UNIVERSIDADE FEDERAL DE SANTA CATARINA
CENTRO DE CIÊNCIAS FÍSICAS E MATEMÁTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA PURA E APLICADA

Francisco Gabriel Klock Campos Vidal

Partial monoid actions on objects in categories with pullbacks and their globalizations

Florianópolis
2024

Francisco Gabriel Klock Campos Vidal

Partial monoid actions on objects in categories with pullbacks and their globalizations

Dissertação submetida ao Programa de Pós-Graduação em Matemática Pura e Aplicada da Universidade Federal de Santa Catarina para a obtenção do título de Mestre em Álgebra.
Orientador: Prof. Eliezer Batista, Dr.
Cooerientador: Prof. Mykola Khrypchenko, Dr.

Florianópolis
2024

Ficha catalográfica gerada por meio de sistema automatizado gerenciado pela BU/UFSC.
Dados inseridos pelo próprio autor.

Klock Campos Vidal, Francisco Gabriel
Partial monoid actions on objects in categories with pullbacks and their globalizations / Francisco Gabriel Klock Campos Vidal ; orientador, Eliezer Batista, coorientador, Mykola Khrypchenko, 2024.
160 p.

Dissertação (mestrado) - Universidade Federal de Santa Catarina, Centro de Ciências Físicas e Matemáticas, Programa de Pós-Graduação em Matemática Pura e Aplicada, Florianópolis, 2024.

Inclui referências.

1. Matemática Pura e Aplicada. 2. Ações Parciais. 3. Monoides. 4. Categorias com Pullbacks. 5. Globalizações. I. Batista, Eliezer. II. Khrypchenko, Mykola. III. Universidade Federal de Santa Catarina. Programa de Pós Graduação em Matemática Pura e Aplicada. IV. Título.

Francisco Gabriel Klock Campos Vidal

Partial monoid actions on objects in categories with pullbacks and their globalizations

O presente trabalho em nível de mestrado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

Prof. Eliezer Batista, Dr.
Universidade Federal de Santa Catarina

Prof. Felipe Lopes Castro, Dr.
Universidade Federal de Santa Catarina

Prof. Joost Vercruysse, Dr.
Université Libre de Bruxelles

Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de mestre em Matemática Pura e Aplicada.

ICP-Edu
Documento assinado digitalmente
Douglas Soares Goncalves
Data: 21/03/2024 13:17:56-0300
CPF: ***.566.318-**
Verifique as assinaturas em https://v.ufsc.br

Prof. Douglas Soares Gonçalves, Dr.
Pós-Graduação

ICP-Edu
Documento assinado digitalmente
Eliezer Batista
Data: 21/03/2024 13:12:44-0300
CPF: ***.541.288-**
Verifique as assinaturas em https://v.ufsc.br

Prof. Eliezer Batista, Dr.
Orientador

Florianópolis, Dia 22 do mês 2 de 2024

AGRADECIMENTOS

Eu gostaria de agradecer inicialmente ao Prof. Mykola Khrypchenko, que me guiou nesta jornada desde o início do mestrado como meu orientador, e posteriormente se tornou meu coorientador por razões técnicas. Fico bastante grato por toda a sua paciência ao me orientar e pelas conversas frequentes que tivemos.

Agradeço ao Prof. Eliezer Batista por aceitar se tornar meu orientador enquanto o Prof. Mykola está afastado, e pelas discussões que auxiliaram no progresso deste trabalho.

Não posso deixar de agradecer ao Prof. Paulinho Demeneghi, que me guiou como meu orientador de TCC e me permitiu cursar o estágio de docência em uma de suas disciplinas.

I'm grateful to Prof. Joost Vercruysse and Dr. Paolo Saracco for the opportunity to have some fruitful discussions that allowed the initial results of this work to be expanded to higher generality.

I thank the members of the examination committee for accepting the invitation to judge this work.

Agradeço a todos os professores que deixaram um impacto positivo em minha carreira acadêmica ao passar pelo meu trajeto, seja pelo conteúdo de suas aulas, pelo modo de as lecionar, ou por outras formas de suporte. Também sou grato aos demais funcionários da universidade, com menção especial ao secretário Marcelo por toda a ajuda e paciência com respeito às questões burocráticas.

Fico extremamente grato à toda minha família pelo apoio e carinho incondicionais, e, em especial, à minha avó, Paulina, por me criar e me cuidar por todos estes anos. Gostaria também de expressar minha gratidão especial à minha mãe, Paula, aos meus irmãos, Yan, Louie e Julia, à minha avó, Ivone, a todos os meus tios, e, finalmente, ao meu pai, Chrystiann, que, ainda que eu não possua lembranças dele, sei que tinha um grande carinho por mim.

Agradeço aos meus colegas Luiz Felipe, Ricardo, Alek e Victor pelos papos amistosos e interessantes que tivemos por estes semestres, além de muitos outros colegas que tornaram meu dia a dia pela faculdade mais agradável.

Agradeço aos meus amigos virtuais do *Nox* pelos anos de amizade, conversas e jogos. Dou um agradecimento especial ao Haruki pela estadia em São Paulo, a qual me permitiu participar de um congresso que serviu como uma boa experiência acadêmica.

Expresso gratidão à Fundação de Amparo à Pesquisa e Inovação do Estado de Santa Catarina (Fapesc) pelo apoio financeiro.

Por fim, agradeço a tudo e todos que me possibilitaram estar aqui hoje, de uma maneira ou de outra.

RESUMO

Sejam M um monoide, $\mathscr{C}$ uma categoria com pullbacks e X um objeto de $\mathscr{C}$. Nós introduzimos os conceitos de ação parcial e ação parcial forte de M em X e estudamos a questão de sua globalização. Se uma ação parcial possui uma reflexão na subcategoria de ações globais, então nós reduzimos o problema à verificação de que um certo diagrama é um pullback em $\mathscr{C}$. Assim, nós damos uma construção de uma tal reflexão em termos de um colimite de um certo funtor com valores em $\mathscr{C}$. Nós especificamos esta construção para o caso de categorias que admitem certos coprodutos e coequalizadores. Nós aplicamos estes resultados nas categorias de conjuntos, espaços topológicos e álgebras.

RESUMO EXPANDIDO

Palavras Chave: ação parcial, monoide, categoria com pullbacks, globalização, reflexão

INTRODUÇÃO

Uma ação parcial de grupo é uma forma mais fraca de uma ação (global) de grupo, em que os elementos do grupo agem apenas em partes de um objeto. O conceito de ação parcial de grupo foi introduzido por Exel no estudo de certas C^*-álgebras em [8], e desde então foi explorado em diversos outros contextos.

Uma grande quantidade de exemplos de ações parciais de grupo provém de restrições de ações globais a subconjuntos apropriados. Assim, um problema bastante estudado é o de determinar sob que condições uma dada ação parcial é a restrição de uma ação global, a qual, neste caso, é dita uma globalização da ação parcial. Esta questão foi inicialmente abordada por Abadie em [1] no caso de ações parciais de um grupo topológico em espaços topológicos e C^*-álgebras.

Ações parciais de monoides foram introduzidas por Megrelishvili e Schröder em [17], onde verificam que as ações parciais de monoides em conjuntos e espaços topológicos são globalizáveis. No artigo [11], Hollings introduz uma definição de ação parcial de monoide que é mais fraca que a de [17], e mostra que as chamadas ações parciais fortes (que correspondem às ações parciais de [17]) são exatamente as ações parciais que possuem uma globalização.

Hu e Vercruysse em [12] definiram o conceito de uma (co)ação parcial de uma (co)álgebra em uma categoria monoidal com pullbacks em um objeto da mesma, chamada de “geometric partial (co)module”. A questão da globalização de tais ações parciais foi posteriormente estudada por Saracco e Vercruysse em [18], em que obtiveram condições necessárias e suficientes em termos de equalizadores e pushouts para que um “geometric partial comodule” seja globalizável.

Apesar de os “geometric partial (co)comodules” abrangerem diversos conceitos de ações parciais vistos na literatura, como ações parciais de monoides topológicos em espaços topológicos, coações parciais de álgebras de Hopf em álgebras e (co)ações parciais de álgebras de Hopf em espaços vetoriais, há certos conceitos de ação parcial que até então não aparentam ser cobertos por esta teoria, como ações parciais de grupos em anéis, álgebras, C^*-álgebras e semigrupos.

OBJETIVOS

O objetivo principal deste trabalho é introduzir uma forma unificada de se estudar diversos tipos ações parciais de monoides e grupos vistos na literatura, por meio da definição de ações parciais (fortes) de monoides em objetos de categorias com pullbacks.

Além disto, também objetivamos oferecer algumas respostas para a questão da globalização de tais ações parciais, dando condições para que uma dada ação parcial neste sentido possua uma globalização (universal), e exibir algumas aplicações para estes resultados em certas categorias, observando também as relações com os resultados de globalização já encontrados na literatura.

METODOLOGIA

Pesquisa bibliográfica por meio do estudo de artigos e outros tipos de trabalhos acadêmicos relacionados ao tema, além de discussões frequentes sobre o trabalho com os orientadores e outros pesquisadores.

RESULTADOS OBTIDOS

Inspirados pelo [12, Lema 1.7], verificamos a relação entre ações parciais (fortes) de monoides em conjuntos e morfismos parciais na categoria de conjuntos por meio de uma certa correspondência.

Com base nesta correspondência, para M um monoide, $\mathscr{C}$ uma categoria e X um objeto de $\mathscr{C}$, definimos *dados de ação parcial*, *ações globais*, *ações parciais* e *ações parciais fortes* de M em X. Definimos uma noção de morfismo entre estes conceitos, bem como a categoria formada pelos mesmos. Estudamos as propriedades de uma ação parcial forte no caso em que o monoide é um grupo.

A partir de uma ação global β de M em um objeto Y de $\mathscr{C}$ e um monomorfismo $\iota : X \to Y$, construímos uma ação parcial forte α de M em X, chamada a *restrição* de β a X via o monomorfismo ι. Com isto, definimos os conceitos de *globalização* e *globalização universal* de um dado de ação parcial. Para um dado de ação parcial α que possui uma reflexão na categoria $M\text{−}\mathbf{Act}_{\mathscr{C}}$, formada pelas ações globais de M em objetos de $\mathscr{C}$, no Teorema 5.2.5 obtivemos condições necessárias e suficientes em termos de pullbacks para que α possua uma globalização (universal).

No Teorema 5.2.15 construímos uma reflexão de α em $M\text{−}\mathbf{Act}_{\mathscr{C}}$ em termos de um colimite de um certo funtor com valores em $\mathscr{C}$. Especificamos este resultado no Corolário 5.2.19 para encontrar uma tal construção em termos de coprodutos e um coequalizador em $\mathscr{C}$. Se $\mathscr{C}$ possui tais coprodutos, no Teorema 5.2.26 exibimos condições necessárias e suficientes para que α possua uma tal reflexão em termos de um coequalizador em $M-\mathbf{Act}_{\mathscr{C}}$.

Observamos que na categoria de conjuntos estes resultados recuperam os resultados de globalização de [11]. Descrevemos as ações parciais globalizáveis na categoria de espaços topológicos. Estudamos as ações parciais de grupos na categoria de álgebras e exploramos as conexões das globalizações universal com as ações envolventes de [7].

CONSIDERAÇÕES FINAIS

Este trabalho proporciona uma forma unificada de se estudar diversos tipos ações parciais de monoides e grupos vistos na literatura. Os principais resultados obtidos são os que dão uma construção de uma reflexão de uma ação parcial na categoria de ações globais, a qual nos permite discernir em termos de pullbacks quando a ação parcial em questão possui uma globalização (universal) ou não. Assim, este trabalho contribui com uma forma unificada de estudar globalizações de ações parciais em diversos contextos.

ABSTRACT

Let M be a monoid, $\mathscr{C}$ a category with pullbacks and X an object of $\mathscr{C}$. We introduce the notions of partial action and strong partial action of M on X and study the question of their globalization. If a partial action admits a reflection in the subcategory of global actions, then we reduce the problem to the verification that a certain diagram is a pullback in $\mathscr{C}$. We then give a construction of such a reflection in terms of a colimit of a certain functor with values in $\mathscr{C}$. We specify this construction to the case of categories admitting certain coproducts and coequalizers. We apply these results to the categories of sets, topological spaces and algebras.

CONTENTS

1 INTRODUCTION

A partial group action is a generalization of a (global) group action that is concerned with symmetries of parts of an object, rather than its whole. This concept was introduced by Exel [8] in order to study certain C^*-algebras as crossed products by partial actions, and were since explored in many other settings. We refer the reader to the survey papers [6, 2] containing an extensive literature on the subject.

Plenty of examples of partial group actions arise from the restriction of a global action to an appropriate subset, in which case the global action is called a globalization or enveloping action of the partial action. The problem whether or not a partial group action has a globalization was first addressed by Abadie in the study of partial actions of a topological group on topological spaces and C^*-algebras in [1], where the author shows that the former are always globalizable and finds conditions for the latter to be globalizable in the commutative case.

In the setting of partial group actions on algebras, Dokuchaev and Exel showed in [7] that if A is a unital (associative) algebra, then a partial group action on A has an enveloping action if and only if each ideal associated to the partial action is unital. In this paper, the authors also considered crossed products by partial actions on algebras, where, unlike in the C^*-algebraic setting, the associativity of a crossed product is not automatic, and used them to relate partial actions with partial representations.

Partial monoid actions were introduced by Megrelishvili and Schröder in [17]. They show that the partial monoid actions on sets and topological spaces are globalizable, and that the latter globalization is well behaved when the partial action is confluent. In [11], Hollings introduces a weaker definition of a partial monoid action on a set and shows that the strong partial actions (which correspond to the partial actions of [17]) are precisely the globalizable ones.

In [12], Hu and Vercruysse introduced the concept of a partial (co)action of a (co)algebra in a monoidal category with pullbacks, called a geometric partial (co)module, in order to describe partial actions of algebraic groups from a Hopf-algebraic point of view. This allowed an unified approach to several kinds of partial actions, such as partial actions of topological monoids on topological spaces, partial coactions of Hopf algebras on algebras and partial (co)actions of Hopf algebras on vector spaces. The question of the globalization of geometric partial comodules was afterwards tackled by Saracco and Vercruysse in [18], where they obtain necessary and sufficient conditions in terms of equalizers and pushouts for such comodules to be globalizable.

The general theory of [12], however, does not seem to encompass certain concepts of partial actions that appear in the literature, such as partial group actions on rings, algebras, C^*-algebras and semigroups.

In this work we propose a parallel unified approach to partial actions that covers these and many other kinds of partial actions of groups and monoids, by defining the concept of a partial action of a monoid on an object in a category with pullbacks. Observe that we do not assume any monoidal structure on the category under consideration, any relation between the category and the monoid or any extra structure on the monoid.

We begin this thesis by briefly recalling in Chapter 2 basic notions and results, and fixing some notations that will be used throughout this work. We recall the definitions of categories, functors, natural transformations, (co)limits, pullbacks, (co)equalizers and (co)products and also review the elementary theory of inverse semigroups.

In Chapter 3, we give a more detailed introduction of spans and partial morphisms in a category $\mathscr{C}$, along with their corresponding categories $\mathbf{span}_{\mathscr{C}}$ and $\mathbf{par}_{\mathscr{C}}$. We introduce restriction and inverse categories and verify that $\mathbf{par}_{\mathscr{C}}$ has an interesting restriction structure that makes it a restriction category.

In Chapter 4 we introduce partial and strong partial monoid actions on sets in terms of partial action data, and describe their connection to partial morphisms through a correspondence similar to [12, Lemma 1.7]. This serves as an inspiration to generalize the notions of *partial action data*, as well as *global*, *partial* and *strong partial* actions, to the context of a monoid acting on an object in a category with pullbacks. We also define morphisms between such partial action data and the corresponding categories that come with this notion. At the end of this chapter we give a description of the strong partial actions in the case where the monoid is a group.

The question of the globalization of these partial actions is tackled in Chapter 5. Given β a global action of a monoid M on an object Y in a category $\mathscr{C}$ and $\iota : X \to Y$ a monomorphism in $\mathscr{C}$, we define the notion of a *restriction* α of β to X via ι and prove that it is a strong partial action of M on X. In this situation, we say that (β, ι) is a *globalization* of α. We further say that (β, ι) is a *universal globalization* of α if it satisfies a certain universal property among the globalizations of α. Unlike [18], we do not require the morphism $\iota : \alpha \to \beta$ to be a reflection of α in the category $M{-}\mathbf{Act}_{\mathscr{C}}$ of global actions of M on objects of $\mathscr{C}$ (see Example 6.3.10 for an example of a partial action that has a universal globalization whose ι is not a reflection), and we also observe that not every strong partial action has a globalization (see Example 6.2.9). However, when such a reflection exists, Theorem 5.2.5 gives necessary and sufficient conditions for α to have a (universal and otherwise) globalization in terms of pullback diagrams in $\mathscr{C}$ that resemble (the dual of) the pushout from [18, Theorem 3.5 (II)]. In the main results of this work, we describe a construction of a reflection of α in $M{-}\mathbf{Act}_{\mathscr{C}}$ in terms of a colimit of a certain functor with values in $\mathscr{C}$ (see Theorem 5.2.15) and in terms of certain coproducts and a certain coequalizer (see Corollary 5.2.19). If $\mathscr{C}$ admits such coproducts, we show in Theorem 5.2.26 that the existence of a reflection of α in $M{-}\mathbf{Act}_{\mathscr{C}}$ is equivalent to the existence of a coequalizer of a certain pair of morphisms in $M\!-\!\mathbf{Act}_{\mathscr{C}}$, giving us a condition similar to (the dual of) [18, Theorem 3.5 (I)]

We apply our general results to certain categories in Chapter 6. We illustrate how one recovers Hollings's results on the globalization of strong partial actions on sets by applying our technique to $\mathscr{C} = \mathbf{Set}$. We study the partial actions on objects in the category of topological spaces and classify the globalizable ones. Finally, we describe the connection between the universal globalizations and the enveloping actions of [7], showing in Proposition 6.3.9 that, in the unital case, the enveloping action of a partial group action on an algebra is a universal globalization of the partial action on an object of a certain subcategory of the category of $\mathbb{K}$-algebras.

2 PRELIMINARIES

In this chapter we introduce notations and recall basic notions of category theory, including those of reflections, functors, natural transformations, (co)limits, pullbacks, (co)equalizers and (co)products. In the last section of this chapter we also review the basic theory of inverse semigroups.

2.1 CATEGORIES, FUNCTORS AND NATURAL TRANSFORMATIONS

Definition 2.1.1. A **category** $\mathscr{C}$ consists of the following data.

(1) A class[1] of objects denoted by $\mathrm{Ob}(\mathscr{C})$ or simply by $\mathscr{C}$ if there is no confusion.

(2) For each $X, Y \in \mathscr{C}$ a class $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of morphisms from X to Y. We write $f : X \to Y$ to mean $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. We require that $\mathrm{Hom}_{\mathscr{C}}(X, Y) \cap \mathrm{Hom}_{\mathscr{C}}(X', Y') = \emptyset$ if $(X, Y) \neq (X', Y')$.

(3) For each $X, Y, Z \in \mathscr{C}$ a map

$$\circ_{XYZ} : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z),$$

called a *composition* of $\mathscr{C}$. We denote $\circ_{XYZ}(g, f)$ by $g \circ f$.

This data is required to satisfy the following properties.

(i) For all $X \in \mathscr{C}$ there exists a morphism $id_X \in \mathrm{End}(X)$, called the *identity morphism* of X, such that

$$f \circ id_X = f = id_Y \circ f$$

for all $f : X \to Y$.

(ii) The composition of $\mathscr{C}$ is associative. That is, for all $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

In a category $\mathscr{C}$, for each $X \in \mathscr{C}$ the set $\mathrm{Hom}_{\mathscr{C}}(X, X)$ is a monoid under the composition composition, called the **monoid of endomorphisms of** X, and we denote it by $\mathrm{End}_{\mathscr{C}}(X)$. Given $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, the **domain** of f is defined to be the object X

[1] Many interesting collections of objects are too big to be sets, and are, rather, proper *classes* [13].

and is denoted by $\operatorname{dom} f = X$, and the **codomain** of f is defined to be the object Y and is denoted by $\operatorname{cdm} f = Y$.

There are several examples of categories, such as

- **Set**, the category whose objects are the sets, the morphisms are the maps between the sets and the composition is given by the usual composition of maps;
- **Sem**, the category whose objects are the semigroups, the morphisms are semigroup homomorphisms between the semigroups and the composition is the usual composition of maps;
- **Mon**, the category formed by monoids and monoid homomorphisms;
- **Grp**, the category formed by groups and group homomorphisms;
- **Top**, the category formed by topological spaces and continuous maps;
- **Poset**, the category formed by partial ordered sets and order-preserving maps;
- **Ring**, the category formed by rings and ring homomorphisms;
- $\mathbf{Vect}_{\mathbb{K}}$, the category formed by vector spaces over a field $\mathbb{K}$ and linear maps;
- $\mathbf{Alg}_{\mathbb{K}}$, the category formed by (associative, not necessarily unital) algebras over a field $\mathbb{K}$ and algebra homomorphisms;
- $\mathbf{C^*}$**-Alg**, the category formed by the C^*-algebras and $*$-homomorphisms;
- given $(X, \leq)$ a partially-ordered set, there is a category $\mathscr{C}$ whose objects are the elements of X, where there is a (unique) morphism from x to y when $x \leq y$ and the composition is given by the transitivity of $\leq$;
- given G a group, there is a category, also denoted by G, with a single object $*$, where $\operatorname{Hom}_G(*, *) = G$ and the composition $h \circ g$ is given by the product $h \cdot g$ in G.

Definition 2.1.2. A category $\mathscr{C}$ is said to be **small** if the class of objects of $\mathscr{C}$ is a set.

Definition 2.1.3. A category $\mathscr{C}$ is said to be **locally small** if the class of morphisms between any two objects of $\mathscr{C}$ is a set.

Throughout this work, we will always assume a category $\mathscr{C}$ to be locally small, unless stated otherwise.

Definition 2.1.4. Let $\mathscr{C}$ be a category. A **subcategory** of $\mathscr{C}$ is a category $\mathscr{D}$ whose class of objects is contained in the class of objects of $\mathscr{C}$ and such that for all $X, Y \in \mathscr{D}$ we have $\operatorname{Hom}_{\mathscr{D}}(X, Y) \subseteq \operatorname{Hom}_{\mathscr{C}}(X, Y)$.

If $\operatorname{Hom}_{\mathscr{D}}(X, Y) = \operatorname{Hom}_{\mathscr{C}}(X, Y)$ for all $X, Y \in \mathscr{C}$, we say that $\mathscr{D}$ is a **full** subcategory of $\mathscr{C}$.

Definition 2.1.5. Let $\mathscr{C}$ be a category, $\mathscr{D}$ a subcategory of $\mathscr{C}$ and $X \in \mathscr{C}$. A **reflection** of X in $\mathscr{D}$ (or a *$\mathscr{D}$-reflection* of X) is a morphism $r : X \to Y$ in $\mathscr{C}$ with $Y \in \mathscr{D}$ such that for any $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ with $Z \in \mathscr{D}$ there is a unique $f' \in \mathrm{Hom}_{\mathscr{D}}(Y, Z)$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "f'"] \\ & Z \end{tikzcd}$$

In this situation, we say that X has a reflection in $\mathscr{D}$.

Definition 2.1.6. Let $\mathscr{C}$ be a category.

- A morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a **monomorphism** if for all $g, h \in \mathrm{Hom}_{\mathscr{C}}(W, X)$ we have
$$f \circ g = f \circ h \implies g = h$$
- A morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is an **epimorphism** if for all $g, h \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$ we have
$$g \circ f = h \circ f \implies g = h$$
- A morphism $\varphi \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is an **isomorphism** if there exists a morphism $\psi \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ such that
$$\psi \circ \varphi = id_X \quad \text{and} \quad \varphi \circ \psi = id_Y.$$

  In this situation, we say that ψ is an **inverse** of φ.

Remark 2.1.7. The inverse of an isomorphism φ is unique, and is denoted by φ^{-1}.

Example 2.1.8. In **Set**, the monomorphisms are precisely the injective maps, the epimorphisms the surjective maps and the isomorphisms the bijective maps.

Definition 2.1.9. Let $\mathscr{C}$ be a category and $Z \in \mathscr{C}$. We say that two monomorphisms $f : X \to Z$ and $g : Y \to Z$ are **equivalent** if there exists an isomorphism $\varphi : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

The equivalence classes formed by this relation on the class of monomorphisms with codomain Z are said to be **subobjects** of Z.

Definition 2.1.10. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A (covariant) **functor** F from $\mathscr{C}$ to $\mathscr{D}$, denoted by $F : \mathscr{C} \to \mathscr{D}$, consists of the following data.

(1) A map F from the class of objects of $\mathscr{C}$ to the class of objects of $\mathscr{D}$.

(2) For $X, Y \in \mathscr{C}$, a map $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$.

This data is required to satisfy the following properties.

(i) For all $X \in \mathscr{C}$,
$$F(id_X) = id_{F(X)}.$$

(ii) For all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$,
$$F(g \circ f) = F(g) \circ F(f).$$

Definition 2.1.11. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be **faithful** if for all $X, Y \in \mathscr{C}$ the map $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is injective.

There are also plenty of examples of functors between categories.

- Let $\mathscr{C} \in \{\mathbf{Set}, \mathbf{Sem}, \mathbf{Mon}, \mathbf{Grp}, \mathbf{Top}, \mathbf{Poset}, \mathbf{Ring}, \mathbf{Vect}_{\mathbb{K}}, \mathbf{Alg}_{\mathbb{K}}, \mathbf{C^*\text{-}Alg}\}$. Then there is a functor $U : \mathscr{C} \to \mathbf{Set}$, called a **forgetful functor**, where $U(X)$ is the underlying set of X for all $X \in \mathscr{C}$ and $U(f) = f$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- There are other kinds of forgetful functors that do not go to **Set** by forgetting less structure. For example, there are forgetful functors from **Grp** to **Mon**, from **Mon** to **Sem**, from **Grp** to **Sem**, from $\mathbf{C^*}$**-Alg** to $\mathbf{Alg}_{\mathbb{K}}$, among many others.
- Let $\mathscr{C}$ be any category. There is a functor $id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, called the identity functor of $\mathscr{C}$, where $id_{\mathscr{C}}(X) = X$ for all $X \in \mathscr{C}$ and $id_{\mathscr{C}}(f) = f$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- Let $\mathscr{D}$ be any category and $\mathscr{C}$ a subcategory of $\mathscr{D}$. Then there is a functor $F : \mathscr{C} \to \mathscr{D}$, called the inclusion functor, where $F(X) = X$ for all $X \in \mathscr{C}$ and $F(f) = f$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.
- Let G be a group and α an action of G on a set X. Then α can be seen as a functor $\alpha : G \to \mathbf{Set}$, where $\alpha(*) = X$ and $\alpha(g)$ is the map from X to X given by the action of g on X, for all $g \in G$.
- There is a functor from **Set** to **Grp** that sends a set X to the free group $F(X)$ generated by X and a map $f : X \to Y$ to the unique map $F(f) : F(X) \to F(Y)$ that restricts to f on the generators of $F(X)$.

Definition 2.1.12. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A **natural transformation** η from F to G, which we denote by $\eta : F \to G$, is a family $\{F(X) \xrightarrow{\eta_X} G(X) : X \in \mathscr{C}\}$ of morphisms in $\mathscr{D}$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ the following diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

2.2 LIMITS AND COLIMITS

Definition 2.2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and $X \in \mathscr{D}$. We define the functor $\Delta(X) : \mathscr{C} \to \mathscr{D}$, called a **constant functor**, that maps all objects in $\mathscr{C}$ to X and all morphisms in $\mathscr{C}$ to id_X.

Definition 2.2.2. Let I and $\mathscr{C}$ be categories (the former is called an index category), and $F : I \to \mathscr{C}$ a functor.

- A **cone** to F is a natural transformation $\eta : \Delta(X) \to F$ for some $X \in \mathscr{C}$.
- A **cocone** to F is a natural transformation $\eta : F \to \Delta(X)$ for some $X \in \mathscr{C}$.
- A **limit** of F is a cone $\eta : \Delta(X) \to F$ such that for all cones $\xi : \Delta(Y) \to F$ there exists a unique morphism $\xi' \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\xi_i"] \arrow[d, "\xi'"'] & F(i) \\ X \arrow[ur, "\eta_i"'] & \end{tikzcd}$$

  commutes for all $i \in I$.
- A **colimit** of F is a cocone $\eta : F \to \Delta(X)$ such that for all cocones $\xi : F \to \Delta(Y)$ there exists a unique morphism $\xi' \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ such that the diagram

$$\begin{tikzcd} F(i) \arrow[r, "\xi_i"] \arrow[dr, "\eta_i"'] & Y \\ & X \arrow[u, "\xi'"'] \end{tikzcd}$$

  commutes for all $i \in I$.

Definition 2.2.3. A category $\mathscr{C}$ is said to be **(co)complete** if every functor from a small category to $\mathscr{C}$ has a (co)limit.

There are some interesting special cases of limits and colimits of functors that we are going to illustrate now.

2.2.1 PULLBACKS

Definition 2.2.4. Let $\mathscr{C}$ be a category, $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$. A **pullback** of f and g is a pair of morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ such that

(1) the diagram

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\vee" very near start] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \tag{2.1}$$

commutes;

(2) whenever $q_1 : Q \to X$ and $q_2 : Q \to Y$ are morphisms such that the diagram

$$\begin{tikzcd} & Q \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes, there exists a unique morphism $\varphi : Q \to P$ such that the following diagram commutes.

$$\begin{tikzcd} & Q \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] \arrow[d, dashed, "\varphi"] & \\ & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\vee" very near start] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

In this situation, we say that diagram (2.1) is a **pullback diagram**, a **pullback square** or simply a pullback.

Remark 2.2.5. Let $\mathscr{C}$ be a category and $f : X \to Z$ and $g : Y \to Z$ morphisms in $\mathscr{C}$. Consider the category I whose class of objects is the set $\{1, 2, 3\}$ and the only nontrivial morphisms in I are $\varphi_{13} : 1 \to 3$ and $\varphi_{23} : 2 \to 3$. Then a pullback of f and g can be seen as a limit of the functor $F : I \to \mathscr{C}$ where $F(\varphi_{13}) = f$ and $F(\varphi_{23}) = g$, and vice-versa.

Definition 2.2.6. Let $\mathscr{C}$ be a category. If every pair of morphisms in $\mathscr{C}$ has a pullback, we will say that $\mathscr{C}$ is a category **with pullbacks**.

Example 2.2.7. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in **Set**. Then the maps $p_1 : P \to X$ and $p_2 : P \to Y$ form a pullback of f and g, where

$$P = \{(x, y) \in X \times Y : f(x) = g(y)\} \tag{2.2}$$

and p_1 and p_2 are given by $p_1(x, y) = x$ and $p_2(x, y) = y$, for all $(x, y) \in P$.

Let $\mathscr{C} \in \{\mathbf{Set}, \mathbf{Sem}, \mathbf{Mon}, \mathbf{Grp}, \mathbf{Top}, \mathbf{Poset}, \mathbf{Ring}, \mathbf{Vect}_{\mathbb{K}}, \mathbf{Alg}_{\mathbb{K}}, \mathbf{C}^*\text{-}\mathbf{Alg}\}$. If the maps f and g are morphisms in $\mathscr{C}$, then the set P, as defined in (2.2), naturally has additional structure that makes it an object of $\mathscr{C}$ and the maps p_1 and p_2 morphisms in $\mathscr{C}$ that form a pullback of f and g in $\mathscr{C}$. Thus, $\mathscr{C}$ is a category with pullbacks.

Example 2.2.8. Let $(X, \leq)$ be a partially-ordered set and $x, y, z \in X$ such that $x, y \leq z$. Let $\mathscr{C}$ be the category associated to $(X, \leq)$, f the unique morphism from x to z in $\mathscr{C}$ and g the unique morphisms from y to z in $\mathscr{C}$.

Consider the set $A = \{a \in X : a \leq x \text{ and } a \leq y\}$. Observe that, given $a \in X$, there exists a morphism ι_a^x from a to x and a morphism ι_a^y from a to y if and only if $a \in A$. In this situation, the morphisms ι_a^x and ι_a^y are such that

$$f \circ \iota_a^x = g \circ \iota_a^y,$$

since there is only one morphism in $\mathrm{Hom}_{\mathscr{C}}(a, z)$.

Thus, a pullback of f and g is precisely a maximum element p of A along with the morphisms $\iota_p^x : p \to x$ and $\iota_p^y : p \to y$ in $\mathscr{C}$.

In particular, if $(X, \leq)$ is a meet-semilattice, $\mathscr{C}$ is a category with pullbacks.

Example 2.2.9. Let $X, Y \in \mathbf{Set}$, $f : X \to Y$ a map and $A \subseteq Y$, with corresponding inclusion map ι. Then the diagram

$$\begin{tikzcd}
& f^{-1}(A) \arrow[dl, "\widehat{\iota}"'] \arrow[dr, "\widehat{f}"] \arrow[dd, phantom, "\vee"] & \\
X \arrow[dr, "f"'] & & A \arrow[dl, "\iota"] \\
& Y &
\end{tikzcd}$$

is a pullback, where $\widehat{\iota}$ is the inclusion of $f^{-1}(A)$ into X and $\widehat{f}$ is given by

$$\widehat{f}(x) = f(x)$$

for all $x \in f^{-1}(A)$.

For our purposes, requiring a category to have pullbacks is a condition that is stronger than necessary, and excludes a category we are going to explore later. Hence, we introduce the following definition, whose nomenclature is inspired by Example 2.2.9.

Definition 2.2.10. Let $\mathscr{C}$ be a category. If every pair of morphisms in $\mathscr{C}$ that includes a monomorphism has a pullback, we will say that $\mathscr{C}$ is a category with **inverse images**.

The following is the previously mentioned example of a category with inverse images that doesn't have all pullbacks.

Definition 2.2.11. Denote by $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ the subcategory of $\mathbf{Alg}_{\mathbb{K}}$ whose morphisms are the $\mathbb{K}$-algebra homomorphisms $f : A \to B$ such that $f(A)$ is an ideal of B.

Proposition 2.2.12. *The category* $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ *has inverse images.*

Proof. Let $f : A \to C$ and $g : B \to C$ be morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, where g is a monomorphism. We will verify that the maps $p_1 : Z \to A$ and $p_2 : Z \to B$ form a pullback of f and g in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, where

$$Z = \{(a, b) \in A \times B : f(a) = g(b)\}$$

as a subalgebra of $A \times B$ and p_1 and p_2 are given by $p_1(a, b) = a$ and $p_2(a, b) = b$, for all $(a, b) \in Z$.

Let us first verify that p_1 and p_2 are morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. It is a simple verification that p_1 and p_2 are algebra homomorphisms, so we will only check that $p_1(Z) \trianglelefteq A$ and $p_2(Z) \trianglelefteq B$.

Let $(a, b) \in Z$ and $a' \in A$. Then

$$f(aa') = f(a)f(a') = g(b)f(a') = g(b') \tag{2.3}$$

for some $b' \in B$, since $g(B)$ is an ideal of C.

By (2.3) it follows that $(aa', b') \in Z$. Therefore,

$$p_1(a, b)a' = aa' = p_1(aa', b') \in p_1(Z).$$

Similarly, $a'p_1(a, b) \in p_1(Z)$. Hence, $p_1(Z)$ is an ideal of A. In a similar fashion, $p_2(Z) \trianglelefteq B$, as desired.

Now let

$$\begin{tikzcd} & W \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{2.4}$$

be a commutative diagram in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$.

Since the diagram

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{2.5}$$

is a pullback in $\mathbf{Alg}_{\mathbb{K}}$, there exists a unique algebra homomorphism $\varphi : W \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} & W \arrow[d, dashed, "\varphi"] \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] & \\ & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{2.6}$$

Observe that φ is given by $\varphi(w) = (q_1(w), q_2(w))$ for all $w \in W$.

We will verify that φ is a morphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. For let $w \in W$ and $(a, b) \in Z$. Since q_1 and q_2 are morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, there exist $w_A, w_B \in W$ such that

$$q_1(w)a = q_1(w_A) \quad \text{and} \quad q_2(w)b = q_2(w_B).$$

Therefore, we have

$$\varphi(w)(a, b) = (q_1(w), q_2(w))(a, b) = (q_1(w)a, q_2(w)b) = (q_1(w_A), q_2(w_B)). \tag{2.7}$$

Now, by the fact that $(q_1(w_A), q_2(w_B)) \in Z$ and the commutativity of (2.4), we have

$$g(q_2(w_A)) = f(q_1(w_A)) = g(q_2(w_B)).$$

Hence, since g is a monomorphism, and, thus, an injective map,

$$q_2(w_A) = q_2(w_B). \tag{2.8}$$

Thus, by (2.7) we have

$$\varphi(w)(a, b) = (q_1(w_A), q_2(w_B)) = (q_1(w_A), q_2(w_A)) = \varphi(w_A),$$

so $\varphi(w)(a, b) \in \varphi(W)$. Therefore, $\varphi(W)$ is an ideal of Z.

Hence, φ is a morphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ such that diagram (2.6) commutes. Since φ is the unique such morphism in $\mathbf{Alg}_{\mathbb{K}}$, its uniqueness in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ also follows. Thus, (2.5) gives a pullback of f and g in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, and, hence, $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ has inverse images, as desired. $\square$

To check that $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ does not in general have pullbacks, we will first prove the following lemma.

Lemma 2.2.13. *Let $f : A \to C$ and $g : B \to C$ be morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Then a pullback of f and g in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ is a pullback of f and g in $\mathbf{Alg}_{\mathbb{K}}$.*

Proof. Let

$$\begin{tikzcd} & W \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{2.9}$$

be a pullback square in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, and let $p_1 : Z \to A$ and $p_2 : Z \to B$, where

$$Z = \{(a, b) \in A \times B : f(a) = g(b)\}$$

as a subalgebra of $A \times B$ and p_1 and p_2 are given by $p_1(a, b) = a$ and $p_2(a, b) = b$, for all $(a, b) \in Z$, so

$$\begin{tikzcd} & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{2.10}$$

is a pullback square in $\mathbf{Alg}_{\mathbb{K}}$, as seen in Example 2.2.7.

The same argument made in Proposition 2.2.12 shows that p_1 and p_2 are morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ (observe that it was not required for f or g to be monomorphisms for this argument).

Then since (2.9) is a pullback and (2.10) is a commmutative diagram in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, there exists a unique morphism φ in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ such that the following diagram commutes.

$$\begin{tikzcd} & Z \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] \arrow[d, dashed, "\varphi"] & \\ & W \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

And since (2.10) is a pullback and (2.9) is a commutative diagram in $\mathbf{Alg}_{\mathbb{K}}$, there exists a unique morphism ψ in $\mathbf{Alg}_{\mathbb{K}}$ such that the following diagram commutes.

$$\begin{tikzcd}
& W \arrow[d, dashed, "\psi"] \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] & \\
& Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\vee" very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

Since (2.10) is a pullback diagram, it is a simple verification that

$$\psi \circ \varphi = id_Z.$$

Hence, $\psi(W) \supseteq id_Z(Z) = Z$, so ψ is a morphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Since (2.9) is a pullback, it is then also a simple verification that

$$\varphi \circ \psi = id_W.$$

Thus, φ is an isomorphism in $\mathbf{Alg}_{\mathbb{K}}$, so it is a straightforward verification that (2.9) is a pullback in $\mathbf{Alg}_{\mathbb{K}}$. $\square$

One can promptly verify that if f and g have a pullback in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, then a pullback diagram in $\mathbf{Alg}_{\mathbb{K}}$ is also a pullback diagram in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Hence, we have the following example of morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ that do not have a pullback.

Example 2.2.14. Consider $f : \mathbb{K}[x] \to \mathbb{K}$ and $g : \mathbb{K}[x] \to \mathbb{K}$ the evaluation homomorphisms given by

$$f(p) = p(0) \quad \text{and} \quad g(p) = p(1),$$

for all $p \in \mathbb{K}[x]$. Observe that f and g are surjective, so they are morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Assume by contradiction that f and g have a pullback in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$.

Then the diagram

$$\begin{tikzcd}
& Z \arrow[dl, "z_1"'] \arrow[dr, "z_2"] \arrow[dd, phantom, "\vee" very near start] & \\
\mathbb{K}[x] \arrow[dr, "f"'] & & \mathbb{K}[x] \arrow[dl, "g"] \\
& \mathbb{K} &
\end{tikzcd} \tag{2.11}$$

is a pullback in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, where

$$Z = \{(p,q) \in \mathbb{K}[x] \times \mathbb{K}[x] : p(0) = q(1)\}$$

and z_1 and z_2 are given by $z_1(p,q) = p$ and $z_2(p,q) = q$, for all $(p,q) \in Z$.

Let W be the ideal of $\mathbb{K}[x]$ generated by $x(x-1)$. Then the corresponding inclusion $\iota : W \to \mathbb{K}[x]$ is such that

$$f \circ \iota = g \circ \iota.$$

Thus, since (2.11) is a pullback diagram, there exists a unique morphism φ in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ such that the following diagram commutes.

$$\begin{tikzcd}
& W \arrow[d, dashed, "\varphi"] \arrow[ddl, bend right, "\iota"'] \arrow[ddr, bend left, "\iota"] & \\
& Z \arrow[dl, "z_1"'] \arrow[dr, "z_2"] \arrow[dd, phantom, "\vee", very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

It is a simple verification that φ must be given by $\varphi(p) = (p,p)$ for all $p \in W$. However, $\varphi(W)$ is not an ideal of Z. Indeed, by taking $p = x$, $q = x-1$ and $r = x(x-1)$, we have

$$\varphi(r)(p,q) = (r,r)(p,q) = (rp, rq) = (x^2(x-1), x(x-1)^2) \notin \varphi(W).$$

Hence, we have a contradiction.

Proposition 2.2.15. *Let $\mathscr{C}$ be a category and consider the following diagram.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "I"] & B \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "II"] & C \arrow[d, "j"] \\
X \arrow[r, "k"'] & Y \arrow[r, "l"'] & Z
\end{tikzcd} \tag{2.12}$$

If the squares I and II in (2.12) *are pullback, then the outermost diagram of* (2.12) *is a pullback.*

Proof. Assume that the squares I and II in (2.12) are pullback. Let $P \in \mathscr{C}$ and $p_1 : P \to C$

and $p_2 : P \to X$ be morphisms such that the following diagram commutes.

$$
\begin{tikzcd}
P \arrow[rrrd, bend left, "p_1"] \arrow[rdd, "p_2"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "I"] & B \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "II"] & C \arrow[d, "j"] \\
& X \arrow[r, "k"'] & Y \arrow[r, "l"'] & Z
\end{tikzcd}
$$

Since the square II is a pullback and

$$j \circ p_1 = l \circ k \circ p_2,$$

there exists a unique morphism $\varphi : P \to B$ such that the following diagram commutes.

$$
\begin{tikzcd}
P \arrow[rrrd, bend left, "p_1"] \arrow[rrd, dashed, bend left, "\varphi"] \arrow[rdd, "p_2"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "I"] & B \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "II"] & C \arrow[d, "j"] \\
& X \arrow[r, "k"'] & Y \arrow[r, "l"'] & Z
\end{tikzcd}
\tag{2.13}
$$

Now, since the square I is a pullback and

$$i \circ \varphi = k \circ p_2,$$

there exists a unique morphism $\psi : P \to A$ such that the following diagram commutes.

$$
\begin{tikzcd}
P \arrow[rrd, dashed, bend left, "\varphi"] \arrow[rd, dashed, "\psi"] \arrow[rdd, "p_2"'] & & \\
& A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "I"] & B \arrow[d, "i"] \\
& X \arrow[r, "k"'] & Y
\end{tikzcd}
\tag{2.14}
$$

In particular, by the commutativity of (2.13) and (2.14) the diagram

$$
\begin{tikzcd}
P \arrow[rrrd, bend left, "p_1"] \arrow[rd, dashed, "\psi"] \arrow[rdd, "p_2"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "I"] & B \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "II"] & C \arrow[d, "j"] \\
& X \arrow[r, "k"'] & Y \arrow[r, "l"'] & Z
\end{tikzcd}
\tag{2.15}
$$

commutes. Let us verify that ψ is the unique morphism such that (2.15) is commutative.

For let ψ' be a morphism such that the diagram

$$\begin{tikzcd}
P \arrow[drrr, bend left, "p_1"] \arrow[dr, dashed, "\psi'"] \arrow[ddr, "p_2"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "I"] & B \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "II"] & C \arrow[d, "j"] \\
& X \arrow[r, "k"'] & Y \arrow[r, "l"'] & Z
\end{tikzcd}$$

commutes. Then since

$$i \circ (f \circ \psi') = k \circ p_2$$

and

$$g \circ (f \circ \psi') = p_1,$$

by the uniqueness of φ in (2.13) it follows that

$$f \circ \psi' = \varphi.$$

Thus, since we also have that

$$h \circ \psi' = p_2,$$

by the uniqueness of ψ in (2.14) it follows that $\psi = \psi'$, as desired. $\square$

Proposition 2.2.16. *Let $\mathscr{C}$ be a category and $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Then f is a monomorphism if and only if the following diagram is a pullback.*

$$\begin{tikzcd}
& X \arrow[dl, "id_X"'] \arrow[dr, "id_X"] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd} \tag{2.16}$$

Proof. Assume that f is a monomorphism and let $g, h : W \to X$ be morphisms such that the diagram

$$\begin{tikzcd}
& W \arrow[dl, "g"'] \arrow[dr, "h"] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd} \tag{2.17}$$

commutes. Then $f \circ g = f \circ h$, so, since f is a monomorphism, $g = h$. In this case, let $\varphi = g = h$.

Then the diagram

$$
\begin{tikzcd}
 & W \arrow[d, "\varphi"] \arrow[ddl, bend right, "g"'] \arrow[ddr, bend left, "h"] & \\
 & X \arrow[dl, "id_X"'] \arrow[dr, "id_X"] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
 & Y &
\end{tikzcd}
\tag{2.18}
$$

commutes. And φ is the unique such morphism, for if φ' is such that $id_X \circ \varphi' = g$ and $id_X \circ \varphi = h$, then $\varphi' = g = \varphi$. Hence, (2.16) is a pullback.

Conversely, assume that (2.16) is a pullback and let $g, h \in \mathrm{Hom}_{\mathscr{C}}(W, X)$ be morphisms such that $f \circ g = f \circ h$. Then (2.17) commutes. Since (2.16) is a pullback, there exists a unique morphism φ such that (2.18) commutes.

By the commutativity of (2.18) we then have

$$h = id_X \circ \varphi = g,$$

so f is a monomorphism, as desired. □

Proposition 2.2.17. *Let $\mathscr{C}$ be a category, $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$. If f is a monomorphism and*

$$
\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & Z &
\end{tikzcd}
\tag{2.19}
$$

is a pullback, then p_2 is a monomorphism.

Proof. Let $h, k : Q \to P$ be morphisms such that

$$p_2 \circ h = p_2 \circ k. \tag{2.20}$$

Then observe that, by the commutativity of (2.19) and by (2.20)

$$f \circ p_1 \circ h = g \circ p_2 \circ h = g \circ p_2 \circ k = f \circ p_1 \circ k.$$

Therefore, since f is a monomorphism,

$$p_1 \circ h = p_1 \circ k.$$

Let

$$\mu = p_1 \circ h = p_1 \circ k : Q \to X \quad \text{and} \quad \nu = p_2 \circ h = p_2 \circ k : Q \to Y. \tag{2.21}$$

Then the diagram

$$\begin{tikzcd}
& Q \arrow[d, "\varphi"] \arrow[ddl, bend right, "\mu"'] \arrow[ddr, bend left, "\nu"] & \\
& P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\vee", very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd} \tag{2.22}$$

commutes for both $\varphi = h$ and $\varphi = k$. Hence, since (2.19) is a pullback, by the uniqueness of φ it follows that $h = k$, so p_2 is a monomorphism, as desired. □

Definition 2.2.18. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F **preserves pullbacks** if whenever

$$\begin{tikzcd}
& P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\vee", very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

is a pullback diagram in $\mathscr{C}$, then so is the following diagram in $\mathscr{D}$.

$$\begin{tikzcd}
& F(P) \arrow[dl, "F(p_1)"'] \arrow[dr, "F(p_2)"] & \\
F(X) \arrow[dr, "F(f)"'] & & F(Y) \arrow[dl, "F(g)"] \\
& F(Z) &
\end{tikzcd}$$

Proposition 2.2.19. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. If F preserves pullbacks, then whenever f is a monomorphism in $\mathscr{C}$, we have that $F(f)$ is a monomorphism in $\mathscr{D}$.*

Proof. Let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ be a monomorphism in $\mathscr{C}$. By Proposition 2.2.16, the diagram

$$\begin{tikzcd}
& X \arrow[dl, "id_X"'] \arrow[dr, "id_X"] \arrow[dd, phantom, "\vee", very near start] & \\
X \arrow[dr, "f"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd} \tag{2.23}$$

is a pullback in $\mathscr{C}$.

Now, since F is a functor, $F(id_X) = id_{F(X)}$. Therefore, since (2.23) is a pullback and F preserves pullbacks, we have the the diagram

$$\begin{tikzcd} & F(X) \arrow[dl, "id_{F(X)}"'] \arrow[dr, "id_{F(X)}"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ F(X) \arrow[dr, "F(f)"'] & & F(X) \arrow[dl, "F(f)"] \\ & F(Y) & \end{tikzcd}$$

is a pullback in $\mathscr{D}$. Hence, by Proposition 2.2.16, $F(f)$ is a monomorphism, as desired. $\square$

2.2.2 EQUALIZERS AND COEQUALIZERS

Definition 2.2.20. Let $\mathscr{C}$ be a category and $f, g : X \to Y$ morphisms in $\mathscr{C}$.

- An **equalizer** of f and g is a morphism $e : E \to X$ such that
  (1) $f \circ e = g \circ e$;
  (2) whenever $h : W \to X$ is a morphism such that $f \circ h = g \circ h$, there exists a unique morphism $h' : W \to E$ such that the triangle in the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ W \arrow[u, dashed, "h'"] \arrow[ur, "h"'] & & \end{tikzcd}$$

- A **coequalizer** of f and g is a morphism $c : Y \to C$ such that
  (1) $c \circ f = c \circ g$
  (2) whenever $h : Y \to Z$ is a morphism such that $h \circ f = h \circ g$, there exists a unique morphism $h' : W \to E$ such that the triangle in the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] \arrow[dr, "h"'] & C \arrow[d, dashed, "h'"] \\ & & Z \end{tikzcd}$$

Remark 2.2.21. Let $\mathscr{C}$ be a category and $f, g : X \to Y$ morphisms in $\mathscr{C}$. Consider the category I whose class of objects is the set $\{1, 2\}$ and the only nontrivial morphisms in I are $\varphi_{12}, \varphi'_{12} : 1 \to 2$, and the functor $F : I \to \mathscr{C}$ where $F(\varphi_{12}) = f$ and $F(\varphi'_{12}) = g$. Then an equalizer of f and g can be seen as a limit of F, and a coequalizer of f and g can be seen as a colimit of F, and vice-versa.

Example 2.2.22. Let $f, g : X \to Y$ be morphisms in **Set**. Consider the set $E = \{x \in X : f(x) = g(x)\}$ and $e : E \to X$ its corresponding inclusion map. Then e is an equalizer of f and g.

Let $\sim$ be the smallest equivalence relation on Y such that $f(x) \sim g(x)$ for all $x \in X$. Then the quotient map $c : Y \to Y/\sim$ is a coequalizer of f and g.

If X and Y are topological spaces and f and g are continuous maps, by imbuing the subspace topology on E and the quotient topology on $Y/\sim$, the maps e and c are, respectively, an equalizer and a coequalizer of f and g in **Top**.

Example 2.2.23. Let $f, g : X \to Y$ be morphisms in $\mathbf{Alg}_{\mathbb{K}}$. Then the set $E = \{x \in X : f(x) = g(x)\}$ is a subalgebra of X and its corresponding inclusion map $e : E \to X$ is an equalizer of f and g.

Let I be the ideal of Y generated by the elements of the form $g(x) - f(x)$ for each $x \in X$. Then the quotient map $c : Y \to Y/I$ is a coequalizer of f and g.

Proposition 2.2.24. *Let $\mathscr{C}$ be a category and $f, g : X \to Y$ morphisms in $\mathscr{C}$.*

- *If e is an equalizer of f and g, then e is a monomorphism.*
- *If c is a coequalizer of f and g, then c is an epimorphism.*

Proof. We will only verify the second item, as the first is analogous. Let $c : Y \to C$ be a coequalizer of f and g and $h, k : C \to C'$ morphisms such that

$$h \circ c = k \circ c.$$

Let

$$\nu = h \circ c = k \circ c. \tag{2.24}$$

Then, since c is a coequalizer of f and g we have

$$\nu \circ f = h \circ c \circ f = h \circ c \circ g = \nu \circ g.$$

Therefore, since c is a coequalizer, there exists a unique morphism $\nu' : C \to C'$ such that the triangle in the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "c"] \arrow[dr, "\nu"'] & C \arrow[d, dashed, "\nu'"] \\ & & C' \end{tikzcd}$$

Hence, by the uniqueness of ν' and by (2.24) it follows that

$$h = \nu' = k,$$

so c is an epimorphism, as desired. □

Proposition 2.2.25. *Let $\mathscr{C}$ be a category, $f, g : X \to Y$ morphisms in $\mathscr{C}$ and $c : Y \to C$ a coequalizer of f and g. If ι is a monomorphism and*

$$c = \iota \circ h \tag{2.25}$$

for some morphism h, then ι is an isomorphism.

Proof. Since c is a coequalizer of f and g, by (2.25), we have

$$\iota \circ h \circ f = c \circ f = c \circ g = \iota \circ h \circ g.$$

Therefore, since ι is a monomorphism,

$$h \circ f = h \circ g.$$

Thus, since c is a coequalizer of f and g, there exists a unique morphism h' such that

$$h = h' \circ c. \tag{2.26}$$

Let us verify that h' is an inverse of ι. Indeed, observe that, by (2.25) and (2.26),

$$\iota \circ h' \circ c = \iota \circ h = c = id_C \circ c. \tag{2.27}$$

Now, by Proposition 2.2.24, c is an epimorphism. Thus, by (2.27) we have

$$\iota \circ h' = id_C. \tag{2.28}$$

On the other hand, observe that by (2.28) we have

$$\iota \circ h' \circ \iota = id_C \circ \iota = \iota = \iota \circ id_{\operatorname{dom} \iota},$$

so, since ι is a monomorphism, we have

$$h' \circ \iota = id_{\operatorname{dom} \iota}.$$

Hence, h' is an inverse of ι and, thus, ι is an isomorphism, as desired. □

2.2.3 PRODUCTS AND COPRODUCTS

Definition 2.2.26. Let $\mathscr{C}$ be a category and $\{X_i\}_{i \in I}$ a family of objects in $\mathscr{C}$.

- A **product** of the family $\{X_i\}_{i\in I}$ is a pair $(P, \{p_i : P \to X_i\})$ formed by an object $P \in \mathscr{C}$ and a family of morphisms $\{p_i : P \to X_i\}$, such that for all such pairs $(W, \{f_i : W \to X_i\})$ there exists a unique morphism $\varphi : W \to P$ such that the diagram
$$\begin{tikzcd} P \arrow[r, "p_i"] & X_i \\ W \arrow[u, dashed, "\varphi"] \arrow[ur, "f_i"'] & \end{tikzcd} \tag{2.29}$$
commutes for all $i \in I$. We denote this unique morphism φ by $\prod_{i\in I} f_i$.

- A **coproduct** of the family $\{X_i\}_{i\in I}$ is a pair $(C, \{u_i : X_i \to C\})$ formed by an object $C \in \mathscr{C}$ and a family of morphisms $\{u_i : X_i \to C\}$, such that for all such pairs $(Y, \{f_i : X_i \to Y\})$ there exists a unique morphism $\varphi : C \to Y$ such that the diagram
$$\begin{tikzcd} X_i \arrow[r, "u_i"] \arrow[dr, "f_i"'] & C \arrow[d, dashed, "\varphi"] \\ & Y \end{tikzcd} \tag{2.30}$$
commutes for all $i \in I$. We denote this unique morphism φ by $\coprod_{i\in I} f_i$.

Remark 2.2.27. Let $\mathscr{C}$ be a category and $\{X_i\}_{i\in I}$ a family of objects in $\mathscr{C}$. Consider the set I as a category whose only morphisms are the identity morphisms and the functor $F : I \to \mathscr{C}$ where $F(i) = X_i$ for all $i \in I$. Then a product of $\{X_i\}_{i\in I}$ can be seen as a limit of F, and a coproduct $\{X_i\}_{i\in I}$ can be seen as a colimit of F, and vice-versa.

Example 2.2.28. Let $\{X_i\}_{i\in I}$ be a family of objects in **Set**. Then the cartesian product $\prod_{i\in I} X_i \coloneqq \{(x_i)_{i\in I} : x_i \in X_i\}$ along with the projections $p_j : \prod_{i\in I} X_i \to X_i$ given by $p_j((x_i)_{i\in I}) = x_j$ for each $j \in I$, form a product of $\{X_i\}_{i\in I}$.

And the disjoint union $\bigsqcup_{i\in I} X_i$ along with the inclusion maps $u_i : X_i \to \bigsqcup_{i\in I} X_i$ form a coproduct of $\{X_i\}_{i\in I}$.

Example 2.2.29. Let $A, B \in \mathbf{Alg}_{\mathbb{K}}$. Then the direct product $A \times B$ along with the projections $p_A : A \times B \to A$ and $p_B : A \times B \to B$ form a product of A and B.

The coproduct of A and B has a less straightforward construction. Consider the vector space
$$T = \bigoplus_{n=1}^{\infty} T_n,$$
where
$$T_1 = A \oplus B, \quad T_2 = (A \otimes A) \oplus (A \otimes B) \oplus (B \otimes A) \oplus (B \otimes B), \quad \dots$$

Define a product in T as follows. Given two generators

$$x_1 \otimes \cdots \otimes x_n \text{ for } n \in \mathbb{N} \text{ and } x_i \in A \cup B, \forall i \in \{1, \ldots, n\},$$

and

$$y_1 \otimes \cdots \otimes y_m \text{ for } m \in \mathbb{N} \text{ and } y_i \in A \cup B, \forall i \in \{1, \ldots, m\},$$

we define

$$(x_1 \otimes \cdots \otimes x_n) \cdot (y_1 \otimes \cdots \otimes y_m) \coloneqq x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m.$$

With this operation, T is an algebra.

Let I be the ideal of T generated by elements of the form

$$a \otimes a' - aa' \quad \text{and} \quad b \otimes b' - bb'$$

for $a, a' \in A$ and $b, b' \in B$.

Then T/I along with the maps $u_A : A \to T/I$ and $u_B : B \to T/I$ given by

$$u_A(a) = a + I, a \in A \quad \text{and} \quad u_B(b) = b + I, b \in B$$

form a coproduct of A and B.

2.3 INVERSE SEMIGROUPS

Recall that a **semigroup** S is a set with an associative operation. We will denote the product of elements of most semigroups in this work simply by juxtaposition.

Also recall that a semigroup is said to be a **monoid** if it has an identity element.

Definition 2.3.1. Let S be a semigroup. We say that S is an **inverse semigroup** (cf. [16]) if for each element $s \in S$ there exists a unique element $s^* \in S$, called the **inverse** of s, satisfying

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*.$$

If S is a monoid, we say that S is an **inverse monoid**.

Example 2.3.2. Let X be a set and $\mathcal{I}(X)$ the set of partial bijections of X, that is, the set of bijective maps between subsets of X. For each $f, g \in \mathcal{I}(X)$, let gf be the partial bijection of X where

$$\operatorname{dom}(gf) = f^{-1}(\operatorname{cdm} f \cap \operatorname{dom} g), \quad \operatorname{cdm}(gf) = g(\operatorname{cdm} f \cap \operatorname{dom} g),$$

and

$$(gf)(x) = g(f(x)),$$

for all $x \in \operatorname{dom}(gf)$. With this operation, $\mathcal{I}(X)$ is an inverse monoid, where the inverse of a partial bijection of X is its usual functional inverse.

Example 2.3.3. Let G be a group. Then it is an inverse monoid, where the inverse of an element $g \in G$ is its usual inverse g^{-1} in the group.

Example 2.3.4. Let S be a semilattice. Then it is an inverse semigroup, where the inverse of $e \in S$ is e itself.

Proposition 2.3.5. *Let S be an inverse semigroup. Then for each element $s \in S$ we have $s = (s^*)^*$.*

Proof. Since s^* satisfies

$$ss^*s = s$$

and

$$s^*ss^* = s^*,$$

by the uniqueness of the inverse of s^* it follows that $s = (s^*)^*$, as desired. □

Idempotent elements form a very important class of elements in an inverse semigroup.

Definition 2.3.6. Let S be a semigroup. An element $e \in S$ is said to be an **idempotent** in S if

$$ee = e.$$

We denote by $\mathcal{E}(S)$ the set of idempotent elements of S.

Observe that inverse semigroups that are not groups have many idempotent elements.

Proposition 2.3.7. *Let S be an inverse semigroup and $s \in S$. Then $s^*s \in \mathcal{E}(S)$.*

Proof. Since $ss^*s = s$, we have

$$(ss^*)(ss^*) = (ss^*s)s^* = ss^*,$$

as desired. □

Lemma 2.3.8. *Let S be an inverse semigroup and $e \in \mathcal{E}(S)$. Then*

$$e^* = e.$$

Proof. Since e is an idempotent, we have

$$eee = ee = e,$$

so $e^* = e$ follows by the uniqueness of e^*. □

Lemma 2.3.9. *Let S be an inverse semigroup and $e, f \in \mathcal{E}(S)$. Then $ef \in \mathcal{E}(S)$.*

Proof. Consider the element $u = f(ef)^*e \in S$. Since $e, f \in \mathcal{E}(S)$,

$$(ef)u(ef) = (ef)(f(ef)^*e)(ef) = e(ff)(ef)^*(ee)f = (ef)(ef)^*(ef) = ef$$

and

$$\begin{aligned} u(ef)u &= (f(ef)^*e)(ef)(f(ef)^*e) = f(ef)^*(ee)(ff)(ef)^*e \\ &= f((ef)^*(ef)(ef)^*)e = f(ef)^*e = u, \end{aligned}$$

so, by the uniqueness of the inverse in an inverse semigroup, $u = (ef)^*$.

Observe that

$$uu = (f(ef)^*e)(f(ef)^*e) = f((ef)^*(ef)(ef)^*)e = f(ef)^*e = u,$$

so $u \in \mathcal{E}(S)$.

By Lemma 2.3.8, since u is an idempotent, $u^* = u$. Thus, since $u = (ef)^*$, by Proposition 3.5.3 we have

$$u = u^* = ((ef)^*)^* = ef.$$

In particular, it follows that ef is an idempotent, as desired. □

Proposition 2.3.10. *Let S be a semigroup. Then S is an inverse semigroup if and only if it satisfies the following.*

(1) For each $s \in S$ there exists a (not necessarily unique) element $t \in S$ satisfying

$$sts = s \tag{2.31}$$

(2) The idempotents of S commute.

Proof. Suppose S is an inverse semigroup. Clearly, S satisfies (1). Let us verify (2).

Let $e, f \in \mathcal{E}(S)$. By Lemma 2.3.9, ef is an idempotent, so

$$(ef)(fe)(ef) = e(ff)(ee)f = (ef)(ef) = ef$$

and

$$(fe)(ef)(fe) = f(ee)(ff)e = (fe)(fe) = fe.$$

Thus, since S is an inverse semigroup, $fe = (ef)^*$. By Lemma 2.3.8, we then have

$$fe = (ef)^* = ef,$$

as desired.

Now assume S satisfies (1) and (2). Observe that, given $s \in S$, if $t \in S$ is an element such that $sts = s$, then $u = tst$ satisfies

$$sus = s(tst)s = (sts)ts = sts = s$$

and

$$usu = (tst)s(tst) = t(sts)tst = tstst = t(sts)t = tst = u.$$

Thus, by (1), for each $s \in S$ there exists an element $t \in S$ such that

$$sts = s \quad \text{and} \quad tst = t. \tag{2.32}$$

So, to verify that S is an inverse semigroup, it suffices to verify that, given $s \in S$, the element $t \in S$ satisfying (2.32) is the unique such element.

For let $t' \in S$ such that

$$st's = s \quad \text{and} \quad t'st' = t'. \tag{2.33}$$

Observe that ts and $t's$ are both idempotents. Indeed, by (2.32) and (2.33) we have

$$(ts)(ts) = (tst)s = ts$$

and

$$(t's)(t's) = (t'st')s = t's.$$

By (2.32) and (2.33), we have

$$t = tst = t(st's)t = (ts)(t's)t. \tag{2.34}$$

Thus, by (2) and (2.32)–(2.34), and since ts and $t's$ are idempotents,

$$t = (ts)(t's)t = (t's)(ts)t = t's(tst) = t'st = (t'st')st = t'(st')(st). \tag{2.35}$$

Similarly to ts and $t's$, we can check that st and st' are also idempotents in S. So, by (2.32), (2.33) and (2.35) and (1) we have

$$t = t'(st')(st) = t'(st)(st') = t'(sts)t' = t'st' = t'.$$

Therefore, the uniqueness of t follows, and S is an inverse semigroup, as desired. □

Proposition 2.3.11. *Let S be an inverse semigroup and $s, t \in S$. Then*

$$(st)^* = t^* s^*.$$

Proof. By Proposition 2.3.7 and Proposition 2.3.10 (2),

$$(st)(t^*s^*)(st) = s(tt^*)(s^*s)t = s(s^*s)(tt^*)t = (ss^*s)(tt^*t) = st$$

and

$$(t^*s^*)(st)(t^*s^*) = t^*(s^*s)(tt^*)s^* = t^*(tt^*)(s^*s)s^* = (t^*tt^*)(s^*ss^*) = t^*s^*.$$

Thus, by the uniqueness of the inverse we have $(st)^* = t^*s^*$, as desired. □

To each group, one associates a very important inverse semigroup, its Exel's semigroup. To introduce this semigroup, let us revise some concepts from the theory of semigroups.

Definition 2.3.12. Let X be a set. The **free semigroup** on X is the semigroup $W(S)$, whose elements are *non-empty* words $x_1x_2 \ldots x_n$ where $x_i \in X$ for all $i = 1, \ldots, n$, and the product of two words $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_m$ is given by juxtaposition:

$$x \cdot y = x_1x_2 \ldots x_n y_1 y_2 \ldots y_m.$$

A free semigroup satisfies the following universal property.

Proposition 2.3.13. *Let X be a set, S a semigroup and $f : X \to S$ a map. Then there exists a unique semigroup homomorphism $\overline{f} : W(X) \to S$ such that*

$$\overline{f}(x) = f(x)$$

for all $x \in X$.

Proof. Let $\overline{f} : W(X) \to S$ be given by

$$\overline{f}(x_1x_2 \ldots x_n) = f(x_1)f(x_2) \ldots f(x_n).$$

Then it is a simple verification that $\overline{f}$ is a semigroup homomorphism and

$$\overline{f}(x) = f(x)$$

for all $x \in X$.

It is unique as such. Indeed, if $h : W(X) \to S$ is a semigroup homomorphism that satisfies

$$h(x) = f(x)$$

for all $x \in X$, then

$$h(x_1x_2 \ldots x_n) = h(x_1)h(x_2)\ldots h(x_n) = f(x_1)f(x_2)\ldots f(x_n) = \overline{f}(x_1x_2\ldots x_n). \qquad \square$$

Definition 2.3.14. Let G be a group with identity e. The **Exel's semigroup** [9] of G is the inverse semigroup $\mathcal{S}(G)$ given by the quotient of the free semigroup on $\{[g] : g \in G\}$ by the relations

$$[g^{-1}][g][h] = [g^{-1}][gh],$$

$$[g][h][h^{-1}] = [gh][h^{-1}],$$

$$[g][e] = [g],$$

and

$$[e][g] = [g],$$

for each $g, h \in G$.

Exel proved in [9] that $\mathcal{S}(G)$ is indeed an inverse semigroup, where the inverse of (the congruence class of) a generator $[g] \in \mathcal{S}(G)$ is (the congruence class of) the element $[g^{-1}]$.

Observe that $\mathcal{S}(G)$ is further an inverse monoid, with identity $[e]$.

The semigroup $\mathcal{S}(G)$ satisfies the following universal property.

Proposition 2.3.15. *Let G be a group with identity e, S a semigroup and $f : G \to S$ a map satisfying*

(1) $f(g)f(h)f(h^{-1}) = f(gh)f(h^{-1})$, *for all* $g, h \in G$*;*

(2) $f(g^{-1})f(g)f(h) = f(g^{-1})f(gh)$, *for all* $g, h \in G$*;*

(3) $f(g)f(e) = f(g)$, *for all* $g \in G$*;*

(4) $f(e)f(g) = f(g)$, *for all* $g \in G$*.*

Then there exists a unique semigroup homomorphism $\overline{f} : \mathcal{S}(G) \to S$ *such that*

$$\overline{f}([g]) = f(g),$$

for all $g \in G$*.*

Proof. A map $f : G \to S$ extends uniquely to a semigroup homomorphism $\tilde{f} : W(G) \to S$.

Now, since f satisfies (1)–(4), $\tilde{f}$ respects the relations that determine $\mathcal{S}(G)$. Thus, by the universal property of the quotient, there exists a unique semigroup homomorphism $\overline{f} : \mathcal{S}(G) \to S$ such that

$$\tilde{f} = \overline{f} \circ q,$$

where $q : W(G) \to \mathcal{S}(G)$ is the appropriate quotient map.

It is then simple to verify that $\overline{f}$ is the unique semigroup homomorphism satisfying

$$\overline{f}([g]) = f(g)$$

for all $g \in G$. □

Proposition 2.3.16. *Let G be a group with identity e, M a monoid and $f : G \to M$ a map satisfying*

(1) $f(g)f(h)f(h^{-1}) = f(gh)f(h^{-1})$, for all $g, h \in G$;

(2) $f(g^{-1})f(g)f(h) = f(g^{-1})f(gh)$, for all $g, h \in G$;

(3) $f(e)$ is the identity of M.

Then there exists a unique monoid homomorphism $\overline{f} : \mathcal{S}(G) \to M$ such that

$$\overline{f}([g]) = f(g)$$

for all $g \in G$.

Proof. Since f satisfies (3), it satisfies Proposition 2.3.15 (3) and (4). Thus, by Proposition 2.3.15 (1)–(4), there exists a unique semigroup homomorphism $\overline{f} : \mathcal{S}(G) \to M$ such that $\overline{f}([g]) = f(g)$ for all $g \in G$.

Since $\overline{f}([e]) = f(e)$ is the identity of M, $\overline{f}$ preserves the identity of $\mathcal{S}(G)$. Thus, $\overline{f}$ is a monoid homomorphism. It is a simple verification that it is unique as such, concluding the proof. □

Lemma 2.3.17. *Let G be a group with identity e, M an inverse monoid and $f : G \to M$ a map satisfying*

(1) $f(g)f(h)f(h^{-1}) = f(gh)f(h^{-1})$, for all $g, h \in G$;

(2) $f(e)$ is the identity of M.

Then f satisfies

$$f(g^{-1})f(g)f(h) = f(g^{-1})f(gh),$$

for all $g, h \in G$.

Proof. First observe that, given $g \in G$, by (1) and (2)

$$f(g)f(g^{-1})f(g) = f(gg^{-1})f(g) = f(e)f(g) = f(g)$$

and

$$f(g^{-1})f(g)f(g^{-1}) = f(g^{-1}g)f(g^{-1}) = f(e)f(g^{-1}) = f(g^{-1}).$$

Thus, $f(g^{-1}) = f(g)^*$ in the inverse monoid M.

Now let $g, h \in G$. Then, by Proposition 2.3.11 and (1),

$$\begin{aligned} f(g^{-1})f(g)f(h) &= (f(h)^* f(g)^* f(g^{-1})^*)^* = (f(h^{-1})f(g^{-1})f(g))^* = (f(h^{-1}g^{-1})f(g))^* \\ &= f(g)^* f(h^{-1}g^{-1})^* = f(g^{-1})f(gh), \end{aligned}$$

as desired. □

Proposition 2.3.18. *Let G be a group with identity e, M an inverse monoid and $f : G \to M$ a map satisfying*

(1) $f(g)f(h)f(h^{-1}) = f(gh)f(h^{-1})$, for all $g, h \in G$;

(2) $f(e)$ is the identity of M.

Then there exists a unique monoid homomorphism $\overline{f} : \mathcal{S}(G) \to M$ such that

$$\overline{f}([g]) = f(g)$$

for all $g \in G$.

Proof. The result follows by Proposition 2.3.16 and Lemma 2.3.17. □

3 SPANS AND PARTIAL MORPHISMS

This chapter gives a more detailed introduction on spans and partial morphisms in a category $\mathscr{C}$, along with their corresponding categories $\mathbf{span}_{\mathscr{C}}$ and $\mathbf{par}_{\mathscr{C}}$, as well as some results on the functors from $\mathbf{par}_{\mathscr{C}}$ to $\mathbf{par}_{\mathscr{D}}$ that are induced by a functor from $\mathscr{C}$ to $\mathscr{D}$.

In the final sections of this chapter it is also introduced the theory of restriction and inverse categories, where we also show how an inverse category can be obtained from any restriction category. There we verify that $\mathbf{par}_{\mathscr{C}}$ has an interesting restriction structure that makes it a restriction category, and study the inverse category $\mathbf{iso}_{\mathscr{C}}$ that comes from it.

3.1 SPANS AND PARTIAL MORPHISMS

We will now work towards introducing partial actions from a categorical perspective. To this end, we define the concept of a span in a category.

Definition 3.1.1. Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. A **span** [3] from X to Y is a triple (A, f, g) where $A \in \mathscr{C}$ and $f : A \to X$ and $g : A \to Y$ are morphisms, as illustrated.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

In particular, we are interested in the concept of a partial morphism:

Definition 3.1.2. Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. A **partial morphism** from X to Y is a span (A, f, g) from X to Y where f is a monomorphism.

Partial morphisms from an object X to an object Y in a category $\mathscr{C}$ — or more precisely, their isomorphism classes, which we shall introduce later — in a way describe morphisms from a subobject of X to Y. Indeed, for example, the partial maps between sets are partial morphisms in **Set**.

Definition 3.1.3. Let X and Y be sets. A **partial map** from X to Y is a map $f : \operatorname{dom} f \to Y$ where $\operatorname{dom} f \subseteq X$.

Example 3.1.4. Let $\mathscr{C} = \mathbf{Set}$, X and Y sets and f a partial map from X to Y. Then $(\operatorname{dom} f, \iota, f)$ is a partial morphism from X to Y, where ι is the inclusion of $\operatorname{dom} f$ into X.

Morphisms in any category can also be described in terms of partial morphisms.

Example 3.1.5. Let $\mathscr{C}$ be a category and $f : X \to Y$ a morphism in $\mathscr{C}$. Then (X, id_X, f) is a partial morphism from X to Y.

Definition 3.1.6. Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and (A, f, g) and (B, h, k) spans from X to Y. A **morphism of spans** from (A, f, g) to (B, h, k) is a morphism $\varphi : A \to B$ in $\mathscr{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[d, "\varphi"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& B \arrow[dl, "h"] \arrow[dr, "k"'] & \\
X & & Y
\end{tikzcd}$$

Lemma 3.1.7. *Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$, $(A, f, g), (B, h, k)$ and (C, m, n) spans from X to Y and $\varphi : (A, f, g) \to (B, h, k)$ and $\psi : (B, h, k) \to (C, m, n)$ morphisms of spans. Then $\psi \circ \varphi$ is a morphism of spans from (A, f, g) to (B, h, k).*

Proof. Consider the following diagram.

$$\begin{tikzcd}
& A \arrow[d, dashed, "\varphi"] \arrow[dddl, "f"'] \arrow[dddr, "g"] & \\
& B \arrow[d, dashed, "\psi"] \arrow[ddl, "h"'] \arrow[ddr, "k"] & \\
& C \arrow[dl, "m"] \arrow[dr, "n"'] & \\
X & & Y
\end{tikzcd} \tag{3.1}$$

Since φ is a span morphism, the two triangles on the top of (3.1) are commutative, and since ψ is a span morphism, the two triangles at the bottom of (3.1) are commutative. Thus, (3.1) commutes, so the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[d, "\psi\circ\varphi"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& C \arrow[dl, "m"] \arrow[dr, "n"'] & \\
X & & Y
\end{tikzcd}$$

Therefore, it follows that $\psi \circ \varphi$ is a morphism of spans from (A, f, g) to (B, h, k), as desired. □

Thus, we can define the following categories.

Definition 3.1.8. Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. We define the category $\mathbf{Span}_{\mathscr{C}}(X, Y)$ **of spans from X to Y**, whose objects are spans from X to Y and whose morphisms are span morphisms between those spans, with the usual composition of $\mathscr{C}$.

Definition 3.1.9. Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. We define the category $\mathbf{Par}_{\mathscr{C}}(X, Y)$ **of partial morphisms from X to Y** as the full subcategory of $\mathbf{Span}_{\mathscr{C}}(X, Y)$ whose objects are the partial morphisms from X to Y.

The identity morphism of a span (or a partial morphism) (A, f, g) in $\mathbf{Span}_{\mathscr{C}}(X, Y)$ (or $\mathbf{Par}_{\mathscr{C}}(X, Y)$) is the morphism id_A.

Example 3.1.10. Let X and Y be objects in a category $\mathscr{C}$, and $X \times Y$ a product of X with Y, with associated projections $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$. Then, given a span (A, f, g) from X to Y, the universal property of $X \times Y$ tells us that there exists a unique morphism $\varphi : A \to X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[d, "\alpha"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& X \times Y \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
X & & Y
\end{tikzcd} .$$

Thus, there exists a unique span morphism from (A, f, g) to $(X \times Y, \pi_1, \pi_2)$, so $(X \times Y, \pi_1, \pi_2)$ is a terminal object in $\mathbf{Span}_{\mathscr{C}}(X, Y)$.

One can also easily verify that any terminal object in $\mathbf{Span}_{\mathscr{C}}(X, Y)$ is a triple that determines a product of X and Y, so products of X and Y correspond to terminal objects in $\mathbf{Span}_{\mathscr{C}}(X, Y)$.

This work deals primarily with isomorphism classes of spans in $\mathbf{Span}_{\mathscr{C}}(X, Y)$, so the following proposition gives us a more straightforward way to determine when two spans are isomorphic in this category.

Proposition 3.1.11. *Let $\mathscr{C}$ be a category and $X, Y \in \mathscr{C}$. A morphism in $\mathbf{Span}_{\mathscr{C}}(X, Y)$ is an isomorphism if and only if it is an isomorphism in $\mathscr{C}$.*

Proof. Let $\varphi : (A, f, g) \to (B, h, k)$ be a morphism in $\mathbf{Span}_{\mathscr{C}}(X, Y)$.

On the one hand, assume that φ is an isomorphism in $\mathbf{Span}_{\mathscr{C}}(X, Y)$ and let $\psi : (B, h, k) \to (A, f, g)$ be its inverse. Then $\psi \circ \varphi = id_A$ and $\varphi \circ \psi = id_B$, so ψ is an inverse of φ in $\mathscr{C}$, and it follows that φ is an isomorphism in $\mathscr{C}$.

On the other hand, assume that φ is an isomorphism in $\mathscr{C}$ and let $\psi : B \to A$ be its inverse. To check that φ is an isomorphism in $\mathbf{Span}_{\mathscr{C}}(X, Y)$, it suffices to verify that ψ is a span morphism from (B, h, k) to (A, f, g), so it will clearly be its inverse in $\mathbf{Span}_{\mathscr{C}}(X, Y)$.

Indeed, since φ is a span morphism, we have $h \circ \varphi = f$ and $k \circ \varphi = g$, so

$$f \circ \psi = h \circ \varphi \circ \psi = h \circ id_B = h$$

and

$$g \circ \psi = k \circ \varphi \circ \psi = k \circ id_B = k,$$

and, thus, the diagram

$$\begin{tikzcd}
 & B \arrow[d, "\psi"] \arrow[ddl, "h"'] \arrow[ddr, "k"] & \\
 & A \arrow[dl, "f"] \arrow[dr, "g"'] & \\
X & & Y
\end{tikzcd}$$

commutes. $\square$

The following proposition shows that $\mathbf{Par}_{\mathscr{C}}(X, Y)$ is a *strictly* full subcategory of $\mathbf{Span}_{\mathscr{C}}(X, Y)$.

Proposition 3.1.12. *Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$, (A, ι, f) a partial morphism from X to Y and (B, h, k) a span from X to Y. If (A, ι, f) and (B, h, k) are isomorphic in $\mathbf{Span}_{\mathscr{C}}(X, Y)$, then (B, h, k) is a partial morphism.*

Proof. Let $\varphi : (A, \iota, f) \to (B, h, k)$ be an isomorphism in $\mathbf{Span}_{\mathscr{C}}(X, Y)$. Then, in particular, $h \circ \varphi = \iota$. Thus, since φ is an isomorphism and ι is a monomorphism (because (A, ι, f) is a partial morphism), it follows that h is a monomorphism, and, thus, (B, h, k) is a partial morphism. $\square$

What follows is one of the fundamental notions in this work.

Definition 3.1.13. Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and $(A, f, g) \in \mathbf{Span}_{\mathscr{C}}(X, Y)$. The **isomorphism class** represented by (A, f, g) is the class $\{P \in \mathbf{Span}_{\mathscr{C}}(X, Y) : P \cong (A, f, g)\}$, denoted by $[(A, f, g)]$ or simply by $[A, f, g]$.

We denote the class formed by the isomorphism classes represented by spans from X to Y by $\mathbf{span}_{\mathscr{C}}(X, Y)$. That is,

$$\mathbf{span}_{\mathscr{C}}(X, Y) = \{[A, f, g] : (A, f, g) \in \mathbf{Span}_{\mathscr{C}}(X, Y)\}.$$

Clearly, by Proposition 3.1.12, if (A, f, g) is a partial morphism, every representative of $[A, f, g]$ is a partial morphism. Thus, we define

Definition 3.1.14. We denote the class formed by the isomorphism classes represented by partial morphisms from X to Y by $\mathbf{par}_{\mathscr{C}}(X, Y)$. That is,

$$\mathbf{par}_{\mathscr{C}}(X, Y) = \{[A, f, g] : (A, f, g) \in \mathbf{Par}_{\mathscr{C}}(X, Y)\}.$$

Proposition 3.1.15. *Let $\mathscr{C}$ be a category and $(A, \iota, f), (A, \iota, g) \in \mathbf{Par}_{\mathscr{C}}(X, Y)$ partial morphisms such that*

$$(A, \iota, f) \cong (A, \iota, g).$$

Then $f = g$.

Proof. Let $\varphi : (A, \iota, g) \to (A, \iota, f)$ be an isomorphism, so the diagram

$$\begin{tikzcd}
& A \arrow[d, "\varphi"] \arrow[ddl, "\iota"'] \arrow[ddr, "g"] & \\
& A \arrow[dl, "\iota"] \arrow[dr, "f"'] & \\
X & & Y
\end{tikzcd} \tag{3.2}$$

commutes. By the commutativity of (3.2) we have

$$\iota \circ \varphi = \iota = \iota \circ id_A,$$

so, since ι is a monomorphism, $\varphi = id_A$.

It then also follows by the commutativity of (3.2) that

$$f = f \circ id_A = f \circ \varphi = g,$$

as desired. □

In **Set** we have the following.

Proposition 3.1.16. *Let X and Y be sets and $(A, f, g) \in \mathbf{Par}_{\mathbf{Set}}(X, Y)$. There exists a unique partial map h from X to Y such that (A, f, g) is isomorphic to $(\operatorname{dom} h, \iota_h, h)$, where ι_h is the inclusion of $\operatorname{dom} h$ into X.*

Proof. Let $\operatorname{dom} h \coloneqq f(A) \subseteq X$, ι_h the inclusion of $\operatorname{dom} h$ into X and $h : \operatorname{dom} h \to Y$ given by $h(x) = g(a)$ if $x = f(a)$. Since f is a monomorphism in **Set**, it is an injective map, so for each $x \in f(A)$ there exists a unique $a \in A$ such that $x = f(a)$ and, thus, h is a well defined map.

Let $\varphi : A \to \operatorname{dom} h$ be the corestriction of f to $\operatorname{dom} h$. Then for all $a \in A$ we have

$$h(\varphi(a)) = h(f(a)) = g(a)$$

by definition of h and φ, so

$$h \circ \varphi = g.$$

Similarly, we have

$$\iota_h \circ \varphi = f.$$

Therefore, the diagram

$$\begin{tikzcd} & A \arrow[ddl, hook, "f"'] \arrow[d, "\varphi"] \arrow[ddr, "g"] & \\ & \operatorname{dom} h \arrow[dl, hook, "\iota_h"] \arrow[dr, "h"'] & \\ X & & Y \end{tikzcd}$$

commutes, so φ is a span morphism from (A, f, g) to $(\operatorname{dom} h, \iota_h, h)$. Further, φ is a bijection, since f is injective, so it is an isomorphism in **Set**. Because of that, by Proposition 3.1.11 it is an isomorphism from (A, f, g) to $(\operatorname{dom} h, \iota_h, h)$, and so the two partial morphisms are isomorphic.

We shall now verify its uniqueness. Let k be a partial map from X to Y (with $\iota_k : \operatorname{dom} k \to X$ the corresponding inclusion map) such that (A, f, g) is isomorphic to $(\operatorname{dom} k, \iota_k, k)$. Then the latter is also isomorphic to $(\operatorname{dom} h, \iota_h, h)$, so there exists a bijection $\varphi : \operatorname{dom} h \to \operatorname{dom} k$ such that the diagram

$$\begin{tikzcd} & \operatorname{dom} h \arrow[ddl, hook, "\iota_h"'] \arrow[d, "\varphi"] \arrow[ddr, "h"] & \\ & \operatorname{dom} k \arrow[dl, hook, "\iota_k"] \arrow[dr, "k"'] & \\ X & & Y \end{tikzcd} \tag{3.3}$$

commutes.

By the commutativity of (3.3) we have $\iota_k \circ \varphi = \iota_h$, so since ι_k and ι_h are inclusion maps it follows that

$$\varphi(x) = \iota_k(\varphi(x)) = \iota_h(x) = x$$

for all $x \in \operatorname{dom} h$. Hence, $\operatorname{dom} h \subseteq \operatorname{dom} k$ and φ is the inclusion map. Since φ is a bijection, it follows that $\operatorname{dom} h = \operatorname{dom} k$, φ is the identity map and, so, $\iota_h = \iota_k$.

Thus, since $(\operatorname{dom} h, \iota_h, h) \cong (\operatorname{dom} k, \iota_k, k)$ and $\iota_h = \iota_k$, by Proposition 3.1.15 we have that $h = k$, and the uniqueness of the partial map follows. □

Thus, every isomorphic class represented by a partial morphism from X to Y has a unique representative that comes from a partial map from X to Y.

Hence, if $A \subseteq X$ and ι_A is the corresponding inclusion map, we may denote an isomorphism class $[A, \iota_A, f] \in \mathbf{par}_{\mathbf{Set}}(X, Y)$ by simply f, its associated partial map.

More generally,

Proposition 3.1.17. *Let $\mathscr{C}$ be a category, $X, Y \in \mathscr{C}$ and I the class of subobjects of X. Let $\{\iota_i\}_{i \in I}$ be a family where $\iota_i : X_i \to X$ is a representative of the subobject i for each $i \in I$. Then for any partial morphism (A, ι, f) from X to Y there exist unique $i \in I$ and $g : X_i \to Y$ such that*

$$(A, \iota, f) \cong (X_i, \iota_i, g). \tag{3.4}$$

Proof. Let (A, ι, f) be a partial morphism from X to Y. Let i be the subobject represented by the monomorphism ι. Since ι and ι_i represent the same subobject i, there exists an isomorphism $\varphi : X_i \to A$ such that

$$\iota_i = \iota \circ \varphi. \tag{3.5}$$

So, let

$$g = f \circ \varphi. \tag{3.6}$$

Then the diagram

$$\begin{tikzcd} & X_i \arrow[d, "\varphi"] \arrow[ddl, "\iota_i"'] \arrow[ddr, "g"] & \\ & A \arrow[dl, "\iota"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

commutes by (3.5) and (3.6), so φ is a span morphism from (X_i, ι_i, g) to (A, ι, f). Since φ is an isomorphism in $\mathscr{C}$, by Proposition 3.1.11 it is an isomorphism between the partial morphisms, and, thus, we have (3.4).

It remains only to verify the uniqueness of i and g. Indeed, suppose we have $j \in I$ and $g' : X_j \to Y$ such that

$$(A, \iota, f) \cong (X_j, \iota_j, g').$$

Then we have

$$(X_i, \iota_i, g) \cong (X_j, \iota_j, g'), \tag{3.7}$$

so let $\psi : (X_j, \iota_j, g') \to (X_i, \iota_i, g)$ be an isomorphism. Then the diagram

$$\begin{tikzcd} & X_j \arrow[d, "\psi"] \arrow[ddl, "\iota_j"'] \arrow[ddr, "g'"] & \\ & X_i \arrow[dl, "\iota_i"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

commutes. In particular, since ψ is an isomorphism and

$$\iota_i \circ \psi = \iota_j,$$

ι_i and ι_j represent the same subobject of X, so $i = j$.

Since there is exactly one representative of the subobject i in $\{\iota_i\}_{i\in I}$, it follows that $\iota_i = \iota_j$. By Proposition 3.1.15 and (3.7) we then have that $g = g'$. Thus, the uniqueness of i and g follows, as desired. □

Remark 3.1.18. Let $\mathscr{C}$ be a category, $X \in \mathscr{C}$ and fix $\{\iota_i : X_i \to X\}_{i\in I}$ a family of representatives of the distinct subobjects of X. Then, by Proposition 3.1.17, $\mathbf{par}_{\mathscr{C}}(X, Y)$ is in bijection with the class of partial morphisms from X to Y of the form (X_i, ι_i, g) for some $i \in I$ and $g : X_i \to Y$. Thus, an element of $\mathbf{par}_{\mathscr{C}}(X, Y)$ is nothing more than a morphism $g : X_i \to Y$ for some $i \in I$, just like in the category **Set** (see Proposition 3.1.16). Hence, instead of $[X_i, \iota_i, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ we may simply write g, if there is no confusion. For the sake of convenience, we assume that $\iota_i = id_X$ whenever ι_i is an isomorphism.

3.2 THE CATEGORIES OF SPANS AND OF PARTIAL MORPHISMS

Definition 3.2.1. Let $\mathscr{C}$ be a category and $X, Y, Z \in \mathscr{C}$. Let (A, f, g) be a span from X to Y and (B, h, k) a span from Y to Z. If g and h have a pullback in $\mathscr{C}$, we say that the ordered pair of spans $((B, h, k), (A, f, g))$ is composable and $(P, f \circ p, k \circ q)$ is a **span composition** of (B, h, k) with (A, f, g), as in the following diagram, whose square is a pullback.

$$\begin{tikzcd} & & P \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\vee" very near start] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X & & Y & & Z \end{tikzcd} \tag{3.8}$$

Remark 3.2.2. Since any two pullbacks of g and h are isomorphic, any two compositions of (B, h, k) with (A, f, g) are also isomorphic as spans, through the isomorphism between the pullbacks.

Notice that any composable pair of spans may have many distinct (albeit isomorphic) compositions, as two morphisms may have plenty of pullbacks. Thus, in order to have a notation for a composition of two spans in a category, we must beforehand fix a choice of composition for each ordered pair of composable spans in the category. So, whenever we compose two spans (B, h, k) and (A, f, g) in a category, unless the choice is specified we will assume that such a choice was made beforehand, and we denote **the** composition of those spans by $(B, h, k) \bullet (A, f, g)$ or simply $(B, h, k)(A, f, g)$.

Remark 3.2.3. If $\mathscr{C}$ is a category with pullbacks, whenever (A, f, g) is a span from X to Y and (B, h, k) is a span from Y to Z in $\mathscr{C}$, there exists a composition of (B, h, k) with (A, f, g).

If (B, h, k) is a partial morphism, for a composition of (B, h, k) with (A, f, g) to exist, it suffices for $\mathscr{C}$ to have inverse images. Indeed, in this situation, the morphisms g and h have a pullback in $\mathscr{C}$ because h is a monomorphism.

Remark 3.2.4. Once we fix a choice of compositions of spans in a category with pullbacks, this choice may not give an associative (partial) operation, and the span (X, id_X, id_X) may not act as an identity. As a result, it is not possible to properly define a category whose morphisms are spans with this composition. Later on, we will explore two ways to address this issue.

Partial morphisms behave well with this composition, as the following proposition shows.

Proposition 3.2.5. *Let $\mathscr{C}$ be a category. A composition of any composable pair of partial morphisms in $\mathscr{C}$ is a partial morphism.*

Proof. Let $((B, h, k), (A, f, g))$ be a composable pair of spans in $\mathscr{C}$, as in Definition 3.2.1, and assume that (A, f, g) and (B, h, k) are partial morphisms, so f and h are monomorphisms. Let $(B, h, k) \bullet (A, f, g)$ be the outer span in the diagram (3.8), whose square is a pullback.

Since h is a monomorphism, by Proposition 2.2.17 we have that p is a monomorphism. Thus, since f is a monomorphism, $f \circ p$ is also a monomorphism. So,

$$(B, h, k) \bullet (A, f, g) = (P, f \circ p, k \circ q)$$

is a partial morphism, as desired. $\square$

It is interesting to observe the composition of partial maps, seen as partial morphisms between objects in **Set**.

Example 3.2.6. Let X, Y and Z be sets, f a partial map from X to Y, g a partial map from Y to Z, and $(\operatorname{dom} f, \iota_f, f)$ and $(\operatorname{dom} g, \iota_g, g)$ their associated partial morphisms.

Then the diagram

$$\begin{tikzcd}
 & f^{-1}(\operatorname{dom} g) \arrow[dl, "\widehat{\iota_g}"'] \arrow[dr, "\widehat{f}"] \arrow[dd, phantom, "\vee"] & \\
\operatorname{dom} f \arrow[dr, "f"'] & & \operatorname{dom} g \arrow[dl, "\iota_g"] \\
 & Y &
\end{tikzcd}$$

is a pullback in **Set**, where $\widehat{\iota_g}$ is the inclusion of $f^{-1}(\operatorname{dom} g)$ into $\operatorname{dom} f$ and $\widehat{f} = f|_{f^{-1}(\operatorname{dom} g)}^{\operatorname{dom} g}$. Thus, a composition of $(\operatorname{dom} g, \iota_g, g)$ with $(\operatorname{dom} f, \iota_f, f)$ is given by the external span in the following diagram.

$$\begin{tikzcd}
 & & f^{-1}(\operatorname{dom} g) \arrow[dl, "\widehat{\iota_g}"'] \arrow[dr, "\widehat{f}"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & \operatorname{dom} f \arrow[dl, "\iota_f"'] \arrow[dr, "f"] & & \operatorname{dom} g \arrow[dl, "\iota_g"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

That is,

$$(f^{-1}(\operatorname{dom} g), \iota_f \circ \widehat{\iota_g}, g \circ \widehat{f})$$

is a composition of $(\operatorname{dom} g, \iota_g, g)$ with $(\operatorname{dom} f, \iota_f, f)$.

Notice that this is precisely the partial morphism associated to the partial morphism $g \circ f|_{f^{-1}(\operatorname{dom} g)}^{\operatorname{dom} g}$ from X to Z. In other words, a span composition of g with f, seen as partial morphisms, may be viewed as the usual composition of maps $g \circ f$ on the largest subset of the domain of f where this composition makes sense.

As we have mentioned, one may not define a category formed by spans and their compositions. However, as we will verify, the *isomorphism classes* of spans behave well with this composition and, along with it, form a category.

Proposition 3.2.7. *Let $\mathscr{C}$ be a category with pullbacks, $X, Y, Z \in \mathscr{C}$, $(A, f, g), (A', f', g') \in \mathbf{Span}_{\mathscr{C}}(X, Y)$ and $(B, h, k), (B', h', k') \in \mathbf{Span}_{\mathscr{C}}(Y, Z)$. If*

$$(A, f, g) \cong (A', f', g') \quad \text{and} \quad (B, h, k) \cong (B', h', k'),$$

then

$$(B, h, k) \bullet (A, f, g) \cong (B', h', k') \bullet (A', f', g').$$

Proof. Consider the diagrams

$$\begin{tikzcd}
 & & P \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & Z
\end{tikzcd} \tag{3.9}$$

and

$$\begin{tikzcd}[column sep=small]
& & P' \arrow[dl, "p'"'] \arrow[dr, "q'"] \arrow[d, phantom, "\vee"] & & \\
& A' \arrow[dl, "f'"'] \arrow[dr, "g'"] & {} & B' \arrow[dl, "h'"'] \arrow[dr, "k'"] & \\
X & & Y & & Z
\end{tikzcd}, \tag{3.10}$$

whose squares are pullback squares, so that

$$(P, f \circ p, k \circ q) = (B, h, k) \bullet (A, f, g)$$

and

$$(P', f' \circ p', k' \circ q') = (B', h', k') \bullet (A', f', g').$$

Since $(A, f, g) \cong (A', f', g')$, there exists an isomorphism $\varphi_A : A \to A'$ such that

$$f' \circ \varphi_A = f \text{ and } g' \circ \varphi_A = g. \tag{3.11}$$

And since $(B, h, k) \cong (B', h', k')$, there exists an isomorphism $\varphi_B : B \to B'$ such that

$$h' \circ \varphi_B = h \text{ and } k' \circ \varphi_B = k. \tag{3.12}$$

Since the square in (3.9) is commutative, $g \circ p = h \circ q$. Then by (3.11) we have

$$g' \circ (\varphi_A \circ p) = (g' \circ \varphi_A) \circ p = g \circ p = h \circ q = (h' \circ \varphi_B) \circ q = h' \circ (\varphi_B \circ q).$$

Thus, by the universal property of the pullback square in (3.10) there exists a unique morphism $\varphi_P : P \to P'$ such that the diagram

$$\begin{tikzcd}
& P \arrow[d, dashed, "\varphi_P"] \arrow[ddl, "\varphi_A \circ p"'] \arrow[ddr, "\varphi_B \circ q"] & \\
& P' \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\
A' & & B'
\end{tikzcd}$$

commutes. This morphism is, even more, an isomorphism, because one can readily see that $\varphi_A \circ p$ and $\varphi_B \circ q$ form a pullback of g' and h'.

The morphism φ_P is, in fact, a span morphism from $(P, f \circ p, k \circ q)$ to $(P', f' \circ p', k' \circ q')$, since

$$(f' \circ p') \circ \varphi_P = f' \circ (p' \circ \varphi_P) = f' \circ (\varphi_A \circ p) = (f' \circ \varphi_A) \circ p = f \circ p$$

and

$$(k' \circ q') \circ \varphi_P = k' \circ (q' \circ \varphi_P) = k' \circ (\varphi_B \circ q) = (k' \circ \varphi_B) \circ q = k \circ q.$$

Thus, by Proposition 3.1.11, since φ_P is a span morphism that is an isomorphism in $\mathscr{C}$, φ_P is an isomorphism from $(B, h, k) \bullet (A, f, g)$ to $(B', h', k') \bullet (A', f', g')$, and, so, the two spans are isomorphic, as desired. $\square$

Thus, whenever $[A, f, g] = [A', f', g']$ and $[B, h, k] = [B', h', k']$ we have that $(B, h, k) \bullet (A, f, g)$ and $(B', h', k') \bullet (A', f', g')$ represent the same isomorphism class. So, the following is well defined.

Definition 3.2.8. Let $\mathscr{C}$ be a category with pullbacks, $X, Y, Z \in \mathscr{C}$, $[A, f, g] \in \mathbf{span}_{\mathscr{C}}(X, Y)$ and $[B, h, k] \in \mathbf{span}_{\mathscr{C}}(Y, Z)$. We define the composition of $[B, h, k]$ with $[A, f, g]$ to be

$$[B, h, k] \bullet [A, f, g] := [(B, h, k) \bullet (A, f, g)]. \tag{3.13}$$

The composition of $[B, h, k]$ with $[A, f, g]$ as in Definition 3.2.8 does not depend on the choice of composition of (B, h, k) with (A, f, g), since any two such compositions are isomorphic, by Remark 3.2.2.

We will sometimes denote $[B, h, k] \bullet [A, f, g]$ by simply $[B, h, k][A, f, g]$.

Definition 3.2.8 gives an associative composition, as the following proposition shows.

Proposition 3.2.9. *Let $\mathscr{C}$ be a category with pullbacks, $X, Y, Z, W \in \mathscr{C}$, $[A, f, g] \in \mathbf{span}_{\mathscr{C}}(X, Y)$, $[B, h, k] \in \mathbf{span}_{\mathscr{C}}(Y, Z)$ and $[C, m, n] \in \mathbf{span}_{\mathscr{C}}(Z, W)$. Then*

$$([C, m, n] \bullet [B, h, k]) \bullet [A, f, g] = [C, m, n] \bullet ([B, h, k] \bullet [A, f, g]).$$

Proof. Consider the diagram

$$\begin{tikzcd}
& & & O \arrow[dl] \arrow[dr] \arrow[dd, phantom, "II"] & & & \\
& & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "I"] & & Q \arrow[dl] \arrow[dr] \arrow[dd, phantom, "III"] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & & C \arrow[dl, "m"'] \arrow[dr, "n"] & \\
X & & Y & & Z & & W
\end{tikzcd} \tag{3.14}$$

whose squares I, II and III are pullback squares.

The largest span with vertex Q in (3.14) is a composition of (C, m, n) with (B, h, k), since square III is a pullback square. Thus, it is a representative of $[C, m, n] \bullet [B, h, k] = [(C, m, n) \bullet (B, h, k)]$.

Since squares I and II are pullbacks, by Proposition 2.2.15, the rectangle they form together is also a pullback. Thus, the outermost span in (3.14) is a composition of $(C, m, n) \bullet (B, h, k)$ with (A, f, g), and, therefore, is a representative of $([C, m, n] \bullet [B, h, k]) \bullet [A, f, g]$.

On the other hand, the largest span with vertex P in (3.14) is a representative of $[B, h, k] \bullet [A, f, g]$, since I is a pullback diagram.

So, since the rectangle formed by the squares II and III is a pullback, because the two squares are pullbacks, it follows that the outermost span in (3.14) is a representative of $[C, m, n] \bullet ([B, h, k] \bullet [A, f, g])$

Thus, as $[C, m, n] \bullet ([B, h, k] \bullet [A, f, g])$ and $([C, m, n] \bullet [B, h, k]) \bullet [A, f, g])$ have the same representative, it follows that the two isomorphism classes are equal, as desired. $\square$

We can then define the following category.

Definition 3.2.10. Let $\mathscr{C}$ be a category with pullbacks. We define the category $\mathbf{span}_{\mathscr{C}}$ as the category whose objects are the objects of $\mathscr{C}$, $\mathrm{Hom}_{\mathbf{span}_{\mathscr{C}}}(X, Y) = \mathbf{span}_{\mathscr{C}}(X, Y)$ for all $X, Y \in \mathscr{C}$, and the composition is given by (3.13).

Observe that the identity morphism of $X \in \mathscr{C}$ is the class represented by (X, id_X, id_X).

Definition 3.2.11. Let $\mathscr{C}$ be a category with pullbacks. We define the category $\mathbf{par}_{\mathscr{C}}$ as the subcategory of $\mathbf{span}_{\mathscr{C}}$ containing the same class of objects, and such that $\mathrm{Hom}_{\mathbf{par}_{\mathscr{C}}}(X, Y) = \mathbf{par}_{\mathscr{C}}(X, Y)$ for all $X, Y \in \mathscr{C}$.

Proposition 3.2.5 assures us that the composition of morphisms in $\mathbf{par}_{\mathscr{C}}$ yields a morphism in $\mathbf{par}_{\mathscr{C}}$.

Remark 3.2.12. The category $\mathbf{par}_{\mathscr{C}}$ in Definition 3.2.11 can be defined regardless of $\mathbf{span}_{\mathscr{C}}$, and this definition only requires $\mathscr{C}$ to be a category with inverse images (see Remark 3.2.3). Throughout this work, all the results regarding $\mathbf{par}_{\mathscr{C}}$ that we prove for a category with pullbacks $\mathscr{C}$ can be adapted to this case.

Remark 3.2.13. While the isomorphism classes of spans in $\mathscr{C}$ form a category, the spans in $\mathscr{C}$ form a *bicategory* [3], whose objects are the objects in $\mathscr{C}$ and whose category of morphisms from an object X to an object Y is $\mathbf{Span}_{\mathscr{C}}(X, Y)$, where the horizontal composition is given by the composition of spans. Similarly, the partial morphisms in $\mathscr{C}$ form a bicategory $\mathbf{Par}_{\mathscr{C}}$.

We choose not to give an exact definition of a bicategory here, as it is quite technical and will not be used throughout the work. The interested reader may consult, for instance [3], for more details.

With the following, one can see a category with pullbacks $\mathscr{C}$ as a subcategory of $\mathbf{par}_{\mathscr{C}}$.

Lemma 3.2.14. *Let $\mathscr{C}$ be a category with pullbacks and $f : X \to Y$ and $g : Y \to Z$ morphisms in $\mathscr{C}$. Then*

$$[Y, id_Y, g] \bullet [X, id_X, f] = [X, id_X, g \circ f].$$

Proof. The square in

$$\begin{tikzcd}
 & & X \arrow[dl, "id_X"'] \arrow[dr, "f"] \arrow[dd, phantom, "\vee", very near start] & & \\
 & X \arrow[dl, "id_X"'] \arrow[dr, "f"] & & Y \arrow[dl, "id_Y"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

is a pullback, so

$$[Y, id_Y, g] \bullet [X, id_X, f] = [X, id_X \circ id_X, g \circ f] = [X, id_X, g \circ f]. \qquad \square$$

Proposition 3.2.15. *Let $\mathscr{C}$ be a category with pullbacks. There is a faithful functor $F : \mathscr{C} \to \mathbf{par}_{\mathscr{C}}$ that is given on the objects by $F(X) = X$ and on the morphisms by $F(f) = [\mathrm{dom}\, f, id_{\mathrm{dom}\, f}, f]$.*

Proof. Observe that id_X is a monomorphism for all $X \in \mathscr{C}$, so $[X, id_X, f]$ is indeed a morphism from X to Y in $\mathbf{par}_{\mathscr{C}}$ for all $f : X \to Y$ in $\mathscr{C}$.

Let us first verify that F is functorial. Indeed, for all $X \in \mathscr{C}$,

$$F(id_X) = [X, id_X, id_X]$$

is the identity of $F(X) = X$ in $\mathbf{par}_{\mathscr{C}}$.

And F preserves the composition. Indeed, if $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathscr{C}$, then by Lemma 3.2.14 we have

$$F(g) \bullet F(f) = [Y, id_Y, g] \bullet [X, id_X, f] = [X, id_X, g \circ f] = F(g \circ f).$$

The faithfulness of F follows by Proposition 3.1.15. $\square$

We give the following definition to name the morphisms in $\mathbf{par}_{\mathscr{C}}$ that come from $\mathscr{C}$.

Definition 3.2.16. Let $\mathscr{C}$ be a category with pullbacks. A morphism in $\mathbf{par}_{\mathscr{C}}$ of the form $[X, id_X, f]$ is said to be **global**.

By Remark 3.1.18, when there is no risk of confusion, we may use both of the notations $[X, id_X, f]$ and f interchangeably. In this situation, observe that, by Lemma 3.2.14, the composition $g \bullet f$ of global morphisms g and f in $\mathbf{par}_{\mathscr{C}}$ is the global morphism $g \circ f$.

Proposition 3.2.17. *Let X and Y be sets and $[A, \iota, f] \in \mathbf{par}_{\mathbf{Set}}(X, Y)$, where $A \subseteq X$ and ι is the corresponding inclusion map. Then $[A, \iota, f]$ is a global morphism if and only if $A = X$ and $\iota = id_X$.*

Proof. Clearly if $A = X$ and $\iota = id_X$ we have that $[A, \iota, f]$ is a global morphism.

Now assume $[A, \iota, f]$ is a global morphism. Then $[A, \iota, f] = [X, id_X, g]$ for some $g : X \to Y$. Since both $\iota : A \to X$ and $id_X : X \to X$ are inclusion maps, and (A, ι, f) and (X, id_X, g) both represent the same isomorphism class, by Proposition 3.1.16 it follows that

$$(A, \iota, f) = (X, id_X, g),$$

so $A = X$ and $\iota = id_X$, as desired. $\square$

Proposition 3.2.18. *Let $\mathscr{C}$ be a category with pullbacks, $X, Y, Z \in \mathscr{C}$, $[A, \iota, f] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$. Then*

$$g \bullet [A, \iota, f] = [A, \iota, g \circ f].$$

Proof. The square in

$$\begin{tikzcd}
& & A \arrow[dl, "id_A"'] \arrow[dr, "f"] \arrow[dd, phantom, "\vee", very near start] & & \\
& A \arrow[dl, "\iota"'] \arrow[dr, "f"] & & Y \arrow[dl, "id_Y"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

is a pullback, so

$$g \bullet [A, \iota, f] = [A, \iota \circ id_A, g \circ f] = [A, \iota, g \circ f]. \qquad \square$$

It should be mentioned that $\mathbf{span}_{\mathscr{C}}$ and $\mathbf{par}_{\mathscr{C}}$ may not be locally small even when $\mathscr{C}$ is locally small (see Proposition 3.2.20 below). We will show a necessary and sufficient condition for $\mathbf{par}_{\mathscr{C}}$ to be locally small.

Definition 3.2.19. A category $\mathscr{C}$ is said to **have few subobjects** if the class of subobjects of every object of $\mathscr{C}$ is a set.

Proposition 3.2.20. *Let $\mathscr{C}$ be a (locally small) category with pullbacks. Then $\mathbf{par}_{\mathscr{C}}$ is locally small if and only if $\mathscr{C}$ has few subobjects.*

Proof. The "if" part. Assume $\mathscr{C}$ has few subobjects and let X and Y be objects in $\mathscr{C}$. We will prove that $\mathbf{par}_{\mathscr{C}}(X, Y)$ is a set.

Let I be the set of subobjects of X and fix a representative $\iota_i : X_i \to X$ for each subobject $i \in I$. By Proposition 3.1.17, each partial morphism from X to Y is isomorphic to a unique partial morphism of the form (X_i, ι_i, g) for some $i \in I$ and $g \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Every element of $\mathbf{par}_{\mathscr{C}}(X, Y)$ is, then, of the form $[X_i, \iota_i, g]$ for some $i \in I$ and $g \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Thus, $\mathbf{par}_{\mathscr{C}}(X, Y)$ is in bijection with the set $\coprod_{i \in I} \mathrm{Hom}_{\mathscr{C}}(X, Y)$, and, so, $\mathbf{par}_{\mathscr{C}}(X, Y)$ is a set.

Since $\mathrm{Hom}_{\mathbf{par}_{\mathscr{C}}}(X, Y)$ is a set for all $X, Y \in \mathscr{C}$, it follows that $\mathbf{par}_{\mathscr{C}}$ is locally small, as desired.

The "only if" part. Assume that $\mathbf{par}_{\mathscr{C}}$ is locally small and let $X \in \mathscr{C}$. Let I be the class of subobjects of X. For each $i \in I$ let ι_i be a representative of i.

Define the following map from I to $\mathbf{par}_{\mathscr{C}}(X, X)$. For each $i \in I$, let $\Psi(i) = [X_i, \iota_i, \iota_i] \in \mathbf{par}_{\mathscr{C}}(X, X)$.

Clearly, Ψ is a well-defined map. We will verify that Ψ is an injection. Let $i, j \in I$ such that $\Psi(i) = \Psi(j)$. Then

$$[X_i, \iota_i, \iota_i] = [X_j, \iota_j, \iota_j],$$

so there is an isomorphism $\varphi : (X_i, \iota_i, \iota_i) \to (X_j, \iota_j, \iota_j)$.

Since φ is a span morphism, in particular we have that

$$\iota_i = \iota_j \circ \varphi,$$

and since φ is an isomorphism, it follows that ι_i and ι_j represent the same subobject. Thus, $i = j$.

By hypothesis, $\mathbf{par}_{\mathscr{C}}$ is locally small, so $\mathbf{par}_{\mathscr{C}}(X, X)$ is a set. Since Ψ is an injection from I to $\mathbf{par}_{\mathscr{C}}(X, X)$, we have that I is also a set. Thus, $\mathscr{C}$ has few subobjects. $\square$

To avoid set-theoretic complications, from this point onward we will always assume a category $\mathscr{C}$ to have few subobjects. We emphasize that every category we are dealing with in this work has few subobjects.

3.3 PULLBACK-PRESERVING FUNCTORS AND SPANS

Proposition 3.3.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks and $F : \mathscr{C} \to \mathscr{D}$ a functor. Then the association that maps an isomorphism class $[A, f, g] \in \mathbf{span}_{\mathscr{C}}(X, Y)$ to the isomorphism class $F([A, f, g]) := [F(A), F(f), F(g)] \in \mathbf{span}_{\mathscr{D}}(F(X), F(Y))$ is well-defined.*

Proof. Let (B, h, k) be a span from X to Y such that $[B, h, k] = [A, f, g]$. We must verify that $[F(A), F(f), F(g)] = [F(B), F(h), F(k)]$.

Since $[B, h, k] = [A, f, g]$, there exists an isomorphism $\varphi : A \to B$ in $\mathscr{C}$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[d, "\varphi"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\ & B \arrow[dl, "h"] \arrow[dr, "k"'] & \\ X & & Y \end{tikzcd} \tag{3.15}$$

Since F is a functor, by the commutativity of (3.15) the diagram

$$\begin{tikzcd} & F(A) \arrow[d, "F(\varphi)"] \arrow[ddl, bend right, "F(f)"'] \arrow[ddr, bend left, "F(g)"] & \\ & F(B) \arrow[dl, "F(h)"'] \arrow[dr, "F(k)"] & \\ F(X) & & F(Y) \end{tikzcd}$$

commutes. By the functoriality of F, since φ is an isomorphism, so is $F(\varphi)$. Thus, it follows that

$$[F(A), F(f), F(g)] = [F(B), F(h), F(k)],$$

as desired. $\square$

Proposition 3.3.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks and $F : \mathscr{C} \to \mathscr{D}$ a functor that preserves pullbacks. Then F induces a functor $F : \mathbf{span}_{\mathscr{C}} \to \mathbf{span}_{\mathscr{D}}$ that is the same on the objects and maps an isomorphism class $[A, f, g] \in \mathbf{span}_{\mathscr{C}}(X, Y)$ to the isomorphism class $F([A, f, g]) \coloneqq [F(A), F(f), F(g)] \in \mathbf{span}_{\mathscr{D}}(F(X), F(Y))$*

Proof. Observe that this functor is well-defined by Proposition 3.3.1.

Since F is a functor, for all $X \in \mathscr{C}$,

$$F([X, id_X, id_X]) = [F(X), F(id_X), F(id_X)] = [F(X), id_{F(X)}, id_{F(X)}],$$

so it preserves identities.

We will now verify that F preserves the composition. Let $[A, f, g] \in \mathbf{span}_{\mathscr{C}}(X, Y)$ and $[B, h, k] \in \mathbf{span}_{\mathscr{C}}(Y, Z)$. Consider the following diagram, whose square is a pullback, so its outermost span is a representative of $[B, h, k] \bullet [A, f, g]$.

$$\begin{tikzcd}
& & P \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & Z
\end{tikzcd} \tag{3.16}$$

Thus, the outermost span of the following diagram is a representative of $F([B, h, k] \bullet [A, f, g])$.

$$\begin{tikzcd}
& & F(P) \arrow[dl, "F(p)"'] \arrow[dr, "F(q)"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
& F(A) \arrow[dl, "F(f)"'] \arrow[dr, "F(g)"] & & F(B) \arrow[dl, "F(h)"'] \arrow[dr, "F(k)"] & \\
F(X) & & F(Y) & & F(Z)
\end{tikzcd} \tag{3.17}$$

On the other hand, since F is a functor that preserves pullbacks, the square in (3.17) is a pullback, so its outermost span is a representative of $[F(B), F(h), F(k)] \bullet [F(A), F(f), F(g)] = F([B, h, k]) \bullet F([A, f, g])$. Thus, we have

$$F([B, h, k]) \bullet F([A, f, g]) = F([B, h, k] \bullet [A, f, g]),$$

so the composition is preserved, as desired. □

Corollary 3.3.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks and $F : \mathscr{C} \to \mathscr{D}$ a functor that preserves pullbacks. Then F induces a functor $F : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$ that is the same on the objects and maps an isomorphism class $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ to the isomorphism class $F([A, f, g]) := [F(A), F(f), F(g)] \in \mathbf{par}_{\mathscr{D}}(F(X), F(Y))$*

Proof. By Proposition 3.3.2 it suffices to verify that the induced functor from $\mathbf{span}_{\mathscr{C}}$ to $\mathbf{span}_{\mathscr{D}}$ sends morphisms in $\mathbf{par}_{\mathscr{C}}$ to morphisms in $\mathbf{par}_{\mathscr{D}}$.

Indeed, let $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$. Since f is a monomorphism and F preserves pullbacks, by Proposition 2.2.19 it follows that $F(f)$ is a monomorphism. Thus,

$$F([A, f, g]) = [F(A), F(f), F(g)] \in \mathbf{par}_{\mathscr{D}}(F(X), F(Y)),$$

as desired. □

Proposition 3.3.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks and $F : \mathscr{C} \to \mathscr{D}$ a faithful functor that preserves pullbacks and satisfies the following property: for all monomorphisms $g \in \mathrm{Hom}_{\mathscr{D}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathscr{D}}(Y, Z)$, if $g = F(g')$ and $g \circ f = F(h)$ for some morphisms*

g', h in $\mathscr{C}$, then $f = F(f')$ for some morphism f' in $\mathscr{C}$ such that $h = g' \circ f'$. Under those assumptions, the induced functor $F : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$ is faithful.

Proof. Let $[A, f, g], [B, h, k] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ such that

$$F([A, f, g]) = F([B, h, k]).$$

Then

$$[F(A), F(f), F(g)] = [F(B), F(h), F(k)],$$

so there exists an isomorphism $\varphi : F(A) \to F(B)$ such that the following diagram commutes.

$$\begin{tikzcd} & F(A) \arrow[d, "\varphi"] \arrow[ddl, bend right, "F(f)"'] \arrow[ddr, bend left, "F(g)"] & \\ & F(B) \arrow[dl, "F(h)"] \arrow[dr, "F(k)"'] & \\ F(X) & & F(Y) \end{tikzcd} \tag{3.18}$$

Since φ and $F(h)$ are monomorphisms and $F(h) \circ \varphi = F(f)$, by hypothesis it follows that $\varphi = F(\varphi')$ for some $\varphi' : A \to B$ such that

$$h \circ \varphi' = f. \tag{3.19}$$

Similarly, since $\psi := \varphi^{-1}$ and $F(f)$ are monomorphisms and $F(f) \circ \psi = F(h)$, we have that $\psi = F(\psi')$ for some $\psi' : B \to A$ such that

$$f \circ \psi' = h. \tag{3.20}$$

Observe that by (3.19) and (3.20) we have

$$f \circ \psi' \circ \varphi' = h \circ \varphi' = f$$

and

$$h \circ \varphi' \circ \psi' = f \circ \psi' = h,$$

so, since f and h are monomorphisms, it follows that $\psi' \circ \varphi' = id_A$ and $\varphi' \circ \psi' = id_B$, and, therefore, φ' is an isomorphism.

Now, observe that by the commutativity of (3.18) we have

$$F(k \circ \varphi') = F(k) \circ F(\varphi') = F(k) \circ \varphi = F(g),$$

so since F is faithful it follows that

$$k \circ \varphi' = g. \tag{3.21}$$

Thus, by (3.19) and (3.21) the diagram

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[d, "\varphi'"] \arrow[dr, "g"] & \\
 & B \arrow[dl, "h"] \arrow[dr, "k"'] & \\
X & & Y
\end{tikzcd}$$

commutes, so since φ' is an isomorphism it follows that $[A, f, g] = [B, h, k]$. Therefore, the induced functor is faithful, as desired. □

3.4 RESTRICTION CATEGORIES

In this section we will define restriction categories, see some of their properties and show that $\mathbf{par}_{\mathscr{C}}$ has a natural restriction structure for any category with pullbacks $\mathscr{C}$.

Definition 3.4.1. A **(right) restriction monoid** is a monoid M together with a unary operation $m \mapsto \overline{m}$, satisfying the following conditions: for all $m, n \in M$,

(R1) $m\overline{m} = m$,

(R2) $\overline{m}\overline{n} = \overline{n}\overline{m}$,

(R3) $\overline{n\overline{m}} = \overline{n}\overline{m}$,

(R4) $\overline{n}m = m\overline{nm}$.

For more details on restriction monoids, see [5] and [10].

A restriction category is a categorical analogue of a restriction monoid (similarly as categories being generalizations of monoids).

Definition 3.4.2. A **restriction structure** on a category $\mathscr{C}$ is an association of a morphism $\overline{f} : X \to X$ to any morphism $f : X \to Y$ in $\mathscr{C}$, satisfying the following conditions. For all $f : X \to Y$, $g : X \to Z$ and $h : Y \to Z$ morphisms in $\mathscr{C}$,

(R1) $f \circ \overline{f} = f$,

(R2) $\overline{f} \circ \overline{g} = \overline{g} \circ \overline{f}$,

(R3) $\overline{g \circ \overline{f}} = \overline{g} \circ \overline{f}$, (R4) $\overline{h} \circ f = f \circ \overline{h \circ f}$.

A **restriction category** [4] is a category $\mathscr{C}$ together with a restriction structure.

Remark 3.4.3. In a locally small restriction category $\mathscr{C}$, for all $X \in \mathscr{C}$ the set $\mathrm{End}_{\mathscr{C}}(X)$ is a (right) restriction monoid [5].

Example 3.4.4. Any category can be trivially seen as a restriction category with the restriction structure that associates to any morphism $f : X \to Y$ the identity morphism id_X.

Example 3.4.5. A basic nontrivial example of a restriction category is that of sets and partial maps. Here, given $f : \mathrm{dom}\, f \subseteq X \to Y$ a partial map, $\overline{f} : \mathrm{dom}\, f \subseteq X \to X$ is the inclusion map of $\mathrm{dom}\, f$ into X.

The above mentioned example is a particular case of the natural restriction structure on $\mathbf{par}_{\mathscr{C}}$ that we are going to introduce below.

Proposition 3.4.6. *Let $\mathscr{C}$ be a category with pullbacks. Then $\mathbf{par}_{\mathscr{C}}$ is a restriction category, with restriction structure that associates to any $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ the isomorphism class $\overline{[A, f, g]} = [A, f, f] \in \mathbf{par}_{\mathscr{C}}(X, X)$.*

Proof. First, we must verify that such an association is well-defined. Indeed, assume $[A, f, g], [B, h, k] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ are such that

$$[A, f, g] = [B, h, k].$$

Then there exists an isomorphism $\varphi : A \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[d, "\varphi"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& B \arrow[dl, "h"] \arrow[dr, "k"'] & \\
X & & Y
\end{tikzcd}$$

In particular, since $h \circ \varphi = f$, the following diagram is commutative.

$$\begin{tikzcd}
& A \arrow[d, "\varphi"] \arrow[ddl, "f"'] \arrow[ddr, "f"] & \\
& B \arrow[dl, "h"] \arrow[dr, "h"'] & \\
X & & X
\end{tikzcd}$$

Then, since φ is an isomorphism, it follows that $(A, f, f) \cong (B, h, h)$, and, so, $[A, f, f] = [B, h, h]$. Thus,

$$\overline{[A, f, g]} = [A, f, f] = [B, h, h] = \overline{[B, h, k]}.$$

Now let us verify the axioms of a restriction category.

(R1). Let $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$. Since f is a monomorphism in $\mathscr{C}$, the square in the following diagram is a pullback, and, thus, its outermost span is a representative of $[A, f, g] \bullet [A, f, f]$.

$$\begin{tikzcd}
& & A \arrow[dl, "id_A"'] \arrow[dr, "id_A"] \arrow[dd, phantom, "\vee" near start] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "f"] & & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & X & & Y
\end{tikzcd}$$

Thus, we have

$$[A, f, g] \bullet \overline{[A, f, g]} = [A, f, g] \bullet [A, f, f] = [A, f \circ id_A, g \circ id_A] = [A, f, g].$$

(R2). Let $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$ and $[B, h, k] \in \mathbf{par}_{\mathscr{C}}(X, Z)$. Consider the following diagram, whose square is a pullback.

$$\begin{tikzcd}
& & P \arrow[dl, "\widehat{h}"'] \arrow[dr, "\widehat{f}"] \arrow[dd, phantom, "\vee" near start] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, "h"'] \arrow[dr, "h"] & \\
X & & X & & X
\end{tikzcd} \tag{3.22}$$

Then the outermost span of (3.22) represents the composition of $[A, f, f]$ with $[B, h, h]$. Thus,

$$\overline{[A, f, g]} \bullet \overline{[B, h, k]} = [A, f, f] \bullet [B, h, h] = [P, f \circ \widehat{h}, h \circ \widehat{f}]. \tag{3.23}$$

Similarly, since

$$\begin{tikzcd}
& & P \arrow[dl, "\widehat{f}"'] \arrow[dr, "\widehat{h}"] \arrow[dd, phantom, "\vee" near start] & & \\
& B \arrow[dl, "h"'] \arrow[dr, "h"] & & A \arrow[dl, "f"'] \arrow[dr, "f"] & \\
X & & X & & X
\end{tikzcd}$$

is a diagram whose square is a pullback, we have that

$$\overline{[B,h,k]} \bullet \overline{[A,f,g]} = [B,h,h] \bullet [A,f,f] = [P, h \circ \widehat{f}, f \circ \widehat{h}]. \tag{3.24}$$

Since the square in (3.22) commutes, we have that $h \circ \widehat{f} = f \circ \widehat{h}$. Thus, by (3.23) and (3.24) it follows that

$$\overline{[A,f,g]} \bullet \overline{[B,h,k]} = \overline{[B,h,k]} \bullet \overline{[A,f,g]}.$$

(R3). Let $[A,f,g] \in \mathbf{par}_{\mathscr{C}}(X,Y)$ and $[B,h,k] \in \mathbf{par}_{\mathscr{C}}(X,Z)$. Consider the following diagram, whose square is a pullback.

$$\begin{tikzcd} & & P \arrow[dl, "\widehat{h}"'] \arrow[dr, "\widehat{f}"] \arrow[dd, phantom, "\vee" very near start] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X & & X & & Z \end{tikzcd} \tag{3.25}$$

The outermost span of (3.25) is a representative of

$$[B,h,k] \bullet \overline{[A,f,g]} = [B,h,k] \bullet [A,f,f].$$

Thus,

$$\overline{[B,h,k] \bullet \overline{[A,f,g]}} = [P, f \circ \widehat{h}, f \circ \widehat{h}]. \tag{3.26}$$

On the other hand, by (3.24) and since the square in (3.25) is commutative, we have

$$\overline{[B,h,k]} \bullet \overline{[A,f,g]} = [P, h \circ \widehat{f}, f \circ \widehat{h}] = [P, f \circ \widehat{h}, f \circ \widehat{h}]. \tag{3.27}$$

Thus,

$$\overline{[B,h,k] \bullet \overline{[A,f,g]}} = \overline{[B,h,k]} \bullet \overline{[A,f,g]}.$$

(R4). Let $[A,f,g] \in \mathbf{par}_{\mathscr{C}}(X,Y)$ and $[B,h,k] \in \mathbf{par}_{\mathscr{C}}(Y,Z)$. Consider the following diagram, whose square is a pullback.

$$\begin{tikzcd} & & P \arrow[dl, "\widehat{h}"'] \arrow[dr, "\widehat{g}"] \arrow[dd, phantom, "\vee" very near start] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\ X & & Y & & Z \end{tikzcd} \tag{3.28}$$

The outermost span of (3.25) is a representative of $[B, h, k] \bullet [A, f, g]$, so

$$\overline{[B, h, k] \bullet [A, f, g]} = [P, f \circ \widehat{h}, f \circ \widehat{h}]. \tag{3.29}$$

Now, the square in

$$\begin{tikzcd}
 & & P \arrow[dl, "id_P"'] \arrow[dr, "\widehat{h}"] \arrow[dd, phantom, "\vee" very near start] & & \\
 & P \arrow[dl, "f\circ\widehat{h}"'] \arrow[dr, "f\circ\widehat{h}"] & & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & X & & Y
\end{tikzcd}$$

is a pullback, since f is a monomorphism. Thus,

$$[A, f, g] \bullet [P, f \circ \widehat{h}, f \circ \widehat{h}] = [P, f \circ \widehat{h}, g \circ \widehat{h}],$$

so, by (3.29),

$$[A, f, g] \bullet \overline{[B, h, k] \bullet [A, f, g]} = [P, f \circ \widehat{h}, g \circ \widehat{h}]. \tag{3.30}$$

On the other hand, since the square in

$$\begin{tikzcd}
 & & P \arrow[dl, "\widehat{h}"'] \arrow[dr, "\widehat{g}"] \arrow[dd, phantom, "\vee" very near start] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "h"] & \\
X & & Y & & Y
\end{tikzcd} \tag{3.31}$$

is a pullback, we have

$$\overline{[B, h, k]} \bullet [A, f, g] = [B, h, h] \bullet [A, f, g] = [P, f \circ \widehat{h}, h \circ \widehat{g}] = [P, f \circ \widehat{h}, g \circ \widehat{h}],$$

where the last equality is due to the commutativity of the square in (3.31).

Thus, by (3.30),

$$\overline{[B, h, k]} \bullet [A, f, g] = [A, f, g] \bullet \overline{[B, h, k] \bullet [A, f, g]}.$$ □

Proposition 3.4.7. *Let $\mathscr{C}$ be a restriction category and $X \in \mathscr{C}$. Then $\overline{id_X} = id_X$.*

Proof. By (R1),

$$id_X \circ \overline{id_X} = id_X.$$

Thus, $\overline{id_X} = id_X$. □

In a restriction category $\mathscr{C}$, for all $X, Y \in \mathscr{C}$ there is a natural partial order on

$\mathrm{Hom}_{\mathscr{C}}(X,Y)$ that comes from the restriction structure.

Definition 3.4.8. Let $\mathscr{C}$ be a restriction category and $X, Y \in \mathscr{C}$. We define the relation $\leq$ on $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ by

$$f \leq g \quad \iff \quad f = g \circ \overline{f}.$$

Proposition 3.4.9. *The relation in Definition 3.4.8 is a partial order.*

Proof. By (R1), for all $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ we have that $f = f \circ \overline{f}$, so

$$f \leq f,$$

and, thus, $\leq$ is a reflexive relation.

For the antisymmetry, let $f, g \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ such that $f \leq g$ and $g \leq f$. Then

$$f = g \circ \overline{f} \quad \text{and} \quad g = f \circ \overline{g},$$

so, by (R2),

$$f = g \circ \overline{f} = f \circ \overline{g} \circ \overline{f} = f \circ \overline{f} \circ \overline{g} = f \circ \overline{g} = g.$$

Finally, for the transitivity, let $f, g, h \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ such that $f \leq g$ and $g \leq h$. Then

$$f = g \circ \overline{f} \quad \text{and} \quad g = h \circ \overline{g},$$

so, by (R3),

$$f = g \circ \overline{f} = h \circ \overline{g} \circ \overline{f} = h \circ \overline{g \circ \overline{f}} = h \circ \overline{f},$$

whence $f \leq h$. □

Proposition 3.4.10. *Let $\mathscr{C}$ be a restriction category and $W, X, Y, Z \in \mathscr{C}$. If $f \leq g \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, $h \in \mathrm{Hom}_{\mathscr{C}}(Y,Z)$ and $k \in \mathrm{Hom}_{\mathscr{C}}(W,X)$ then*

$$h \circ f \leq h \circ g \tag{3.32}$$

and

$$f \circ k \leq g \circ k. \tag{3.33}$$

Proof. Since $f \leq g$, we have

$$f = g \circ \overline{f}. \tag{3.34}$$

Then

$$h \circ g \circ \overline{h \circ f} \overset{(3.34)}{=} h \circ g \circ \overline{h \circ g \circ \overline{f}} \overset{(R3)}{=} h \circ g \circ \overline{h \circ g} \circ \overline{f} \overset{(R1)}{=} h \circ g \circ \overline{f} \overset{(3.34)}{=} h \circ f,$$

and thus (3.32) follows.

And

$$g \circ k \circ \overline{f \circ k} = g \circ (k \circ \overline{f \circ k}) \overset{(R4)}{=} g \circ (\overline{f} \circ k) = (g \circ \overline{f}) \circ k \overset{(3.34)}{=} f \circ k,$$

so (3.33) follows. □

Corollary 3.4.11. *Let $\mathscr{C}$ be a restriction category and $X, Y, Z \in \mathscr{C}$. If $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$, $h \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ and $g \circ f \leq h$, then*

$$g \circ f \leq h \circ \overline{f}. \tag{3.35}$$

Proof. By Proposition 3.4.10, since $g \circ f \leq h$ we have

$$g \circ f \circ \overline{f} \leq h \circ \overline{f},$$

so (3.35) follows by (R1). □

Proposition 3.4.12. *Let $\mathscr{C}$ be a restriction category and $X, Y \in \mathscr{C}$. Then for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and any morphism g with $\operatorname{dom} g = X$ we have*

$$f \circ \overline{g} \leq f. \tag{3.36}$$

Proof. By (R3) we have

$$f \circ \overline{\overline{f} \circ \overline{g}} = f \circ \overline{f} \circ \overline{g} = f \circ \overline{g},$$

so (3.36) follows. □

Proposition 3.4.13. *Let $\mathscr{C}$ be a category with pullbacks and $[A, f, g], [B, h, k] \in \mathbf{par}_{\mathscr{C}}(X, Y)$. Then $[A, f, g] \leq [B, h, k]$ if and only if there exists a span morphism from (A, f, g) to (B, h, k).*

Proof. The "only if" part. Assume $[A, f, g] \leq [B, h, k]$. Then

$$[A, f, g] = [B, h, k] \bullet \overline{[A, f, g]} = [B, h, k] \bullet [A, f, f]. \tag{3.37}$$

Consider the following diagram

$$\begin{tikzcd}
& & P \arrow[dl, "\widehat{h}"'] \arrow[dr, "\widehat{f}"] \arrow[d, phantom, "\vee"] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & X & & Z
\end{tikzcd} \tag{3.38}$$

whose square is a pullback, so

$$[B, h, k] \bullet [A, f, f] = [P, f \circ \widehat{h}, k \circ \widehat{f}]. \tag{3.39}$$

Since the square in (3.38) is commutative, by (3.37) and (3.39) it follows that

$$[A, f, g] = [B, h, k] \bullet [A, f, f] = [P, f \circ \widehat{h}, k \circ \widehat{f}] = [P, h \circ \widehat{f}, k \circ \widehat{f}],$$

and, thus

$$(A, f, g) \cong (P, h \circ \widehat{f}, k \circ \widehat{f}).$$

So, there exists an isomorphism $\varphi : A \to P$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "g"] & \\ X & P \arrow[l, "h \circ \widehat{f}"] \arrow[r, "k \circ \widehat{f}"'] & X \end{tikzcd}$$

It follows, then, that $\psi := \widehat{f} \circ \varphi$ is a span morphism from (A, f, g) to (B, h, k), since the diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\psi"] \arrow[dr, "g"] & \\ X & B \arrow[l, "h"] \arrow[r, "k"'] & X \end{tikzcd}$$

commutes.

The "if" part. Assume there is a span morphism ψ from (A, f, g) to (B, h, k). Than is, ψ is such that the diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, dashed, "\psi"] \arrow[dr, "g"] & \\ X & B \arrow[l, "h"] \arrow[r, "k"'] & X \end{tikzcd}$$

commutes, and, in particular,

$$f = h \circ \psi \tag{3.40}$$

and

$$g = k \circ \psi. \tag{3.41}$$

Since h is a monomorphism and we have (3.40), the square in the following diagram is a pullback.

$$\begin{tikzcd}
& & A \arrow[dl, "id_A"'] \arrow[dr, "\psi"] \arrow[dd, phantom, "\vee" very near start] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & X & & Y
\end{tikzcd}$$

Thus,

$$[B, h, k] \bullet \overline{[A, f, g]} = [B, h, k] \bullet [A, f, f] = [A, f \circ id_A, k \circ \psi],$$

and so, by (3.41),

$$[B, h, k] \bullet \overline{[A, f, g]} = [A, f, g].$$

Therefore,

$$[A, f, g] \leq [B, h, k].$$

□

Definition 3.4.14. Let $\mathscr{C}$ be a restriction category. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be a **total morphism** if $\overline{f} = id_X$.

Proposition 3.4.15. *Let $\mathscr{C}$ be a category with pullbacks. The total morphisms of* $\mathbf{par}_{\mathscr{C}}$ *are the global morphisms.*

Proof. Clearly, if $[X, id_X, f]$ is a global morphism in $\mathbf{par}_{\mathscr{C}}$ then

$$\overline{[X, id_X, f]} = [X, id_X, id_X],$$

so it is a total morphism.

On the other hand, let $[A, f, g] : X \to Y$ be a total morphism in $\mathbf{par}_{\mathscr{C}}$. Then

$$[A, f, f] = \overline{[A, f, g]} = [X, id_X, id_X].$$

Thus, $(X, id_X, id_X) \cong (A, f, f)$, so there exists an isomorphism $\varphi : X \to A$ such that the following diagram commutes.

$$\begin{tikzcd}
& X \arrow[ddl, "id_X"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "id_X"] & \\
& A \arrow[dl, "f"] \arrow[dr, "f"'] & \\
X & & X
\end{tikzcd}$$

Hence, φ is an isomorphism such that the diagram

$$\begin{tikzcd}
& X \arrow[dl, "id_X"'] \arrow[d, dashed, "\varphi"] \arrow[dr, "g\circ\varphi"] & \\
& A \arrow[l, "f"] \arrow[r, "g"'] & \\
X & & Y
\end{tikzcd}$$

commutes, and so $(A, f, g) \cong (X, id_X, g \circ \varphi)$, and, therefore

$$[A, f, g] = [X, id_X, g \circ \varphi],$$

so $[A, f, g]$ is a global morphism. $\square$

Definition 3.4.16. Let $\mathscr{C}$ and $\mathscr{D}$ be restriction categories. A **restriction functor** from $\mathscr{C}$ to $\mathscr{D}$ is a functor $F : \mathscr{C} \to \mathscr{D}$ satisfying

$$F(\overline{f}) = \overline{F(f)}$$

for all morphisms f in $\mathscr{C}$.

Proposition 3.4.17. *Let $F : \mathscr{C} \to \mathscr{D}$ be a restriction functor. If $f \leq g$ in* $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, *then*

$$F(f) \leq F(g)$$

is $\mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$.

Proof. Since $f \leq g$, we have $f = g \circ \overline{f}$; Therefore, since F is a restriction functor it follows that

$$F(f) = F(g \circ \overline{f}) = F(g) \circ F(\overline{f}) = F(g) \circ \overline{F(f)},$$

and so we have $F(f) \leq F(g)$, as desired. $\square$

Proposition 3.4.18. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks and $F : \mathscr{C} \to \mathscr{D}$ a functor that preserves pullbacks. Then the induced functor F : $\mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$ is a restriction functor.*

Proof. Let $[A, f, g] \in \mathbf{par}_{\mathscr{C}}(X, Y)$. Then

$$F(\overline{[A, f, g]}) = F([A, f, f]) = [F(A), F(f), F(f)] = \overline{[F(A), F(f), F(g)]} = \overline{F([A, f, g])},$$

so F preserves the restriction structure, as desired. $\square$

3.5 INVERSE CATEGORIES

Definition 3.5.1. Let $\mathscr{C}$ be a category. We say $\mathscr{C}$ is an **inverse category** [19][1] if for each morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ there exists a unique morphism $f^* \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, called the **inverse**[2] of f, satisfying

$$f \circ f^* \circ f = f$$

and

$$f^* \circ f \circ f^* = f^*.$$

Remark 3.5.2. The monoid of endomorphisms of an object X in a locally small inverse category $\mathscr{C}$ is an inverse monoid, where the inverse of $f \in \mathrm{End}_{\mathscr{C}}(X)$ is its inverse f^* in $\mathscr{C}$.

Proposition 3.5.3. *Let $\mathscr{C}$ be an inverse category. Then for each morphism f in $\mathscr{C}$ we have $f = (f^*)^*$.*

Proof. The proof is analogous to the proof of Proposition 2.3.5. □

Just like idempotents are important elements in an inverse monoid (recall Definition 2.3.6), they also manifest great importance in inverse categories.

Definition 3.5.4. Let $\mathscr{C}$ be a category and $X \in \mathscr{C}$. We denote by $\mathcal{E}(X)$ the set $\mathcal{E}(\mathrm{End}_{\mathscr{C}}(X))$, formed by the idempotent elements of the monoid $\mathrm{End}_{\mathscr{C}}(X)$. A morphism f in $\mathscr{C}$ is an idempotent morphism if $f \in \mathcal{E}(X)$ for some $X \in \mathscr{C}$.

The following proposition could be proved in a similar way to Proposition 2.3.10, but we opted for another proof using

Proposition 3.5.5. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is an inverse category if and only if it satisfies the following.*

(1) *For each $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ there exists a (not necessarily unique) morphism $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ satisfying*

$$f \circ g \circ f = f. \tag{3.42}$$

(2) *For each $X \in \mathscr{C}$ the elements of $\mathcal{E}(X)$ commute in $\mathrm{End}_{\mathscr{C}}(X)$.*

Proof. The proof is analogous to the proof of Proposition 2.3.10. □

Definition 3.5.6. Let $\mathscr{C}$ be a restriction category. We define $\mathbf{inv}(\mathscr{C})$ to be the subclass of the class of morphisms of $\mathscr{C}$ formed by the morphisms $f : X \to Y$ in $\mathscr{C}$ such that there

---

1 Most articles that deal with inverse categories reference [14]. However, we could not find a way to access this article.

2 This is not the standard inverse of the morphism in the category. However, it is worth noting that when f is an isomorphism we have $f^* = f^{-1}$.

exists a morphism $g : Y \to X$ in $\mathscr{C}$ satisfying

$$\overline{f} = g \circ f \quad \text{and} \quad \overline{g} = f \circ g.$$

Remark 3.5.7. Observe that the morphism g in Definition 3.5.6 belongs to $\mathbf{inv}(\mathscr{C})$ by symmetry.

Proposition 3.5.8. *Let $\mathscr{C}$ be a restriction category. Then $\mathbf{inv}(\mathscr{C})$ is a subcategory of $\mathscr{C}$.*

Proof. The identity morphism of each object in $\mathscr{C}$ is a morphism in $\mathbf{inv}(\mathscr{C})$ by Proposition 3.4.7. We must then verify that the composition of morphisms in $\mathbf{inv}(\mathscr{C})$ is still a morphism in $\mathbf{inv}(\mathscr{C})$.

Indeed, let $f : X \to Y$ and $f' : Y \to Z$ be morphisms in $\mathbf{inv}(\mathscr{C})$. Let $g : Y \to X$ and $g' : Z \to Y$ be morphisms in $\mathscr{C}$ (and, thus, in $\mathbf{inv}(\mathscr{C})$, by Remark 3.5.7) such that

$$\overline{f} = g \circ f, \quad \overline{g} = f \circ g, \quad \overline{f'} = g' \circ f' \quad \text{and} \quad \overline{g'} = f' \circ g'. \tag{3.43}$$

Then observe that, by (R1)–(R4) and (3.43),

$$g \circ g' \circ f' \circ f = g \circ \overline{f'} \circ f = g \circ f \circ \overline{f' \circ f} = \overline{f} \circ \overline{f' \circ f} = \overline{f' \circ f} \circ \overline{f} = \overline{f' \circ f \circ \overline{f}} = \overline{f' \circ f}$$

and

$$f' \circ f \circ g \circ g' = f' \circ \overline{g} \circ g' = f' \circ g' \circ \overline{g \circ g'} = \overline{g'} \circ \overline{g \circ g'} = \overline{g \circ g'} \circ \overline{g'} = \overline{g \circ g' \circ \overline{g'}} = \overline{g \circ g'}.$$

Thus, $f' \circ f$ is a morphism in $\mathbf{inv}(\mathscr{C})$, and, thus, $\mathbf{inv}(\mathscr{C})$ is a subcategory of $\mathscr{C}$. □

Definition 3.5.9. Let $\mathscr{C}$ be a category. The **core groupoid** of $\mathscr{C}$ is the subcategory of $\mathscr{C}$ whose objects are the objects of $\mathscr{C}$ and whose morphisms are the isomorphisms in $\mathscr{C}$.

Example 3.5.10. If $\mathscr{C}$ is a category with the trivial restriction structure, then $\mathbf{inv}(\mathscr{C})$ is the core groupoid of $\mathscr{C}$.

We will verify that $\mathbf{inv}(\mathscr{C})$ is an inverse category by using Proposition 3.5.5. To do so, we have the following lemma.

Lemma 3.5.11. *Let $\mathscr{C}$ be a restriction category and e an idempotent morphism of $\mathscr{C}$. If $e \in \mathbf{inv}(\mathscr{C})$, then $\overline{e} = e$.*

Proof. Let $e \in \mathrm{End}_{\mathscr{C}}(X)$ be an idempotent morphism of $\mathscr{C}$ in $\mathbf{inv}(\mathscr{C})$, and let $f \in \mathrm{End}_{\mathscr{C}}(X)$ be a morphism such that

$$\overline{e} = f \circ e \tag{3.44}$$

and

$$\overline{f} = e \circ f. \tag{3.45}$$

By (R1) and (R4) and the fact that e is an idempotent morphism, observe that

$$\overline{e} \circ e = e \circ \overline{\overline{e} \circ e} = e \circ \overline{e} = e. \tag{3.46}$$

And by (R1) and (3.44) and (3.45), observe that

$$\overline{f} \circ e = e \circ f \circ e = e \circ \overline{e} = e. \tag{3.47}$$

Then, by (R1) and (R2) and (3.44), (3.45) and (3.47),

$$\begin{aligned} e &= e \circ e = e \circ \overline{e} \circ \overline{f} \circ e = e \circ \overline{f} \circ \overline{e} \circ e = e \circ e \circ f \circ f \circ e \circ e = e \circ f \circ f \circ e \\ &= \overline{f} \circ \overline{e} = \overline{e} \circ \overline{f} = f \circ e \circ e \circ f = f \circ e \circ f = f \circ \overline{f} = f. \end{aligned}$$

Thus, by (3.44),

$$\overline{e} = f \circ e = e \circ e = e,$$

as desired. □

Proposition 3.5.12. *Let $\mathscr{C}$ be a restriction category. Then $\mathbf{inv}(\mathscr{C})$ is an inverse category, where the inverse of a morphism f in $\mathbf{inv}(\mathscr{C})$ is the morphism g in $\mathscr{C}$ satisfying*

$$\overline{f} = g \circ f \quad \text{and} \quad \overline{g} = f \circ g.$$

Proof. We will verify items (1) and (2) of Proposition 3.5.5.

To this end, let $f : X \to Y$ be a morphism in $\mathbf{inv}(\mathscr{C})$. Then there exists a morphism $g : Y \to X$ in $\mathscr{C}$ (and in $\mathbf{inv}(\mathscr{C})$, by Remark 3.5.7) such that

$$\overline{f} = g \circ f \quad \text{and} \quad \overline{g} = f \circ g. \tag{3.48}$$

Now, by (3.48) and (R1), g satisfies

$$f \circ g \circ f = f \circ \overline{f} = f \quad \text{and} \quad g \circ f \circ g = g \circ \overline{g} = g, \tag{3.49}$$

so Proposition 3.5.5 (1) follows.

To verify Proposition 3.5.5 (2), let $e, f \in \mathrm{Hom}_{\mathbf{inv}(\mathscr{C})}(X, X)$ be idempotent morphisms. By Lemma 3.5.11, $e = \overline{e}$ and $f = \overline{f}$. So, by (R2),

$$e \circ f = \overline{e} \circ \overline{f} = \overline{f} \circ \overline{e} = f \circ e.$$

Thus, idempotent morphisms commute in $\mathbf{inv}(\mathscr{C})$.

Therefore, by Proposition 3.5.5, $\mathbf{inv}(\mathscr{C})$ is an inverse category. □

Definition 3.5.13. Let $\mathscr{C}$ be a category with pullbacks. We define $\mathbf{iso}_{\mathscr{C}}$ as the inverse category $\mathbf{inv}(\mathbf{par}_{\mathscr{C}})$.

Proposition 3.5.14. *Let $\mathscr{C}$ be a category with pullbacks. The morphisms in $\mathbf{iso}_{\mathscr{C}}$ are precisely the isomorphism classes $[A, f, g]$ in $\mathbf{par}_{\mathscr{C}}$ such that g is a monomorphism in $\mathscr{C}$.*

The inverse of a morphism $[A, f, g]$ in $\mathbf{iso}_{\mathscr{C}}$ is the isomorphism class

$$[A, f, g]^* = [A, g, f].$$

Proof. Let $[A, f, g]$ be an isomorphism class in $\mathbf{par}_{\mathscr{C}}(X, Y)$ such that g is a monomorphism. Observe that since g is a monomorphism, the diagram

$$\begin{tikzcd}
 & A \arrow[dl, "id_A"'] \arrow[dr, "id_A"] \arrow[dd, phantom, "\vee"] & \\
A \arrow[dr, "g"'] & & A \arrow[dl, "g"] \\
 & Y &
\end{tikzcd}$$

is a pullback, so $[A, g, f] \bullet [A, f, g]$ is the isomorphism class represented by the outermost partial morphism in the following diagram.

$$\begin{tikzcd}
 & & A \arrow[dl, "id_A"'] \arrow[dr, "id_A"] \arrow[dd, phantom, "\vee"] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\
X & & Y & & X
\end{tikzcd}$$

Thus, we have

$$[A, g, f] \bullet [A, f, g] = [A, f, f] = \overline{[A, f, g]}.$$

Similarly, we have

$$[A, f, g] \bullet [A, g, f] = [A, g, g] = \overline{[A, g, f]},$$

so, by Definition 3.5.6, $[A, f, g] \in \mathbf{iso}_{\mathscr{C}}$.

Now let $[A, f, g] \in \mathbf{iso}_{\mathscr{C}}(X, Y)$. By Definition 3.5.6, there exists an isomorphism class $[B, h, k] \in \mathbf{par}\,\mathscr{C}(Y, X)$ such that

$$\overline{[A, f, g]} = [B, h, k] \bullet [A, f, g] \tag{3.50}$$

and

$$\overline{[B, h, k]} = [A, f, g] \bullet [B, h, k]. \tag{3.51}$$

By (3.50), there exists an isomorphism φ such that the following diagram, whose square is a pullback, is commutative.

$$\begin{tikzcd}
 & & A \arrow[d, dashed, "\varphi"] \arrow[dddll, bend right, "f"'] \arrow[dddrr, bend left, "f"] & & \\
 & & P \arrow[dl, "\widehat{h}"'] \arrow[dr, "\widehat{g}"] \arrow[dd, phantom, "\vee"] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & X
\end{tikzcd} \tag{3.52}$$

And by (3.51) there exists an isomorphism ψ such that the following diagram, whose square is a pullback, is commutative.

$$\begin{tikzcd}
 & & B \arrow[d, dashed, "\psi"] \arrow[dddll, bend right, "h"'] \arrow[dddrr, bend left, "h"] & & \\
 & & Q \arrow[dl, "\widehat{f}"'] \arrow[dr, "\widehat{k}"] \arrow[dd, phantom, "\vee"] & & \\
 & B \arrow[dl, "h"'] \arrow[dr, "k"] & & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
Y & & X & & Y
\end{tikzcd} \tag{3.53}$$

By the commutativity of (3.52), we have

$$f \circ \widehat{h} \circ \varphi = f,$$

so, since f is a monomorphism, $\widehat{h} \circ \varphi = id_A$. Thus, also by the commutativity of (3.52), we get

$$h \circ \widehat{g} \circ \varphi = g \circ \widehat{h} \circ \varphi = g \circ id_A = g. \tag{3.54}$$

Since (3.53) commutes, we have

$$g \circ \widehat{k} \circ \psi = h. \tag{3.55}$$

By (3.54) and (3.55), it follows that

$$h = h \circ \widehat{g} \circ \varphi \circ \widehat{k} \circ \psi,$$

so, since h is a monomorphism,

$$\widehat{g} \circ \varphi \circ \widehat{k} \circ \psi = id_B. \tag{3.56}$$

A similar analysis yields

$$\widehat{k} \circ \psi \circ \widehat{g} \circ \varphi = id_A. \tag{3.57}$$

Thus, by (3.56) and (3.57), $\widehat{g} \circ \varphi$ is an isomorphism in $\mathscr{C}$. Since φ is an isomorphism, it then follows that so is $\widehat{g}$. Therefore, by (3.54), since $\widehat{g}$ and φ are isomorphisms and h is a monomorphism, g is a monomorphism, as desired. $\square$

4 PARTIAL ACTIONS ON OBJECTS IN CATEGORIES WITH PULLBACKS

We begin the discussion of partial actions in this chapter. Section 4.1 is a review of partial actions of monoids (and groups) on sets, while Section 4.2 relates such partial actions with partial morphisms in the category of sets.

Inspired by this relationship between partial actions and partial morphisms, we introduce in Section 4.3 the partial actions of monoids on objects in categories with pullbacks, as well as the corresponding strong partial actions and global actions. There we show that many notions of partial action seen in the literature are covered by this concept. In Section 4.4 we give the appropriate definition of a morphism between these concepts, along with the corresponding categories that come with it.

Finally, in Section 4.5 we study the case in which the monoid is a group, showing equivalent descriptions of a strong partial action in this situation.

For the remainder of this chapter, if not stated otherwise, M will be a monoid with multiplication $\mu : M \times M \to M$ and identity e.

4.1 PARTIAL MONOID ACTIONS ON SETS

Throughout this and the following section, X and Y will be sets, if not otherwise stated.

Recall that a **transformation** of X is a map from X to X, and the set $\mathcal{T}_X$ of all transformations of X is a monoid under the composition of maps.

Definition 4.1.1. An **action** of M on X is a monoid homomorphism from M to $\mathcal{T}_X$.

Whenever α is an action of M on X, we will usually denote the map $\alpha(m) : X \to X$ by simply α_m.

Taking an approach similar to that of [12], we will define partial actions of a monoid in terms of partial action data.

Definition 4.1.2. A *partial action datum* of M on X is a family of maps $\{\alpha_m : \operatorname{dom}\alpha_m \to X\}_{m\in M}$ where $\operatorname{dom}\alpha_m \subseteq X$, for all $m \in M$.

Before defining a partial action, observe that the actions of a monoid can be seen as partial action data in the following way.

Proposition 4.1.3. *There is a one-to-one correspondence between the actions of M on X and the partial action data $\{\alpha_m\}_{m\in M}$ of M on X satisfying*

(GA1) $\alpha_e = id_X$,

(GA2) $\operatorname{dom}\alpha_m = X$ *for all* $m \in M$ *and*

$$\alpha_m \circ \alpha_n = \alpha_{nm}, \tag{4.1}$$

for all $m, n \in M$.

Proof. To each action $\alpha : M \to \mathcal{T}_X$ we associate the partial action datum $\{\alpha(m)\}_{m\in M}$. Observe that $\{\alpha(m)\}_{m\in M}$ satisfies (GA1) because α preserves the identity and satisfies (GA2) because it is a map with values in $\mathcal{T}_X$ and it preserves the product of M.

And to each partial action datum $\{\alpha_m\}_{m\in M}$ satisfying (GA1) and (GA2) we associate the map $\alpha : M \to \mathcal{T}_X$ where $\alpha(m) = \alpha_m$ for each $m \in M$.

Observe that each α_m is indeed a transformation in $\mathcal{T}_X$ because $\operatorname{dom}\alpha_m = X$, by (GA2). Then (GA1) states that α preserves the identity and (GA2) that α preserves the composition. Hence, α is a monoid homomorphism, and, thus, an action.

It is then a straightforward verification that the two associations are inverse to one another, and so we have a one-to-one correspondence, as desired. □

We will then interchange the definition of a monoid action with that of a partial action datum satisfying (GA1) and (GA2).

A *partial* action of a monoid is, in a way, a generalization of the concept of an action of a monoid, where the elements of M do not have to interact with every element of X.

Definition 4.1.4. A *partial action* of M on X is a partial action datum $\{\alpha_m\}_{m\in M}$ of M on X, such that:

(PA1) $\operatorname{dom}\alpha_e = X$ and $\alpha_e = id_X$;

(PA2) $\alpha_m^{-1}(\operatorname{dom}\alpha_n) \subseteq \operatorname{dom}\alpha_{nm}$, for all $m, n \in M$;

(PA3) $\alpha_n \circ \alpha_m = \alpha_{nm}$ on $\alpha_m^{-1}(\operatorname{dom}\alpha_n)$, for all $m, n \in M$.

Observe that axiom (PA3) makes sense because of axioms (PA2).

Remark 4.1.5. Axiom (PA1) is equivalent to [11, Definition 2.2 (PA1)] and axioms (PA2) and (PA3) together are equivalent to [11, Definition 2.2 (PA2')].

To distinguish an action of a monoid from a partial action, we may also call the former a *global* action.

Definition 4.1.6. A partial action $\{\alpha_m\}_{m\in M}$ of M on a set X is said to be *strong* if instead of (PA2) we have the following stronger condition:

(PA2') $\alpha_m^{-1}(\operatorname{dom}\alpha_n) = \operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m$, for all $m, n \in M$.

Remark 4.1.7. The concept of a strong partial monoid action from Definition 4.1.6 is equivalent to that of [11, Definition 2.4] and [17, Definition 2.3] (although in the latter definition it is called just a partial action).

Example 4.1.8. Let $M = (\mathbb{N}, +)$ and $X = \mathbb{N}$. Then the partial action datum $\{\alpha_n\}_{n\in\mathbb{N}}$ is a partial action of M on X, where for each $n \in \mathbb{N}$,

$$\operatorname{dom}\alpha_n = \{z \in \mathbb{N} : z \leq n\} \quad \text{and } \alpha_n(z) = z + n,$$

for all $z \in \operatorname{dom}\alpha_n$. Moreover, $\{\alpha_n\}_{n\in\mathbb{N}}$ is not a strong partial action.

Example 4.1.9. Let $M = (\mathbb{N}, +)$ and $X = \mathbb{Z}^-$. Then the partial action datum $\{\alpha_n\}_{n\in\mathbb{N}}$ is a strong partial action of M on X, where for each $n \in \mathbb{N}$,

$$\operatorname{dom}\alpha_n = \{z \in \mathbb{Z}^- : z + n \leq 0\} \quad \text{and } \alpha_n(z) = z + n,$$

for all $z \in \operatorname{dom}\alpha_n$.

Proposition 4.1.10. *Let α be a global action of M on X. Then the partial action datum $\{\alpha_m\}_{m\in M}$ of M on X is a strong partial action.*

Proof. Since α is a monoid homomorphism, it preserves the identity of M, and so $\{\alpha_m\}_{m\in M}$ satisfies (PA1).

Because $\alpha_m = \alpha(m)$ is a transformation of X, $\operatorname{dom}\alpha_m = X$ for all $m \in M$, so (PA2') follows trivially.

Finally, $\{\alpha_m\}_{m\in M}$ satisfies (PA3) because α preserves the operation of M. □

We may, in fact, construct many strong partial actions from global actions, by restricting them to subsets.

Definition 4.1.11. Let β be a global action of M on a set Y and $X \subseteq Y$. The **restriction** of β to X is the partial action datum α of M on X, where

$$\alpha = \{\alpha_m : \operatorname{dom}\alpha_m = X \cap \beta_m^{-1}(X) \to X\}_{m\in M}, \quad \alpha_m(x) = \beta_m(x), \forall x \in \operatorname{dom}\alpha_m.$$

Proposition 4.1.12. *Let β be a global action of M on a set Y and $X \subseteq Y$. The restriction α of β to X is a strong partial action.*

Proof. Since β is a global action, $\beta_e = id_Y$, so

$$\operatorname{dom}\alpha_e = X \cap \beta_e^{-1}(X) = X \cap X = X$$

and $\alpha_e(x) = \beta_e(x) = x$ for all $x \in X$, so α satisfies (PA1).

To verify (PA2'), let $m, n \in M$. Let $x \in \alpha_m^{-1}(\operatorname{dom} \alpha_n)$. Clearly, $x \in \operatorname{dom} \alpha_m$. Furthermore, we have

$$\beta_{nm}(x) = \beta_n \circ \beta_m(x) = \beta_n(\beta_m(x)) = \beta_n(\alpha_m(x)),$$

so, since $\alpha_m(x) \in \operatorname{dom} \alpha_n$, $\beta_n(\alpha_m(x)) = \alpha_n(\alpha_m(x))$. In particular, it follows that $\beta_{nm}(x) \in X$, so $x \in \beta_{nm}^{-1}(X)$. Thus, $x \in X \cap \beta_{nm}^{-1}(X) = \operatorname{dom} \alpha_{nm}$. Therefore, $\alpha_m^{-1}(\operatorname{dom} \alpha_n) \subseteq \operatorname{dom} \alpha_{nm} \cap \operatorname{dom} \alpha_m$.

Now let $x \in \operatorname{dom} \alpha_{nm} \cap \operatorname{dom} \alpha_m$. Observe that

$$\beta_n(\alpha_m(x)) = \beta_n(\beta_m(x)) = \beta_n \circ \beta_m(x) = \beta_{nm}(x) \in X,$$

since $x \in \operatorname{dom} \alpha_{nm}$. Thus, $\alpha_m(x) \in X \cap \beta_n^{-1}(X) = \operatorname{dom} \alpha_n$. Therefore, it follows that $x \in \alpha_m^{-1}(\operatorname{dom} \alpha_n)$, so $\operatorname{dom} \alpha_{nm} \cap \operatorname{dom} \alpha_m \subseteq \alpha_m^{-1}(\operatorname{dom} \alpha_n)$.

Hence, $\alpha_m^{-1}(\operatorname{dom} \alpha_n) = \operatorname{dom} \alpha_{nm} \cap \operatorname{dom} \alpha_m$, and, thus, α satisfies (PA2').

Finally, let $x \in \alpha_m^{-1}(\operatorname{dom} \alpha_n)$. Then

$$\alpha_n(\alpha_m(x)) = \beta_n(\beta_m(x)) = \beta_n \circ \beta_m(x) = \beta_{nm}(x) = \alpha_{nm}(x),$$

so α satisfies (PA3), as desired. $\square$

Example 4.1.9 is an example of a strong partial action that comes from the restriction of a global action. Indeed, let β be the global action of $\mathbb{N}$ on $\mathbb{Z}$ given by $\beta_n(z) = z + n$ for all $n \in \mathbb{N}$ and $z \in \mathbb{Z}$. Then the partial action α in Example 4.1.9 is the restriction of β to $\mathbb{Z}^-$.

Megrelishvili and Schröder [17], and, later on, Hollings [11], proved a converse of Proposition 4.1.12. That is, every strong partial action of a monoid on a set can be obtained as a restriction of some global action. In fact, they showed, even more, that, in a way, each strong partial action has a minimal global action that restricts to it.

In Proposition 6.1.3 we will also prove this fact, by using the machinery we develop in Chapter 5.

The natural concept of a morphism between partial action data is the following.

Definition 4.1.13. Let M be a monoid and X and Y sets. Let $\alpha = \{\alpha_m\}_{m \in M}$ and $\beta = \{\beta_m\}_{m \in M}$ be partial action data of M on X and Y, respectively. A *datum morphism* from α to β is a map $f : X \to Y$ such that

(DM1) $f(\operatorname{dom} \alpha_m) \subseteq \operatorname{dom} \beta_m$, for all $m \in M$;

(DM2) $\beta_m \circ f = f \circ \alpha_m$ on $\operatorname{dom} \alpha_m$ for all $m \in M$.

4.1.1 PARTIAL GROUP ACTIONS ON SETS

Throughout this section, let G be a group with identity e. A partial action of a *group* on a set is defined in [12] in terms of partial action data as follows.

Definition 4.1.14. A *partial action* of G on X is a partial action datum $\{\alpha_g\}_{g\in G}$ of G on X, such that:

(PGA1) $\operatorname{dom}\alpha_e = X$ and $\alpha_e = id_X$;

(PGA2) $\alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}) \subseteq \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}}$, for all $g, h \in G$;

(PGA3) $\alpha_h \circ \alpha_g = \alpha_{hg}$ on $\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{hg}$, for all $g, h \in G$.

Observe that axiom (PGA3) makes sense because of (PGA2). Indeed, by (PGA2), for each $g, h \in G$ we have

$$\begin{aligned}\alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{hg}) &= \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{(g^{-1}h^{-1})^{-1}}) \subseteq \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gg^{-1}h^{-1})^{-1}}\\ &= \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_h \subseteq \operatorname{dom}\alpha_h.\end{aligned}$$

Remark 4.1.15. Definition 4.1.14 is equivalent to the classical definition of a partial action of a group on a set, such as the definition found in [7, Definition 1.1].

Proposition 4.1.16. *Let $\{\alpha_g\}_{g\in G}$ be a partial action of G on X. Then*

$$\alpha_g(\operatorname{dom}\alpha_g) \subseteq \operatorname{dom}\alpha_{g^{-1}} \tag{4.2}$$

and $\alpha_{g^{-1}} \circ \alpha_g = id_X$ on $\operatorname{dom}\alpha_g$ for all $g \in G$.

Proof. Let $g \in G$. By taking $h = e$ on (PGA2) we obtain

$$\alpha_g(\operatorname{dom}\alpha_g) = \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{e^{-1}}) \subseteq \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(ge)^{-1}} = \operatorname{dom}\alpha_{g^{-1}},$$

so (4.2) follows.

Then the fact that $\alpha_{g^{-1}} \circ \alpha_g = id_X$ on $\operatorname{dom}\alpha_g$ follows by (PGA1) and (PGA3). □

Corollary 4.1.17. *Let $\{\alpha_g\}_{g\in G}$ be a partial action of G on X. Then α_g is an injective map for all $g \in G$.*

Proof. Let $g \in G$ and $x, y \in \operatorname{dom}\alpha_g$ such that $\alpha_g(x) = \alpha_g(y)$. Then by Proposition 4.1.16

$$x = id_X(x) = \alpha_{g^{-1}} \circ \alpha_g(x) = \alpha_{g^{-1}} \circ \alpha_g(y) = id_Y(y) = y,$$

so the injectivity of α_g follows. □

The following results will relate a partial action of a group, in the sense of Definition 4.1.14, with a (strong) partial action of the group seen as a monoid, in the sense of Definitions 4.1.4 and 4.1.6.

Lemma 4.1.18. *Let $\{\alpha_g\}_{g\in G}$ be a partial action of G on X. Then we have*

$$\alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}) = \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}},$$

for all $g, h \in G$.

Proof. Let $g, h \in G$. The inclusion $\alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}) \subseteq \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}}$ follows from (PGA2), so it suffices to verify that

$$\operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}} \subseteq \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}).$$

Now, by (PGA2),

$$\alpha_{g^{-1}}(\operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}}) \subseteq \operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{(g^{-1}gh)^{-1}} = \operatorname{dom}\alpha_g \circ \operatorname{dom}\alpha_{h^{-1}}.$$

Therefore, by Proposition 4.1.16,

$$\begin{aligned}\operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}} &= id_X(\operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}}) = \alpha_g \circ \alpha_{g^{-1}}(\operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gh)^{-1}})\\ &\subseteq \alpha_g(\operatorname{dom}\alpha_g \circ \operatorname{dom}\alpha_{h^{-1}}),\end{aligned}$$

as desired. □

Proposition 4.1.19. *Let $\{\alpha_g\}_{g\in G}$ be a partial action datum of G on X. Then the following are equivalent.*

(1) $\{\alpha_g\}_{g\in G}$ is a partial action of G on X;

(2) $\{\alpha_g\}_{g\in G}$ is a partial action of G seen as a monoid on X and

$$\alpha_g(\operatorname{dom}\alpha_g) \subseteq \operatorname{dom}\alpha_{g^{-1}} \tag{4.3}$$

for all $g \in G$;

(3) $\{\alpha_g\}_{g\in G}$ is a strong partial action of G seen as a monoid on X.

Proof. Let $\alpha = \{\alpha_g\}_{g\in G}$.

(1) $\Rightarrow$ (2). Suppose α is a partial action of G on X. Then (4.3) follows by Proposition 4.1.16. Let us verify that α satisfies (PA1)–(PA3).

Clearly, (PGA1) implies (PA1).

Now, to verify (PA2), let $g, h \in G$. Then by Lemma 4.1.18 and (4.3) we have

$$\begin{aligned}\alpha_g(\alpha_g^{-1}(\operatorname{dom}\alpha_h)) &\subseteq \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_h = \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(h^{-1})^{-1}} \\ &= \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(gg^{-1}h^{-1})^{-1}} = \operatorname{dom}\alpha_{g^{-1}} \cap \operatorname{dom}\alpha_{(g(hg)^{-1})^{-1}} \\ &= \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{((hg)^{-1})^{-1}}) = \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{hg}).\end{aligned}$$

By Corollary 4.1.17, α_g is an injective map. Therefore we have

$$\alpha_g^{-1}(\operatorname{dom}\alpha_h) \subseteq \operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{hg} \subseteq \operatorname{dom}\alpha_{hg}.$$

Hence, α satisfies (PA2).

Axiom (PA3) then follows by (PGA3) because of (PA2).

(2) $\Rightarrow$ (3). Suppose α is a partial action of the monoid G on X. Since α satisfies (PA1) and (PA3), all that remains is to verify (PA2').

Let $g, h \in G$. Clearly, $\alpha_g^{-1}(\operatorname{dom}\alpha_h) \subseteq \operatorname{dom}\alpha_g$, so

$$\alpha_g^{-1}(\operatorname{dom}\alpha_h) \subseteq \operatorname{dom}\alpha_{hg} \cap \operatorname{dom}\alpha_g$$

by (PA2).

On the other hand, let $x \in \operatorname{dom}\alpha_{hg} \cap \operatorname{dom}\alpha_g$. By (4.3), since $x \in \operatorname{dom}\alpha_g$, $x \in \alpha_g^{-1}(\operatorname{dom}\alpha_{g^{-1}})$. Therefore, by (PA1) and (PA3) we have

$$\alpha_{g^{-1}}(\alpha_g(x)) = \alpha_e(x) = x. \tag{4.4}$$

Since $x \in \operatorname{dom}\alpha_{hg}$, by (4.4) and (PA2) we have

$$\alpha_g(x) \in \alpha_{g^{-1}}^{-1}(\operatorname{dom}\alpha_{hg}) \subseteq \operatorname{dom}\alpha_{hgg^{-1}} = \operatorname{dom}\alpha_h.$$

Thus, $x \in \alpha_g^{-1}(\operatorname{dom}\alpha_h)$. Hence,

$$\operatorname{dom}\alpha_{hg} \cap \operatorname{dom}\alpha_{hg} \cap \operatorname{dom}\alpha_g \subseteq \alpha_g^{-1}(\operatorname{dom}\alpha_h).$$

Therefore, (PA2') follows, and α is a strong partial action, as desired.

(3) $\Rightarrow$ (1). Suppose α is a strong partial action of the monoid G on X. Then (PGA1) follows by (PA1).

For (PGA2), let $g, h \in G$. Then let $x \in \operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}$. By (PA2'),

$$x \in \operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}} = \operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}g^{-1}g} = \operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{(gh)^{-1}g} = \alpha_g^{-1}(\operatorname{dom}\alpha_{(gh)^{-1}}).$$

Thus, $\alpha_g(x) \in \operatorname{dom}\alpha_{(gh)^{-1}}$, so

$$\alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}) \subseteq \operatorname{dom}\alpha_{(gh)^{-1}}. \tag{4.5}$$

Also, by (PA2') and (PA1) we have

$$\begin{aligned}\alpha_g(\operatorname{dom}\alpha_g) &= \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_e) = \alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{g^{-1}g})\\ &= \alpha_g(\alpha_g^{-1}(\operatorname{dom}\alpha_{g^{-1}})) \subseteq \operatorname{dom}\alpha_{g^{-1}},\end{aligned}$$

so

$$\alpha_g(\operatorname{dom}\alpha_g \cap \operatorname{dom}\alpha_{h^{-1}}) \subseteq \alpha_g(\operatorname{dom}\alpha_g) \subseteq \operatorname{dom}\alpha_{g^{-1}}. \tag{4.6}$$

Therefore, (PGA2) follows by (4.5) and (4.6).

Then (PGA3) follows by (PA3), because of (PA2'). Hence, α is a partial action in the sense of Definition 4.1.14, as desired. □

4.2 PARTIAL MORPHISMS AND PARTIAL ACTION DATA

Proposition 4.2.1. *There exists a bijection between*

(1) the set of partial action data of M on X;

(2) $\mathbf{par}_{\mathbf{Set}}(M \times X, X)$;

(3) the set of maps from M to $\mathbf{par}_{\mathbf{Set}}(X, X)$.

Proof. (1) $\leftrightarrow$ (2). Given $\{\alpha_m\}_{m\in M}$ a partial action datum of M on X, let

$$M \bullet X := \{(m, x) \in M \times X : x \in \operatorname{dom}\alpha_m\}.$$

Let $\alpha : M \bullet X \to X$ be given by $\alpha(m, x) = \alpha_m(x)$ and ι be the inclusion of $M \bullet X$ into $M \times X$.

We associate to $\{\alpha_m\}_{m\in M}$ the isomorphism class $[M \bullet X, \iota, \alpha] \in \mathbf{par}_{\mathbf{Set}}(M \times X, X)$, whose representative is illustrated as follows.

$$\begin{tikzcd} & M \bullet X \arrow[dl, "\iota"'] \arrow[dr, "\alpha"] & \\ M \times X \arrow[rr] & & X \end{tikzcd}$$

Given $[A, f, g] \in \mathbf{par}_{\mathbf{Set}}(M \times X, X)$, let $(M \bullet X, \iota, \alpha)$ be the unique representative of $[A, f, g]$ where $M \bullet X \subseteq M \times X$ and ι is the corresponding inclusion map, given by Proposition 3.1.16.

For each $m \in M$, let

$$\operatorname{dom}\alpha_m \coloneqq \{x \in X : (m, x) \in M \bullet X\}$$

and $\alpha_m : \operatorname{dom}\alpha_m \to X$ be given by $\alpha_m(x) = \alpha(m, x)$. We associate to $[A, f, g]$ the partial action datum $\{\alpha_m\}_{m\in M}$ of M on X.

It is a straightforward verification that the two associations are inverse to one another, and, thus, induce bijections between (1) and (2).

(1) $\leftrightarrow$ (3). Given a partial action datum $\{\alpha_m\}_{m\in M}$ of M on X, let ι_m be the inclusion of $\operatorname{dom}\alpha_m$ into X.

We associate to $\{\alpha_m\}_{m\in M}$ the map $\alpha : M \to \mathbf{par}_{\mathbf{Set}}(X, X)$ given by $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$. The representative for $\alpha(m)$ is illustrated as follows.

$$\begin{tikzcd} & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] & \\ X \arrow[rr] & & X \end{tikzcd}$$

Given $\alpha : M \to \mathbf{par}_{\mathbf{Set}}(X, X)$, for each $m \in M$ let $(\operatorname{dom}\alpha_m, \iota_m, \alpha_m)$ be a representative of the isomorphism class $\alpha(m)$ where $\operatorname{dom}\alpha_m \subseteq X$ and ι_m is the corresponding inclusion map, given by Proposition 3.1.16.

We associate to α the partial action datum $\{\alpha_m\}_{m\in M}$ of M on X.

It is also a simple verification that these associations are inverse to each other and induce a bijection between (1) and (3). $\square$

Proposition 4.2.2. *Let $\{\alpha_m\}_{m\in M}$ be a partial action datum of M on X. The following are equivalent.*

(1) $\{\alpha_m\}_{m\in M}$ satisfies (PA1);

(2) The associated isomorphism class $[M \bullet X, \iota, \alpha]$ from Proposition 4.2.1 is such that

$$[X, \eta, id_X] \leq [M \bullet X, \iota, \alpha],$$

where $\eta : X \to M \times X$ is given by

$$\eta(x) = (e, x);$$

(3) The associated map $\alpha : M \to \mathbf{par}_{\mathbf{Set}}(X, X)$ from Proposition 4.2.1 is such that

$$\alpha(e) = [X, id_X, id_X].$$

Proof. (1) $\Rightarrow$ (2). Assume $\{\alpha_m\}_{m\in M}$ satisfies (PA1). Then $\operatorname{dom}\alpha_e = X$ and $\alpha_e = id_X$. Then the map $\varphi : X \to M \bullet X$ given by $\varphi(x) = (e, x)$ is well defined and the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\eta"'] \arrow[d, "\varphi"] \arrow[dr, "id_X"] & \\
M \times X & M \bullet X \arrow[l, "\iota"] \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

commutes, because

$$\iota(\varphi(x)) = \iota(e,x) = (e,x) = \eta(x)$$

and

$$\alpha(\varphi(x)) = \alpha(e,x) = \alpha_e(x) = x = id_X(x),$$

for all $x \in X$.

Thus, φ is a span morphism from (X, η, id_X) to $(M \bullet X, \iota, \alpha)$, so (2) follows by Proposition 3.4.13.

(2) $\Rightarrow$ (1). Assume that

$$[X, \eta, id_X] \leq [M \bullet X, \iota, \alpha].$$

By Proposition 3.4.13, there exists a span morphism φ from (X, η, id_X) to $(M \bullet X, \iota, \alpha)$. That is, a morphism from X to $M \bullet X$ such that the following diagram commutes.

$$\begin{tikzcd}
 & X \arrow[dl, "\eta"'] \arrow[d, "\varphi"] \arrow[dr, "id_X"] & \\
M \times X & M \bullet X \arrow[l, "\iota"] \arrow[r, "\alpha"'] & X
\end{tikzcd} \tag{4.7}$$

By the commutativity of the left triangle of (4.7), for all $x \in X$ we have

$$\varphi(x) = \iota(\varphi(x)) = \eta(x) = (e,x). \tag{4.8}$$

In particular, it follows that $(e,x) \in M \bullet X = \{(m,x) \in M \times X : x \in \operatorname{dom} \alpha_m\}$ for all $x \in X$, so $\operatorname{dom} \alpha_e = X$.

By (4.8) and by the commutativity of the right triangle of (4.7) it follows that

$$\alpha_e(x) = \alpha(e,x) = \alpha(\varphi(x)) = id_X(x)$$

for all $x \in X$, so $\alpha_e = id_X$.

Thus, (1) follows.

(1) $\Rightarrow$ (3). Suppose $\{\alpha_m\}_{m \in M}$ satisfies (PA1), so $\operatorname{dom} \alpha_e = X$ and $\alpha_e = id_X$.

Since the associated map $\alpha : M \to \mathbf{par}_{\mathbf{Set}}(X,X)$ is such that, for all $g \in G$,

$\alpha(g) = [\operatorname{dom}\alpha_g, \iota_g, \alpha_g]$, where ι_g is the corresponding inclusion map of $\operatorname{dom}\alpha_g$ into X, it follows that

$$\alpha(e) = [\operatorname{dom}\alpha_e, \iota_e, \alpha_e] = [X, id_X, id_X],$$

and so we have (3).

(3) $\Rightarrow$ (1). Assume (3). Since by definition of α we have

$$\alpha(e) = [\operatorname{dom}\alpha_e, \iota_e, \alpha_e]$$

and by hypothesis we have

$$\alpha(e) = [X, id_X, id_X],$$

it follows that

$$(\operatorname{dom}\alpha_e, \iota_e, \alpha_e) \cong (X, id_X, id_X).$$

Thus, there exists an isomorphism φ from $(\operatorname{dom}\alpha_e, \iota_e, \alpha_e)$ to (X, id_X, id_X). Since φ is a span morphism, the following diagram commutes.

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_e \arrow[dl, "\iota_e"'] \arrow[d, "\varphi"] \arrow[dr, "\alpha_e"] & \\
 & X \arrow[dl, "id_X"] \arrow[dr, "id_X"'] & \\
X & & X
\end{tikzcd}$$

In particular, we have

$$\iota_e = \varphi = \alpha_e. \tag{4.9}$$

By (4.9), since φ is an isomorphism, so is ι_e. Thus, since ι_e is an inclusion of a subset of X on X, it follows that $\operatorname{dom}\alpha_e = X$ and $\iota_e = id_X$. It then also follows by (4.9) that $\alpha_e = id_X$. Therefore, we have (1). $\square$

Remark 4.2.3. Let $\{\alpha_m\}_{m\in M}$ a partial action datum of M on X and $[M \bullet X, \iota, \alpha]$ its associated isomorphism class from Proposition 4.2.1. In the following propositions, we will at times denote $[M \bullet X, \iota, \alpha]$ simply by α (recall Remark 3.1.18), and by $id_M \times \alpha$ we mean the isomorphism class represented by the partial morphism

$$\begin{tikzcd}
 & M \times (M \bullet X) \arrow[dl, "id_M \times \iota"'] \arrow[dr, "id_M \times \alpha"] & \\
M \times M \times X & & M \times X,
\end{tikzcd}$$

and by $\mu \times id_X$ we mean the isomorphism class represented by the partial morphism

$$\begin{tikzcd} & M \times M \times X \arrow[dl, "id_{M\times M\times X}"'] \arrow[dr, "\mu\times id_X"] & \\ M \times M \times X & & M \times X. \end{tikzcd}$$

Proposition 4.2.4. *Let $\{\alpha_m\}_{m\in M}$ be a partial action datum of M on X. The following are equivalent.*

(1) $\{\alpha_m\}_{m\in M}$ satisfies (GA2);

(2) The associated isomorphism class $[M \bullet X, \iota, \alpha]$ from Proposition 4.2.1 is a global morphism such that

$$\alpha \bullet (id_M \times \alpha) = \alpha \bullet (\mu \times id_X) \tag{4.10}$$

in $\mathbf{par_{Set}}(M \times M \times X, X)$;

(3) The associated map $\alpha : M \to \mathbf{par_{Set}}(X, X)$ from Proposition 4.2.1 is such that $\alpha(m)$ is a global morphism for all $m \in M$ and

$$\alpha(n) \bullet \alpha(m) = \alpha(nm), \tag{4.11}$$

for all $m, n \in M$.

Proof. (1) $\Leftrightarrow$ (2). First we will verify that $\operatorname{dom}\alpha_m = X$ for all $m \in M$ if and only if $[M \bullet X, \iota, \alpha]$ is a global morphism.

Indeed, if $\operatorname{dom}\alpha_m = X$ for all $m \in M$, we have

$$M \bullet X = \{(m, x) \in M \times X : x \in \operatorname{dom}\alpha_m\} = \{(m, x) \in M \times X : x \in X\} = M \times X$$

and $\iota = id_{M\times X}$, so

$$[M \bullet X, \iota, \alpha] = [M \times X, id_{M\times X}, \alpha]$$

is a global morphism.

And if $[M \bullet X, \iota, \alpha]$ is a global morphism, by Proposition 3.2.17 it follows that $M \bullet X = M \times X$ and $\iota = id_{M\times X}$. Thus, given $m \in M$ and $x \in X$ we have $(m, x) \in M \bullet X$, so $x \in \operatorname{dom}\alpha_m$, so it follows that $\operatorname{dom}\alpha_m = X$.

In this case, since $M \bullet X = M \times X$, we have $M \times (M \bullet X) = M \times M \times X$. Thus, $\alpha \bullet (id_M \times \alpha)$ and $\alpha \bullet (\mu \times X)$ are $\alpha \circ (id_M \times \alpha)$ and $\alpha \circ (\mu \times id_X)$, seen as global morphisms. So (4.10) is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc} M \times M \times X & \xrightarrow{id_M\times\alpha} & M \times X \\ {\scriptstyle \mu\times id_X}\downarrow & & \downarrow{\scriptstyle \alpha} \\ M \times X & \xrightarrow{\quad\alpha\quad} & X \end{array} \tag{4.12}$$

Now, for all $m, n \in M$ and $x \in X$ we have

$$\alpha \circ (id_M \times \alpha)(n, m, x) = \alpha(n, \alpha_m(x)) = \alpha_n(\alpha_m(x)) \tag{4.13}$$

and

$$\alpha \circ (\mu \times id_X)(n, m, x) = \alpha(nm, x) = \alpha_{nm}(x), \tag{4.14}$$

so (4.10) holds if and only if $\alpha_n \circ \alpha_m = \alpha_{nm}$ for all $m, n \in M$, and we have what was desired.

(1) $\Leftrightarrow$ (3). Let $m \in M$. Since ι_m is the inclusion of $\operatorname{dom}\alpha_m$ into X, by Proposition 3.2.17 we have that $\operatorname{dom}\alpha_m = X$ if and only if $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ is a global morphism. In this case, $\alpha(m)$ can be identified with α_m.

Thus, for all $m, n \in M$, $\alpha(n) \bullet \alpha(m)$ is just $\alpha_n \circ \alpha_m$. Hence, (4.11) is equivalent to (4.1), and we have what was desired. $\square$

Proposition 4.2.5. *Let $\{\alpha_m\}_{m \in M}$ be a partial action datum of M on X. The following are equivalent.*

(1) $\{\alpha_m\}_{m \in M}$ satisfies (PA2) and (PA3);

(2) The associated isomorphism class $[M \bullet X, \iota, \alpha]$ from Proposition 4.2.1 is such that

$$\alpha \bullet (id_M \times \alpha) \leq \alpha \bullet (\mu \times id_X)$$

in $\mathbf{par}_{\mathbf{Set}}(M \times M \times X, X)$;

(3) The associated map $\alpha : M \to \mathbf{par}_{\mathbf{Set}}(X, X)$ from Proposition 4.2.1 is such that

$$\alpha(n) \bullet \alpha(m) \leq \alpha(nm),$$

for all $m, n \in M$.

Proof. (1) $\Leftrightarrow$ (2). Denote $(id_M \times \alpha)^{-1}(M \bullet X)$ by $M \bullet (M \bullet X)$, i.e.,

$$M \bullet (M \bullet X) \coloneqq \{(n, m, x) \in M \times M \times X : x \in \alpha_m^{-1}(\operatorname{dom}\alpha_n)\}. \tag{4.15}$$

Then the square in the diagram

$$\begin{tikzcd}
& & M \bullet (M \bullet X) \arrow[dl, hook', "\iota_{M\bullet(M\bullet X)}"'] \arrow[dr, "id_M \bullet \alpha"] \arrow[d, phantom, "\vee"] & & \\
& M \times (M \bullet X) \arrow[dl, "id_M \times \iota"'] \arrow[dr, "id_M \times \alpha"] & {} & M \bullet X \arrow[dl, hook', "\iota"'] \arrow[dr, "\alpha"] & \\
M \times M \times X & & M \times X & & X
\end{tikzcd} \tag{4.16}$$

is a pullback, where $\iota_{M\bullet(M\bullet X)}$ is the inclusion of $M\bullet(M\bullet X)$ into $M\times(M\bullet X)$ and $id_M\bullet\alpha$ is given by

$$(id_M\bullet\alpha)(n,m,x)=(n,\alpha_m(x)). \tag{4.17}$$

Observe that if $(n,m,x)\in M\bullet(M\bullet X)$ then $x\in\alpha_m^{-1}(\operatorname{dom}\alpha_n)$, so $\alpha_m(x)\in\operatorname{dom}\alpha_n$ and $(n,\alpha_m(x))$ is an element of $M\bullet X$.

Thus, we have

$$\alpha\bullet(id_M\times\alpha)=[M\bullet(M\bullet X),(id_M\times\iota)\circ\iota_{M\bullet(M\bullet X)},\alpha\circ(id_M\bullet\alpha)]. \tag{4.18}$$

Now denote $(\mu\times id_X)^{-1}(M\bullet X)$ by $(M\times M)\bullet X$, i.e.,

$$(M\times M)\bullet X\coloneqq\{(n,m,x)\in M\times M\times X : x\in\operatorname{dom}\alpha_{nm}\}. \tag{4.19}$$

Then the square in the diagram

$$\begin{tikzcd}
& & (M\times M)\bullet X \arrow[dl, "\iota_{(M\times M)\bullet X}"'] \arrow[dr, "\mu\bullet id_X"] \arrow[dd, phantom, "\vee" very near start] & & \\
& M\times M\times X \arrow[dl, "id_{M\times M\times X}"'] \arrow[dr, "\mu\times id_X"] & & M\bullet X \arrow[dl, "\iota"'] \arrow[dr, "\alpha"] & \\
M\times M\times X & & M\times X & & X
\end{tikzcd} \tag{4.20}$$

is a pullback, where $\iota_{(M\times M)\bullet X}$ is the inclusion of $(M\times M)\bullet X$ into $M\times M\times X$ and $\mu\bullet id_X$ is given by

$$(\mu\bullet id_X)(n,m,x)=(nm,x). \tag{4.21}$$

Observe that if $(n,m,x)\in(M\times M)\bullet X$, then $x\in\operatorname{dom}\alpha_{nm}$, so $(nm,x)\in M\bullet X$.

Thus,

$$\alpha\bullet(\mu\times id_X)=[(M\times M)\bullet X,\iota_{(M\times M)\bullet X},\alpha\circ(\mu\bullet id_X)]. \tag{4.22}$$

Notice that $\{\alpha_m\}_{m\in M}$ satisfies (PA2) if and only if $M\bullet(M\bullet X)\subseteq(M\times M)\bullet X$ (recall (4.15) and (4.19)). In this case, $\{\alpha_m\}_{m\in M}$ satisfies (PA3) if and only if

$$\alpha\circ(id_M\bullet\alpha)=\alpha\circ(\mu\bullet id_X)\text{ on }M\bullet(M\bullet X). \tag{4.23}$$

Indeed, observe that for all $(n,m,x)\in M\bullet(M\bullet X)$, by (4.17),

$$\alpha\circ(id_M\bullet\alpha)(n,m,x)=\alpha(n,\alpha_m(x))=\alpha_n(\alpha_m(x)),$$

and, by (4.21),

$$\alpha\circ(\mu\bullet id_X)=\alpha(nm,x)=\alpha_{nm}(x),$$

so (4.23) if and only if

$$\alpha_n \circ \alpha_m(x) = \alpha_{nm}(x)$$

for all $n, m \in M$ and $x \in \alpha_m^{-1}(\operatorname{dom} \alpha_n)$, which is precisely (PA3).

Conversely, observe that by (4.18) and (4.22) and Proposition 3.4.13 we have that

$$\alpha \bullet (id_M \times \alpha) \leq \alpha \bullet (\mu \times id_X).$$

if and only if there is a span morphism from the outermost span of (4.16) to the outermost span of (4.20).

Thus, to check the equivalence (1) $\Leftrightarrow$ (2) it suffices to show that there is a span morphism from the outermost span of (4.16) to the outermost span of (4.20) if and only if we have $M \bullet (M \bullet X) \subseteq M \times (M \bullet X)$ and (4.23).

Assume that there is a span morphism φ from the outermost span of (4.16) to the outermost span of (4.20). That is, φ is a morphism such that the following commutes.

$$\begin{tikzcd}
 & M \bullet (M \bullet X) \arrow[ddl, "(id_M \times \iota)\circ \iota_{M\bullet(M\bullet X)}"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "\alpha\circ(id_M \bullet \alpha)"] & \\
 & (M \times M) \bullet X \arrow[dl, hook, "\iota_{(M\times M)\bullet X}"] \arrow[dr, "\alpha\circ(\mu \bullet id_X)"'] & \\
M \times M \times X & & X
\end{tikzcd} \tag{4.24}$$

Since the morphisms $\iota_{(M\times M)\bullet X}$ and $(id_M \times \iota) \circ \iota_{M\bullet(M\bullet X)}$ are inclusions, it follows from the commutativity of the left triangle of (4.24) that $M \bullet (M \bullet X) \subseteq (M \times M) \bullet X$ and φ is the corresponding inclusion map. In this case, the commutativity of the right triangle of (4.24) gives us (4.23).

And by assuming that we have $M \bullet (M \bullet X) \subseteq M \times (M \bullet X)$ and (4.23), let φ be the inclusion map of $M \bullet (M \bullet X)$ into $M \times (M \bullet X)$. It is then straightforward to check that φ is such that the diagram (4.24) commutes, completing the proof.

(1) $\Leftrightarrow$ (3). Observe that, for each $m, n \in M$, the square in the diagram

$$\begin{tikzcd}
 & & \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[dl, "\widehat{\iota_n}"'] \arrow[dr, "\widehat{\alpha_m}"] \arrow[dd, phantom, "\vee"] & & \\
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] & & \operatorname{dom}\alpha_n \arrow[dl, "\iota_n"'] \arrow[dr, "\alpha_n"] & \\
X & & X & & X
\end{tikzcd} \tag{4.25}$$

is a pullback, where $\widehat{\iota_n}$ is the inclusion map of $\alpha_m^{-1}(\operatorname{dom}\alpha_n)$ into $\operatorname{dom}\alpha_m$ and $\widehat{\alpha_m}$ is given by

$$\widehat{\alpha_m}(x) = \alpha_m(x),$$

for all $x \in \alpha_m^{-1}(\operatorname{dom}\alpha_n)$. Thus, the outermost span of (4.25) is a representative of $\alpha(n) \bullet \alpha(m)$. That is,

$$\alpha(n) \bullet \alpha(m) = [\alpha_m^{-1}(\operatorname{dom}\alpha_n), \iota_m \circ \widehat{\iota_n}, \alpha_n \circ \widehat{\alpha_m}]. \tag{4.26}$$

Then, by Proposition 3.4.13, we have that $\alpha(n) \bullet \alpha(m) \leq \alpha(nm)$ if and only if there exists a span morphism from $(\alpha_m^{-1}(\operatorname{dom}\alpha_n), \iota_m \circ \widehat{\iota_m}, \alpha_n \circ \widehat{\alpha_m})$ to $(\operatorname{dom}\alpha_m, \iota_m, \alpha_m)$, which is to say that there exists a map $\varphi : \alpha_m^{-1}(\operatorname{dom}\alpha_n) \to \operatorname{dom}\alpha_{nm}$ such that the following diagram commutes.

$$\begin{tikzcd}
& \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[ddl, "\iota_m\circ\widehat{\iota_n}"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "\alpha_n\circ\widehat{\alpha_m}"] & \\
& \operatorname{dom}\alpha_{nm} \arrow[dl, "\iota_{nm}"] \arrow[dr, "\alpha_{nm}"'] & \\
X & & X
\end{tikzcd} \tag{4.27}$$

So, for the "only if" part assume (1) and let $m, n \in M$. By (PA2), $\alpha_m^{-1}(\operatorname{dom}\alpha_n) \subseteq \operatorname{dom}\alpha_{nm}$, so let φ be the associated inclusion map. It is then immediate that φ is such that the left triangle of (4.27) commutes, and the commutativity of the right triangle of (4.27) follows easily from (PA3).

And for the "if" part, assume (3), so for each $m, n \in M$ there exists a map φ such that (4.27) commutes. Then it is easy to see that for each $m, n \in M$ the commutativity of the left triangle of (4.27) implies that $\alpha_m^{-1}(\operatorname{dom}\alpha_n) \subseteq \operatorname{dom}\alpha_{nm}$ with φ being the corresponding inclusion map, and that the commutativity of the right triangle of (4.27) implies that $\alpha_n \circ \alpha_m = \alpha_{nm}$ on $\alpha_m^{-1}(\operatorname{dom}\alpha_n)$, so both (PA2) and (PA3) follow. □

Proposition 4.2.6. *Let $\{\alpha_m\}_{m\in M}$ be a partial action datum of M on X. The following are equivalent.*

(1) $\{\alpha_m\}_{m\in M}$ satisfies (PA2') and (PA3);

(2) The associated isomorphism class $[M \bullet X, \iota, \alpha]$ from Proposition 4.2.1 is such that

$$\alpha \bullet (id_M \times \alpha) = \alpha \bullet (\mu \times id_X) \bullet \overline{(id_M \times \alpha)} \tag{4.28}$$

in $\mathbf{par_{Set}}(M \times M \times X, X)$;

(3) The associated map $\alpha : M \to \mathbf{par_{Set}}(X, X)$ from Proposition 4.2.1 is such that

$$\alpha(n) \bullet \alpha(m) = \alpha(nm) \bullet \overline{\alpha(m)},$$

for all $m, n \in M$.

Proof. (1) ⇔ (2). Denote $(M \times (M \bullet X)) \cap ((M \times M) \bullet X)$ by $(M \bullet M) \bullet X$, i.e.,

$$(M \bullet M) \bullet X = \{(n, m, x) \in M \times M \times X : x \in \operatorname{dom} \alpha_{nm} \cap \operatorname{dom} \alpha_m\}. \tag{4.29}$$

Then the square in the diagram

$$\begin{tikzcd}
 & & (M \bullet M) \bullet X \arrow[dl, "\widehat{\iota}"'] \arrow[dr, "\iota_{M\bullet M}\bullet id_X"] \arrow[dd, phantom, "\vee"] & & \\
 & M \times (M \bullet X) \arrow[dl, "id_M\times\iota"'] \arrow[dr, "id_M\times\iota"] & & (M \times M) \bullet X \arrow[dl, hook', "\iota_{(M\times M)\bullet X}"'] \arrow[dr, "\alpha\circ(\mu\bullet id_X)"] & \\
M \times X \times X & & M \times M \times X & & X
\end{tikzcd} \tag{4.30}$$

is a pullback, where $\widehat{\iota}$ and $\iota_{M\bullet M} \bullet id_X$ are inclusion maps. Thus, by (4.22), the outermost span of (4.30) is a representative of $\alpha \bullet (\mu \times id_X) \bullet \overline{(id_M \times \alpha)}$. That is,

$$\alpha \bullet (\mu \times id_X) \bullet \overline{(id_M \times \alpha)} = [(M \bullet M) \bullet X, (id_M \times \iota) \circ \widehat{\iota}, \alpha \circ (\mu \bullet id_X) \circ (\iota_{M\bullet M} \bullet id_X)] \tag{4.31}$$

Assume (1). In particular we have Proposition 4.2.5 (1), so Proposition 4.2.5 gives us that

$$\alpha \bullet (id_M \times \alpha) \leq \alpha \bullet (\mu \times id_X).$$

By Corollary 3.4.11 it then follows that

$$\alpha \bullet (id_M \times \alpha) \leq \alpha \bullet (\mu \times id_X) \bullet \overline{(id_M \times \alpha)}. \tag{4.32}$$

By (4.18), (4.31) and (4.32) and Proposition 3.4.13 there is a morphism φ such that the following diagram commutes.

$$\begin{tikzcd}
 & M \bullet (M \bullet X) \arrow[ddl, "(id_M\times\iota)\circ\iota_{M\bullet(M\bullet X)}"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "\alpha\circ(id_M\bullet\alpha)"] & \\
 & (M \bullet M) \bullet X \arrow[dl, hook', "(id_M\times\iota)\circ\widehat{\iota}"] \arrow[dr, "\alpha\circ(\mu\bullet id_X)\circ(\iota_{M\bullet M}\bullet id_X)"'] & \\
M \times M \times X & & X
\end{tikzcd} \tag{4.33}$$

Since $(id_M \times \iota) \circ \iota_{M\bullet(M\bullet X)}$ and $(id_M \times \iota) \circ \widehat{\iota}$ are both inclusion maps, by the commutativity of the left triangle of (4.33) it follows that $M \bullet (M \bullet X) \subseteq (M \bullet M) \bullet X$ and φ is the corresponding inclusion map.

Let $(n, m, x) \in (M \bullet M) \bullet X$. Then, by (4.29), $x \in \operatorname{dom} \alpha_{nm} \cap \operatorname{dom} \alpha_m$. Since $\{\alpha_m\}_{m\in M}$ satisfies (PA2'), it follows that $x \in \alpha_m^{-1}(\operatorname{dom} \alpha_n)$. So, by (4.15), we have $(n, m, x) \in M \bullet (M \bullet X)$. Hence, $(M \bullet M) \bullet X \subseteq M \bullet (M \bullet X)$ and φ is a bijection.

Since φ is a bijection, an isomorphism in **Set**, by Proposition 3.1.11 it follows that the spans in (4.33) are isomorphic. Thus, by (4.18) and (4.31) we have (2).

Now assume (2). In particular, we have Proposition 4.2.5 (2), which is equivalent to Proposition 4.2.5 (1). It suffices, then, to verify that $\operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m\subseteq\alpha_m^{-1}(\operatorname{dom}\alpha_n)$ for all $m,n\in M$.

By (4.18), (4.28) and (4.31) there exists an isomorphism φ such that diagram (4.33) commutes.

Similar to a previous argument, the commutativity of the left triangle of (4.33) implies that $M\bullet(M\bullet X)\subseteq(M\bullet M)\bullet X$ and φ is the corresponding inclusion map. Since φ is an isomorphism in **Set**, it is a bijection, so we have

$$M\bullet(M\bullet X)=(M\bullet M)\bullet X. \tag{4.34}$$

Let $m,n\in M$ and $x\in\operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m$. Then by (4.29) we have $(n,m,x)\in(M\bullet M)\bullet X$. Thus, by (4.34) $(n,m,x)\in M\bullet(M\bullet X)$, and, so, by (4.15) it follows that $x\in\alpha_m^{-1}(\operatorname{dom}\alpha_n)$. Therefore, $\operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m\subseteq\alpha_m^{-1}(\operatorname{dom}\alpha_n)$, as desired.

(1) $\Leftrightarrow$ (3). Observe that the square in

$$\begin{tikzcd}
& & \operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m \arrow[dl, "\overline{\iota_{nm}}"'] \arrow[dr, "\overline{\iota_m}"] \arrow[d, phantom, "\vee"] & & \\
& \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\iota_m"] & {} & \operatorname{dom}\alpha_{nm} \arrow[dl, "\iota_{nm}"'] \arrow[dr, "\alpha_{nm}"] & \\
X & & X & & X
\end{tikzcd} \tag{4.35}$$

is a pullback, where $\overline{\iota_m}$ and $\overline{\iota_{nm}}$ are inclusion maps. Thus,

$$\alpha(nm)\bullet\overline{\alpha(m)}=[\operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m,\alpha_{nm}\circ\overline{\iota_m},\iota_m\circ\overline{\iota_{nm}}]=[\operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m,\alpha_{nm}\circ\overline{\iota_m},\iota_{nm}\circ\overline{\iota_m}], \tag{4.36}$$

where the last equality follows by the commutativity of the square in (4.35).

Firstly, assume (1). By Proposition 4.2.5, for all $m,n\in M$ we have

$$\alpha(n)\bullet\alpha(m)\leq\alpha(nm),$$

and, thus, by Corollary 3.4.11,

$$\alpha(n)\bullet\alpha(m)\leq\alpha(nm)\bullet\overline{\alpha(m)}$$

So, by Proposition 3.4.13 and (4.26) and (4.36), for all $m,n\in M$ there exists a map $\varphi : \alpha_m^{-1}(\mathrm{dom}\,\alpha_n) \to \mathrm{dom}\,\alpha_{nm} \cap \mathrm{dom}\,\alpha_m$ such that the following diagram commutes.

$$
\begin{tikzcd}
& \alpha_m^{-1}(\mathrm{dom}\,\alpha_n) \arrow[ddl, "\iota_m \circ \widehat{\iota_n}"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "\alpha_n \circ \widehat{\alpha_m}"] & \\
& \mathrm{dom}\,\alpha_{nm} \cap \mathrm{dom}\,\alpha_m \arrow[dl, "\iota_{nm} \circ \overline{\iota_m}"] \arrow[dr, "\alpha_{nm} \circ \overline{\iota_m}"'] & \\
X & & X
\end{tikzcd}
\tag{4.37}
$$

Since the left triangle in (4.37) commutes, it follows that φ is the inclusion map of $\alpha_m^{-1}(\mathrm{dom}\,\alpha_n)$ into $\mathrm{dom}\,\alpha_{nm} \cap \mathrm{dom}\,\alpha_m$. By (PA2') it follows that φ is, in fact, a bijection.

Thus, since (4.37) commutes, its top and bottom spans are isomorphic, and, so, by (4.26) and (4.36) we have (3).

Now assume (3). In particular, it follows that $\alpha(n) \bullet \alpha(m) \leq \alpha(nm) \bullet \overline{\alpha(m)}$ for all $m, n \in M$, and, thus, we have (3), so by Proposition 4.2.5 $\{\alpha_m\}_{m \in M}$ satisfies (PA2) and (PA3). Thus, to show (1), it suffices to verify that

$$\mathrm{dom}\,\alpha_{nm} \cap \mathrm{dom}\,\alpha_m \subseteq \alpha_m^{-1}(\mathrm{dom}\,\alpha_n) \tag{4.38}$$

for all $m, n \in M$.

Indeed, let $m, n \in M$. Since $\alpha(n) \bullet \alpha(m) = \alpha(nm) \bullet \overline{\alpha(m)}$, by (4.26) and (4.36) there exists an isomorphism φ in **Set** such that (4.37) commutes.

By the commutativity of the left triangle of (4.37) we have that φ is the inclusion map of α_m^{-1} into $\mathrm{dom}\,\alpha_{nm} \cap \mathrm{dom}\,\alpha_m$. Since φ is a bijection, (4.38) follows. □

Proposition 4.2.7. *Let $\{\alpha_m\}_{m \in M}$ and $\{\beta_m\}_{m \in M}$ be partial action data of M on X and Y, respectively, and let $f : X \to Y$ be a map. The following are equivalent.*

(1) f is a datum morphism from $\{\alpha_m\}_{m \in M}$ to $\{\beta_m\}_{m \in M}$;

(2) f is such that

$$f \bullet \alpha \leq \beta \bullet (id_M \times f)$$

in $\mathbf{par_{Set}}(M \times Y, Y)$, where $[M \bullet X, \iota, \alpha] \in \mathbf{par_{Set}}(M \times X, X)$ and $[M \bullet Y, \kappa, \beta] \in \mathbf{par_{Set}}(M \times Y, Y)$ are the isomorphism classes from Proposition 4.2.1, associated to $\{\alpha_m\}_{m \in M}$ and $\{\beta_m\}_{m \in M}$, respectively;

(3) f is such that

$$f \bullet \alpha(m) \leq \beta(m) \bullet f$$

for all $m \in M$, where $\alpha : M \to \mathbf{par_{Set}}(X, X)$ and $\beta : M \to \mathbf{par_{Set}}(Y, Y)$ are the maps from Proposition 4.2.1, associated to $\{\alpha_m\}_{m \in M}$ and $\{\beta_m\}_{m \in M}$, respectively.

Proof. (1) $\Leftrightarrow$ (2). By Proposition 3.2.18,

$$f \bullet \alpha = [M \bullet X, \iota, f \circ \alpha]. \tag{4.39}$$

The following diagram illustrates a representative of $f \bullet \alpha$.

$$\begin{tikzcd} & M \bullet X \arrow[dl, "\iota"'] \arrow[dr, "f\circ\alpha"] & \\ X & & Y \end{tikzcd}$$

Denote $(id_M \times f)^{-1}(M \bullet Y)$ by $M \bullet f^{-1}(Y)$, i.e.,

$$M \bullet f^{-1}(Y) = \{(m, x) \in M \times X : x \in f^{-1}(\operatorname{dom} \beta_m)\}.$$

Then the square in

$$\begin{tikzcd} & & M \bullet f^{-1}(Y) \arrow[dl, "\widehat{\kappa}"'] \arrow[dr, "\widehat{id_M \times f}"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\ & M \times X \arrow[dl, "id_{M\times X}"'] \arrow[dr, "id_M \times f"] & & M \bullet Y \arrow[dl, "\kappa"'] \arrow[dr, "\beta"] & \\ M \times X & & M \times Y & & Y \end{tikzcd}$$

is a pullback, where $\widehat{\kappa}$ is the inclusion map of $M \bullet f^{-1}(Y)$ into $M \times X$ and $\widehat{id_M \times f}$ is the appropriate restriction and corestriction of $id_M \times f$, so

$$\beta \bullet (id_M \times f) = [M \bullet f^{-1}(Y), \widehat{\kappa}, \beta \circ \widehat{id_M \times f}]. \tag{4.40}$$

By Proposition 3.4.13 and (4.39) and (4.40), $f \bullet \alpha \leq \beta \bullet (id_M \times f)$ if and only if there exists a map φ such that the following diagram commutes.

$$\begin{tikzcd} & M \bullet X \arrow[ddl, "\iota"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "f\circ\alpha"] & \\ & M \bullet f^{-1}(Y) \arrow[dl, "\widehat{\kappa}"] \arrow[dr, "\beta\circ\widehat{id_M\times f}"'] & \\ X & & Y \end{tikzcd} \tag{4.41}$$

Let us assume (1), so f satisfies (DM1) and (DM2).

Let $(m, x) \in M \bullet X$. Then $x \in \operatorname{dom} \alpha_m$. By (DM1), it follows that $f(x) \in \operatorname{dom} \beta_m$, so $(id_M \times f)(m, x) = (m, f(x)) \in M \bullet Y$. Therefore, $(m, x) \in M \bullet f^{-1}(Y)$, and, thus, $M \bullet X \subseteq M \bullet f^{-1}(Y)$. Let φ be the associated inclusion map.

It is immediate that φ is such that the left triangle of (4.41) commutes, since it if formed by inclusion maps.

Now let $(m,x) \in M \bullet X$. Then

$$\beta \circ \widehat{id_M \times f} \circ \varphi(m,x) = \beta \circ \widehat{id_M \times f}(m,x) = \beta(m, f(x)) = \beta_m(f(x)). \tag{4.42}$$

Since $(m,x) \in M \bullet X$, $x \in \operatorname{dom} \alpha_m$, so by (DM2) we have $\beta_m(f(x)) = f(\alpha_m(x))$. Thus, by (4.42),

$$\beta \circ \widehat{id_M \times f} \circ \varphi(m,x) = f(\alpha_m(x)) = f \circ \alpha(m,x),$$

so the right triangle of (4.41) commutes.

Thus, since φ is such that (4.41) commutes, we have (2).

Now assume (2), so there exists a map φ such that (4.41) commutes.

The commutativity of the left triangle of (4.41) implies that

$$M \bullet X \subseteq M \bullet f^{-1}(Y) \tag{4.43}$$

and φ is the associated inclusion map.

Let $m \in M$ and $x \in \operatorname{dom} \alpha_m$. Then $(m,x) \in M \bullet X$, so by (4.43) $(m,x) \in M \bullet f^{-1}(Y)$, and, thus, $x \in f^{-1}(\operatorname{dom} \beta_m)$. So, it follows that $f(\operatorname{dom} \alpha_m) \subseteq \operatorname{dom} \beta_m$. Therefore, f satisfies (DM1).

And by the commutativity of the right triangle of (4.41) it follows that for all $(m,x) \in M \bullet X$

$$\beta_m(f(x)) = \beta(m, f(x)) = \beta \circ \widehat{id_M \times f} \circ \varphi(m,x) = f \circ \alpha(m,x) = f(\alpha_m(x)),$$

so f also satisfies (DM2). Thus, (1) follows.

(1) $\Leftrightarrow$ (3). Let $m \in M$. By Proposition 3.2.18,

$$f \bullet \alpha(m) = [\operatorname{dom} \alpha_m, \iota_m, f \circ \alpha_m]. \tag{4.44}$$

The following diagram illustrates a representative of $f \bullet \alpha(m)$.

$$\begin{tikzcd} & \operatorname{dom} \alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "f \circ \alpha_m"] & \\ X & & Y \end{tikzcd}$$

Let us denote by κ_m the inclusion map of $\operatorname{dom} \beta_m$ into Y. Then also observe that the square in

$$
\begin{tikzcd}
& & f^{-1}(\operatorname{dom}\beta_m) \arrow[dl, "\widehat{\kappa_m}"'] \arrow[dr, "\widehat{f}"] \arrow[d, phantom, "\vee"] & & \\
& X \arrow[dl, "id_X"'] \arrow[dr, "f"] & {} & \operatorname{dom}\beta_m \arrow[dl, "\kappa_m"'] \arrow[dr, "\beta_m"] & \\
X & & Y & & Y
\end{tikzcd}
\tag{4.45}
$$

is a pullback, where $\widehat{\kappa_m}$ is the inclusion map of $f^{-1}(\operatorname{dom}\beta_m)$ into X and $\widehat{f}$ is the appropriate restriction and corestriction of f, so

$$
\beta(m) \bullet f = [f^{-1}(\operatorname{dom}\beta_m), \widehat{\kappa_m}, \beta_m \circ \widehat{f}]. \tag{4.46}
$$

Thus, by Proposition 3.4.13 and (4.44) and (4.46), $f \bullet \alpha(m) \leq \beta(m) \bullet f$ if and only if there exists a map φ such that the following diagram commutes.

$$
\begin{tikzcd}
& \operatorname{dom}\alpha_m \arrow[ddl, "\iota_m"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "f\circ\alpha_m"] & \\
& f^{-1}(\operatorname{dom}\beta_m) \arrow[dl, "\widehat{\kappa_m}"] \arrow[dr, "\beta_m\circ\widehat{f}"'] & \\
X & & Y
\end{tikzcd}
\tag{4.47}
$$

Assume (1) first and let $m \in M$. Then f satisfies (DM1) and (DM2). By (DM1), it follows that $\operatorname{dom}\alpha_m \subseteq f^{-1}(\operatorname{dom}\beta_m)$, so let φ be the corresponding inclusion map.

Then we can see that the left triangle of (4.47) commutes immediately, and the right triangle of (4.47) commutes by (DM2). Therefore, $f \bullet \alpha(m) \leq \beta(m) \bullet f$, and, so, (3) follows.

Now assume (3) and let $m \in M$. Then there exists a map φ such that (4.47) commutes.

By the commutativity of the left triangle of (4.47) it follows that $\operatorname{dom}\alpha_m \subseteq f^{-1}(\operatorname{dom}\beta_m)$ and φ is the corresponding inclusion map. Hence, $f(\operatorname{dom}\alpha_m) \subseteq \operatorname{dom}\beta_m$, so (DM1) follows.

And by the commutativity of the right triangle of (4.47) it follows that $\beta_m \circ f = f \circ \alpha_m$ on $\operatorname{dom}\alpha_m$, so (DM2) follows. Consequently, we have (1). $\square$

4.3 PARTIAL MONOID ACTIONS ON OBJECTS IN CATEGORIES WITH PULLBACKS

For the remainder of this chapter, let $\mathscr{C}$ be a category with pullbacks (or only with inverse images) and M a monoid, whose identity we denote by e.

Motivated by the correspondences proved in Section 4.2, we define partial action data of a monoid M on an object of $\mathscr{C}$.

Definition 4.3.1. A **partial action datum** of M on $X \in \mathscr{C}$ is a map $\alpha : M \to \mathbf{par}_{\mathscr{C}}(X, X)$.

A straightforward definition of a global action of M on an object X of $\mathscr{C}$ would be that of a monoid homomorphism from M to $\mathrm{End}_{\mathscr{C}}(X)$ (see, for example, the definition of an M-object in [20], or, in the group case, the definition of a G-object in [21]). However, we will give a different (albeit equivalent) definition, to see the global actions as partial action data.

Definition 4.3.2. A **global action** of M on $X \in \mathscr{C}$ is a partial action datum α of M on X such that:

(CGA1) $\alpha(e) = [X, id_X, id_X]$;

(CGA2) $\alpha(m)$ is a global morphism from all $m \in M$ and $\alpha(n) \bullet \alpha(m) = \alpha(nm)$, for all $n, m \in M$.

Remark 4.3.3. By Propositions 4.2.2 and 4.2.4, in **Set** axiom (CGA1) corresponds to (GA1) and axiom (CGA2) corresponds to (GA2).

Proposition 4.3.4. *A partial action datum $\alpha(m) = [\operatorname{dom}\alpha_m = X, id_X, \alpha_m]$ of M on $X \in \mathscr{C}$ is a global action if and only if the map $\overline{\alpha} : M \to \mathrm{End}_{\mathscr{C}}(X)$ given by $\overline{\alpha}(m) = \alpha_m$ is a monoid homomorphism.*

Proof. Observe that $\overline{\alpha}(m)$ is well-defined by Proposition 3.1.15.

Now, since $\overline{\alpha}(e) = \alpha_e$, α satisfies (CGA1) if and only if $\overline{\alpha}(e) = id_X$.

And by Proposition 3.1.15 and Lemma 3.2.14 α satisfies (CGA2) if and only if

$$\overline{\alpha}(n) \circ \overline{\alpha}(m) = \overline{\alpha}(nm),$$

for all $m, n \in M$.

Hence, α satisfies (CGA1) and (CGA2) if and only if $\overline{\alpha}(m) = \alpha_m$ preserves the identity and the product of M, as desired. □

The following definitions were inspired by Propositions 4.2.2, 4.2.5 and 4.2.6.

Definition 4.3.5. A **partial action** of M on $X \in \mathscr{C}$ is a partial action datum α of M on X such that

(CPA1) $\alpha(e) = [X, id_X, id_X]$;

(CPA2) $\alpha(n) \bullet \alpha(m) \leq \alpha(nm)$ for all $m, n \in M$.

Definition 4.3.6. A **strong partial action** of M on $X \in \mathscr{C}$ is a partial action datum α of M on X such that

(SCPA1) $\alpha(e) = [X, id_X, id_X]$;

(SCPA2) $\alpha(n) \bullet \alpha(m) = \alpha(nm) \bullet \overline{\alpha(m)}$ for all $m, n \in M$.

Remark 4.3.7. A (strong) partial action of M on $X \in \mathscr{C}$ is a (strong) premorphism [11, Definitions 2.7 and 2.9] from M to the restriction monoid $\mathbf{par}_{\mathscr{C}}(X, X)$.

Remark 4.3.8. For each $m \in M$, let $(\operatorname{dom}\alpha_m, \iota_m, \alpha_m)$ be a representative of $\alpha(m)$. Axiom (CPA2) says that for all $m, n \in M$, if

$$\begin{tikzcd}
 & \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[dl, "\hat{\iota}_n^m"'] \arrow[dr, "\widehat{\alpha}_m^n"] \arrow[d, phantom, "\vee"] & \\
\operatorname{dom}\alpha_m \arrow[dr, "\alpha_m"'] & {} & \operatorname{dom}\alpha_n \arrow[dl, "\iota_n"] \\
 & X &
\end{tikzcd} \tag{4.48}$$

is a pullback (recall diagram (4.25)), then there exists a morphism $\varphi : \alpha_m^{-1}(\operatorname{dom}\alpha_n) \to \operatorname{dom}\alpha_{nm}$ such that the diagram

$$\begin{tikzcd}
 & \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[ddl, "\iota_m \circ \hat{\iota}_n^m"'] \arrow[ddr, "\alpha_n \circ \widehat{\alpha}_m^n"] \arrow[d, dashed, "\varphi"] & \\
 & \operatorname{dom}\alpha_{nm} \arrow[dl, "\iota_{nm}"] \arrow[dr, "\alpha_{nm}"'] & \\
X & & X
\end{tikzcd} \tag{4.49}$$

commutes.

Remark 4.3.9. For each $m \in M$, let $(\operatorname{dom}\alpha_m, \iota_m, \alpha_m)$ be a representative of $\alpha(m)$. Axiom (SCPA2) says that for all $m, n \in M$, if

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m \arrow[dl, "\bar{\iota}_{nm}^m"'] \arrow[dr, "\bar{\iota}_m^{nm}"] \arrow[d, phantom, "\vee"] & \\
\operatorname{dom}\alpha_m \arrow[dr, "\iota_m"'] & {} & \operatorname{dom}\alpha_{nm} \arrow[dl, "\iota_{nm}"] \\
 & X &
\end{tikzcd} \tag{4.50}$$

and (4.48) are pullbacks (recall diagrams (4.25) and (4.35)), then there exists an isomorphism $\varphi : \alpha_m^{-1}(\operatorname{dom}\alpha_n) \to \operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m$ such that the diagram

$$\begin{tikzcd}
 & & \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[lld, "\iota_m \circ \hat{\iota}_n^m"'] \arrow[d, dashed, "\theta"] \arrow[rrd, "\alpha_n \circ \widehat{\alpha}_m^n"] & & \\
 & & \operatorname{dom}\alpha_m \cap \operatorname{dom}\alpha_{nm} \arrow[lld, "\iota_{nm} \circ \bar{\iota}_m^{nm}"] \arrow[rrd, "\alpha_{nm} \circ \bar{\iota}_m^{nm}"'] & & \\
X & & & & X
\end{tikzcd} \tag{4.51}$$

commutes.

Remark 4.3.10. By Propositions 4.2.2, 4.2.5 and 4.2.6, in **Set** axioms (CPA1) and (SCPA1) correspond to (PA1), axiom (CPA2) corresponds to (PA2) and (PA3) and axiom (SCPA2) corresponds to (PA2') and (PA3).

Hence, the set-theoretic (strong) partial actions of M on a set X correspond to the (strong) partial actions of M on the object X in **Set**.

Proposition 4.3.11. *Let α be a partial action datum of M on $X \in \mathscr{C}$. The following statements hold.*

1. *If α is a global action, then it is a strong partial action;*
2. *If α is a strong partial action, then it is a partial action.*

Proof. First assume that α is a global action. Then (SCPA1) follows by (CGA1).

To verify (SCPA2), let $m, n \in M$. Since α is a global action, $\alpha(m)$ is a global morphism, so $\overline{\alpha(m)} = [X, id_X, id_X]$. Thus, by (CGA2) we have

$$\alpha(n) \bullet \alpha(m) = \alpha(nm) = \alpha(nm) \bullet \overline{\alpha(m)},$$

so it follows that α is a strong partial action.

Now assume that α is a strong partial action. Let us verify that it is a partial action. Obviously, (CPA1) follows by (SCPA1).

And by (SCPA2), (CPA2) is equivalent to

$$\alpha(nm) \bullet \overline{\alpha(m)} \leq \alpha(nm)$$

for all $m, n \in M$, which follows by Proposition 3.4.12. □

For the following definition, recall Proposition 3.3.1.

Definition 4.3.12. Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks, $F : \mathscr{C} \to \mathscr{D}$ a functor and $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ a partial action datum of M on $X \in \mathscr{C}$. We define the partial action datum $F(\alpha)$ of M on $F(X)$ in $\mathscr{D}$ given by

$$(F(\alpha))(m) = F(\alpha(m)) = [F(\operatorname{dom}\alpha_m), F(\iota_m), F(\alpha_m)],$$

for each $m \in M$.

Observe that by Proposition 3.3.1 the partial action datum in Definition 4.3.12 is well-defined.

Proposition 4.3.13. *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with pullbacks, $F : \mathscr{C} \to \mathscr{D}$ a functor that preserves pullbacks and $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ a partial action datum of M on $X \in \mathscr{C}$. Then*

(1) If α is a partial action (resp. strong partial action), then $F(\alpha)$ is a partial action (resp. strong partial action);

(2) If F the induced functor $F : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$ is faithful and $F(\alpha)$ is a partial action (resp. strong partial action), then α is a partial action (resp. strong partial action);

Proof. We will only verify (1) and (2) for strong partial actions.

(1). Assume that α is a strong partial action and denote $\beta = F(\alpha)$. Since F is a functor and α satisfies (SCPA1) we have

$$\beta(e) = [F(X), F(id_X), F(id_X)] = [F(X), id_{F(X)}, id_{F(X)}],$$

so β satisfies (SCPA1).

Now recall that since F preserves pullbacks it induces a functor from $\mathbf{par}_{\mathscr{C}}$ to $\mathbf{par}_{\mathscr{D}}$, as in Proposition 3.3.2, which, by Proposition 3.4.18 is a restriction functor between the two restriction categories. Thus, since α satisfies (SCPA2), for each $m, n \in M$ we have

$$\begin{aligned}\beta(n) \bullet \beta(m) = F(\alpha(n)) \bullet F(\alpha(m)) = F(\alpha(n) \bullet \alpha(m)) = F(\alpha(nm) \bullet \overline{\alpha(m)})\\ = F(\alpha(nm)) \bullet \overline{F(\alpha(m))} = \beta(nm) \bullet \overline{\beta(m)}.\end{aligned}$$

(2). Assume $\beta = F(\alpha)$ is a strong partial action. Once again, since F preserves pullbacks it induces a restriction functor $F : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$.

Because β satisfies (SCPA1) we have

$$F(\alpha(e)) = \beta(e) = [F(X), id_{F(X)}, id_{F(X)}] = F([X, id_X, id_X]).$$

Since $F : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$ is faithful, it follows that $\alpha(e) = [X, id_X, id_X]$. Therefore, α satisfies (SCPA1).

Now let $m, n \in M$. Since β satisfies (SCPA2), we have

$$F(\alpha(n) \bullet \alpha(m)) = F(\alpha(n)) \bullet F(\alpha(m)) = F(\alpha(nm)) \bullet \overline{F(\alpha(m))}$$
$$= F(\alpha(nm)) \bullet F(\overline{\alpha(m)}) = F(\alpha(nm) \bullet \overline{\alpha(m)}),$$

so, since $F : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathscr{D}}$ is faithful, α satisfies (SCPA2). Thus, α is a strong partial action, as desired. □

In particular in many concrete categories the (strong) partial actions coincide with the set-theoretical (strong) partial actions.

Proposition 4.3.14. *Let $\mathscr{C}$ be a concrete category whose associated forgetful functor $U : \mathscr{C} \to \mathbf{Set}$ preserves pullbacks and induces a faithful functor $U : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathbf{Set}}$. Let $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ be a partial action datum of M on $X \in \mathscr{C}$ where $\operatorname{dom}\alpha_m \subseteq X$ and ι_m is the corresponding inclusion map of $\operatorname{dom}\alpha_m$ into X. Then α is a partial action (resp. strong partial action) if and only if the set-theoretic partial action datum $\{\alpha_m\}_{m\in M}$ of M on X is a partial action (resp. strong partial action).*

Proof. By Proposition 4.3.13, since the forgetful functor U preserves pullbacks and the induced functor $U : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathbf{Set}}$ is faithful, α is a partial action if and only if $F(\alpha)$ is a partial action.

Now, by Remark 4.3.10, $F(\alpha)$ satisfies (CPA1) and (CPA2) if and only if the corresponding partial action datum $\{\alpha_m\}_{m\in M}$ of M on $U(X) = X$ satisfies (PA1)–(PA3).

Thus, α is a partial action if and only if $\{\alpha_m\}_{m\in M}$ is a partial action, as desired.

Similarly, α is a strong partial action if and only if $\{\alpha_m\}_{m\in M}$ is a strong partial action. □

In particular,

Corollary 4.3.15. *Let $\mathscr{C} \in \{\mathbf{Set}, \mathbf{Sem}, \mathbf{Mon}, \mathbf{Grp}, \mathbf{Ring}, \mathbf{Vect}_{\mathbb{K}}, \mathbf{Alg}_{\mathbb{K}}, \mathbf{C^*\text{-}Alg}\}$ and $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ a partial action datum of M on $X \in \mathscr{C}$ where $\operatorname{dom}\alpha_m \subseteq X$ and ι_m is the corresponding inclusion map of $\operatorname{dom}\alpha_m$ into X. Then α is a partial action (resp. strong partial action) if and only if the set-theoretic partial action datum $\{\alpha_m\}_{m\in M}$ of M on X is a partial action (resp. strong partial action).*

Proof. The forgetful functor U from $\mathscr{C}$ to **Set** preserves pullbacks in each of those cases. It is a straightforward verification that U also satisfies the hypothesis of Proposition 3.3.4, so the induced functor $U : \mathbf{par}_{\mathscr{C}} \to \mathbf{par}_{\mathbf{Set}}$ is faithful. Hence, the result follows from Proposition 4.3.14. □

Remark 4.3.16. If M is a group, by Proposition 4.1.19, the partial action datum α in Proposition 4.3.14 or in Corollary 4.3.15 is a strong partial action if and only if the set-theoretic partial action datum $\{\alpha_m\}_{m\in M}$ of M on X is a partial action in the sense of Definition 4.1.14.

However, Corollary 4.3.15 does not apply to certain concrete categories whose associated forgetful functor preserves pullbacks, such as **Top** and **Poset**, as the following example illustrates.

Example 4.3.17. Let M be the trivial monoid and X a set with at least two elements. Consider τ the indiscrete topology on X and τ' the discrete topology on X.

Consider the partial action datum α of M on the object (X,τ) in **Top**, where $\alpha(e)$ is the isomorphism class represented by the following partial morphism.

$$\begin{tikzcd} & (X,\tau') \arrow[dl, "id_X"'] \arrow[dr, "id_X"] & \\ (X,\tau) & & (X,\tau) \end{tikzcd}$$

Observe that id_X is a continuous map from (X,τ') to (X,τ) that is a monomorphism in **Top**, so $\alpha(e)$ is indeed an element of $\mathbf{par_{Top}}(X,X)$.

However, id_X is not an isomorphism in **Top**. Therefore, $\alpha(e) \neq [(X,\tau), id_{(X,\tau)}, id_{(X,\tau)}]$, so α does not satisfy (CPA1), and is, thus, not a partial action.

Nonetheless, α, seen as a partial action datum of M on the object X in **Set**, is a partial action.

A similar example can be found in **Poset**, by considering partial orders $\leq$ and $\leq'$ on the set $X=\{a,b\}$, where $a\leq b$, but $a \not\leq' b$. In this situation, id_X is an order preserving map from $(X,\leq')$ to $(X,\leq)$ that is not an isomorphism in **Poset**.

The following illustrates what are the partial actions on objects of a category coming from a meet-semilattice.

Example 4.3.18. Let $(X,\leq)$ be a meet-semilattice, $\mathscr{C}$ its corresponding category and $x\in X$. Then a partial morphism from x to x in $\mathscr{C}$ is a diagram of the form

$$\begin{tikzcd} & a \arrow[dl, "\iota_a^x"'] \arrow[dr, "\iota_a^x"] & \\ x & & x \end{tikzcd}$$

in $\mathscr{C}$. That is, each partial morphism from x to x corresponds to an element $a\in X$ such that $a\leq x$. By reflexivity of $\leq$, (a,ι_a^x,ι_a^x) is the only representative of its isomorphism class.

Since the pullbacks in $\mathscr{C}$ are given by the meet of the elements of X, we have

$$[b,\iota_b^x,\iota_b^x]\bullet[a,\iota_a^x,\iota_a^x] = [b\wedge a, \iota_{b\wedge a}^x, \iota_{b\wedge a}^x].$$

Also, by Proposition 3.4.13 the partial order in $\mathbf{par}_{\mathscr{C}}(X,X)$ is given by $[a,\iota_a^x,\iota_a^x] \leq [b,\iota_b^x,\iota_b^x]$ if and only if $a \leq b$.

Hence, the restriction monoid (recall Remark 3.4.3) $\mathbf{par}_{\mathscr{C}}(X,X)$ is isomorphic to the semilattice $x^{\downarrow} = \{a \in X : a \leq x\}$.

Therefore, the partial actions of M on x correspond to the maps $\alpha : M \to x^{\downarrow}$ where

$$\alpha(e) = x \quad \text{and} \quad \alpha(n) \wedge \alpha(m) \leq \alpha(nm),$$

and the strong partial actions of M on x correspond to the maps $\alpha : M \to x^{\downarrow}$ where

$$\alpha(e) = x \quad \text{and} \quad \alpha(n) \wedge \alpha(m) = \alpha(nm) \wedge \alpha(m).$$

4.4 DATUM MORPHISMS AND THE CATEGORY OF PARTIAL ACTION DATA

The following definition was inspired by Proposition 4.2.7.

Definition 4.4.1. Let α and β be partial action data of M on objects X and Y in $\mathscr{C}$, respectively. A **datum morphism** from α to β is a morphism $f : X \to Y$ in $\mathscr{C}$ such that

(CDM1) $f \bullet \alpha(m) \leq \beta(m) \bullet f$ for all $m \in M$.

Remark 4.4.2. For each $m \in M$, let $(\operatorname{dom}\alpha_m, \iota_m, \alpha_m)$ be a representative of $\alpha(m)$ and $(\operatorname{dom}\beta_m, \kappa_m, \beta_m)$ be a representative of $\beta(m)$. Axiom (CDM1) says that for all $m \in M$, if

$$\begin{tikzcd}
 & f^{-1}(\operatorname{dom}\beta_m) \arrow[dl, "\widehat{\kappa_m}"'] \arrow[dr, "\widehat{f}"] \arrow[dd, phantom, "\vee"] & \\
X \arrow[dr, "f"'] & & \operatorname{dom}\beta_m \arrow[dl, "\kappa_m"] \\
 & Y &
\end{tikzcd} \tag{4.52}$$

is a pullback (recall diagram (4.45)), then there exists a morphism $\varphi : \operatorname{dom}\alpha_m \to f^{-1}(\operatorname{dom}\beta_m)$ such that the following diagram commutes.

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[ddl, "\iota_m"'] \arrow[d, dashed, "\varphi"] \arrow[ddr, "f\circ\alpha_m"] & \\
 & f^{-1}(\operatorname{dom}\beta_m) \arrow[dl, "\widehat{\kappa_m}"] \arrow[dr, "\beta_m\circ\widehat{f}"'] & \\
X & & Y
\end{tikzcd} \tag{4.53}$$

Proposition 4.4.3. *Let $\alpha(m) = [\mathrm{dom}\,\alpha_m, \iota_m, \alpha_m]$ and $\beta(m) = [\mathrm{dom}\,\beta_m, \kappa_m, \beta_m]$ be partial action data of M on objects X and Y in $\mathscr{C}$, respectively. Then a morphism $f : X \to Y$ is a datum morphism from α to β if and only if for all $m \in M$ there exists a morphism $f_m : \mathrm{dom}\,\alpha_m \to \mathrm{dom}\,\beta_m$ such that the following diagram commutes.*

$$\begin{tikzcd}
& \mathrm{dom}\,\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, "f_m"] & \\
X \arrow[dd, "f"'] & & X \arrow[dd, "f"] \\
& \mathrm{dom}\,\beta_m \arrow[dl, "\kappa_m"'] \arrow[dr, "\beta_m"] & \\
Y & & Y
\end{tikzcd} \tag{4.54}$$

Proof. Firstly, assume f is a datum morphism from α to β and let $m \in M$. Then by Remark 4.4.2 there exists a morphism $\varphi : \mathrm{dom}\,\alpha_m \to f^{-1}(\mathrm{dom}\,\beta_m)$ such that (4.53) commutes, where (4.52) is a pullback.

Then observe that $f_m := \widehat{f} \circ \varphi$ is such that (4.54) commutes. Indeed, by the commutativity of (4.52) and (4.53) we have

$$\kappa_m \circ f_m = \kappa_m \circ \widehat{f} \circ \varphi = f \circ \widehat{\kappa_m} \circ \varphi = f \circ \iota_m$$

and

$$\beta_m \circ f_m = \beta_m \circ \widehat{f} \circ \varphi = f \circ \alpha_m.$$

Conversely, assume that for each $m \in M$ there exists a morphism f_m such that (4.54) commutes, and let $m \in M$. Since the left square of (4.54) commutes and (4.52) is a pullback, there exists a unique morphism $\varphi : \mathrm{dom}\,\alpha_m \to f^{-1}(\mathrm{dom}\,\beta_m)$ such that the following diagram commutes.

$$\begin{tikzcd}
& \mathrm{dom}\,\alpha_m \arrow[d, "\varphi"] \arrow[ddl, "\iota_m"'] \arrow[ddr, "f_m"] & \\
& f^{-1}(\mathrm{dom}\,\beta_m) \arrow[dl, "\widehat{\kappa_m}"] \arrow[dr, "\widehat{f}"'] & \\
X & & \mathrm{dom}\,\beta_m
\end{tikzcd} \tag{4.55}$$

Then φ is such that (4.53) commutes. Indeed, the commutativity of the left triangle of (4.53) follows from the commutativity of the left triangle of (4.55), and the commutativity of the right triangle of (4.53) follows from the commutativity of (4.54) and (4.55), because

$$\beta_m \circ \widehat{f} \circ \varphi = \beta_m \circ f_m = f \circ \alpha_m.$$

□

When β is a global action, datum morphisms $f : \alpha \to \beta$ admit a simpler description.

Lemma 4.4.4. *Let $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ be a partial action datum on $X \in \mathscr{C}$ and $\beta(m) = [Y, id_Y, \beta_m]$ be a global action of M on $Y \in \mathscr{C}$. Then a morphism f from X to Y in $\mathscr{C}$ is a datum morphism from α to β if and only if the following diagram commutes for all $m \in M$.*

$$\begin{tikzcd}
& \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] & \\
X \arrow[dr, "\beta_m \circ f"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd} \tag{4.56}$$

Proof. If f is a datum morphism from α to β, then, by Proposition 4.4.3, for each $m \in M$ there exists a morphism $f_m : \operatorname{dom}\alpha_m \to Y$ such that the diagram

$$\begin{tikzcd}
& \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, dashed, "f_m"] & \\
X \arrow[dd, "f"] & & X \arrow[dd, "f"] \\
& Y \arrow[dl, "id"'] \arrow[dr, "\beta_m"] & \\
Y & & Y
\end{tikzcd} \tag{4.57}$$

commutes. The commutativity of the left square of (4.57) yields $f_m = f \circ \iota_m$, which together with the commutativity of the right square yields $\beta_m \circ f \circ \iota_m = \beta_m \circ f_m = f \circ \alpha_m$, so the commutativity of (4.56) follows.

Conversely, if (4.56) commutes for all $m \in M$, then the morphism $f_m \coloneqq f \circ \iota_m$ makes the diagram (4.57) commute, and hence, by Proposition 4.4.3, f is a datum morphism. □

Corollary 4.4.5. *Let $\alpha(m) = [X, id_X, \alpha_m]$ and $\beta(m) = [Y, id_Y, \beta_m]$ be global actions on objects X and Y in $\mathscr{C}$, respectively. Then a morphism f from X to Y in $\mathscr{C}$ is a datum morphism from α to β if and only if*

$$f \circ \alpha_m = \beta_m \circ f,$$

for all $m \in M$.

Proof. Follows immediately from Lemma 4.4.4. □

Proposition 4.4.6. *Let $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$, $\beta(m) = [\operatorname{dom}\beta_m, \kappa_m, \beta_m]$ and $\gamma(m) = [\operatorname{dom}\gamma_m, \lambda_m, \gamma_m]$ be partial action data of M on objects X, Y and Z in $\mathscr{C}$, respectively. If f is a datum morphism from α to β and g is a datum morphism from β to γ. Then $g \circ f$ is a datum morphism from α to γ.*

Proof. Let $m \in M$. Since f is a datum morphism from α to β, by Proposition 4.4.3 there exists a morphism $f_m : \operatorname{dom}\alpha_m \to \operatorname{dom}\beta_m$ such that the following diagram commutes.

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, "f_m"] & \\
X \arrow[dd, "f"'] & & X \arrow[dd, "f"] \\
 & \operatorname{dom}\beta_m \arrow[dl, "\kappa_m"'] \arrow[dr, "\beta_m"] & \\
Y & & Y
\end{tikzcd} \tag{4.58}$$

And since g is a datum morphism from β to γ, by Proposition 4.4.3 there exists a morphism g_m such that the following diagram commutes.

$$\begin{tikzcd}
 & \operatorname{dom}\beta_m \arrow[dl, "\kappa_m"'] \arrow[dr, "\beta_m"] \arrow[dd, "g_m"] & \\
Y \arrow[dd, "g"'] & & Y \arrow[dd, "g"] \\
 & \operatorname{dom}\gamma_m \arrow[dl, "\lambda_m"'] \arrow[dr, "\gamma_m"] & \\
Z & & Z
\end{tikzcd} \tag{4.59}$$

Then consider the following diagram.

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[d, "f_m"] & \\
X \arrow[d, "f"'] & \operatorname{dom}\beta_m \arrow[dl, "\kappa_m"'] \arrow[dr, "\beta_m"] \arrow[d, "g_m"] & X \arrow[d, "f"] \\
Y \arrow[d, "g"'] & \operatorname{dom}\gamma_m \arrow[dl, "\lambda_m"'] \arrow[dr, "\gamma_m"] & Y \arrow[d, "g"] \\
Z & & Z
\end{tikzcd} \tag{4.60}$$

Observe that by the commutativity of (4.58) the top two squares of (4.60) commute, and by the commutativity of (4.59) the bottom two squares of (4.60) commute. Thus,

$g_m \circ f_m$ is a morphism such that the following diagram commutes.

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, "g_m\circ f_m"] & \\
X \arrow[dd, "g\circ f"'] & & X \arrow[dd, "g\circ f"] \\
 & \operatorname{dom}\gamma_m \arrow[dl, "\lambda_m"'] \arrow[dr, "\gamma_m"] & \\
Z & & Z
\end{tikzcd}$$

Hence, by Proposition 4.4.3, $g \circ f$ is a datum morphism from α to γ. $\square$

Proposition 4.4.7. *Let $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ be a partial action datum of M on $X \in \mathscr{C}$. Then id_X is a datum morphism from α to α.*

Proof. For each $m \in M$, the morphism $id_{\operatorname{dom}\alpha_m}$ makes the diagram

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, "id_{\operatorname{dom}\alpha_m}"] & \\
X \arrow[dd, "id_X"'] & & X \arrow[dd, "id_X"] \\
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] & \\
X & & X
\end{tikzcd}$$

commute. Thus, id_X is a datum morphism from α to α. $\square$

We can then define the categories of partial action data, partial actions and global actions.

Definition 4.4.8. Denote by $M-\mathbf{Datum}_{\mathscr{C}}$ the category whose objects are partial action data of M on objects in $\mathscr{C}$ and whose morphisms are the datum morphisms between those objects, where the composition is inherited from $\mathscr{C}$. Moreover, let

1. $M-\mathbf{pAct}_{\mathscr{C}}$ denote the full subcategory of $M-\mathbf{Datum}_{\mathscr{C}}$ formed by the partial actions of M on objects in $\mathscr{C}$.

2. $M-\mathbf{Act}_{\mathscr{C}}$ denote the full subcategory of $M-\mathbf{Datum}_{\mathscr{C}}$ formed by the global actions of M on objects in $\mathscr{C}$.

3. $M-\mathbf{spAct}_{\mathscr{C}}$ denote the full subcategory of $M-\mathbf{Datum}_{\mathscr{C}}$ formed by the strong partial actions of M on objects in $\mathscr{C}$.

4.5 PARTIAL GROUP ACTIONS ON OBJECTS IN CATEGORIES WITH PULLBACKS

If M is a group, there are some equivalent descriptions of its strong partial actions, which will be explored in this section. Throughout this section, let G be a group, that will be treated as its underlying monoid whenever necessary, and recall that $\mathscr{C}$ is a category with pullbacks.

Recall that $\mathbf{iso}_{\mathscr{C}}$ is the inverse category induced by the restriction structure in $\mathbf{par}_{\mathscr{C}}$, which, by Proposition 3.5.14, is composed of the isomorphism classes $[A, f, g]$ in $\mathbf{par}_{\mathscr{C}}$ such that g is a monomorphism.

From this point onward, for each $X \in \mathscr{C}$ we will denote $\mathbf{iso}_{\mathscr{C}}(X, X)$ by $\mathcal{I}(X)$.

Lemma 4.5.1. *Let α be a strong partial action of G on $X \in \mathscr{C}$. Then*

$$\alpha(g^{-1}) \bullet \alpha(g) = \overline{\alpha(g)}, \tag{4.61}$$

for all $g \in G$.

Proof. Let $g \in G$. By (SCPA1) and (SCPA2), we get

$$\alpha(g^{-1}) \bullet \alpha(g) = \alpha(g^{-1}g) \bullet \overline{\alpha(g)} = \alpha(e) \bullet \overline{\alpha(g)} = \overline{\alpha(g)},$$

as desired. □

Corollary 4.5.2. *Let α be a strong partial action of G on $X \in \mathscr{C}$. Then*

$$\alpha(g) \bullet \alpha(g^{-1}) \bullet \alpha(g) = \alpha(g), \tag{4.62}$$

for all $g \in G$.

Proof. Let $g \in G$. Then, by Corollary 4.5.3 and (R1), we have

$$\alpha(g) \bullet \alpha(g^{-1}) \bullet \alpha(g) = \alpha(g) \bullet \overline{\alpha(g)} = \alpha(g)$$

as desired. □

Corollary 4.5.3. *Let α be a strong partial action of G on $X \in \mathscr{C}$. Then $\alpha(G) \subseteq \mathcal{I}(X)$.*

Proof. Let $g \in G$. By Lemma 4.5.1, $\alpha(g^{-1})$ is an isomorphism class in $\mathbf{par}_{\mathscr{C}}(X, X)$ such that

$$\alpha(g^{-1}) \bullet \alpha(g) = \overline{\alpha(g)}$$

and

$$\alpha(g) \bullet \alpha(g^{-1}) = \overline{\alpha(g^{-1})}.$$

Thus, by Definition 3.5.6 $\alpha(g) \in \mathcal{I}(X)$. Therefore, $\alpha(G) \subseteq \mathcal{I}(X)$ as desired. □

Theorem 4.5.4. *A partial action datum α of G on $X \in \mathscr{C}$ is a strong partial action if and only if*

(1) $\alpha(G) \subseteq \mathcal{I}(X)$,

(2) $\alpha(e) = [X, id_X, id_X]$,

(3) $\alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}) = \alpha(hg) \bullet \alpha(g^{-1})$ *for all* $g, h \in G$.

Proof. ($\Rightarrow$). Assume that α is a strong partial action. Then (1) follows by Corollary 4.5.3 and (2) follows by (SCPA1).

To verify (3), let $g, h \in G$. By (SCPA2) and Lemma 4.5.1 we have

$$\alpha(h) \bullet \alpha(g) = \alpha(hg) \bullet \overline{\alpha(g)} = \alpha(hg) \bullet \alpha(g^{-1}) \bullet \alpha(g). \tag{4.63}$$

By (4.63) and Corollary 4.5.2 it follows that

$$\alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}) = \alpha(hg) \bullet \alpha(g^{-1}) \bullet \alpha(g) \bullet \alpha(g^{-1}) = \alpha(hg) \bullet \alpha(g^{-1}),$$

so we have (3).

($\Leftarrow$). Now assume (1)–(3). By (2), α satisfies (SCPA1), so it suffices to verify (SCPA2).

By (2) and (3),

$$\alpha(g) \bullet \alpha(g^{-1}) \bullet \alpha(g) = \alpha(e) \bullet \alpha(g) = \alpha(g)$$

for all $g \in G$.

Let $g \in G$. Since $\alpha(g) \bullet \alpha(g^{-1}) \bullet \alpha(g) = \alpha(g)$ and $\alpha(g^{-1})\alpha(g)\alpha(g^{-1} = \alpha(g^{-1})$, and $\mathcal{I}(X)$ is an inverse monoid, by (1) we have $\alpha(g)^* = \alpha(g^{-1})$. In particular, by Proposition 3.5.12, we get

$$\alpha(g^{-1}) \bullet \alpha(g) = \overline{\alpha(g)}. \tag{4.64}$$

Now let $g, h \in G$. By (3) and (4.64) it follows that

$$\alpha(h) \bullet \alpha(g) = \alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}) \bullet \alpha(g) = \alpha(hg) \bullet \alpha(g^{-1}) \bullet \alpha(g) = \alpha(hg) \bullet \overline{\alpha(g)}.$$

Thus, α satisfies (SCPA2), and is a strong partial action, as desired. □

We then have the following relationship between the strong partial actions of a group and the Exel's semigroup of the group.

Theorem 4.5.5. *Let $X \in \mathscr{C}$. There is a correspondence between the*

(1) strong partial actions of G on X;

(2) monoid homomorphisms from $\mathcal{S}(G)$ to $\mathcal{I}(X)$.

Proof. The monoid homomorphisms from $\mathcal{S}(G)$ to $\mathcal{I}(X)$ are in correspondence with the maps from G to $\mathcal{I}(X)$ satisfying Proposition 2.3.18 (1) and (2). Those, in turn, correspond to the strong partial actions of G on X, by Theorem 4.5.4. □

As a final characterization of strong partial actions of groups we have the following.

Theorem 4.5.6. *A partial action datum α of G on $X \in \mathscr{C}$ is a strong partial action if and only if*

(1) α is a partial action,

(2) $\alpha(G) \subseteq \mathcal{I}(X)$,

(3) $\alpha(g^{-1}) = \alpha(g)^*$ for all $g \in G$.

Proof. If α is a strong partial action, then (1) follows by Proposition 4.3.11, and (2) and (3) follow by Corollaries 4.5.2 and 4.5.3.

Now assume that α satisfies (1)–(3). By (2), α satisfies Theorem 4.5.4 (1), and by (CPA1), α satisfies Theorem 4.5.4 (2). Let us verify Theorem 4.5.4 (3).

Let $g, h \in G$. By (CPA2),

$$\alpha(h) \bullet \alpha(g) \leq \alpha(hg),$$

so, by Proposition 3.4.10,

$$\alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}) \leq \alpha(hg) \bullet \alpha(g^{-1}). \tag{4.65}$$

On the other hand, also by (CPA2) we have

$$\alpha(hg) \bullet \alpha(g^{-1}) \leq \alpha((hg)g^{-1}) = \alpha(h).$$

So, using the definition of the partial order $\leq$ and by (R1)–(R3) we have

$$\begin{aligned}\alpha(hg) \bullet \alpha(g^{-1}) &= \alpha(h) \bullet \overline{\alpha(hg) \bullet \alpha(g^{-1})} = \alpha(h) \bullet \overline{\alpha(hg) \bullet \alpha(g^{-1}) \bullet \overline{\alpha(g^{-1})}} \\ &= \alpha(h) \bullet \overline{\alpha(hg) \bullet \alpha(g^{-1})} \bullet \overline{\alpha(g^{-1})} = \alpha(h) \bullet \overline{\alpha(g^{-1})} \bullet \overline{\alpha(hg) \bullet \alpha(g^{-1})}.\end{aligned}$$

By (3), it follows that $\overline{\alpha(g^{-1})} = \alpha(g) \bullet \alpha(g^{-1})$. Thus, by Proposition 3.4.12 we have

$$\alpha(hg) \bullet \alpha(g^{-1}) = \alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}) \bullet \overline{\alpha(hg) \bullet \alpha(g^{-1})} \leq \alpha(h) \bullet \alpha(g) \bullet \alpha(g^{-1}). \tag{4.66}$$

Therefore, Theorem 4.5.4 (3) follows by (4.65) and (4.66), and, so, α is a strong partial action by Theorem 4.5.4. $\square$

5 RESTRICTIONS AND GLOBALIZATIONS OF PARTIAL MONOID ACTIONS

The problem of the globalization of the partial actions of monoids on objects in categories with pullbacks is tackled in this chapter. In Section 5.1 we introduce the restriction of a global action of a monoid on an object in a category with pullbacks to a subobject, and show that it is a strong partial action.

Section 5.2 contains the main results of this work, where we introduce globalizations of partial actions in this categorical context and find conditions for a given partial action to be globalizable.

In Theorem 5.2.5 we show necessary and sufficient conditions in terms of pullback diagrams for a given partial action to have a (universal) globalization, under the assumption that it has a reflection in $M-\mathbf{Act}_{\mathscr{C}}$. We then show in Theorem 5.2.15 and Corollary 5.2.19 that if the category $\mathscr{C}$ has a certain colimit, or if it has certain coproducts and coequalizer, the partial action has such a reflection, which allows the application of Theorem 5.2.5. Assuming the existence of the previous coproducts, in Theorem 5.2.26 we show necessary and sufficient conditions for such a reflection to exist, in terms of a coequalizer in $M-\mathbf{Act}_{\mathscr{C}}$.

5.1 RESTRICTIONS OF GLOBAL ACTIONS

Throughout this chapter, M will be a monoid with identity e and $\mathscr{C}$ a category with pullbacks, and whenever β is a global action of M on an object $Y \in \mathscr{C}$, we will assume that $\beta(m) = [Y, id_Y, \beta_m]$ for all $m \in M$.

In this categorical context, we can also construct (strong) partial actions from global actions, similar to Definition 4.1.11.

Definition 5.1.1. Let β be a global action of M on $Y \in \mathscr{C}$ and $\iota : X \to Y$ a monomorphism in $\mathscr{C}$. The **restriction** of β to X (via ι) is the partial action datum $\alpha(m) = [\operatorname{dom} \alpha_m, \iota_m, \alpha_m]$, where for each $m \in M$ the following diagram is a pullback.

$$\begin{tikzcd} & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, phantom, "\vee" very near start] & \\ X \arrow[dr, "\beta_m \circ \iota"'] & & X \arrow[dl, "\iota"] \\ & Y & \end{tikzcd} \tag{5.1}$$

Remark 5.1.2. Observe that since ι is a monomorphism, each ι_m is a monomorphism, so $[\operatorname{dom} \alpha_m, \iota_m, \alpha_m] \in \mathbf{par}_{\mathscr{C}}(X, X)$ and α is indeed a partial action datum.

Also, α does not depend on the choice of a pullback of $\beta_m \circ \iota$ and ι, since any two such pullbacks are always isomorphic spans.

Remark 5.1.3. Notice that by Lemma 4.4.4 the morphism $\iota : X \to Y$ in Definition 5.1.1 is a datum morphism from α to β.

The following proposition gives an equivalent way to describe restrictions of global actions.

Recall that any morphism $\iota : X \to Y$ in $\mathscr{C}$ can be seen as the isomorphism class $[X, id_X, \iota] \in \mathbf{par}_{\mathscr{C}}(X, Y)$. Moreover, if ι is a monomorphism, it can be seen as a morphism in the inverse category $\mathbf{iso}_{\mathscr{C}}$. In this case, by Proposition 3.5.14, $[X, id_X, \iota]^*$, which we will denote only by ι^*, is the isomorphism class $[X, \iota, id_X] \in \mathbf{par}_{\mathscr{C}}(Y, X)$.

Proposition 5.1.4. *Let β be a global action of M on $Y \in \mathscr{C}$, $\iota : X \to Y$ a monomorphism in $\mathscr{C}$ and $\alpha(m) = [\operatorname{dom} \alpha_m, \iota_m, \alpha_m]$ the restriction of β to X via ι. Then*

$$\alpha(m) = \iota^* \bullet \beta(m) \bullet \iota,$$

for all $m \in M$.

Proof. Observe that the square in

$$\begin{tikzcd}
 & & X \arrow[dl, "id_X"'] \arrow[dr, "\iota"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & X \arrow[dl, "id_X"'] \arrow[dr, "\iota"] & & Y \arrow[dl, "id_Y"'] \arrow[dr, "\beta_m"] & \\
X & & Y & & Y
\end{tikzcd}$$

is a pullback, so

$$\beta(m) \bullet \iota = [X, id_X, \beta_m \circ \iota]. \tag{5.2}$$

Now, since α is the restriction of β to X via ι, the square in the diagram

$$\begin{tikzcd}
 & & \operatorname{dom} \alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & X \arrow[dl, "id_X"'] \arrow[dr, "\beta_m \circ \iota"] & & X \arrow[dl, "\iota"'] \arrow[dr, "id_X"] & \\
X & & Y & & X
\end{tikzcd}$$

is a pullback, so

$$\iota^* \bullet [X, id_X, \beta_m \circ \iota] = [\operatorname{dom} \alpha_m, id_X \circ \iota_m, id_X \circ \alpha_m] = [\operatorname{dom} \alpha_m, \iota_m, \alpha_m]. \tag{5.3}$$

Thus, by (5.2) and (5.3),

$$\iota^* \bullet \beta(m) \bullet \iota = \iota^* \bullet [X, id_X, \beta_m \circ \iota] = [\operatorname{dom} \alpha_m, \iota_m, \alpha_m] = \alpha(m),$$

as desired. □

If $\mathscr{C} = \mathbf{Set}$ and ι is the inclusion of a subset X into Y, the restriction of β to X via ι from Definition 5.1.1 gives us the construction seen in Definition 4.1.11, as shown in the sequel.

Proposition 5.1.5. *Let β be a global action of M on $Y \in \mathbf{Set}$ and $X \subseteq Y$ with associated inclusion map ι. Then the restriction of β to X in the sense of Definition 4.1.11 corresponds (as in Proposition 4.2.1) to the restriction of β to X via ι in the sense of Definition 5.1.1.*

Proof. The restriction of β to X in the sense of Definition 4.1.11 is the partial action $\{\alpha_m\}_{m \in M}$ where, for each $m \in M$

$$\operatorname{dom} \alpha_m = X \cap \beta_m^{-1}(X)$$

and $\alpha_m : \operatorname{dom} \alpha_m \to X$ is given by

$$\alpha_m(x) = \beta_m(x) \tag{5.4}$$

for each $x \in \operatorname{dom} \alpha_m$.

By Proposition 4.2.1 the family $\{\alpha_m\}_{m \in M}$ corresponds to $\alpha(m) = [\operatorname{dom} \alpha_m, \iota_m, \alpha_m]$, where ι_m is the inclusion of $\operatorname{dom} \alpha_m$ into X.

Notice that, by (5.4), diagram (5.1) commutes for all $m \in M$. We are going to verify that it is a pullback diagram.

Let $Z \in \mathbf{Set}$ and $p_1, p_2 : Z \to X$ such that

$$\beta_m \circ \iota \circ p_1 = \iota \circ p_2. \tag{5.5}$$

Given $z \in Z$, since $\beta_m(\iota(p_1(z))) = \iota(p_2(z)) \in \iota(X) = X$, we have $p_1(z) = \iota(p_1(z)) \in \beta_m^{-1}(X)$, so $p_1(z) \in \operatorname{dom} \alpha_m$. This way, define $\varphi : Z \to \operatorname{dom} \alpha_m$ by

$$\varphi(z) = p_1(z). \tag{5.6}$$

Then we have for all $z \in Z$

$$\iota_m(\varphi(z)) = \iota_m(p_1(z)) = p_1(z)$$

and, by (5.4)–(5.6),

$$\alpha_m(\varphi(z)) = \beta_m(p_1(z)) = p_2(z).$$

Thus φ makes the diagram

$$\begin{tikzcd}
& Z \arrow[d, dashed, "\varphi"] \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
& \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"] \arrow[dr, "\alpha_m"'] & \\
A \arrow[dr, "\beta_m \circ \iota"'] & & A \arrow[dl, "\iota"] \\
& B &
\end{tikzcd}$$

commute, and it is unique as such because ι_m is a monomorphism in **Set**. It follows that (5.1) is a pullback diagram.

By Definition 5.1.1, we have that α is the restriction of β to X via ι, as desired. $\square$

We will prove below that any restriction of a global action is a partial action. To this end, for the remainder of this section, assume that we are in the setting of Definition 5.1.1.

Lemma 5.1.6. *The restriction α of β to X via ι in Definition 5.1.1 is a partial action.*

Proof. We first check (CPA1). Since β is a global action, $\beta_e = \mathrm{id}_Y$, so $\beta_e \circ \iota = \iota$. Thus, because ι is a monomorphism, by Proposition 2.2.16 the following diagram is a pullback.

$$\begin{tikzcd}
& X \arrow[dl, "id_X"'] \arrow[dr, "id_X"] \arrow[dd, phantom, "\lrcorner", very near start] & \\
X \arrow[dr, "\beta_e \circ \iota"'] & & X \arrow[dl, "\iota"] \\
& Y &
\end{tikzcd}$$

Therefore,

$$\alpha(e) = [X, id_X, id_X].$$

We will verify (CPA2) by using Remark 4.3.8. Let $m, n \in M$. Our goal is to, given pullback squares (4.48) and (4.50), construct an isomorphism $\varphi : \alpha_m^{-1}(\operatorname{dom}\alpha_n) \to \operatorname{dom}\alpha_{nm}$ which makes diagram (4.51) commute.

Notice that, since β is a global action, and by the commutativity of the diagrams (4.48) and (5.1), we have

$$\begin{aligned}
(\beta_{nm} \circ \iota) \circ (\iota_m \circ \hat{\iota}_n^m) &= (\beta_n \circ \beta_m) \circ \iota \circ \iota_m \circ \hat{\iota}_n^m = \beta_n \circ (\beta_m \circ \iota \circ \iota_m) \circ \hat{\iota}_n^m \\
&= \beta_n \circ (\iota \circ \alpha_m) \circ \hat{\iota}_n^m = \beta_n \circ \iota \circ (\alpha_m \circ \hat{\iota}_n^m) = \beta_n \circ \iota \circ (\iota_n \circ \widehat{\alpha}_n^m) \\
&= (\beta_n \circ \iota \circ \iota_n) \circ \widehat{\alpha}_n^m = (\iota \circ \alpha_n) \circ \widehat{\alpha}_n^m = \iota \circ (\alpha_n \circ \widehat{\alpha}_n^m).
\end{aligned}$$

Thus, the diagram

$$
\begin{tikzcd}
& \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[dl, "\iota_m\circ\hat{\iota}_n^m"'] \arrow[dr, "\alpha_n\circ\widehat{\alpha}_n^m"] & \\
X \arrow[dr, "\beta_{nm}\circ\iota"'] & & X \arrow[dl, "\iota"] \\
& Y &
\end{tikzcd}
$$

commutes, and so, by the universal property of the pullback (5.1), there exists a unique morphism φ such that the diagram

$$
\begin{tikzcd}
& \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[ddl, "\iota_m\circ\hat{\iota}_n^m"'] \arrow[ddr, "\alpha_n\circ\widehat{\alpha}_n^m"] \arrow[d, dashed, "\varphi"] & \\
& \operatorname{dom}\alpha_{nm} \arrow[dl, "\iota_{nm}"] \arrow[dr, "\alpha_{nm}"'] \arrow[dd, phantom, "\vee" very near start] & \\
X \arrow[dr, "\beta_{nm}\circ\iota"'] & & X \arrow[dl, "\iota"] \\
& Y &
\end{tikzcd}
$$

commutes, and so φ makes (4.49) commute. Therefore, α satisfies (CPA2), as desired. □

Proposition 5.1.7. *The restriction α of β to X via ι in Definition 5.1.1 is a strong partial action.*

Proof. By Lemma 5.1.6, α is a partial action. In particular, α satisfies (CPA1), and, thus, (SCPA1).

We will now verify (SCPA2) by using Remark 4.3.9. To do so, let $m, n \in M$.

Since α is a partial action, there exists a morphism $\varphi : \alpha_m^{-1}(\operatorname{dom}\alpha_n) \to \operatorname{dom}\alpha_{nm}$ that makes diagram (4.49) commute.

Notice that, by the commutativity of (4.49),

$$\iota_{nm} \circ \varphi = \iota_m \circ \hat{\iota}_n^m,$$

so since (4.50) is a pullback there exists a unique morphism $\theta : \alpha_m^{-1}(\operatorname{dom}\alpha_n) \to \operatorname{dom}\alpha_{nm} \cap$

$\operatorname{dom}\alpha_m$ that makes the following diagram commute.

$$
\begin{tikzcd}
& & \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[d, dashed, "\theta"] \arrow[ddll, "\hat{\iota}_n^m"'] \arrow[ddrr, "\varphi"] & & \\
& & \operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m \arrow[dll, "\bar{\iota}_{nm}^m"] \arrow[drr, "\bar{\iota}_m^{nm}"'] \arrow[dd, phantom, "\vee" near start] & & \\
\operatorname{dom}\alpha_m \arrow[drr, "\iota_m"'] & & & & \operatorname{dom}\alpha_{nm} \arrow[dll, "\iota_{nm}"] \\
& & X & &
\end{tikzcd}
\tag{5.7}
$$

The morphism θ makes (4.51) commute, because, by the commutativity of (4.50) and (5.7),

$$(\iota_{nm}\circ\bar{\iota}_m^{nm})\circ\theta = (\iota_m\circ\bar{\iota}_{nm}^m)\circ\theta = \iota_m\circ(\bar{\iota}_{nm}^m\circ\theta) = \iota_m\circ\hat{\iota}_n^m$$

and by the commutativity of (4.49) and (5.7),

$$(\alpha_{nm}\circ\bar{\iota}_m^{nm})\circ\theta = \alpha_{nm}\circ(\bar{\iota}_m^{nm}\circ\theta) = \alpha_{nm}\circ\varphi = \alpha_n\circ\widehat{\alpha}_m^n.$$

Let us verify that θ is an isomorphism by exhibiting its inverse. Notice that by the commutativity of (4.50) and (5.1) we have

$$
\begin{aligned}
\iota\circ(\alpha_{nm}\circ\bar{\iota}_m^{nm}) &= (\iota\circ\alpha_{nm})\circ\bar{\iota}_m^{nm} = (\beta_{nm}\circ\iota\circ\iota_{nm})\circ\bar{\iota}_m^{nm} \\
&= (\beta_n\circ\beta_m)\circ\iota\circ(\iota_{nm}\circ\bar{\iota}_m^{nm}) = \beta_n\circ\beta_m\circ\iota\circ(\iota_m\circ\bar{\iota}_{nm}^m) \\
&= \beta_n\circ(\beta_m\circ\iota\circ\iota_m)\circ\bar{\iota}_{nm}^m = \beta_n\circ(\iota\circ\alpha_m)\circ\bar{\iota}_{nm}^m \\
&= \beta_n\circ\iota\circ(\alpha_m\circ\bar{\iota}_{nm}^m).
\end{aligned}
$$

so since (5.1) is a pullback, there exists a unique morphism $\eta : \operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m \to \alpha_m^{-1}(\operatorname{dom}\alpha_n)$ such that the following diagram commutes.

$$
\begin{tikzcd}
& & \operatorname{dom}\alpha_{nm}\cap\operatorname{dom}\alpha_m \arrow[d, dashed, "\eta"] \arrow[ddll, "\alpha_m\circ\bar{\iota}_{nm}^m"'] \arrow[ddrr, "\alpha_{nm}\circ\bar{\iota}_m^{nm}"] & & \\
& & \operatorname{dom}\alpha_n \arrow[dll, "\iota_n"] \arrow[drr, "\alpha_n"'] \arrow[dd, phantom, "\vee" near start] & & \\
X \arrow[drr, "\beta_n\circ\iota"'] & & & & X \arrow[dll, "\iota"] \\
& & Y & &
\end{tikzcd}
\tag{5.8}
$$

In particular, by the commutativity of (5.8) it follows that

$$\iota_n \circ \eta = \alpha_m \circ \bar{\iota}^m_{nm},$$

so since (4.48) is a pullback, there exists a unique morphism $\psi : \operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m \to \alpha_m^{-1}(\operatorname{dom}\alpha_n)$ such that the following diagram commutes.

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m \arrow[ddl, "\bar{\iota}^m_{nm}"'] \arrow[d, dashed, "\psi"] \arrow[ddr, "\eta"] & \\
 & \alpha_m^{-1}(\operatorname{dom}\alpha_n) \arrow[dl, "\hat{\iota}^m_n"] \arrow[dr, "\widehat{\alpha}^n_m"'] \arrow[dd, phantom, "\vee"] & \\
\operatorname{dom}\alpha_m \arrow[dr, "\alpha_m"'] & & \operatorname{dom}\alpha_n \arrow[dl, "\iota_n"] \\
 & X &
\end{tikzcd} \tag{5.9}$$

Let us verify that ψ is the inverse of θ. Notice that by the commutativity of (5.7) and (5.9) we have

$$\bar{\iota}^m_{nm} \circ \theta \circ \psi = \hat{\iota}^m_n \circ \psi = \bar{\iota}^m_{nm} = \bar{\iota}^m_{nm} \circ id_{\operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m},$$

so, since $\bar{\iota}^m_{nm}$ is a monomorphism (because (4.50) is a pullback for all $m, n \in M$ and ι_{nm} is a monomorphism), we have $\theta \circ \psi = id_{\operatorname{dom}\alpha_{nm} \cap \operatorname{dom}\alpha_m}$.

Similarly, the commutativity of (5.7) and (5.9) gives us

$$\hat{\iota}^m_n \circ \psi \circ \theta = \bar{\iota}^m_{nm} \circ \theta = \hat{\iota}^m_n = \hat{\iota}^m_n \circ id_{\alpha_m^{-1}(\operatorname{dom}\alpha_n)},$$

so, since $\hat{\iota}^m_n$ is a monomorphism (because ι_n is a monomorphism in (4.48)), we have $\psi \circ \theta = id_{\alpha_m^{-1}(\operatorname{dom}\alpha_n)}$. Thus, ψ is an isomorphism, as desired.

Therefore, α satisfies (SCPA2), and is, thus, a strong partial action, as desired. □

Remark 5.1.8. Another proof that α satisfies (SCPA2) using Proposition 5.1.4 goes as follows. Let $m, n \in M$. Then by the fact that β is a global action and Proposition 5.1.4 and (R4)

$$\begin{aligned}
\alpha(n) \bullet \alpha(m) &= (\iota^* \bullet \beta(n) \bullet \iota) \bullet (\iota^* \bullet \beta(m) \bullet \iota) = \iota^* \bullet \beta(n) \bullet (\iota \bullet \iota^* \bullet \beta(m) \bullet \iota) \\
&= \iota^* \bullet \beta(n) \bullet (\overline{\iota^*} \bullet \beta(m) \bullet \iota) = \iota^* \bullet \beta(n) \bullet (\beta(m) \bullet \iota \bullet \overline{\iota^* \bullet \beta(m) \bullet \iota}) \\
&= \iota^* \bullet (\beta(n) \bullet \beta(m)) \bullet \iota \bullet \overline{\alpha(m)} = \iota^* \bullet \beta(nm) \bullet \iota \bullet \overline{\alpha(m)} = \alpha(nm) \bullet \overline{\alpha(m)}.
\end{aligned}$$

Inspired by Proposition 5.1.4, one could in fact define restrictions of any partial action data to a subobject. The following proposition illustrates what happens if one were to restrict partial actions and strong partial actions in such a way.

Proposition 5.1.9. *Let β be a partial action datum of M on $Y \in \mathscr{C}$, $\iota : X \to Y$ a monomorphism in $\mathscr{C}$ and consider the partial action datum $\alpha(m) = \iota^* \bullet \beta(m) \bullet \iota$ of M on X. If β is a (strong) partial action, then α is a (strong) partial action.*

Proof. First, observe that, by (R4), for all $m, n \in M$ we have

$$\begin{aligned}\alpha(n) \bullet \alpha(m) &= (\iota^* \bullet \beta(n) \bullet \iota) \bullet (\iota^* \bullet \beta(m) \bullet \iota) = \iota^* \bullet \beta(n) \bullet (\iota \bullet \iota^* \bullet \beta(m) \bullet \iota) \\ &= \iota^* \bullet \beta(n) \bullet (\overline{\iota^*} \bullet \beta(m) \bullet \iota) = \iota^* \bullet \beta(n) \bullet (\beta(m) \bullet \iota \bullet \overline{\iota^* \bullet \beta(m) \bullet \iota})\end{aligned}$$

so,

$$\alpha(n) \bullet \alpha(m) = \iota^* \bullet \beta(n) \bullet \beta(m) \bullet \iota \bullet \overline{\alpha(m)}, \tag{5.10}$$

Assume, then, that β is a partial action. Since β satisfies (CPA1), $\beta(e) = [Y, id_Y, id_Y]$ is the identity morphism of Y in $\mathbf{par}_{\mathscr{C}}$. Thus, we have

$$\alpha(e) = \iota^* \bullet \beta(e) \bullet \iota = \iota^* \bullet \iota = \overline{\iota} = [X, id_X, id_X],$$

so α satisfies (CPA1).

Now let $m, n \in M$. Then, since β satisfies (CPA2), and by Propositions 3.4.10 and 3.4.12 and (5.10), we have

$$\begin{aligned}\alpha(n) \bullet \alpha(m) &= \iota^* \bullet (\beta(n) \bullet \beta(m)) \bullet \iota \bullet \overline{\alpha(m)} \leq \iota^* \bullet \beta(nm) \bullet \iota \bullet \overline{\alpha(m)} \\ &= \alpha(nm) \bullet \overline{\alpha(m)} \leq \alpha(nm).\end{aligned}$$

Thus, α is a partial action.

Now assume that β is a strong partial action. Similar to the previous case, it then follows that α satisfies (SCPA1), so let us verify (SCPA2).

Let $m, n \in M$. Then, since β satisfies (SCPA2), and by (R2)–(R4) and (5.10) we have

$$\begin{aligned}\alpha(n) \bullet \alpha(m) &= \iota^* \bullet (\beta(n) \bullet \beta(m)) \bullet \iota \bullet \overline{\alpha(m)} = \iota^* \bullet (\beta(nm) \bullet \overline{\beta(m)}) \bullet \iota \bullet \overline{\alpha(m)} \\ &= \iota^* \bullet \beta(nm) \bullet (\overline{\beta(m)} \bullet \iota) \bullet \overline{\alpha(m)} = \iota^* \bullet \beta(nm) \bullet (\iota \bullet \overline{\beta(m) \bullet \iota}) \bullet \overline{\alpha(m)} \\ &= (\iota^* \bullet \beta(nm) \bullet \iota) \bullet (\overline{\beta(m) \bullet \iota} \bullet \overline{\alpha(m)}) = \alpha(nm) \bullet (\overline{\alpha(m)} \bullet \overline{\beta(m) \bullet \iota}) \\ &= \alpha(nm) \bullet \overline{\alpha(m) \bullet \overline{\beta(m) \bullet \iota}} = \alpha(nm) \bullet \overline{\alpha(m)},\end{aligned}$$

where the final equality follows by (R1), since $\alpha(m) = \iota^* \bullet \beta(m) \bullet \iota$. Thus, α satisfies (SCPA2). □

Proposition 5.1.10. *Let β be a global action of M on $Y \in \mathscr{C}$ and $\iota : X \to Y$ and $\iota' : X' \to Y$ monomorphisms. Let α be the restriction of β to X via ι and α' be the* *restriction of β to X' via ι'. If ι and ι' represent the same subobject of Y, then α and α' are isomorphic in $M-\mathbf{Datum}_{\mathscr{C}}$.*

Proof. Let us say that $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ and $\alpha'(m) = [\operatorname{dom}\alpha'_m, \iota'_m, \alpha'_m]$ for all $m \in M$.

Assume that ι and ι' represent the same subobject of Y. Then there exists an isomorphism $\varphi : X \to X'$ such that

$$\iota' \circ \varphi = \iota. \tag{5.11}$$

Let us verify that φ is an isomorphism from α to α' in $M-\mathbf{Datum}_{\mathscr{C}}$. We will first verify that φ is a datum morphism from α to α'.

Let $m \in M$. Observe that since α is the restriction of β to X via ι, the diagram (5.1) is a pullback, and, in particular, commutes. With this and (5.11) we thus have

$$\beta_m \circ \iota' \circ \varphi \circ \iota_m = \beta_m \circ \iota \circ \iota_m = \iota \circ \alpha_m = \iota' \circ \varphi \circ \alpha_m. \tag{5.12}$$

Now, since α' is the restriction of β to X' via ι', the diagram

$$\begin{tikzcd}
 & \operatorname{dom}\alpha'_m \arrow[dl, "\iota'_m"'] \arrow[dr, "\alpha'_m"] \arrow[dd, phantom, "\vee"] & \\
X' \arrow[dr, "\beta_m \circ \iota'"'] & & X \arrow[dl, "\iota'"] \\
 & Y &
\end{tikzcd}$$

is a pullback. Then, by (5.12), there exists a unique morphism $\varphi_m : \operatorname{dom}\alpha_m \to \operatorname{dom}\alpha'_m$ such that the diagram

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[d, "\varphi_m"] \arrow[dr, "\alpha_m"] & \\
X \arrow[d, "\varphi"'] & \operatorname{dom}\alpha'_m \arrow[dl, "\iota'_m"'] \arrow[dr, "\alpha'_m"] \arrow[dd, phantom, "\vee"] & X \arrow[d, "\varphi"] \\
X' \arrow[dr, "\beta_m \circ \iota'"'] & & X' \arrow[dl, "\iota'"] \\
 & Y &
\end{tikzcd}$$

commutes. In particular, the diagram

$$\begin{tikzcd}
 & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dd, "\varphi_m"] \arrow[dr, "\alpha_m"] & \\
X \arrow[dd, "\varphi"'] & & X \arrow[dd, "\varphi"] \\
 & \operatorname{dom}\alpha'_m \arrow[dl, "\iota'_m"'] \arrow[dr, "\alpha'_m"] & \\
X' & & X'
\end{tikzcd}$$

commutes for all $m \in M$, so φ is a datum morphism from α to α' by Proposition 4.4.3.

Now let ψ be the inverse of φ. A similar verification, utilizing the universal property of (5.1), shows us that ψ is a datum morphism from α' to α. It is immediate, then, that ψ is an inverse of φ in $M{-}\mathbf{Datum}_{\mathscr{C}}$, so φ is an isomorphism from α to α' in $M{-}\mathbf{Datum}_{\mathscr{C}}$, as desired. □

5.2 GLOBALIZATIONS OF PARTIAL ACTIONS

We can now define the notion which is in some sense inverse to the restriction of a global action.

Definition 5.2.1. Let α be a partial action datum of M on $X \in \mathscr{C}$. A **globalization** of α is a pair (β, ι) formed by a global action β of M on an object $Y \in \mathscr{C}$ and a monomorphism $\iota : X \to Y$, such that α is the restriction of β to X via ι.

If α has a globalization, we say that α is **globalizable**.

By Remark 5.1.3, the morphism ι in Definition 5.2.1 is a datum morphism from α to β.

Example 5.2.2. Let $(X, \leq)$ be a meet-semilattice, $\mathscr{C}$ its associated category and α a globalizable partial action of M on $x \in \mathscr{C}$. Then α is a global action.

Indeed, for all $y \in \mathscr{C}$ the only global morphism in $\mathbf{par}_{\mathscr{C}}(y, y)$ is $[y, id_y, id_y]$, so any global action of M on y is trivial. Hence, if (β, ι) is a globalization of α where, say, β acts on $y \in \mathscr{C}$, then, by Proposition 5.1.4,

$$\alpha(m) = \iota^* \bullet \beta(m) \bullet \iota = \iota^* \bullet [y, id_y, id_y] \bullet \iota = \iota^* \bullet \iota = [x, id_x, id_x],$$

for all $m \in M$.

Definition 5.2.3. Let α be a partial action datum of M on $X \in \mathscr{C}$. A **universal globalization** of α is a pair (β, ι) such that:

(UG1) (β, ι) is a globalization of α;

(UG2) whenever (γ, κ) is a globalization of α, there exists a unique morphism $\kappa' : \beta \to \gamma$ such that the following diagram commutes.

$$\begin{tikzcd} \alpha \arrow[r, "\iota"] \arrow[dr, "\kappa"'] & \beta \arrow[d, dashed, "\kappa'"] \\ & \gamma \end{tikzcd} \tag{5.13}$$

Remark 5.2.4. Observe that our concept of a universal globalization slightly differs from that of a globalization defined in [18] because we do not require the datum morphism ι in Definition 5.2.3 to be a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$.

Nevertheless, whenever a reflection ι of α in $M-\mathbf{Act}_{\mathscr{C}}$ exists, it gives us a necessary and sufficient condition for α to have a (universal) globalization.

Theorem 5.2.5 ([15, Theorem 4.4])**.** *Let $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ be a partial action datum of M on $X \in \mathscr{C}$. Assume that α has a reflection $\iota : \alpha \to \beta$ in $M-\mathbf{Act}_{\mathscr{C}}$, with, say, β acting on $Y \in \mathscr{C}$. Then the following are equivalent:*

(1) (β, ι) is a globalization of α;

(2) (β, ι) is a universal globalization of α;

(3) α has a universal globalization;

(4) α has a (not necessarily universal) globalization;

(5) for all $m \in M$ the following diagram is a pullback diagram in $\mathscr{C}$.

$$\begin{tikzcd} & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, phantom, "\vee"] & \\ X \arrow[dr, "\beta_m \circ \iota"'] & & X \arrow[dl, "\iota"] \\ & Y & \end{tikzcd} \tag{5.14}$$

Proof. Implication (1) $\Rightarrow$ (2) follows because (β, ι), being a globalization, satisfies (UG1), and (UG2) is a consequence of the fact that ι is a reflection.

Implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate and (5) $\Rightarrow$ (1) follows from the definition of a globalization, so it remains to check (4) $\Rightarrow$ (5).

Assume thus that α has a globalization (γ, κ), with, say, γ acting on $Z \in \mathscr{C}$. By Definition 5.2.1, α is the restriction of γ to X via κ. That is, for all $m \in M$ the diagram

$$\begin{tikzcd} & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] \arrow[dd, phantom, "\vee" very near start] & \\ X \arrow[dr, "\gamma_m\circ\kappa"'] & & X \arrow[dl, "\kappa"] \\ & Z & \end{tikzcd}$$

is a pullback diagram.

Since ι is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$ and γ is a global action with κ a datum morphism from α to γ, there exists a unique datum morphism $\kappa' : \beta \to \gamma$ such that

$$\kappa = \kappa' \circ \iota. \tag{5.15}$$

Fix $m \in M$. As κ' and ι are datum morphisms, by Lemma 4.4.4 we have

$$\gamma_m \circ \kappa' = \kappa' \circ \beta_m \text{ and } \beta_m \circ \iota \circ \iota_m = \iota \circ \alpha_m. \tag{5.16}$$

This way, by (5.15) and (5.16), the diagram

$$\begin{tikzcd} & \operatorname{dom}\alpha_m \arrow[dl, "\iota_m"'] \arrow[dr, "\alpha_m"] & \\ X \arrow[dr, "\beta_m\circ\iota"] \arrow[ddr, "\gamma_m\circ\kappa"'] & & X \arrow[dl, "\iota"'] \arrow[ddl, "\kappa"] \\ & Y \arrow[d, "\kappa'"] & \\ & Z & \end{tikzcd}$$

commutes. Now, since its perimeter is a pullback diagram, it is customary to check that the inner square is also a pullback. □

Remark 5.2.6. Since ι is a datum morphism from α to β, diagram (5.14) is already a commutative diagram, by Lemma 4.4.4.

In Example 6.3.10 we will present a universal globalization (β, ι) whose ι is not a reflection.

Observe that universal globalizations are unique up to isomorphism.

Proposition 5.2.7. *Let α be a partial action datum of M on $X \in \mathscr{C}$ and (β, ι) and (γ, κ) universal globalizations of α. Then there exists an isomorphism $\varphi : \beta \to \gamma$ such that $\varphi \circ \iota = \kappa$.*

Proof. Since (β, ι) is a universal globalization of α and (γ, κ) is a globalization of α, by (UG2) there exists a unique morphism φ such that the following diagram commutes.

$$\begin{tikzcd} \alpha \arrow[r, "\iota"] \arrow[dr, "\kappa"'] & \beta \arrow[d, dashed, "\varphi"] \\ & \gamma \end{tikzcd}$$

In particular, φ is such that $\varphi \circ \iota = \kappa$. Let us verify that it is an isomorphism.

In a similar way, since (γ, κ) is a universal globalization of α, by (UG2) there exists a unique morphism $\psi : \gamma \to \beta$ such that $\psi \circ \kappa = \iota$.

Then observe that

$$\psi \circ \varphi \circ \iota = \psi \circ \kappa = \iota = id_\beta \circ \iota,$$

so since (β, ι) satisfies (UG2) it follows that $\psi \circ \varphi = id_\beta$.

Similarly, since (γ, κ) satisfies (UG2) we have $\varphi \circ \psi = id_\gamma$. Hence, φ is an isomorphism, as desired. □

Corollary 5.2.8. *Let α be a partial action datum of M on $X \in \mathscr{C}$ and (β, ι) an universal globalization of α. If α has a reflection in $M-\mathbf{Act}_{\mathscr{C}}$, then $\iota : \alpha \to \beta$ is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$.*

Proof. Let $r : \alpha \to \gamma$ be a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$. Since α has a globalization, by Theorem 5.2.5, (γ, r) is a universal globalization of α. Hence, since (β, ι) and (γ, r) are universal globalizations of α, the result follows by Proposition 5.2.7. □

5.2.1 REFLECTION IN TERMS OF A COLIMIT

Now we are going to provide conditions for a partial action datum to have a reflection in $M-\mathbf{Act}_{\mathscr{C}}$. To this end, for the remainder of this subsection fix a partial action datum $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ of M on $X \in \mathscr{C}$.

Define the category I with $\mathrm{Ob}(I) = (M \times M) \sqcup M$, where for each $(m, n) \in \mathrm{Ob}(I)$ there is a morphism from (m, n) to mn and a morphism from (m, n) to m, and there are no other non-trivial morphisms.

Definition 5.2.9. The **functor**[1] **associated to** α is the functor $F : I \to \mathscr{C}$ defined as follows. Given $m, n \in M$, it maps

- (m, n) to $\operatorname{dom}\alpha_n$,
- m to X,

[1] Strictly speaking, the functor is not unique, since it depends on the choice of representatives of the isomorphism classes $\alpha(m)$, $m \in M$.

- the morphism $(m,n) \to mn$ to ι_n, and
- the morphism $(m,n) \to m$ to α_n,

as illustrated.

$$\begin{tikzcd}[column sep=small] & (m,n) \arrow[dl] \arrow[dr] & & & & \operatorname{dom}\alpha_n \arrow[dl, "\iota_n"'] \arrow[dr, "\alpha_n"] & \\ mn & & m \arrow[rr, rightsquigarrow, "F", shift left=12] & & X & & X \end{tikzcd}$$

We are going to show that a colimit of F induces a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$. To this end, recall that for all $Y \in \mathscr{C}$ we denote by $\Delta(Y)$ the constant functor from I to $\mathscr{C}$ that maps all objects in I to Y and all morphisms in I to id_Y.

Lemma 5.2.10. *Let $\eta = \{F(i) \xrightarrow{\eta_i} Y : i \in I\}$ be a natural transformation from F to $\Delta(Y)$. Then for each $m \in M$ the family $\eta^m = \{F(i) \xrightarrow{\eta_i^m} Y : i \in I\}$, where*

$$\eta_i^m = \begin{cases} \eta_{(ms,t)}, & \text{if } i = (s,t) \in M \times M, \\ \eta_{ms}, & \text{if } i = s \in M, \end{cases} \tag{5.17}$$

is also a natural transformation from F to $\Delta(Y)$.

Proof. Fix $m \in M$. Since the only non-trivial morphisms in I are $(s,t) \to st$ and $(s,t) \to s$ for each $s,t \in M$, to verify that η^m is a natural transformation from F to $\Delta(Y)$, it suffices to check that the diagrams

$$\begin{tikzcd} \operatorname{dom}\alpha_t \arrow[r, "\eta^m_{(s,t)}"] \arrow[d, "\iota_t"'] & Y \arrow[d, "id"] \\ X \arrow[r, "\eta^m_{st}"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{dom}\alpha_n \arrow[r, "\eta^m_{(s,t)}"] \arrow[d, "\alpha_t"'] & Y \arrow[d, "id"] \\ X \arrow[r, "\eta^m_{s}"] & Y \end{tikzcd} \tag{5.18}$$

commute for each $s,t \in M$.

Let $s,t \in M$. Since η is a natural transformation from F to $\Delta(Y)$, $\eta_{m(st)} \circ \iota_t = \eta_{(ms)t} \circ \iota_t = \eta_{(ms,t)}$. So, we have the commutativity of the left diagram of (5.18), in view of (5.17).

The commutativity of the right diagram of (5.18) follows similarly. □

Assume that there exists a colimit $\eta : F \to \Delta(Y)$ of the functor F associated to α and let η^m be the corresponding natural transformation from Lemma 5.2.10. Fixed $m \in M$, by the universal property of η there exists a unique natural transformation β_m from $\Delta(Y)$ to $\Delta(Y)$ such that $\eta^m = \beta_m \circ \eta$.

That is, for each $m \in M$ there exists a unique morphism $Y \xrightarrow{\beta_m} Y$ such that

$$\beta_m \circ \eta_{(s,t)} = \eta^m_{(s,t)} = \eta_{(ms,t)} \tag{5.19}$$

for all $s, t \in M$, and

$$\beta_m \circ \eta_s = \eta_s^m = \eta_{ms} \tag{5.20}$$

for all $s \in M$.

Consider then the partial action datum β of M on Y given by

$$\beta(m) = [Y, id_Y, \beta_m] \tag{5.21}$$

for each $m \in M$.

Lemma 5.2.11. *The partial action datum β defined in* (5.21) *is a global action of M on Y.*

Proof. We will verify that the map $\overline{\beta} : M \to \mathrm{End}_{\mathscr{C}}(Y)$, where $\overline{\beta}(m) = \beta_m$ is a monoid homomorphism.

By (5.17),

$$id_Y \circ \eta_{(s,t)} = \eta_{(s,t)} = \eta_{(es,t)} = \eta^e_{(s,t)} = \beta_e \circ \eta_{(s,t)}$$

for all $s, t \in M$, and

$$id_Y \circ \eta_s = \eta_s = \eta_{es} = \eta_s^e = \beta_e \circ \eta_s$$

for all $s \in M$. Thanks to the uniqueness of β_e, it follows that $\beta_e = id_Y$. Therefore, $\overline{\beta}$ preserves the identity.

Now let $m, n \in M$. By (5.19) and (5.20), for all $s, t \in M$ we have

$$\begin{aligned}(\beta_n \circ \beta_m) \circ \eta_{(s,t)} &= \beta_n \circ (\beta_m \circ \eta_{(s,t)}) = \beta_n \circ \eta_{(ms,t)} = \eta_{(n(ms),t)} \\ &= \eta_{((nm)s,t)} = \beta_{nm} \circ \eta_{(s,t)}\end{aligned}$$

and

$$(\beta_n \circ \beta_m) \circ \eta_s = \beta_n \circ (\beta_m \circ \eta_s) = \beta_n \circ \eta_{ms} = \eta_{n(ms)} = \eta_{(nm)s} = \beta_{nm} \circ \eta_s.$$

Thus, by the uniqueness of β_{nm} we have $\beta_n \circ \beta_m = \beta_{nm}$. Therefore, $ol\beta$ preserves the product of M.

Thus, $\overline{\beta}$ is a monoid homomorphism. Hence, β is a global action by Proposition 4.3.4. $\square$

Definition 5.2.12. Let η be a colimit of the functor associated to α. By the **global action associated to** η we mean $\beta \in M{-}\mathbf{Act}_{\mathscr{C}}$ given by (5.21).

Proposition 5.2.13. *Let $\eta : F \to \Delta(Y)$ be a colimit of the functor F associated to α, and let β be the global action associated to η. Then $\eta_e : X \to Y$ is a datum morphism from α to β.*

Proof. Given $m \in M$, since η is a natural transformation from F to $\Delta(Y)$, we have $\eta_m \circ \iota_m = \eta_{em} \circ \iota_m = \eta_{(e,m)} = \eta_e \circ \alpha_m$. By (5.20) we have $\beta_m \circ \eta_e = \eta_m$. Thus,

$$(\beta_m \circ \eta_e) \circ \iota_m = \eta_m \circ \iota_m = \eta_e \circ \alpha_m,$$

and so, by Lemma 4.4.4, η_e is a datum morphism as desired. □

Lemma 5.2.14. *Let (γ, f) be a pair formed by a global action γ of M on $Z \in \mathscr{C}$ and a datum morphism $f : \alpha \to \gamma$. Then the family $\xi = \{F(i) \xrightarrow{\xi_i} Z : i \in I\}$, where*

$$\xi_i = \begin{cases} \gamma_{mn} \circ f \circ \iota_n, & \text{if } i = (m,n) \in M \times M, \\ \gamma_m \circ f, & \text{if } i = m \in M, \end{cases} \tag{5.22}$$

is a natural transformation from F to $\Delta(Z)$.

Proof. We shall verify that for each $m, n \in M$ the diagrams

$$\begin{array}{ccc} \operatorname{dom}\alpha_n & \xrightarrow{\xi_{(m,n)}} & Z \\ {\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle id} \\ X & \xrightarrow{\xi_{mn}} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{dom}\alpha_n & \xrightarrow{\xi_{(m,n)}} & Z \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle id} \\ X & \xrightarrow{\xi_{m}} & Z \end{array} \tag{5.23}$$

commute.

The commutativity of the left diagram of (5.23) follows directly by (5.22).

For the second diagram, by Lemma 4.4.4 we have

$$f \circ \alpha_n = \gamma_n \circ f \circ \iota_n.$$

Thus, since γ is a global action, we have

$$\begin{aligned} \xi_m \circ \alpha_n = (\gamma_m \circ f) \circ \alpha_n &= \gamma_m \circ (f \circ \alpha_n) = \gamma_m \circ (\gamma_n \circ f \circ \iota_n) \\ &= (\gamma_m \circ \gamma_n) \circ f \circ \iota_n = \gamma_{mn} \circ f \circ \iota_n = \xi_{(m,n)}, \end{aligned}$$

giving us the commutativity of the right diagram of (5.23). □

Theorem 5.2.15 ([15, Theorem 4.15])**.** *Let $F \xrightarrow{\eta} \Delta(Y)$ be a colimit of the functor F associated to α and β the global action associated to η. Then $\alpha \xrightarrow{\eta_e} \beta$ is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$.*

Proof. Let (γ, f) be a pair formed by a global action γ of G on $Z \in \mathscr{C}$ and a datum morphism $f : \alpha \to \gamma$. We must show that there exists a unique datum morphism $f' : \beta \to \gamma$

such that the diagram

$$\begin{tikzcd} \alpha \arrow[r, "\eta_e"] \arrow[dr, "f"'] & \beta \arrow[d, dashed, "f'"] \\ & \gamma \end{tikzcd} \tag{5.24}$$

commutes.

Let ξ be the natural transformation (5.22) from F to $\Delta(Z)$ constructed in Lemma 5.2.14.

By the universal property of η, there exists a unique morphism $Y \xrightarrow{f'} Z$ such that

$$\xi_i = f' \circ \eta_i \tag{5.25}$$

for each $i \in I$. Since $\gamma_e = id_Z$, by (5.22) we have $\xi_e = f$, whence $f' \circ \eta_e = f$ by (5.25). So, diagram (5.24) commutes modulo the verification that f' is a datum morphism from β to γ, which we are going to do now. By Lemma 4.4.4, this will be accomplished if we show that for each $m \in M$

$$f' \circ \beta_m = \gamma_m \circ f'. \tag{5.26}$$

To this end, fix $m \in M$ and consider the natural transformation $\xi^m = \{F(i) \xrightarrow{\xi_i^m} Z : i \in I\}$ from F to $\Delta(Z)$, where

$$\xi_i^m = \begin{cases} \xi_{(ms,t)}, & \text{if } i = (s,t) \in M \times M, \\ \xi_{ms}, & \text{if } i = s \in M, \end{cases} \tag{5.27}$$

constructed from ξ as in Lemma 5.2.10.

By the universal property of η, there exists a unique morphism $Y \xrightarrow{\overline{\xi^m}} Z$ such that for each $i \in I$

$$\xi_i^m = \overline{\xi^m} \circ \eta_i. \tag{5.28}$$

Since γ is a global action, for all $s \in M$ we have

$$\gamma_m \circ \gamma_s = \gamma_{ms}. \tag{5.29}$$

Thus, for all $s, t \in M$,

$$\gamma_m \circ f' \circ \eta_s \overset{(5.25)}{=} \gamma_m \circ \xi_s \overset{(5.22)}{=} \gamma_m \circ \gamma_s \circ f \overset{(5.29)}{=} \gamma_{ms} \circ f \overset{(5.22)}{=} \xi_{ms} \overset{(5.27)}{=} \xi_s^m \overset{(5.28)}{=} \overline{\xi^m} \circ \eta_s$$

and

$$\begin{aligned} \gamma_m \circ f' \circ \eta_{(s,t)} &\overset{(5.25)}{=} \gamma_m \circ \xi_{(s,t)} \overset{(5.22)}{=} \gamma_m \circ \gamma_{st} \circ f \circ \iota_t \overset{(5.29)}{=} \gamma_{m(st)} \circ f \circ \iota_t \\ &= \gamma_{(ms)t} \circ f \circ \iota_t \overset{(5.22)}{=} \xi_{(ms,t)} \overset{(5.27)}{=} \xi_{(s,t)}^m \overset{(5.28)}{=} \overline{\xi^m} \circ \eta_{(s,t)}, \end{aligned}$$

so the uniqueness of $\overline{\xi^m}$ in (5.28) gives

$$\gamma_m \circ f' = \overline{\xi^m}. \tag{5.30}$$

On the other hand, for all $s, t \in M$ we have

$$f' \circ \beta_m \circ \eta_s \overset{(5.20)}{=} f' \circ \eta_{ms} \overset{(5.25)}{=} \xi_{ms} \overset{(5.27)}{=} \xi^m_s \overset{(5.28)}{=} \overline{\xi^m} \circ \eta_s$$

and

$$f' \circ \beta_m \circ \eta_{(s,t)} \overset{(5.19)}{=} f' \circ \eta_{(ms,t)} \overset{(5.25)}{=} \xi_{(ms,t)} \overset{(5.27)}{=} \xi^m_{(s,t)} \overset{(5.28)}{=} \overline{\xi^m} \circ \eta_{(s,t)},$$

so the uniqueness of $\overline{\xi^m}$ in (5.28) also gives

$$f' \circ \beta_m = \overline{\xi^m}. \tag{5.31}$$

This way, (5.30) and (5.31) complete the proof of (5.26), so f' is a datum morphism from β to γ, as desired.

Finally, let us check the uniqueness of f' as a datum morphism from β to γ. To do so, let f'' be a datum morphism from β to γ such that

$$f = f'' \circ \eta_e. \tag{5.32}$$

Since f'' is a datum morphism, by Lemma 4.4.4 we have

$$f'' \circ \beta_m = \gamma_m \circ f''. \tag{5.33}$$

Thus, given $s, t \in G$, we have

$$f'' \circ \eta_s \overset{(5.20)}{=} f'' \circ \beta_s \circ \eta_e \overset{(5.33)}{=} \gamma_s \circ f'' \circ \eta_e \overset{(5.32)}{=} \gamma_s \circ f \overset{(5.22)}{=} \xi_s. \tag{5.34}$$

Since η is a natural transformation from F to $\Delta(Y)$, for each $s, t \in M$ we have

$$\eta_{(s,t)} = \eta_{st} \circ \iota_t. \tag{5.35}$$

Hence,

$$f'' \circ \eta_{(s,t)} \overset{(5.35)}{=} f'' \circ \eta_{st} \circ \iota_t \overset{(5.34)}{=} \xi_{st} \circ \iota_t \overset{(5.22)}{=} \gamma_{st} \circ f \circ \iota_t \overset{(5.22)}{=} \xi_{(s,t)}. \tag{5.36}$$

So, by (5.34) and (5.36), $\xi_i = f'' \circ \eta_i$ for all $i \in I$. Since f' is the unique morphism satisfying (5.25), we have $f'' = f'$, as desired. □

Corollary 5.2.16. *Let $\mathscr{C}$ be a cocomplete category. Then α has a strong universal globalization if and only if α has a (not necessarily strong universal) globalization.*

Proof. Since $\mathscr{C}$ is cocomplete, the functor F associated to α has a colimit. In this case, by Theorem 5.2.15, α has a reflection in $M-\mathbf{Act}_{\mathscr{C}}$. Thus, the result follows by Theorem 5.2.5. $\square$

5.2.2 REFLECTION IN TERMS OF COPRODUCTS AND A COEQUALIZER

One particular case of a colimit of the functor F gives us a stronger but more tangible condition for a partial action to have a reflection in $M-\mathbf{Act}_{\mathscr{C}}$, where we assume that certain coproducts and a certain coequalizer exist in $\mathscr{C}$.

Fix a partial action datum $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ of M on $X \in \mathscr{C}$ and assume that the coproducts $\coprod_{m\in M} X$ and $\coprod_{(m,n)\in M\times M} \operatorname{dom}\alpha_n$ exist in $\mathscr{C}$. For each $m, n \in M$, denote the associated inclusion morphisms by

$$u_m : X \to \coprod_{m\in M} X \text{ and } u_{(m,n)} : \operatorname{dom}\alpha_n \to \coprod_{(m,n)\in M\times M} \operatorname{dom}\alpha_n, \tag{5.37}$$

and all the coproducts of morphisms in this subsection will be with respect to one of the two families in (5.37).

Consider the morphisms $p, q : \coprod_{(m,n)\in M\times M} \operatorname{dom}\alpha_n \to \coprod_{m\in M} X$ given as follows:

$$p = \coprod_{(m,n)\in M\times M} (u_{mn} \circ \iota_n) \text{ and } q = \coprod_{(m,n)\in M\times M} (u_m \circ \alpha_n). \tag{5.38}$$

We shall now work towards verifying that a coequalizer of p and q induces a colimit of the functor F associated to α.

Lemma 5.2.17. *Let $Z \in \mathscr{C}$ and $\xi = \{F(i) \xrightarrow{\xi_i} Z : i \in I\}$ be a natural transformation $F \to \Delta(Z)$. Then the coproduct $\coprod_{m\in M} \xi_m : \coprod_{m\in M} X \to Z$ satisfies*

$$\left(\coprod_{m\in M} \xi_m\right) \circ p = \left(\coprod_{m\in M} \xi_m\right) \circ q. \tag{5.39}$$

Proof. To prove (5.39), it suffices to verify that for all $m, n \in M$ we have

$$\left(\coprod_{m\in M} \xi_m\right) \circ p \circ u_{(m,n)} = \left(\coprod_{m\in M} \xi_m\right) \circ q \circ u_{(m,n)}.$$

Denote, for simplicity, $\coprod_{m\in M} \xi_m$ by Ξ. Fix $m, n \in M$. Then we have

$$\Xi \circ (p \circ u_{(m,n)}) = \Xi \circ (u_{mn} \circ \iota_n) = (\Xi \circ u_{mn}) \circ \iota_n = \xi_{mn} \circ \iota_n = \xi_{(m,n)},$$

where the last equality follows from the fact that ξ is a natural transformation (see (5.23)).

Similarly, we have

$$\Xi \circ (q \circ u_{(m,n)}) = \Xi \circ (u_m \circ \alpha_n) = (\Xi \circ u_m) \circ \alpha_n = \xi_m \circ \alpha_n = \xi_{(m,n)}. \qquad \square$$

Proposition 5.2.18. *Assume that there exists a coequalizer $\coprod_{m\in M} X \xrightarrow{c} Y$ of p and q. Then the family $\eta = \{F(i) \xrightarrow{\eta_i} Y : i \in I\}$ such that*

$$\eta_i = \begin{cases} c \circ p \circ u_{(s,t)} = c \circ q \circ u_{(s,t)}, & \text{if } i = (s,t) \in M \times M, \\ c \circ u_s, & \text{if } i = s \in M, \end{cases} \tag{5.40}$$

is a colimit of the functor F associated to α.

Proof. Firstly, note that η is a natural transformation from F to $\Delta(Y)$, since for each $m, n \in G$ the diagrams

$$\begin{tikzcd} \operatorname{dom}\alpha_n \arrow[r, "\eta_{(m,n)}"] \arrow[d, "\iota_n"'] & Y \arrow[d, "id"] \\ X \arrow[r, "\eta_{mn}"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{dom}\alpha_n \arrow[r, "\eta_{(m,n)}"] \arrow[d, "\alpha_n"'] & Y \arrow[d, "id"] \\ X \arrow[r, "\eta_{m}"] & Y \end{tikzcd}$$

commute by (5.38) and (5.40).

Given $Z \in \mathscr{C}$, let $\xi = \{F(i) \xrightarrow{\xi_i} Z : i \in I\}$ be a natural transformation from F to $\Delta(Z)$. Let us show that there exists a unique morphism $\varphi : Y \to Z$ such that

$$\xi = \varphi \circ \eta. \tag{5.41}$$

By Lemma 5.2.17, we have $(\coprod_{m\in M} \xi_m) \circ p = (\coprod_{m\in M} \xi_m) \circ q$, so, by the universal property of c as a coequalizer of p and q, there exists a unique morphism $\varphi : Y \to Z$ such that

$$\coprod_{m\in M} \xi_m = \varphi \circ c. \tag{5.42}$$

Now, φ satisfies (5.41), since for all $m, n \in M$ by (5.40) and (5.42) we have

$$\varphi \circ \eta_m = \varphi \circ (c \circ u_m) = (\varphi \circ c) \circ u_m = \left(\coprod_{m\in M} \xi_m\right) \circ u_m = \xi_m$$

and by (5.38), (5.40) and (5.42) together with the fact that ξ is a natural transformation

$$\begin{aligned} \varphi \circ \eta_{(m,n)} &= \varphi \circ (c \circ p \circ u_{(m,n)}) = (\varphi \circ c) \circ (p \circ u_{(m,n)}) \\ &= \left(\coprod_{m\in M} \xi_m\right) \circ (u_{mn} \circ \iota_n) = \xi_{mn} \circ \iota_n = \xi_{(m,n)}. \end{aligned}$$

It remains to show that φ is the unique morphism satisfying (5.41). For, assume that φ' is a morphism such that $\xi = \varphi' \circ \eta$. Then for each $m \in M$ using (5.40) we have

$$\varphi' \circ c \circ u_m = \varphi' \circ \eta_m = \xi_m,$$

so that $\varphi' \circ c = \coprod_{m \in M} \xi_m$. Since φ is the unique morphism that satisfies (5.42), we conclude that $\varphi' = \varphi$. $\square$

Let $\beta(m) = [Y, id_Y, \beta_m]$ be the global action associated (see Definition 5.2.12) to the colimit η from Proposition 5.2.18. Notice that for each $m \in M$, β_m can be described precisely as the unique morphism such that for all $s, t \in M$

$$\beta_m \circ c \circ u_s = c \circ u_{ms}, \tag{5.43}$$

since, in this case,

$$\beta_m \circ c \circ p \circ u_{s,t} = c \circ p \circ u_{ms,t}$$

automatically follows from (5.43).

As a consequence of Proposition 5.2.18 and Theorem 5.2.15, we get the following.

Corollary 5.2.19 ([15, Corollary 4.20])**.** *Let $\coprod_{m \in M} X \xrightarrow{c} Y$ be a coequalizer of p and q. Then $c \circ u_e : \alpha \to \beta$ is a reflection of α in $M{-}\mathbf{Act}_{\mathscr{C}}$.*

So, in this case, we can work with a universal globalization of α in terms of coproducts and a coequalizer, due to Theorem 5.2.5.

In a final approach to finding conditions for α to have a reflection in $M{-}\mathbf{Act}_{\mathscr{C}}$, we shall define structures of global actions on the coproducts we worked with so far, in order to find necessary and sufficient conditions in terms of a coequalizer in $M{-}\mathbf{Act}_{\mathscr{C}}$.

For each $m \in M$, consider the morphisms

$$\varphi_m = \coprod_{(s,t) \in M \times M} u_{(ms,t)} : \coprod_{(s,t) \in M \times M} \operatorname{dom} \alpha_t \to \coprod_{(s,t) \in M \times M} \operatorname{dom} \alpha_t \tag{5.44}$$

and

$$\psi_m = \coprod_{s \in M} u_{ms} : \coprod_{s \in M} X \to \coprod_{s \in M} X. \tag{5.45}$$

It is a simple verification that the partial action data

$$\varphi(m) = [\textstyle\coprod_{(s,t) \in M \times M} \operatorname{dom} \alpha_t, id, \varphi_m] \text{ and } \psi(m) = [\textstyle\coprod_{s \in M} X, id, \psi_m]$$

are global actions of M on $\coprod_{(m,n) \in M \times M} \operatorname{dom} \alpha_n$ and $\coprod_{m \in M} X$, respectively.

Proposition 5.2.20. *The morphisms $p, q : \coprod_{(m,n) \in M \times M} \operatorname{dom} \alpha_n \to \coprod_{m \in M} X$ in (5.38) are morphisms from φ to ψ in $M{-}\mathbf{Act}_{\mathscr{C}}$.*

Proof. We shall only verify that q is a datum morphism, as the verification for p is analogous. By Lemma 4.4.4, it suffices to show that

$$q \circ \varphi_m = \psi_m \circ q,$$

for each $m \in M$. The latter is equivalent to

$$q \circ \varphi_m \circ u_{(s,t)} = \psi_m \circ q \circ u_{(s,t)} \text{ for each } (s,t) \in M \times M.$$

Indeed, fixed $m \in M$, by (5.38), (5.44) and (5.45), for all $s, t \in M$ we have

$$\begin{aligned} q \circ \varphi_m \circ u_{(s,t)} &= q \circ u_{(ms,t)} = u_{ms} \circ \alpha_t = (\psi_m \circ u_s) \circ \alpha_t \\ &= \psi_m \circ (u_s \circ \alpha_t) = \psi_m \circ q \circ u_{(s,t)}. \end{aligned}$$

□

Lemma 5.2.21. *Let γ be a global action of M on $Z \in \mathscr{C}$ and $X \xrightarrow{f} Z$ a morphism in $\mathscr{C}$. Then $\coprod_{m \in M} (\gamma_m \circ f)$ is a morphism from ψ to γ in $M\!-\!\mathbf{Act}_{\mathscr{C}}$.*

Proof. For the simplicity of notation, let $\Gamma = \coprod_{m \in M} (\gamma_m \circ f)$. Then for each $s \in M$ we have

$$\Gamma \circ u_s = \gamma_s \circ f.$$

Therefore, given $m, s \in M$, by (5.45) and the fact that γ is a global action we have

$$\begin{aligned} (\Gamma \circ \psi_m) \circ u_s &= \Gamma \circ (\psi_m \circ u_s) = \Gamma \circ u_{ms} = \gamma_{ms} \circ f = (\gamma_m \circ \gamma_s) \circ f \\ &= \gamma_m \circ (\gamma_s \circ f) = \gamma_m \circ (\Gamma \circ u_s) = (\gamma_m \circ \Gamma) \circ u_s, \end{aligned}$$

so that $\Gamma \circ \psi_m = \gamma_m \circ \Gamma$.

Thus, by Lemma 4.4.4, Γ is a datum morphism from ψ to γ, as desired. □

In view of Lemma 5.2.21 we can define the following map.

Definition 5.2.22. Let γ be a global action of M on $Z \in \mathscr{C}$. Define $H_{X,\gamma} : \mathrm{Hom}_{\mathscr{C}}(X, Z) \to \mathrm{Hom}_{M-\mathbf{Act}_{\mathscr{C}}}(\psi, \gamma)$ by

$$H_{X,\gamma}(f) = \coprod_{m \in M} (\gamma_m \circ f), \tag{5.46}$$

for any $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$.

Proposition 5.2.23. *Let γ be a global action of M on $Z \in \mathscr{C}$. Then $H_{X,\gamma}$ given by* (5.46) *is a bijection whose inverse is*

$$\mathrm{Hom}_{M-\mathbf{Act}_{\mathscr{C}}}(\psi, \gamma) \ni \Gamma \mapsto \Gamma \circ u_e \in \mathrm{Hom}_{\mathscr{C}}(X, Z). \tag{5.47}$$

Proof. To simplify the notation, denote $H_{X,\gamma}$ by H. Let us verify that the map G defined in (5.47) is the inverse of H.

Let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$. Then by (5.46) and (5.47)

$$G(H(f)) = \coprod_{m \in M} (\gamma_m \circ f) \circ u_e = \gamma_e \circ f = f.$$

On the other hand, let $\Gamma \in \mathrm{Hom}_{M-\mathbf{Act}_{\mathscr{C}}}(\psi, \gamma)$. We have by (5.46) and (5.47)

$$H(G(\Gamma)) = \coprod_{m \in M} (\gamma_m \circ (\Gamma \circ u_e)). \tag{5.48}$$

As Γ is a datum morphism, for each $m \in M$ we have $\Gamma \circ \psi_m = \gamma_m \circ \Gamma$, so, by (5.45) and (5.48),

$$\Gamma \circ u_m = \Gamma \circ \psi_m \circ u_e = \gamma_m \circ \Gamma \circ u_e = H(G(\Gamma)) \circ u_m.$$

Thus, $\Gamma = H(G(\Gamma))$. □

Lemma 5.2.24. *Let β and γ be global actions of M on $Y \in \mathscr{C}$ and $Z \in \mathscr{C}$, respectively, $g \in \mathrm{Hom}_{M-\mathbf{Act}_{\mathscr{C}}}(\beta, \gamma)$ and $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Then*

$$H_{X,\gamma}(g \circ f) = g \circ H_{X,\beta}(f). \tag{5.49}$$

Proof. Let $m \in M$. By Lemma 4.4.4 we have $\gamma_m \circ g = g \circ \beta_m$, so by (5.46)

$$H_{X,\gamma}(g \circ f) \circ u_m = \gamma_m \circ g \circ f = g \circ \beta_m \circ f = g \circ H_{X,\beta}(f) \circ u_m,$$

whence (5.49). □

Lemma 5.2.25. *Let γ be a global action of M on $Z \in \mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$. Then f is a datum morphism from α to γ if and only if*

$$H_{X,\gamma}(f) \circ p = H_{X,\gamma}(f) \circ q. \tag{5.50}$$

Proof. Write $H = H_{X,\gamma}$ for short. Assume that f is a datum morphism from α to γ. For all $m \in M$, by Lemma 4.4.4 we have

$$\gamma_m \circ f \circ \iota_m = f \circ \alpha_m, \tag{5.51}$$

and by (5.46) we have

$$H(f) \circ u_m = \gamma_m \circ f. \tag{5.52}$$

By (5.38) and (5.52) we get

$$H(f) \circ q \circ u_{(s,t)} = H(f) \circ u_s \circ \alpha_t = \gamma_s \circ f \circ \alpha_t. \tag{5.53}$$

Therefore, by (5.38) and (5.51)–(5.53), and since γ is a global action, for each $(s,t) \in M \times M$,

$$
\begin{aligned}
H(f) \circ q \circ u_{(s,t)} &\overset{(5.53)}{=} \gamma_s \circ f \circ \alpha_t = \gamma_s \circ (f \circ \alpha_t) \overset{(5.51)}{=} \gamma_s \circ (\gamma_t \circ f \circ \iota_t) \\
&= (\gamma_s \circ \gamma_t) \circ f \circ \iota_t = \gamma_{st} \circ f \circ \iota_t = (\gamma_{st} \circ f) \circ \iota_t \\
&\overset{(5.52)}{=} (H(f) \circ u_{st}) \circ \iota_t = H(f) \circ (u_{st} \circ \iota_t) \overset{(5.38)}{=} H(f) \circ p \circ u_{(s,t)},
\end{aligned}
$$

whence (5.50).

Conversely, assume that $\Gamma \coloneqq H(f)$ satisfies

$$\Gamma \circ p = \Gamma \circ q. \tag{5.54}$$

By Proposition 5.2.23,

$$f = H(f) \circ u_e = \Gamma \circ u_e. \tag{5.55}$$

It follows from (5.38), (5.52), (5.54) and (5.55) that for all $m \in M$

$$
\begin{aligned}
(\gamma_m \circ f) \circ \iota_m &\overset{(5.52)}{=} (\Gamma \circ u_m) \circ \iota_m = \Gamma \circ (u_m \circ \iota_m) \overset{(5.38)}{=} \Gamma \circ (p \circ u_{(e,m)}) \\
&= (\Gamma \circ p) \circ u_{(e,m)} \overset{(5.54)}{=} (\Gamma \circ q) \circ u_{(e,m)} = \Gamma \circ (q \circ u_{(e,m)}) \\
&\overset{(5.38)}{=} \Gamma \circ (u_e \circ \alpha_m) = (\Gamma \circ u_e) \circ \alpha_m \overset{(5.55)}{=} f \circ \alpha_m,
\end{aligned}
$$

and, thus, by Lemma 4.4.4, f is a datum morphism from α to γ. □

Theorem 5.2.26 ([15, Theorem 4.27])**.** *The following statements hold:*

(1) If $\alpha \xrightarrow{r} \beta$ is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$, then $H_{X,\beta}(r)$ is a coequalizer of p and q in $M-\mathbf{Act}_{\mathscr{C}}$.

(2) If $\psi \xrightarrow{c} \beta$ is a coequalizer of p and q in $M-\mathbf{Act}_{\mathscr{C}}$, then $c \circ u_e$ is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$.

In particular, α has a reflection in $M-\mathbf{Act}_{\mathscr{C}}$ if and only if p and q have a coequalizer in $M-\mathbf{Act}_{\mathscr{C}}$.

Proof. (1) Assume $\alpha \xrightarrow{r} \beta$ is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$. Let us check that $H_{X,\beta}(r)$ is a coequalizer of p and q in $M-\mathbf{Act}_{\mathscr{C}}$.

By Lemmas 5.2.21 and 5.2.25, $H_{X,\beta}(r)$ is a datum morphism from ψ to β such that $H_{X,\beta}(r) \circ p = H_{X,\beta}(r) \circ q$.

Let $\psi \xrightarrow{f} \gamma$ be a datum morphism such that $f \circ p = f \circ q$. We must show that there exists a unique datum morphism $f' : \beta \to \gamma$ such that

$$f = f' \circ H_{X,\beta}(r). \tag{5.56}$$

By Lemma 5.2.25, $H^{-1}_{X,\gamma}(f) = f \circ u_e$ is a datum morphism from α to γ. Since r is a reflection of α in $M{-}\mathbf{Act}_{\mathscr{C}}$, there exists a unique datum morphism $f' : \beta \to \gamma$ such that

$$f' \circ r = f \circ u_e. \tag{5.57}$$

This way, by Proposition 5.2.23, (5.57), and Lemma 5.2.24 we have

$$f = H_{X,\gamma}(f \circ u_e) = H_{X,\gamma}(f' \circ r) = f' \circ H_{X,\beta}(r).$$

Note that f' is the unique datum morphism satisfying (5.56). Indeed, if f'' is a datum morphism from β to γ such that $f'' \circ H_{X,\beta}(r) = f$, then by Proposition 5.2.23

$$f'' \circ r = f'' \circ (H_{X,\beta}(r) \circ u_e) = (f'' \circ H_{X,\beta}(r)) \circ u_e = f \circ u_e,$$

and by the uniqueness of f' in (5.57), we have $f'' = f'$. Thus, $H_{X,\beta}(r)$ is a coequalizer of p and q in $M{-}\mathbf{Act}_{\mathscr{C}}$.

(2) Let c be a coequalizer of p and q in $M{-}\mathbf{Act}_{\mathscr{C}}$. We shall verify that $c \circ u_e$ is a reflection of α in $M{-}\mathbf{Act}_{\mathscr{C}}$.

By Lemma 5.2.25, $H^{-1}(c) = c \circ u_e$ is a datum morphism from α to β, since c is a datum morphism such that $c \circ p = c \circ q$.

So, let $\alpha \xrightarrow{f} \gamma$ be a datum morphism. We must show that there exists a unique datum morphism f' from β to γ such that

$$f = f' \circ (c \circ u_e). \tag{5.58}$$

Lemma 5.2.25 tells us that $H_{X,\gamma}(f)$ is a datum morphism from ψ to γ such that $H_{X,\gamma}(f) \circ p = H_{X,\gamma}(f) \circ q$. So, since c is a coequalizer of p and q, there exists a unique datum morphism f' from β to γ such that

$$H_{X,\gamma}(f) = f' \circ c. \tag{5.59}$$

This morphism is such that

$$f = H_{X,\gamma}(f) \circ u_e = (f' \circ c) \circ u_e = f' \circ (c \circ u_e).$$

Moreover, f' is the unique datum morphism satisfying (5.58). Indeed, if there were f'' with $f = f'' \circ (c \circ u_e)$, then by Lemma 5.2.24 and Proposition 5.2.23

$$H_{X,\gamma}(f) = H_{X,\gamma}(f'' \circ (c \circ u_e)) = f'' \circ H_{X,\beta}(c \circ u_e) = f'' \circ c,$$

and, due to the uniqueness of f' in (5.59), we would have $f'' = f'$. □

Observe that if a coequalizer of p and q exists in $\mathscr{C}$, it is also a coequalizer in $M-\mathbf{Act}_{\mathscr{C}}$.

Proposition 5.2.27. *Let* $\coprod_{m\in M} X \xrightarrow{c} Y$ *be a coequalizer of* p *and* q *in* $\mathscr{C}$ *and* β *the global action of* M *on* Y *satisfying* (5.43)*. Then* c *is a datum morphism from* ψ *to* β *that is a coequalizer of* p *and* q *in* $M-\mathbf{Act}_{\mathscr{C}}$*.*

Proof. We will first verify that c is a morphism from ψ to β. By Corollary 4.4.5 it suffices to verify that $c \circ \psi_m = \beta_m \circ c$ for all $m \in M$.

Let $s \in M$. Then, by (5.43),

$$\beta_m \circ c \circ u_s = c \circ u_{ms} = c \circ \psi_m \circ u_s$$

for all $s \in M$. Hence, it follows that

$$\beta_m \circ c = c \circ \psi_m,$$

as desired.

Now let us verify that $c : \psi \to \beta$ is a coequalizer of p and q in $M-\mathbf{Act}_{\mathscr{C}}$. Let $\psi \xrightarrow{f} \gamma$ be a morphism in $M-\mathbf{Act}_{\mathscr{C}}$ such that

$$f \circ p = f \circ q. \tag{5.60}$$

The global action γ acts on, say, an object $Z \in \mathscr{C}$. Since f is a morphism in $\mathscr{C}$ satisfying (5.60) and c is a coequalizer of p and q, there exists a unique morphism $f' : Y \to Z$ such that

$$f' \circ c = f. \tag{5.61}$$

Let us verify that f' is a morphism from β to γ by Corollary 4.4.5. For let $m \in M$. Then, by Corollary 4.4.5 and (5.61),

$$\gamma_m \circ f' \circ c = \gamma_m \circ f = f \circ \psi_m = f' \circ c \circ \psi_m = f' \circ \beta_m \circ c.$$

Since c is a coequalizer in $\mathscr{C}$, it is an epimorphism in $\mathscr{C}$. Hence, it follows that

$$\gamma_m \circ f' = f' \circ \beta_m,$$

as desired.

Clearly, f' is the unique morphism from ψ to γ satisfying (5.61), because it is the unique morphism in $\mathscr{C}$ satisfying such equation. Hence, c is a coequalizer of p and q in $M-\mathbf{Act}_{\mathscr{C}}$, as desired. $\square$

We will now verify a certain property that a reflection coming from a coequalizer satisfies.

Lemma 5.2.28. *Let (β, ι) be a universal globalization of α and assume that p and q have a coequalizer in $\mathscr{C}$. Then $\coprod_{m\in M}(\beta_m \circ \iota)$ is a coequalizer of p and q in $\mathscr{C}$.*

Proof. By Corollary 5.2.19, since p and q have a coequalizer c in $\mathscr{C}$, α has a reflection in $M-\mathbf{Act}_{\mathscr{C}}$. Thus, by Corollary 5.2.8, it follows that $\iota : \alpha \to \beta$ is a reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$.

Therefore, by Theorem 5.2.26 (1), $H_{X,\beta}(\iota) = \coprod_{m\in M}(\beta_m \circ \iota)$ is a coequalizer of p and q in $M-\mathbf{Act}_{\mathscr{C}}$, as desired. □

Theorem 5.2.29. *Let (β, ι) be a universal globalization of α, where β acts on an object $Y \in \mathscr{C}$, and $v : Y' \to Y$ a monomorphism, and assume that p and q have a coequalizer in $\mathscr{C}$. If $\beta_m \circ \iota$ factors through v for all $m \in M$, then v is an isomorphism.*

Proof. For each $m \in M$, let $v_m : X \to Y'$ be the morphism such that

$$v \circ v_m = \beta_m \circ \iota.$$

Consider then the morphism $\coprod_{m\in M} v_m : \coprod_{m\in M} X \to Y'$. Observe that

$$\coprod_{m\in M}(\beta_m \circ \iota) = v \circ \coprod_{m\in M} v_m, \tag{5.62}$$

since for each $m \in M$ we have

$$\left(\coprod_{m\in M} \beta_m \circ \iota\right) \circ u_m = \beta_m \circ \iota = v \circ v_m = v \circ (\left(\coprod_{m\in M} v_m\right) \circ u_m) = \left(v \circ \coprod_{m\in M} v_m\right) \circ u_m.$$

By Lemma 5.2.28, $\coprod_{m\in M}(\beta_m \circ \iota)$ is a coequalizer of p and q in $\mathscr{C}$. Hence, since v is a monomorphism, by Proposition 2.2.25 and (5.62) it is an isomorphism, as desired. □

Theorem 5.2.29 tells us that the category-theoretic union of the family $\{\beta_m \circ \iota\}_{m\in M}$ of subobjects of Y is the subobject id_Y.

6 RESULTS AND EXAMPLES IN CERTAIN CATEGORIES

In this chapter we apply the results obtained in Chapter 5 to certain categories.

We verify in Section 6.1 that, in **Set**, Corollary 5.2.19 and Theorem 5.2.5 recover Hollings's results on the globalization of strong partial actions on sets.

In Section 6.2 we study the partial actions on objects in **Top** and classify the globalizable ones in Proposition 6.2.4.

Finally, in Section 6.3 we consider the partial actions of groups on algebras in the sense of [7] and in Proposition 6.3.9 we observe that, in the unital case, the enveloping action of such a partial action α is a universal globalization of α, seen as a partial action on an object in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$.

6.1 THE CATEGORY OF SETS

Fix $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ a partial action datum of M on $X \in \mathbf{Set}$, where $\operatorname{dom}\alpha_m \subseteq X$ and ι_m is the corresponding inclusion map (recall that every partial morphism in **Set** has such a representative, by Proposition 3.1.16).

Denote by $\approx$ the equivalence relation on $M \times X$ generated by $\sim$, where

$$(m, x) \sim (n, y) \iff \exists m' \in M \text{ such that } m = nm', x \in \operatorname{dom}\alpha_{m'} \text{ and } y = \alpha_{m'}(x). \quad (6.1)$$

Let $Y = (M \times X)/{\approx}$ and denote by $[m, x]$ the $\approx$-equivalence class of (m, x).

Lemma 6.1.1. *The maps $\beta_n : Y \to Y$ given by*

$$\beta_n([m, x]) = [nm, x] \quad (6.2)$$

define a global action β of M on Y and the map $\iota : X \to Y$ given by

$$\iota(x) = [e, x]$$

is a reflection $\alpha \to \beta$ of α in $M{-}\mathbf{Act}_{\mathbf{Set}}$.

Proof. Consider the coproduct $\coprod_{m\in M} X = M \times X$ with inclusions $u_m : X \ni x \mapsto (m, x) \in M \times X$, and the coproduct $\coprod_{(m,n)\in M\times M} \operatorname{dom}\alpha_n = \{(m, n, x) : x \in \operatorname{dom}\alpha_n\} =: M^2 \bullet X$ with inclusions $u_{(m,n)} : \operatorname{dom}\alpha_n \ni x \mapsto (m, n, x) \in M^2 \bullet X$. Then the maps $p, q : M^2 \bullet X \to M \times X$

from (5.38) are given by

$$p(m,n,x) = (mn, \iota_n(x)) = (mn, x)$$

and

$$q(m,n,x) = (m, \alpha_n(x)).$$

The canonical projection c of $M \times X$ onto its quotient by the equivalence relation generated by $\sim = \{(p(m,n,x), q(m,n,x)) : (m,n,x) \in M^2 \bullet X\}$ is a coequalizer of p and q. It is a simple verification that $\sim$ coincides with (6.1), so c is precisely the natural projection of $M \times X$ onto Y.

Then the global action β of M on Y from Corollary 5.2.19 (see (5.43)) is given precisely by (6.2), with $\iota = c \circ u_e : \alpha \to \beta$ being a reflection of α in $M{-}\mathbf{Act}_{\mathbf{Set}}$. □

Lemma 6.1.2. *Let ι be the reflection of α in $M{-}\mathbf{Act}_{\mathbf{Set}}$ as in Lemma 6.1.1. If $\alpha \in M{-}\mathbf{spAct}_{\mathbf{Set}}$, then diagram (5.14) is a pullback for all $m \in M$.*

Proof. Let $m \in M$. Since ι is a datum morphism, by Lemma 4.4.4 diagram (5.14) commutes. Consider the pullback square

$$\begin{tikzcd}
 & Z \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\vee"] & \\
X \arrow[dr, "\beta_m \circ \iota"'] & {} & X \arrow[dl, "\iota"] \\
 & Y &
\end{tikzcd}$$

where $Z = \{(x,y) \in X \times X : \beta_m(\iota(x)) = \iota(y)\} = \{(x,y) \in X \times X : [m,x] = [e,y]\}$ and p_1 and p_2 are the corresponding projections. There exists $\varphi : \operatorname{dom} \alpha_m \to Z$, $\varphi(x) = (x, \alpha_m(x))$ such that $\varphi \circ p_1 = \iota_m$ and $\varphi \circ p_2 = \alpha_m$. We are going to show that φ is a bijection. It is clearly an injective map, so we will verify that it is surjective.

Observe that since $\alpha \in M{-}\mathbf{spAct}_{\mathbf{Set}}$, (6.1) implies that

$$\text{if } (m,x) \sim (n,y), \text{ then } x \in \operatorname{dom} \alpha_m \iff y \in \operatorname{dom} \alpha_n, \tag{6.3}$$

and, in this case, $\alpha_m(x) = \alpha_n(y)$. Indeed, by Corollary 4.3.15 the partial action datum $\{\alpha_m\}_{m \in M}$ satisfies (PA2') and (PA3), so we have

$$x \in \alpha_{m'}^{-1}(\operatorname{dom} \alpha_n) = \operatorname{dom} \alpha_{m'} \cap \operatorname{dom} \alpha_{nm'} = \operatorname{dom} \alpha_{m'} \cap \operatorname{dom} \alpha_m \subseteq \operatorname{dom} \alpha_m$$

and

$$\alpha_m(x) = \alpha_{nm'}(x) = \alpha_n(\alpha_{m'}(x)) = \alpha_n(y).$$

Now, let $(x,y) \in Z$. Then $(m,x) \approx (e,y)$, so, since $\approx$ is the smallest equivalence relation containing $\sim$, there exists a sequence $(m,x) = (m_1,x_1), \ldots, (m_k,x_k) = (e,y)$ such that either

$$(m_i, x_i) \sim (m_{i+1}, x_{i+1}) \text{ or } (m_{i+1}, x_{i+1}) \sim (m_i, x_i) \tag{6.4}$$

for all $i \in \{1, \ldots, k-1\}$. Since $y \in \operatorname{dom}\alpha_e$ (because $\operatorname{dom}\alpha_e = X$, by (PA1)), in either of the cases of (6.4), by (6.3) we have $x_{k-1} \in \operatorname{dom}\alpha_{k-1}$ and $\alpha_{m_{k-1}}(x_{k-1}) = \alpha_e(y) = y$. Recursively, we have for all i that $x_i \in \operatorname{dom}\alpha_{m_i}$ and $\alpha_{m_i}(x_i) = \alpha_{m_{i+1}}(x_{i+1}) = \cdots = \alpha_{m_{k-1}}(x_{k-1}) = y$. In particular, taking $i = 1$ we get $x \in \operatorname{dom}\alpha_m$ and $\alpha_m(x) = y$.

Hence, $(x, y) = (x, \alpha_m(x)) = \varphi(x)$, and it follows that φ is surjective. Thus, φ is a bijection and (5.14) is a pullback, as desired. □

Proposition 6.1.3. *A partial action datum α in $M-\mathbf{Datum}_{\mathbf{Set}}$ has a universal globalization if and only if $\alpha \in M-\mathbf{spAct}_{\mathbf{Set}}$.*

Proof. If α has a universal globalization, then $\alpha \in M-\mathbf{spAct}_{\mathbf{Set}}$ by Proposition 5.1.7.

Conversely, by Lemma 6.1.1 α has a reflection in $M-\mathbf{Act}_{\mathscr{C}}$, which, since $\alpha \in M-\mathbf{spAct}_{\mathbf{Set}}$, by Lemma 6.1.2, is such that (5.14) is a pullback for all $m \in M$. Thus, by Theorem 5.2.5 α has a universal globalization. □

6.2 THE CATEGORY OF TOPOLOGICAL SPACES

The coequalizers in **Top** are, those from **Set** equipped with a suitable topology.

Fix $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ a partial action datum of M on $X \in \mathbf{Top}$, where $\operatorname{dom}\alpha_m \subseteq X$ and ι_m is the corresponding inclusion map.

Consider the topological space $M \times X$ with the product topology, where M has the discrete topology. Denote by $\approx$ the equivalence relation on $M \times X$ generated by $\sim$, where

$$(m, x) \sim (n, y) \iff \exists m' \in M \text{ such that } m = nm', x \in \operatorname{dom}\alpha_{m'} \text{ and } y = \alpha_{m'}(x). \tag{6.5}$$

Let $Y = (M \times X)/{\approx}$ with the quotient topology and denote by $[m, x]$ the $\approx$-equivalence class of (m, x).

Lemma 6.2.1. *The maps $\beta_n : Y \to Y$ given by*

$$\beta_n([m, x]) = [nm, x] \tag{6.6}$$

define a global action β of M on Y and the map $\iota : X \to Y$ given by

$$\iota(x) = [e, x]$$

is a reflection $\alpha \to \beta$ of α in $M-\mathbf{Act}_{\mathbf{Top}}$.

Proof. Consider the coproduct $\coprod_{m\in M} X = M \times X$ with inclusions $u_m : X \ni x \mapsto (m, x) \in M \times X$, and the coproduct $\coprod_{(m,n)\in M\times M} \operatorname{dom}\alpha_n = \{(m, n, x) : x \in \operatorname{dom}\alpha_n\} =: M^2 \bullet X$ with inclusions $u_{(m,n)} : \operatorname{dom}\alpha_n \ni x \mapsto (m, n, x) \in M^2 \bullet X$. Then the maps $p, q : M^2 \bullet X \to M \times X$ from (5.38) are given by

$$p(m, n, x) = (mn, \iota_n(x)) = (mn, x)$$

and

$$q(m, n, x) = (m, \alpha_n(x)).$$

The canonical projection c of $M \times X$ onto its quotient by the equivalence relation generated by $\sim = \{(p(m, n, x), q(m, n, x)) : (m, n, x) \in M^2 \bullet X\}$ is a coequalizer of p and q. It is a simple verification that $\sim$ coincides with (6.5), so c is precisely the natural projection of $M \times X$ onto Y.

Then the global action β of M on Y from Corollary 5.2.19 (see (5.43)) is given precisely by (6.6), while the map ι is precisely the morphism $c \circ u_e : \alpha \to \beta$ in Corollary 5.2.19, which is a reflection of α in $M-\mathbf{Act}_{\mathbf{Top}}$. □

The following lemma characterizes the pullbacks in **Top**, which will be helpful for the description of the (universally) globalizable partial actions on objects of **Top**.

Lemma 6.2.2. *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in* **Top**. *Then a diagram*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \tag{6.7}$$

in **Top** *is a pullback if and only if it is a pullback diagram in* **Set** *and the topology τ of P is the smallest topology on P such that p_1 and p_2 are continuous maps.*

Proof. Assume that (6.7) is a pullback in **Top**. Then it is a pullback diagram in **Set** because the forgetful functor from **Top** to **Set** preserves pullbacks.

Let us verify that τ is the smallest topology on P such that p_1 and p_2 are continuous. Let τ' be any such topology on P. Then the diagram

$$\begin{tikzcd} & (P, \tau') \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

is a commutative diagram in **Top**. Since (6.7) is a pullback in **Top**, there exists a unique continuous map $\varphi : (P, \tau') \to (P, \tau)$ such that the diagram

$$\begin{tikzcd}
 & (P,\tau') \arrow[d, "\varphi"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] & \\
 & (P,\tau) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & Z &
\end{tikzcd}$$

commutes. Since (6.7) is a pullback in **Set**, the map φ can easily be verified to be equal to id_P. Hence, id_P is a continuous map from (P, τ') to (P, τ). Thus, τ' contains τ, and, so, τ is a the smallest topology on P such that p_1 and p_2 are continuous, as desired.

Now assume that (6.7) is a pullback diagram in **Set** and τ is the smallest topology on P such that p_1 and p_2 are continuous. Let Q be a topological space and q_1 and q_2 be continuous maps such that the diagram

$$\begin{tikzcd}
 & Q \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & Z &
\end{tikzcd} \tag{6.8}$$

commutes.

Since (6.7) is a pullback in **Set**, there exists a unique map $\varphi : Q \to P$ such that diagram

$$\begin{tikzcd}
 & Q \arrow[d, "\varphi"] \arrow[ddl, bend right, "q_1"'] \arrow[ddr, bend left, "q_2"] & \\
 & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & Z &
\end{tikzcd} \tag{6.9}$$

commutes in **Set**. Let us verify that φ is a continuous.

By hypothesis, the topology τ is generated by sets of the form $p_1^{-1}(U)$, for some open $U \subseteq X$ and $p_2^{-1}(V)$ for some open $V \subseteq Y$. Therefore, it suffices to verify that the inverse image of each of those sets by φ is open in Q. Indeed, given an open $U \subseteq X$,

$$\varphi^{-1}(p_1^{-1}(U)) = (p_1 \circ \varphi)^{-1}(U) = q_1^{-1}(U),$$

which is open in Q, since q_1 is continuous. Similarly, $\varphi^{-1}(p_2^{-1}(V))$ is open in Q for all open $V \subseteq Y$.

Hence, φ is a morphism in **Top** such that diagram (6.9) commutes, and it is unique as mentioned above. Therefore, (6.7) is a pullback in **Top**, as desired. □

Lemma 6.2.3. *Let ι be the reflection of α in $M-\mathbf{Act}_{\mathbf{Top}}$ as in Lemma 6.2.1. If $\alpha \in M-\mathbf{spAct}_{\mathbf{Set}}$ and the topology on $\operatorname{dom}\alpha_m$ is the smallest topology such that ι_m and α_m are continuous maps for all $m \in M$, then diagram* (5.14) *is a pullback for all $m \in M$.*

Proof. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ preserves pullbacks. Thus, by Proposition 4.3.13 (1), we have that $U(\alpha)$ is a strong partial action of M on X in **Set**.

By applying the forgetful functor U to (5.14) we obtain a diagram in **Set**, which is a pullback by Lemma 6.1.2.

Therefore, given $m \in M$, since $\operatorname{dom}\alpha_m$ has the smallest topology such that ι_m and α_m are continuous maps, (5.14) is a pullback by Lemma 6.2.2. □

Proposition 6.2.4. *A partial action datum α in $M-\mathbf{Datum}_{\mathbf{Set}}$ has a universal globalization if and only if $\alpha \in M-\mathbf{spAct}_{\mathbf{Set}}$ and the topology on $\operatorname{dom}\alpha_m$ is the smallest topology such that ι_m and α_m are continuous maps for all $m \in M$.*

Proof. If α has a universal globalization, then $\alpha \in M-\mathbf{spAct}_{\mathbf{Set}}$ by Proposition 5.1.7. And, by Theorem 5.2.5, for all $m \in M$ diagram (5.14) is a pullback, so the topology on $\operatorname{dom}\alpha_m$ is the smallest topology such that ι_m and α_m are continuous maps, by Lemma 6.2.2.

Conversely, let ι be the reflection of α in $M-\mathbf{Act}_{\mathscr{C}}$ given by Lemma 6.2.1. Then, since the topology on $\operatorname{dom}\alpha_m$ is the smallest topology such that ι_m and α_m are continuous maps for all $m \in M$, by Lemma 6.2.3 we have that diagram (5.14) is a pullback for all $m \in M$. Thus, by Theorem 5.2.5 α has a universal globalization. □

Definition 6.2.5. An **embedding** of a topological space X into a topological space Y is an injective continuous map $f : X \to Y$ that is a homeomorphism onto its image.

Corollary 6.2.6. *Let $\alpha(m) = [\operatorname{dom}\alpha_m, \iota_m, \alpha_m]$ be a strong partial action of M on $X \in \mathbf{Top}$, where $\operatorname{dom}\alpha_m \subseteq X$ and ι_m is the corresponding inclusion map. If ι_m is an embedding for all $m \in M$, then α has a universal globalization.*

Proof. Let $m \in M$. Since ι_m is an embedding, $\operatorname{dom}\alpha_m$ has the smallest topology such that ι_m is a continuous map. Hence, $\operatorname{dom}\alpha_m$ has the smallest topology such that ι_m and α_m are continuous maps.

Therefore, α has a universal globalization by Proposition 6.2.4. □

The following example illustrates that the converse of Corollary 6.2.6 does not hold.

Example 6.2.7. Let $M = \mathbb{Z}_2 = \{0, 1\}$ and $X = \{x, y\}$ with the topology whose only non-trivial open set is $\{x\}$. Consider the partial action datum α of M on X in **Top**, where

$$\alpha(0) = [X, id_X, id_X] \text{ and } \alpha(1) = [X^{\text{disc}}, id_X, f],$$

where X^{disc} is the set X with the discrete topology and $f : X^{\text{disc}} \to X$ is given by $f(x) = y$ and $f(y) = x$.

Then α is a strong partial action where the topology on $\operatorname{dom}\alpha_m$ is the smallest topology such that ι_m and α_m are continuous maps for all $m \in M$, so, by Proposition 6.2.4, it has a universal globalization.

However, $\iota_1 : X^{\text{disc}} \to X$ is not an embedding.

Example 6.2.8. Let α as in Example 6.2.7, and let X^{indisc} be the set X with the indiscrete topology. Consider β the global action of M on X^{indisc} where

$$\beta_1(x) = y \quad \text{and} \quad \beta_1(y) = x, \tag{6.10}$$

and

$$\iota = id_X : X \to X^{\text{indisc}}. \tag{6.11}$$

Then (β, ι) is a universal globalization of α.

Indeed, consider the coproduct $\coprod_{m \in M} X = M \times X$ (with the discrete topology on M). Then the equivalence relation (6.5) on $M \times X$ has exactly the following non-trivial relations.

$$(0, x) \sim (1, y), \quad (1, y) \sim (0, x), \quad (0, y) \sim (1, x) \quad \text{and} \quad (1, x) \sim (0, y).$$

Hence, $Y = (M \times X)/\approx\, = \{[0, x], [0, y]\} \cong \{x, y\}$ with the indiscrete topology. That is, $Y \cong X^{\text{indisc}}$. In this situation, the global action β and map ι from Lemma 6.2.1 are, up to the homeomorphism $Y \cong X^{\text{indisc}}$, given by (6.10) and (6.11).

The following is an example of a strong partial action that does not have a universal globalization.

Example 6.2.9. Let $M = \mathbb{Z}_2 = \{0, 1\}$ and $X = \{x, y\}$ with the topology whose only non-trivial open set is $\{x\}$. Consider the partial action datum α of M on X in **Top**, where

$$\alpha(0) = [X, id_X, id_X] \text{ and } \alpha(1) = [X^{\text{disc}}, id_X, id_X],$$

where X^{disc} is the set X with the discrete topology.

Then α is a strong partial action of M on X. However, the discrete topology is not the smallest topology on X that makes $id_X : X \to X$ continuous. Hence, by Proposition 6.2.4 the strong partial action α doesn't have a universal globalization.

6.3 THE CATEGORIES OF ASSOCIATIVE ALGEBRAS

Throughout this section, let $\mathbb{K}$ be a field and G a group with identity e. Every algebra in this section is assumed to be an associative and not necessarily unital $\mathbb{K}$-algebra.

The concept of a partial action of G on an algebra, as defined by Dokuchaev and Exel in [7], is the following.

Definition 6.3.1. A partial action of G on an algebra A is a partial action $\{\alpha_g\}_{g\in G}$ of G on the underlying set of A where $\operatorname{dom}\alpha_g$ is an ideal of A and α_g is a homomorphism of algebras for each $g \in G$.

Observe that, by Corollary 4.3.15 and Proposition 4.1.19, if $\{\alpha_g\}_{g\in G}$ is a partial action of G on an algebra A, then the partial action datum $\alpha(g) = [\operatorname{dom}\alpha_g, \iota_g, \alpha_g]$ of the monoid G on the object $A \in \mathbf{Alg}_{\mathbb{K}}$, where ι_g is the inclusion map of $\operatorname{dom}\alpha_g$ into A for all $g \in G$, is a strong partial action of G on the object A in $\mathbf{Alg}_{\mathbb{K}}$.

At times we will interchange the notation a partial action $\{\alpha_g\}_{g\in G}$ in the sense of Definition 6.3.1 with its corresponding strong partial action α in the sense of Definition 4.3.6.

We distinguish the two concepts of partial actions by saying that one is *classical*, while the other is *categorical*.

However, given a strong partial action α of a group on an object X in $\mathbf{Alg}_{\mathbb{K}}$, it may not come from a partial action of the group on an algebra, as the domains of the corresponding partial maps of α may be subalgebras of X that are not ideals.

In [7] the authors define the concept of an *enveloping action* of a partial action of a group on an algebra, which may be described as follows.

Definition 6.3.2. Let α be a classical partial action of a group G on an algebra A. An *enveloping action* of α is a pair (β, ι), where β is a global action of G on an algebra B and $\iota : A \to B$ is an injective $\mathbb{K}$-algebra homomorphism whose image is an ideal of B, satisfying the following.

(EA1) $\iota(\operatorname{dom}\alpha_g) = \iota(A) \cap \beta_{g^{-1}}(\iota(A))$;

(EA2) $\iota \circ \alpha_g(x) = \beta_g \circ \iota(x)$ for all $x \in \operatorname{dom}\alpha_g$;

(EA3) B is generated by $\bigcup_{g\in G} \beta_g(\iota(A))$.

Proposition 6.3.3. *Let α be a categorical partial action of a group G on an algebra A. A pair (β, ι), where β is a global action of G on an algebra B and $\iota : A \to B$ is an injective*

$\mathbb{K}$*-algebra homomorphism, satisfies (EA1) and (EA2) if and only if it is a globalization of* α*, seen as a categorical partial action.*

Proof. Let $\alpha(g) = [\operatorname{dom}\alpha_g, \iota_g, \alpha_g]$, where $\operatorname{dom}\alpha_g \subseteq A$ and ι_g is the corresponding inclusion of each $g \in G$.

First assume that (β, ι) is a globalization of α. Then for each $g \in G$ the diagram

$$\begin{tikzcd} & \operatorname{dom}\alpha_g \arrow[dl, "\iota_g"'] \arrow[dr, "\alpha_g"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "\beta_g\circ\iota"'] & & A \arrow[dl, "\iota"] \\ & B & \end{tikzcd} \tag{6.12}$$

is a pullback in $\mathbf{Alg}_{\mathbb{K}}$.

It is a simple verification that (EA2) follows from the commutativity of (6.12) for each $g \in G$.

Let us now verify (EA1). Let $g \in G$. Consider the subalgebra

$$P = \{(a, b) \in A \times A : \beta_g(\iota(a)) = \iota(b)\}$$

of $A \times A$. Then the diagram

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr, "\beta_g\circ\iota"'] & & A \arrow[dl, "\iota"] \\ & B & \end{tikzcd} \tag{6.13}$$

is a pullback in $\mathbf{Alg}_{\mathbb{K}}$, where p_1 and p_2 are given by $p_1(a, b) = a$ and $p_2(a, b) = b$, for all $(a, b) \in P$.

Since both $\operatorname{dom}\alpha_g$ and P form pullbacks of $\beta_g \circ \iota$ and ι, there exists an isomorphism $\varphi : \operatorname{dom}\alpha_g \to P$ such that the diagram

$$\begin{tikzcd} & \operatorname{dom}\alpha_g \arrow[d, "\varphi"] \arrow[ddl, bend right, "\iota_g"'] \arrow[ddr, bend left, "\alpha_g"] & \\ & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & A \end{tikzcd} \tag{6.14}$$

commutes. It is a simple verification that φ is given by

$$\varphi(a) = (a, \alpha_g(a)). \tag{6.15}$$

Let $x \in \iota(\operatorname{dom}\alpha_g)$. Then $x = \iota(a)$ for some (unique) $a \in \operatorname{dom}\alpha_g$. In particular, $x \in \iota(A)$, since $a \in A$. Also, observe that, by the commutativity of (6.12),

$$x = \iota(a) = \iota \circ \iota_g(a) = \beta_{g^{-1}} \circ \beta_g \circ \iota \circ \iota_g(a) = \beta_{g^{-1}} \circ \iota \circ \alpha_g(a) = \beta_{g^{-1}}(\iota(\alpha_g(a))).$$

Thus, since $\alpha_g(a) \in A$, $x \in \beta_{g^{-1}}(\iota(A))$. Hence, $x \in \iota(A) \cap \beta_{g^{-1}}(\iota(A))$.

On the other hand, let $x \in \iota(A) \cap \beta_{g^{-1}}(\iota(A))$. Then there exist $a, b \in A$ such that

$$x = \iota(a) = \beta_{g^{-1}}(\iota(b)).$$

In this situation, observe that

$$\beta_g(\iota(a)) = \iota(b).$$

Thus, $(a, b) \in P$. Since $\varphi : \operatorname{dom}\alpha_g \to P$ is an isomorphism of algebras, in particular it is a surjective map. Therefore, there exists $c \in \operatorname{dom}\alpha_g$ such that $\varphi(c) = (a, b)$. Hence, by (6.15),

$$(a, b) = \varphi(c) = (c, \alpha_g(c)),$$

so $a = c \in \operatorname{dom}\alpha_g$, and, consequently, $x = \iota(a) \in \iota(\operatorname{dom}\alpha_g)$.

Thus, $\iota(\operatorname{dom}\alpha_g) = \iota(A) \cap \beta_{g^{-1}}(\iota(A))$ so (EA1) also follows.

Now assume that (β, ι) satisfies (EA1) and (EA2). To verify that (β, ι) is a globalization of α we must show that diagram (6.12) is a pullback for all $g \in G$.

Fix $g \in G$. By (EA2), (6.12) is commutative. Hence, since (6.13) is a pullback, there exists a unique $\mathbb{K}$-algebra homomorphism $\varphi : \operatorname{dom}\alpha_g \to P$ such that (6.14) commutes. To verify that (6.12) is a pullback, it then suffices to prove that φ is an isomorphism of algebras.

It is an easy verification that φ is given by the formula (6.15) and that it is injective, so all that remains is to check that φ is surjective.

Let $(a, b) \in P$. Then $\beta_g(\iota(a)) = \iota(b)$, so $\iota(a) = \beta_{g^{-1}}(\iota(b))$. Hence, since $a, b \in A$, $\iota(a) \in \iota(A) \cap \beta_{g^{-1}}(\iota(A))$.

By (EA1) it follows that $\iota(a) \in \iota(\operatorname{dom}\alpha_g)$. Since ι is an injective map, we have $a \in \operatorname{dom}\alpha_g$.

By (EA2), since $a \in \operatorname{dom}\alpha_g$ we then have

$$\iota(b) = \beta_g(\iota(a)) = \iota(\alpha_g(a)).$$

Thus, by the injectivity of ι, $b = \alpha_g(a)$.

Therefore, we have

$$\varphi(a) = (a, \alpha_g(a)) = (a, b).$$

Hence, φ is surjective, as desired. $\square$

Proposition 6.3.4. *Let α be a categorical partial action of G on an algebra A and (β, ι) a universal globalization of α. Then (β, ι) satisfies (EA3).*

Proof. Let us say that β acts on an algebra B. Consider the subalgebra $\sum_{m \in M} \beta_m(\iota(A))$ of B. Then clearly it contains the subalgebra $\beta_m(\iota(A))$ of B for each $m \in M$.

Thus, $\beta_m \circ \iota$ factors through the inclusion map v of $\sum_{m \in M} \beta_m(\iota(A))$ into B for all $m \in M$. Therefore, by Theorem 5.2.29, v is an isomorphism, so $\sum_{m \in M} \beta_m(\iota(A)) = B$. Hence, (β, ι) satisfies (EA3), as desired. $\square$

Observe, however, that Propositions 6.3.3 and 6.3.4 do not imply that a universal globalization (β, ι) of a classical partial action of a group on an algebra is an enveloping action, since the image of ι may not be an ideal of the algebra on which β acts.

The following example illustrates this fact, by providing a partial action that has an enveloping action and a universal globalization that are not isomorphic.

Example 6.3.5. Let $G = \mathbb{Z}_2 = \{0, 1\}$ and A any non-trivial algebra. Consider the global action β of G on $B = A \times A$, where

$$\beta_1(a, b) = (b, a),$$

for all $(a, b) \in B$.

Let $\iota : A \to B$ be given by

$$\iota(a) = (a, 0),$$

for all $a \in A$. Clearly, it is a monomorphism in $\mathbf{Alg}_{\mathbb{K}}$. Let, then, α be the restriction of β to A via ι.

It is a simple verification that α is the categorical partial action of G on $A \in \mathbf{Alg}_{\mathbb{K}}$ given by

$$\alpha(1) = [0, 0, 0]. \tag{6.16}$$

This partial action comes from the classical partial action $\{\alpha_g\}_{g \in G}$ of G on the algebra A where $\operatorname{dom} \alpha_1 = \{0\}$ and α_1 is the zero map.

Since (β, ι) is a globalization of α, by Proposition 6.3.3, it satisfies (EA1) and (EA2). Since

$$B = A \times A = A \times \{0\} + \{0\} \times A = \iota(A) + \beta_g(\iota(A)),$$

the pair (β, ι) also satisfies (EA3). Thus, since $\iota(A)$ is an ideal of B, (β, ι) is an enveloping action of α.

However, (β, ι) is not a universal globalization of α. Consider $C = \coprod_{g \in G} A$ with corresponding inclusion morphisms u_0 and u_1 (recall Example 2.2.29). Let γ be the global action of G on C where γ_1 is the unique morphism in $\mathbf{Alg}_{\mathbb{K}}$ with $\gamma_1 \circ u_0 = u_1$ and $\gamma_1 \circ u_1 = u_0$, given by the universal property of the coproduct. Also let $\kappa = u_0 : A \to C$.

Since α has a globalization, Theorem 5.2.5 and the construction in Corollary 5.2.19 show us that the pair (γ, κ) is a universal globalization of α.

Now, assume by contradiction that (β, ι) is a universal globalization of α. Then, by Proposition 5.2.7, there exists an isomorphism $\varphi : \beta \to \gamma$ such that $\varphi \circ \iota = \kappa$. Since $\iota(A)$ is an ideal of B and φ is an isomorphism, it then follows that $\kappa(A) = \varphi(\iota(A))$ is an ideal of C.

However, since A is not the trivial algebra, $\kappa(A)$ is not an ideal of C. Indeed, $\kappa(a)u_1(a) \notin \kappa(A)$ for any $a \neq 0$ in A. Hence, (β, ι) is not a universal globalization of α.

As we have observed, the partial actions of a group on objects in $\mathbf{Alg}_{\mathbb{K}}$ generalize *properly* the classical partial actions of the group on algebras. A more appropriate setting to deal with these partial actions is, then, the category $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ (recall Definition 2.2.11).

Proposition 6.3.6. *Let α be a partial action datum of a monoid M on an object $A \in \mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Then α is a (strong) partial action if and only if it is a (strong) partial action of M on A in $\mathbf{Alg}_{\mathbb{K}}$.*

Proof. By Lemma 2.2.13, the inclusion functor of $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ into $\mathbf{Alg}_{\mathbb{K}}$ preserves pullbacks. It is a simple verification that it also satisfies the hypothesis of Proposition 3.3.4, so the result follows from Proposition 4.3.13. □

Thus, the classical partial actions of a group on algebras correspond to strong partial actions of the group on objects in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. And, in this case, the converse is also true, since the morphisms in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ that are inclusion maps are inclusions of ideals.

The next proposition will show a certain relationship between the enveloping actions and the universal globalizations in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Its proof was heavily inspired by the proof of the uniqueness part of [7, Theorem 4.5].

Lemma 6.3.7. *Let $\{\alpha_g\}_{g\in G}$ be a classical partial action of a group G on an algebra A where each ideal $\operatorname{dom}\alpha_g$ is a unital algebra with unity element 1_g. Then $\alpha_g(1_g) = 1_{g^{-1}}$ for each $g \in G$.*

Proof. Let $g \in G$. Observe that α_g is injective by Corollary 4.1.17. And by taking $h = e$ in Lemma 4.1.18 we obtain

$$\alpha_g(\operatorname{dom}\alpha_g) = \operatorname{dom}\alpha_{g^{-1}}. \tag{6.17}$$

Hence, α_g induces a bijective map, and, thus, an isomorphism of algebras, from $\operatorname{dom}\alpha_g$ to $\operatorname{dom}\alpha_{g^{-1}}$. Therefore, $\alpha_g(1_g) = 1_{g^{-1}}$, as desired. □

Lemma 6.3.8. *Let A be an algebra and $\{A_i\}_{i\in I}$ a finite family of unital ideals of A such that $A = \sum_{i\in I} A_i$. Then A is a unital algebra.*

Proof. First we prove that if A is the sum of two ideals I and J, with unity elements 1_I and 1_J, respectively, then A is a unital algebra.

Let $1_A = 1_I + 1_J - 1_I 1_J$. Let $a \in I$. Then we have

$$a1_A = a(1_I + 1_J - 1_I 1_J) = a1_I + a1_J - a1_I 1_J = a + a1_J - a1_J = a$$

and, similarly, $1_A a = a$. Likewise, $a1_A = a = 1_A a$ for all $a \in J$. Hence, since $A = I + J$, 1_A is a unity element of A.

Then the general result follows by induction. □

Proposition 6.3.9. *Let $\{\alpha_g\}_{g\in G}$ be a classical partial action of a group G on an algebra A where each ideal* $\operatorname{dom}\alpha_g$ *is a unital algebra, and (β, ι) an enveloping action of α. Then (β, ι) is a universal globalization of the strong partial action α of G on A in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$.*

Proof. For each $g \in G$ denote by 1_g the unity element of the ideal $\operatorname{dom}\alpha_g$.

Since the pullbacks in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ are the same as in $\mathbf{Alg}_{\mathbb{K}}$ and (β, ι) is an enveloping action of α, by Proposition 6.3.3 it is a globalization of α. So, it suffices to verify that (β, ι) satisfies (UG2).

Let (γ, κ) be a globalization of α. We must show that there exists a unique morphism $\kappa' : \beta \to \gamma$ such that diagram (5.13) commutes. Let us say that γ acts on an algebra C.

Observe that if such κ' exists, by the commutativity of (5.13) and by Corollary 4.4.5, we must have that

$$\kappa'(\beta_g(\iota(a))) = \gamma_g(\kappa'(\iota(a))) = \gamma_g(\kappa(a)), \tag{6.18}$$

for all $g \in G$ and $a \in A$.

Now, by (EA3), B is generated by $\bigcup_{g\in G} \beta_g(\iota(A))$. Hence, B is generated as a vector space by elements of the form $\beta_g(\iota(a))$, for $g \in G$ and $a \in A$. So, once we establish the existence of κ', we have its uniqueness, as it must be given by (6.18).

Define, then, the map $\kappa' : B \to C$ given on $\beta_g(\iota(A))$ by (6.18) for each $g \in G$.

Observe that if κ' is a well-defined morphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$, then it is a morphism from β to γ. Indeed, for each $g \in G$, and each generator $x = \beta_h(\iota(a))$ of B, by (6.18) and the fact that β and γ are global actions, we have

$$\begin{aligned}\kappa'(\beta_g(x)) &= \kappa'(\beta_g(\beta_h(\iota(a)))) = \kappa'(\beta_{gh}(\iota(a))) = \gamma_{gh}(\kappa(a)) = \gamma_g(\gamma_h(\kappa(a))) \\ &= \gamma_g(\kappa'(\beta_h(\iota(a)))) = \gamma_g(\kappa'(x)),\end{aligned}$$

so $\kappa' \circ \beta_g = \gamma_g \circ \kappa'$. Thus, by Corollary 4.4.5, κ' is a datum morphism from β to γ.

Also, observe that, in this case, diagram (5.13) commutes, since, by (6.18),

$$\kappa'(\iota(a)) = \kappa'(\beta_e(\iota(a))) = \gamma_e(\kappa(a)) = \kappa(a)$$

for all $a \in A$.

All that remains is to show that κ' is a well-defined homomorphism of algebras whose image is an ideal of C.

First, let us verify that κ' is a well-defined linear map. Let $\{a_g\}_{g\in G} \subseteq A$ with only a finite number of non-zero elements such that

$$\sum_{g\in G} \beta_g(\iota(a_g)) = 0. \tag{6.19}$$

We must verify that $\sum_{g\in G} \gamma_g(\kappa(a_g)) = 0$.

Let $h \in G$ and $a \in A$. Then, by (6.19), $\sum_{g\in G} \beta_g(\iota(a_g))\beta_h(\iota(a)) = 0$. By applying $\beta_{h^{-1}}$ to this equality we get

$$\sum_{g\in G} \beta_{h^{-1}g}(\iota(a_g))\iota(a) = 0. \tag{6.20}$$

Let $g \in G$. Since $\iota(A)$ and $\beta_{h^{-1}g}(\iota(A))$ are ideals of B, we have $\beta_{h^{-1}g}(\iota(a_g))\iota(a) \in \iota(A) \cap \beta_{h^{-1}g}(\iota(A))$. Thus, by (EA1) and the fact that $\operatorname{dom}\alpha_{g^{-1}h}$ is a unital ideal of A,

$$\beta_{h^{-1}g}(\iota(a_g))\iota(a) \in \iota(\operatorname{dom}\alpha_{g^{-1}h}) = \iota(A1_{g^{-1}h}) = \iota(A)\iota(1_{g^{-1}h}).$$

The algebra $\iota(A)\iota(1_{g^{-1}h})$ is a unital ideal of B, with unit element $\iota(1_{g^{-1}h})$.

Then, by Lemma 6.3.7 and (EA2)

$$\begin{aligned}
\beta_{h^{-1}g}(\iota(a_g))\iota(a) &= \beta_{h^{-1}g}(\iota(a_g))\iota(a)\iota(1_{g^{-1}h}) = \beta_{h^{-1}g}(\iota(a_g))\iota(1_{g^{-1}h})\iota(a) \\
&= \beta_{h^{-1}g}(\iota(a_g))\iota(\alpha_{h^{-1}g}(1_{h^{-1}g}))\iota(a) = \beta_{h^{-1}g}(\iota(a_g))\beta_{h^{-1}g}(\iota(1_{h^{-1}g}))\iota(a) \\
&= \beta_{h^{-1}g}(\iota(a_g 1_{h^{-1}g}))\iota(a) = \iota(\alpha_{h^{-1}g}(a_g 1_{h^{-1}g}))\iota(a) = \iota(\alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a).
\end{aligned} \tag{6.21}$$

Now, (γ, κ) is a globalization of α. Therefore, by Proposition 6.3.3 it satisfies (EA1) and (EA2). Hence, similarly we have

$$\gamma_{h^{-1}g}(\kappa(a_g))\kappa(a) = \kappa(\alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a). \tag{6.22}$$

By (6.20) and (6.21),

$$\iota\left(\sum_{g\in G} \alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a\right) = \sum_{g\in G} \iota(\alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a) = \sum_{g\in G} \beta_{h^{-1}g}(\iota(a_g))\iota(a) = 0,$$

so, since ι is an injective map, we have

$$\sum_{g\in G} \alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a = 0. \tag{6.23}$$

Then, by (6.22) and (6.23), we have

$$\sum_{g\in G}\gamma_{h^{-1}g}(\kappa(a_g))\kappa(a)=\sum_{g\in G}\kappa(\alpha_{h^{-1}g}(a_g1_{h^{-1}g})a)=\kappa\left(\sum_{g\in G}\alpha_{h^{-1}g}(a_g1_{h^{-1}g})a\right)=\kappa(0)=0. \tag{6.24}$$

By applying γ_h to (6.24) we get

$$\sum_{g\in G}\gamma_g(\kappa(a_g))\gamma_h(\kappa(a))=0. \tag{6.25}$$

Since (6.25) holds for all $h\in G$ and $a\in A$, the element $\sum_{g\in G}\gamma_g(\kappa(a_g))$ annihilates $\gamma_h(\kappa(A))$ for all $h\in G$.

Let G' be the finite set $\{g\in G : a_g\neq 0\}\subseteq G$ and C_1 be the algebra generated by $\bigcup_{g\in G'}\gamma_g(\kappa(A))$. Since $\gamma_g(\kappa(A))$ is an ideal of C for each $g\in G'$, it is also an ideal of C_1. Clearly each $\gamma_g(\kappa(A))$ is unital, because A is unital. Therefore, as $C_1=\sum_{g\in G'}\gamma_g(\kappa(A))$, by Lemma 6.3.8 we have that C_1 is a unital algebra. Let

$$1_{C_1}=\sum_{h\in G'}\gamma_h(\kappa(a_h^1))$$

be its unit.

Now, $\sum_{g\in G}\gamma_g(\kappa(a_g))\in C_1$. Thus, since $\sum_{g\in G}\gamma_g(\kappa(a_g))$ annihilates each $\gamma_h(\kappa(A))$, we have

$$\begin{aligned}\sum_{g\in G}\gamma_g(\kappa(a_g))&=\sum_{g\in G}\gamma_g(\kappa(a_g))1_{C_1}=\sum_{g\in G}\gamma_g(\kappa(a_g))\sum_{h\in G'}\gamma_h(\kappa(a_h^1))\\&=\sum_{h\in G'}(\sum_{g\in G}\gamma_g(\kappa(a_g))\gamma_h(\kappa(a_h^1)))=\sum_{h\in G'}0=0.\end{aligned}$$

Hence, the well-definition of κ' follows.

We now have a linear map $\kappa' : B\to C$. Let us verify that it preserves the product of the algebras.

It suffices to do so on two generators of B. Let $g,h\in G$ and $a_g,a_h\in A$. We will check that $\kappa'(\beta_g(\iota(a_g))\beta_h(\iota(a_h)))=\kappa'(\beta_g(\iota(a_g)))\kappa'(\beta_h(\iota(a_h)))$.

By (6.21), we have $\beta_{h^{-1}g}(\iota(a_g))\iota(a_h)=\iota(\alpha_{h^{-1}g}(a_g1_{h^{-1}g})a_h)$, so, by applying β_h we obtain

$$\beta_g(\iota(a_g))\beta_h(\iota(a_h))=\beta_h(\iota(\alpha_{h^{-1}g}(a_g1_{h^{-1}g})a_h)) \tag{6.26}$$

Similarly, by (6.22) we have

$$\gamma_g(\kappa(a_g))\gamma_h(\kappa(a_h))=\gamma_h(\kappa(\alpha_{h^{-1}g}(a_g1_{h^{-1}g})a_h)) \tag{6.27}$$

Hence, by (6.18), (6.26) and (6.27),

$$\begin{aligned}\kappa'(\beta_g(\iota(a_g))\beta_h(\iota(a_h))) &= \kappa'(\beta_h(\iota(\alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a_h))) = \gamma_h(\kappa(\alpha_{h^{-1}g}(a_g 1_{h^{-1}g})a_h)) \\ &= \gamma_g(\kappa(a_g))\gamma_h(\kappa(a_h)) = \kappa'(\beta_g(\iota(a_g)))\kappa'(\beta_h(\kappa(a_h))),\end{aligned}$$

as desired.

So, κ' is an algebra morphism from B to C. Finally, κ' is a morphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Indeed, by (EA3) and (6.18),

$$\kappa'(B) = \kappa'\left(\sum_{g\in G}\beta_g(\iota(A))\right) = \sum_{g\in G}\kappa'(\beta_g(\iota(A))) = \sum_{g\in G}\gamma_g(\kappa(A))$$

is a sum of ideals of C, since each $\gamma_g \circ \kappa$ is a morphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. □

Unfortunately, $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ does not have all colimits, so we cannot apply the results from Theorem 5.2.15 or Corollary 5.2.19 to partial actions in this category.

In fact, the category $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ gives us an example of a universal globalization that does not come from a reflection.

Example 6.3.10. Let $G = \mathbb{Z}_2 = \{0, 1\}$ and A be a non-trivial unital algebra. Let β and ι as in Example 6.3.5. Observe that $\iota(A)$ is an ideal of $B = A \times A$. We can then interpret β as a global action on an object of $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ and ι as a monomorphism in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$.

Let α be the restriction of β to A via ι in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Clearly, since the pullbacks involving monomorphisms of $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$ are the same of $\mathbf{Alg}_{\mathbb{K}}$ (by Lemma 2.2.13), α is given by (6.16).

As we have verified, (β, ι) is an enveloping action of α. Hence, since each of the ideals of α is unital, by Proposition 6.3.9, (β, ι) is a universal globalization of α in $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}$. Let us verify that ι is not a reflection of α in $M{-}\mathbf{Act}_{\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}}$.

To do so, consider the global action γ of M on A given by $\gamma_1 = id_A$ and the morphism $\kappa = id_A$.

Notice that, by Lemma 4.4.4, κ is a morphism in $M{-}\mathbf{pAct}_{\mathbf{Alg}_{\mathbb{K}}^{\mathbf{Id}}}$ from α to γ, since $\gamma_0 \circ \kappa \circ \iota_0 = id_A = \kappa \circ \alpha_0$ and $\gamma_1 \circ \kappa \circ \iota_1 = 0 = \kappa \circ \alpha_1$.

Suppose that there is a morphism $\kappa' : \beta \to \gamma$ such that

$$\kappa' \circ \iota = \kappa.$$

Then it as a simple verification that that κ' must be given by

$$\kappa'(a, b) = \kappa(a) + \gamma_1(\kappa(b)) = a + b,$$

which is not an algebra morphism, since

$$\kappa'(1,0)\kappa'(0,1) = 1 \neq 0 = \kappa'((1,0)(0,1)).$$

REFERENCES

1 ABADIE, Fernando. Enveloping actions and Takai duality for partial actions. English. **J. Funct. Anal.**, v. 197, n. 1, p. 14–67, 2003. ISSN 0022-1236. DOI: 10.1016/S0022-1236(02)00032-0. Cit. on pp. 6, 12.

2 BATISTA, Eliezer. Partial actions: what they are and why we care. English. **Bull. Belg. Math. Soc. - Simon Stevin**, v. 24, n. 1, p. 35–71, 2017. ISSN 1370-1444. Cit. on p. 12.

3 BÉNABOU, Jean. **Introduction to bicategories**. English. [S.l.: s.n.], 1967. Bénabou, Jean et al., Reports of the Midwest Category Seminar. Lect. Notes Math. 47, 1-77 (1967). DOI: 10.1007/BFb0074299. Cit. on pp. 43, 55.

4 COCKETT, J. R. B.; LACK, Stephen. Restriction categories. I: Categories of partial maps. English. **Theor. Comput. Sci.**, v. 270, n. 1-2, p. 223–259, 2002. ISSN 0304-3975. DOI: 10.1016/S0304-3975(00)00382-0. Cit. on p. 63.

5 CORNOCK, Claire. **Restriction semigroups: structure, varieties and presentations**. 2011. PhD thesis – University of York. Cit. on pp. 62, 63.

6 DOKUCHAEV, M. Recent developments around partial actions. English. **São Paulo J. Math. Sci.**, v. 13, n. 1, p. 195–247, 2019. ISSN 1982-6907. DOI: 10.1007/s40863-018-0087-y. Cit. on p. 12.

7 DOKUCHAEV, M.; EXEL, R. Associativity of crossed products by partial actions, enveloping actions and partial representations. English. **Trans. Am. Math. Soc.**, v. 357, n. 5, p. 1931–1952, 2005. ISSN 0002-9947. DOI: 10.1090/S0002-9947-04-03519-6. Cit. on pp. 8, 12, 14, 82, 142, 149, 153.

8 EXEL, Ruy. Circle actions on C^*-algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence. English. **J. Funct. Anal.**, v. 122, n. 2, p. 361–401, 1994. ISSN 0022-1236. DOI: 10.1006/jfan.1994.1073. Cit. on pp. 6, 12.

9 ________. Partial actions of groups and actions of inverse semigroups. English. **Proc. Am. Math. Soc.**, v. 126, n. 12, p. 3481–3494, 1998. ISSN 0002-9939. DOI: 10.1090/S0002-9939-98-04575-4. Cit. on p. 40.

10 GOULD, VICTORIA. Notes on restriction semigroups and related structures; formerly (weakly) left E-ample semigroups, 2010. Available from: <http://www-users.york.ac.uk/~varg1/restriction.pdf>. Cit. on p. 62.

11 HOLLINGS, Christopher. Partial actions of monoids. English. **Semigroup Forum**, v. 75, n. 2, p. 293–316, 2007. ISSN 0037-1912. DOI: 10.1007/s00233-006-0665-7. Cit. on pp. 6, 8, 12, 79–81, 101.

12 HU, Jiawei; VERCRUYSSE, Joost. Geometrically partial actions. English. **Trans. Am. Math. Soc.**, v. 373, n. 6, p. 4085–4143, 2020. ISSN 0002-9947. DOI: 10.1090/tran/8058. Cit. on pp. 6, 7, 12, 13, 78, 82.

13 JECH, Thomas. **Set theory.** The third millennium edition, revised and expanded. [S.l.]: Berlin: Springer, 2003. (Springer Monogr. Math.). ISBN 3-540-44085-2. DOI: 10.1007/3-540-44761-X. Cit. on p. 15.

14 KASTL, J. Inverse categories. English, 1979. Cit. on p. 72.

15 KHRYPCHENKO, Mykola; KLOCK, Francisco. **Partial monoid actions on objects in categories with pullbacks and their globalizations**. [S.l.: s.n.], 2023. arXiv: 2309.11300v2 [math.CT]. Cit. on pp. 125, 130, 135, 138.

16 LAWSON, Mark. **Inverse semigroups. The theory of partial symmetries.** [S.l.]: Singapore: World Scientific, 1998. ISBN 981-02-3316-7. Cit. on p. 35.

17 MEGRELISHVILI, Michael; SCHRÖDER, Lutz. Globalization of confluent partial actions on topological and metric spaces. English. **Topology Appl.**, v. 145, n. 1-3, p. 119–145, 2004. ISSN 0166-8641. DOI: 10.1016/j.topol.2004.06.006. Cit. on pp. 6, 12, 80, 81.

18 SARACCO, Paolo; VERCRUYSSE, Joost. Globalization for geometric partial comodules. English. **J. Algebra**, v. 602, p. 37–59, 2022. ISSN 0021-8693. DOI: 10.1016/j.jalgebra.2022.03.013. Cit. on pp. 6, 12–14, 125.

19 SCHWAB, Emil; SCHWAB, Emil Daniel. On inverse categories with split idempotents. English. **Arch. Math., Brno**, v. 51, n. 1, p. 13–25, 2015. ISSN 0044-8753. DOI: 10.5817/AM2015-1-13. Cit. on p. 72.

20 SCHWEDE, Stefan. Categories and orbispaces. English. **Algebr. Geom. Topol.**, v. 19, n. 6, p. 3171–3215, 2019. ISSN 1472-2747. DOI: 10.2140/agt.2019.19.3171. Cit. on p. 100.

21 TONDEUR, Philippe. Introduction to Lie groups and transformation groups. English. Springer, Cham, v. 7, 1965. ISSN 0075-8434. DOI: 10.1007/BFb0097829. Cit. on p. 100.