimes C)\overline{\otimes} F(M)})\overline{\otimes} F^{l.a.}(F(M)), C'))(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))\overline{\otimes} F(M),M'}(ev_{F(M),F(M')})\circ$$

$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}) \circ (\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),(C'\otimes C)\overline{\otimes} F(M))}(Hom_{\mathcal{M}}(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ$$

$$(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)})\circ$$

$$c_{(C'\otimes C)\overline{\otimes} F(M)})\overline{\otimes} F^{l.a.}(F(M)), C')(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))\overline{\otimes} F(M),M'}(ev_{F(M),F(M')})\circ$$

$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)})) \circ (\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),(C'\otimes C)\overline{\otimes} F(M))}(\phi^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))\overline{\otimes} F(M),M'}(ev_{F(M),F(M')})\circ$$

$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)} \circ (\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$

$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)})\overline{\otimes} id_{F^{l.a.}(F(M))}))\circ$$

$$(\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$$\overset{(3)}{=} (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),(C'\otimes C)\overline{\otimes} F(M))}(e_{M'} \circ F^{l.a.}(ev_{F(M),F(M')})\circ$$

$$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)} \circ (\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$

$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)})\overline{\otimes} id_{F^{l.a.}(F(M))}))\circ$$

$$(\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)})$$

$= (Hom_{\mathcal{C}}((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes}F(M)}), C')\circ$
$(\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),(C'\otimes C)\overline{\otimes}F(M))})(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')})\circ$
$d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}\circ(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ$
$(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}id_{F^{l.a.}(F(M))}))$

$\overset{(p)}{=} ((\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}\circ Hom_{\mathcal{M}}((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}$
$(id_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}F^{l.a.}(F(M)), M'))(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}\circ$
$(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$
$m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}id_{F^{l.a.}(F(M))}))$

$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(Hom_{\mathcal{M}}((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}$
$(id_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}F^{l.a.}(F(M)), M')(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}\circ$
$(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$
$m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}id_{F^{l.a.}(F(M))})))$

$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}\circ$
$(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$
$m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}id_{F^{l.a.}(F(M))})\circ$
$((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes}F(M)})\overline{\otimes}id_{F^{l.a.}(F(M))}))$

$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')})\circ d^{-1}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}\circ$
$((\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$
$m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\circ((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}$
$(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes}F(M)})))\overline{\otimes}id_{F^{l.a.}(F(M))}))$

$\overset{(f)}{=} (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')})\circ$
$F^{l.a.}((\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$
$m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\circ((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}$
$(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes}F(M)})))\overline{\otimes}id_{F(M)})\circ d^{-1}_{C'\otimes C,F(M)})$

$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'}\circ F^{l.a.}(ev_{F(M),F(M')}\circ$
$((\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'}\circ(\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes}id_M)\circ(id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$
$m_{C',C,F^{l.a.}(F(M))}\circ d_{C'\otimes C,F(M)})\circ c_{(C'\otimes C)\overline{\otimes}F(M)})\circ((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}$
$(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes}F(M)})))\overline{\otimes}id_{F(M)}))\circ d^{-1}_{C'\otimes C,F(M)})$

$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'} \circ F^{l.a.}(ev_{F(M),F(M')}\circ$$
$$(\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M})\circ$$
$$m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)})\overline{\otimes} id_{F(M)}) \circ (((\Psi^{(C'\otimes C)\overline{\otimes} F(M)}$$
$$(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}(id_{(C'\otimes C)\overline{\otimes} F(M)}))\overline{\otimes} id_{F(M)})) \circ d^{-1}_{C'\otimes C,F(M)})$$
$$\overset{(q)}{=} (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'} \circ F^{l.a.}(F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes} F(M)}\circ$$
$$ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)} \circ (((\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}$$
$$(id_{(C'\otimes C)\overline{\otimes} F(M)}))\overline{\otimes} id_{F(M)})) \circ d^{-1}_{C'\otimes C,F(M)})$$
$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'} \circ F^{l.a.}(F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)})) \circ F^{l.a.}(c_{(C'\otimes C)\overline{\otimes} F(M)})\circ$$
$$F^{l.a.}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)} \circ (((\Psi^{(C'\otimes C)\overline{\otimes} F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes} F(M)}))^{-1}_{C'\otimes C}$$
$$(id_{(C'\otimes C)\overline{\otimes} F(M)}))\overline{\otimes} id_{F(M)})) \circ d^{-1}_{C'\otimes C,F(M)})$$
$$\overset{(r)}{=} (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(e_{M'} \circ F^{l.a.}(F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)})) \circ F^{l.a.}(c_{(C'\otimes C)\overline{\otimes} F(M)})\circ$$
$$F^{l.a.}(id_{(C'\otimes C)\overline{\otimes} F(M)}) \circ d^{-1}_{C'\otimes C,F(M)})$$
$$\overset{(s)}{=} (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)} \circ e_{F^{l.a.}((C'\otimes C)\overline{\otimes} F(M))}\circ$$
$$F^{l.a.}(c_{(C'\otimes C)\overline{\otimes} F(M)}) \circ d^{-1}_{C'\otimes C,F(M)})$$
$$\overset{(t)}{=} (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)} \circ id_{F^{l.a.}((C'\otimes C)\overline{\otimes} F(M))}\circ$$
$$d^{-1}_{C'\otimes C,F(M)})$$
$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e^{-1}_{M}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))}) = \nu.$$

The naturalities of $\Psi^{F(M')}(ev_{F(M),F(M')})$ and m are used in the equalities labeled with (i) and (j), respectively. The equalities labeled with (k) are valid by equation (35), i.e.,

$$ev_{F(M),F(M')} \circ (\xi_{F(M),M'}\overline{\otimes} id_{F(M)}) = \phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M')\overline{\otimes} F(M),M'}(ev_{F^{l.a.}(F(M)),M'} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M'),F(M)}),$$

and

$$ev_{F(M),F(M)} \circ (\xi_{F(M),M}\overline{\otimes} id_{F(M)}) = \phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M)\overline{\otimes} F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M),F(M)}).$$

The equality (l) holds by using the equation (3) while noticing that $e_M = \phi^{-1}_{F(M),M}(id_{F(M)})$: $(F^{l.a.} \circ F)(M) \to M$ is an isomorphism (in $\mathcal{M}$), i.e.,

$$e_M \circ F^{l.a.}(\phi_{C\overline{\otimes} F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{C,F(M)})) = \phi^{-1}_{C\overline{\otimes} F(M),M}(\phi_{C\overline{\otimes} F(M),M}(ev_{F^{l.a.}(F(M)),M} \circ d_{C,F(M)})).$$

Using the definition of the morphism $\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')$ in $\mathcal{C}$ we get (m), and via the pentagon diagram of the $\mathcal{C}$-module functor $(F^{r.a.}, d)$[8] we get (n).

The functor $\_\,\overline{\otimes}F(M) : \mathcal{C} \to \mathcal{N}$ is left adjoint to $\underline{Hom}_{\mathcal{N}}(F(M), \_\,) : \mathcal{N} \to \mathcal{C}$, and the natural isomorphism is given by the family

$$\{(\Psi^N(ev_{F(M),N}))_X^{-1} : Hom_{\mathcal{N}}(X\overline{\otimes}F(M), N) \to Hom_{\mathcal{C}}(X, \underline{Hom}_{\mathcal{N}}(F(M), N))\}_{(X,N)\in\mathcal{C}^{op}\times\mathcal{N}}$$

(see Proposition 4.2.1). By fixing $X = C' \otimes C \in \mathcal{C}$ we obtain the naturality of

$$\{(\Psi^N(ev_{F(M),N}))_{C'\otimes C}^{-1} : Hom_{\mathcal{N}}((C'\otimes C)\overline{\otimes}F(M), N) \to Hom_{\mathcal{C}}(C'\otimes C, \underline{Hom}_{\mathcal{N}}(F(M), N))\}_{N\in\mathcal{N}}$$

which implies the equality (o).

The equalities labeled with (p) and (q) hold by the naturality of $\Psi^{M'}(ev_{F^{l.a}(F(M)),M'})$ and the definition of the morphism

$$\underline{Hom}_{\mathcal{N}}(F(M), F(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes}id_M) \circ (id_{C'}\overline{\otimes}ev_{F^{l.a.}(F(M)),M})\circ$$
$$m_{C',C,F^{l.a.}(F(M))} \circ d_{C'\otimes C,F(M)}) \circ c_{(C'\otimes C)\overline{\otimes}F(M)}),$$

respectively.

The equality (r) is slight similar to (g), i.e.,

$$\begin{aligned} id_{(C'\otimes C)\overline{\otimes}F(M)} &= \Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)})_{C'\otimes C} \\ &\quad ((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))_{C'\otimes C}^{-1}(id_{(C'\otimes C)\overline{\otimes}F(M)})) \\ &= ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)} \circ (((\Psi^{(C'\otimes C)\overline{\otimes}F(M)}(ev_{F(M),(C'\otimes C)\overline{\otimes}F(M)}))_{C'\otimes C}^{-1}(id_{(C'\otimes C)\overline{\otimes}F(M)}))\overline{\otimes}id_{F(M)}). \end{aligned}$$

At last, (s) is valid by the naturality of the unit e and (t) is via the equality

$$id_{F^{l.a.}((C'\otimes C)\overline{\otimes}F(M))} = e_{F^{l.a.}((C'\otimes C)\overline{\otimes}F(M))} \circ F^{l.a.}(c_{(C'\otimes C)\overline{\otimes}F(M)})$$

(see Proposition 1.3.8). Next, the second rectangle is commutative because

$$\begin{aligned} &\underline{Hom}_{\mathcal{M}}(e_M, M') \circ \mu_{M,M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)) \\ &= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{\underline{Hom}_{\mathcal{M}}(M,M')\otimes\underline{Hom}_{\mathcal{M}}(M,M)}(ev_{M,M'}\circ \\ &\quad (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}ev_{M,M}) \circ m_{\underline{Hom}_{\mathcal{M}}(M,M'),\underline{Hom}_{\mathcal{M}}(M,M),M})\circ \\ &\quad (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)) \\ &= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{\underline{Hom}_{\mathcal{M}}(M,M')\otimes A}(ev_{M,M'} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}ev_{M,M})\circ \\ &\quad m_{\underline{Hom}_{\mathcal{M}}(M,M'),A,M}) \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)) \\ &= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (Hom_{\mathcal{C}}(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M), \underline{Hom}_{\mathcal{M}}(M, M'))\circ \\ &\quad (\Psi^{M'}(ev_{M,M'}))^{-1}_{\underline{Hom}_{\mathcal{M}}(M,M')\otimes A})(ev_{M,M'} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}ev_{M,M}) \circ m_{\underline{Hom}_{\mathcal{M}}(M,M'),A,M}) \\ &\overset{(u)}{=} \underline{Hom}_{\mathcal{M}}(e_M, M') \circ ((\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}\circ \\ &\quad Hom_{\mathcal{M}}((\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M))\overline{\otimes}M, M'))(ev_{M,M'}\circ \\ &\quad (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}ev_{M,M}) \circ m_{\underline{Hom}_{\mathcal{M}}(M,M'),A,M}) \end{aligned}$$

8 Definition 2.2.6.

$$= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}$$
$$(Hom_{\mathcal{M}}((\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M))\overline{\otimes} M, M')(ev_{M,M'}\circ$$
$$(id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes} ev_{M,M}) \circ m_{\underline{Hom}_{\mathcal{M}}(M,M'),A,M}))$$
$$= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes} ev_{M,M})\circ$$
$$m_{\underline{Hom}_{\mathcal{M}}(M,M'),A,M} \circ ((\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \otimes \underline{Hom}_{\mathcal{M}}(e_M^{-1}, M))\overline{\otimes} id_M))$$
$$\overset{(j)}{=} \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes} ev_{M,M})\circ$$
$$(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes}(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)\overline{\otimes} id_M)) \circ m_{C',C,M})$$
$$= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} ev_{M,M})\circ$$
$$(id_{C'}\overline{\otimes}(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)\overline{\otimes} id_M)) \circ m_{C',C,M})$$
$$= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes}(ev_{M,M} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)\overline{\otimes} id_M))) \circ m_{C',C,M})$$
$$\overset{(v)}{=} \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes}(ev_{F^{l.a.}(F(M)),M} \circ (id_C\overline{\otimes} e_M^{-1}))) \circ m_{C',C,M})$$
$$= \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ (id_{C'}\overline{\otimes}(id_C\overline{\otimes} e_M^{-1})) \circ m_{C',C,M})$$
$$\overset{(j)}{=} \underline{Hom}_{\mathcal{M}}(e_M, M') \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ ((id_{C'} \otimes id_C)\overline{\otimes} e_M^{-1}))$$
$$= (Hom_{\mathcal{C}}(C' \otimes C, hi_{\mathcal{M}}(e_M, M')) \circ (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C})(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ ((id_{C'} \otimes id_C)\overline{\otimes} e_M^{-1}))$$
$$= (Hom_{\mathcal{C}}(C' \otimes C, hi_{\mathcal{M}}(e_M, M')) \circ \gamma'_M)(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ ((id_{C'} \otimes id_C)\overline{\otimes} e_M^{-1}))$$
$$\overset{(w)}{=} (\gamma'_{F^{l.a}(F(M))} \circ Hom_{\mathcal{M}}((C' \otimes C)\overline{\otimes} e_M, M'))(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M)\circ$$
$$(id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ ((id_{C'} \otimes id_C)\overline{\otimes} e_M^{-1}))$$
$$= ((\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C} \circ Hom_{\mathcal{M}}((C' \otimes C)\overline{\otimes} e_M, M'))(ev_{M,M'}\circ$$
$$(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ ((id_{C'} \otimes id_C)\overline{\otimes} e_M^{-1}))$$
$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(Hom_{\mathcal{M}}((C' \otimes C)\overline{\otimes} e_M, M')(ev_{M,M'}\circ$$
$$(\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M}) \circ m_{C',C,F^{l.a.}(F(M))} \circ (id_{C'\otimes C}\overline{\otimes} e_M^{-1})))$$
$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M})\circ$$
$$m_{C',C,F^{l.a.}(F(M))} \circ (id_{C'\otimes C}\overline{\otimes} e_M^{-1}) \circ (id_{C'\otimes C}\overline{\otimes} e_M))$$
$$= (\Psi^{M'}(ev_{F^{l.a.}(F(M)),M'}))^{-1}_{C'\otimes C}(ev_{M,M'} \circ (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M')\overline{\otimes} id_M) \circ (id_{C'}\overline{\otimes} ev_{F^{l.a.}(F(M)),M})\circ$$
$$m_{C',C,F^{l.a.}(F(M))}) = \nu,$$

where the equality (u) holds by the naturality of $\Psi^{M'}(ev_{M,M'})$ and (v) is via the definition of the morphism $\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M)$. From the definition of the natural isomorphism

$$\gamma' = \{\gamma'_M := (\Psi^{M'}(ev_{M,M'}))^{-1}_{C'\otimes C} : Hom_{\mathcal{M}}((C'\otimes C)\overline{\otimes}M, M') \to Hom_{\mathcal{C}}(C'\otimes C, \underline{Hom}_{\mathcal{M}}(M, M'))\}_{M\in\mathcal{M}^{op}}$$

(see Lemma 6.1) it follows the equality (w).

Therefore, the diagram

$$\begin{array}{ccccc}
\underline{Hom}_{\mathcal{N}}(F(M), F(M')) \otimes B & \xrightarrow{\xi^{-1}_{F(M),M'}\otimes\xi^{-1}_{F(M),M}} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') \otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M) & \xrightarrow{\underline{Hom}_{\mathcal{M}}(e_M^{-1},M')\otimes\underline{Hom}_{\mathcal{M}}(e_M^{-1},M)} & \underline{Hom}_{\mathcal{M}}(M, M') \otimes A \\
\downarrow{\scriptstyle \mu_{F(M),F(M),F(M')}} & & & & \downarrow{\scriptstyle \mu_{M,M,M'}} \\
\underline{Hom}_{\mathcal{N}}(F(M), F(M')) & \xleftarrow[\xi_{F(M),M'}]{} & \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') & \xleftarrow[\underline{Hom}_{\mathcal{M}}(e_M,M')]{} & \underline{Hom}_{\mathcal{M}}(M, M')
\end{array}$$

commutes, that is,

$$\begin{aligned}
\mu_{F(M),F(M),F(M')} = \xi_{F(M),M'} \circ \underline{Hom}_{\mathcal{M}}(e_M, M') \circ \mu_{M,M,M'} \circ \\
((\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M') \circ \xi^{-1}_{F(M),M'}) \otimes (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M) \circ \xi^{-1}_{F(M),M}))
\end{aligned}$$

and particularly, for $M' = M$ we get

$$\begin{aligned}
\mu_{F(M),F(M),F(M)} &= \xi_{F(M),M} \circ \underline{Hom}_{\mathcal{M}}(e_M, M) \circ \mu_{M,M,M} \circ \\
&\quad ((\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M) \circ \xi^{-1}_{F(M),M}) \otimes (\underline{Hom}_{\mathcal{M}}(e_M^{-1}, M) \circ \xi^{-1}_{F(M),M})) \\
&= \beta_M \circ \mu \circ (\beta_M^{-1} \otimes \beta_M^{-1})
\end{aligned}$$

as wanted. Hence, the algebra structure (multiplication and unit) of $\underline{Hom}_{\mathcal{N}}(F(M), F(M))$ given in terms of the algebra $\underline{Hom}_{\mathcal{M}}(M, M)$ is the same as the one defined in Section 5.2, that is,

$$\beta_M \circ \mu \circ (\beta_M^{-1} \otimes \beta_M^{-1}) = \mu_{F(M),F(M),F(M)}$$

and

$$\beta_M \circ u = (\Psi^{F(M)}(ev_{F(M),F(M)}))^{-1}_{\mathbf{1}}(I_{F(M)}).$$

Next, by Theorem 5.4.1, $\mathcal{N}$ is equivalent to $\mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))}$ as $\mathcal{C}$-module categories, and the equivalence is given by

$$\begin{aligned}
F_2 : \mathcal{N} &\longrightarrow \mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))} \\
N' &\longmapsto F_2(N') = (\underline{Hom}_{\mathcal{N}}(F(M), N'), \rho_{\underline{Hom}_{\mathcal{N}}(F(M),N')}) \\
h &\longmapsto F_2(h) = \underline{Hom}_{\mathcal{N}}(F(M), h).
\end{aligned}$$

Notice that $F(M) \neq 0$ because otherwise $0 \neq M \cong F^{l.a.}(F(M)) \cong 0$ leading to a contradiction[9].

[9] In the second isomorphism we are using the additive property of the functor $F^{l.a.}$ (see Remark 1.1.20).

Therefore, the composition $F_2^{-1} \circ G \circ F_1 : \mathcal{M} \to \mathcal{N}$ is an equivalence of categories which admits a structure of $\mathcal{C}$-module functor[10] and, hence, $\mathcal{M}$ and $\mathcal{N}$ are equivalent as $\mathcal{C}$-module categories.

The other case (in which $c_N = \phi_{N,F^{l.a.}(N)}(id_{F^{l.a.}(N)}) : N \to (F \circ F^{l.a.})(N)$ is an isomorphism in $\mathcal{N}$ for some $0 \neq N \in \mathcal{N}$) follows analogously. ■

As we could see, the equivalence between the $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ of the previous theorem is given by the $\mathcal{C}$-module functor composition $F_2^{-1} \circ G \circ F_1 : \mathcal{M} \to \mathcal{N}$, and not necessarily by F. The next corollary shows that there is a natural isomorphism of $\mathcal{C}$-module functors from $F_2^{-1} \circ G \circ F_1$ to F implying that F is also an equivalence of $\mathcal{C}$-module categories.

Before beginning, let us consider some morphisms we are going to use. The $\mathcal{C}$-module functor structure d of $F^{l.a.} : \mathcal{N} \to \mathcal{M}$ is given by equation (19) as

$$d = \{d_{X,N} := e_{X \overline{\otimes} F^{l.a.}(N)} \circ F^{l.a.}(b^{-1}_{X,F^{l.a.}(N)}) \circ F^{l.a.}(id_X \overline{\otimes} c_N)\}_{(X,N) \in \mathcal{C}^{op} \times \mathcal{N}}.$$

The functor $\_\,\overline{\otimes} M : \mathcal{C} \to \mathcal{M}$ is left adjoint to $\underline{Hom}_{\mathcal{M}}(M, \_\,) : \mathcal{M} \to \mathcal{C}$ (see Proposition 4.2.1) and the natural isomorphism of this adjunction is given by the family

$$\{(\Psi^{M'}(ev_{M,M'}))^{-1}_X : Hom_{\mathcal{M}}(X \overline{\otimes} M, M') \to Hom_{\mathcal{C}}(X, \underline{Hom}_{\mathcal{M}}(M, M'))\}_{(X,M') \in \mathcal{C}^{op} \times \mathcal{M}}.$$

So via Proposition 1.3.8, the counit $\overline{e}$ and unit $\overline{c}$ of this adjunction are

$$\overline{e} = \{\overline{e}_{M'} = \Psi^{M'}(ev_{M,M'})_{\underline{Hom}_{\mathcal{M}}(M,M')}(id_{\underline{Hom}_{\mathcal{M}}(M,M')}) = ev_{M,M'}\}_{M' \in \mathcal{M}}, \quad \text{and}$$
$$\overline{c} = \{\overline{c}_X = (\Psi^{X \overline{\otimes} M}(ev_{M,X \overline{\otimes} M}))^{-1}_X (id_{X \overline{\otimes} M})\}_{X \in \mathcal{C}},$$

respectively. The $\mathcal{C}$-module functor structure d' of $F_1 : \mathcal{M} \to \mathcal{C}_{\underline{Hom}_{\mathcal{M}}(M,M)}$ is given by the $\mathcal{C}$-module structure of $\underline{Hom}_{\mathcal{M}}(M, \_\,) : \mathcal{M} \to \mathcal{C}$ as we could see in Proposition 5.3.2, and its inverse[11]

$$d'^{-1} = \{d'^{-1}_{X,M'} = \underline{Hom}_{\mathcal{M}}(M, id_X \overline{\otimes} ev_{M,M'}) \circ \underline{Hom}_{\mathcal{M}}(M, m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}) \circ \overline{c}_{X \otimes \underline{Hom}_{\mathcal{M}}(M,M')}\}_{(X,M') \in \mathcal{C} \times \mathcal{M}}$$

comes from equation (24).

Notice that

$$d'^{-1}_{X,M'} = (\Psi^{X \overline{\otimes} M'}(ev_{M,X \overline{\otimes} M'}))^{-1}_{X \otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X \overline{\otimes} ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}).$$

Indeed,

$$\begin{aligned} d'^{-1}_{X,M'} &= \underline{Hom}_{\mathcal{M}}(M, id_X \overline{\otimes} ev_{M,M'}) \circ \underline{Hom}_{\mathcal{M}}(M, m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}) \circ \overline{c}_{X \otimes \underline{Hom}_{\mathcal{M}}(M,M')} \\ &= \underline{Hom}_{\mathcal{M}}(M, (id_X \overline{\otimes} ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}) \circ \\ &\quad (\Psi^{(X \otimes \underline{Hom}_{\mathcal{M}}(M,M')) \overline{\otimes} M}(ev_{M,(X \otimes \underline{Hom}_{\mathcal{M}}(M,M')) \overline{\otimes} M}))^{-1}_{X \otimes \underline{Hom}_{\mathcal{M}}(M,M')} \\ &\quad (id_{(X \otimes \underline{Hom}_{\mathcal{M}}(M,M')) \overline{\otimes} M}) \end{aligned}$$

[10] By Proposition 2.2.10.

[11] We'll use the inverse because it is much simpler that d'.

$$= (Hom_{\mathcal{C}}(X \otimes \underline{Hom}_{\mathcal{M}}(M, M'), \underline{Hom}_{\mathcal{M}}(M, (id_X \overline{\otimes} ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M})) \circ$$
$$(\Psi^{(X\otimes \underline{Hom}_{\mathcal{M}}(M,M'))\overline{\otimes} M}(ev_{M,(X\otimes \underline{Hom}_{\mathcal{M}}(M,M'))\overline{\otimes} M}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')})$$
$$(id_{(X\otimes \underline{Hom}_{\mathcal{M}}(M,M'))\overline{\otimes} M})$$
$$= ((\Psi^{X\overline{\otimes} M'}(ev_{M,X\overline{\otimes} M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')} \circ Hom_{\mathcal{M}}((X \otimes \underline{Hom}_{\mathcal{M}}(M, M'))\overline{\otimes} M,$$
$$(id_X \overline{\otimes} ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}))(id_{(X\otimes \underline{Hom}_{\mathcal{M}}(M,M'))\overline{\otimes} M})$$
$$= (\Psi^{X\overline{\otimes} M'}(ev_{M,X\overline{\otimes} M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}(Hom_{\mathcal{M}}((X \otimes \underline{Hom}_{\mathcal{M}}(M, M'))\overline{\otimes} M,$$
$$(id_X \overline{\otimes} ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M})(id_{(X\otimes \underline{Hom}_{\mathcal{M}}(M,M'))\overline{\otimes} M}))$$
$$= (\Psi^{X\overline{\otimes} M'}(ev_{M,X\overline{\otimes} M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X \overline{\otimes} ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M})$$

in which we use the natural isomorphism of the previous adjunction in the fourth equality.

Similarly, the functor $\_\,\overline{\otimes} F(M) : \mathcal{C} \to \mathcal{N}$ is left adjoint to $\underline{Hom}_{\mathcal{N}}(F(M), \_\,) : \mathcal{N} \to \mathcal{C}$, so the $\mathcal{C}$-module functor structure d'' of $F_2 : \mathcal{N} \to \mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))}$ is given by the $\mathcal{C}$-module structure of $\underline{Hom}_{\mathcal{N}}(F(M), \_\,) : \mathcal{N} \to \mathcal{C}$ and it has inverse defined as

$$d''^{-1} = \{d''^{-1}_{X,N} = (\Psi^{X\overline{\otimes} N}(ev_{F(M),X\overline{\otimes} N}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{N}}(F(M),N)}((id_X \overline{\otimes} ev_{F(M),N}) \circ$$
$$m_{X,\underline{Hom}_{\mathcal{N}}(F(M),N),F(M)})\}_{(X,N)\in \mathcal{C}\times\mathcal{N}}.$$

Finally, the functor $\_\,\overline{\otimes} F^{l.a.}(F(M)) : \mathcal{C} \to \mathcal{M}$ is left adjoint to $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), \_\,) : \mathcal{M} \to \mathcal{C}$, thus the $\mathcal{C}$-module functor structure $\widetilde{d}$ of $\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), \_\,) : \mathcal{M} \to \mathcal{C}$ has inverse

$$\widetilde{d}^{-1} = \{\widetilde{d}^{-1}_{X,M'} = (\Psi^{X\overline{\otimes} M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes} M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M')}((id_X \overline{\otimes} ev_{F^{l.a.}(F(M)),M'}) \circ$$
$$m_{X,\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),M'),F^{l.a.}(F(M))})\}_{(X,M')\in \mathcal{C}\times\mathcal{M}}.$$

Corollary 6.4. *If there is an object $0 \neq M \in \mathcal{M}$ (respectively, $0 \neq N \in \mathcal{N}$) and a $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{N}$ admitting an adjunction $(F^{l.a.}, F, \phi)$ such that*

$$e_M : (F^{l.a.} \circ F)(M) \to M$$

(respectively, $c_N : N \to (F \circ F^{l.a.})(N)$) is an isomorphism then $F : \mathcal{M} \to \mathcal{N}$ (then $F^{l.a.} : \mathcal{N} \to \mathcal{M}$) is an equivalence of categories.

Proof. Before we begin, set $A = \underline{Hom}_{\mathcal{M}}(M, M)$, $B = \underline{Hom}_{\mathcal{N}}(F(M), F(M))$, $C = \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M)$ and $C' = \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M')$ as we've done previously. This proof is done by showing that there is a natural isomorphism of $\mathcal{C}$-module functors between $F_2^{-1} \circ G \circ F_1$ and F, that is, $F_2^{-1} \circ G \circ F_1$ and F are equivalent as $\mathcal{C}$-module functors. We begin by constructing a natural isomorphism of $\mathcal{C}$-module functors from

$$G \circ F_1 : \mathcal{M} \to \mathcal{C}_{\underline{Hom}_{\mathcal{M}}(M,M)} \to \mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))}$$

to

$$F_2 \circ F : \mathcal{M} \to \mathcal{N} \to \mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))}.$$

Affirmation 1: $\beta = \{\beta_{M'} = \xi_{F(M),M'} \circ \underline{Hom}_{\mathcal{M}}(e_M, M') : \underline{Hom}_{\mathcal{M}}(M, M') \to \underline{Hom}_{\mathcal{N}}(F(M), F(M'))\}_{M' \in \mathcal{M}}$ is a natural isomorphism between the functors $\underline{Hom}_{\mathcal{M}}(M, \_\,)$ and $\underline{Hom}_{\mathcal{N}}(F(M), \_\,) \circ F$.

In fact, let $f : M' \to M''$ be a morphism in $\mathcal{M}$ and notice that

$$\begin{aligned}
\underline{Hom}_{\mathcal{N}}(F(M), F(f)) \circ \beta_{M'} &= \underline{Hom}_{\mathcal{N}}(F(M), F(f)) \circ \xi_{F(M),M'} \circ \underline{Hom}_{\mathcal{M}}(e_M, M') \\
&= \xi_{F(M),M''} \circ \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), f) \circ \underline{Hom}_{\mathcal{M}}(e_M, M') \\
&= \xi_{F(M),M''} \circ \underline{Hom}_{\mathcal{M}}(e_M, M'') \circ \underline{Hom}_{\mathcal{M}}(M, f) \\
&= \beta_{M''} \circ \underline{Hom}_{\mathcal{M}}(M, f)
\end{aligned}$$

where the naturality of ξ is used in the second equality and the fact that $\underline{Hom}_{\mathcal{M}}(\_\,,\,\_\,)$ is a bifunctor in the third.

We may as well notice that for every $M' \in \mathcal{M}$,

$$\begin{aligned}
(G \circ F_1)(M') = G(F_1(M')) &= G(\underline{Hom}_{\mathcal{M}}(M, M'), \rho_{\underline{Hom}_{\mathcal{M}}(M,M')}) \\
&= (\underline{Hom}_{\mathcal{M}}(M, M'), \rho'_{\underline{Hom}_{\mathcal{M}}(M,M')}) \\
&= (\underline{Hom}_{\mathcal{M}}(M, M'), \rho_{\underline{Hom}_{\mathcal{M}}(M,M')} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')} \otimes \beta_M^{-1}))
\end{aligned}$$

and

$$(F_2 \circ F)(M') = F_2(F(M')) = (\underline{Hom}_{\mathcal{N}}(F(M), F(M')), \rho_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}).$$

Affirmation 2: For any $M' \in \mathcal{M}$, $\beta_{M'}$ is a morphism in $\mathcal{C}_{\underline{Hom}_{\mathcal{N}}(F(M),F(M))}$ from $(G \circ F_1)(M')$ to $(F_2 \circ F)(M')$.

This fact can be verified via the commutativity of the diagram

$$\begin{array}{ccc}
\underline{Hom}_{\mathcal{M}}(M, M') \otimes B & \xrightarrow{\beta_{M'} \otimes id_B} & \underline{Hom}_{\mathcal{N}}(F(M), F(M')) \otimes B \\
{\scriptstyle \rho_{\underline{Hom}_{\mathcal{M}}(M,M')} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')} \otimes \beta_M^{-1})} \downarrow & & \downarrow {\scriptstyle \rho_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}} \\
\underline{Hom}_{\mathcal{M}}(M, M') & \xrightarrow[\beta_{M'}]{} & \underline{Hom}_{\mathcal{N}}(F(M), F(M')).
\end{array}$$

It commutes given that

$$\begin{aligned}
&\beta_{M'} \circ \rho_{\underline{Hom}_{\mathcal{M}}(M,M')} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')} \otimes \beta_M^{-1}) \circ (\beta_{M'}^{-1} \otimes id_B) \\
&= \beta_{M'} \circ \rho_{\underline{Hom}_{\mathcal{M}}(M,M')} \circ (\beta_{M'}^{-1} \otimes \beta_M^{-1}) \\
&= \beta_{M'} \circ \mu_{M,M,M'} \circ (\beta_{M'}^{-1} \otimes \beta_M^{-1}) \\
&\overset{(36)}{=} \mu_{F(M),F(M),F(M')} \\
&= \rho_{\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}.
\end{aligned}$$

Affirmation 3: $\beta = \{\beta_{M'}\}_{M' \in \mathcal{M}}$ is a natural isomorphism of $\mathcal{C}$-module functors from $G \circ F_1$ to $F_2 \circ F$.

The $\mathcal{C}$-module functor structure of a composition of $\mathcal{C}$-module functors is given by Proposition 2.2.10. For the composition $G \circ F_1$ it is

$$\{id_{X,\underline{Hom}_{\mathcal{M}}(M,M')} \circ G(d'_{X,M'}) = d'_{X,M'}\}_{(X,M') \in \mathcal{C} \times \mathcal{M}},$$

and for $F_2 \circ F$, it is

$$\{d''_{X,F(M')} \circ F_2(b_{X,M'}) = d''_{X,F(M')} \circ \underline{Hom}_{\mathcal{N}}(F(M), b_{X,M'})\}_{(X,M') \in \mathcal{C} \times \mathcal{M}}.$$

We know that β is already a natural isomorphism from the functor $G \circ F_1$ to $F_2 \circ F$ by the Affirmations 1 and 2. It remains to prove that this is a natural transformation of $\mathcal{C}$-module functors, i.e., the commutativity of the diagram[12]

$$\begin{tikzcd}
\underline{Hom}_{\mathcal{M}}(M, X\overline{\otimes}M') \arrow[r, "\beta_{X\overline{\otimes}M'}"] \arrow[d, "d'_{X,M'}"'] & \underline{Hom}_{\mathcal{N}}(F(M), F(X\overline{\otimes}M')) \arrow[d, "d''_{X,F(M')} \circ \underline{Hom}_{\mathcal{N}}(F(M), b_{X,M'})"] \\
X \otimes \underline{Hom}_{\mathcal{M}}(M, M') \arrow[r, "id_X \otimes \beta_{M'}"'] & X \otimes \underline{Hom}_{\mathcal{N}}(F(M), F(M')).
\end{tikzcd}$$

To make this easier, let us check that this equivalent diagram

$$\begin{tikzcd}
\underline{Hom}_{\mathcal{M}}(M, X\overline{\otimes}M') \arrow[rr, "\beta_{X\overline{\otimes}M'}"] \arrow[dr, "\underline{Hom}_{\mathcal{M}}(e_M, X\overline{\otimes}M')"] & & \underline{Hom}_{\mathcal{N}}(F(M), F(X\overline{\otimes}M')) \arrow[dd, "\underline{Hom}_{\mathcal{N}}(F(M), b_{X,M'})"] \\
& \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), X\overline{\otimes}M') \arrow[ur, "\xi_{F(M), X\overline{\otimes}M'}"] & \\
& & \underline{Hom}_{\mathcal{N}}(F(M), X\overline{\otimes}F(M')) \\
& X \otimes \underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)), M') \arrow[uu, "\tilde{d}^{-1}_{X,M'}"] \arrow[dr, "id_X \otimes \xi_{F(M),M'}"] & \\
X \otimes \underline{Hom}_{\mathcal{M}}(M, M') \arrow[uuuu, "d'^{-1}_{X,M'}"] \arrow[ur, "id_X \otimes \underline{Hom}_{\mathcal{M}}(e_M, M')"] \arrow[rr, "id_X \otimes \beta_{M'}"'] & & X \otimes \underline{Hom}_{\mathcal{N}}(F(M), F(M')) \arrow[uu, "d''^{-1}_{X,F(M')}"']
\end{tikzcd}$$

commutes by commuting its smaller diagrams.

The triangles on the top and bottom of this diagram commute simply via the

[12] See Definition 2.2.9.

definition of β. The diagram on the left commutes since

$$
\begin{aligned}
&\widetilde{d}^{-1}_{X,M'} \circ (id_X \otimes \underline{Hom}_{\mathcal{M}}(e_M, M')) \\
&= (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}ev_{F^{l.a.}(F(M)),M'}) \circ m_{X,C',F^{l.a.}(F(M))})\circ \\
&\quad (id_X \otimes \underline{Hom}_{\mathcal{M}}(e_M, M')) \\
&\overset{(a)}{=} (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}ev_{F^{l.a.}(F(M)),M'})\circ \\
&\quad m_{X,C',F^{l.a.}(F(M))} \circ ((id_X \otimes \underline{Hom}_{\mathcal{M}}(e_M, M'))\overline{\otimes}id_{F^{l.a.}(F(M))})) \\
&\overset{(b)}{=} (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}ev_{F^{l.a.}(F(M)),M'})\circ \\
&\quad (id_X\overline{\otimes}(\underline{Hom}_{\mathcal{M}}(e_M, M')\overline{\otimes}id_{F^{l.a.}(F(M))})) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),F^{l.a.}(F(M))}) \\
&= (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}(ev_{F^{l.a.}(F(M)),M'}\circ \\
&\quad (\underline{Hom}_{\mathcal{M}}(e_M, M')\overline{\otimes}id_{F^{l.a.}(F(M))}))) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),F^{l.a.}(F(M))}) \\
&\overset{(c)}{=} (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}(ev_{M,M'} \circ (id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}e_M)))\circ \\
&\quad m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),F^{l.a.}(F(M))}) \\
&= (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}ev_{M,M'}) \circ (id_X\overline{\otimes}(id_{\underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}e_M))\circ \\
&\quad m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),F^{l.a.}(F(M))}) \\
&\overset{(b)}{=} (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}\circ \\
&\quad ((id_X \otimes id_{\underline{Hom}_{\mathcal{M}}(M,M')})\overline{\otimes}e_M)) \\
&= (\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}ev_{M,M'}) \circ m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}\circ \\
&\quad (id_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}\overline{\otimes}e_M)) \\
&\overset{(d)}{=} \underline{Hom}_{\mathcal{M}}(e_M, X\overline{\otimes}M') \circ (\Psi^{X\overline{\otimes}M'}(ev_{M,X\overline{\otimes}M'}))^{-1}_{X\otimes \underline{Hom}_{\mathcal{M}}(M,M')}((id_X\overline{\otimes}ev_{M,M'})\circ \\
&\quad m_{X,\underline{Hom}_{\mathcal{M}}(M,M'),M}) \\
&= \underline{Hom}_{\mathcal{M}}(e_M, X\overline{\otimes}M') \circ d'^{-1}_{X,M'}
\end{aligned}
$$

where the equalities labeled with (a) and (b) hold by the naturality of $\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'})$ and m, respectively. The equality (c) is valid by the definition of the morphism $\underline{Hom}(e_M, M')$, and (d) is by the naturality coming from Lemma 6.1.

And for the diagram on the right side notice that

$$
\begin{aligned}
&\underline{Hom}_{\mathcal{N}}(F(M), b_{X,M'}) \circ \xi_{F(M),X\overline{\otimes}M'} \circ \widetilde{d}^{-1}_{X,M'} \\
&= \underline{Hom}_{\mathcal{N}}(F(M), b_{X,M'}) \circ (\Psi^{F(X\overline{\otimes}M')}(ev_{F(M),F(X\overline{\otimes}M')}))^{-1}_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M')} \\
&\quad (\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M')\overline{\otimes}F(M),X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M'),F(M)}))\circ \\
&\quad \widetilde{d}^{-1}_{X,M'}
\end{aligned}
$$

$$\stackrel{(e)}{=} \underline{Hom}_{\mathcal{N}}(F(M), b_{X,M'}) \circ (\Psi^{F(X\overline{\otimes}M')}(ev_{F(M),F(X\overline{\otimes}M')}))^{-1}_{X\otimes C'}$$
$$(\phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M')\overline{\otimes}F(M),X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M'),F(M)})\circ$$
$$(\widetilde{d}^{-1}_{X,M'}\overline{\otimes} id_{F(M)}))$$

$$\stackrel{(f)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M')\overline{\otimes}F(M),X\overline{\otimes}M'}$$
$$(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'} \circ d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M'),F(M)}) \circ (\widetilde{d}^{-1}_{X,M'}\overline{\otimes} id_{F(M)}))$$

$$\stackrel{(g)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}\circ$$
$$d_{\underline{Hom}_{\mathcal{M}}(F^{l.a.}(F(M)),X\overline{\otimes}M'),F(M)} \circ F^{l.a.}(\widetilde{d}^{-1}_{X,M'}\overline{\otimes} id_{F(M)})))$$

$$\stackrel{(h)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}\circ$$
$$(\widetilde{d}^{-1}_{X,M'}\overline{\otimes} id_{F^{l.a.}(F(M))}) \circ d_{X\otimes C',F(M)}))$$

$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}$$
$$(\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'})_{X\otimes C'}(\widetilde{d}^{-1}_{X,M'}) \circ d_{X\otimes C',F(M)}))$$

$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}$$
$$(\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'})_{X\otimes C'}((\Psi^{X\overline{\otimes}M'}(ev_{F^{l.a.}(F(M)),X\overline{\otimes}M'}))^{-1}_{X\otimes C'}$$
$$((id_X\overline{\otimes} ev_{F^{l.a.}(F(M)),M'}) \circ m_{X,C',F^{l.a.}(F(M))})) \circ d_{X\otimes C',F(M)}))$$

$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}$$
$$((id_X\overline{\otimes} ev_{F^{l.a.}(F(M)),M'}) \circ m_{X,C',F^{l.a.}(F(M))} \circ d_{X\otimes C',F(M)}))$$

$$\stackrel{(i)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}$$
$$((id_X\overline{\otimes} ev_{F^{l.a.}(F(M)),M'}) \circ (id_X\overline{\otimes} d_{C',F(M)}) \circ d_{X,C'\otimes F(M)} \circ F^{l.a.}(m_{X,C',F(M)})))$$

$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{(X\otimes C')\overline{\otimes}F(M),X\overline{\otimes}M'}$$
$$((id_X\overline{\otimes}(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)})) \circ d_{X,C'\otimes F(M)} \circ F^{l.a.}(m_{X,C',F(M)})))$$

$$\stackrel{(g)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ \phi_{X\overline{\otimes}(C'\overline{\otimes}F(M)),X\overline{\otimes}M'}$$
$$((id_X\overline{\otimes}(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)})) \circ d_{X,C'\otimes F(M)}) \circ m_{X,C',F(M)})$$

$$\stackrel{(2)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}(b_{X,M'} \circ F(id_X\overline{\otimes}(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)}))\circ$$
$$F(d_{X,C'\otimes F(M)}) \circ c_{X\overline{\otimes}(C'\overline{\otimes}F(M))} \circ m_{X,C',F(M)})$$

$$\stackrel{(j)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))} \circ F(d_{X,C'\otimes F(M)}) \circ c_{X\overline{\otimes}(C'\overline{\otimes}F(M))} \circ m_{X,C',F(M)})$$

$$\stackrel{(k)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'} \circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))} \circ F(e_{X\overline{\otimes}F^{l.a.}(C'\overline{\otimes}F(M))} \circ F^{l.a.}(b^{-1}_{X,F^{l.a.}(C'\overline{\otimes}F(M))})\circ$$
$$F^{l.a.}(id_X\overline{\otimes} c_{C'\overline{\otimes}F(M)})) \circ c_{X\overline{\otimes}(C'\overline{\otimes}F(M))} \circ m_{X,C',F(M)})$$

$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ F(e_{X\overline{\otimes}F^{l.a.}(C'\overline{\otimes}F(M))})\circ F(F^{l.a.}(b^{-1}_{X,F^{l.a.}(C'\overline{\otimes}F(M))}))\circ$$
$$F(F^{l.a.}(id_X\overline{\otimes}c_{C'\overline{\otimes}F(M)}))\circ c_{X\overline{\otimes}(C'\overline{\otimes}F(M))}\circ m_{X,C',F(M)})$$
$$\overset{(l)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ F(e_{X\overline{\otimes}F^{l.a.}(C'\overline{\otimes}F(M))})\circ F(F^{l.a.}(b^{-1}_{X,F^{l.a.}(C'\overline{\otimes}F(M))}))\circ$$
$$c_{X\overline{\otimes}F(F^{l.a.}(C'\overline{\otimes}F(M)))}\circ (id_X\overline{\otimes}c_{C'\overline{\otimes}F(M)})\circ m_{X,C',F(M)})$$
$$\overset{(l)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ F(e_{X\overline{\otimes}F^{l.a.}(C'\overline{\otimes}F(M))})\circ c_{F(X\overline{\otimes}F^{l.a.}(C'\overline{\otimes}F(M)))}\circ$$
$$b^{-1}_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ (id_X\overline{\otimes}c_{C'\overline{\otimes}F(M)})\circ m_{X,C',F(M)})$$
$$\overset{(m)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ id_{F(X\overline{\otimes}F^{l.a.}(C'\overline{\otimes}F(M)))}\circ b^{-1}_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ (id_X\overline{\otimes}c_{C'\overline{\otimes}F(M)})\circ$$
$$m_{X,C',F(M)})$$
$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$b_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ b^{-1}_{X,F^{l.a.}(C'\overline{\otimes}F(M))}\circ (id_X\overline{\otimes}c_{C'\overline{\otimes}F(M)})\circ m_{X,C',F(M)})$$
$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$(id_X\overline{\otimes}c_{C'\overline{\otimes}F(M)})\circ m_{X,C',F(M)})$$
$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}F(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)})\circ c_{C'\overline{\otimes}F(M)})\circ$$
$$m_{X,C',F(M)})$$
$$\overset{(2)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}\phi_{C'\overline{\otimes}F(M),M'}(ev_{F^{l.a.}(F(M)),M'}\circ d_{C',F(M)}))\circ$$
$$m_{X,C',F(M)})$$
$$\overset{(35)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}(ev_{F(M),F(M')}\circ (\xi_{F(M),M'}\overline{\otimes}id_{F(M)})))\circ$$
$$m_{X,C',F(M)})$$
$$= (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}ev_{F(M),F(M')})\circ (id_X\overline{\otimes}(\xi_{F(M),M'}\overline{\otimes}id_{F(M)}))\circ$$
$$m_{X,C',F(M)})$$
$$\overset{(b)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes C'}((id_X\overline{\otimes}ev_{F(M),F(M')})\circ m_{X,\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)}\circ$$
$$((id_X\otimes\xi_{F(M),M'})\overline{\otimes}id_{F(M)}))$$
$$\overset{(n)}{=} (\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')}))^{-1}_{X\otimes\underline{Hom}_{\mathcal{N}}(F(M),F(M'))}((id_X\overline{\otimes}ev_{F(M),F(M')})\circ$$
$$m_{X,\underline{Hom}_{\mathcal{N}}(F(M),F(M')),F(M)})\circ (id_X\otimes\xi_{F(M),M'})$$
$$= d''^{-1}_{X,F(M')}\circ (id_X\otimes\xi_{F(M),M'}).$$

The equality (e) holds by the naturality of $\Psi^{F(X\overline{\otimes}M')}(ev_{F(M),F(X\overline{\otimes}M')})$. We know that the functor $\_\,\overline{\otimes}F(M) : \mathcal{C} \to \mathcal{N}$ is left adjoint to $\underline{Hom}_{\mathcal{N}}(F(M), \_\,) : \mathcal{N} \to \mathcal{C}$ by Proposition 4.2.1, so using the natural isomorphism of this adjunction we get (f). The naturalities of ϕ and d are used in (g) and (h), respectively. The pentagon diagram of the $\mathcal{C}$-module functor $F^{l.a.}$ is used in (i), the naturality of b in (j), the definition of d in (k) and the naturality of c in (l). In (m) we used an identity present on Proposition 1.3.8, and in (n) the naturality of $\Psi^{X\overline{\otimes}F(M')}(ev_{F(M),X\overline{\otimes}F(M')})$.

This means that the diagram

$$\begin{array}{ccc} \underline{Hom}_{\mathcal{M}}(M, X\overline{\otimes}M') & \xrightarrow{\beta_{X\overline{\otimes}M'}} & \underline{Hom}_{\mathcal{N}}(F(M), F(X\overline{\otimes}M')) \\ \Big\downarrow {\scriptstyle d'_{X,M'}} & & \Big\downarrow {\scriptstyle d''_{X,F(M')}\circ \underline{Hom}_{\mathcal{N}}(F(M),b_{X,M'})} \\ X\otimes \underline{Hom}_{\mathcal{M}}(M,M') & \xrightarrow[id_X\otimes\beta_{M'}]{} & X\otimes \underline{Hom}_{\mathcal{N}}(F(M),F(M')). \end{array}$$

is commutative and therefore $\beta : G\circ F_1 \to F_2\circ F$ is a natural isomorphism of $\mathcal{C}$-module functors.

Affirmation 4: There is a natural isomorphism of $\mathcal{C}$-module funtors from $F_2^{-1}\circ G\circ F_1$ to F.

The $\mathcal{C}$-module functor F_2 is an equivalence of categories, so F_2^{-1} has a $\mathcal{C}$-module functor structure[13] $(F_2^{-1}, \overline{\overline{d}})$ and there is a natural isomorphism of $\mathcal{C}$-module functors $\varphi : F_2^{-1}\circ F_2 \to Id_{\mathcal{N}}$.

Let us define

$$\varepsilon_{M'} := \varphi_{F(M')}\circ F_2^{-1}(\beta_{M'}) : (F_2^{-1}\circ G\circ F_1)(M') \xrightarrow{F_2^{-1}(\beta_{M'})} (F_2^{-1}\circ F_2\circ F)(M') \xrightarrow{\varphi_{F(M')}} F(M')$$

and check that $\varepsilon = \{\varepsilon_{M'}\}_{M'\in\mathcal{M}}$ is a natural isomorphism of $\mathcal{C}$-module functors from $F_2^{-1}\circ G\circ F_1$ to F. Using Proposition 2.2.10, we may define the $\mathcal{C}$-module functor structure κ of the composition $F_2^{-1}\circ F_2$ as

$$\kappa = \{\kappa_{X,N} := \overline{\overline{d}}_{X,F_2(N)}\circ F_2^{-1}(d''_{X,N})\}_{(X,N)\in\mathcal{C}\times\mathcal{N}},$$

and the $\mathcal{C}$-module functor structure σ of $F_2^{-1}\circ G\circ F_1$ is

$$\sigma = \{\sigma_{X,M'} := \overline{\overline{d}}_{X,G(F_1(M'))}\circ F_2^{-1}(d'_{X,M'})\}_{(X,M')\in\mathcal{C}\times\mathcal{M}}.$$

Finally, the diagram

$$\begin{array}{ccc} (F_2^{-1}\circ G\circ F_1)(X\overline{\otimes}M') & \xrightarrow{\varepsilon_{X\overline{\otimes}M'}} & F(X\overline{\otimes}M') \\ \Big\downarrow {\scriptstyle \sigma_{X,M'}} & & \Big\downarrow {\scriptstyle b_{X,M'}} \\ X\overline{\otimes}(F_2^{-1}\circ G\circ F_1)(M') & \xrightarrow[id_X\overline{\otimes}\varepsilon_{M'}]{} & X\overline{\otimes}F(M') \end{array}$$

13 See Proposition 2.2.12.

commutes since

$$
\begin{aligned}
b_{X,M'} \circ \varepsilon_{X\overline{\otimes}M'} &= b_{X,M'} \circ \varphi_{F(X\overline{\otimes}M')} \circ F_2^{-1}(\beta_{X\overline{\otimes}M'}) \\
&\overset{(o)}{=} \varphi_{X\overline{\otimes}F(M')} \circ (F_2^{-1} \circ F_2)(b_{X,M'}) \circ F_2^{-1}(\beta_{X\overline{\otimes}M'}) \\
&= id_{X\overline{\otimes}F(M')} \circ \varphi_{X\overline{\otimes}F(M')} \circ (F_2^{-1} \circ F_2)(b_{X,M'}) \circ F_2^{-1}(\beta_{X\overline{\otimes}M'}) \\
&\overset{(p)}{=} (id_X\overline{\otimes}\varphi_{F(M')}) \circ \kappa_{X,F(M')} \circ (F_2^{-1} \circ F_2)(b_{X,M'}) \circ F_2^{-1}(\beta_{X\overline{\otimes}M'}) \\
&= (id_X\overline{\otimes}\varphi_{F(M')}) \circ \overline{\overline{d}}_{X,F_2(F(M'))} \circ F_2^{-1}(d''_{X,F(M')}) \circ (F_2^{-1} \circ F_2)(b_{X,M'}) \circ \\
&\qquad F_2^{-1}(\beta_{X\overline{\otimes}M'}) \\
&= (id_X\overline{\otimes}\varphi_{F(M')}) \circ \overline{\overline{d}}_{X,F_2(F(M'))} \circ F_2^{-1}(d''_{X,F(M')} \circ F_2(b_{X,M'}) \circ \beta_{X\overline{\otimes}M'}) \\
&\overset{(q)}{=} (id_X\overline{\otimes}\varphi_{F(M')}) \circ \overline{\overline{d}}_{X,F_2(F(M'))} \circ F_2^{-1}((id_X\overline{\otimes}\beta_{M'}) \circ d'_{X,M'}) \\
&= (id_X\overline{\otimes}\varphi_{F(M')}) \circ \overline{\overline{d}}_{X,F_2(F(M'))} \circ F_2^{-1}(id_X\overline{\otimes}\beta_{M'}) \circ F_2^{-1}(d'_{X,M'}) \\
&\overset{(r)}{=} (id_X\overline{\otimes}\varphi_{F(M')}) \circ (id_X\overline{\otimes}F_2^{-1}(\beta_{M'})) \circ \overline{\overline{d}}_{X,G(F_1(M'))} \circ F_2^{-1}(d'_{X,M'}) \\
&= (id_X\overline{\otimes}(\varphi_{F(M')} \circ F_2^{-1}(\beta_{M'}))) \circ \overline{\overline{d}}_{X,G(F_1(M'))} \circ F_2^{-1}(d'_{X,M'}) \\
&= (id_X\overline{\otimes}\varepsilon_{M'}) \circ \overline{\overline{d}}_{X,G(F_1(M'))} \circ F_2^{-1}(d'_{X,M'}) \\
&= (id_X\overline{\otimes}\varepsilon_{M'}) \circ \sigma_{X,M'}
\end{aligned}
$$

in which the equality (o) holds by the naturality of φ and in (p) is used its $\mathcal{C}$-module natural structure. The $\mathcal{C}$-module natural structure of β is used in (q) and the naturality of $\overline{\overline{d}}$ in (r).

Since the $\mathcal{C}$-module functor $F_2^{-1} \circ G \circ F_1 : \mathcal{M} \to \mathcal{N}$ is an equivalence of categories and there is a natural isomorphism of $\mathcal{C}$-module functors from $F_2^{-1} \circ G \circ F_1$ to F, then $(F, b) : \mathcal{M} \to \mathcal{N}$ is an equivalence of $\mathcal{C}$-module categories.

The other case can be done analogously. ■

A small application of this theorem can be seen in the following example. Here $\Bbbk$ is assumed to be an algebraically closed field of characteristic zero, and all module product bifunctors to be $\Bbbk$-bilinear and biexact.

Example 6.5. Let us consider the main theorem (Theorem 3.8) of the work [19]: Let G be a finite group acting on a fusion category $\mathcal{C}$, H be a subgroup of G and $\mathcal{M}$ a semisimple[14] indecomposable module category over $\mathcal{C}$. For every simple object $N \in \mathcal{M}$, the ($\mathcal{C}^G$-module) functor $L_N : \mathcal{M}^{H_N} \to \mathcal{M}^H$ (which is left adjoint to the forgetful functor $F_N : \mathcal{M}^H \to \mathcal{M}^{H_N}$, see Proposition 3.6 in [19]) induces a bijective correspondence between isomorphism classes of

[14] Any semisimple module category is exact since any object in a semisimple category is projective (see page 138 of [4]).

(i) simple objects $(Y, \mu) \in \mathcal{M}^{H_N}$ such that $Hom_{\mathcal{M}}(N, Y) \neq 0$, and

(ii) simple objects $(M, \nu) \in \mathcal{M}^{H}$ such that $Hom_{\mathcal{M}}(N, M) \neq 0$.

Affirmation: If there exists a simple object $(M, \nu) \in \mathcal{M}^{H}$ satisfying condition (ii) such that $F_N(M, \nu)$ is simple in $\mathcal{M}^{H_N}$, then the categories $\mathcal{M}^{H}$ and $\mathcal{M}^{H_N}$ are equivalent as $\mathcal{C}^G$-module categories.

In fact, from the correspondence of this theorem consider a simple object $(Y, \mu) \in \mathcal{M}^{H_N}$ satisfying $L_N(Y, \mu) \cong (M, \nu)$ (and $Hom_{\mathcal{M}}(N, Y) \neq 0$). This implies that $F_N(L_N(Y, \mu)) \cong F_N(M, \mu)$[15] and thus $F_N(L_N(Y, \mu))$ is a simple object in $\mathcal{M}^{H_N}$.

The functor L_N is left adjoint to F_N, so we may consider a counit $e : L_N \circ F_N \to Id_{\mathcal{M}^H}$ and unit $c : Id_{\mathcal{M}^{H_N}} \to F_N \circ L_N$ of this adjunction which particularly satisfies (See Proposition 1.3.8)

$$id_{L_N(Y,\mu)} = e_{L_N(Y,\mu)} \circ L_N(c_{(Y,\mu)}).$$

Let us now verify that $c_{(Y,\mu)} : (Y, \mu) \to (F_N \circ L_N)(Y, \mu)$ is a nonzero morphism in $\mathcal{M}^{H_N}$.

The object $L_N(Y, \mu)$ is not the zero object of the category $\mathcal{M}^{H}$. Indeed, $0 \neq (Y, \mu) \in \mathcal{M}^{H_N}$ and the functor L_N is a nonzero[16] additive[17] $\mathcal{C}$-module functor which implies $0 \neq L_N(Y, \mu) \in \mathcal{M}^{H}$ by Proposition 4.4.1, and thus $id_{L_N(Y,\mu)} \neq 0$ via Remark 1.1.8.

Suppose that $c_{(Y,\mu)} = 0$. Then $L_N(c_{(Y,\mu)}) = L_N(0) = 0$[18] and therefore,

$$0 \neq id_{L_N(Y,\mu)} = e_{L_N(Y,\mu)} \circ L_N(c_{(Y,\mu)}) = 0$$

which is a contradiction. Hence, $c_{(Y,\mu)}$ is a nonzero morphism between the simple objects (Y, μ) and $(F_N \circ L_N)(Y, \mu)$ in $\mathcal{M}^{H_N}$, and thus an isomorphism in $\mathcal{M}^{H_N}$ (by Corollary 1.2.10).

So we can use Theorem 6.3 to conclude that the categories $\mathcal{M}^{H_N}$ and $\mathcal{M}^{H}$ are equivalent as $\mathcal{C}^G$-module categories and also, by Corollary 6.4, that the functor $L_N : \mathcal{M}^{H_N} \to \mathcal{M}^{H}$ is an equivalence of $\mathcal{C}^G$-module categories.

[15] Any functor maps isomorphisms in isomorphisms.
[16] Since it maps simple objects of $\mathcal{M}^{H_N}$ in simple objects of $\mathcal{M}^{H}$.
[17] It is right exact via Proposition 1.4.5 and thus additive.
[18] L_N is additive.

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