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An Extension of The Lewis Development Model With Endogenous Savings

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Thales Luis Benites-Soria

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Abstract

This work proposes extending a Lewis development model with endogenous savings rates based on the need for subsistence income in a Stone-Geary function. Such an extension opens the possibility of studying the dynamics around multiple equilibria, including stability around a lower equilibrium level, analogous to an economic trap. We first explore the origins and effects of this extension as suggested by King and Rebelo (1993) in a Solow-Swan model context, where its dynamics results in an unstable poverty trap. Then we defend our hypotheses and explore the consequences of our modification, including the dynamics through the capital accumulation process in a dual-sector economy, which sets an dynamic bifurcation. A lower capital stock equilibrium emerges in a dual-sector context before the Lewis turning point, this equilibrium could attract or repulsive. Thus, the economy may never fully mature, which would characterize an economic trap. If the lower equilibrium is at the turning point, given conditions of the dynamic accumulation around the bifurcation then the economy would mature, with no labor surplus but unable to enter a virtuous cycle of capital accumulation, which sets an middle income trap.

Key-words: Poverty trap. Endogenous saving rates. Lewis development model. Economic Dynamics. Middle-income trap.

Resumo

Este trabalho propõe estender um modelo de desenvolvimento de Lewis com taxas de poupança endógenas baseadas na necessidade de renda de subsistência em uma função Stone-Geary. Tal extensão abre a possibilidade de estudar a dinâmica em torno do equilíbrios múltiplos, incluindo estabilidade em torno de um nivel inferior, análoga a uma armadilha econômica. Primeiro exploramos as origens e os efeitos desta extensão como sugerido por King e Rebelo (1993) em um contexto de modelo Solow-Swan, onde sua dinâmica resulta em uma armadilha de pobreza instável. Em seguida, defendemos nossas hipóteses e exploramos as consequências de nossa modificação, incluindo a dinâmica através do processo de acumulação de capital em uma economia de dois setores. Um equilíbrio de estoque de capital mais baixo surge em um contexto de setor dual antes do*turning point* de Lewis, esse equilíbrio pode atrair ou repulsar. Assim, a economia pode nunca amadurecer plenamente, o que caracterizaria uma armadilha econômica. Se o equilíbrio inferior estiver no turning point, dadas as condições de acumulação dinâmica em torno da bifurcação, a economia amadureceria, sem trabalho excedente, mas incapaz de entrar em um ciclo virtuoso de acumulação de capital, que configura uma armadilha de renda média.

Palavras-chave: Armadilha de pobreza. Taxas de poupança endógena. Modelo de desenvolvimento de Lewis. Dinâmica Econômica. Armadilha de renda media.

Resumo Expandido

Introdução

Do ponto de vista teórico, a pobreza é um problema desafiador, especialmente quando ela surge como uma possibilidade em um artigo sobre modelagem de crescimento econômico e acaba impulsionando uma parte significativa da pesquisa. Por que alguns países são mais prósperos do que outros? Por que algumas pessoas são pobres? Como as pessoas podem alcançar uma melhor qualidade de vida? Ou há algo inerente que torna algumas pessoas ou países mais pobres do que outras? São perguntas interessantes e a modelagem econômica pode fornecer algumas respostas pontuais

Eesta dissertação discute uma proposta de extensão para o modelo de desenvolvimento de Lewis de duplo setor, no qual as taxas de poupança são endógenas à renda de subsistência e aos gastos não relacionados à subistencia, modelados por meio de uma função de utilidade Stone-Geary. Nessa extensão, são apresentados dois equilíbrios no contexto do modelo de King e Rebelo1993. Os autores demonstram a dinâmica desses equilíbrios por meio da análise em torno de seus respectivos pontos críticos.

Metodologia

Primeiramente é apresentado o modelo com poupança endongena no modelo de Solow-Swan. Tem-se dois equilibrios. O equilíbrio inferior, k_ψ, é instável e repulsivo em seu entorno, enquanto o equilíbrio superior, k^*, atua como uma bacia de atração. Se a economia estiver em k_ψ, ela fica impedida de atingir o equilíbrio superior, configurando uma armadilha da pobreza.

Na extensão proposta para o modelo de Lewis, é mostrada a existência de uma bifurcação dinâmica transcítica, com um equilíbrio de longo prazo na fase madura e outro na fase de transição. O equilíbrio k_μ na fase de transição pode ser estável em um ramo convergente ou instável em um ramo divergente, dependendo das configurações dos parâmetros.

Uma fase excedente divergente atua como uma armadilha da pobreza, assim como no modelo de King e Rebelo, impedindo que a economia atinja níveis mais altos. Mesmo sendo estável, também atua como uma barreira para a acumulação de capital se estiver abaixo do ponto de virada de Lewis, . Se o estoque de capital pudesse atingir κ , a economia amadureceria completamente, com todos os trabalhadores empregados nos setores capitalistas e salários crescendo igual ao seu produto marginal. Em ambos os casos, o setor capitalista não atinge a maturidade e a economia fica presa em uma mistura de setores modernos e tradicionais, onde os salários ainda são pagos com um prêmio f.

A diferença econômica fundamental entre uma fase transitória divergente e uma convergente é que, na primeira, para que a economia fique presa, o setor capitalista deve surgir em um k específico. Qualquer choque externo que empurre a economia acima ou abaixo do equilíbrio desencadeia uma corrida em direção ao ponto de virada de Lewis ou a zero. Porém, uma fase transitória convergente significa que o setor de capital poderia surgir em qualquer nível abaixo do ponto de virada, sendo impedido em k_μ , e somente um choque externo suficientemente forte faria com que a dinâmica ultrapassasse a bifurcação e alcançasse a fase madura de acumulação.

Conclusão

Foi demonstrado que mais do que a emergência de um setor capitalista sob determinadas condições é necessário para que uma economia atinja a maturidade. Isso não é uma conclusão trivial. Existem requisitos iniciais (todas as condições estruturais) para a taxa de lucro no setor moderno que garantem uma fase madura.

As condições iniciais do setor capitalista determinam suas taxas de lucro ao longo da fase excedente de trabalho, determinando a convergência na fase de desenvolvimento. Se a estrutura não fizer com que a acumulação ultrapasse o ponto de virada, ela convergirá para um equilíbrio estável $k_\mu < \kappa$. Esse cenário configura uma armadilha da pobreza de setores diversos.

Em seguida, é analisado o processo de acumulação na fase madura e a existência de outros dois equilíbrios. Um deles é instável em torno de seu ponto crítico, e o outro é um atrator. Esses dois equilíbrios são análogos ao modelo de King e Rebelo. O equilíbrio inferior é instável, enquanto o equilíbrio superior é uma bacia de atração.

Se o ponto de virada estiver precisamente na armadilha de renda da fase madura, um choque externo pode fazer com que a dinâmica retorne à fase de acumulação excedente. Se o período de transição divergir, a economia se moverá novamente em direção a κ . Se o ramo de transição for estável em k, a economia convergirá em direção ao seu equilíbrio inferior, e ocorrerá algum processo de desindustrialização, retornando a uma mistura de setores tradicionais e capitalistas.

Em outro caso, se κ for igual ao equilíbrio inferior na fase madura, sofrer um choque que o empurre abaixo do ponto de virada fará com que o ramo excedente divergente ou convergente tenha $k_\mu > \kappa$. A economia retornará a κ, mas será incapaz de crescer. Essa dinâmica configura uma armadilha de renda média, na qual a economia atingiu plena maturidade, não há excedente de trabalho, mas os salários continuam sendo pagos com f porque a economia parece incapaz de ultrapassar o ponto de virada. Essa armadilha de renda média é sempre estável.

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1 Introduction

From a theoretical perspective, poverty is a challenging problem. Especially when it appears as a possibility in a paper about economic growth modeling and ends ups driving a significant part of the research. Why are countries more prosperous than others? Why are some people poor? How can people achieve a higher quality of life? Or is there something inherent that makes people or countries poorer than other people and or countries? Many interesting questions and economic modeling can provide some punctual answers (Ros, 2013).

On an academic level, why and how economies grow out of poverty has always been a question that has troubled economists since the conception of our science. We can attribute the first growth theories back to the first classical economists. An underlying message of Adam Smith in *Wealth of Nations* is that the division of labor among workers into specialized tasks increases productivity and consequently was one of the driving forces for economic growth and material prosperity. And if we ask ourselves, how can an economy increase its productivity? We can also follow another classical economist, David Ricardo (1817), and specialize in what an economy has as a comparative advantage over others, what it can produce at a lower opportunity cost and then trade for products that inherently have a higher opportunity cost. Although we might think of these theories as growth models in their own right, they are not intended to be mathematical models.

Today's established theories in macroeconomic literature are based on hypotheses, assumptions, and conclusions, modeled through mathematical constructs, and benefit from mathematical rigor. Some of the continuous premises are capital accumulation and technical progress. The savings rate plays an essential role in capital accumulation and, along with technological growth, is often considered the backbone of economic modeling (Sato, 1964). Much of the literature on economic development deals with endogenous saving rates. Some models are now landmarks in economic growth modeling, such as the infinite horizon growth model, a formalized version of the Ramsey model (1928), which is related to the work of Cass (1965) and Koopmans (1965) and is the precursor of all real business cycle models, or Diamond's (1965) overlapping generations model, often used in public pension and social security planning problems.

Robust models with solid bases and structures became part of the main framework of modern macroeconomics. Extensions to these and other models have often been presented in the literature, taking advantage of a model's robust structure to illuminate a different aspect when put in a different light. Elegant extensions that focus on the environmental context have been presented by Brock and Taylor (2010) in the Solow green model and more recently by de Oliveira and Lima (2020) in a Lewis green model. Galor (1992) presents a sophisticated combination of overlapping generations and a two-sector consumption/production economy.

In the spirit of collaborating with the current literature and modeling economic growth, we want to present a two-sector model with endogenous savings rates based on the need for subsistence consumption per capita. Such an extension opens the possibility of studying the dynamics around multiples equilibrium, including a lower equilibrium analogous to an economic trap.

We will use the term economic trap analogous to a poverty trap, where there are self-reinforcing mechanisms that keep individuals or communities trapped in poverty. Low income, poor access to healthcare and education, and few opportunities contribute to maintaining a vicious cycle (Azariadis; Stachurski, 2005). Such conditions have been observed through the analyses of histograms, where countries with better initial levels of wealth and over a determined threshold have grown somewhat together and accelerated at the same rate. In contrast, countries below that threshold grow slowly and suffer stagnation altogether. This pattern inflates initial discrepancies and gives origin to convergence clubs (Desdoigts, 1999).

With this said, why propose a dual-sector extension? Why not focus on and explore the dynamic of endogenous savings in a classical model environment? As suggested by King and Rebelo (1993), in certain conditions, it could also lead up to a poverty trap and gives plenty of room to explore, that as we know, no work as yet investigated the dynamics as we have done in this paper.

For those questions, we have three answers and motivations. The first one comes from empirical observation that some countries experience periods of economic growth and social development that can last years, seem set on a good path toward prosperity, and then incur an economic downturn that overwrites a significant part of the economic and social development achieved. Such as Zimbabwe, which experienced strong economic growth after gaining independence in 1980 (Munangagwa, 2009), where negative years had roots in the severe drought that affected the region, but in the middle of the 1990s started an economic collapse (Coltart, 2008), that ended in one of the largest hyperinflation ever recorded, forcing the government to abandon control of its currency and to rely in the American dollar (Coomer; Gstraunthaler, 2011).

We can also look at Nicaragua, 1992, after years of civil conflicts, made some political and economic reforms that leaned toward a more market-based economy, enjoyed economic growth, and managed poverty reduction (Rios-Morales, 2006). But after fifteen years of steady growth, it has never recovered from the 2007 global crisis and has seen a drastic slowdown in poverty reduction.

Or more recently, Brazil, which emerged relatively unharmed from the turmoil of the 2007 global crisis (Brainard; Martinez-Diaz, 2009), and with major economies still recovering, was seen as resilient and with optimism[1]. But in the first half of the 2010s, the euphoria was over, Brazil's structural deficiencies started to appear (Amann; Baer, 2012), and confidence yielded place to doubt[2]. And more recently, the promising roaring 10's has been reviewed as a lost decade[3].

Those countries could not simply converge to a higher level of wealth after starting their journey. They had particular reasons to stop climbing and had idiosyncrasies to their downfall. But we want to investigate if there could be an endogenous mechanism that could explain why a country fails to reach prosperity. Discuss what structural reason could block their transition. As the Lewis development model gives us the tools to analyze an economy in two phases, a mature and a transitional period, we can introduce King and Rebelo's idea and mathematically model the dynamics in each period.

Another motivation here is somewhat consensus in a significant part of the development literature, especially in why its called Developmentalism theory, that a nation is poor or peripheral because it has not yet industrialized and has yet to be integrated into a global market as an equal partner (Dirlik, 2014). Once a country develops a capitalist sector and is included as a peer in world trading, it acquires industrialized characteristics. Then enters a virtuous cycle of growth and expansion reflected in social changes. For this, it's justified political interventions to implement particular development strategies to achieve industrialization. The Lewis development model could provide a theoretical and formal base for this argument (Oreiro; Silva; Dávila-Fernández, 2020). There is a phase of underdevelopment in the economy, with lower wages and surplus labor, and a mature stage in which capital expands and salaries rise.

We recognize that a significant part of Developmentalism theories verses on why underdeveloped countries are included in the world market as secondary partners, generally as a cheap labor market or for the export of primary goods. Still, we propose a counterargument, with endogenous savings rates as modeled here, a nation, even with standard big pushes, could still be stuck in a middle-income trap. Or if this economic trap happens after the surplus economy is industrialized *a lá* Lewis as desired, it could still not reap its full benefits. A crucial underlying message could be that pushing industrialization does not guarantee virtuous economic growth. Some conditions can hold back the flow of prosperity. It's interesting to explore those particular cases.

Finally, our last motivation relates to the purpose of economic modeling. Economic models are descriptive constructs. Their goal is to represent an often convoluted system in a

[1] See the cover of The Economist magazine (Nov 12, 2009): *Brazil takes Off.*
[2] See the cover of The Economist magazine (Sep 27, 2013): *Has Brazil Blown It?*
[3] See the cover of The Economist magazine (Jun 21, 2021): *Brazil's Dismal Decade.*

simpler version of itself, sacrificing complexity to gain specificity. As a 1:1 scale map would be pretty pointless, a model intended to represent the entire economy would be useless, if not unfeasible. No model will always answer all questions or serve all purposes. A model is a representation of a system, not the system itself, yet a good model with robust hypotheses might give some good answers and some good direction of where things are going.

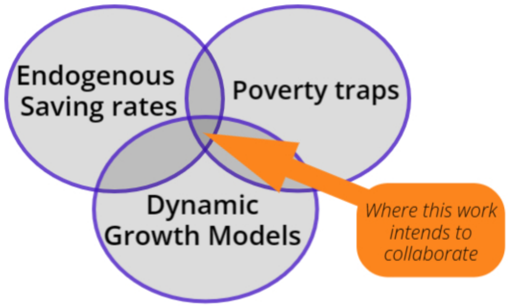

Figure 1 – Where the current research fits in the literature

From an academic perspective, an extension that considers subsistence needs and livelihood costs could strengthen the Lewis development model. A model focused on transitioning from a subsistence economy, where all activities are devoted to obtaining food and shelter, to a market-based economy would consider subsistence needs throughout this process. Subsistence needs do not disappear just because the sector changes. A human body needs approximately the same calories to maintain biological functions and keep its capabilities (Sen, 1999); this remains true despite sector shift.

To this end, we propose extending a Lewis development model with endogenous savings rates based on the need for substance income in a Stone-Geary function. Figure 1 shows a diagram where this paper might tie in with the current literature. We also aim to provide formal mathematical support for a savings rate modeled on a Stone-Geary function proposed by King and Rebelo and explore its multiple equilibria implications in the Solow-Swan context.

As far as we know, such an extension still needs to be done. The idea for this work naturally emerged when countries accumulate capital but are still stuck in a middle-income trap, unable to mature fully. On the contrary, what we could find in works such as Rosenstein-Rodan (1943), and in King and Rebelo (1993) that inspired our work.

This dissertation is not strictly about poverty but about a growth model with endogenous savings that leads to an economic trap. We intend to defend our hypotheses and explore the consequences of our modification, including the dynamics through the capital accumulation process.

1.1 Structure of Content

This dissertation is divided into three main chapters, plus one introduction and one conclusion section. The present section introduces the reader to this work's context, primary objective, and specific goals.

Chapter 2 is a literature review that focuses on the history of economic growth models with an endogenous savings rate that became staples and their general contribution to the literature on economic growth.

Chapter 3 invites the reader to dive into endogenous saving rates at a subsistence level of consumption, modeled from a Stone-Geary utility function, and their dynamic consequences in the classical economic growth model.

In Chapter 4, the debate on multiple equilibria caused by endogenous saving rates in the classical model *a lá* King and Rebelo are transferred to the Lewis development model and analyzed in this context, where an extension is presented.

The general conclusion and suggestions for future extensions or research are in the last section.

2 Literature Survey

Before delving into models with endogenous savings rates *per se*, it is beneficial to take a step back and introduce the first models constructed to answer how and why economies grow formally. And How they differ from models with an exogenous saving rate and other growth theories.

1. Theories without capital accumulation

Although, as mentioned before, savings rates are considered the engine of capital accumulation and are constantly present in macroeconomic models, some theories emphasize other processes for economic growth. In contrast to the classical approach, Schumpeter's (1911) theory of innovative expansion emphasizes innovation and entrepreneurship in disregarding capital accumulation as a factor of economic development.

For the Schumpeterians, the economy is in a stable equilibrium, and monopoly profits stimulate the entrepreneur, who is naturally a risk-taking individual, to innovate. This innovation may take a new form of production, incremental changes, or entirely new goods. An emerging innovation breaks the equilibrium state of the economy, and the entrepreneur benefits from a temporary monopoly until new players enter the market and the economy accommodates at a new higher level.

Capital is involved in this cycle of creative destruction. The innovator needs an initial investment for his project, so he looks for a business partner to support him financially. This businessman (i.e., a capitalist) accepts the partnership with the entrepreneur because he has the foresight and expects greater returns from the innovation than investing in a steady-state circular economy. That argument is why Schumpeterians emphasize stable institutions and robust financial markets so much. Capital is present, but the engine of growth is the partnership between the entrepreneur and the businessman, driven by innovation and the expectation of higher profits than in a circular economy.

In another theory focused on institutions but sometimes criticized (Tsiang, 1964) as being vague in what effectively drives the economy upward, Rostow (1960) identified a five-stage model of economic growth. The first stage is a traditional society in which producers consume production with little or no trade. Any increase in production is achieved by a slight, almost random rise in labor productivity through improved labor methods. In the second stage, specialization has increased, allowing product surpluses and, thus, the emergence of direct trade. Specialized labor and work lead to output growth. In the third phase, the trade surplus becomes an investment and drives industrialization. Output growth is now self-sustaining, as investment leads to higher income levels, leading to more increased investment.

In the fourth stage, when the economy is mature and there is no room for expansion in the number of industries, investment is channeled into research and development. As technological progress grows, so does productivity and production. Finally, in the fifth stage, economic growth is led by cycles of mass consumption.

According to Rostow's theory, the economy would be driven by the material surplus from the previous phase to the next step, which happens mechanically if institutions remain free. Again, capital is present (from the third stage onward), but its accumulation is an almost natural consequence of the health of institutions.

2. Exogenous saving rates

After the great depression of the 1920s, Keynes in the *General Theory* (1936) formulated a macroeconomic theory where that demand was more significant than supply, which was traditionally thought to be the main factor in growth by classical economists. In his critique of the classics, Keynes was convinced that economic growth is unbalanced. Investment is the primary driver of growth and, thus, the main source of imbalance. Harrod (1939) and Domar (1946) independently tried to dynamize the Keynesian short-term model to find a long-run equilibrium.

In the formalized, now known as the Harrod-Domar model (Sato, 1964), three growth rates must be equal to be sustainable: the actual growth rate, the guaranteed growth rate, and the natural growth rate. Such a situation guaranteed full use of labor and capital, or a steady growth patch, which Harrod referred to as *a golden period* of economic expansion. This equilibrium, however, requires a balance between household-dependent savings and capitalist-controlled investment. The savings rate and population growth, or the guaranteed growth rate and the natural growth rate, which comes on their independent dynamic, appear determined outside the model or given as exogenous.

The model presupposes a fixed ratio of capital to labor, blocking any factor substitution in the economy. As a result, no system is placed to balance the three growth rates, and equilibrium happens almost by chance. This unstable growth or, at *knife's edge* was a source of criticism among economists, including Robert Solow (1956).

In response to what he called unsatisfactory results obtained from the Harrod-Domar model, which lead to constant employment, or rising inflation, Solow proposed a long-term economic growth model that later along Trevor Swan (1956) similar model, became known as the neoclassical growth model. The Solow-Swan model's main objective was to demonstrate how an economy can experience sustainable growth over time. The Harrod-Domar model's two issues, economic instability and the impossibility of fully utilizing labor outside the golden age, were resolved in the Solow-Swan model by adding the assumption of factor substitution between labor and capital, which eliminated the assumption of a constant ratio of capital to output.

Diminishing capital returns affected the economy *per capita* behavior and guaranteed a steady growth path in the long run, eliminating the knife-edge problem. Another fundamental change was that family savings S and firms investments I are determined *a priori* the model, and at any time, a rise in savings implies an increase in investment. While Keynesian models maintain this identity, the investment determines how much saving is needed.

Another group of models with exogenous saving rates is the endogenous AK models of growth. The first AK model was formulated by Frankel (1962) with the contribution of Arrow (1971). Frankel stated that conventional neoclassical exogenous growth models were theoretically inadequate to investigate long-term growth because, by neglecting technological changes, they predicted that economies would eventually converge to a steady state with zero *per capita* growth. What was seen as a good thing in the Solow-Swan model now lacked an explanation for long-term growth (McGrattan et al., 1998). The Neoclassical model holds that growth in production per worker is ultimately driven by increases in total factor productivity (given by technological progress). Still, they do not explain how these increases occur, as they are exogenously determined.

A fundamental change introduced by the AK models is removing the capital diminishing returns, with a technological stock $A > 0$. The product per worker would continually be expanding at the same rate $\frac{dA}{dt}/A$ technological progress is increasing, implying no product per labor convergence to a steady state. Frankel's model first assumed constant exogenous saving rates, then when this group of models was rediscovered in the 1980s, Romer (1986) presented a model with consumer utility maximization. Other contributions to endogenous models were given by Lucas (1988), where knowledge is created and transmitted through the development of human capital.

3. Ramsey-Cass-Koopman and Diamond's overlapping generations

Frank Ramsey (1928) addressed the issue of the ideal level of savings and savings rates, leading to another neoclassical growth model. Although mathematically robust, Ramsey's contributions received little attention or support at his time. It was much later revisited, and now this model is sometimes referred to as the Ramsey-Cass-Koopmans model since it was later refined by Cass (1965) and Koopmans (1965).

The fundamental contribution is that saving rates are not given exogenously but are endogenous and determined by families' intertemporal preferences. While initially formulated as a social planner model, the established model in macroeconomics accounts for a decentralized model. A market clearing condition covers the absence of a social planner: given prices and wages, firms maximize profits, rent capital and hire labor from families, and offer goods and services, while families offer capital and labor and consume an optimal of goods to maximize their utility.

A difference from the standard neoclassical model is that, although the savings rates are identical in the long run to the exogenous saving rates in a Solow-Swan context, this cannot be the case throughout the accumulation process. Families, who are all-knowing about the economy at any point, expand their horizons over an infinite period. As $t \to \infty$ in a utility function $u(c_t)$, they decide their optimal consumption c_t, which affects savings in $t+1$, future consumption c_{t+1} and sequentially.

As Ramsey and his collaborators, Peter Diamond (1965) elaborated a model with endogenous saving rates. The key difference between Ramsey's infinite horizon model and the Diamond model is that although the economy will go on forever, families will not, and neither the representative agent. The representative agent is short-lived and looks to maximize his utility throughout his lifetime. The representative agent has two periods: when his young, he works, receives wages, and saves part of his income, and when older, he stops earning wages but lives off the interest of his savings when younger. Given this framework, the representative agent decides his savings throughout his lifetime.

Dimond's model considers savings as cautionary motives, where workers knowing they won't earn wages in their retirement, save and invest for the future in the face of uncertainty. But the driving reason for considering his savings is to smooth consumption throughout his lifetime. Suppose they consume all their income earned in t in the same period c_t. Their consumption behavior would be high in t and null in $t+1$. Then they rearrange their saving and spending behavior to smooth that consumption.

With a continuous influx of new workers living together with now retired pensioners, Diamond's model offers a unique take on *overlapping generations* from this interaction between multiple generations alive simultaneously. Because of this particularity, this model has often been used ins discussions about pension funds and governmental social planning (Hviding; Mérette, 1998).

4. The Cambridge Equation

Kaldor 1955 also searched for a solution to the Harrod and Domar's knife's edge instability problem. Alternatively to Solow, Kaldor didn't take the saving rate as exogenous and looked into the actual sources of savings in the economy. He considered not all agents behaved the same way toward savings and presented an endogenous saving rates theory. Kaldor's research led him to conclude that the economy's growth rate was explained by its profit rate. He considered that every family in the economy earned wages from the labor of his members and interest from lending capital to firms, and firms earned profits by selling goods to families.

Later, Pasinetti (1962), following Kaldor's studies, presented a class division, dividing the economy into two classes, workers and capitalists. Where workers exclusively earn wages, and capitalists solely make profits. Pasinetti concluded that the economy's savings rate is not given by the average savings of all families in both classes but solely from the savings rates of the capitalist. The propensity to save of capitalists is the factor that determines the growth rate of the economy as stated by kaldorian economists *capitalists earn the interest they spend, and workers spend the wages they earn.*

The Cambridge equation is the synthesis of Pasinetti and Kaldor's works. A mathematical expression that connects the rate profit and income distribution to the economy's growth rate through the different propensities to save wages and profits. Aggregate income Q is divided into wages w and profits π. The investment I of the economy is the sum of workers' savings $s_w w$ and capitalists' savings $s_\pi \pi$. And as profit share in the economy is given by the aggregate income less the wages share, as $\frac{\pi}{Q} = 1 - \frac{w}{Q}$, then the results hold

$$\frac{I}{Q} = (s_\pi - s_w)\frac{\pi}{Q} + s_w \implies \frac{\pi}{Q} = \frac{I}{Q}\frac{1}{(s_\pi - s_w)} + \frac{s_w}{(s_\pi - s_w)}, \tag{2.1}$$

which is the Cambridge equation. From this concludes that, given wages savings rates for workers and the profits of the capitalists, the ratio between the investment/income of the economy and the share of profit s determined

5. A remark on Lewis, plus King and Rebelo

A more in-depth analysis of a Solow-Swan model with endogenous savings rates *a lá* King and Rebelo (1993) is discussed in Chapter 3. A Stone-Geary utility function with subsistence income models endogenous savings rates. Since we will propose an extension of endogenous savings rates to a two-sector economy, a denser presentation of a Lewis (1954, 1955, 1958) development model is explored in Chapter 4 before our additional assumptions.

3 Diving into Endogenous Saving Rates

We begin by analyzing the implications of King and Rebelo's (1993) change in the hypothesis from an exogenous to an endogenous savings rate and its consequences for the classical economic growth model. Although the authors focus on their original paper to demonstrate various simulations of capital growth, they introduce the exciting idea of endogenous saving rates modeled through a Stone-Geary function with a subsistence income. They show the existence of a poverty trap through simulation but need to discuss the derivation of the endogenous saving rate and its bifurcation on the dynamics of accumulation. We must take a few steps back and spend some work delving into it, especially since we intend to use it later in our model.

1. General framework

As a general framework, we will take an analogous economy to the Solow-Swan model. There is an aggregation of continuous firms and families. All agents are price takers and know the general aspect of the economy. We guarantee marginal productivity and Inada conditions (Definition 1) is set. This model could be solved through the existence of an all-knowing social planner. But this would imply starting the analysis in a defined aggregate consumption and aggregate savings, as presented in Ros (2013). Alternatively, we will begin by setting the preference of every worker and then deduce a consumption from their utility maximization.

On an individual level, suppose each worker knows how much they need to consume and save for the next period and wants to maximize his utility. The utility maximization problem for workers is given by a Stone-Geary function constrained by the product per worker, as follows:

$$\max U = (c_t - \tilde{c})^{\phi}(z_t - \tilde{z})^{1-\phi}, \text{ subject to } q_t = c_t + z_t, \tag{3.1}$$

where c_t is consumption per worker, $tildec$ is livelihood consumption per worker, z is savings per worker, $tildez$ is savings per worker consistent with a continuous subsistence level of income, q_t is product/income per worker, and ϕ is the propensity to consume beyond subsistence income. To make the model mathematically consistent in the economically relevant domain, we need to place some constraints on these values: $c_t, z_t, q_t > 0 \in \mathbb{R}$, these values, if 0 or less, are economically innocuous and irrelevant; $\tilde{c}$, $tildez \geq 0 \in \mathbb{R}$, the subsistence levels can be set to 0 to return to the classical case with no subsistence needs, and $\phi \subset [0, 1] \in \mathbb{R}$.

The first-order conditions derivatives proceeds as

$$\frac{\partial U}{\partial c_t} = 0 \longrightarrow \phi(c_t - \tilde{c})^{\phi-1}(z_t - \tilde{z})^{1-\phi} = 0$$
$$\frac{\partial U}{\partial z_t} = 0 \longrightarrow (1-\phi)(c_t - \tilde{c})^{\phi}(z_t - \tilde{z})^{-\phi} = 0,$$

which gives

$$\phi\,(c_t - \tilde{c})^{\phi-1}(z_t - \tilde{z})^{1-\phi} = (1-\phi)(c_t - \tilde{c})^{\phi}(z_t - \tilde{z})^{-\phi}$$
$$\phi\,(z_t - \tilde{z}) = (1-\phi)(c_t - \tilde{c}),$$

rearranging to isolates consumption c_t, and retaking $q_t = c_t + z_t$, we obtain

$$c_t - \phi c_t - \phi\tilde{c} + \tilde{c} = \phi\,(z_t - \tilde{z}) \longrightarrow c_t = \tilde{c} + \phi(c_t + z_t - \tilde{c} - \tilde{z})$$
$$c_t = \tilde{c} + \phi(q_t - \tilde{c} - \tilde{z}). \tag{3.2}$$

The result of this equation tells us what consumption c_t the worker will choose if he maximizes his preferences, always giving priority to subsistence consumption $\tilde{c}$ and savings $tildez$, which at least ensure subsistence income in the next period, repeating the cycle. We can also define subsistence income as $\psi = \tilde{c} + \tilde{z}$. Thus, we have a consumption level in t defined in terms of subsistence income plus the propensity to consume from a non-subsistence income:

$$\text{as } \psi = \tilde{c} + \tilde{z} \Longrightarrow c_t = \tilde{c} + \phi(q_t - \psi), \tag{3.3}$$

this equation will also be our starting point, but we will go much further in our investigation. So far, all we have done is express a functional form for a Stone-Geary preference and determine a consumption equation.

Equation (3.3) gives us the tools to find an endogenous savings rate. We begin by defining the identity that totals saving in the economy as total income minus total consumption as $S_t = Q_t - C_t$. Since total income Q and total consumption C_t are, correspondingly, income per worker and consumption per worker times the number of workers L_t in the economy, we obtain

$$S_t = Q_t - C_t \longrightarrow S_t = q_t L_t - c_t L_t$$
$$S_t = q_t L_t - (\tilde{c} + \phi(q_t - \psi))L_t,$$

but following the classical model, we can still define the total savings of the economy as a part s_t of the income Q_t at a given time, i.e

$$S_t = s_t Q_t \longrightarrow s_t Q_t = q_t L_t - (\tilde{c} + \phi(q - \psi))L_t$$
$$s_t q_t L_t = q_t L_t - (\tilde{c} + \phi(q - \psi))L_t$$
$$s_t q_t = q_t - \tilde{c} + \phi(q_t - \psi),$$

where subsistence income is again expressed as $\psi = \tilde{c} + \tilde{z}$, we can reformulate the above equation as

$$\begin{aligned} s_t q_t &= \tilde{z} + (1-\phi)(q_t - \psi) \\ s_t &= \frac{\tilde{z}}{q_t} + (1-\phi)(1 - \frac{\psi}{q_t}). \end{aligned} \tag{3.4}$$

Thus, we have successfully obtained an endogenous saving rate s_t via depreciation over the subsistence capital stock z_ψ and a saving propensity $(1-\phi)$ out of a non-subsistence income $q_t - \psi$. The authors in their original paper imply this equation in their simulation but don't formally express it nor the initial Stone-Geary utility function. The social planner manner that Ros used also describes this equation, taking it as guaranteed, but doesn't show its derivation.

2. Emergence of a poverty trap in a Solow-Swan context

Through Equation (3.4) one may ask, but for what values is s_t defined, and for what values $\tilde{z}$ does it hold? As we defined at the beginning, *tildez* is the savings each worker retains to ensure at least the next subsistence level of income. A good analogy is a hunter who lives paycheck to paycheck and who, before investing in new equipment, has only enough to maintain and sharpen his old knives; or a farming community that sets aside a certain amount of grain each harvest in order to have the same amount of crops at the next season.

If we take the capital dynamics in the standard Solow-Swan model, we get

$$\frac{dk}{dt} = \bar{s}q_t - (n+\delta)k_t, \tag{3.5}$$

where n is the labor force growth rate, δ is the capital depreciation rate, $\bar{s}$ is the exogenous saving rates, and k_t and q_t are, as before, the capital stock and the product per unit of effective labor as $q = F(\frac{K}{L}, 1) = f(k)$, respectively. And now let $\bar{s}$ not be exogenous but endogenous, as given in Equation (3.4), then

$$\dot{k} = \left[\frac{\tilde{z}}{q_t} + (1-\phi)(1-\frac{\psi}{q_t})\right] q_t - (n+\delta)k_t, \tag{3.6}$$

and assuming we are in a poverty trap scenario where $q = \psi$ and $k = k_\psi$, where k_ψ is the capital stock per worker consistent with the subsistence level of income ψ. By definition, there is no capital accumulation in a poverty trap, then

$$\begin{aligned} \dot{k} = 0 \longrightarrow 0 &= \left[\frac{\tilde{z}}{\psi} + (1-\phi)(1-\frac{\psi}{\psi})\right] \psi - (n+\delta)k_\psi \\ 0 &= \left[\frac{\tilde{z}}{\psi}\right] \psi - (n+\delta)k_\psi \\ \tilde{z} &= (n+\delta)k_\psi, \end{aligned} \tag{3.7}$$

which is analogous to the steady-state equilibrium in the classical model, and as cruel as it sounds, *tildez* is the saving per capita that *breaks even poverty.*

3. A note on average propensity to save

Although the authors' original paper doesn't describe or explore the following possibility, Equation (3.4) also holds a compelling message. One of the main questions haunting economic growth and development theory is how and why the savings rate tends to increase in industrial societies. It's a stylized fact in the development literature that, in the long run, the proportion of savings to national income determines economic equilibrium or at least a steady growth path. Any relevant structural change is influenced by and reflects differences in this ratio.

There could be some cultural explanations; perhaps people save more in modern economies because they invest more or simply because they can manage their money better and have access to a variety of financial instruments. But that's putting too much faith in a general, uniform change in behavior that does not apply to a relevant degree in most advanced economies (just out of curiosity, today the average American savings account balance is about five thousand dollars[1]). One can argue that these behaviors are true *after* the modern financial system was set up to meet the demand for more diverse forms of investment, but that does not explain how or why the propensity to save arises *before* those incentives.

There may be a generational component, as studied in Diamond's (1965) model with overlapping generations. Working people tend to save more for retirement and accumulate debt in their youth to smooth consumption over their lifetime. But here we would like to bring a more straightforward explanation: not only do people save more because they have more, but they also save *disproportionately* more when they have more.

Equation (3.4) provides a mathematical expression for this argument. As capital expands and the product per capita q_t grows, the ratios $\frac{\tilde{z}}{q_t}$ and $\frac{\psi}{q_t}$ tend to zero, and as these values decrease, s_t approaches $(1-\phi)$. Indeed, q_t will never be infinitely larger than *tildez* or ψ, but even if moving towards some $q^* > \psi$ of equilibrium, s_t will gradually increase to its saving rate equilibrium, as shown below

$$\begin{array}{lll} & \text{given} & q_t \longrightarrow s_t = \dfrac{\tilde{z}}{q_t} + (1-\phi)(1-\dfrac{\psi}{q_t}), \\ (i) & \text{at subsistence level} & q_t = \psi \longrightarrow s_\psi = \dfrac{\tilde{z}}{\psi}, \\ (ii) & \text{at equilibrium} & q_t = q^* \longrightarrow s^* = \dfrac{\tilde{z}}{q^*} + (1-\phi)(1-\dfrac{\psi}{q^*}), \\ & \text{then} & s^* > s_t \text{ as } \lim q_t \to q^*. \end{array} \tag{3.8}$$

[1] Source: Survey of Consumer Finances 1989 - 2019, by the Federal Reserve Board of Governors.

We recognize that for the Equation (3.8) to be valid, we should impose the condition that $(1-\phi) > \frac{tildez}{\psi}$, but given the context of our analysis, this is not a problem. As capital expands, product growth, wages, and profits rise, and people get richer, the share of income devoted to subsistence becomes more paltry. Most spending is driven by nonliving expenses, guided by the propensity to save above the subsistence level.

Further down into our model, we will consider a separation between profits and wages, but this does not invalidate our argument. It strengthens it. Using Equation (3.4), we can explain why workers tend to spend all their wages proportionally and why the bulk of savings in any economy is generated by the top earners, whose income comes from profits and rents. This argument is on par with Kaldor's and Passineti's assumption that workers' marginal propensity to consume is greater than that of capitalists and that they often have negligible savings (Pasinetti, 1962; Kaldor, 1957). We will revisit these assumptions in the next chapter.

Previously, we defined $tildez$ as the savings per capita consistent with a continuous subsistence minimum. In Equation (3.7), we then showed that this is the amount of saving that manages to keep capital constant through a population growth n and that covers the depreciated capital δ in a subsistence minimum. This mechanism is strongly reminiscent of the classical model in which, in the long run

$$\bar{s}q^* = (n+\delta)k^*, \tag{3.9}$$

where $\bar{s}$ is the exogenous saving rate, n and δ are population growth and the capital depreciation rate, and q^* and k^* are the product and capital stock per worker in the long run equilibrium.

A fundamental difference is that we have two equilibria instead of one by rearranging Equation (3.4). One is the lower equilibrium in the poverty trap, $s_\psi\psi$, scenario (i) in Equation (3.8), leading to Equation (3.7), and the other a higher and desirable equilibrium s^*q^* shown in the limit of Equation (3.8), in scenario (ii).

3.1 Multiples Equilibrium and Dynamic Analysis

As mentioned in the previous argument, endogenous saving rates based on the consumption of a basket of subsistence goods $tildec$ result in multiples equilibrium of savings amount, namely $s_\psi\psi$ and s^*q^*. And in turn, it implies the existence of a multiple equilibrium level of capital per worker k_t. This is by no means a trivial implication.

Here and in the subsequent arguments, we would like to deal with three critical aspects: a) finding the equilibria/stable points, b) proving the existence of these equilibria, and c) analyzing the equilibrium stability and its economic effects. For the first aspect, we use Equation (3.6), the dynamic equation for the accumulation of capital per worker, the definition of a critical point (Definition 2), and a direct corollary.

Corollary 1. *In the capital accumulation equation per worker terms, critical points of $\dot{k}$ are the equilibrium level of k_t. A null accumulation $\dot{k} = 0$ implies stationary levels of k_0.*

Although we can find hints and mentions of the existence of k_t equilibrium through earlier arguments - in particular by rearranging and substituting the Equation (3.7) in scenarios (i) and (ii) of Equation (3.8), if we want to convince ourselves that such critical points can be reached, we can express the production function as a Cobb-Douglas function $Q = K^\alpha L^{1-\alpha}$, with $\alpha \subset [0, 1] \in \mathbb{R}$, which guarantees all the necessary Inada conditions. If we write the product per worker in intensive form as $\frac{Q}{L} = q = k^\alpha$ and follow the required steps in the Equations (3.2) to (3.6), we obtain

$$\dot{k} = \left[\frac{\tilde{z}}{{k_t}^\alpha} + (1-\phi)(1-\frac{\psi}{{k_t}^\alpha})\right]{k_t}^\alpha - (n+\delta)k_t, \tag{3.10}$$

which allows us to find the critical points k_0 that null the accumulation as

$$\begin{aligned} \dot{k} = 0 \implies 0 &= \left[\frac{\tilde{z}}{{k_0}^\alpha} + (1-\phi)(1-\frac{\psi}{{k_0}^\alpha})\right]{k_0}^\alpha - (n+\delta)k_0, \\ (n+\delta)k_0 &= \left[\frac{\tilde{z}}{{k_0}^\alpha} + (1-\phi)(1-\frac{\psi}{{k_0}^\alpha})\right]{k_0}^\alpha, \\ k_0 &= \left[\left(\frac{\tilde{z}}{{k_0}^\alpha} + (1-\phi)(1-\frac{\psi}{{k_0}^\alpha})\right)(\frac{1}{n+\delta})\right]^{\frac{1}{1-\alpha}}. \end{aligned} \tag{3.11}$$

While the result of this equation is nonintuitive and not easy to grasp at first glance, it is simply the extended version of the multiple equilibria we have already dealt with. This time, instead of going through a particular scenario and looking for an s_ψ or an s^*, we are looking for a specific set of k_0 that causes these equilibria. We risk being redundant because the same scenario that causes the stability of s_t and q_t is also the critical point of k_t, except that this stability now comes from the zero capital movement. Equation (3.11) gives rise to two possible critical points,

$$\begin{aligned} &\text{given} && \dot{k} \longrightarrow k_0, \\ (i)\quad &\text{at subsistence level} && k_0 = k_\psi \longrightarrow k_\psi = \left[\frac{s_\psi}{(n+\delta)}\right]^{\frac{1}{1-\alpha}}, \\ (ii)\quad &\text{in the long run} && k_0 = k^* \longrightarrow k^* = \left[\left(\frac{\tilde{z}}{k^{*\alpha}} + (1-\phi)(1-\frac{\psi}{k^{*\alpha}})\right)(\frac{1}{n+\delta})\right]^{\frac{1}{1-\alpha}}, \end{aligned} \tag{3.12}$$

one in scenario (i) at the subsistence level where $k_0 = k_\psi$, where the amount of capital per worker is just enough to grant the subsistence level as ${k_\psi}^\alpha = \psi$, which is repeated in the next period and leads into Equation (3.7) where the poverty trap occurs. And scenario (ii), correlated with Equation (3.8), since $\lim_{s \to s^*} k_t = k^*$, except that we now seek a saving rate s^*, which is correlated to a capital stock per worker in equilibrium. Scenarios (i) and (ii) are both equilibria in the sense that if the economy is at one of these levels and is not touched, it rests at these levels.

These two critical points can be better identified and explored if shown in a diagram. Figure 2 represents the two parts of the dynamics of k. The investment per unit of labor is the expression $s_t f(k)$, shown as the blue line, where s_t are the endogenous savings rates, as discussed earlier, and the orange line is the break-even investment. If the investment per unit of labor is higher than the break-even investment, $\dot{k}$ is positive and grants a higher level of k in the next period. Similarly, a lower investment per unit of labor than the break-even investment causes a falling level of k. If both values are equal, this is a critical point, and k is constant.

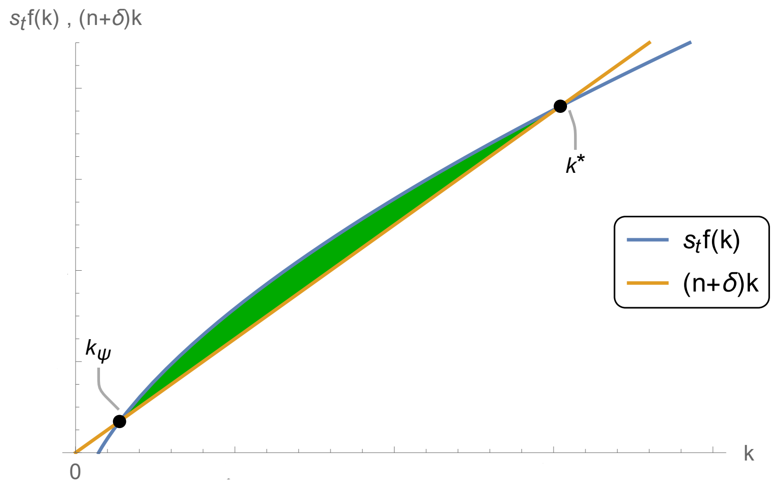

Figure 2 – Classical model diagram with endogenous saving rates.

Since $f(k_\psi) = \psi$ is subsistence income and, as we have explored so far, subsistence income induces a savings rate and subsistence savings income $s_\psi \psi = \bar{z}$, which equals $(n+\delta)k_\psi$ in the result of Equation (3.7), capital expansion at this level is nullified. Thus, this low-level equilibrium represents a poverty trap.

The other critical point rests at k^*, a higher equilibrium level analogous to the steady state equilibrium in the classical model. This particular equilibrium arises from our modeling hypotheses. By the Inada conditions, we know that for a low level of k, its marginal productivity will be high (namely $\lim_{x \to 0} f'(k) = \infty$), in this case, high enough for the investment per unit of labor to be greater than the break-even investment at any level lower than k^*, this is true for both exogenous saving rates and endogenous saving rates.

As k rises, under Inada's conditions, productivity falls ($\lim_{x\to\infty} f'(k) = 0$), causing investment per worker to cross the break-even point at k^*. In the classical model, this would represent a smooth dynamic transition to a steady growth path; in our case, it leads to unused growth potential. The area highlighted in green represents the potential capital accumulation from k_ψ to k^*, which is dammed by the poverty trap.

4. Existence and stability

A reasonable question arises from the last argument. If k_ψ acts like a dam on capital growth, preventing k from reaching a high steady-state level, could k^* also obstruct the flow of growth toward an even higher equilibrium? To provide an answer, we can verify the existence of each of these critical points using the intermediate value theorem and then show that if there can be no additional third point, there is a single long-run equilibrium at k^*.

The intermediate value theorem (Theorem 1) allows us to enunciate the following corollary:

Corollary 2. *We can show the existence of k_ψ and k^* in the capital accumulation function. If we can show that after k^*, there is no other possible critical point where $\dot{k} = 0$, then k^* is the unique long-run equilibrium.*

First, we will determine the level of k at which the expansion function peaks. For simplicity and without loss of generality, we can use the Cobb-Douglas expression and calculate by the first order of Equation (3.10),

$$\frac{\partial \dot{k}}{\partial k} = 0 \implies -\alpha k^{-\alpha-1}(z + \psi(\phi - 1)) - (n + \delta) = 0,$$

and solving for k returns:

$$k = \left[-\frac{\alpha(z + (\phi - 1)\psi)}{n + \delta}\right]^{\frac{1}{\alpha+1}}, \tag{3.13}$$

we denote this particular level as k^{max}. Since k^{max} is a maximum, we can be sure that the nearest left derivative is positive, since $\lim_{k\to k^{max\text{-}}} \frac{\partial \dot{k}}{\partial k}|_k > 0$. And similarly, we can be certain that reaching k^{max} from the right will give a negative derivative, since $\lim_{k\to k^{max+}} \frac{\partial \dot{k}}{\partial k}|_k < 0$, we can call these limits as k^{max-} and k^{max+}, respectively.

We now take the interval $(0, k^{max-}]$. Evaluating capital at zero is irrelevant and would give an undefined derivative at this level, but approaching zero through the right side, such as 0^+, gives a positive derivative and also causes the accumulation to tend toward negative infinity, or $\lim_{k\to 0^+} \frac{\partial \dot{k}}{\partial k}|_k > 0$, and $\lim_{k\to 0^+} \dot{k}(k) = -\infty$. Since we know that $\dot{k}(k^{max}) > 0$, by the Intermediate Value Theorem, there is a point $k \in (0, k^{max-}]$ where

$\dot{k} = 0$, which by definition is a critical point, and as we showed earlier is k_ψ. Additionally, since the function is strictly increasing over the entire interval, we can be sure that k_ψ is the only low-level equilibrium, and there is no equilibrium under the poverty trap.

The same logical sequence can be applied to the interval $[k^{max+}, +\infty)$. As capital grows toward infinity, the break-even investment becomes so much higher than the potential saving per unit of labor that capital accumulation tends toward negative infinity, and its derivative is also negative. Since $k^{max} > 0$ and $lim_{k\to+\infty}\dot{k}(k) = -\infty$, there is a point $k \in [k^{max+}, +\infty]$ at which $\dot{k} = 0$, which, as shown, is k^*. And since $\lim_{k\to+\infty} \frac{\partial \dot{k}}{\partial k}|_k < 0$, the function is decreasing throughout the interval, so k^* is unique.

As $(0, k^{max-}] \cup \{k^{max}\} \cup [k^{max+}, +\infty)$ exhaust all possible points in $\dot{k}$, we have shown that there is no third possible critical point. Then k^* is the highest equilibrium, and no higher stable level can be reached after it.

Now that we have convinced ourselves that there are only two equilibria, we can verify stability around these critical points. For this purpose, we first define a Lyapunov function (Definition 4) and can invoke a powerful theorem about locally stable equilibrium (Theorem 2).

Equipped with these mathematical instruments, we can investigate the stability of these points. First, let's take the Taylor expansion of the dynamical equation of capital around these points as

$$\dot{k} \simeq \dot{k}|_{k_0} + \frac{\partial \dot{k}}{\partial k}|_{k_0}\ (k - k_0), \tag{3.14}$$

expanding to

$$\begin{aligned}
\dot{k} &\simeq \frac{\partial \dot{k}}{\partial k}|_{k_\psi}\ (k - k_\psi) \to \dot{k} \simeq (\alpha(1-\phi)k_\psi^{\alpha-1} - (n+\delta))(k - k_\psi) \\
\dot{k} &\simeq \frac{\partial \dot{k}}{\partial k}|_{k^*}\ (k - k^*) \to \dot{k} \simeq (\alpha(1-\phi)k^{*\alpha-1} - (n+\delta))(k - k^*).
\end{aligned}$$

From the parametric configuration, we have constrained through our argumentation, and if we take k_ψ and k^* as we define it in Equation (3.12), then $\frac{\partial \dot{k}}{\partial k}|_{k^*}$ will be negative, then $\frac{\partial \dot{k}}{\partial k}|_{k^*}\ (k-k^*)$ zero if $k = k^*$, and is strictly negative for all $k > k^*$, which by Theorem 2 implies asymptotically stable. In contrast, $\frac{\partial \dot{k}}{\partial k}|_{k_\psi}$ is positive or negative, depending on the values of our parameters, so k_ψ has no stable neighborhood in $k \in \Omega_\psi$.

Since accumulation is positive when approaching the long run equilibrium from the left, as $\lim_{k^{*-}} \dot{k}(k) > 0$, and negative from the right as $\lim_{k^{*+}} \dot{k}(k) < 0$, then we can say it is locally stable. We can check that k^* is not globally stable. Take any point $k \leq k_\psi$ which breaks global stability as previously shown, $\dot{k} \leq 0$.

From this, we can conclude that the critical point k^* acts like a basin of attraction for the interval $(k_\psi, +\infty)$, while k_ψ is repulsive at any point $\epsilon > 0$ except for itself. These dynamics can be better seen in a phase portrait, as in Figure 2.

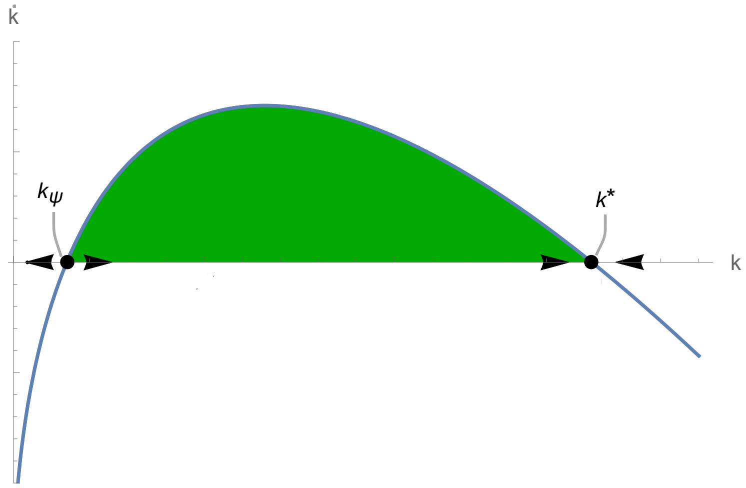

Figure 3 – Phase portrait with endogenous saving rates.

5. *Economic implications*

Besides the mathematical digression, we should do some economic reasoning for the instability around k_ψ. A good way to think about this is asking, what if, for some reason, there's an exogenous shock on any of the subsistence parameters?

We know that when the economy is trapped at k_ψ, per capita income ψ can sustain per capita savings *tildez* and subsistence per worker *tildec*. Now suppose there is a shock to the basket of subsistence goods that $\tilde{c}' > \tilde{c}$ and subsequently increase subsistence income needed to at least stay at that level, or an increase in the parameters $(n + \delta)' > (n + \delta)$. In both cases, the sequence of events is similar.

Since workers cannot reduce their consumption to increase their savings, they will save less than necessary to receive the same income tomorrow. Unless another shock reduces $\tilde{c}'$ or $(n + \delta)'$, savings will be even lower in the next period, generating even less income and causing a downward spiral toward zero.

This hypothetical downturn would not occur in the higher equilibrium k^*. An exogenous shock that increases the cost of living, or a reallocation of capital, would be absorbed in a lower equilibrium. In our example, an increase in *tildec* would also increase subsistence income. However, the worker would save less since the revenue generated in k^* is higher than subsistence income. Still, not less than the new subsistence income[2].These

[2] We admit that this is not the case if the increasing shock in *tildec* is so large that the income in q^* becomes the new subsistence income. But this is extremely unlikely. Any economy where this could happen would have structural problems far beyond savings and investment, perhaps in the context of war or famine. We like to think of this as an exceptional case, confined to academic papers.

processes are analogous to exogenous shocks in the classical model. Still, since the level of k also determines the saving rate, any increase/decrease would have a more substantial impact in a context with endogenous saving rates.

6. Final remarks

This concludes our analysis of endogenous savings rates conditional on subsistence income modeled from a Stone-Geary utility function as proposed by King and Rebelo in 1993. Much, if not all, of this chapter has had to be researched and constructed from scratch because, as far as we know, no work has yet explored this idea as we have done here. We hope we have done justice to the authors' ideas and paved the way for future research.

Much effort was necessary because we want to reintroduce this idea in a dual sector context *a lá* Lewis and explore the conditions under which an economy can be fully industrialized and still be trapped in a lower equilibrium. This extended model could provide a mathematical model for middle-income traps in developing economies.

4 A Lewis Development Model with Endogenous Savings

Now that we fully grasp endogenous saving rates and their effects on the long-run dynamics in a modified classical model, we can advance and apply its concept in a dual-sector context. The primary motivation for this extension is that developing nations are indeed in an economic trap, unable to grow. However, the trap there are in is more flexible and less rigid than defined in the King and Rebelo model. They often grow, then often regress, and sometimes fall under the previous level, to return and repeat this cycle. There is capital movement, and it is not automatically towards a greater level of capital, and neither vanishes to zero. There is some *stability* in their lower income level. To explore this discussion, we will follow a standard Lewis development model (1954, 1955, 1958) as presented by Ros (2013) with some complements, and then introduce our proposed extension.

1. Context and environment

Suppose an economy which consists of two sectors, sector A, analogous to a subsistence agricultural sector, where more traditional technique prevails, and M, a modern sector with industrial and capitalist characteristics. Both sectors produce the same good. The total product of the economy is the sum of the products of the two sectors, $Q_t = A_t + M_t$. All workers are employed in one of the two sectors at any given time, so $L = L_A + L_M$, or if normalized to be a continuous labor, it is more useful expressed as $1 = (1-l)L + lL$, where $l \subset [0,1] \in \mathbb{R}$ is the proportion of workers in the capitalist sector.

The subsistence sector is labor-intensive, has a surplus of labor, and uses a negligible level of capital stock $\bar{K}$, with all workers equally compensated by a wage ω_A. A fundamental aspect of this part of the economy is that capital is not reproducible here, nor is there the possibility of accumulation and growth. It has the traits of a traditional and subsistence economy *a lá* Lewis. The subsistence economy is not defined by the non-existence of capital but by the lack of possibility of accumulation, and capital fully depreciates after producing a good.

The second sector corresponds to a modern or mature capitalist economy *a lá* Solow-Swan, with a product defined by $M = f(K, L_M)$, in which all classical assumptions hold, in particular positive and diminishing returns and the Inada conditions (Definition 1), which in return guarantee the possibility of capital accumulation and the stability of an economic growth path. Here we make no distinction between a private or public capitalist sector. For the second sector to be capitalist, it must meet five defined characteristics:

a) it must be capital-intensive, b) capital must be able to expand, c) the existence of income profits from the use of capital, d) part - or all - of the profit is reinvested in capital expansion, e) the capitalist sector wants to maximize its profits.

We like to refer to the production function of the traditional sector with a Leontief technology as $A = min\{\bar{K}, aL_A\}$ because it settles the idea that it does not matter how many workers L_A it has since $\bar{K}$ is fixed for any amount of labor. And following the Leontief cost minimization problem, the wage per worker is given by $\omega_A = \frac{A}{L_A}$, and since for any given time $A = aL_A$, wages are equal to a, which is constant as shown:

$$\text{since } A = \min\{\bar{K}, aL_A\} \longrightarrow A = aL_A = \bar{K},$$
$$\text{and } \omega_A = \frac{A}{L_A} \longleftrightarrow \omega_A = \frac{aL_A}{L_A} = a \ \forall \ A, L_A, \bar{K},$$

as it is innocuous, we set $a = 1$ for the rest of this work. Then we also set wages in the traditional sector as $\omega_A = 1$.

In the modern sector, we have a first crucial difference with the classical model over wages. Under perfect competition and the market clearing rule, wages should equal the marginal productivity of labor. Then again, we are in a context of excess labor in the traditional sector, so the capitalist sector can easily attract workers by paying a premium over the current wage. We can define the premium rate $f > 1$. As Ros (2013) states, the premium f is defined as the minimum amount the capitalist sector can pay above the subsistence wage to ensure the inflow of labor. Thus, while the modern sector succeeds in drawing labor from the subsistence sector, we set its wage per worker as

$$\omega_M = f\omega_A \longrightarrow \omega_M = f. \tag{4.1}$$

We should also be aware that when we divide the economy into two sectors, we should not think of a unique subsistence firm nor a unique modern firm but of a conglomerate of subsistence families and a conglomerate of modern firms. There is still competition between firms in the capitalist sector. They momentarily take advantage of the surplus labor in the traditional sector; then, they can act as a group and offer wages they could not provide if competing against each other.

We continue by describing the labor demand of the capitalist sector. For simplicity, we consider its production function a Cobb-Douglas $M_t = K_t^{\alpha} L_{Mt}^{\beta}$. Since this is outside the scope of this work, we disregard technical progress and its effects or implications. Labor demand proceeds as follows

$$\frac{\partial M_t}{\partial L_{Mt}} = \omega_M \longrightarrow \omega_M = (1-\alpha)(\frac{K}{L_{Mt}})^{\alpha} \tag{4.2}$$
$$L_{Mt} = (\frac{1-\alpha}{\omega_M})^{\frac{1}{\alpha}} K_t,$$

since the modern sector can pay a premium on the wages of the first sector, we take $\omega_M = f\omega_A$, which further gives

$$L_{Mt} = (\frac{1-\alpha}{f\omega_A})^{\frac{1}{\alpha}} K_t \longrightarrow L_{Mt} = (\frac{1-\alpha}{f})^{\frac{1}{\alpha}} K_t \tag{4.3}$$

for any stock of K_t. We also assume there are no movement restrictions for workers in the economy. It means that a worker can always choose to stay in or go to the sector that is financially more attractive to him. Thus, labor demand L_{Mt} is always satisfied as the second sector pays higher wages than the traditional sector.

The capitalist sector expands as capital accumulates, drawing labor out of the traditional economy. Thus, there is a level of capital per capita where the capitalist sector absorbs all workers from the economy. At this point, $l = 1$, the first sector ceases to exist, and the economy consists of one unique sector, now $Q = M$. From this *turning point* level of capital per capita κ, as all workers are employed in the mature sector, and there is no longer a surplus of labor for the capitalist sector to exploit, firms in the mature economy are obligated to start paying wages equal to wages under perfect competition, so they pay workers based on the marginal productivity of labor $\omega_M = MP_L$. From then on, the economy behaves similarly to the classical model *a lá* Solow-Swan.

From this, we have two clearly defined periods of wage formulation. The first transitional period, or surplus accumulation phase, consists between the emergence of the capitalist sector and Lewis' turning point κ. Modern and traditional economies coexist during this interval, capitalist firms take advantage of existing surplus labor, and wages are set as f. The second period starts after the turning point; all workers are employed in the second sector, perfect competition over labor is set, and the marginal productivity of labor gives wages as $\omega_M = MP_L$. Figure 4 shows a diagram of ω_M over k through these two phases.

Wages in the modern sector before the turning point are fixed at f and do not depend on the level of capital per capita. In the surplus accumulation phase, capital expands but does not affect wages. After the transitional period, and κ is reached, the first sector is absorbed, and capitalist firms must compete for workers in the mature economy. As seen in equation (4.2), the marginal productivity of labor depends on the capital stock per capita; as capital grows, wages follow. As in the classical model, the mature accumulation phase reaches a steady state in k^*, as does wages in ω^*.

Also note that similarly from Equation (4.3), we can find an equation for short-term wages ω_M in the modern sector after the transitional period as

$$\omega_M = (1-\alpha)k^\alpha, \tag{4.4}$$

and also the interest rate of the second sector through all phases of the economy as

$$\frac{\partial M_t}{\partial K_t} = r \longrightarrow r = \alpha(\frac{K_t}{L_M})^{\alpha-1}$$

and taking labor demand as defined in Equation (4.3), we can verify that r depends on ω_M as

$$r = \alpha(\frac{1-\alpha}{\omega_M})^{\frac{1-\alpha}{\alpha}}. \tag{4.5}$$

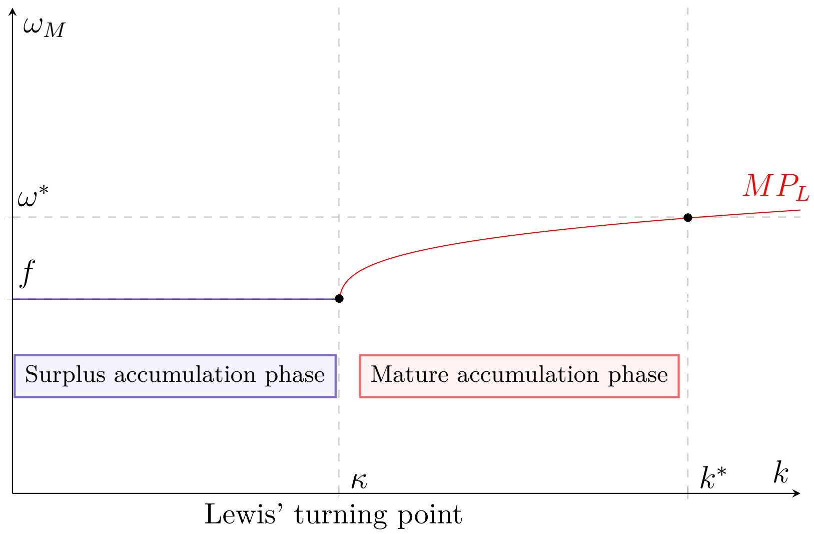

Figure 4 – Wages behavior in the standard Lewis development model.

From Figure 4, one can ask if the turning point κ is calculable. Although the presentation (Ros, 2013) that we are following doesn't give a direct expression, we can express the conditions for its existence and find what level of k they are satisfied. By definition, the turning point is when all workers are employed in the second sector, then $l = 1 \to L_M = L$. But since the first sector has just been extinguished, there is no effect on wages yet, then $\omega_M = f\omega_A = f$. If we take the Equation (4.3) and impose these conditions, and solve for $k = \kappa$, which returns

$$L = (\frac{1-\alpha}{f\omega_A})^{\frac{1}{\alpha}} K \longrightarrow 1 = (\frac{1-\alpha}{f})^{\frac{1}{\alpha}} \kappa$$
$$\kappa = (\frac{f}{1-\alpha})^{\frac{1}{\alpha}}. \tag{4.6}$$

2. Endogenous assumptions

With the Lewis' turning point defined, it's interesting to see how capital accumulates through the economy and what mechanisms act before and after κ. As we have two sectors, capital behavior will be different in each one.

By definition, the first sector does not accumulate capital. Capital is present but in a negligible quantity. A good analogy is smallholdings typically found in developing countries, where all family members are in this small-scale farming by default. This activity focuses primarily on meeting survival needs for the next year, with little to no surplus available for trade. Suppose one family member is hired to work in a factory in town. In that case, the other members cannot simply take over the equipment of the family member who has left and works twice as hard to maintain the same level of applied labor on a farm, so survival production decreases but is compensated by the new salary coming from outside (Wharton, 2017). As more people leave the first sector, less maintenance is done, and subsistence capital naturally depreciates.

Thus, we shifted our focus to the capitalist sector, and here we would like to propose our model extension. As discussed before, the capitalist industry behaves as a Solow-Swan economy. We assume capital stock in this sector accumulates through an investment minus a depreciation rate, $\dot{K} = I_t - \delta K_t$. Additionally, investment comes from the aggregate savings of the second sector, given by the identity $I \equiv S_M$. Here we will make a crucial hypothesis we have already explored in Chapter 2: we will assume that workers in the capitalist sector have no savings, and the capitalists make all the savings. On top of that, we will take capitalists' behavior toward savings $s_{\pi t} rK$ is denoted by a utility function with a subsistence income *a lá* King and Rebelo. With this, we can define

$$\frac{\dot{K}}{K} = s_{\pi t} r - \delta \longrightarrow (\frac{\tilde{z}}{q_t} + (1-\phi)(1-\frac{\psi}{q_t}))r - \delta,$$

and in per capita terms,

$$\frac{dk}{dt} = (\frac{\tilde{z}}{q_t} + (1-\phi)(1-\frac{\psi}{q_t}))rk - (\delta + n)k. \tag{4.7}$$

Here, we will borrow the concept of labor capacity to defend our extension. We want the reader to think of this process as calories. The first calories your body consumes to maintain its vital functions and renew the loss of tissue. If you have eaten enough to survive, the next calories provide energy for your daily activities (Dasgupta, 1997). These calories needed to maintain bodily functions do not change if a person is a worker in the subsistence sector or a capitalist living off profits in the mature sector. Duflo and Banerjee (2011) have explored this concept through an S-shaped capacity curve.

Figure 5 translate the nutrition problem into an income problem. The **Q** line is the current income that guarantees the same payment tomorrow, the **N** curve denotes actual remuneration, and **P** is the break-even point out of the poverty trap. Below the **P** point, people earn less than enough to do proper work tomorrow and make even less in the future, moving from A1 to A2, A3, and so on. After **P** point, there is an exponential consumption of energy capacity until the energy per calorie begins to dwindle. Then as people earn more, they can buy more food and gain strength, working more effectively and earning more from B1 to B2, B3. etc.

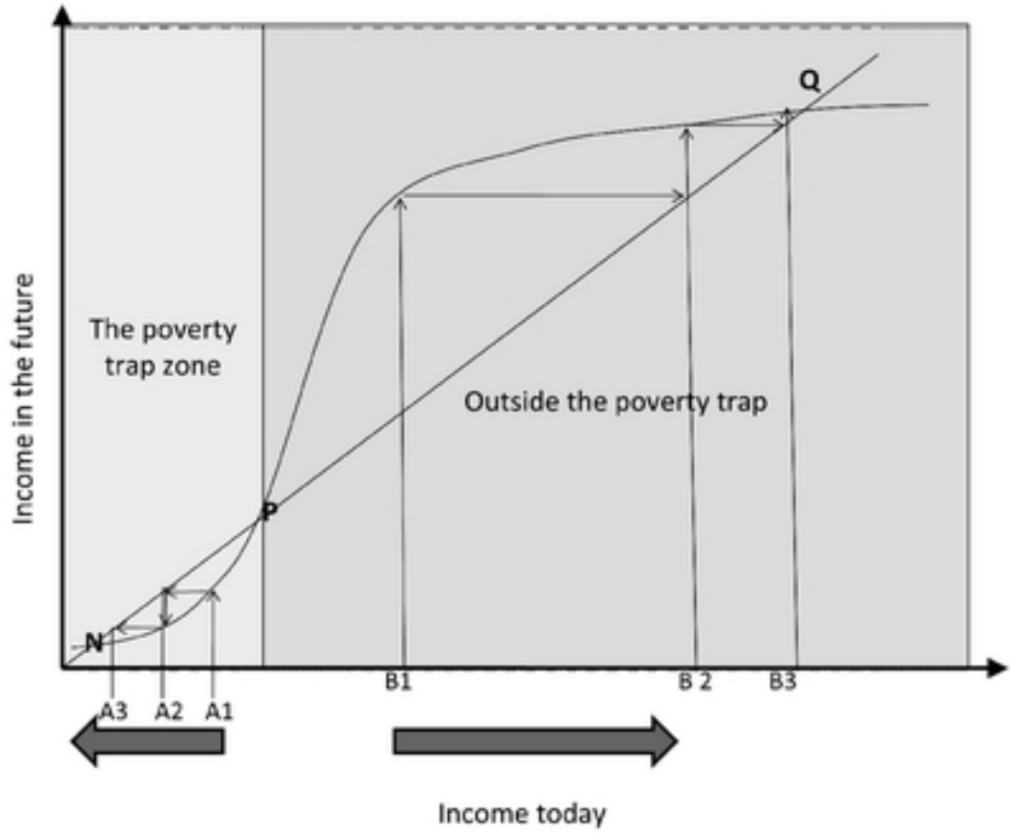

Figure 5 – S-Shape capacity curve. (Duflo; Banerjee, 2011)

Although Duflo and Banerjee do not explicitly refer to the need for subsistence savings, the idea behind the S-shaped curve is similar to the one discussed in Chapter 3. But instead of calories and energy expenditure, we have implied subsistence consumption and savings to cover depreciation. And Instead of the biological factor, we have explained this curve bumpiness by the Inada conditions (Definition 1) on the marginal productivity of capital.

The argument is that in sectors A and M, workers and capitalists must consider a certain level of livelihood consumption before non-essential expenditures. This is the primary concern of workers in the first sector, as noted in smallholding agriculture. But even in industrialized economies, where wages are higher than the subsistence income, subsistence is consumed first, then comes additional spending.

3. Defining an economic trap in a Lewis development model

Before the long-term behavior analyses, we must make a clear distinction: the subsistence sector in a Lewis development model is not an economic trap. We imply by economic trap a capital level of k at which the economy is dam to reach superior equilibrium, at which, if the gates are open, capital flows towards a higher stable level. An economic trap only exists if a higher level of capital and income can be achieved but is restrained to attain. The first sector in the Lewis development model has no such characteristics, its product is constant, and the capital stock is negligible. There is no sleeping potential to accumulate; there is no pile of straw waiting for a spark to ignite a fire. Capital cannot accumulate for structural reasons, not because it is hindered at a lower level.

In this context, it makes no sense to ask whether the first sector with wage level ω_A is in an economic trap determined by endogenous savings, similar to a poverty trap *a lá* King and Rebelo because, by our definition, this cannot be the case. However, a good question (or we think it is a good question) and the basis of this work is to ask whether there could be a low equilibrium in the transition phase and, more precisely, if the turning point κ could be an economic trap. And what are the necessary conditions?

After all, it meets all the requirements for the emergence of an economic trap, the capital *can* grow, and it is possible to reach a higher level of capital k^*. This possibility could hold an interesting fact: an economy could reach full maturity, be industrialized, and still not benefit from this structural change, with only a marginal wage increase. With this, we hope to offer a formal approach to *middle-income trap* present in development literature.

4.1 Long Term Behavior and Dynamic Analysis

As defined in Equation (4.7), capital movement depends on profits, and note that according to Equation (4.5), profits r always depend on the wage paid in the modern sector. Then we can define the capital movement over wages ω_M, first in the development phase before κ with $l < 0$, and in the mature phase $l = 1$ after the turning point. Then

$$\begin{array}{lll} & \text{given} & \dot{k} = s_\pi r k - (n+\delta)k, \\ (i) & \text{at developing phase} & \dot{k} = s_\pi \left[\alpha(\frac{1-\alpha}{f})^{\frac{1-\alpha}{\alpha}}\right] k - (n+\delta)k, \\ (ii) & \text{fully mature} & \dot{k} = s_\pi \left[\alpha k^{\alpha-1}\right] k - (n+\delta)k, \end{array} \tag{4.8}$$

We can extend s_π at each stage to be determined by endogenous subsistence income and non-subsistence expenditure. During the surplus phase, the profits rate is fixed as $\bar{r}$ given initial parameters f and α, as shown in (i) and Equation (4.5), and stay constant for any level of k before the turning point. Now we have an equation system for the transition phase set as

$$\frac{dk}{dt} = \left(\frac{\tilde{z}}{k^\alpha} + (1-\phi)(1-\frac{\psi}{k^\alpha})\right)\bar{r}k - (n+\delta)k,$$

where we can find the critical point k_0 according to Definition 2. We have set that wages are constant. We can find the level of k_0 that implies zero accumulation as

$$\dot{k} = 0 \implies (\frac{\tilde{z}}{k_0^\alpha} + (1-\phi)(1-\frac{\psi}{k_0^\alpha}))\bar{r}k_0 - (n+\delta)k_0 = 0$$

by solving for k_0, and denoting it as k_μ returns

$$k_\mu = \left[\frac{(\tilde{z} - (1-\phi)\psi)}{\frac{n+\delta}{\bar{r}} - (1-\phi)} \right]^{\frac{1}{\alpha}}, \tag{4.9}$$

where all components are exogenous and give a unique solution. Plotting this equation into a phase portrait, we obtain the mapping in Figure 7 or Figure 6 depending on the parameters we set[1] [2].

In the first diverging dynamic, k_μ is analogous to the k_ψ equilibrium in King-Rebelo's model. There (Figure 3), the economy is trapped at the subsistence level, and the accumulation intensifies shortly after escaping k_ψ because of marginal productivity of capital, and converges to the equilibrium k^* as more savings is required to cover depreciation. Here (Figure 6), this branch has no second equilibrium so capital accumulation would accelerate indefinitely. But as discussed before, the transitional phase comes to an end after it reaches the turning point κ and enters another dynamic of accumulation, which we will discuss later; this mature branch behaves as a Solow-Swan model and stops at k^* (similar to wages in Diagram 4).

In the second case (Figure 7), economically, a stable critical point on k_μ offers nontrivial consequences. As expressed in equation (4.9), given a fixed profit rate $\bar{r}$ (that depends on an established premium f), the dynamics converge to its designed k_μ.

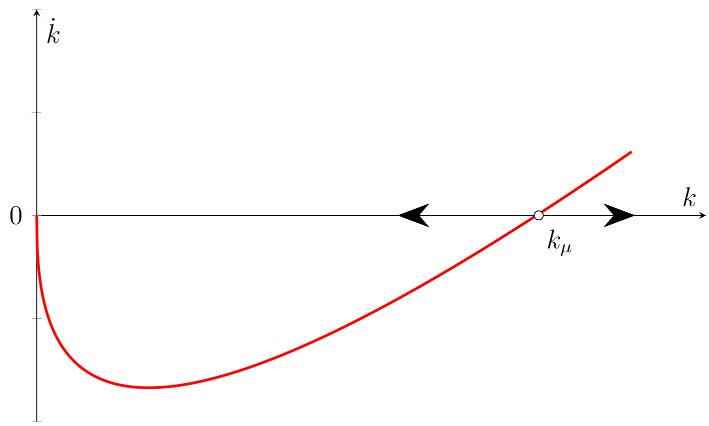

Figure 6 – Transitional phase with a diverging dynamic.

However, from the previous discussion, we know that this first accumulation process stops at the turning point κ, and then the mature accumulation phase starts. We have calculated κ in Equation (4.6). Then we have two points of k, one where capital movement changes and the other where capital movement converges. If it turns out that the turning point is higher than the convergence level as $k_\mu < \kappa$, then capital accumulation will stop before the capitalist sector exhausts the first sector, and the economy will never fully

[1] Possible parameters for a converging dynamics: $\alpha = 0.8, \psi = 117, \tilde{z}, \phi = 0.82, n = 0.055, f = 1.3$.

[2] Possible parameters for a diverging dynamics: $\alpha = 0.58, \psi = 41.82, \tilde{z} = 18.56, \phi = 0.38, n = 0.01, f = 2.21$.

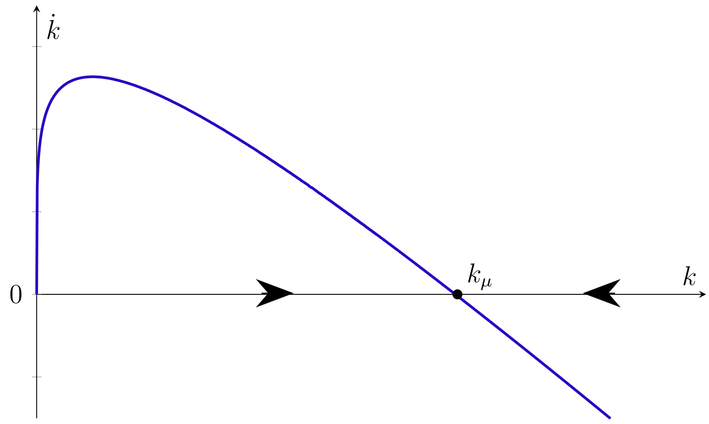

Figure 7 – Transitional phase with a converging dynamic.

mature. The economy will be stuck in a mix of traditional manufacturing and capitalist conglomerates with a labor population ratio $0 < l < 1$, and workers being paid wages 1 and f accordingly.

When the turning point is lower than or equal to the level the system converges parametrically, such as $k_\mu \geq \kappa$, the economy reaches the turning point before convergence. It enters a new dynamic of capital accumulation. Labor proportion is then $l = 1$, and wages are paid by their marginal productivity MP_L.

In both cases, there is a stable or unstable equilibrium in the transitional period, a dynamic that can reach or not reach *kappa*, and another equilibrium in the mature phase k^*. In the dynamical fields of mathematics, this exchange of stabilities and systems from some determined point denotes a transcritical bifurcation. The first equilibrium in this model is k_μ; after reaching the bifurcation point κ, the dynamics change its convergence towards k^*. And in both cases, this configures an economic trap; there is another dynamic of accumulation to enter and a higher equilibrium to attain, but the economy could remain trapped in the surplus accumulation period.

Before continuing and diving into the consequences of a bifurcation system, we have to make a disclaimer. Although the surplus labor phase could converge toward k_μ, there is no configuration where it's not below κ. With this, we conclude that any surplus phase with a stable equilibrium would act as a standard Lewis development model, where the capitalist sector emerges at any point k_e under the turning point and mechanically runs towards k_μ, crossing the bifurcation point automatically, and entering the mature phase. But this would make a higher equilibrium in the mature phase unstable, as shown in Figure 9, which defeats the purpose of a mature phase over the surplus labor phase.

If, for any chance, the convergent surplus labor phase is inferior at the turning point κ, this would imply that if the capitalist sector emerges in k_e between the equilibrium and the bifurcation, and as profit rate $r(k^*) > r(f)$ (from Equation 4.5, this would imply the capitalists' voluntary disinvestment of capital stock to converge to k_μ. Which it cannot be.

This being said, there are no conditions where a stable surplus labor phase corresponds to an economic trap, and as it just reinforces a standard Lewis development model, we will disregard its implication and instead focus on the diverging equilibrium.

4. Consequences of bifurcation

The previous argument shows us that it is possible that the process of capital accumulation in the Lewis Development Model doesn't always imply a mature phase. The emergence of a capitalist sector is not a guarantee of the extinction of the traditional manufacturing sector.

We want to know what parametric configurations would lead to the economy reaching the turning point. Since we have the condition that the convergences k_μ must be greater than or equal to κ, Equations (4.6) and (4.9), give us

$$k_\mu \geq \kappa \implies \left[\frac{(\tilde{z} - (1-\phi)\psi)}{\frac{n+\delta}{r} - (1-\phi)}\right]^{\frac{1}{\alpha}} \geq (\frac{f\omega_A}{1-\alpha})^{\frac{1}{\alpha}},$$

solving for the profit rate r gives

$$r_\kappa \geq \frac{f(\delta+n)}{f(\phi-1)+(\alpha-1)(z+\psi(\phi-1))}, \tag{4.10}$$

where all components are exogenous. In the case that the surplus period is divergent, if a capitalist sector manages to be set with a profit rate r^i lower than r_κ, the economy will stay trapped at k_μ until an external shock happens, converging to 0 or towards the turning point. It is also important to note that, in the diverging case, the capitalist sector *has* to be set on a valid k^i_μ. If the dynamic is divergent, and the capitalist emerges in a $k > k^i_\mu$, then it automatically escapes the poverty trap and runs towards the turning point; if it sets in a capital level $k^i_\mu < k^i_\mu$, it will asymptotically run towards zero. Another essential characteristic is that in a diverging transitional phase, any external capital inflow that pushes the economy over k_μ converges the system to κ and subsequently to k^*, and to its respectively ω^* as shown in Figure 8 diagram.

The argument we want to reinforce here is that it is not because wages are constant in the surplus phase that capital accumulation is not subject to forces like depreciation and redistribution of population per capita $n+\delta$. The expansion would not run indefinitely, and this process would stop. It stops at a level where the economy has not matured. It would have been subject to other dynamics if it had passed the bifurcation point.

The same applies to wages during the transition period. Suppose wages change due to an external stimulus to a higher premium $\omega'_M = f'$ without changing the capital level, and the capital level remains below κ'. In that case, the economy converges to its defined k'_μ

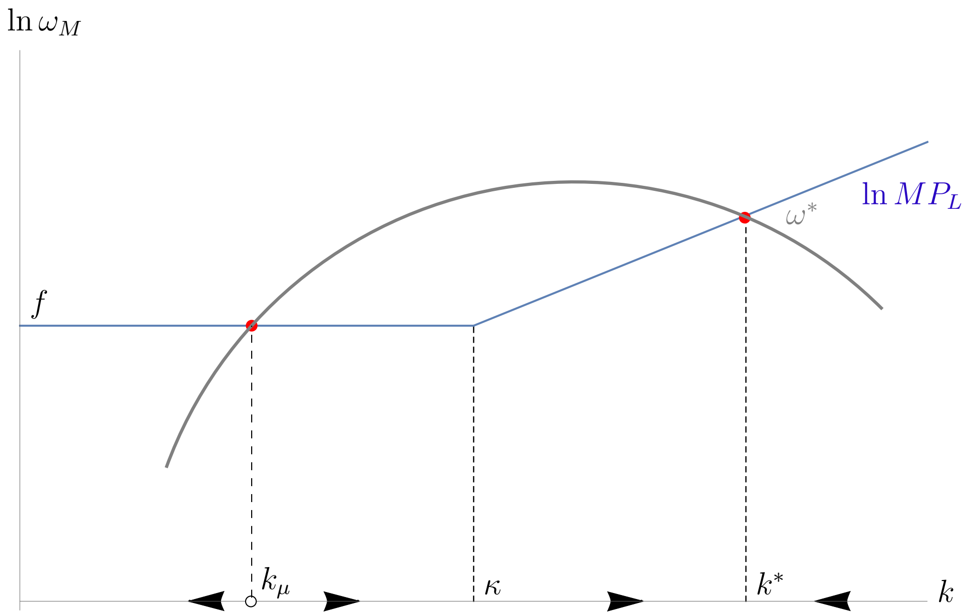

Figure 8 – Long-term behavior with a diverging transitional period.

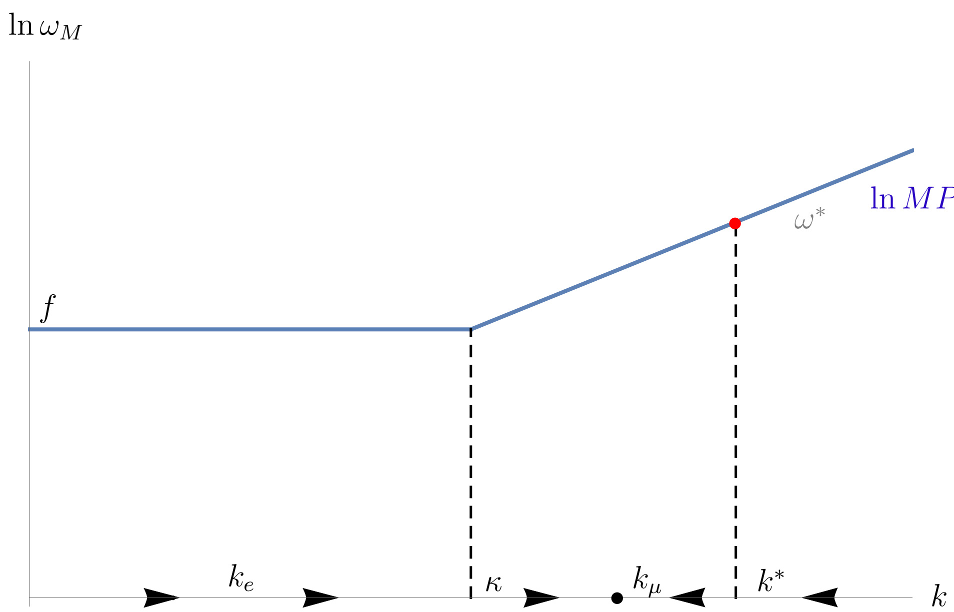

Figure 9 – Wage diagram with a stable equilibrium in the surplus phase, higher than the turning point, causing the mature equilibrium to be unstable.

and does not enter the mature phase. The premium f, in this case, changes the level but not the final behavior.

Figure 8 reflects the economic trap in a wages and capital diagram. The blue line is the short-term wage paid at any level of k_t. From Ros, we can calculate the long-term wages as in Equation (4.4) and evaluate the k of long-term equilibrium in gray. As we also have an equilibrium in the transitional period, then

$$w^*(k_\mu) < w^*(k^*), \tag{4.11}$$

where k_μ is the lower equilibrium, and k^* a equilibrium in the mature phase.

The main idea we want to conceive is, given a structural framework, where parameters are set, k_μ and κ are defined, and the dynamics of the transitional period are given. In this scenario, the capitalist sector can emerge in a capital per capita level k_i under ever k_μ, and by definition, always under the turning point; if it happens to emerge over κ, then the economy is already mature, and it's not a dual sector model.

If the transitional period is diverging, and $k_i < k_\mu$, the capitalist sector will converge to zero; if $k_i > k_\mu$, the economy will gradually advance towards maturity and enter a long-term behavior, the only way the economy is trapped in a lower equilibrium in a diverging transitional period, is if it emerges precisely at k_μ. To better understand this idea, take Figure 8, and take an emergence point k_e right, left, or over of k_u, and it will indicate the surplus period dynamics. This mechanism is analogous to the poverty trap in Chapter 3; any inflow of external capital that moves the capital stock to a k_ψ^+ is sufficient to open the gates to higher levels.

5. *The mature phase of accumulation and middle-income trap*

Until now, we have discussed economic traps in lower-level equilibrium in the transitional period. Where the economy is trapped before the turning point, doesn't mature, and is in a mix of surplus labor in the traditional sector and a modern sector that cannot absorb more workers, and wages are always kept at a premium level.

Now we want to focus on the idea that the economy could fully mature, the modern sector absorbs all workers, and the economy *should* keep its growth path towards a higher capital level and attain higher wages. Still, it is then trapped at the bifurcation point. The economy has all the characteristics of a modern and mature model, a lá Solow-Swan, but doesn't manage to reach its benefits. To this end, we will focus on the long-run behavior dynamics.

In Equation (4.8) item (*ii*), we have capital accumulation in the mature period. If we expand to hold our assumption in (4.7) and given that wages are affected by capital movement in this period, we have the following dynamics

$$\frac{dk}{dt} = \left(\frac{\tilde{z}}{k^\alpha} + (1-\phi)(1-\frac{\psi}{k^\alpha})\right)\left[\alpha k^{\alpha-1}\right]k - (n+\delta)k,$$

with critical points equal to

$$k_0 = \left[\alpha\left(\frac{\tilde{z}}{{k_0}^\alpha} + (1-\phi)(1-\frac{\psi}{{k_0}^\alpha})\right)(\frac{1}{n+\delta})\right]^{\frac{1}{1-\alpha}}, \tag{4.12}$$

which is analogous to the multiple equilibria in King and Rebelo's model. As before, the lower critical point is unstable in this dynamic, and the higher critical point acts as an attractor. We will denote the higher equilibrium in the mature phase as k_m^* and the lower as $k_{m\psi}$.

Because the lower equilibrium $k_{m\psi}$ is unstable, if the economy gets a push of additional capital, it will flow towards the higher stable equilibrium k_m^*. But if external shock makes the economy go below $k_{m\psi}$, it will go toward the origin of the dynamics, in this case, κ. Those are all mechanism remnants from King and Rebelo's model, except that the origin is zero.

Here enters an interesting aspect of the dynamics that have exciting consequences for economic reasoning. Take that the economy is in a mature phase; now, there's a momentary external sharp burn of capital so intense that toss the economy to a capital level per capita k_i under the turning point. Now we have two scenarios. Suppose the transitional phase that gave the origin mature period diverges, and the shock is not intense enough to put k_i under k_μ. In that case, the economy will again run toward the turning point.

But suppose the surplus phase is converging towards k_μ, any shock external shot that pushes k_i under κ. In that case, the economy can be attracted back to the lower equilibrium, which could imply a return of the traditional sector, which configures a deindustrialization process. A middle-income trap is set when the lower equilibrium of the mature phase is at the turning point, so $\kappa = k_{m\psi}$, and the dynamics of the surplus phase runs toward the bifurcation point; in any case where it's diverging under κ, but also the case where it's diverging and $k_\mu > \kappa$ (Figure 10 illustrates that condition. In this case, the economy is at a turning point; the second sector has fully matured, the traditional sector is exhausted, and there is no more labor surplus. However, wages are still locked at f and unable to grow because capital is trapped at the bifurcation.

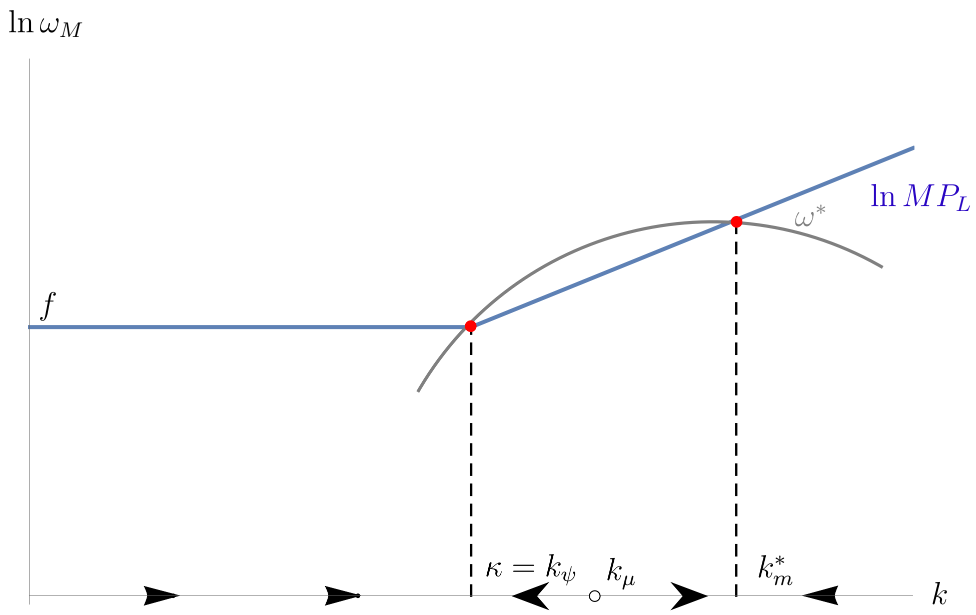

Figure 10 – Diagram with κ being a middle income trap

At the beginning of this chapter, we defended our extension by an analogy of calories and Duflo and Banerjee's S-shaped capacity curve. We explained why the traditional and the capitalist sectors are subject to the subsistence effect. Now we must defend why an economy could reach a capital stock that allows maturity, with initial conditions that guarantee $k_\mu \geq \kappa$, pass the turning point, and immediately ceases to accumulate, being hindered precisely at the beginning of a new process of accumulation. This process sounds counterintuitive. An endogenous saving rate at a lower k is sufficient to keep capital growing. When a better process emerges, with higher wages, the accumulation is trapped.

But unlike King and Rebelo's model, only capitalist contributes to the saving rates of the economy. And saving rates depends on the profit rates that capitalist earn. The profits rate r^i ends up dictating k^i_μ, and in the surplus phase, the economy will converge. As shown before, if $k^i_\mu \geq \kappa$, the economy will enter the mature phase. If not, it will stay in the developing stage, from Equation (4.5), the profit rates are determined by the wages paid in the modern sector.

Given some parametric initial conditions, we argue that the profit rate r^i only allowed capital expansion in the surplus phase because wages are fixed and constant. The moment workers would earn over f^i, the now endogenous r^i is insufficient to allow a saving rate s_π that accumulates capital higher than κ. In this specific case, the turning point κ **is** a critical point of the mature accumulation phase.

6. Final remarks

This Chapter ends our extension proposal and the emergence of a stable economic trap. We argue that this *middle-income trap* offers a more general approach to economic underdevelopment than King and Rebelo's poverty trap. For the economy to be trapped in King and Rebelo model, the economy must be precisely at k_ψ, so it must start at the poverty trap or have exogenous shocks that specifically hinder capital accumulation at a new point. Any configuration over k_ψ grows, and any structure under k_ψ vanishes to zero. For the economy to exist and not grow, it should be at that precise point, as a knife's edge.

Here those assumptions are not necessary. We imply that the capital accumulation could begin, the economy could grow toward maturity, and then stop in equilibrium in the surplus labor phase, not because of an exogenous shock but because of the initial structural conditions that won't let the economy reach the turning point. And unfortunately, only a strong enough influx of external capital that pushes the economy over κ will affect the long-term equilibrium and converges towards k^*_m.

5 Conclusion

In this work, we offered an extension proposal to the dual sector Lewis development model, where saving rates are endogenous to subsistence income and nonliving expenses, modeled through a Stone-Geary utility function. First, we investigate the effect of this extension in a Solo-Swan context, as formulated by King and Rebelo. We showed that in this scenario appear two equilibriums. We showed the dynamics of those equilibria by the analysis around their respective critical points. The lower equilibrium k_ψ is unstable and repulsive in his neighborhood, and the higher equilibrium k^* acts like a basin of attraction.

If the economy is at k_ψ, then it is hindered from reaching the higher equilibrium, which configures a poverty trap. We demonstrated that those critical points are unique, and the dynamics constantly diverge to the lower equilibrium and converge to the higher equilibrium when dynamics are set in motion. Any external shock that disrupts the economy at the lower level goes to zero or towards the higher and stable equilibrium.

In our proposed extension applied in a Lewis development model, we showed the existence of a transcritical dynamic bifurcation, with a long-term equilibrium in the mature phase and another in the transitional phase. The equilibrium k_μ in the transitional phase can be stable in a converging branch or unstable, in a diverging branch, depending on parametric configurations. The parametric configuration for a stable labor surplus phase goes against economic intuition and beyond this dissertation's motivation.

A diverging surplus phase acts like King and Rebelo's poverty trap, hindering the economy from reaching higher levels. If the capital stock could reach κ, the economy would fully mature, all workers employed in the capitalist sectors and wages growing equal to their marginal product. In both cases, the capitalist sector doesn't reach maturity, and the economy is stuck in a mix of modern and traditional sectors, where wages are still paid at a premium f.

The economically main difference between a diverging transitional patch and a converging one is that in the first, for the economy to be trapped, the capitalist sector has to emerge at that precise k_μ. Any external shock that pushes the economy over or under the equilibrium sparks a run toward the Lewis turning point or zero. But a converging transitional phase means that the capital sector could emerge at any level under the turning point, it will be hindered at k_μ, and only a strong enough external shock would make the dynamics go over the bifurcation and after the mature phase of accumulation.

Thus, we have shown that more than the emergence of a capitalist sector under certain conditions is needed for an economy to reach maturity. This is not a trivial conclusion.

There are initial requirements (all structural conditions) on the profit rate in the modern sector that guarantees a mature phase.

The capitalist sector's initial conditions determine its profit rates along the surplus labor phase, determining the convergence in the developing stage. Suppose its structure doesn't make accumulation go over the turning point. It will converge to a stable equilibrium $k_u < \kappa$. This scenario configures a poverty trap of diverse sectors.

Then we analyzed the process accumulation in the mature phase and the existence of another two equilibriums. One is unstable around its critical point, and another attractor. Those two equilibria are analogous to King and Rebelo model. The lower equilibrium is unstable, and the higher equilibrium is a basin of attraction.

If the turning point is precisely at the income trap of the mature phase, An external shock could make the dynamics return to the surplus accumulation phase. If the transitional period diverges, the economy will run again towards κ. Suppose the transitional branch is stable in k_μ. In that case, the economy will converge toward its lower equilibrium, and some process of deindustrialization occurs, returning to some mix of the traditional and capital sectors.

In another case, if the turning κ equals the lower equilibrium in the mature phase, it suffers a shock pushing it under the turning point, and the surplus branch diverges or converges with a $k_\mu > \kappa$. The economy will return to κ but will be unable to grow. These dynamics configure a middle-income trap, where the economy has fully matured, the is no labor surplus, but wages remain paid at f because the economy doesn't seem mage to grow passe the turning point. This middle-income trap is always stable.

1. Suggestions for future research

We have dealt with subsistence income based on live hood consumption and the savings necessary to maintain that consumption. We have to restrain our attention on its effects on the saving rates. This focus allowed us to design and analyze Lewis's development model, where the economy cannot attain maturity by structural factors, not because of external forces. The inner mechanism hinders the economy in the transitional phase. Given the suitable condition, another crucial conclusion is that the turning point could act as a damn toward the mature equilibrium.

But we have not exhausted all the possibilities the Lewis development gives a model with subsistence consumption allows. As far as we know, few works have taken this approach. We expect to look at how this affects people's behavior and what exactly changes with a level of income over subsistence.

Suggestions for future research will try to model this behavior change and how they would affect the dynamics in capital accumulation, primarily through endogenous population growth as $n_t = n(k_t)$. It's intuitive to think that a community where the main focus is surviving for the next day would have fewer conditions to bear children and a higher rate of child mortality than another community with a higher income level. The Lewis development model could offer a great structure to this context.

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APPENDIX A – Definitions

Definition 1. Inada conditions. *Take* $\boldsymbol{T}$ *a transformation from a space* $\mathbb{R}^n$ *to* $\mathbb{R}$*, that is represented by a continuously differentiable function* $f : X \to Y$*, where* $X = \{x \mid x \in \mathbb{R}^n_+\}$ *and* $Y = \{y \mid y \in \mathbb{R}_+\}$*. If* f *is an production function, we say it satisfies the Inada conditions if:*

1. f *in* 0 *is the* 0 *of the vector transformation as:* $f(\mathbf{x})$ *and* $\mathbf{x} = \mathbf{0} \to f(\mathbf{0}) = 0$*.*
2. f *is concave in* $\mathbf{x}$*, this implies positive but decreasing returns in* x_i*, as:* $\frac{\partial f(\mathbf{x})}{\partial x_i} > 0$*, and* $\frac{\partial^2 f(\mathbf{x})}{\partial {x_i}^2} < 0$*.*
3. *The limit of the derivative of* f *as* x *approaches zero tends to infinity,* $\lim_{x_i \to 0} \frac{\partial f(\mathbf{x})}{\partial x_i} = \infty$*.*
4. *The limit of the derivative of* f *as* x *approaches infinity tends to zero,* $\lim_{x_i \to \infty} \frac{\partial f(\mathbf{x})}{\partial x_i} = 0$*.*

Definition 2. Critical Point. *Let* $\dot{x} = f(x)$ *be a differential equation system, with* $x \in \mathbb{R}^n$*. We say* x_0 *is a critical point of* $\dot{x}$*, if* $f(x_0) = 0$*.*

Definition 3. Lyapunov function. *Let* $\dot{x} = f(x)$ *be a differential equation system, with* $x \in \mathbb{R}^n$*, and let* x_0 *be a critical point of* $\dot{x}$*. Take a function* $V(x) : \mathbb{R}^n \to \mathbb{R}$ *defined in a closed set* Ω*, where* Ω *are all the points around* x_0 *in a* $\epsilon > 0$ *distance. If* $V(x)$ *has the follow proprieties: i)* $V(x) = 0$ *when* $x = x_0$ *and, ii)* $V(x) > 0$ *when* $x \neq x_0$ $\forall x \in \Omega$*, then we say* $V(x)$ *is a Lyapunov function.*

Definition 4. Transcription

APPENDIX B – Theorems

Theorem 1. Intermediate value theorem and unique solutions. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function in the whole interval. If there exists $L \in [f(a), f(b)]$, then there exists $c \in [a, b]$, where $f(c) = L$. If furthermore the function is strictly monotonic on $[a, b]$, then the equation $f(x) = L$ admits an unique solution in c.*

Theorem 2. Locally asymptotically stable equilibrium *Let $\dot{x} = f(x)$ be a differential equation system, with $x \in \mathbb{R}^n$, and let x_0 be a critical point of $\dot{x}$. Let $V(x)$ be a Lyapunov function of $\dot{x}$. Take $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x}$. If $\dot{V}(x) < 0 \ \forall x \in \Omega$, then $\dot{x}$ is a asymptotically local stable equilibrium.*