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Programa de Pós-Graduação em Matemática Pura e Aplicada

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Some extremal problems in Hilbert spaces of entire functions

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Ricardo Machado da Motta

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Ricardo Machado da Motta

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Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de Mestre em Análise.

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To my parents

Romario & Hilda

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"Oh, the little more, and how much it is!
And the little less, and what worlds away!"

Robert Browning

Resumo

O presente trabalho apresenta alguns problemas extremais em Análise de Fourier. Inicialmente, são discutidos alguns conceitos fundamentais da teoria de funções inteiras, tais como o princípio de Phragmén-Lindelöf, funções da classe de Pólya e funções de tipo limitado. Utilizando esses conceitos, os espaços de De Branges são definidos e suas propriedades, como espaços de Hilbert com núcleo reprodutor, são exploradas. Em seguida, são enunciados alguns problemas extremais relacionados à imersão de espaços de Paley-Wiener e a variações de problemas extremais clássicos com restrições de monotonicidade. Por meio da poderosa teoria dos espaços de De Branges, é possível encontrar as constantes ótimas para tais problemas.

Palavras-chaves: Análise de Fourier. Funções inteiras. Espaços de De Branges. Espaços de Paley-Wiener. Problemas extremais.

Resumo Expandido

Introdução

Os espaços de Paley-Wiener (ou simplesmente $\mathcal{PW}$), estudados por R. Paley e N. Wiener, desempenham um papel fundamental na análise harmônica clássica. Nesses espaços de Hilbert de funções inteiras, a norma de uma função pode expressar muitas de suas propriedades quantitativas, como controle pontual.

Isso ocorre porque $\mathcal{PW}$ possuem, devido ao teorema da representação de Riesz, um núcleo reprodutor, ou seja, uma função no espaço que permite recuperar o valor pontual via um produto interno. Além disso, os elementos desses espaços têm restrições no suporte de sua transformada de Fourier e podemos dar, devido a uma equivalência, uma definição alternativa para $\mathcal{PW}$ usando o tipo exponencial (que é uma forma de medir o crescimento de uma função inteira em todo o plano complexo).

Devido às boas propriedades dos espaços de Paley-Wiener, alguns matemáticos procuraram estender esse conceito de espaço. Talvez uma das contribuições mais relevantes nessa direção tenha sido feita por Louis de Branges, que utilizou a poderosa teoria de funções inteiras para introduzir e trabalhar sistematicamente com espaços de Hilbert que agora levam seu nome.

Os espaços de De Branges (ou simplesmente $\mathcal{H}(E)$) são generalizações naturais dos espaços de Paley-Wiener. Assim como $\mathcal{PW}$, os espaços de De Branges são espaços de Hilbert de funções inteiras com núcleos reprodutores e, por causa disso, temos representações explícitas para funções e suas normas usando os núcleos reprodutores. Com isso em mente, os matemáticos aproveitaram essa teoria poderosa para resolver problemas dos tipos mais diversos, como otimização de Fourier e teoria analítica dos números. Entre esses trabalhos, podemos mencionar [HV96], [Car+14] e [Car+23].

Objetivos

Este trabalho investiga os espaços de De Branges e problemas extremais na análise de Fourier, explorando suas propriedades como espaços de Hilbert para resolver desafios como o *one-delta problem* com restrições de monotonicidade. Além disso, visa introduzir esses conceitos de forma acessível para leitores não familiarizados com espaços de núcleo reprodutor e problemas extremais na análise harmônica.

Metodologia

A metodologia adotada baseou-se em uma abordagem colaborativa com o orientador, envolvendo reuniões regulares para discussão e refinamento das ideias. Além disso, foram realizadas apresentações de seminários, garantindo assim o aprimoramento contínuo do trabalho. A pesquisa também incluiu uma extensa revisão bibliográfica, abrangendo artigos e livros relevantes para embasar teoricamente o estudo.

Resultados e Discussões

Esta dissertação de mestrado tem como objetivo abordar a teoria dos espaços de De Branges e alguns problemas extremais na análise de Fourier. Para conseguir isso, começamos com uma breve revisão dos resultados clássicos da análise harmônica no Capítulo 1. No Capítulo 2 apresentamos alguns fatos essenciais da teoria das funções analíticas, como o princípio de Phragmén-Lindelöf, fatoração de funções na classe Pólya e funções do tipo limitado. No Capítulo 3, usamos esses novos conceitos para definir espaços de De Branges e estudar suas propriedades como espaços de Hilbert.

Essencialmente, os capítulos 2 e 3 constituem um estudo parcial dos dois primeiros capítulos do livro [Bra68]. Consequentemente, algumas proposições ao longo do texto podem parecer monótonas e desinteressantes para leitores já familiarizados com os conceitos de análise. Porém, isso ocorre porque essas proposições são exercícios do referido livro e serão empregadas posteriormente como lemas ou resultados auxiliares nas provas dos teoremas centrais.

O capítulo 4 é dedicado à investigação de alguns problemas extremais, relacionados à generalização do clássico *one-delta problem* com restrições e ao cálculo da norma de um operador entre dois espaços multidimensionais de Paley-Wiener. Adicionalmente, no final do Capítulo 4, introduzimos um problema extremal relativo aos espaços de De Branges. É importante notar que este último problema garante a existência de soluções para outros problemas neste capítulo, devido à rica teoria por trás desses espaços de nucleo reprodutor. A referência deste capítulo é o artigo [Car+23].

Considerações Finais

Em resumo, esta dissertação explora a teoria dos espaços de De Branges e problemas extremais na análise de Fourier. Ao investigar conceitos fundamentais da teoria das funções analíticas, delineamos as propriedades dos espaços de De Branges como espaços de Hilbert. A investigação de problemas extremais destaca não apenas a profundidade teórica, mas também a aplicabilidade prática das poderosas ferramentas abordadas neste estudo de espaços de Hilbert de funções inteiras.

Palavras-chaves: Análise de Fourier. Funções inteiras. Espaços de De Branges. Espaços de Paley-Wiener. Problemas extremais.

Abstract

The present work addresses extremal problems in Fourier Analysis. Initially, fundamental concepts of the theory of entire functions are discussed, such as the Phragmén-Lindelöf principle, functions from the class of Pólya, and functions of bounded type. Utilizing these concepts, de Branges spaces are defined, and their properties, such as being Hilbert spaces with reproducing kernels, are explored. Subsequently, extremal problems related to embedding Paley-Wiener spaces and variations of classical extremal problems with monotonicity constraints are stated. Through the powerful theory of de Branges spaces, it is possible to determine the sharp constants for such problems.

Keywords: Fourier Analysis. Entire functions. De Branges spaces. Weighted Paley-Wiener spaces. Extremal problems.

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Introduction

Systematically studied for the first time by R. Paley and N. Wiener[1], the Paley-Wiener spaces (or simply $\mathcal{PW}$) play a fundamental role in classical harmonic analysis. In these Hilbert spaces of entire functions, the norm of a function can express many of its quantitative properties, such as pointwise control.

This happens because $\mathcal{PW}$ has, due to the Riesz Representation Theorem, a reproducing kernel, which is a function in the space that allows recovering the pointwise value via an inner product. Furthermore, the elements of these spaces have restrictions on the support of their Fourier transform, and we can provide an alternative definition to $\mathcal{PW}$ using exponential type (which is a way of measuring the growth of an entire function across the entire complex plane) because of an equivalence.

Due to the good properties of Paley-Wiener spaces, some mathematicians sought to extend this concept of space. Perhaps one of the most relevant contributions in this direction was made by Louis de Branges, who utilized the powerful theory of entire functions to introduce and work with Hilbert spaces that now bear his name.

The de Branges spaces (or simply $\mathcal{H}(E)$) are natural generalizations of the Paley-Wiener spaces. Just like $\mathcal{PW}$, de Branges spaces are Hilbert spaces of entire functions with reproducing kernels, and because of this, we have explicit representations for functions and their norms using the reproducing kernels. With this in mind, mathematicians have taken advantage of this powerful theory to solve problems of the most diverse types, such as in Fourier optimization and analytical number theory. Among these works, we can mention [HV96], [Car+14], and [Car+23].

This master's thesis aims to address the theory of de Branges spaces and some extremal problems in Fourier analysis. To achieve this, we start with a brief review of classical results in harmonic analysis in Chapter 1. In Chapter 2, we present some essential facts from the theory of analytic functions, such as the Phragmén-Lindelöf principle, factorization of functions in the Pólya class, and functions of bounded type. In Chapter 3, we use these new concepts to define de Branges spaces and study their properties as Hilbert spaces.

Essentially, Chapters 2 and 3 constitute a partial study of the first two chapters of de Branges's book [Bra68]. Consequently, some propositions throughout the text may appear uninteresting to readers already familiar with analysis concepts. However, this is because these propositions are problems from the mentioned book and will be

[1] See for example [PW34].

employed later as lemmas or auxiliary results in the proofs of central theorems.

Chapter 4 is dedicated to the investigation of some extremal problems related to the generalization of the classic *one-delta problem* with additional restrictions and the calculation of an operator norm between two weighted Paley-Wiener spaces.

Additionally, at the end of Chapter 4, we introduce an extremal problem concerning de Branges spaces. It is worth noting that this extremal problem ensures the existence of solutions to other problems in this chapter, owing to the rich theory behind these reproducing kernel spaces. The reference for this chapter is the article [Car+23].

Finally, we use Appendix A to establish specific results employed throughout the text. If the reader is familiar with the basic theory of classical harmonic analysis, we recommend starting the reading from Chapter 2.

Notation

We will use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the sets of natural numbers, integers, real numbers, and complex numbers, respectively. Furthermore, we use $\mathbb{Z}_+$ for nonnegative integers, $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ for the upper half-plane, and $\overline{\mathbb{C}^+} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ for the closed upper half-plane. We write $z = x + iy$ for the complex variable, where x and y are real. This will often be implicit in the text.

Let $F : \mathbb{C}^d \to \mathbb{C}$ be a function. It is said to be real entire if it is entire and its restriction to $\mathbb{R}^d$ is real-valued. We define the function $F^* : \mathbb{C}^d \to \mathbb{C}$ by $F^*(z) \stackrel{\text{def}}{=} \overline{F(\overline{z})}$. If F is entire, then F^* is also entire and coincides with the conjugate of F on $\mathbb{R}^d$. If an entire function F is real, then F coincides with F^* on $\mathbb{R}^d$, and hence, by analytic continuation, in all $\mathbb{C}^d$. Thus, an entire function F is real if and only if $F^* = F$.

An entire function $F : \mathbb{C}^d \to \mathbb{C}$ is called radial if its restriction to $\mathbb{R}^d$ is radial. Similarly, we say that an entire function $F \in L^p(\mathbb{R}^d, \mu)$ if its restriction to $\mathbb{R}^d$ belongs to this space. The notation $B_r(p)$ represents the ball of radius $r > 0$ centered at a point p. If $p = 0$, we will simply write B_r.

For $w \in \mathbb{C}^d$, we define the translation operator $(\tau_w F)(z) = F(z-w)$ and the reflection operator $\breve{F}(z) = F(-z)$. Moreover, given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where[2] $\alpha_i \in \mathbb{Z}_+$, we define the polynomials $x^\alpha \stackrel{\text{def}}{=} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}$ and the partial differentials $\partial^\alpha \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$. As usual, $C_0(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ that tend to zero at infinity and $C_c^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions with compact support.

[2] The order of the multi-index α is denoted by $|\alpha|$, where $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$.

Chapter 1

Background

This chapter reviews fundamental concepts that will be used throughout the text. However, it is important to note that this text assumes a certain level of familiarity from the reader with complex analysis, measure theory, and functional analysis.

In the first section, we establish the definition of the Fourier transform, and later we will extend this notion to tempered distributions. The second section introduces some results on Fourier series, in particular, the Poisson summation formula. In the last section, we explore the definition of exponential type in higher dimensions and present several versions of the Paley-Wiener theorem. The references for the first two sections are [Fol99, Chapter 8] and [SW71, Chapter 1§3]. For the final section, we draw on [SW71, Chapter III§4], [Fol99, Chapter 9], and [Hör64, Chapter 1].

1.1 Fourier transform

For $f \in L^1(\mathbb{R}^d)$, we define its Fourier transform $\widehat{f}$ by

$$\widehat{f}(\xi) \stackrel{def}{=} \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx.$$

Our first proposition outlines the fundamental properties of the Fourier transform.

Proposition 1.1. Let $f, g \in L^1(\mathbb{R}^d)$ and $k \in \mathbb{N}$. Then

(i) $\|\widehat{f}\|_\infty \leq \|f\|_1$.

(ii) $(\tau_y f)^\wedge(\xi) = e^{-2\pi i\xi\cdot y}\widehat{f}(\xi)$ and $\tau_\eta(\widehat{f}) = \widehat{h}$, where $h(x) = e^{2\pi i\eta\cdot x} f(x)$.

(iii) If T is an invertible linear transformation on $\mathbb{R}^d$ and $S = (T^*)^{-1}$ is the inverse of its transpose, then $(f \circ T)^\wedge = |\det T|^{-1}\widehat{f} \circ S$. In particular

- if T is a rotation, we have $(f \circ T)^\wedge = \widehat{f} \circ T$;
- if $Tx = t^{-1}x$ $(t > 0)$, then $(f \circ T)^\wedge(\xi) = t^d\widehat{f}(t\xi)$.

(iv) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

(v) If $x^\alpha f \in L^1(\mathbb{R}^d)$ for $|\alpha| \leq k$, then $\widehat{f} \in C^k(\mathbb{R}^d)$ and $\partial^\alpha\widehat{f} = [(-2\pi i x)^\alpha f]^\wedge$.

(vi) If $f \in C^k(\mathbb{R}^d)$, $\partial^\alpha f \in L^1(\mathbb{R}^d)$ for $|\alpha| \leq k$, and $\partial^\alpha f \in C_0(\mathbb{R}^d)$ for $|\alpha| \leq k-1$, then $\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi)$.

(vii) **(Riemann-Lebesgue lemma)** We have that $\widehat{f} \in C_0(\mathbb{R}^d)$. In particular, $\widehat{f}$ is uniformly continuous.

(viii) **(Multiplication formula)**

$$\int_{\mathbb{R}^d} f(x)\widehat{g}(x)\,dx = \int_{\mathbb{R}^d} \widehat{f}(x)g(x)\,dx. \tag{1.1}$$

Proof. See [Fol99, Theorem 8.22] and [Fol99, Lemma 8.25]. □

We now turn to investigate the inverse problem of obtaining f from its Fourier transform. For this purpose, we define the inverse Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ as

$$\check{f}(\xi) \stackrel{def}{=} \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(x)\,dx.$$

Theorem 1.2 (Inversion formula). If f and $\widehat{f}$ belong to $L^1(\mathbb{R}^d)$, then

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\,d\xi,$$

for almost every point $x \in \mathbb{R}^d$.

Proof. For more information, refer to [Fol99, Theorem 8.26]. □

Although the integral defining the Fourier transform does not generally make sense for a function $f \in L^2(\mathbb{R}^d)$, there exists a simple and elegant way to extend the theory to the L^2 context. The key point for this extension is the following result.

Theorem 1.3. If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $\|\widehat{f}\|_2 = \|f\|_2$.

The set $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, implying the existence of a unique bounded extension, denoted by $\mathcal{F}$, of the Fourier transform to the space $L^2(\mathbb{R}^d)$. We will continue using the notation $\widehat{f} = \mathcal{F}(f)$ when $f \in L^2(\mathbb{R}^d)$. In particular, if χ_{B_k} is the characteristic function of the ball of radius $k > 0$, $f \in L^2(\mathbb{R}^d)$, and we denote $f_k = f\chi_{B_k}$, then $\widehat{f}$ is an L^2 limit of $\widehat{f_k}$.

Theorem 1.4 (Plancherel). The operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is unitary, meaning it is linear, isometric, and surjective. Additionally, its inverse $\mathcal{F}^{-1}$ can be obtained as

$$(\mathcal{F}^{-1}(g))(x) = \mathcal{F}(g)(-x),$$

for any $g \in L^2(\mathbb{R}^d)$.

The proof of the last two results can be found in [Fol99, Theorem 8.29].

In order to generalize the notion of the Fourier transform to the context of distributions, we will first define the appropriate class of test functions. To freely explore the relationship between differentiation and multiplication by polynomials peculiar to the Fourier transform (see Proposition 1.1 (v) and (vi)), we consider the class of functions $f \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} |x^\alpha (\partial^\beta f)(x)| < \infty \tag{1.2}$$

for any multi-indices α and β, i.e., the set of infinitely differentiable functions such that any derivative decays faster than any polynomial. The Schwartz space is a Fréchet space, denoted by $\mathcal{S}(\mathbb{R}^d)$ (or simply $\mathcal{S}$ when clear from the context), defined as the set of functions with the topology given by the semi-norms on the left-hand side of (1.2).

Proposition 1.5. Let $\mathcal{S}$ be the Schwartz space and let $f, g \in \mathcal{S}$.

(i) $\widehat{f} \in \mathcal{S}$ and $f * g \in \mathcal{S}$;

(ii) The Fourier transform is an isomorphism from the Schwartz space $\mathcal{S}$ onto itself.

Proof. See [Fol99, Corollary 8.28]. □

The dual space $\mathcal{S}'$ formed by all continuous linear functionals on $\mathcal{S}$ is called the space of tempered distributions. In fact, this dual space contains a wide variety of elements, as we will see in the following examples.

Example 1.6 (Polynomial growth functions). We say that a function f has[1] polynomial growth L^p, if $f(x)(1+|x|^2)^{-k} \in L^p(\mathbb{R}^d)$ for some nonnegative integer k and some p with $1 \le p \le \infty$. We define the functional $L_f : \mathcal{S} \to \mathbb{R}$ by

$$L_f(\varphi) \overset{def}{=} \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx.$$

As L_f is linear, to verify its continuity, it is sufficient to consider the case $\varphi \to 0$. In this case, we use the Hölder inequality to conclude that

$$\begin{aligned} |Lf(\varphi)| = \left|\int_{\mathbb{R}^d} f(x)\varphi(x)\,dx\right| &\le \int_{\mathbb{R}^d} \left|\frac{f(x)}{(1+|x|^2)^k}\right| \left|(1+|x|^2)^k\varphi(x)\right|\,dx \\ &\le \left\|\frac{f(x)}{(1+|x|^2)^k}\right\|_p \left\|(1+|x|^2)^k\varphi(x)\right\|_{p'} \xrightarrow{\varphi\to 0} 0. \end{aligned}$$

[1] Or simply polynomial growth, if $p = \infty$.

Example 1.7 (Polynomial growth measures). We say that a complex Borel measure μ has polynomial growth (or tempered measure) if

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^k}\, d|\mu|(x) < \infty$$

for some nonnegative integer k. Analogously to Example 1.6, we verify that

$$L_\mu(\varphi) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x)$$

is continuous and thus a tempered distribution.

Example 1.8 (Evaluation functionals). For each pair of multi-indices (α, β) and each $x_0 \in \mathbb{R}^d$, we can consider the functional

$$L_{\alpha,\beta,x_0}(\varphi) = x_0^\alpha \partial^\beta \varphi(x_0),$$

which, directly from the definition of the metric on $\mathcal{S}$, is continuous.

We can extend various operations of analysis, such as multiplication, translation, Fourier transform, and convolution, to the context of tempered distributions.

- **Multiplication by a function.** Let $u \in \mathcal{S}'$ and $\psi \in C^\infty(\mathbb{R}^d)$ be a function such that ψ and all its derivatives have polynomial growth. We define the multiplication $u\psi \in \mathcal{S}'$ as

  $$(u\psi)(\varphi) \overset{\text{def}}{=} u(\psi\varphi),$$

  for any $\varphi \in \mathcal{S}$.

- **Translation.** If $y \in \mathbb{R}^d$ and $u \in \mathcal{S}'$, we define[2] $\tau_y u \in \mathcal{S}'$ as

  $$(\tau_y u)(\varphi) \overset{\text{def}}{=} u(\tau_{-y}\varphi),$$

  for any $\varphi \in \mathcal{S}$.

- **Differentiation.** For u and φ in $\mathcal{S}$, using integration by parts, we have

  $$\int_{\mathbb{R}^d} (\partial^\alpha u)(x)\varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} u(x)(\partial^\alpha \varphi)(x)\, dx.$$

  If $u \in \mathcal{S}'$, the linear functional $\varphi \mapsto (-1)^{|\alpha|} u(\partial^\alpha \varphi)$ is continuous, and this will be the α-th partial derivative of the distribution u, denoted by $(\partial^\alpha u) \in \mathcal{S}'$. Therefore

  $$(\partial^\alpha u)(\varphi) \overset{\text{def}}{=} (-1)^{|\alpha|} u(\partial^\alpha \varphi),$$

  for each $\varphi \in \mathcal{S}$.

[2] See the Notation section in the Introduction.

- **Fourier transform.** Now, using the multiplication formula in (1.1) to motivate the definition. If $u, \varphi \in \mathcal{S}$, we have

$$\int_{\mathbb{R}^d} \varphi(x)\widehat{u}(x) = \int_{\mathbb{R}^d} u(x)\widehat{\varphi}(x)\, dx.$$

  If $u \in \mathcal{S}'$, we define its Fourier transform $\widehat{u} \in \mathcal{S}'$ as

$$\widehat{u}(\varphi) \stackrel{\text{def}}{=} u(\widehat{\varphi}),$$

  for each $\varphi \in \mathcal{S}$. It is easy to see that $\widehat{u} \in \mathcal{S}'$ as it is the composition of two continuous functions. It is also easy to see that the Fourier transform defines an isomorphism of the topological vector space $\mathcal{S}'$ onto itself.

- **Convolution.**
  Option 1. If $u, \varphi \in \mathcal{S}$, we write

$$u * \varphi(x) = \int_{\mathbb{R}^d} u(y)\varphi(x-y)\, dy = \int_{\mathbb{R}^d} u(y)\tau_x\breve{\varphi}(y)\, dy.$$

  Thus, if $u \in \mathcal{S}'$, we can define the convolution of u with a function $\varphi \in \mathcal{S}$ as a new function given by

$$(u * \varphi)(x) \stackrel{\text{def}}{=} u(\tau_x\breve{\varphi}).$$

  Option 2. If $u, \varphi, \psi \in \mathcal{S}$, we have

$$\int_{\mathbb{R}^d}(u * \varphi)(x)\psi(x)\, dx = \int_{\mathbb{R}^d} u(x)(\breve{\varphi} * \psi)(x)\, dx.$$

  Thus, we can define the convolution of a tempered distribution $u \in \mathcal{S}'$ with a function $\varphi \in \mathcal{S}$ as the element $u * \varphi \in \mathcal{S}'$ defined by

$$(u * \varphi)(\psi) = u(\breve{\varphi} * \psi) \quad \text{for any} \quad \psi \in \mathcal{S}. \tag{1.3}$$

Both definitions for the convolution $u * \varphi$, with $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, are equivalent, and the proof of this fact can be found in [SW71, Theorem 3.13]. Finally, note that the basic properties of the Fourier transform in Proposition 1.1 (see (ii)-(vi)) continue to hold in this sense.

Proposition 1.9. If $u \in \mathcal{S}'$ we have

(ii') $(\tau_y u)^\wedge = e^{-2\pi i\xi\cdot y}\widehat{u}$ and $\tau_\eta(\widehat{u}) = [e^{2\pi i\eta\cdot x}u]^\wedge$.

(iii') If T is an invertible linear transformation in $\mathbb{R}^d$ and $S = (T^*)^{-1}$ is the inverse of its transpose, then $(u \circ T)^\wedge = |\det T|^{-1}\widehat{u} \circ S$.

(iv') $\widehat{u * \varphi} = \widehat{u} \cdot \widehat{\varphi}$, where $\varphi \in \mathcal{S}$.

(v') $\partial^\alpha \widehat{u} = [(-2\pi i x)^\alpha u]^\wedge$ and $(\partial^\alpha u)^\wedge = (2\pi i\xi)^\alpha \widehat{u}$.

1.2 Fourier series

We say that a function $f : \mathbb{R}^d \to \mathbb{C}$ is periodic[3] if $f(x+m) = f(x)$ for all $x \in \mathbb{R}^d$ and all $m \in \mathbb{Z}^d$. Periodic functions in $\mathbb{R}^d$ can then be identified with functions defined on the compact metric space $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

A set $\mathcal{D} \subset \mathbb{R}^d$ is said to be a fundamental domain if each point in $\mathbb{R}^d$ has exactly one translated point, with respect to $\mathbb{Z}^d$, in $\mathcal{D}$. Clearly, a periodic function in $\mathbb{R}^d$ is entirely determined by its restriction to a fundamental domain.

In most cases, it will be convenient to use the cube $Q^d = \{x \in \mathbb{R}^d; -1/2 \leq x_i < 1/2\}$ as the fundamental domain. Therefore, we identify measurability and integration on the torus $\mathbb{T}^d$ with measurability and integration on the cube Q^d with respect to the Lebesgue measure

$$\int_{\mathbb{T}^d} f(x)\, dx \overset{\text{def}}{=} \int_{Q^d} f(x)\, dx.$$

The spaces $L^p(\mathbb{T}^d)$ and $C^j(\mathbb{T}^d)$ (functions in Q^d whose periodic extension to $\mathbb{R}^d$ is of class C^j) are well-defined this way. If $f \in L^1(\mathbb{T}^d)$, we define its k-th Fourier coefficient (where k is an element of $\mathbb{Z}^d$) as

$$\widehat{f}(k) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i k\cdot x}\, dx.$$

Our goal in this section is to understand when we can express a periodic function f as a superposition of $e^{2\pi i m\cdot x}$, i.e.,

$$f(x) = \sum_{m\in\mathbb{Z}^d} a_m e^{2\pi i m\cdot x} \tag{1.4}$$

Formally, if we multiply (1.4) by $e^{-2\pi i k\cdot x}$ and integrate over a period Q^d (using the orthonormality of the system $\{e^{2\pi i k\cdot x}; k \in \mathbb{Z}^d\}$), we obtain $a_k = \widehat{f}(k)$. Therefore, we would like to know when the equality

$$f(x) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k\cdot x} \tag{1.5}$$

holds. At first, we know nothing about the convergence of the right-hand side of (1.5). Therefore, we limit ourselves to just formally defining the Fourier series $S[f]$ of a function $f \in L^1(\mathbb{T}^d)$ as

$$S[f] = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k\cdot x}.$$

We now present the mechanism that relates the theory of the Fourier transform in the case of $\mathbb{R}^d$ to its periodic version: the so-called Poisson summation formula. Setting

[3] Modulo the lattice $\mathbb{Z}^d$.

aside convergence issues momentarily, given a function $f : \mathbb{R}^d \to \mathbb{C}$, we consider its periodic extension given by

$$F(x) = \sum_{m \in \mathbb{Z}^d} f(x+m). \tag{1.6}$$

For the right-hand side of (1.6) to make sense pointwise or at least almost everywhere, we need some decay in f. Indeed, if $f \in L^1(\mathbb{R}^d)$, we have that

$$\begin{aligned} \int_{\mathbb{T}^d} |F(x)|\, dx = \int_{Q^d} \left| \sum_{m \in \mathbb{Z}^d} f(x+m) \right| dx \leq \int_{Q^d} \sum_{m \in \mathbb{Z}^d} |f(x+m)|\, dx \\ = \sum_{m \in \mathbb{Z}^d} \int_{Q^d} |f(x+m)|\, dx = \sum_{m \in \mathbb{Z}^d} \int_{Q^d+m} |f(y)|\, dy = \int_{\mathbb{R}^d} |f(y)|\, dy. \end{aligned}$$

If $f \in L^1(\mathbb{R}^d)$, the above calculation shows that $F \in L^1(\mathbb{T}^d)$, and hence the sum (1.6) is absolutely convergent almost everywhere $x \in \mathbb{T}^d$. We can then compute the Fourier coefficients of the periodic function F, namely

$$\begin{aligned} \widehat{F}(k) &= \int_{\mathbb{T}^d} F(x) e^{-2\pi i k \cdot x}\, dx \\ &= \int_{Q^d} \left(\sum_{m \in \mathbb{Z}^d} f(x+m) \right) e^{-2\pi i k \cdot x}\, dx = \sum_{m \in \mathbb{Z}^d} \int_{Q^d} f(x+m) e^{-2\pi i k \cdot x}\, dx \\ &= \sum_{m \in \mathbb{Z}^d} \int_{Q^d+m} f(y) e^{-2\pi i k \cdot y}\, dy = \int_{\mathbb{R}^d} f(y) e^{-2\pi i k \cdot y}\, dy = \widehat{f}(k), \end{aligned} \tag{1.7}$$

where the interchange between the integral and the sum is justified by (1.7). Therefore, we can conclude that

$$S[F](x) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k \cdot x}.$$

If f and $\widehat{f}$ have well-controlled decay, not only in the integral sense but also in a pointwise sense, we can establish convergence at every point.

Theorem 1.10 (Poisson summation formula). Suppose f and $\widehat{f}$ satisfy the pointwise estimates[4]

$$|f(x)| \leq \frac{C}{(1+|x|)^{d+\varepsilon}} \quad \text{and} \quad |\widehat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{d+\varepsilon}},$$

for some $C > 0$ and $\varepsilon > 0$. Then, the identity

$$\sum_{m \in \mathbb{Z}^d} f(x+m) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k \cdot x}$$

holds for any point $x \in \mathbb{T}^d$. In particular,

$$\sum_{m \in \mathbb{Z}^d} f(m) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k).$$

[4] In particular, we can assume f and $\widehat{f}$ are continuous by Fourier inversion.

The proof of this result can be found in [Fol99, Theorem 8.36]. Now we present a refinement of the pointwise convergence result in the case of dimension $d = 1$, whose proof follows from [Fol99, Theorem 8.43].

Theorem 1.11 (Poisson summation formula - one dimension). Let $f \in L^1(\mathbb{R})$ be a normalized function of bounded variation. Then, the following holds[5]

$$\sum_{m\in\mathbb{Z}} f(x+m) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$$

for every point $x \in \mathbb{R}$.

1.3 Exponential type and Paley-Wiener theorem

Definition 1.12 (Exponential type). An entire function $F : \mathbb{C} \to \mathbb{C}$ is said to be of exponential type if

$$\tau(F) \stackrel{def}{=} \limsup_{|z|\to\infty} |z|^{-1} \log |F(z)| < \infty. \tag{1.8}$$

In this case, the number $\tau(F)$ (or simply τ, if that does not cause confusion) is called the exponential type of F. In the limsup language, F has exponential type τ if for each $\varepsilon > 0$, there exists a constant C_ε such that

$$|F(z)| \leq C_\varepsilon e^{(\tau+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$. One of the most classical results connecting the Fourier transform with exponential type is the Paley-Wiener theorem. This theorem states that for an entire function $F : \mathbb{C} \to \mathbb{C}$ such that $F \in L^2(\mathbb{R})$, the conditions

- F has exponential type $2\pi\delta$.
- The Fourier transform of F is supported on $[-\delta, \delta]$.

are equivalent and we will prove this in the first section of Chapter 3. For now, we will extend the concept of exponential type to d dimensions. In the context of $\mathbb{R}^d$, any norm $||\cdot||$ is equivalent to the Euclidean norm. Selecting a norm $\|\cdot\|$, we define a convex, compact, and symmetric[6] set $K = \left\{x \in \mathbb{R}^d \,:\, ||x|| \leq 1\right\}$ as the unit sphere with respect to this norm. When such a subset has a nonempty interior, it is referred to as a symmetric body.

[5] Both sums are taken in the sense $\lim_{N\to\infty} \sum_{-N}^{N}$.

[6] That is, if $x \in K$ then $-x \in K$.

For any subset K of $\mathbb{R}^d$, we define the polar set

$$K^* \stackrel{def}{=} \left\{ y \in \mathbb{R}^d \, : \, x \cdot y \leq 1 \text{ for all } x \in K \right\}.$$

If K is a symmetric body, then K^* is also a symmetric body. This makes K^* the unit sphere with respect to a norm $|| \cdot ||^*$, known as the dual norm to the one defining K as the unit sphere. An equivalent way of introducing the dual norm is

$$||y||^* = \sup_{x \in K} |x \cdot y|. \tag{1.9}$$

The next lemma is simple but useful. Its proof can be found in [SW71, Lemma 4.7].

Lemma 1.13. Suppose $K \subset \mathbb{R}^d$ is convex, closed, and $0 \in K$. Then

$$K^{**} = (K^*)^* = K.$$

Applying this result and the relation (1.9), we can therefore obtain

$$||x|| = ||x||^{**} = \sup_{y \in K^*} |x \cdot y|.$$

If $z \in \mathbb{C}^d$, it is natural to define $||z||$ by extending

$$||z|| \stackrel{def}{=} \sup_{y \in K^*} |z \cdot y|.$$

We now say that an entire function $F : \mathbb{C}^d \to \mathbb{C}$ is of exponential type δ with respect to a symmetric body K if for each $\varepsilon > 0$ there exists a positive constant C such that

$$|F(z)| \leq C_\varepsilon e^{(\delta+\varepsilon)||z||}. \tag{1.10}$$

This simplifies to something similar to (1.8) using the concept of lim sup, since it is essentially equivalent to

$$\tau(F) \stackrel{def}{=} \limsup_{\|z\| \to \infty} \|z\|^{-1} \log |F(z)| = \delta.$$

The class of all functions of exponential type $2\pi\delta$ with respect to a symmetric body K is denoted by $\mathfrak{E}(2\pi\delta; K)$. In this case, the number $\tau(F)$ is called the exponential type of F. A d-dimensional version of the Paley-Wiener theorem can be stated, with the proof available in [SW71, Theorem 4.9].

Theorem 1.14 (Paley-Wiener - d-dimensional version). Suppose $F \in L^2(\mathbb{R}^d)$. Then F is the Fourier transform of a function vanishing outside a symmetric body[7] δK^* if and only if F is the restriction to $\mathbb{R}^d$ of a function in $\mathfrak{E}(2\pi\delta; K)$.

[7] The symmetric body δK^* is obtained by dilating K^* by a factor of δ.

It is interesting to learn about this general construction of exponential type. However, from now until the end of our work, when we refer to a function in several dimensions having exponential type, we will be referring to it as defined in (1.10) with respect to the norm

$$\|z\| \stackrel{def}{=} \sup\left\{\left|\sum_{n=1}^{d} z_n t_n\right| \,:\, t\in\mathbb{R}^d \text{ and } |t|\leq 1\right\}.$$

Thus, K^* is the ball in the usual Euclidean norm. As before, where we extended results of functions in $L^2(\mathbb{R}^d)$ to distributions, we can do the same for Paley-Wiener's theorem. To achieve this, we need to revisit some basic concepts of distribution theory. We typically have the inclusions $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{S}(\mathbb{R}^d)\subseteq C^\infty(\mathbb{R}^d)$.

Additionally, a topology induced by norms can be imposed on each of these spaces, transforming them into Fréchet spaces. If we denote the topological dual of these spaces by $(C^\infty(\mathbb{R}^d))'=\mathcal{E}'(\mathbb{R}^d)$ and $(C_c^\infty(\mathbb{R}^d))'=\mathcal{D}'(\mathbb{R}^d)$, then

$$\mathcal{E}'(\mathbb{R}^d)\subseteq\mathcal{S}'(\mathbb{R}^d)\subseteq\mathcal{D}'(\mathbb{R}^d).$$

Therefore, based on the results discussed in Section 1.1, we can define the Fourier transform of each element $u\in\mathcal{E}'(\mathbb{R}^d)$.

Furthermore, according to [Hör64, Theorem 1.5.2], the space $\mathcal{E}'(\mathbb{R}^d)$ is precisely the space of distributions in $\mathbb{R}^d$ with[8] compact support. The proof of the following results can be found in [Hör64, Theorem 1.7.5 and Theorem 1.7.7].

Lemma 1.15. The Fourier transform of a distribution $u\in\mathcal{E}'(\mathbb{R}^d)$ is the function

$$\widehat{u}(\xi)=u_x(e^{-2\pi i\xi\cdot x}). \tag{1.11}$$

The right-hand side is also defined for every complex vector $\xi\in\mathbb{C}^d$ and is an entire function of ξ, called the Fourier-Laplace transform of u.

Theorem 1.16 (Paley-Wiener-Schwartz). An entire analytic function $F(z)$ is the Fourier-Laplace transform of a distribution with support in the ball $B_\delta\subset\mathbb{R}^d$ if and only if, for some positive constants C and N, we have

$$|F(z)|\leq C(1+|z|)^N e^{2\pi\delta|\operatorname{Im}(z)|}.$$

Moreover, F is the Fourier-Laplace transform of a function f in $\mathcal{C}_c^\infty(B_\delta)$ if and only if, for every integer N, there exists a positive constant C_N such that

$$|F(z)|\leq C_N(1+|z|)^{-N} e^{2\pi\delta|\operatorname{Im}(z)|}.$$

[8] If $u\in\mathcal{D}'(\mathbb{R}^d)$, the support of u is defined as the set of points in $\mathbb{R}^d$ which have no neighborhood where u is equal to the distribution 0. We shall say that two distributions u_1 and u_2 in $\mathcal{D}'(\mathbb{R}^d)$ are equal in a neighborhood of a point $x\in\mathbb{R}^d$ if the restrictions of u_1 and u_2 to some open neighborhood of x are equal (remember that we can define the restriction of a distribution to an open set $\mathcal{O}\subset\mathbb{R}^d$ simply by restricting the domain of definition of the linear form u to $\mathcal{C}_c^\infty(\mathcal{O})$).

Chapter 2

Entire functions

In this chapter, we present some essential facts about the theory of analytic functions, a fundamental prerequisite for understanding the theory of de Branges spaces. In the first section, we will prove the Phragmén-Lindelöf theorem. In the second section, we will present the Stieltjes inversion and the Poisson representation, aiming to construct and characterize analytic functions with nonnegative real parts in the upper half-plane.

In sections 2.3 and 2.4, we will present the concepts of Pólya class and bounded type, respectively, later proving some factorization theorems for these classes of functions. Furthermore, in each of these two sections, we will prove some properties that are useful for constructing de Branges spaces. In the end, we will prove Krein's theorem, which establishes a connection between the concepts of exponential type and mean type. The reference for this chapter is [Bra68, Chapter 1].

2.1 Phragmén–Lindelöf principle

If a function $f(z)$ is analytic in the unit disk, $|z| < 1$, and has a continuous extension to the closed disk, then $|f(z)|$ must attain a maximum value in the closed disk. According to the maximum principle, this maximum value does not occur within the interior of the disk when $f(z)$ is not a constant. Therefore, if $|f(z)|$ remains bounded by 1 along the boundary of the disk, it is also bounded by 1 within the disk itself.

Remark 2.1. Let $G \subset \mathbb{C}$ be a bounded and connected open set. Let $f : \overline{G} \to \mathbb{C}$ be continuous on the closure $\overline{G}$ of G and analytic on G. Then

$$\max_{z \in \overline{G}} |f(z)| = \max_{z \in \partial G} |f(z)|.$$

On the other hand, the scenario differs in the upper half-plane, where a maximum might not exist within the closure of an unbounded region. Consider a function $f(z)$ that is analytic in the upper half-plane, continuous in the closed half-plane, and bounded by 1 along the real axis. Our objective is to determine the boundedness of $|f(z)|$ by 1 in the upper half-plane. However, the example $f(z) = e^{-iz}$ illustrates the necessity of certain additional hypotheses, because $f(iy) = e^y$ for real y.

The Phragmén–Lindelöf principle asserts the validity of this conclusion if $|f(z)|$ remains bounded within $\mathbb{C}^+$ or if it satisfies a similar, but weaker, hypothesis.

Theorem 2.2 (Phragmén–Lindelöf). Assume that $f(z)$ is analytic in the upper half-plane, that $|f(z)|$ has a continuous extension to the closed half-plane, and that

$$\liminf_{a\to\infty} a^{-1}\int_0^\pi \log^+|f(ae^{i\theta})|\sin\theta\,d\theta = 0. \tag{2.1}$$

If $|f(z)|$ is bounded by 1 on the real axis, then $|f(z)|\le 1$ in the upper half-plane.

Proof. The idea is to prove a boundedness on the unit disk and extend it to a disk of arbitrary radius with a dilation argument. We start with an estimate of $|f(z)|$ in the upper half-disk, $|z|<1$ and $y>0$, based on the knowledge that $|f(z)|$ is bounded by 1 on the real part of the boundary of the half-disk. If $h(\theta)$ is any continuous, real-valued function of real θ that is periodic with period 2π, there exists a function $g(z)$ that is analytic for $|z|<1$ such that $\operatorname{Re} g(z)$ has a continuous extension to the closed half-disk and $h(\theta)=\operatorname{Re} g(e^{i\theta})$ for all real θ. It is given by[1]

$$g(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}h(\theta)\,d\theta$$

for $|z|<1$ and

$$\operatorname{Re} g(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}h(\theta)\,d\theta=\int_0^{2\pi}K_\rho(\alpha-\theta)h(\theta)\,d\theta$$

where $z=\rho e^{i\alpha}$ and $K_\rho(\theta)$ is the Poisson kernel of the disk

$$K_\rho(\theta)=\frac{1}{2\pi}\frac{1-\rho^2}{1-2\rho\cos(\theta)+\rho^2}.$$

We use this construction with $h(\theta)=\log^+|f(e^{i\theta})|$ for $0\le\theta\le\pi$. Extend $h(\theta)$ to be an odd periodic function of real θ which is periodic with period 2π. This is possible because $|f(z)|$ is bounded by 1 on the real axis, so $h(0)=h(\pi)=h(-\pi)=0$. Therefore

$$h(\theta)\overset{\text{def}}{=}\begin{cases}\log^+|f(e^{i\theta})|, & \text{if } 0\le\theta\le\pi;\\ -\log^+|f(e^{-i\theta})|, & \text{if } -\pi\le\theta\le 0.\end{cases}$$

Since

$$g(z)=\frac{1}{2\pi}\int_0^\pi\frac{e^{i\theta}+z}{e^{i\theta}-z}\,h(\theta)\,d\theta-\frac{1}{2\pi}\int_0^\pi\frac{e^{-i\theta}+z}{e^{-i\theta}-z}\,h(\theta)\,d\theta$$

for $|z|<1$, we obtain

$$\begin{aligned}\operatorname{Re} g(z)&=\frac{1-|z|^2}{2\pi}\int_0^\pi\frac{h(\theta)\,d\theta}{|e^{i\theta}-z|^2}-\frac{1-|z|^2}{2\pi}\int_0^\pi\frac{h(\theta)\,d\theta}{|e^{-i\theta}-z|^2}\\&=\frac{1-|z|^2}{2\pi}\int_0^\pi\frac{4yh(\theta)\sin\theta\,d\theta}{|e^{i\theta}-z|^2|e^{-i\theta}-z|^2}.\end{aligned}\tag{2.2}$$

[1] The analyticity follows from Morera's Theorem.

Note that $\operatorname{Re} g(z)$ vanishes on the real part of the unit disk, just set $y = 0$ in (2.2). Since we assume that $|f(z)| \leq 1$ on the real axis and by construction

$$\operatorname{Re} g(e^{i\theta}) = h(\theta) = \log^+ |f(e^{i\theta})|$$

we have $|f(z)|e^{-\operatorname{Re} g(z)}$ bounded by 1 on the boundary of the upper half-disk. By the maximum principle, the function is bounded by 1 in the interior of the half-disk. Explicitly,

$$\log |f(z)| \leq \frac{1 - |z|^2}{2\pi} \int_0^\pi \frac{4y \log^+ |f(e^{i\theta})| \sin\theta}{|e^{i\theta} - z|^2 |e^{-i\theta} - z|^2} d\theta \tag{2.3}$$

for $|z| < 1$ and $y > 0$. The same argument applies with $f(z)$ replaced by $f(az)$ when $a > 0$. If z is replaced by z/a in (2.3), it reads

$$\log |f(z)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log^+ |f(ae^{i\theta})| \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta \tag{2.4}$$

for $|z| < a$ and $y > 0$. If $|z| < \varepsilon a$ where $0 < \varepsilon < 1$, then

$$|ae^{i\theta} - z| \geq a(1 - |z/a|) \geq a(1 - \varepsilon).$$

For each fixed z in the upper half-plane, we have

$$\log |f(z)| \leq \frac{2y}{\pi}(1 - \varepsilon)^{-4} \liminf_{a\to\infty} a^{-1} \int_0^\pi \log^+ |f(ae^{i\theta})| \sin\theta \, d\theta.$$

Since ε is arbitrary, we take $\varepsilon \to 0$

$$\log |f(z)| \leq (2y/\pi) \liminf_{a\to\infty} a^{-1} \int_0^\pi \log^+ |f(ae^{i\theta})| \sin\theta \, d\theta$$

and the theorem follows.

□

Corollary 2.3. Given $a > 0$, we have

$$ay = \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ya \sin^2\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta$$

for $|z| < a$ and $y > 0$.

Proof. Take $a = 1$. Here we copy the formula of (2.2) in Theorem 2.2, using the fact that $-iz$ solves the Dirichlet problem in the unit disk with boundary condition $h(\theta) = \sin(\theta)$. Since the solution of this problem is unique, the result follows in this case. The general case follows by a dilation argument.

□

Remark 2.4. Every entire function with exponential type zero satisfies condition (2.1). The same holds true for analytical bounded functions on the upper half-plane.

Corollary 2.5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of zero exponential type. If f is bounded on the real axis in the complex plane, then f is a constant function.

Proof. The function satisfies the Phragmén–Lindelöf condition (2.1). Since the function is bounded over the real line, it is also bounded in the upper half-plane. We can replace f with f^* and reach the same conclusion. Therefore, f is bounded, and by Liouville's theorem, it is constant. □

2.2 Stieltjes inversion and Poisson representation

To apply the Phragmén–Lindelöf principle, we need to construct functions that are analytic in the upper half-plane and have a given modulus at points on the real axis. The following theorem outlines this process.

Theorem 2.6. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(x) \geq 1$ for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \frac{\log h(t)}{1+t^2}\, dt < \infty.$$

The formula

$$f(z) = \exp\left(\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2}\frac{\log h(t)}{t-z} dt + \frac{z}{\pi i}\int_{-\infty}^{\infty} \frac{\log h(t)}{1+t^2} dt\right)$$

defines a function $f(z)$ that is analytic in the upper half-plane. It satisfies $|f(z)| \geq 1$, $|f(z)|$ has a continuous extension to $\overline{\mathbb{C}^+}$, and $|f(x)| = h(x)$ for all real x.

Proof. It is clear that $\log f(z)$ is a well-defined function that is analytic in $\mathbb{C}^+$. Since

$$\operatorname{Re}\left(\frac{1\,+\,z^2}{i(t-z)} + \frac{z}{i}\right) = \frac{1}{2}\left(\frac{1\,+\,z^2}{i(t-z)} - \frac{1\,+\,\bar{z}^2}{i(t-\bar{z})} + \frac{z\,-\,\bar{z}}{i}\right) = y\,\frac{1\,+\,t^2}{|t-z|^2}$$

we obtain for $y > 0$

$$\log|f(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log h(t)}{(t-x)^2+y^2} dt = (P_y * \log h)(x),$$

where $P_y(t) = y/(\pi(x^2+y^2))$ is the Poisson kernel for the upper half-plane. Here we cannot apply the classical theory of approximation of identity because $\log h$ does not necessarily belong to $L^1(\mathbb{R})$, but we can apply similar ideas to prove the result.

Since we assume that $h(x) \geq 1$ for all real x, it follows that $|f(z)| \geq 1$ for $y > 0$. The main problem is to show that $|f(z)|$ is continuous in the closed half-plane and has $h(x)$ as a boundary value function. Let u be a real number. We need to prove that

$$\log h(u) = \lim_{z\to u} \operatorname{Re}\log f(z)$$

as $z \to u$ through nonreal values. Explicitly, the problem is to show that

$$\log h(u) = \lim_{z\to u} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log h(t)}{(t-x)^2+y^2} dt.$$

Given that

$$1 = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{dt}{(t-x)^2+y^2},$$

it is sufficient to show that

$$0 = \lim_{z\to u} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|\log h(t) - \log h(u)|}{(t-x)^2+y^2} dt.$$

If $\varepsilon > 0$ is given, choose $\delta > 0$ so that $|\log h(t) - \log h(u)| \leq \varepsilon/2$ whenever $|t-u| \leq \delta$. If $|u-x| \leq \delta/2$, then

$$\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|\log h(t) - \log h(u)|}{(t-x)^2+y^2} dt \leq \frac{1}{2}\varepsilon + \frac{y}{\pi} \int_{|u-t|>\delta} \frac{|\log h(t) - \log h(u)|}{(t-u+\frac{1}{2}\delta)^2} dt < \varepsilon$$

when y is sufficiently small. □

Remark 2.7. The Theorem 2.6 can be written as follows. Let $h(x)$ be a continuous function of real x such that $h(x) \geq 0$ for all real x and $\int_{-\infty}^{\infty} h(t)(1+t^2)^{-1}\, dt < \infty$. Then the formula

$$g(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2} \frac{h(t)}{t-z} dt + \frac{z}{\pi i} \int_{-\infty}^{\infty} \frac{h(t)}{1+t^2} dt$$

defines a function $g(z)$, that is analytic in the upper half-plane, such that

$$\operatorname{Re} g(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(t)}{(t-x)^2+y^2} dt \geq 0, \tag{2.5}$$

$\operatorname{Re} g(z)$ has a continuous extension to $\overline{\mathbb{C}^+}$ and $\operatorname{Re} g(x) = h(x)$ for all real x.

Note that if we sum $-ipz$ for $p \geq 0$ on the right side of (2.5), the function $g(z)$ still retains the same properties. The idea now is to make the opposite construction: given a function g that has the nonnegative real part in the upper half-plane, we want to characterize this function. We want, morally, to prove that it is something similar to the formula of (2.5). For this, we need some auxiliary propositions.

Proposition 2.8. Let $f(z)$ be a function that is analytic and has a nonnegative real part in the upper half-plane. Assume that $\operatorname{Re} f(z)$ has a continuous extension to the closed half-plane and that $h(x)$ is a bounded, continuous function of real x such that $0 \leq h(x) \leq \operatorname{Re} f(x)$ for all real x. Then for $y > 0$

$$\operatorname{Re} f(x+iy) \geq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(t)}{(t-x)^2+y^2} dt. \tag{2.6}$$

Proof. As $h(x)$ is bounded, we can use Remark 2.7 and construct g such that

$$\operatorname{Re} g(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{h(t)}{(t-x)^2+y^2}dt = h * P_y(x) \leq M, \tag{2.7}$$

for some $M > 0$. Define $\varphi(z) = e^{g(z)-f(z)}$ in the upper half-plane. This function is analytic in this domain and $|\varphi(z)| = e^{\operatorname{Re}(g(z)-f(z))}$ is continuous in $\overline{\mathbb{C}^+}$, because $\operatorname{Re} g(z)$ and $\operatorname{Re} f(z)$ are already. Since $\operatorname{Re} f(z) \geq 0$, it follows that

$$|\varphi(z)| = e^{\operatorname{Re} g(z) - \operatorname{Re} f(z)} \leq e^M.$$

Furthermore, using (2.7) and approximations of identity, we have

$$|\varphi(x)| \leq e^{\operatorname{Re} g(x) - \operatorname{Re} f(x)} = e^{h(x) - \operatorname{Re} f(x)} \leq 1.$$

So, by Remark 2.4 and Theorem 2.2, the result follows. □

Proposition 2.9. Let $f(z)$ be a function that is analytic and has a nonnegative real part in the upper half-plane. If $\operatorname{Re} f(z)$ has a continuous extension to the closed half-plane, then there exists a function $g(z)$ that is analytic and has a nonnegative real part in the upper half-plane, such that

$$\operatorname{Re} f(x+iy) = \operatorname{Re} g(x+iy) + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2+y^2}dt \tag{2.8}$$

for $y > 0$. Moreover, $\operatorname{Re} g(z)$ is continuous in $\overline{\mathbb{C}^+}$ and $\operatorname{Re} g(x) = 0$ for all real x.

Proof. The idea here is to use Proposition 2.8. Construct a sequence $\{h_n(x)\}$ of continuous and bounded functions on the real line such that $0 \leq h_n(x) \leq \operatorname{Re} f(x)$ and $h_n(x) \nearrow \operatorname{Re} f(x)$ when $n \to \infty$. Using Monotone Convergence Theorem in (2.6)

$$\operatorname{Re} f(x+iy) \geq \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2+y^2}dt. \tag{2.9}$$

In particular, for $x = 0$ and $y = 1$, we observe that $\operatorname{Re} f(x)$ satisfies the condition of Remark 2.7. So, we can construct an analytic function $\varphi(z)$ in the upper half-plane such that

$$\operatorname{Re} \varphi(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2+y^2}dt.$$

Let $g \overset{\text{def}}{=} f - \varphi$. Note that g is analytic in $\mathbb{C}^+$ and continuous in the closed half-plane, because f and φ are already. Moreover, g satisfies the condition (2.8) by definition. Finally, by (2.9), $\operatorname{Re} g(z) \geq 0$, and if x is real, $\operatorname{Re} \varphi(x) = \operatorname{Re} h(x)$ by Remark 2.7. We obtain that $\operatorname{Re} g(x) = 0$. □

Proposition 2.10. Let $g(z)$ be an analytic function with nonnegative real part in the upper half-plane. Suppose that $\operatorname{Re} g(z)$ is continuous in the closed upper half-plane and that $\operatorname{Re} g(x) = 0$ for all real x. Then $\operatorname{Re} g(x+iy) = py$, where p is a constant.

Proof. We will prove that

$$\operatorname{Re} g(z) = \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \operatorname{Re} g(ae^{i\theta}) \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta \tag{2.10}$$

when $a > 0$, $|z| < a$, and $y > 0$. To prove (2.10), recall that the function on the right-hand side is a solution of the Dirichlet problem on the disk, as proved in the proof of Theorem 2.2. Since this disk solution solves the problem for the upper half-disk, the result follows by the uniqueness of the solution of the associated Dirichlet problem. When $a > 0$, $|z| < a$ and $y > 0$

$$\frac{\operatorname{Re} g(x+iy)}{y} = \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4a \operatorname{Re} g(ae^{i\theta}) \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta. \tag{2.11}$$

As the left-hand side of (2.11) does not depend on a, we conclude that for a fixed z in the upper half-plane, the limit as $a \to \infty$ of the right-hand side exists. We need to show that the limit is the same for each z. First, for $x = 0$, we have by the Dominated Convergence Theorem

$$\begin{aligned} p \stackrel{\text{def}}{=} \lim_{y\to 0} \frac{\operatorname{Re} g(iy)}{y} &= \lim_{y\to 0} \frac{2a(a^2 - y^2)}{\pi} \int_0^\pi \frac{\operatorname{Re} g(ae^{i\theta}) \sin\theta}{|ae^{i\theta} - iy|^2 |ae^{-i\theta} - iy|^2} d\theta \\ &= \frac{2}{\pi a} \int_0^\pi \operatorname{Re} g(ae^{i\theta}) \sin\theta \, d\theta. \end{aligned}$$

The last equality is valid for all $a > 0$. Fixed $z \in \mathbb{C}^+$, we obtain

$$\begin{aligned} \frac{\operatorname{Re} g(x+iy)}{y} &= \lim_{a\to\infty} \frac{a(a^2 - |z|^2)}{a^3} \frac{2}{\pi a} \int_0^\pi \frac{\operatorname{Re} g(ae^{i\theta}) \sin\theta}{|e^{i\theta} - z/a|^2 |e^{-i\theta} - z/a|^2} d\theta \\ &= \lim_{a\to\infty} \frac{2}{\pi a} \int_0^\pi \frac{\operatorname{Re} g(ae^{i\theta}) \sin\theta}{|e^{i\theta} - z/a|^2 |e^{-i\theta} - z/a|^2} d\theta = p \end{aligned}$$

and the result follows.

□

Now we can state the characterization for functions with nonnegative real part in the upper half-plane.

Theorem 2.11. Let $f(z)$ be a function that is analytic and has a nonnegative real part in the upper half-plane. If $\operatorname{Re} f(z)$ has a continuous extension to the closed half-plane, then

$$\operatorname{Re} f(x+iy) = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2 + y^2} dt \tag{2.12}$$

for $y > 0$. Moreover, we have that p is nonnegative and

$$p = \lim_{y\to\infty} \operatorname{Re} f(iy)/y. \tag{2.13}$$

Proof. By Propositions 2.9 and 2.10, we only need to show (2.13). Using these propositions, we have

$$\frac{\operatorname{Re} f(iy)}{y} = p + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{t^2+y^2}dt \tag{2.14}$$

for $y > 0$. The functions $\{h_y\}_{y\geq 1}$ defined by

$$h_y(t) \stackrel{\text{def}}{=} \frac{1}{\pi}\frac{\operatorname{Re} f(t)}{t^2+y^2}$$

satisfy $h_y(t) \leq h_1(t)$ for every $t \in \mathbb{R}$ and $y > 1$. Moreover, by (2.9), we have $h_1 \in L^1(\mathbb{R})$ and for each fixed t, $\lim_{y\to\infty} h_y(t) = 0$. Therefore, by the Dominated Convergence Theorem and (2.14), the equality follows. □

A more general approach to Remark 2.7 involves replacing a function with a measure. The treatment of boundary behavior is contained in the following theorem.

Theorem 2.12 (Stieltjes inversion formula). Let $\mu(x)$ be a nondecreasing function of real x such that $\int_{-\infty}^{\infty} (1+t^2)^{-1}\, d\mu(t)$ is finite. Then

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2}\frac{d\mu(t)}{t-z} + \frac{z}{\pi i}\int_{-\infty}^{\infty} \frac{1}{1+t^2}d\mu(t)$$

is analytic and has a nonnegative real part in the upper half-plane. If a and b are points of continuity of $\mu(x)$, with $a < b$, then

$$\mu(b) - \mu(a) = \lim_{y\searrow 0}\int_a^b \operatorname{Re} f(x+iy)\, dx \tag{2.15}$$

Proof. As in the proof of Theorem 2.6, $f(z)$ is a well-defined function that is analytic in the upper half-plane. The real part of $f(z)$ is given by

$$\operatorname{Re} f(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \geq 0.$$

With a change in the order of integration

$$\begin{aligned}\int_a^b \operatorname{Re} f(x+iy)\, dx &= \frac{y}{\pi}\int_{-\infty}^{\infty}\int_a^b \frac{1}{(t-x)^2+y^2}dx\, d\mu(t)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right] d\mu(t).\end{aligned}$$

The interchange is justified by Fubini-Tonelli's theorem, since the integrand is nonnegative. To complete the proof, we must show that

$$\mu(b) - \mu(a) = \lim_{y\searrow 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right] d\mu(t),$$

but to prove the desired limit, we need to remember some of the properties of the $\arctan(x)$ function. Note that since $\arctan(x)$ is increasing, so

$$0 \le \arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \le \pi \tag{2.16}$$

for all real t, and

$$\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} = \arctan\frac{y(b-a)}{y^2+(a-t)(b-t)} \tag{2.17}$$

when $a-t$ and $b-t$ have the same sign, because

$$\tan(v-u) = \frac{\tan(v)-\tan(u)}{1+\tan(v)\tan(u)}.$$

Also note that for $x>0$

$$\arctan x = \int_0^x \frac{1}{1+t^2}dt \le \int_0^x dt \le x. \tag{2.18}$$

If $\varepsilon > 0$ is given, choose $\delta > 0$ by the continuity of $\mu(x)$ at a and b so that

$$\mu(a+\delta)-\mu(a-\delta) < \frac{\varepsilon}{5} \quad \text{and} \quad \mu(b+\delta)-\mu(b-\delta) < \frac{\varepsilon}{5}. \tag{2.19}$$

It follows that

$$\begin{aligned}
I \overset{\text{def}}{=} \Big|\mu(b)-\mu(a)-&\frac{1}{\pi}\int_{-\infty}^{\infty}\Big[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\Big]\,d\mu(t)\Big| \\
&\le \frac{1}{\pi}\int_b^{\infty}\Big[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\Big]\,d\mu(t) \\
&\quad + \frac{1}{\pi}\int_a^{b}\Big[\pi - \arctan\frac{b-t}{y} + \arctan\frac{a-t}{y}\Big]\,d\mu(t) \\
&\quad + \frac{1}{\pi}\int_{-\infty}^{a}\Big[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\Big]\,d\mu(t) \\
&\overset{\text{def}}{=} I_1 + I_2 + I_3.
\end{aligned} \tag{2.20}$$

Now, split the integrals as follows:

$$I_1 = \int_b^{b+\delta} + \int_{b+\delta}^{\infty} \overset{\text{def}}{=} I_{1,1} + I_{1,2}; \qquad I_2 = \int_a^{a+\delta} + \int_{a+\delta}^{b-\delta} + \int_{b-\delta}^{b} \overset{\text{def}}{=} I_{2,1} + I_{2,2} + I_{2,3};$$

$$I_3 = \int_{-\infty}^{a-\delta} + \int_{a-\delta}^{a} \overset{\text{def}}{=} I_{3,1} + I_{3,2}.$$

By the two inequalities in (2.16), we have

$$I_{1,1} \le \mu(b+\delta)-\mu(b) \quad \text{and} \quad I_{2,3} \le \mu(b)-\mu(b-\delta).$$

Similarly, we obtain

$$I_{2,1} \le \mu(a+\delta)-\mu(a) \quad \text{and} \quad I_{3,2} \le \mu(a)-\mu(a-\delta).$$

Combining these inequalities with (2.19), (2.20) becomes

$$I \leq \frac{2\varepsilon}{5} + I_{1,2} + I_{2,2} + I_{3,1}. \tag{2.21}$$

Since arctan is increasing and μ is nondecreasing,

$$I_{2,2} \leq \frac{1}{\pi} \int_{a+\delta}^{b-\delta} \left[\pi - 2\arctan\frac{\delta}{y}\right] d\mu(t) \leq \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan\frac{\delta}{y}\right)(\mu(b) - \mu(a)). \tag{2.22}$$

Moreover, we have by (2.17) and (2.18) that

$$I_{1,2} \leq \frac{1}{\pi} \int_{b+\delta}^{\infty} \arctan\frac{y(b-a)}{y^2 + (a-t)(b-t)}\, d\mu(t) \leq \frac{1}{\pi} \int_{b+\delta}^{\infty} \frac{y(b-a)}{y^2 + (a-t)(b-t)}\, d\mu(t) \tag{2.23}$$

and by similar argument

$$I_{3,1} \leq \frac{1}{\pi} \int_{-\infty}^{a-\delta} \arctan\frac{y(b-a)}{y^2 + (a-t)(b-t)}\, d\mu(t) \leq \frac{1}{\pi} \int_{-\infty}^{a-\delta} \frac{y(b-a)}{y^2 + (a-t)(b-t)}\, d\mu(t). \tag{2.24}$$

Finally, when y is small, it follows from (2.21), (2.22), (2.23), and (2.24) that $I < \varepsilon$. As $\varepsilon > 0$ is arbitrary, the result follows. □

Remark 2.13. If $p \geq 0$, then the function

$$f(z) \stackrel{def}{=} -ipz + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2} \frac{d\mu(t)}{t-z} + \frac{z}{\pi i} \int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2}$$

satisfies the same properties as $f(z)$ in Theorem 2.12, i.e., it is analytic, has a nonnegative real part in the upper half-plane, and satisfies the limit (2.15). The difference is that for $z \in \mathbb{C}^+$, we have

$$\operatorname{Re} f(z) = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}.$$

The Phragmén–Lindelöf principle and the Stieltjes inversion formula are used to obtain the Poisson representation of functions that are analytic and have nonnegative real parts in the upper half-plane. However, proving this result requires some real analysis techniques, one of which is contained in the following lemma.

Lemma 2.14 (Helly selection principle). Suppose $\{\mu_j\}_{j=1}^{\infty}$ is a uniformly bounded sequence of increasing functions on an interval I. Then there exists a subsequence that converges pointwise to an increasing function.

Proof. Let $\{r_i\}_{i\in\mathbb{N}}$ represent a dense sequence in the interval I, such as an enumeration of the rational numbers within I. Utilizing Bolzano-Weierstrass's theorem, we can select a subsequence $\{\mu_{1j}\}_{j=1}^{\infty}$ from the original sequence $\{\mu_j\}_{j=1}^{\infty}$ such that $\mu_{1j}(r_1)$ converges. Similarly, we may choose a subsequence $\{\mu_{2j}\}_{j=1}^{\infty}$ of $\{\mu_{1j}\}_{j=1}^{\infty}$ such that $\mu_{2j}(r_2)$ converges. As a subsequence of $\{\mu_{1j}(r_1)\}_{j=1}^{\infty}$, $\{\mu_{2j}(r_1)\}_{j=1}^{\infty}$ also converges.

By continuing this process iteratively, we generate a sequence of sequences $\{\mu_{kj}\}_{j=1}^{\infty}$, where each μ_{kj} is a subsequence of its predecessor, and $\mu(r_n) = \lim_{j\to\infty} \mu_{kj}(r_n)$ exists for $n \leq k$. Thus, for every n, the sequence $\{\mu_{jj}(r_n)\}_{j=1}^{\infty}$ converges to $\mu(r_n)$ as $j \to \infty$, since it is a subsequence of $\{\mu_{nj}(r_n)\}_{j=1}^{\infty}$ from $j = n$ onward.

As μ is monotonically increasing, if $x \in I$ but $x \neq r_n$ for any n, we can choose an increasing subsequence $\{r_{jk}\}_{k=1}^{\infty}$ of $\{r_j\}_{j=1}^{\infty}$ converging to x and define $\mu(x) = \lim_{k\to\infty} \mu(r_{jk})$. Consider a point of continuity x for μ. When $r_k < x < r_n$, the following inequalities hold

$$\mu_{jj}(r_k) - \mu(r_n) \leq \mu_{jj}(x) - \mu(x) \leq \mu_{jj}(r_n) - \mu(r_k).$$

For any $\varepsilon > 0$, we can choose k and n such that $\mu(r_n) - \mu(r_k) < \varepsilon$. Consequently,

$$-\varepsilon \leq \liminf_{j\to\infty}(\mu_{jj}(x) - \mu(x)) \leq \limsup_{j\to\infty}(\mu_{jj}(x) - \mu(x)) \leq \varepsilon.$$

The sequence $\{\mu_{jj}\}_{j=1}^{\infty}$ converges pointwise to μ, except possibly at points of discontinuity. Since μ has countable discontinuities due to its monotonicity, we can repeat the process of extracting subsequences and employing a "diagonal" sequence to obtain a subsequence of the original sequence that converges everywhere in I. □

Theorem 2.15 (Poisson representation). If $f(z)$ is analytic and has a nonnegative real part in the upper half-plane, then there exists a nonnegative number p and a nondecreasing function $\mu(x)$ of real x such that

$$\operatorname{Re} f(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2}.$$

Proof. By Theorem 2.11, we can write

$$\operatorname{Re} f(z+i\varepsilon) = p(\varepsilon)y + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t+i\varepsilon)}{(t-x)^2+y^2}dt$$

for $y > 0$ when $\varepsilon > 0$, where $p(\varepsilon)$ is a nonnegative constant. Since

$$p(\varepsilon) = \lim_{y\to\infty} \operatorname{Re} f(iy+i\varepsilon)/y = \lim_{y\to\infty} \operatorname{Re} f(iy)/y,$$

$p(\varepsilon) = p$ is independent of ε. In terms of

$$\mu_\varepsilon(x) = \int_0^x \operatorname{Re} f(t+i\varepsilon)\, dt$$

the representation becomes

$$\operatorname{Re} f(z+i\varepsilon) = py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu_\varepsilon(t)}{(t-x)^2+y^2}. \tag{2.25}$$

Taking $x = 0$ and $y = 1$ in (2.25), it follows that the numbers $\int_{-\infty}^{\infty}(1+t^2)^{-1}d\mu_\varepsilon(t)$ are uniformly bounded in ε for $0 < \varepsilon < 1$. Given that

$$\mu_\varepsilon(a) - \mu_\varepsilon(-a) \leq (1+a^2)\int_{-\infty}^{\infty}(1+t^2)^{-1}\, d\mu_\varepsilon(t)$$

and $\mu_\varepsilon(-a) \le 0 \le \mu_\varepsilon(a)$ for all $a > 0$, we can conclude that

$$|\mu_\varepsilon(x)| \le (1+a^2)M \qquad \text{for all } x \in [-a, a],$$

where M independently of ε for $0 < \varepsilon < 1$ and for each fixed a. By Lemma 2.14, there exists a decreasing sequence $\{\varepsilon_n\}$ of positive numbers such that $\mu(x) = \lim \mu_\varepsilon(x)$ exists for all real x as $\varepsilon \to 0$ through the sequence $\{\varepsilon_n\}$. We will just use $\varepsilon \to 0$ to simplify the notation. Being a pointwise limit of nondecreasing functions the limit $\mu(x)$ is a nondecreasing function of x, and we will prove that

$$\int_a^b \frac{d\mu(t)}{(t-x)^2+y^2} = \lim_{\varepsilon\to 0} \int_a^b \frac{d\mu_\varepsilon(t)}{(t-x)^2+y^2} \tag{2.26}$$

for $y > 0$ and $-\infty < a < b < \infty$. In fact, let $s(x)$ be a nondecreasing function on $[a, b]$. Given $\delta > 0$, there exists a partition P of $[a, b]$ given by $a = t_0 < t_1 < \ldots < t_k = b$ such that

$$I \overset{\text{def}}{=} \int_a^b s(t)\, d\mu(t) \le \delta + S^-_{P,\mu},$$

where

$$S^-_{P,\mu} \overset{\text{def}}{=} \sum_{i=0}^{k} s(t_i)(\mu(t_{i+1}) - \mu(t_i)).$$

Then

$$I \le \delta + \lim_{\varepsilon\to 0} \sum_{i=0}^{k} s(t_i)(\mu_\varepsilon(t_{i+1}) - \mu_\varepsilon(t_i)) \le \delta + \liminf_{\varepsilon\to 0} S^-_{P,\mu_\varepsilon} \le \delta + \liminf_{\varepsilon\to 0} \int_a^b s(t)\, d\mu_\varepsilon(t)$$

and we can take $\delta \to 0$. Similarly, we can show that

$$\limsup_{\varepsilon\to 0} \int_a^b s(t)\, d\mu_\varepsilon(t) \le I$$

and we conclude that

$$\lim_{\varepsilon\to 0} \int_a^b s(t)\, d\mu_\varepsilon(t) = \int_a^b s(t)\, d\mu(t)$$

for a nondecreasing function $s(x)$ in $[a, b]$. The general result follows from the fact that every function with locally bounded variation can be locally written as the difference of two nondecreasing functions. This proves (2.26). Since by (2.25)

$$\operatorname{Re} f(z + i\varepsilon) \ge py + \frac{y}{\pi} \int_a^b \frac{d\mu_\varepsilon(t)}{(t-x)^2+y^2},$$

it follows that

$$\operatorname{Re} f(z) \ge py + \frac{y}{\pi} \int_a^b \frac{d\mu(t)}{(t-x)^2+y^2}.$$

By the arbitrariness of a and b,

$$\operatorname{Re} f(z) \ge py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2}.$$

Consider the function

$$g(z) = -ipz + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2}\frac{d\mu(t)}{t-z} + \frac{z}{\pi i}\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2}$$

which is analytic and has a nonnegative real part for $y > 0$. Given that

$$\operatorname{Re} g(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2},$$

we have $\operatorname{Re} f(z) \geq \operatorname{Re} g(z)$ for $y > 0$. The function $h(z) = f(z) - g(z)$ is analytic and has a nonnegative real part in the upper half-plane. If a and b are points of continuity of $\mu(x)$, with $a < b$, then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_a^b \operatorname{Re} h(x+i\varepsilon)\,dx &= \lim_{\varepsilon\to 0}\int_a^b \operatorname{Re} f(x+i\varepsilon)\,dx - \lim_{\varepsilon\to 0}\int_a^b \operatorname{Re} g(x+i\varepsilon)\,dx \\ &= [\mu(b)-\mu(a)] - [\mu(b)-\mu(a)] = 0\end{aligned}$$

by the definition of $\mu(x)$ and the Stieltjes inversion formula. The same conclusion follows for all a and b since any interval (a, b) is contained in an interval (c, d) whose endpoints are points of continuity of $\mu(x)$. Since $p = \lim_{y\to\infty} \operatorname{Re} g(iy)/y$ by Theorem 2.11, we obtain $\lim_{y\to\infty} \operatorname{Re} h(iy)/y = 0$. By the representation (2.12),

$$\operatorname{Re} h(z+i\varepsilon) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} h(t+i\varepsilon)}{(t-x)^2+y^2}dt$$

for $y > 0$ if $\varepsilon > 0$. If $-\infty < a < b < \infty$, then we have

$$0 \leq \frac{y}{\pi}\int_a^b \frac{\operatorname{Re} h(t+i\varepsilon)}{(t-x)^2+y^2}dt \leq \frac{1}{\pi y}\int_a^b \operatorname{Re} h(t+i\varepsilon)\,dt,$$

where $\lim_{\varepsilon\to 0}\int_a^b \operatorname{Re} h(t+i\varepsilon)\,dt = 0$. It follows that

$$\operatorname{Re} h(z) = \lim_{\varepsilon\to 0}\left[\frac{y}{\pi}\int_{-\infty}^{a} \frac{\operatorname{Re} h(t+i\varepsilon)}{(t-x)^2+y^2}dt + \frac{y}{\pi}\int_{b}^{\infty} \frac{\operatorname{Re} h(t+i\varepsilon)}{(t-x)^2+y^2}dt\right]$$

If x_1 and x_2 are points in the interval (a, b) and if t lies outside the interval,

$$\begin{aligned}\frac{(t-x_2)^2+y^2}{(t-x_1)^2+y^2} &= \left|\frac{t-x_2-iy}{t-x_1-iy}\right|^2 \\ &= \left|1 - \frac{x_2-x_1}{t-x_1-iy}\right|^2 \\ &\leq \left[1 + \frac{|x_2-x_1|}{\min(|x_1-a|, |x_1-b|)}\right]^2.\end{aligned}$$

In this way, we obtain

$$\operatorname{Re} h(x_1+iy) \leq \operatorname{Re} h(x_2+iy)[1 + |x_2-x_1|/\min(|x_1-a|, |x_1-b|)]^2.$$

By the arbitrariness of a and b, $\operatorname{Re} h(x_1+iy) \leq \operatorname{Re} h(x_2+iy)$. Equality holds since x_1 and x_2 can be interchanged, so for a fixed $y>0$ we have that $\operatorname{Re} h(x+iy)$ is a constant function of x. Therefore, if $h = u+iv$ with u and v real-valued, we obtain

$$h'(z) = \frac{\partial}{\partial x} u(x+iy) + i\frac{\partial}{\partial x} v(x+iy) = i\frac{\partial}{\partial x} v(x+iy).$$

This means that $h'(z)$ maps the upper half-plane onto the imaginary axis iy. By the Open Mapping Theorem, we conclude that $\operatorname{Im} h(x+iy)$ is a real constant. This implies that $h'(z)$ is an imaginary constant for $y>0$. So, $h(z) = icz+d$, with $c \in \mathbb{R}$ and $d \in \mathbb{C}$. Since $\lim_{y\to\infty} \operatorname{Re} h(iy)/y = 0$, the constant c is zero. Thus, $h(z) = d$. Since $\lim_{\varepsilon\to 0} \int_a^b \operatorname{Re} h(t+i\varepsilon)\, dt = 0$, the real part of d is zero. □

Proposition 2.16. Let $\varphi(z)$ be a function that is analytic and has a nonnegative real part in the upper half-plane. Extend it to the lower half-plane so that $\varphi^*(z) = -\varphi(z)$. Let p be a nonnegative number and $\mu(x)$ be a nondecreasing function of x such that

$$\operatorname{Re} \varphi(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

when $y>0$. Then for all nonreal z and w

$$\frac{\varphi(z)+\overline{\varphi(w)}}{\pi i(\bar{w}-z)} = \frac{p}{\pi} + \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-z)(t-\bar{w})}.$$

Proof. By Theorem 2.15 and Remark 2.13, we know that

$$\varphi(z) = -ipz + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2}\frac{d\mu(t)}{t-z} + \frac{z}{\pi i}\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2}$$

for $z \in \mathbb{C}^+$, so for z and w nonreal

$$\begin{aligned} &\varphi(z)+\overline{\varphi(w)} \\ &= ip(\bar{w}-z) + \frac{1}{\pi i}\int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{1+z^2}{t-z} - \frac{1+\bar{w}^2}{t-\bar{w}}\right)\frac{d\mu(t)}{1+t^2} + \frac{(z-\bar{w})}{\pi i}\int_{-\infty}^{\infty}\frac{d\mu(t)}{1+t^2} \\ &= ip(\bar{w}-z) + \frac{1}{\pi i}\operatorname{Re}\left(\int_{-\infty}^{\infty}\left(\frac{1+z^2}{t-z} - \frac{1+\bar{w}^2}{t-\bar{w}} + z - \bar{w}\right)\frac{d\mu(t)}{1+t^2}\right). \end{aligned}$$

As for each t in $\mathbb{R}$,

$$\frac{1+z^2}{t-z} - \frac{1+\bar{w}^2}{t-\bar{w}} + z - \bar{w} = \frac{1}{(t-z)(t-\bar{w})}(1+t^2)(z-\bar{w}),$$

we have that

$$\begin{aligned} \varphi(z)+\overline{\varphi(w)} &= ip(\bar{w}-z) + \frac{z-\bar{w}}{\pi i}\int_{-\infty}^{\infty}\frac{d\mu(t)}{(t-z)(t-\bar{w})} \\ &= \pi i(\bar{w}-z)\left(\frac{p}{\pi} + \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{d\mu(t)}{(t-z)(t-\bar{w})}\right). \end{aligned}$$

□

Proposition 2.17. Let $f(z)$ be a function that is analytic in the complex plane except for simple poles at points $\{t_n\}$ on the real axis. Suppose that $f^*(z) = f(z)$ and that $\operatorname{Re}(-if(z)) > 0$ for $y > 0$. Then there exist positive numbers p_n and a nonnegative number p such that

$$[f(z) - \bar{f}(w)]/(z - \bar{w}) = p + \sum p_n(t_n - z)^{-1}(t_n - \bar{w})^{-1}$$

when z and w are nonreal. Moreover, for all n

$$p_n = \lim_{z \to t_n} (t_n - z) f(z).$$

Proof. Let $G(z) \stackrel{\text{def}}{=} -if(z)$. Note that G is analytic in the upper half-plane with $\operatorname{Re} G(z) > 0$. Furthermore,

$$G(z) = -G^*(z) \tag{2.27}$$

for all $z \in \mathbb{C}^+$. It follows from Proposition 2.16 that for z and w nonreal

$$\frac{G(z) + \overline{G(w)}}{i(\bar{w} - z)} = p + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t - z)(t - \bar{w})},$$

with μ given by the Poisson representation of G in Theorem 2.15

$$\mu(x) = \lim_{\varepsilon \to 0} \int_0^x \operatorname{Re} G(t + i\epsilon)\, dt,$$

for any real x. Now we simply apply the Residue Theorem. Without loss of generality, suppose that 0 and x are not poles of G, with x positive. Then, for $\varepsilon > 0$, define $\Gamma_{x,\varepsilon}$ as a curve given by the boundary of the rectangle with vertices $x + i\varepsilon$, $x - i\varepsilon$, $-i\varepsilon$, and $i\varepsilon$, oriented in the anticlockwise direction. Therefore,

$$\oint_{\Gamma_{x,\varepsilon}} G(z)\, dz = 2\pi i \sum_{0<t_n<x} \operatorname{Res}(G, t_n) = 2\pi \sum_{0<t_n<x} p_n.$$

Note that

$$\oint_{\Gamma_{x,\varepsilon}} G(z)\, dz = \int_0^x G(t + i\epsilon)\, dt - \int_{-\varepsilon}^{\varepsilon} G(x + it)\, dt - \int_0^x G(t - i\epsilon)\, dt + \int_{-\varepsilon}^{\varepsilon} G(it)\, dt$$

and from (2.27), it follows that

$$\oint_{\Gamma_{x,\varepsilon}} G(z)\, dz = \int_{-\varepsilon}^{\varepsilon} G(it)\, dt - \int_{-\varepsilon}^{\varepsilon} G(x + it)\, dt + 2\int_0^x \operatorname{Re} G(t + i\epsilon)\, dt.$$

The first two integrals tend to zero when $\varepsilon \to 0^+$, so

$$\mu(x) = \lim_{\varepsilon \to 0^+} \int_0^x \operatorname{Re} G(t + i\epsilon)\, dt = \frac{1}{2} \lim_{\varepsilon \to 0^+} \oint_{\Gamma_{x,\varepsilon}} G(z)\, dz = \sum_{0<t_n<x} \pi p_n.$$

A proof of this fact is analogous for x negative. This implies that $\mu(x)$ is constant in an interval without poles of f and has a jump of continuity of weight πp_n at the points t_n. Due to the fact that μ is nondecreasing, we have $p_n \geq 0$ and it follows that

$$\frac{f(z) - \overline{f(w)}}{z - \bar{w}} = \frac{G(z) + \overline{G(w)}}{i(\bar{w} - z)} = p + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t - z)(t - \bar{w})} = p + \sum \frac{p_n}{(t_n - z)(t_n - \bar{w})}.$$

$\square$

2.3 Factorization of Pólya class

Definition 2.18 (Pólya class). An entire function $E : \mathbb{C} \to \mathbb{C}$ is said to be of Pólya class if it has no zeros in the upper half-plane, if $|E^*(z)| \leq |E(z)|$ for $z \in \mathbb{C}^+$, and if $|E(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed x. We denote the Pólya class as $\mathcal{P}$.

Remark 2.19. We can rewrite the last two conditions of the Pólya class definition in an equivalent form, namely

$$|E(\bar{z}+ih)| \leq |E(z+ih)| \tag{2.28}$$

for all $z \in \mathbb{C}^+$ and for all $h \geq 0$.

Remark 2.20. Using (2.28), we conclude that $\mathcal{P}$ is closed under multiplication. Moreover, it is easy to check that a necessary and sufficient condition for a polynomial to be in the Pólya class is that it has no zeros in the upper half-plane. Finally, $\mathcal{P}$ is closed under uniform convergence, i.e., if $\{f_n\} \subset \mathcal{P}$ and $f_n \to f$ uniformly in $\mathbb{C}$, then $f \in \mathcal{P}$. This is true because (2.28) is trivial to check and f cannot be zero in the upper half-plane by Hurwitz's Theorem for uniform convergence.

Proposition 2.21. Let $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $\{a_i\}_{i=1}^n \subset \mathbb{C}$.

(i) If we define the elementary factors as

$$E_n(z) \stackrel{def}{=} (1-z)\exp[z + z^2/2 + \ldots + z^n/n], \tag{2.29}$$

then

$$|E_n(z) - 1| \leq \exp\left(|z|^{n+1}\right) - 1;$$

(ii) We have that

$$1 + \left|\left(\prod_{i=1}^n a_i\right) - 1\right| \leq \prod_{i=1}^n (1 + |a_i - 1|).$$

Proof. (i) First of all, by induction over n, we can prove that for $x > 0$

$$x + \frac{1}{2}x^2 + \ldots + \frac{1}{n}x^n \leq x^{n+1} + \log(1+n) - \frac{1}{n+3/2} \leq x^{n+1} + \log(1+n). \tag{2.30}$$

If $f(z) = (1-z)\exp[z + z^2/2 + \ldots + z^n/n]$, then

$$f'(z) = -z^n \exp\left[z + z^2/2 + \ldots + z^n/n\right].$$

In particular, we have

$$\begin{aligned}
|E_n(z) - 1| = |f(z) - f(0)| &\leq \int_0^{|z|} |f'(w)|\, dw \\
&\leq \int_0^{|z|} |w|^n \exp\left[|w| + |w|^2/2 + \ldots + |w|^n/n\right] dw \\
&\leq \exp(|z|^{n+1}) - 1,
\end{aligned}$$

where in the last inequality we use (2.30).

(ii) This inequality is an immediate consequence of the triangle inequality, by induction.

□

The last proposition, as we will see, serves to construct the infinite products necessary for the development of the theory.

Proposition 2.22. Let $\{z_n\} \subset \overline{\mathbb{C}^+}$ be a sequence such that

$$\sum_1^\infty \frac{1+y_n}{x_n^2+y_n^2} < \infty. \tag{2.31}$$

Then the product

$$E(z) \stackrel{\text{def}}{=} \prod_1^\infty (1 - z/\bar{z}_n)\, e^{h_n z}$$

converges uniformly on bounded sets of the complex plane if $h_n \stackrel{\text{def}}{=} x_n/\,(x_n^2+y_n^2)$. Moreover, the limit is an entire function of the Pólya class.

Proof. With the notation of (2.29), consider the partial products

$$Q_n(z) = \prod_{k=1}^n E_1(z/\bar{z}_k).$$

We will prove that $\{Q_n(z)\}_n$ is uniformly Cauchy on compact sets. Fix a compact set $K \subset \mathbb{C}$ such that $|z| \le M$ for $z \in K$. If $n, m \in \mathbb{N}$, with $n > m$, we have

$$|Q_n(z) - Q_m(z)| = \prod_{k=1}^m |E_1(z/\bar{z}_k)| \left| \prod_{k=m+1}^n E_1(z/\bar{z}_k) - 1 \right|.$$

Note that, by Proposition 2.21 (i), we have for $z \in K$

$$\begin{aligned}
\prod_{k=1}^m |E_1(z/\bar{z}_k)| &\le \prod_{k=1}^m (1 + |E_1(z/\bar{z}_k) - 1|) \\
&\le \prod_{k=1}^m \exp(|z/\bar{z}_k|^2) \le \prod_{k=1}^m \exp(M^2/|z_k|^2) \\
&\le \exp\left(M^2 \sum_{k=1}^m \frac{1+y_k}{x_k^2+y_k^2} \right) \le \exp\left(M^2 \sum_{k\ge 1} \frac{1+y_k}{x_k^2+y_k^2} \right) C < \infty.
\end{aligned}$$

So, this product is uniformly bounded for $z \in K$ and $m \in \mathbb{N}$. By Proposition 2.21 (ii)

$$\begin{aligned}
\left| \prod_{k=m+1}^n E_1(z/\bar{z}_k) - 1 \right| &\le -1 + \prod_{k=m+1}^n (1 + |E_1(z/\bar{z}_k) - 1|) \\
&\le -1 + \prod_{k=m+1}^n \exp(M^2/|z_k|^2) \\
&\le -1 + \exp\left(M^2 \sum_{k=m+1}^n \frac{1+y_k}{x_k^2+y_k^2} \right)
\end{aligned} \tag{2.32}$$

and the last inequality is true by the same argument as before. As the right-hand side of (2.32) is Cauchy and does not depend on z, we conclude that $\{Q_n(z)\}$ is uniformly Cauchy in K. Therefore, $\{Q_n(z)\}$ converges uniformly in compacts to an entire function $Q(z)$. Let us define

$$P_n(z) = \prod_{k=1}^{n} (1 - z/\bar{z}_k) e^{h_k z}.$$

If

$$R_n(z) = \exp\left(-iz \sum_{k=1}^{n} \frac{y_k}{x_k^2 + y_k^2} \right),$$

then $P_n(z) = Q_n(z) R_n(z)$. Note that R_n converges uniformly in compacts to an entire function R because the condition (2.31) implies that

$$\sum_{k \geq 1} \frac{y_k}{x_k^2 + y_k^2} < \infty.$$

So it follows that $\{P_n(z)\}$ converges uniformly in compacts to an entire function $P(z)$. We will now prove that $P \in \mathcal{P}$. Note that P_n are functions in the Pólya class by Proposition 2.20 (because they are the product of Pólya functions). This fact, along with the uniform convergence and the same proposition, concludes that $P \in \mathcal{P}$.

□

Proposition 2.23. Let $\{z_n\}$ be a sequence of nonzero numbers such that $\lim 1/z_n = 0$. With the notation of (2.29), the Weierstrass product

$$P(z) = \prod_{n=1}^{\infty} E_n(z/z_n)$$

converges uniformly on every bounded subset of $\mathbb{C}$.

Proof. The proof is similar to that of the previous proposition. We prove that the partial products are uniformly Cauchy in compacts using Proposition 2.21, but now we use the fact that $|z_n| \to \infty$ when $n \to \infty$. Details will be omitted.

□

Proposition 2.24. Let $\{x_n\}$ be a sequence of real numbers that has no finite limit point. Then there exists a real entire function $S(z)$ which has the sequence $\{x_n\}$ as its sequence of zeros.

Proof. Let $S(z) \stackrel{\text{def}}{=} z^m P(z)$, with m the number of times that 0 appears in $\{x_n\}$ and $P(z)$ given by Proposition 2.23 for the sequence $\{x_n\} \setminus \{0\}$. By construction, S is a real entire function because $\{x_n\} \subset \mathbb{R}$. So, we need to prove that S has the desired property over the zeros.

Fix $R > 0$, and suppose that z belongs to the disc $|z| < R$. We shall prove that f has all the desired properties in this disc, and since R is arbitrary, this will prove the result. We can consider two types of factors in the formula defining f, with the choice depending on whether $|z_n| \leq 2R$ or $|z_n| > 2R$. There are only finitely many terms of the first kind because $|a_n| \to \infty$, and we see that the finite product vanishes at all $z = z_n$ with $|z_n| < R$. If $|z_n| \geq 2R$, we have $|z/z_n| \leq 1/2$, hence $|1 - E_n(z/z_n)| \leq C/2^{n+1}$. Therefore, the product

$$\prod_{|a_n| \geq 2R} E_n\left(\frac{z}{a_n}\right)$$

defines a holomorphic function when $|z| < R$ and does not vanish in that disk. This shows that the function f has the desired properties and the proof is complete.

$\square$

The Phragmén-Lindelöf principle is also used to obtain a factorization theorem for functions of the Pólya class.

Lemma 2.25. If an entire function $E(z)$ is of Pólya class and has a zero $\bar{w}$ for $w \in \overline{\mathbb{C}^+}$, then $E(z)/\,(z - \bar{w})$ is of Pólya class.

Proof. Since $E(z)$ is of Pólya class, the condition (2.28) states that

$$|E(x + ih - iy)| \leq |E(x + ih + iy)|$$

when $y \geq 0$ and $h \geq 0$. For each fixed h,

$$G_h(x + iy) \stackrel{\text{def}}{=} \overline{E(x + ih - iy)}/E(x + ih + iy) = \overline{E(\bar{z} + ih)}/E(z + ih)$$

is analytic and bounded for $y > 0$. Let $F(z) = E(z)/\,(z - \bar{w})$. Since $\bar{w}$ is a zero of $E(z)$, $F(z)$ is an entire function. Since $E(z)$ has no zeros in the upper half-plane, neither does $F(z)$. So the function

$$\frac{\overline{F(\bar{z} + ih)}}{F(z + ih)} = \frac{\overline{E(\bar{z} + ih)}}{E(z + ih)} \cdot \frac{(z + ih - \bar{w})}{(z - ih - w)}$$

is analytic, because the only possible singularities that cancel are zeros of F that cancel out, and bounded in the upper half-plane. It is bounded by 1 on the real axis. By the Phragmén–Lindelöf principle in Theorem 2.2, it is bounded by 1 in the upper half-plane. It follows that

$$|F(\bar{z} + ih)| \leq |F(z + ih)|$$

for $z \in \mathbb{C}^+$ and $h \geq 0$, and this implies that $F(z)$ is of Pólya class.

$\square$

Lemma 2.26. Let $E : \mathbb{C} \to \mathbb{C}$ be an entire function. If $E \in \mathcal{P}$ and has no zeros, then

$$E(z) = E(0)e^{-az^2 - ibz}$$

where $a \geq 0$ and $\operatorname{Re} b \geq 0$.

Proof. Since $E(z)$ has no zeros, we can express it as $E(z) = E(0)\exp[F(z)]$, where $F(z)$ is an entire function with a zero at the origin. Our goal is to show that F is a polynomial of degree two. Given that $E(z)$ is of Pólya class, the condition (2.28) can be rephrased as

$$\operatorname{Re} F(x + ih - iy) \leq \operatorname{Re} F(x + ih + iy)$$

for $y \geq 0$ and $h \geq 0$. If $h \geq 0$, then $G_h(z) \overset{\text{def}}{=} F(z + ih) - \overline{F(\bar{z} + ih)}$ is an entire function whose real part is nonnegative in the upper half-plane and zero on the real axis. By Proposition 2.10,

$$\operatorname{Re} G_h(z) = p(h)y$$

for $y \geq 0$, where $p(h)$ is a nonnegative constant. Since the real part of the entire function $R_h(z) \overset{\text{def}}{=} G_h(z) + ip(h)z$ vanishes in the upper half-plane, this function is a constant by the Open Mapping Theorem. Since the second derivative of the function must vanish identically,

$$F''(z + ih) = \overline{F''(\bar{z} + ih)} \tag{2.33}$$

for every $z \in \mathbb{C}$. When $y = 0$, we have

$$2 \operatorname{Im} F''(x + ih) = F''(x + ih) - \overline{F''(x + ih)} = 0$$

for all $h \geq 0$. From (2.33), we find that $F''(z)$ is a real-valued entire function. By the Open Mapping Theorem, $F''(z)$ is a real constant. Since $F(z)$ vanishes at the origin, we have $F(z) = -az^2 - ibz$ for some numbers a and b, with a real. Given that

$$\operatorname{Re} G_h(z) = 4ahy + 2y \operatorname{Re} b \geq 0$$

for all $h \geq 0$ and $y \geq 0$, we must have $a \geq 0$ and $\operatorname{Re} b \geq 0$.

□

Remark 2.27. Let f be analytic in the upper half-plane without zeros. Then we have $f(z) = \exp[g(z)]$, where g is analytic. Write as usual

$$g(x + iy) = u(x, y) + iv(x, y)$$

with u, v real-valued. Then, by the Cauchy-Riemann equations,

$$\frac{f'(z)}{f(z)} = (\log f(z))' = g'(z) = \frac{1}{i}\left(\frac{\partial u}{\partial y}(x, y) + i\frac{\partial v}{\partial y}(x, y)\right).$$

Therefore

$$\operatorname{Re}\left(i\frac{f'(z)}{f(z)}\right) = \frac{\partial u}{\partial y}(x, y) = \frac{\partial}{\partial y} \log |f(x + iy)|.$$

Theorem 2.28 (Factorization of Pólya class). Let $E : \mathbb{C} \to \mathbb{C}$ be an entire function. If E belongs to the Pólya class and has a zero of order r at the origin, then

$$E(z) = E^{(r)}(0)(z^r/r!)e^{-az^2}e^{-ibz}\prod_{n=1}^{\infty}(1 - z/\bar{z}_n)e^{h_n z} \tag{2.34}$$

where $a \geq 0$ and $\operatorname{Re} b \geq 0$, $\{\bar{z}_n\}$ is the sequence of nonzero zeros of $E(z)$, and

$$h_n \stackrel{\text{def}}{=} x_n/\left(x_n^2 + y_n^2\right).$$

Proof. By repeated application of Lemma 2.25, we can write

$$E(z) = E^{(r)}(0)(z^r/r!)F(z),$$

where $F(z)$ is an entire function of Pólya class and $F(0) = 1$. If $F(z)$ has no zeros, the theorem follows from Lemma 2.26. Otherwise, let $\overline{z_0}$ be the choice of a zero of $F(z)$ nearest to the origin. By Lemma 2.25, $F(z)/(1 - z/\overline{z_0})$ is of Pólya class.

Let $h_0 = x_0/(x_0^2 + y_0^2)$. Since the modulus of $e^{h_0 z}$ is constant on every vertical line, $F_1(z) = F(z)e^{-h_0 z}/(1 - z/\bar{z}_0)$ is of Pólya class. If $F_1(z)$ has no zeros, the theorem follows from Lemma 2.26. Otherwise, continue inductively in the obvious way. At the n-th stage, $F_n(z)$ will be an entire function of Pólya class which has value 1 at the origin. The theorem follows immediately if this function has no zeros. Otherwise, let $\overline{z_n}$ be the choice of a zero of $F_n(z)$ nearest the origin, and let

$$F_{n+1}(z) = F_n(z)e^{-h_n z}/(1 - z/\bar{z}_n)$$

where $h_n = x_n/(x_n^2 + y_n^2)$. In the worst case, $F_n(z)$ is defined for every $n \in \mathbb{N}$. Let

$$P_n(z) = \prod_{k=0}^{n}(1 - z/\bar{z}_k)e^{h_k z}.$$

Then $P_n(z)$ is of Pólya class, and $F(z) = P_n(z)F_{n+1}(z)$. Taking the logarithmic derivative

$$iF'(z)/F(z) = iP_n'(z)/P_n(z) + iF_{n+1}'(z)/F_{n+1}(z)$$

where each term is analytic in the upper half-plane, because the functions in the quotients do not have zeros in the upper half-plane. Since $F_{n+1}(z)$ is of Pólya class,

$$\operatorname{Re}\left(iF_{n+1}'(z)/F_{n+1}(z)\right) = \partial/\partial y \log|F_{n+1}(x+iy)| \geq 0$$

for $y > 0$ (see Remark 2.27). It follows that

$$\operatorname{Re}\left(iP_n'(z)/P_n(z)\right) \leq \operatorname{Re}\left(iF'(z)/F(z)\right)$$

for $y > 0$. Since

$$\operatorname{Re}\left(\frac{iP_n'(z)}{P_n(z)}\right) = \operatorname{Re}\left(i(\log P_n(z))'\right) = \sum_{k=0}^{n}\frac{y + y_k}{(x - x_k)^2 + (y + y_k)^2}$$

and n is arbitrary, we have that

$$\sum_{k=0}^{\infty} \frac{y+y_k}{(x-x_k)^2+(y+y_k)^2} \leq \operatorname{Re} \frac{iF'(z)}{F(z)}$$

for $y>0$. Taking $x=0$ and $y>1$, we can prove that $x_k^2+(y+y_k)^2 \leq C(x_k^2+y_k^2)$ for some[2] positive constant. It follows that

$$\sum_{n=1}^{\infty} \frac{1+y_n}{x_n^2+y_n^2} < \infty.$$

By Proposition 2.22, $P_\infty(z) = \lim P_n(z)$ converges uniformly on bounded sets and the limit is an entire function of Pólya class. It follows that $\lim F_n(z) = F_\infty(z)$ exists uniformly on bounded sets. Since $F_n(z)$ is of Pólya class for every n, $F_\infty(z)$ is an entire function of Pólya class. Given that we always select $\overline{z}_n$ as the zero of $F_n(z)$ closest to the origin, our construction guarantees that we have exhausted all zeros of F and that $F_\infty(z)$ has no zeros. The theorem can now be deduced from Lemma 2.26 because $F(z) = P_\infty(z)F_\infty(z)$.

□

Proposition 2.29. If E is a given entire function of Pólya class, there exists a sequence of polynomials $\{P_n\} \subset \mathcal{P}$ such that $E(z) = \lim_{n\to\infty} P_n(z)$ uniformly on compact sets.

Proof. The proof idea utilizes the decomposition (2.34) from Lemma 2.38. Recall that $p_n(z) = (1+z/n)^n$ implies $p_n \to e^z$ uniformly on every compact subset of $\mathbb{C}$. Consequently, for any given $a>0$, the sequence $p_n(-az^2) \to e^{-az^2}$ uniformly on compacts and each polynomial has only real zeros. By Remark 2.20, each of these polynomials belongs to the Pólya class.

Similarly, for a given number b with $\operatorname{Re} b > 0$, the sequence $p_n(-ibz) \to e^{-ibz}$ uniformly on compacts and each polynomial is in the Pólya class.

Examine decomposition (2.34). Based on the preceding remarks, we can express the first terms as a uniform limit of polynomials. If each term is a limit of polynomials in the Pólya class, we can take a common subsequence and conclude the result. Now, for the infinite product P, we can represent P as a limit in n of $P_n \cdot \exp\left(z \sum_{1\leq k\leq n} h_k\right) = \lim_{m\to\infty} P_n Q_{n,m}$, with uniform convergence and $P_n Q_{n,m}$ being a polynomial in the Pólya class. By taking a common subsequence, we conclude that $P = \lim_{k\to\infty} P_{n_k} Q_{n_k,m_k}$ uniformly, and the result follows. □

[2] This is an application of the Cauchy-Schwarz inequality and the fact that zero is not an accumulation point of zeros of F (there exists a positive δ such that $\delta < x_k^2 + y_k^2$ for every k).

Proposition 2.30. Let $E : \mathbb{C} \to \mathbb{C}$ be an entire function that has no zeros in $\mathbb{C}^+$ and satisfies $|E^*(z)| \leq |E(z)|$ for $z \in \mathbb{C}^+$. Then $|E^*(z)| < |E(z)|$ for $z \in \mathbb{C}^+$ unless $E(z)$ and $E^*(z)$ are linearly dependent.

Proof. If $E^*(z) = cE(z)$ for all z, we choose z to be real and deduce that c is a complex number with modulus 1. Suppose there exists a $z_0 \in \mathbb{C}^+$ such that $|E^*(z_0)| = |E(z_0)|$. According to the hypothesis, the function $F(z) = E^*(z)/E(z)$ is analytic (as E has no zeros in this domain) and is bounded by 1 in the upper half-plane. Since $F(z_0) = c$, where c is a complex number with modulus 1, we apply the maximum principle to a neighborhood B around z_0 and conclude that F is constant in B. Therefore, $F(z) = c$ and $E^*(z) = cE(z)$ in B. Through analytical continuation, we infer that $E^*(z) = cE(z)$ for all complex z. $\square$

Proposition 2.31. Let $E : \mathbb{C} \to \mathbb{C}$ be an entire function of Pólya class such that $|E^*(z)| < |E(z)|$ for $z \in \mathbb{C}^+$. Then $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are real entire functions of Pólya class.

Proof. By Proposition 2.30 E and E^* are linearly independent functions over $\mathbb{C}$. Let us define

$$A(z) = \frac{1}{2}\left(E(z) + E^*(z)\right) \quad \text{and} \quad B(z) = \frac{i}{2}\left(E(z) - E^*(z)\right).$$

To see that A and B are Pólya class, note that A and B are nonzero in the upper half-plane because they are linear combinations of E and E^*. The second condition of Pólya class is obvious by construction. By Proposition 2.29, we have $E = \lim P_n$, with $\{P_n\} \subset \mathcal{P}$ polynomials and the limit uniform in compact sets. Construct for each n

$$A_n(z) = \frac{1}{2}\left(P_n(z) + P_n^*(z)\right) \quad \text{and} \quad B_n(z) = \frac{i}{2}\left(P_n(z) - P_n^*(z)\right).$$

Given $z_0 \in \mathbb{C}^+$, by the monotonicity of limit,

$$|P_n^*(z_0)| < |P_n(z_0)|$$

for all $n > N$ large enough. Based on Proposition 2.30, we can conclude that

$$|P_n^*(z)| < |P_n(z)|$$

for all $z \in \mathbb{C}^+$ and for all $n > N$. Using the initial argument, A_n and B_n are in the Pólya class, as stated in Remark 2.20. Since $|A(x + iy)|$ and $|B(x + iy)|$ are the pointwise limits of nondecreasing functions with respect to the variable $y > 0$, the third condition of $\mathcal{P}$ follows. $\square$

Proposition 2.32. Let $E : \mathbb{C} \to \mathbb{C}$ be an entire function of Pólya class that is not constant. Then $E'(z)$ is also of Pólya class.

Proof. By Proposition 2.29, it is sufficient to prove the result for polynomials (because $P_n \to F$ uniformly implies $P_n' \to F'$ uniformly). Moreover, by Remark 2.20, it is enough to prove that if P is a polynomial in the Pólya class, then P' is nonzero in the upper half-plane. For this, consider

$$P(z) = c \prod_{k=1}^{n} (z - \bar{z}_i) \in \mathcal{P},$$

where $z_i \in \overline{\mathbb{C}^+}$. Let $f(z) = P'(z)/P(z)$ be the logarithmic derivative of P. The zeros of P' that are nonzeros of P are also zeros of f. According to the Gauss-Lucas theorem, the zeros of f do not belong to the upper half-plane.

□

2.4 Functions of bounded type

Definition 2.33 (Bounded type). A function $F(z)$, analytic in the domain Ω, is said to be of bounded type in Ω if it can be expressed as $F(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are analytic and bounded in Ω, and $Q(z)$ is not identically zero. We denote the set of functions of bounded type in Ω as $\mathcal{B}(\Omega)$.

Remark 2.34. Equivalently, asking for P and Q to be bounded is the same as requesting that both P and Q be bounded by 1. Note that every bounded and analytic function in Ω has bounded type.

Proposition 2.35. The following results are valid:

(i) For a domain Ω, $\mathcal{B}(\Omega)$ is closed under sum and product.

(ii) A function $F(z)$, which is analytic in the upper half-plane, belongs to $\mathcal{B}(\mathbb{C}^+)$ if its real part is nonnegative in $\mathbb{C}^+$.

(iii) Any polynomial is a function of bounded type in the upper half-plane.

Proof. (i) This item is trivial; we assume that two functions are in $\mathcal{B}(\Omega)$ and perform the algebraic manipulation to conclude.
(ii) Note that we can simply take

$$P(z) = \frac{F(z)}{(F(z) + \operatorname{Re} F(z) + 1)} \quad \text{and} \quad Q(z) = \frac{1}{(F(z) + \operatorname{Re} F(z) + 1)},$$

because both are analytic and bounded in $\mathbb{C}^+$ and $Q(z)$ is not the zero function.
(iii) For the monomial z, we set $P(z) = z/(z+i)$ and $Q = 1/(z+i)$, then apply item (i) to conclude the result for arbitrary polynomials. $\square$

Theorem 2.36. Let $\{z_n\}$ be a sequence of numbers in the upper half-plane with

$$\sum_{k=1}^{\infty} \frac{y_k}{x_k^2 + y_k^2} < \infty.$$

Define the Blaschke product as

$$B(z) \stackrel{def}{=} \prod_{k=1}^{\infty} \frac{(1 - z/z_k)}{(1 - z/\bar{z}_k)}. \tag{2.35}$$

Then the Blaschke product converges uniformly on every bounded set that is at a positive distance from the numbers $\bar{z}_n$. Moreover, $B(z)$ is an analytic function bounded by 1 in the upper half-plane, and $B(z)B^*(z) = 1$.

Proof. Let

$$B_n(z) = \prod_1^n (1 - z/z_k)/(1 - z/\bar{z}_k).$$

We will prove that $B_n(z)$ is uniformly Cauchy on bounded sets with a positive distance from $\bar{z}_k$, and because of this, the product converges uniformly to an analytic function $B(z)$ in this case. Let K be a compact set[3], such that $|z| \le M$ for $z \in K$ and $dist(K, \bar{z}_k) = \varepsilon > 0$. Define

$$\rho(z) = \inf_k |1/z - 1/\bar{z}_k|.$$

First, note that $1/\rho(z) \le C$ for all $z \in K$. This is obvious if $0 \in K$ or $\bar{z}_k = 0$ for some k. Otherwise, for any $k \in \mathbb{N}$ and $z \in K$

$$\begin{aligned} |z\bar{z}_k| \le (|z - \bar{z}_k| + |z|)|z| &\le (|z - \bar{z}_k| + M)M \\ &\le \left(|z - \bar{z}_k| + \varepsilon\frac{M}{\varepsilon}\right) M \\ &\le (|z - \bar{z}_k| + |z - \bar{z}_k| M/\varepsilon)M \\ &\le |z - \bar{z}_k|(1 + M/\varepsilon)M, \end{aligned}$$

so

$$\left|\frac{1}{z} - \frac{1}{\bar{z}_k}\right| = \frac{|z - \bar{z}_k|}{|z\bar{z}_k|} \ge \frac{|z - \bar{z}_k|}{|z - \bar{z}_k|(1 + M/\varepsilon)M} = \frac{1}{M(1 + M/\varepsilon)} \stackrel{def}{=} \frac{1}{C} > 0$$

and this implies that $1/\rho(z) \le C$. Now, we will show that

$$|(1 - z/z_k)/(1 - z/\bar{z}_k)| \le 1 + |(1 - z/z_k)/(1 - z/\bar{z}_k) - 1| \le \exp\left(\frac{2}{\rho(z)} \frac{y_k}{x_k^2 + y_k^2}\right).$$

[3] We start with K bounded, but we can suppose that K is also closed because the distance from $\bar{z}_k$ is equal to the distance from its closure

The first inequality is obvious, and the second follows from $1+x \leq e^x$ for $x \geq 0$ and by the fact that

$$|(1-z/z_k)/(1-z/\bar{z}_k)-1| \leq \frac{1}{\rho(z)}|(1/z-1/z_k)-(1/z-1/\bar{z}_k)| = \frac{2}{\rho(z)}\frac{y_k}{x_k^2+y_k^2}.$$

Note that, if $m, n \in \mathbb{N}$ and $m < n$,

$$|B_n(z)-B_m(z)| = |B_m(z)|\left|\prod_{k=m+1}^{n}(1-z/z_k)/(1-z/\bar{z}_k)-1\right|.$$

However,

$$\begin{aligned}|B_m(z)| &= \prod_{k=1}^{m}|(1-z/z_k)/(1-z/\bar{z}_k)| \\ &\leq \prod_{k=1}^{m}\exp\left(\frac{2}{\rho(z)}\frac{y_k}{x_k^2+y_k^2}\right) \leq \exp\left(2C\sum_{k=1}^{m}\frac{y_k}{x_k^2+y_k^2}\right) \quad (2.36)\end{aligned}$$

and the right-hand side of (2.36) is uniformly bounded in m because the corresponding series is convergent by hypothesis. Applying Proposition 2.21, we obtain that

$$\begin{aligned}\left|\prod_{k=m+1}^{n}(1-z/z_k)/(1-z/\bar{z}_k)-1\right| &\leq -1+\prod_{k=m+1}^{n}(1+|(1-z/z_k)/(1-z/\bar{z}_k)-1|) \\ &\leq -1+\prod_{k=m+1}^{n}\exp\left(\frac{2}{\rho(z)}\frac{y_k}{x_k^2+y_k^2}\right) \leq -1+\exp\left(2C\sum_{k=m+1}^{n}\frac{y_k}{x_k^2+y_k^2}\right)\end{aligned}$$

and the right-hand side is uniformly convergent to zero. This proves the first part of the theorem. Because

$$|(1-z/z_k)/(1-z/\bar{z}_k)| \leq 1$$

for all k and $z \in \mathbb{C}^+$, we obtain that $|B_n(z)| \leq 1$ and this estimate is also true for $B(z)$. Finally, if z is not a pole or zero of B, the truncated product $B_n(z)B_n^*(z)$ is equal to 1 and then tends to 1 in the limit. Therefore, $B(z)B^*(z) = 1$ at these points. For a zero or pole, we use analytic continuation and the result follows. □

The last theorem allows us to prove some important factorizations for the theory. A first interesting application is to guarantee the factorization of positive functions. Remember that if an entire function $P(z)$ is of the form $P(z) = Q(z)Q^*(z)$ for some entire function $Q(z)$, then the values of $P(z)$ are nonnegative on the real axis. The next theorem proves that the converse is true if the zeros of $P(z)$ are sufficiently near the real axis.

Theorem 2.37. Let $P : \mathbb{C} \to \mathbb{C}$ be an entire function that has nonnegative values on the real axis and does not vanish identically. Let $\{z_n\} \subset \mathbb{C}^+$ be the zeros of P in the upper half-plane repeated according to multiplicity. If

$$\sum_{n=1}^{\infty}\frac{y_n}{x_n^2+y_n^2} < \infty,$$

then $P(z) = Q(z)Q^*(z)$ for some entire function $Q(z)$, which has no zeros in the upper half-plane and satisfies $|Q^*(z)| \le |Q(z)|$ for $z \in \mathbb{C}^+$.

Proof. By Theorem 2.36, the Blaschke product

$$B(z) = \prod(1 - z/z_n)/(1 - z/\bar{z}_n)$$

converges. Since the zeros of $B(z)$ are also zeros of $P(z)$, the function $F(z) = P(z)/B(z)$ is entire and has no zeros on $\mathbb{C}^+$. Since $P^* = P$ (because P is real on the real line), the non-real zeros of $P(z)$ occur in conjugate pairs. From this and the product formula of B, it follows that the nonreal zeros $\overline{z_n}$ of $F(z)$ have even multiplicities.

Since P is nonnegative on the real axis, its real zeros have even multiplicities too. In fact, suppose P has a real zero at $x = x_k$ with an odd multiplicity n. By the Taylor series expansion around x_k, we can express $P(x)$ as $P(x) = (x - x_k)^n h(x)$, where $h(x_k) \neq 0$. If n were not even, we would have a contradiction in the sign of $P(x)$ at x_k due to the odd power $(x - x_k)^n$.

Therefore, all zeros of F have even multiplicities, and we can write $F(z) = Q(z)^2$ for some entire function $Q(z)$. This occurs because we can use the factorization of F as a product

$$F(z) = e^{a(z)} z^{2\nu_0} \prod \left(1 - \tfrac{z}{b_k}\right)^{2\nu_k} e^{p_k(z)},$$

and then conclude that

$$Q(z) = e^{a(z)/2} z^{\nu_0} \prod \left(1 - \tfrac{z}{b_k}\right)^{\nu_k} e^{p_k(z)/2}.$$

Since the zeros of $B(z)$ exhaust the zeros of $P(z)$ in the upper half-plane, $Q(z)$ has no zeros in the half-plane. Since $P^*(z) = P(z)$, $B^*(z)B(z) = 1$ and $B(z)$ is bounded by 1 in the upper half-plane, $|Q^*(z)| \le |Q(z)|$ for $z \in \mathbb{C}^+$. By construction, $P(z)^2 = [Q(z)Q^*(z)]^2$. Since $P(z)$ and $Q(z)Q^*(z)$ are nonnegative on the real axis, they are identical in the complex plane by analytical continuation. □

Another interesting application of Theorem 2.36 is to guarantee a factorization of functions of bounded type. In fact, if a function is analytic and of bounded type in the upper half-plane and has no zeros in a neighborhood of the origin, then its zeros coincide with those of a Blaschke product.

Lemma 2.38. Let $F(z)$ be a function that is analytic and of bounded type in the upper half-plane, such that the origin is not a limit point of zeros of $F(z)$. Then

$$F(z) = G(z)B(z),$$

where $B(z)$ is the Blaschke product defined in (2.35) with $\{z_n\}$ being the zeros of $F(z)$ in $\mathbb{C}^+$, repeated according to multiplicity, and $G(z)$ is a function that is analytic and of bounded type in the upper half-plane and has no zeros in the half-plane.

Remark 2.39. If $F(z)$ has no zeros in $\mathbb{C}^+$, the Blaschke product is taken to be equal to 1. If there are an infinite number of zeros, then we show in the proof that the convergence condition

$$\sum_{1}^{\infty} \frac{y_n}{x_n^2 + y_n^2} < \infty$$

is satisfied for using Theorem 2.36.

Proof of Lemma 2.38. Since $F(z)$ is assumed to be of bounded type in the upper half-plane, there exists a nonzero function $Q(z)$, which is analytic and bounded by 1 in the upper half-plane, such that $Q(z)F(z)$ is analytic and bounded by 1 in the upper half-plane. Since the zeros of a nonzero analytic function are isolated and have finite multiplicities, they are countable.

Let $\{z_n\}$ be an enumeration of the zeros of $F(z)$ in the upper half-plane, repeated according to multiplicity. The theorem is immediate when $F(z)$ has no zeros and is easily obtained when $F(z)$ has only a finite number of zeros. Define a sequence $\{F_n(z)\}$ of analytic functions inductively by $F_1(z) = F(z)$ and

$$F_{n+1}(z) = F_n(z)(1 - z/\bar{z}_n)/(1 - z/z_n).$$

We will show by induction that $Q(z)F_n(z)$ is bounded by 1 in the upper half-plane. We know that $Q(z)F(z)$ is bounded by 1. Assume it is known that $Q(z)F_n(z)$ is bounded by 1. We show that $Q(z)F_{n+1}(z)$ is bounded by 1. By construction,

$$Q(z)F_{n+1}(z) = Q(z)F_n(z)(1 - z/\bar{z}_n)/(1 - z/z_n),$$

where the last factor is bounded on any set that lies at a positive distance from the point z_n. Since $Q(z)F_{n+1}(z)$ is bounded in a neighborhood of z_n by continuity, it is also bounded in the upper half-plane. On the line $y = h$, where $h > 0$, it is bounded by

$$\begin{aligned}
&\max \left| 1 - \frac{(x+ih)}{\overline{z_n}} \right| \left| 1 - \frac{(x+ih)}{z_n} \right|^{-1} \\
&= \max \left(1 + \frac{4hy_n}{(x-x_n)^2 + (h-y_n)^2} \right)^{1/2} = \left(1 + \frac{4hy_n}{(h-y_n)^2} \right)^{1/2} = \frac{h+y_n}{|h-y_n|}. \quad (2.37)
\end{aligned}$$

By the Phragmén-Lindelöf principle in Theorem 2.2, $Q(z)F_{n+1}(z)$ has the same bound in the half-plane $y > h$. Due to the arbitrariness of h, $Q(z)F_{n+1}(z)$ is bounded by 1 in the upper half-plane.

The theorem follows immediately if there are only a finite number of zeros. In the infinite case, we show that the convergence condition of Theorem 2.36 is satisfied. Write $F(z) = B_n(z)F_{n+1}(z)$, where

$$B_n(z) = \prod_{k=1}^{n} (1 - z/z_k)/(1 - z/\bar{z}_k).$$

Since $Q(z)F_{n+1}(z)$ is bounded by 1 in the upper half-plane, $|Q(z)F(z)| \leq |B_n(z)|$, or equivalently,

$$\begin{aligned} -\log|Q(z)F(z)| &\geq \sum_{k=1}^{n} \log\left|\frac{1 - z/\bar{z}_k}{1 - z/z_k}\right| \\ &\geq \frac{1}{2}\sum_{k=1}^{n} \log\left\{1 + \frac{4yy_k}{(x - x_k)^2 + (y - y_k)^2}\right\}. \end{aligned}$$

By the arbitrariness of n,

$$\sum_{1}^{\infty} \log\left\{1 + 4yy_k/[(x - x_k)^2 + (y - y_k)^2]\right\} \leq -2\log|Q(z)F(z)|.$$

Since $Q(z)F(z)$ does not vanish identically and since $\log(1+x) \sim x$ for small positive x, it follows that

$$\sum_{k=1}^{\infty} \frac{yy_k}{x_k^2 + (y - y_k)^2} < \infty \tag{2.38}$$

for some $y > 0$. Given that the origin is not a limit point of zeros[4] of $F(z)$, the inequality (2.38) implies that

$$\sum_{k=1}^{\infty} \frac{y_k}{x_k^2 + y_k^2} < \infty.$$

By Theorem 2.36, $\lim_{n\to\infty} B_n(z) = B(z)$ exists uniformly on any bounded set at a positive distance from the real axis. The limit function is analytic and bounded by 1 in the upper half-plane, and $F(z) = B(z)G(z)$ where $G(z) = \lim_{n\to\infty} F_n(z)$ uniformly on any bounded set at a positive distance from the real axis.

It follows that $G(z)$ is analytic in the upper half-plane and that $Q(z)G(z)$ is bounded by 1. Since $Q(z)$ is bounded by 1, $G(z)$ is of bounded type in the upper half-plane. Since the sequence $\{z_n\}$ is chosen to exhaust the zeros of $F(z)$, $G(z)$ has no zeros in the half-plane. □

Proposition 2.40. Let $F(z)$ be a function that is analytic and of bounded type in the upper half-plane. Then there exists a function $G(z)$, which is analytic and bounded by 1 and has no zeros in the upper half-plane, such that $L(z) = G(z)F(z)$ is bounded by 1 in the half-plane.

---

[4] The condition over the zeros implies by Cauchy–Schwarz that

$$y_k/\left(x_k^2 + y_k^2\right) \leq Cyy_k/\left(x_k^2 + (y - y_k)^2\right)$$

for some positive constant C universal for every k.

Proof. If F is the null function, the result follows. Since F is of bounded type, we can write $F(z)Q(z) = P(z)$, with P, Q analytic and bounded by 1 in the upper half-plane. If Q has no zeros, the result follows.

Otherwise, since the zeros of a nonzero analytic function are isolated and have finite multiplicities, they are countable. Let $\{z_n\}$ be an enumeration of the zeros of $Q(z)$ in the upper half-plane, repeated according to multiplicity. Note that every zero of Q is a zero of P. Let us divide these zeros into two parts: $\{z'_n\}$ are the zeros inside the unitary upper half-disk and $\{z''_n\}$ are the zeros outside this set. Using the proof of Lemma 2.38, we can remove the zeros $\{z'_n\}$ from Q and obtain

$$Q(z) = S(z)B(z)$$

with $|S(z)| \leq 1$ and S having zeros only on the unitary upper half-disk. We can apply the same argument with P and obtain $P(z) = U(z)B(z)$, with U bounded by 1 and U having zeros only on the unitary upper half-disk. So

$$F(z)S(z)B(z) = F(z)Q(z) = P(z) = U(z)B(z)$$

and therefore $F(z)S(z) = U(z)$. To remove the zeros of S inside the unitary upper half-disk and obtain G, apply the same argument with $F_1(z) = F(z+2)$, $S_1(z) = S(z+2)$, and $U_1(z) = U(z+2)$. □

These results are used to obtain Nevanlinna's factorization of functions in $\mathcal{B}(\mathbb{C}^+)$.

Notation 2.41. Given some real-valued function $\mu(x)$, we define

$$\int_{-\infty}^{\infty} \frac{|d\mu|(t)}{1+t^2} \stackrel{\text{def}}{=} \sup_P \sum_{k=1}^{n} \frac{|\mu(t_k) - \mu(t_{k-1})|}{1 + \max(t_{k-1}^2, t_k^2)}$$

over all partitions $P = \{t_1 < t_2 < \ldots < t_n\}$ of the real line.

Lemma 2.42. Let $\mu(x)$ be a real-valued function of real x such that

$$\int_{-\infty}^{\infty} \frac{|d\mu|(t)}{1+t^2} < \infty.$$

Then $\mu(x) = \sigma(x) - \nu(x)$ for some nondecreasing functions $\sigma(x)$ and $\nu(x)$ such that

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{d\nu(t)}{1+t^2} < \infty.$$

The idea behind the proof of this lemma is to utilize the fact that any locally bounded variation function can be locally expressed as the difference of two nondecreasing functions. The reader who wishes to delve into the detailed proof can refer to [Bra68, Lemma 3], and we will omit the details here.

Theorem 2.43 (Nevanlinna's factorization). Let $F(z)$ be a function that is analytic in the upper half-plane and does not have the origin as a limit point of zeros. A necessary and sufficient condition for $F(z)$ to be of bounded type in the upper half-plane is that

$$F(z) = B(z)\exp(-ihz)\exp G(z) \tag{2.39}$$

where $B(z)$ is a Blaschke product, h is a real number, and $G(z)$ is a function analytic in the upper half-plane such that

$$\operatorname{Re} G(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \tag{2.40}$$

for some real-valued function $\mu(x)$ such that

$$\int_{-\infty}^{\infty} \frac{|d\mu|(t)}{1+t^2} < \infty.$$

Proof. By Lemma 2.38, we write $F(z) = B(z)L(z)$, where $B(z)$ is a Blaschke product and $L(z)$ is an analytic function of bounded type with no zeros in the upper half-plane. According to Proposition 2.40, $L(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are analytic functions bounded by 1 and with no zeros in the upper half-plane. Therefore, we can express $P(z) = \exp(-V(z))$ and $Q(z) = \exp(-U(z)$, where $U(z)$ and $V(z)$ are analytic functions with nonnegative real parts in the upper half-plane.

By the Poisson representation in Theorem 2.15, there exist nonnegative numbers p and q and nondecreasing functions $\sigma(x)$ and $\nu(x)$ such that

$$\operatorname{Re} U(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-x)^2+y^2},$$

and

$$\operatorname{Re} V(x+iy) = qy + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\nu(t)}{(t-x)^2+y^2},$$

for $y > 0$. Let $h = p - q$ and let $G(z) = ihz + U(z) - V(z)$. Then, we have $F(z) = B(z)\exp(-ihz)\exp G(z)$ and

$$\begin{aligned}\operatorname{Re} G(x+iy) &= -hy + \operatorname{Re} U(x+iy) - \operatorname{Re} V(x+iy) \\ &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-x)^2+y^2} - \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\nu(t)}{(t-x)^2+y^2} \\ &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2},\end{aligned}$$

where $\mu(x) = \sigma(x) - \nu(x)$. If $t_0 < t_1 < \ldots < t_r$ is a finite subset of the real line, then

$$\begin{aligned}\sum_1^r \frac{|\mu(t_k)-\mu(t_{k-1})|}{1+\max(t_{k-1}^2, t_k^2)} &\le \sum_1^r \frac{\sigma(t_k)-\sigma(t_{k-1})}{1+\max(t_{k-1}^2, t_k^2)} + \sum_1^r \frac{\nu(t_k)-\nu(t_{k-1})}{1+\max(t_{k-1}^2, t_k^2)} \\ &\le \int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} + \int_{-\infty}^{\infty} \frac{d\nu(t)}{1+t^2},\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{|d\mu|(t)}{1+t^2} = \sup_P \sum_{k=1}^{r} \frac{|\mu(t_k) - \mu(t_{k-1})|}{1 + \max(t_{k-1}^2, t_k^2)} < \infty.$$

Conversely, let G be as in (2.40). Since h is real, we can write $h = p - q$ where p and q are nonnegative. By Lemma 2.42, $\mu(x) = \sigma(x) - \nu(x)$ where $\sigma(x)$ and $\nu(x)$ are nondecreasing functions of real x such that

$$\int_{-\infty}^{\infty} (1+t^2)^{-1}\, d\sigma(t) < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} (1+t^2)^{-1}\, d\nu(t) < \infty.$$

As in Remark 2.13, let $U(z)$ and $V(z)$ be functions that are analytic in the upper half-plane such that

$$\operatorname{Re} U(x+iy) = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\sigma(t)}{(t-x)^2 + y^2},$$

and

$$\operatorname{Re} V(x+iy) = qy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(t-x)^2 + y^2},$$

for $y > 0$. We can assume that $G(z) = ihz + U(z) - V(z)$. Otherwise, since the functions on each side of the equality have the same real part, the difference of imaginary parts is a constant by the Open Mapping Theorem. Therefore, we can add constants in U or V and rename them to guarantee equality. Let $P(z) = \exp(-V(z))$ and $Q(z) = \exp(-U(z))$.

Since the real parts of $U(z)$ and $V(z)$ are nonnegative in the upper half-plane, $P(z)$ and $Q(z)$ are bounded by 1 in the half-plane. Since $F(z) = B(z)P(z)/Q(z)$ where $B(z)$, $P(z)$, and $Q(z)$ are analytic and bounded by 1 in the upper half-plane, $F(z)$ is of bounded type in the half-plane. □

Proposition 2.44. A function F is of bounded type in $\mathbb{C}^+$ if and only if $\log|F(z)|$ has a harmonic and positive majorant in $\mathbb{C}^+$.

Proof. If F is of bounded type in $\mathbb{C}^+$, it follows from the proof of the Nevanlinna factorization in Theorem 2.43 that $F(z) = B(z)e^{-ihz}\exp(Q(z) - P(z))$, where Q and P are analytic in the upper half-plane with nonnegative real parts. In particular

$$\log|F(z)| \le hy + \operatorname{Re} Q(z) \le |h|y + 1 + \operatorname{Re} Q(z)$$

and the function on the right-hand side is harmonic and positive.

On the other hand, if $\log|F(z)| \le U(z)$, with U harmonic and positive, there is a function G analytic in $\mathbb{C}^+$ such that $\operatorname{Re} G(z) = U(z)$. So $|F(z)| \le |\exp\{G(z)\}|$ and we can write $F(z) = P(z)/Q(z)$ with P and Q bounded and analytic given by

$$P(z) = F(z)\exp(-G(z)) \qquad \text{and} \qquad P(z) = \exp(-G(z)).$$

□

In working with functions of bounded type, it is frequently necessary to refer to the number h associated with $F(z)$ by Theorem 2.43.

Definition 2.45 (Mean type). Let F be a function that satisfies the conditions of Nevanlinna's factorization theorem. We define the mean type of F in the upper half-plane, denoted by $v(F)$, as the real number h given in (2.39).

Note that by the statement of the Nevanlinna factorization, the mean type has been defined only for functions that are nonzero in a neighborhood of the origin. If a function F is analytic and of bounded type in the upper half-plane, then the mean type of $F(z+i\varepsilon)$ is defined for every positive ε, and it does not depend on ε.

This number is equal to the mean type of $F(z)$ if the origin is not a limit point of zeros of $F(z)$. Otherwise, we take it as the definition of the mean type of $F(z)$. The mean type of the function, which is identically zero, is taken to be $-\infty$.

Remark 2.46. If $F(z)$ is a function that is analytic and of bounded type in the upper half-plane, according to Nevanlinna factorization, the mean type of $F(z-a)$ is equal to the mean type of $F(z)$ for every real number a.

Two useful formulas for the mean type are known. One of these expresses the mean type as an average radial limit in the upper half-plane. The second formula proves that the mean type is determined purely by what occurs on the imaginary axis.

Theorem 2.47. Consider Nevanlinna's factorization (2.39) in Theorem 2.40. Then,

$$h = \lim_{r\to\infty} \frac{2}{\pi r}\int_0^{\pi} \log|F(re^{i\theta})| \sin\theta \, d\theta \tag{2.41}$$

and

$$h = \limsup_{y\to\infty} y^{-1} \log|F(iy)|. \tag{2.42}$$

Corollary 2.48. Every nonzero polynomial, by (2.42), has zero mean type in $\mathbb{C}^+$.

Corollary 2.49. Let $F(z)$ and $G(z)$ be functions that are analytic and of bounded type in the upper half-plane. Then, according to (2.41), the mean type of $F(z)G(z)$ is the sum of the mean types of $F(z)$ and $G(z)$.

Remark 2.50. Let $F(z)$ be analytic and of bounded type in the upper half-plane. If F is bounded in $\mathbb{C}^+$, we can apply Corollary 2.49 and conclude that the mean type of F is nonpositive.

By Nevanlinna's factorization, the formulas (2.41) and (2.42) have easy proofs when $F(z)$ has no zeros. Therefore, we need three lemmas to show that the presence of the Blaschke product has a negligible effect on the limits.

Lemma 2.51. Let $F(z)$ be a function that is analytic and bounded by 1 in the upper half-plane and does not vanish identically. If

$$h = \liminf_{r\to\infty} \frac{2}{\pi r} \int_0^\pi \log|F(re^{i\theta})| \sin\theta \, d\theta,$$

then $h > -\infty$ and $|F(z)\exp(ihz)| \leq 1$ in the upper half-plane.

Proof. If $\varepsilon > 0$ and $p > 0$, the function $F(z + i\varepsilon)\exp(-ipz)$ is analytic in the upper half-plane. It has a continuous extension to the closed half-plane and is bounded by 1 on the real axis. By (2.4) in the proof of the Phragmén–Lindelöf principle, the inequality

$$\begin{aligned} &\log|F(z + i\varepsilon)\exp(-ipz)| \\ &\qquad \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log^+ |F(ae^{i\theta} + i\varepsilon)\exp(-ipae^{i\theta})| \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta \end{aligned}$$

is valid for $|z| < a$ and $y > 0$. According to Corollary 2.3, the inequality can be written as

$$\log|F(z + i\varepsilon)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \max\{-pa\sin\theta, \log|F(ae^{i\theta} + i\varepsilon)|\} \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta.$$

By the arbitrariness of p, we can take $p \to \infty$, and Fatou's Lemma implies

$$\log|F(z + i\varepsilon)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log|F(ae^{i\theta} + i\varepsilon)| \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta.$$

Let $\varepsilon \to 0$. Again, by Fatou's Lemma, we have

$$\log|F(z)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log|F(ae^{i\theta})| \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - z|^2} d\theta$$

for $|z| < a$ and $y > 0$. Let $a \to \infty$ with z fixed. If $|z/a| < \varepsilon$, where $0 < \varepsilon < 1$, then

$$(1 - \varepsilon^2)^{-1}(1 + \varepsilon)^4 \log|F(z)| \leq (2y/\pi)a^{-1} \int_0^\pi \log|F(ae^{i\theta})| \sin\theta \, d\theta.$$

Due to the arbitrariness of a and ε, we can take the limits when $\varepsilon \to 0$ and $a \to \infty$ to obtain $\log|F(z)| \leq hy$. $\square$

Lemma 2.52. Let $F(z)$ be a function that is analytic and bounded by 1 in the upper half-plane and does not vanish identically. If

$$h = \limsup_{y\to\infty} y^{-1} \log|F(iy)|,$$

then $h > -\infty$ and $|F(z)\exp(ihz)| \leq 1$ in the upper half-plane.

Proof. Explicit proof is restricted to the special case in which $F(z)$ is continuous in the closed half-plane. The general case follows by considering $F(z+\varepsilon)$ where $\varepsilon > 0$. The result is immediate if $h = 0$. If $h < 0$, consider any number $a > 0$ such that $h < -a$. According to the definition of h

$$y^{-1} \log |F(iy)| < -a$$

for large values of y, let us say $y > N$. For these values of y, we have $e^{ay} |F(iy)| < 1$. Since $F(iy)$ is bounded by 1, there exists a number $M \geq 1$ such that $|F(iy)| \leq Me^{-ay}$ for $y > 0$. In fact, we can take $M = e^{aN}$ for this inequality to hold true. We use this inequality to obtain a rough estimate of $F(z)$ in the upper half-plane.

When $y > 0$, let $\sqrt{z}$ be the choice of square root which lies in the first quadrant. TThen,

$$G(z) = M^{-1} F(\sqrt{z}) \exp(-ia\sqrt{z})$$

is analytic in the upper half-plane, has a continuous extension to the closed half-plane, and is bounded by 1 on the real axis. Since $F(\sqrt{z})$ is bounded by 1, we have

$$\log^+ |G(re^{i\theta})| \leq a\sqrt{r} \sin(\tfrac{1}{2}\theta)$$

and since

$$\lim_{r\to\infty} r^{-1} \int_0^\pi \sqrt{r} \sin(\tfrac{1}{2}\theta) \sin\theta \, d\theta = 0,$$

we obtain

$$\lim_{r\to\infty} r^{-1} \int_0^\pi \log^+ |G(re^{i\theta})| \sin\theta \, d\theta = 0.$$

By the Phragmén–Lindelöf principle in Theorem 2.2, $G(z)$ is bounded by 1 in the upper half-plane. It follows that $F(z)\exp(-iaz)$ is bounded by M in the first quadrant. The same argument applied to $F^*(-z)$ will show that $F(z)\exp(-iaz)$ is bounded by M in the second quadrant. So $F(z)\exp(-iaz)$ is bounded by M in the upper half-plane, is continuous in $\overline{\mathbb{C}^+}$ and it is bounded by 1 on the real axis. By the Phragmén–Lindelöf principle, it is bounded by 1 in the upper half-plane. The lemma follows by the arbitrariness of a. □

Lemma 2.53. If $B(z)$ is a Blaschke product and h is a real number such that $|B(z)\exp(ihz)|$ is bounded by 1 in the upper half-plane, then $h \geq 0$.

Proof. If $B(z) = \prod_k (1 - z/z_k)/(1 - z/\bar{z}_k)$, let

$$B_n(z) = \prod_{k>n} (1 - z/z_k)/(1 - z/\bar{z}_k).$$

By the convergence of the product for $B(z)$, $\lim B_n(z) = 1$ as $n \to \infty$. Since

$$B(z) = B_1(z)(1 - z/z_1)/(1 - z/\bar{z}_1)$$

and $B(z)\exp(ihz)$ is bounded by 1 in the upper half-plane, $B_1(z)\exp(ihz)$ is bounded in the upper half-plane.

If $\varepsilon > 0$, then $B_1(z+i\varepsilon)\exp(ihz)$ is bounded in the upper half-plane and continuous in the closed half-plane. According to (2.37) from the proof of Lemma 2.38, this function is bounded by $(\varepsilon - y_1)/|\varepsilon - y_1|$ on the real axis. By the Phragmén-Lindelöf principle, the function has the same bound in the upper half-plane. Due to the arbitrariness of ε, $B_1(z)\exp(ihz)$ is bounded by 1 in the upper half-plane.

Continuing inductively in the obvious way, since $B_n(z)\exp(ihz)$ is bounded by 1 in the upper half-plane for every n, $\exp(ihz)$ is bounded by 1 in the upper half-plane, and the result follows. □

Proof of Theorem 2.47. By the proof of Theorem 2.43, we can write

$$F(z) = B(z)\exp(-ihz)P(z)/Q(z)$$

where $P(z)$ and $Q(z)$ are functions that are analytic and bounded by 1 in the upper half-plane, have no zeros in the half-plane, and satisfy the following equations

$$-\log|Q(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\sigma(t)}{(t-x)^2+y^2}$$

and

$$-\log|P(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\nu(t)}{(t-x)^2+y^2}$$

for some nondecreasing functions $\sigma(x)$ and $\nu(y)$. As $B(z)$ is bounded by 1, we have by Lemmas 2.51, 2.52, and 2.53

$$0 \le \lim_{r\to\infty}(2/\pi)r^{-1}\int_0^{\pi}\log|B(re^{i\theta})|\sin\theta\,d\theta \le 0$$

and

$$0 \le \limsup_{y\to\infty} y^{-1}\log|B(iy)| \le 0.$$

Therefore

$$\limsup_{y\to\infty} y^{-1}\log|B(iy)| = \lim_{r\to\infty}(2/\pi)r^{-1}\int_0^{\pi}\log|B(re^{i\theta})|\sin\theta\;d\theta = 0.$$

The theorem follows once we show that

$$0 = \lim_{y\to\infty} y^{-1}\log|P(iy)|$$

and

$$0 = \lim_{r\to\infty}(2/\pi)r^{-1}\int_0^{\pi}\log|P(re^{i\theta})|\sin\theta\;d\theta, \tag{2.43}$$

and that the same formulas hold with $P(z)$ replaced by $Q(z)$. The first of these formulas is true by the proof of Theorem 2.11. The second formula follows for the following reason. If we define h as the right-hand side of (2.43) and $h < 0$, then by Lemma 2.51, we have $-\infty < h$ and $|P(z)e^{ihz}| \leq 1$. But this implies for $z = iy$ that $|P(iy)|e^{-hy} \leq 1$. So

$$|y|^{-1} \log |P(iy)| \leq h$$

and when we take the lim sup on both sides, we have a contradiction with the first formula. □

Proposition 2.54. Let $F(z)$ be a function that is analytic and of bounded type in the upper half-plane. Assume that $|F(z)|$ has a continuous extension to the closed half-plane. If $\mu(x)$ and $G(z)$ are defined as in Theorem 2.43, then

$$\operatorname{Re} G(x+iy) \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |F(t)|}{(t-x)^2 + y^2} dt$$

for $y > 0$ and equality holds if $|F(x)| \neq 0$ for all real x.

The proof of this proposition can be found in [Bra68, Problem 27]. The following condition for bounded type is commonly used.

Theorem 2.55 (Conditions for bounded type). Let $F(z)$ be a function that is analytic in the upper half-plane, and assume that $|F(z)|$ has a continuous extension to the closed half-plane. Then, $F(z)$ is of bounded type in the half-plane if the following conditions hold:

$$\int_{-\infty}^{\infty} (1+t^2)^{-1} \log^+ |F(t)| \, dt < \infty, \tag{2.44}$$

and

$$\liminf_{r \to \infty} \frac{1}{r^2} \int_0^{\pi} \log^+ \left| F(re^{i\theta}) \right| \sin\theta \, d\theta = 0, \tag{2.45}$$

and

$$\limsup_{y \to \infty} y^{-1} \log |F(iy)| < \infty. \tag{2.46}$$

Proof. There exists[5] a function Q, which is analytic and has no zeros for $y > 0$, such that

$$-\log |Q(x+iy)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |F(t)|}{(t-x)^2 + y^2} dt$$

for $y > 0$. The function $Q(z)$ is bounded by 1 in the upper half-plane, $|Q(z)|$ has a continuous extension to the closed half-plane and

$$|1/Q(x)| = \max(1, |F(x)|) \tag{2.47}$$

for all real x. Let h be a real number chosen such that

$$h > \limsup_{y \to \infty} y^{-1} \log |F(iy)| \,. \tag{2.48}$$

[5] Consider $Q(z) = \exp(-g(z))$, with g given by Remark 2.7.

The function

$$P(z) \stackrel{\text{def}}{=} Q(z)\exp(ihz)F(z)$$

is analytic in the upper half-plane, $|P(z)|$ is continuous in the closed half-plane and $|P(x)| \le 1$ for all real x by (2.47). Since

$$0 = \lim_{y\to\infty} y^{-1}\log|Q(iy)|$$

by Theorem 2.11, we have by definition of P, that[6]

$$\limsup_{y\to\infty} y^{-1}\log|P(iy)| < 0.$$

It follows that $P(z)$ is bounded on the imaginary axis. Since $Q(z)$ is bounded by 1, the hypotheses (2.45) imply that

$$\liminf_{r\to\infty} \frac{1}{r^2}\int_0^\pi \log^+\left|P(re^{i\theta})\right|\sin\theta\, d\theta = 0. \tag{2.49}$$

When $y > 0$, let $\sqrt{z}$ be the choice of square root which lies in the first quadrant. Then $G(z) \stackrel{\text{def}}{=} P(\sqrt{z})$ is analytic in the upper half-plane. Its modulus is continuous in the closed half-plane and is bounded on the real axis. We want to apply the Phragmén–Lindelöf principle. For this, we need to verify that

$$\liminf_{r\to\infty} r^{-1}\int_0^\pi \log^+|G(e^{i\theta})|\ \sin\theta\, d\theta = 0. \tag{2.50}$$

This follows because, taking $r = \sqrt{r}$ and $\theta = \alpha/2$ in (2.49), we have

$$\begin{aligned}
0 = \liminf_{r\to\infty} r^{-1}\int_0^{2\pi} \log^+|P(\sqrt{r}e^{i\alpha/2})|\sin(\alpha/2)\,\frac{d\alpha}{2} \\
&\ge \liminf_{r\to\infty} r^{-1}\int_0^{\pi} \log^+|P(\sqrt{r}e^{i\alpha/2})|\sin(\alpha/2)\,\frac{d\alpha}{2} \\
&= \liminf_{r\to\infty} r^{-1}\int_0^{\pi} \log^+|P(\sqrt{r}e^{i\alpha/2})|\frac{\sin(\alpha)}{\cos(\alpha/2)}\,\frac{d\alpha}{4} \\
&\qquad\ge (1/4)\liminf_{r\to\infty} r^{-1}\int_0^{\pi} \log^+|G(re^{i\alpha})|\sin(\alpha)\,d\alpha
\end{aligned}$$

and for the last inequality, we only use that $1/\cos(\alpha/2)$ is bounded below by 1 in this domain of integration. So (2.50) follows. Thus, by the Phragmén-Lindelöf principle, $P(\sqrt{z})$ is bounded in the upper half-plane, implying that $P(z)$ is bounded in the first quadrant.

The same argument with $P(z)$ replaced by $P^*(-z)$ will show that $P(z)$ is bounded in the upper half-plane. Since $|P(z)|$ is continuous in the closed half-plane and is bounded by 1 on the real axis. According to the Phragmén-Lindelöf principle, $P(z)$ is bounded by 1 in $\mathbb{C}^+$. □

[6] Otherwise, we have a contradiction with (2.48) by the definition of lim sup.

Bounded type theory is utilized to prove Cauchy's formula in the upper half-plane.

Theorem 2.56 (Cauchy's formula in a half-plane). Let $f(z)$ be a function that is analytic and of bounded type in the upper half-plane, and has a continuous extension to the closed half-plane. Assume that the mean type of $f(z)$ is not positive and that $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$. Then

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (t - z)^{-1} f(t)\, dt$$

for $y > 0$, and

$$0 = \int_{-\infty}^{\infty} (t - z)^{-1} f(t)\, dt$$

for $y < 0$.

Proof. By Proposition 2.54, the hypotheses imply that

$$\log |f(x + iy)| \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t - x)^2 + y^2} dt$$

for $y > 0$. Since the inequality

$$\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t - x)^2 + y^2} dt \leq \log \left(\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{(t - x)^2 + y^2} \right) dt$$

holds by Jensen's inequality, we obtain for $y > 0$

$$|f(x + iy)| \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{(t - x)^2 + y^2} dt.$$

Since $f \in L^2(\mathbb{R})$, we can apply Remark 2.7 to conclude that there exists a function $g(z)$, which is analytic in the upper half-plane, such that

$$\operatorname{Re} g(x + iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{(t - x)^2 + y^2} dt$$

for $y > 0$. Then $|f(x + iy)| \leq \operatorname{Re} g(x + iy)$ for $y > 0$. Therefore, the function $g(z) - f(z)$ is analytic and has a nonnegative real part in the upper half-plane. By Remark 2.7, $\operatorname{Re} g(z)$ has a continuous extension to the closed half-plane, and $|f(x)| = \lim_{y \to 0^+} \operatorname{Re} g(x + iy)$ for all real x. Since $f(z)$ has a continuous extension to the closed half-plane, $\operatorname{Re}[g(z) - f(z)]$ has a continuous extension to the closed half-plane and is equal to $|f(x)| - \operatorname{Re} f(x)$ on the boundary. By Theorem 2.11, there is a number $p \geq 0$ such that

$$\operatorname{Re}[g(z) - f(z)] = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)| - \operatorname{Re} f(t)}{(t - x)^2 + y^2} dt$$

for $y > 0$. For the same reasons, there is a number $q \geq 0$ such that

$$\operatorname{Re}[g(z) + f(z)] = qy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)| - \operatorname{Re} f(t)}{(t - x)^2 + y^2} dt$$

for $y > 0$. By the definition of $g(z)$, we can conclude that $p = q = 0$ and that

$$\operatorname{Re} f(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2+y^2}\,dt$$

for $y > 0$. Since the functions $g(z) + if(z)$ and $g(z) - if(z)$ are analytic in the upper half-plane, and since the real parts of the functions are nonnegative, the same argument will show that

$$\operatorname{Re} if(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} if(t)}{(t-x)^2+y^2}\,dt$$

for $y > 0$. It follows that

$$\begin{aligned}\operatorname{Re} f(x+iy) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2+y^2}\,dt \\ &= (2\pi i)^{-1}\int_{-\infty}^{\infty}(t-z)^{-1}f(t)\,dt - (2\pi i)^{-1}\int_{-\infty}^{\infty}(t-\bar{z})^{-1}f(t)\,dt\end{aligned}$$

for $y > 0$. But the first integral represents a function analytic in the upper half-plane, and the second integral represents a function whose complex conjugate is analytic. If a function and its conjugate are both analytic, then the derivative of the function vanishes identically, and the function is a constant. Since $\int_{-\infty}^{\infty}(t-\bar{z})^{-1}f(t)\,dt$ is a constant for $y > 0$ and

$$\left|\int_{-\infty}^{\infty}(t-z)^{-1}f(t)\,dt\right|^2 \le \int_{-\infty}^{\infty}|f(t)|^2\,dt\int_{-\infty}^{\infty}|t-z|^{-2}\,dt \le (\pi/y)\,\|f\|_2^2\,,$$

the integral vanishes identically. □

2.5 A theorem of Krein

Remember that an entire function $F : \mathbb{C} \to \mathbb{C}$ is said to be of exponential type τ if

$$\limsup_{|z|\to\infty} |z|^{-1}\log|F(z)| = \tau < \infty.$$

As we will see in this section, according to a theorem by Mark G. Krein, an entire function is of exponential type if it is of bounded type in the upper and lower half-planes. In this case, the exponential type of the function is equal to the maximum of its mean types in the upper and lower half-planes. Thus, we can see that mean type is a generalization of exponential type to functions that are not necessarily entire.

Theorem 2.57 (Krein). Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function. The following are equivalent:

(i) F has exponential type and

$$\int_{\mathbb{R}} \frac{\log^+|F(t)|}{1+t^2}\,dt < \infty; \tag{2.51}$$

(ii) F and F^* are of bounded type in the upper half-plane.

If one of these statements is true, and therefore both, we have

$$\tau(F) = \max\{v(F), v(F^*)\}. \tag{2.52}$$

Proof. The first direction follows from the conditions for the bounded type in Theorem 2.55. Conversely, suppose that (ii) is true. In this part of the proof, we will show that (ii) implies (i) and that (2.52) is valid.

We start by proving (2.51) using Nevanlinna's factorization. By Theorem 2.43, we have

$$F(z) = B(z)\exp(-ihz)F_1(z), \tag{2.53}$$

where $B(z)$ is a Blaschke product, h is a real number, and $F_1(z)$ is a function analytic and without zeros in the upper half-plane such that

$$\log|F_1(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\mu(t)}{(t-x)^2+y^2} \tag{2.54}$$

for some real-valued function $\mu(x)$ such that

$$\int_{-\infty}^{\infty}\frac{|d\mu|(t)}{1+t^2} < \infty.$$

Claim 1. For any real number x,

$$\mu(x) = \int_0^x \log|F_1(t)|\,dt.$$

Proof of Claim 1. We have $F_1(z) = \exp[G(z)] = \exp[U(z) - V(z)]$ by the proof of the Nevanlinna factorization, where U and V are analytic functions with a nonnegative real part in the upper half-plane,

$$\operatorname{Re}U(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\sigma(t)}{(t-x)^2+y^2},$$

and

$$\operatorname{Re}V(x+iy) = qy + \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{d\nu(t)}{(t-x)^2+y^2},$$

for $p, q \geq 0$, $\sigma(x)$ and $\nu(x)$ are real nondecreasing functions, and $\mu(x) \overset{\text{def}}{=} \sigma(x) - \nu(x)$. By the proof of the Poisson representation in Theorem 2.15, we know that

$$\sigma(x) = \lim_{\varepsilon\to 0}\sigma_\varepsilon(x) = \int_0^x \operatorname{Re}U(t+i\varepsilon)\,dt \tag{2.55}$$

and

$$\nu(x) = \lim_{\varepsilon\to 0}\nu_\varepsilon(x) = \int_0^x \operatorname{Re}V(t+i\varepsilon)\,dt \tag{2.56}$$

for some subsequences. For any positive ε, we have

$$\log|F_1(z+i\varepsilon)| = \operatorname{Re} U(z+i\varepsilon) - \operatorname{Re} V(z+i\varepsilon) \quad \forall z \in \overline{\mathbb{C}^+}.$$

By taking a common subsequence in (2.55) and (2.56), we obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^x \log|F_1(t+i\varepsilon)|\,dt &= \lim_{\varepsilon\to 0}\int_0^x \operatorname{Re} U(t+i\varepsilon)\,dt - \lim_{\varepsilon\to 0}\int_0^x \operatorname{Re} V(t+i\varepsilon)\,dt \\ &= \sigma(x) - \nu(x) = \mu(x).\end{aligned}$$

Let

$$\mu_\varepsilon(x) \stackrel{def}{=} \int_0^x \log|F_1(t+i\varepsilon)|\,dt.$$

We will now prove that

$$\mu(x) = \lim_{\varepsilon\to 0}\mu_\varepsilon(x) = \int_0^x \log|F_1(t)|\,dt.$$

By analytic continuation, F_1 is analytic in an open set V such that $\overline{\mathbb{C}^+} \subset V$. This is true because $B(z)$ is analytic for an open set V. Through analytic continuation on V, $F_1(z)$ can have only a finite number of zeros in $[0,x]$. Because of this, we write

$$F_1(z) = F_2(z)\prod_{k=1}^{N}(z-x_k),$$

where F_2 is analytical and nonzero in $(-x-1, x+1)\cup\mathbb{C}^+$. So

$$\log|F_1(z)| = \log|F_2(z)| + \sum_{k=1}^{K}\log|z-x_k|$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\mu_\varepsilon(x) &= \lim_{\varepsilon\to 0}\int_0^x \log|F_2(t+i\varepsilon)| + \sum_{k=1}^{K}\log|t-x_k+i\varepsilon|\,dt \\ &= \int_0^x \log|F_2(t)|\,dt + \lim_{\varepsilon\to 0}\int_0^x \sum_{k=1}^{K}\log|t-x_k+i\varepsilon|\,dt. \quad (2.57)\end{aligned}$$

The problem here is the sum over the zeros. To prove that we can take a limit over the sum over zeros, consider one zero x_k and suppose, without loss of generality, that x_k is equal to 0. If ε is positive and sufficiently small, we have

$$|\log|t+i\varepsilon|| \le |\log|t||$$

for $|t| < \delta$ with δ small. Since $\log|t+i\varepsilon|$ is bounded in $\delta < |t| \le x$, we have, by the Dominated Convergence Theorem, that

$$\lim_{\varepsilon\to 0}\int_0^x \log|t+i\varepsilon|\,dt = \int_0^x \log|t|\,dt.$$

Once we have proven the case $x_k = 0$, the other cases follow analogously. We can then conclude using (2.57) that

$$\mu(x) = \lim_{\varepsilon \to 0} \mu_\varepsilon(x) = \int_0^x \log|F_2(t)|\,dt + \sum_{k=1}^{K} \int_0^x \log|t - x_k|\,dt = \int_0^x \log|F_1(t)|\,dt.$$

This proves *Claim* 1 and implies that

$$d\mu(t) = \log|F_1(t)|\,dt. \tag{2.58}$$

Furthermore, from (2.53),

$$\log|F(z)| = \log|B(z)| + hy + \log|F_1(z)|.$$

For z real, the first two terms on the right-hand side are zero, so

$$\log|F(x)| = \log|F_1(x)|$$

and by (2.58)

$$\int_{-\infty}^{\infty} \frac{\log^+|F(t)|}{t^2+1}\,dt \leq \int_{-\infty}^{\infty} \frac{|\log|F(t)||}{t^2+1}\,dt = \int_{-\infty}^{\infty} \frac{|d\mu|(t)}{t^2+1} < \infty.$$

This proves the condition (2.51).

We will now prove that F has finite exponential type. The strategy to prove this is as follows:

(i) The "exponential type" of F on the line $z = iy$ implies the "exponential type" of F on a cone V_δ^+.

(ii) The entire function F has an order less than or equal to 2.

(iii) Apply the Phragmén–Lindelöf principle to prove the theorem.

We start with (i). Define, for $\delta > 0$, the cones

$$V_\delta^+ = \{z \in \mathbb{C} \,:\, \delta \leq \arg(z) \leq \pi - \delta\}; \quad V_\delta^- = \{z \in \mathbb{C} \,:\, -\delta \leq \arg(z) \leq -\pi + \delta\};$$
$$W_\delta^+ = \{z \in \mathbb{C} \,:\, -\delta \leq \arg z \leq \delta\}; \qquad W_\delta^- = \{z \in \mathbb{C} \,:\, \pi - \delta \leq \arg z \leq \pi + \delta\}.$$

Consider the Nevanlinna factorization of F given in (2.53), which implies that

$$\log|F(z)| \leq \log|\exp(-ihz)F_1(z)|.$$

Let $z = re^{i\theta}$, then (2.54) in polar coordinates becomes

$$\log|F_1(re^{i\theta})| = \frac{r\sin\theta}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{t^2 - 2tr\cos\theta + r^2}. \tag{2.59}$$

Because $|\cos\theta| < 1 - \eta_\delta$ in V_δ^+, we have

$$t^2 - 2tr\cos\theta + r^2 \geq t^2 - 2tr(1-\eta_\delta) + r^2 \geq t^2 + r^2 - (t^2+r^2)(1-\eta_\delta) \\ = \eta_\delta(t^2+r^2),$$

and (2.59) becomes

$$\log|F_1(re^{i\theta})| \leq \frac{r\sin\theta}{\pi\eta_\delta}\int_{\mathbb{R}}\frac{d\mu(t)}{t^2+r^2}.$$

According to the Dominated Convergence Theorem

$$\limsup_{r\to\infty}\frac{\log|F_1(re^{i\theta})|}{r\sin\theta} \leq \limsup_{r\to\infty}\frac{1}{\pi\eta_\delta}\int_{\mathbb{R}}\frac{d\mu(t)}{t^2+r^2} = 0.$$

Given $\varepsilon > 0$, there exists $C = C(\varepsilon,\delta)$ such that $|F_1(z)| \leq C\exp(\varepsilon y)$ for all $z \in V_\delta^+$. In particular, this means that F_1 has mean type 0 in V_δ^+, and by the definition of F_1, we can conclude that

$$|F(z)| \leq e^{hy}|F_1(z)| \leq Ce^{y(\varepsilon+h)} \quad \text{for all} \quad z \in V_\delta^+.$$

Applying the same argument to F^*, we obtain

$$|F(z)| \leq Ce^{|y|(\tau+\varepsilon)} \tag{2.60}$$

for all $z \in V_\delta^+ \cup V_\delta^-$ and $\tau \stackrel{\text{def}}{=} \max\{h, h^*\}$. By (2.60), item (i) is proved. For now, assume item (ii), i.e., that there exist $\alpha, \beta > 0$ such that for z in $\mathbb{C}$

$$|F(z)| \leq \alpha e^{\beta|z|^2}.$$

Claim 2. If

$$\begin{cases} |F(z)| \leq Ce^{\tau|y|}, \text{ for } z \in \partial W_\delta^+; \\ |F(z)| \leq \alpha\exp\beta|z|^2; \\ \delta < \pi/4, \end{cases}$$

then $|F(z)| \leq Ce^{\tau|y|}$ in W_δ^+.

Proof of Claim 2. Let $G(z) = F(z)e^{-(\tau+2\varepsilon)z}$. Since $x > |y|$ for $z \in W_\delta^+$, we obtain

$$\begin{cases} |G(z)| \leq C, \text{ for } z \in \partial W_\delta^+; \\ |G(z)| \leq \alpha\exp\beta|z|^2; \\ \delta < \pi/4. \end{cases} \tag{2.61}$$

Let $\varphi_1 : \mathbb{C} \to \mathbb{C}$ be a clockwise rotation by an angle δ in the complex plane. Define $\varphi_2 : \mathbb{C} \to \mathbb{C}$ as $\varphi(z) = z^{2\delta/\pi}$, mapping the cone $\varphi_1(W_\delta^+)$ to the upper half-plane. Define $G_1((\varphi_2\circ\varphi_1)(z)) = G(z)$. For G_1, the conditions of (2.61) are

$$\begin{cases} |G_1(z)| \leq C, \text{ for } z \in \mathbb{R}; \\ |G_1(z)| \leq \alpha\exp\beta|z|^{4\delta/\pi}, \end{cases}$$

and because $4\delta/\pi < 1$, we have

$$\liminf_{r\to\infty} r^{-1}\int_0^{\pi} \log^+ |G_1(re^{i\theta})| \sin\theta \, d\theta = 0.$$

By the Phragmén–Lindelöf principle in Theorem 2.2, we conclude that

$$|G_1(z)| \le C \text{ for } z \in \mathbb{C}^+$$

and *Claim* 2 follows.

In particular, this implies[7], together with the conclusion of item (i), that for all $z \in \mathbb{C}$

$$|F(z)| \le C_\varepsilon e^{|y|(\tau+\varepsilon)} \le C_\varepsilon e^{|z|(\tau+\varepsilon)}.$$

Therefore, F has exponential type and $\tau(F) \le \tau$. Since τ is reached in a direction of the imaginary axis, by definition of the mean type, we obtain that $\tau(F) = \tau$. Finally, we will prove item (ii), i.e., F has order less than or equal to 2.

Claim 3.

$$L \stackrel{def}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\log^+ |F(z)|}{|z|^4+1}\, dx\, dy < \infty. \tag{2.62}$$

Suppose for a moment that (2.62) is valid.

Since $\log|F(w)|$ is subharmonic (because F is an entire function), then $\log^+|F(w)|$ is subharmonic since it is the maximum of two subharmonic functions. By the mean value property over the ball centered at w with radius $|w|$,

$$\begin{aligned} \log^+|F(w)| &\le \frac{1}{\pi|w|^2}\int_{B_{|w|}(w)} \log^+|F(z)|\, dxdy \\ &\le \frac{1}{\pi|w|^2}\int_{B_{|w|}(w)} \frac{\log^+|F(z)|(|z|^4+1)}{|z|^4+1}\, dx\, dy, \end{aligned}$$

and because $|z|^4+1 \le (2|w|)^4+1$ in $B_{|w|}(w)$, we have for $|w| \ge 1$

$$\log^+|F(w)| \le \frac{1}{\pi|w|^2}(16|w|^4+1)L \le \frac{17|w|^4}{\pi|w|^2}L = C|w|^2.$$

So F has order 2 for $|w| \ge 1$, and taking α positive and large enough to control F in $B(0,1)$ (entire function, so it has a maximum), we have

$$|F(w)| \le \alpha e^{\beta|w|^2} \quad \text{for all} \quad w \in \mathbb{C}.$$

Proof of Claim 3. Let[8] $\psi : \mathbb{C}^+ \to \mathbb{D}$ be a conformal map given by

$$\psi(z) = \frac{z-i}{z+i},$$

[7] We can do an analogous proof to conclude this bound in the cone W_δ^-.

[8] Here $\mathbb{D} \subset \mathbb{C}$ is the unitary open disk.

with inverse $\varphi : \mathbb{D} \to \mathbb{C}^+$ also being a conformal map given by

$$\varphi(s) = \frac{i(1+s)}{1-s}.$$

Define, for $s \in \mathbb{D}$, the function $f(s) = F(\varphi(s))$. If $\log|F(z)| \leq U(z)$, where U is a positive majorant that is harmonic (by Remark 2.44), then $u(s) = U(\varphi(s))$ satisfies

$$\log|f(s)| \leq u(s). \tag{2.63}$$

As the composition of a harmonic function and a conformal map is a harmonic function, u is harmonic and positive. From (2.63) we have

$$\log^+|f(s)| \leq u(s).$$

Let $s = a + ib$, with $r < 1$. Then, by the mean value property, we have

$$\int_{B_r} \log^+|f(s)|\, da\, db \leq \int_{B_r} u(s)\, da\, db = \pi r^2 u(0).$$

Using the Monotone Convergence Theorem, we can show that

$$\int_{\mathbb{D}} \log^+|f(s)|\, da\, db \leq \pi u(0).$$

Since $\psi'(z) = 2i/|z+i|^2$, by changing variables we have

$$\begin{aligned}\pi u(0) \geq \int_{\mathbb{D}} \log^+|F(\varphi(s))|\, da\, db &= \int_{-\infty}^{\infty}\int_0^{\infty} \log^+|F(z)||\det\psi(z)|^2\, dy\, dx \\ &= 4\int_{-\infty}^{\infty}\int_0^{\infty} \frac{\log^+|F(z)|}{|z+i|^4}\, dy\, dx.\end{aligned}$$

Because[9] $|z+i|^4 \leq C(|z|^4+1)$ in the upper half-plane for some positive constant C, we have proved that

$$\int_{-\infty}^{\infty}\int_0^{\infty} \frac{\log^+|F(z)|}{|z|^4+1}\, dy\, dx < \infty,$$

the analogous argument proves that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{\log^+|F(z)|}{|z|^4+1}\, dy\, dx < \infty,$$

and the result follows. □

Corollary 2.58. Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function. If $F(z)$ and $F^*(z)$ are of bounded type in the upper half-plane and F is bounded on the imaginary axis, then F is constant.

Proof. By Theorem 2.57, F is determined to have exponential type. Given that F is bounded over the imaginary axis, the formula (2.42) implies that F has exponential type 0. Since $F(iz)$ also has exponential type 0 and is bounded along the real axis, this conclusion follows from Corollary 2.5. □

[9] Split into $|z| \leq 1$ and $|z| > 1$, and use the maximum to obtain the desired inequality.

Chapter 3

De Branges spaces

In this chapter, we will provide a brief introduction to de Branges spaces, denoted by $\mathcal{H}(E)$, using the concepts developed in Chapter 2. In the first section, we will prove the Paley-Wiener theorem without employing the powerful tools from harmonic analysis, but rather utilizing the machinery of entire functions from the last chapter.

Moving on to the second section, we will define spaces $\mathcal{H}(E)$ and demonstrate that they possess a reproducing kernel. In the third section, we will observe that de Branges spaces have an alternative definition and are Hilbert spaces. Finally, we will identify orthogonal sets in $\mathcal{H}(E)$ and provide an alternative characterization of these spaces based on three universal properties for Hilbert spaces of entire functions.

3.1 Construction of Paley-Wiener spaces

Definition 3.1 (Paley-Wiener spaces). Let $\delta > 0$. The Paley-Wiener space $\mathcal{PW}(2\pi\delta)$ consists of entire functions $F : \mathbb{C} \to \mathbb{C}$ of exponential type at most $2\pi\delta$ that are square integrable on the real axis.

Theorem 3.2 (Reproducing kernel). The space $\mathcal{PW}(2\pi\delta)$ is a Hilbert space with the norm

$$\|F\|^2_{\mathcal{PW}(2\pi\delta)} = \int_{\mathbb{R}} |F(t)|^2 \, dt.$$

The function

$$K(w, z) \stackrel{def}{=} \frac{\sin[2\pi\delta(z - \bar{w})]}{\pi(z - \bar{w})}$$

belongs to the space for every complex number w and the identity

$$F(w) = \int_{-\infty}^{\infty} F(t)\overline{K(w, t)} \, dt$$

holds for every element $F(z)$ of the space.

Proof. If $F \in \mathcal{PW}(2\pi\delta)$, then

$$\int_{\mathbb{R}} (1 + |F(t)|^2)(1 + t^2)^{-1} \, dt < \infty$$

and by Jensen's inequality with the normalized measure $d\mu(t) = \frac{1}{\pi}(1+t^2)^{-1}dt$, we have

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}} \frac{\log(1+|F(t)|^2)}{1+t^2}\,dt &= \int_{\mathbb{R}} \log(1+|F(t)|^2)\,d\mu(t)\\ &\leq \log\left(\int_{\mathbb{R}} 1+|F(t)|^2\,d\mu(t)\right) = \log\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{1+|F(t)|^2}{1+t^2}\,dt\right) < \infty.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}} (1+t^2)^{-1}\log^+|F(t)|\,dt < \infty. \tag{3.1}$$

Since $F(z)$ is of exponential type, it follows from (3.1) and from Theorem 2.57 that $F(z)$ is of bounded type in the upper half-plane. From (2.52), the mean type of $F(z)$ in the upper half-plane is at most $2\pi\delta$, and $e^{2\pi i\delta z}F(z)$ has mean type nonpositive. So Cauchy's formula in Theorem (2.56) implies that

$$2\pi i e^{i2\pi\delta z}F(z) = \int_{-\infty}^{\infty} (t-z)^{-1}e^{i2\pi\delta t}F(t)\,dt$$

for $y > 0$ and

$$0 = \int_{-\infty}^{\infty} (t-z)^{-1}e^{i2\pi\delta t}F(t)\,dt$$

for $y < 0$. From (2.52) the same formulas hold with $F(z)$ replaced by $F^*(z)$. Therefore, we can combine these two formulas for $y > 0$ and $y < 0$ to get

$$2\pi i F(z) = \int_{\mathbb{R}} (t-z)^{-1}[e^{i2\pi\delta t}e^{-i2\pi\delta z} - e^{-i2\pi\delta t}e^{i2\pi\delta z}]F(t)\,dt$$

when z is nonreal. The formula can be written as

$$F(z) = \int_{-\infty}^{\infty} F(t)\overline{K(z,t)}\,dt.$$

In principle, it is valid only when z is nonreal, but this relation remains true in the other case. It is also valid in the limit of real x_0 by the Dominated Convergence Theorem[1]. It is easily verified that the function $K(w,z)$ belongs to the Paley-Wiener space for every complex number w. We can, therefore, apply the Cauchy-Schwarz inequality to obtain

$$|F(z)|^2 \leq ||F||^2_{\mathcal{PW}(2\pi\delta)}\frac{\sin(2\pi\delta(z-\bar{z}))}{\pi(z-\bar{z})}$$

for every element $F(z)$ of the Paley-Wiener space. Completeness follows from this inequality. In fact, if $\{F_n(z)\}$ is a Cauchy sequence in the space, then

$$|F_n(z) - F_k(z)|^2 \leq ||F_n - F_k||^2_{\mathcal{PW}(2\pi\delta)}\frac{\sin(2\pi\delta(z-\bar{z}))}{\pi(z-\bar{z})}$$

[1] To establish this, divide the integral into two parts: one over the interval (x_0-1, x_0+1) and the other over its complement. In the first part, the integrand is continuous and bounded; in the second part, we can apply the Cauchy-Schwarz inequality and the Dominated Convergence Theorem.

for all complex z. The sequence $F_n(z)$ converges uniformly on bounded sets, implying that the limit function $F(z)$ is entire and

$$\int_{-\infty}^{\infty} |F(t)|^2\, dt \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} |F_n(t)|^2\, dt < \infty$$

by Fatou's lemma. Since

$$|F(z)|^2 \leq \lim_{n\to\infty} ||F_n||^2_{\mathcal{PW}(2\pi\delta)} \frac{\sin(2\pi\delta(z-\bar{z}))}{\pi(z-\bar{z})} = \lim_{n\to\infty} ||F_n||^2_{\mathcal{PW}(2\pi\delta)} \frac{\sin(4\pi i\delta y)}{\pi(2iy)}$$

the limit function $F(z)$ is of exponential type at most[2] $2\pi\delta$. By Fatou's lemma

$$\int_{-\infty}^{\infty} |F(t) - F_k(t)|^2\, dt \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} |F_n(t) - F_k(t)|^2\, dt.$$

Assuming that the sequence $\{F_n\}$ is Cauchy, it converges to $F(z)$ in the metric of the Paley-Wiener space.

□

Theorem 3.3 (Paley-Wiener). Let $\delta > 0$. If $f \in L^2(\mathbb{R})$ and vanishes outside a finite interval $(-\delta, \delta)$, then

$$F(z) = \int_{-\infty}^{+\infty} e^{-2\pi itz} f(t)\, dt$$

is an entire function of exponential type at most $2\pi\delta$ such that

$$\int_{\mathbb{R}} |F(t)|^2\, dt = \int_{\mathbb{R}} |f(t)|^2\, dt.$$

Moreover, every entire function of exponential type at most $2\pi\delta$ that is square integrable on the real axis is of this form.

Proof. If $f_w(x) = e^{2\pi ix\bar{w}}\chi_{(-\delta,\delta)}(x)$, then

$$K(w,z) = \frac{\sin(2\pi\delta(z-\bar{w}))}{\pi(z-\bar{w})} = \int_{-\delta}^{\delta} e^{-2\pi itz} e^{2\pi it\bar{w}}\, dt$$

belongs to the Paley-Wiener space. By Theorem 3.2, the identity

$$\begin{aligned}\int_{\mathbb{R}} \frac{\sin(2\pi\delta(t-\bar{w}_1))}{\pi(t-\bar{w}_1)} \frac{\sin(2\pi\delta(t-w_2))}{\pi(t-w_2)}\, dt &= \langle K(w_1,\cdot), K(w_2,\cdot)\rangle \\ &= K(w_1,w_2) = \int_{-\delta}^{\delta} e^{-2\pi itw_2} e^{2\pi it\bar{w}_1}\, dt\end{aligned}$$

holds for all complex numbers w_1 and w_2. Let

$$M(\delta) \stackrel{def}{=} \{f : \mathbb{R} \to \mathbb{C} \,:\, f \in \operatorname{span}\{f_w\}\,,\ \text{with } w \in \mathbb{C}\}\,.$$

[2] Take the square root and use the argument of sin with a factor of 2 to achieve the correct exponential type.

If $f \in M(\delta)$, write $f(x) = \sum_{k=1}^{N} c_k f_{w_k}(x)$ where w_k are complex numbers, and we can conclude that

$$F(z) = \int_{-\delta}^{\delta} e^{-2\pi i t z} f(t)\,dt = \sum_{k=1}^{N} c_k \int_{-\delta}^{\delta} e^{-2\pi i t z} f_{w_k}(t)\,dt = \sum_{k=1}^{N} c_k K(w_k, z)$$

belongs to the Paley-Wiener space and that

$$\begin{aligned}\int_{\mathbb{R}} |F(t)|^2 dt &= \langle \sum_{k=1}^{N} c_k K(w_k, \cdot), \sum_{i=1}^{N} c_i K(w_i, \cdot)\rangle = \sum_{k=1}^{N}\sum_{i=1}^{N} c_k \overline{c_i} \langle K(w_k, \cdot), K(w_i, \cdot)\rangle \\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} c_k \overline{c_i} \int_{-\delta}^{\delta} e^{-2\pi i t w_i} e^{2\pi i t \bar{w}_k} dt = \langle f, f\rangle = \int_{\mathbb{R}} |f(t)|^2 dt.\end{aligned}$$

The same conclusion holds when f is in the closure of $M(\delta)$ by continuity. In fact, let $I \stackrel{def}{=} [-\delta, \delta]$. If there exists a sequence $\{f_n\} \subset M(\delta)$ such that $f_n \to f$ in $L^2(I)$, then by the isometry, $\{F_n\}$ is a Cauchy sequence in $PW(2\pi\delta)$. Since this space is complete, we have $F_n \to G$ in $\mathcal{PW}(2\pi\delta)$ and

$$\int_{\mathbb{R}} |f(t)|^2\,dt = \lim_{n\to\infty} \int_{\mathbb{R}} |f_n(t)|^2\,dt = \lim_{n\to\infty} \int_{\mathbb{R}} |F_n(t)|^2\,dt = \int_{\mathbb{R}} |G(t)|^2\,dt.$$

We know that $F(z) = \int_{-\delta}^{\delta} e^{-2\pi i t z} f(t)\,dt \in \mathcal{PW}(2\pi\delta)$ by Theorem 3.2. Finally, because $G(z) = \lim_{n\to\infty} F_n(z)$ (see the proof of Theorem 3.2), we obtain for any $z \in \mathbb{C}$ that

$$|G(z) - F(z)| = \lim_{n\to\infty} |F_n(z) - F(z)| \le C \lim_{n\to\infty} \|F_n - F\| \le C \lim_{n\to\infty} \|f_n - f\| = 0,$$

because the last inequality follows from the Cauchy–Schwarz inequality. According to the Stone–Weierstrass theorem, we know that $\overline{M(\delta)} = C(I)$ in the uniform topology. Due to this, $\overline{M(\delta)} = L^2(I)$. Consequently, $T : L^2(I) \to \mathcal{PW}(2\pi\delta)$ is an isometry, and the range of T is a closed subspace.

To prove that T is an isomorphism, it is sufficient to show that T is surjective. Let F be orthogonal to $T(\overline{M(\delta)})$. In particular, $\langle K(w, \cdot), F\rangle = 0$ for all $w \in \mathbb{C}$ and by the reproducing kernel property, we have $F(w) = 0$ for every w.

□

3.2 Basic theory of de Branges spaces

A generalization of Fourier analysis is obtained when the Paley-Wiener spaces are replaced by more general Hilbert spaces of entire functions. A de Branges space is associated with any entire function $E : \mathbb{C} \to \mathbb{C}$ which satisfies the inequality

$$|E^*(z)| < |E(z)| \quad \text{for all } z \in \mathbb{C}^+. \tag{3.2}$$

Definition 3.4 (Hermite-Biehler). An entire function $E : \mathbb{C} \to \mathbb{C}$ that satisfies (3.2) is called a Hermite-Biehler function.

Definition 3.5 (De Branges spaces). The de Branges space $\mathcal{H}(E)$ associated with a function E that satisfies the Hermite-Biehler condition is the set of all entire functions $F : \mathbb{C} \to \mathbb{C}$ such that

$$\int_{\mathbb{R}} |F(t)|^2 |E(t)|^{-2}\, dt < \infty \tag{3.3}$$

and such that both ratios F/E and F^*/E are of bounded type and of nonpositive mean type in the upper half-plane.

The space $\mathcal{H}(E)$ is a vector space over the complex numbers and an inner product is defined in the space by

$$\langle F, G\rangle_{\mathcal{H}(E)} = \int_{\mathbb{R}} F(t)\overline{G(t)}|E(t)|^{-2}\, dt.$$

Example 3.6. The Paley-Wiener space $\mathcal{PW}(2\pi\delta)$ is $\mathcal{H}(E)$ in the case that $E(z) = e^{-2\pi i\delta z}$. In fact, let F be entire. Since E has modulus 1 over the real line, we observe that the integrability condition (3.3) is the same as F being square integrable. Suppose that $F \in \mathcal{H}(E)$. Since E has no zeros, both F/E and F^*/E are entire functions of bounded type. Furthermore, given that E has bounded type, it follows from Corollary 2.49 that both F and F^* also have bounded type, with

$$v(F) \leq \max\{v(F/E), v(E)\} = v(E) = 2\pi\delta$$

and

$$v(F^*) \leq \max\{v(F^*/E), v(E)\} = 2\pi\delta.$$

According to Krein's Theorem 2.57, F has exponential type and

$$\tau(F) = \max\{v(F), v(F^*)\} \leq 2\pi\delta.$$

Therefore, $\mathcal{H}(E) \subset \mathcal{PW}(2\pi\delta)$. Similarly, we have $\mathcal{PW}(2\pi\delta) \subset \mathcal{H}(E)$.

In the space $\mathcal{H}(E)$, we denote $E(z) = A(z) - iB(z)$, where $A(z)$ and $B(z)$ are real entire functions that are real for real z. These functions are given by

$$A(z) = \frac{1}{2}[E(z) + E^*(z)] \quad \text{and} \quad B(z) = \frac{i}{2}[E(z) - E^*(z)]. \tag{3.4}$$

Remark 3.7. If $\mathcal{H}(E)$ is a given space, the function

$$K(w, w) \overset{\text{def}}{=} \frac{B(w)\bar{A}(w) - A(w)\bar{B}(w)}{\pi(w - \bar{w})}$$

is a continuous function of w.

We will show that any space $\mathcal{H}(E)$ contains nonzero elements.

Theorem 3.8 (Reproducing kernel for de Branges spaces). Let $E : \mathbb{C} \to \mathbb{C}$ be a given entire function that satisfies the Hermite-Biehler condition (3.2). Then

$$K(w,z) \stackrel{def}{=} \frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{\pi(z-\bar{w})} \tag{3.5}$$

belongs to $\mathcal{H}(E)$ as a function of z for every complex number w and for every $F \in \mathcal{H}(E)$

$$F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}(E)}.$$

Remark 3.9. We can rewrite $K(w,z)$ in terms of E, so

$$K(w,z) = \frac{E(z)\bar{E}(w) - E^*(z)E(\bar{w})}{2\pi i(\bar{w}-z)}. \tag{3.6}$$

Proof of Theorem 3.8. Given w and z complex numbers, we have

$$2\pi i(\bar{w}-z)K(w,z) = E(z)\bar{E}(w) - E^*(z)E(\bar{w}),$$

where the function $E^*(z)/E(z)$ is bounded by 1 in the upper half-plane due to the Hermite-Biehler condition. In particular, according to Remark 2.34, this function has bounded type. So

$$G(z) = \frac{2\pi i(\bar{w}-z)K(w,z)}{E(z)}$$

is bounded in the upper half-plane. By Proposition 2.35 (iii), $2\pi i(\bar{w}-z)$ is of bounded type in the upper half-plane. Therefore, the quotient $K(w,z)/E(z)$ is also of bounded type in this domain. The mean type of the quotient is nonpositive according to Corollary 2.49 since a bounded function has nonpositive mean type (Remark 2.50), and a nonzero polynomial has zero mean type (Corollary 2.48). For the same reasons,

$$K^*(w,z)/E(z) = K(\bar{w},z)/E(z)$$

is of bounded type and of nonpositive mean type for $z \in \mathbb{C}^+$. Note that, for any w, the function $K(w,z)$ is entire in the variable z due to its expansion in terms of E. Therefore, $K(w,z)/E(z)$ and $K^*(w,z)/E(z)$ are of bounded type in variable z with nonpositive mean type.

When w is not real, there exists a positive constant C such that $|\bar{w}-t| \geq C$ for all $t \in \mathbb{R}$. So,

$$\int_{\mathbb{R}} \left|\frac{K(w,t)}{E(t)}\right|^2 dt = \int_{\mathbb{R}} \left|\frac{E(t)\bar{E}(w) - E^*(t)E(\bar{w})}{2\pi i(\bar{w}-t)E(t)}\right|^2 dt < \infty$$

in this case, because the integrability of $1/|t-\bar{w}|^2$ at infinity implies

$$\int_{\mathbb{R}} \left|\frac{E(t)\bar{E}(w)}{2\pi i(\bar{w}-t)E(t)}\right|^2 dt < \infty \tag{3.7}$$

and

$$\int_{\mathbb{R}} \left| \frac{E^*(t)E(\bar{w})}{2\pi i(\bar{w}-t)E(t)} \right|^2 dt < \infty. \tag{3.8}$$

Note that $E(t)$ can be equal to zero, but the singularities cancel in the integral.

If $w = \bar{w}$ is a real number, the integral is finite because the integrand is a continuous function of t in the interval $(w-1, w+1)$. In fact, if $E(\bar{w}) \neq 0$, the result holds true as the numerator cancels the singularity of the denominator. Otherwise, we consider a neighborhood of w where E is zero only at w. We express $E(x) = (x-w)^r h(x)$ and the singularity of the denominator is canceled again by the zero of the numerator. Since the last two integrals (3.7) and (3.8) converge in the complement of $(w-1, w+1)$ for the same reason as in the case of nonreal w, solely due to the integrability of $1/|t-\bar{w}|^2$ at infinity, it follows that $K(w,z)$ belongs to $\mathcal{H}(E)$ as a function of z for every w.

If $F(z)$ belongs to $\mathcal{H}(E)$, then $F(z)/E(z)$ is analytic in the upper half-plane and continuous in the closed half-plane except for possible singularities at the real zeros of $E(z)$. However, it is continuous even at these points. In fact, if $E(z)$ has a zero of order $r > 0$ at a real point w_0, then $E(z) = (z-w_0)^r G(z)$ for some entire function $G(z)$ which has a nonzero value in a neighborhood $(w_0 - \varepsilon, w_0 + \varepsilon)$ of w_0. Since

$$\|F\|^2_{\mathcal{H}(E)} = \int_{\mathbb{R}} |(t-w_0)^{-r} F(t)/G(t)|^2 \, dt < \infty$$

and

$$\int_{w_0-\varepsilon}^{w_0+\varepsilon} (t-w_0)^{-2k} \, dt = \infty$$

for every positive integer k, $F(z)$ must have a zero of order r or more at w_0. So the ratio $F(z)/E(z)$ is therefore continuous at w_0 and has no singularities on the real axis. By Cauchy's formula in Theorem 2.56,

$$F(z)/E(z) = (2\pi i)^{-1} \int_{\mathbb{R}} (t-z)^{-1} F(t)/E(t) \, dt$$

and

$$0 = \int_{\mathbb{R}} (t-\bar{z})^{-1} F(t)/E(t) \, dt$$

for $z \in \mathbb{C}^+$. The formulas are also valid when $F(z)$ is replaced by $F^*(z)$. The four formulas thus obtained imply, using the same argument as the proof of Theorem 3.2, that

$$F(w) = \langle F, K(w,\cdot) \rangle_{\mathcal{H}(E)} \tag{3.9}$$

for all nonreal w. If w is real, choose a sequence $\{w_n\}$ of nonreal numbers such that $w = \lim w_n$. Then

$$K(w,x)/E(x) = \lim K(w_n,x)/E(x)$$

uniformly on every bounded subset of the real axis. If $-\infty < a < b < \infty$, we have

$$\int_a^b |K(w,t) - K(w_n,t)|^2 |E(t)|^{-2}\, dt = \lim_{k\to\infty} \int_a^b |K(w_k,t) - K(w_n,t)|^2 |E(t)|^{-2}\, dt.$$

Since

$$\begin{aligned}\int_{\mathbb{R}} &|K(w_k,t) - K(w_n,t)|^2 |E(t)|^{-2}\, dt \\ &= \langle K(w_k,t) - K(w_n,t), K(w_k,t) - K(w_n,t)\rangle \\ &= K(w_k,w_k) - K(w_n,w_k) - K(w_k,w_n) + K(w_n,w_n)\end{aligned}$$

it follows from Remark 3.7 that

$$\begin{aligned}\int_a^b &|K(w,t) - K(w_n,t)|^2 |E(t)|^{-2}\, dt \\ &\leq K(w,w) - K(w_n,w) - K(w,w_n) + K(w_n,w_n)\end{aligned}$$

Given that a and b are arbitrary, the limit $K(w,z) = \lim K(w_n,z)$ occurs in the metric of $\mathcal{H}(E)$. From equation (3.9), it can be deduced that for every F in $\mathcal{H}(E)$

$$F(w) = \lim F(w_n) = \lim \langle F, K(w_n,\cdot)\rangle_{\mathcal{H}(E)} = \langle F, K(w,\cdot)\rangle_{\mathcal{H}(E)}.$$

□

Remark 3.10. Some authors, in the definition of de Branges spaces, do not require strict inequality in (3.2) for the function E, but require that E has no zeros in $\mathbb{C}^+$ and $|E^*(z)| \leq |E(z)|$. However, in the case of equality $|E^*(z_0)| = |E(z_0)|$ for some $z_0 \in \mathbb{C}^+$, it follows that the associated space $\mathcal{H}(E)$ is the null space.

In fact, in this case, by analytic continuation (similar to the approach used in Proposition 2.30 with the maximum principle), we find that $E^*(z) = e^{i\theta}E(z)$ for some $\theta \in \mathbb{R}$. Therefore, $G(z) = e^{i\theta/2}E(z)$ satisfies $G^*(z) = G(z)$ and $\mathcal{H}(E) = \mathcal{H}(G)$ for obvious reasons. Following the same steps as in the proof of Theorem 3.8, we can conclude that if $F \in \mathcal{H}(G)$, then $F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}(G)}$. Since G is an entire real function, from (3.6), we have $K(w,w) = 0$ for complex nonreal w, thus

$$0 = K(w,w) = \langle K(w,\cdot), K(w,\cdot)\rangle_{\mathcal{H}(G)} = \|K(w,\cdot)\|^2_{\mathcal{H}(G)}.$$

Because the function $K(w,t)$ is continuous with respect to t, it follows that $K(w,t) = 0$ for real t. Consequently, $F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}(G)} = 0$, implying that F is null in the upper half-plane, and therefore, $F(z) = 0$ for all $z \in \mathbb{C}$ through analytical continuation. Therefore, $\mathcal{H}(G)$ is the null space.

Proposition 3.11. If $\mathcal{H}(E)$ is a given space, then $E(z) = S(z)E_0(z)$ whenever $\mathcal{H}(E_0)$ exists, E_0 has no real zeros and S is a real entire function. Moreover, the operator $\mathcal{J} : \mathcal{H}(E_0) \to \mathcal{H}(E)$ given by $\mathcal{J}(F)(z) = S(z)F(z)$ is an isometric map.

Proof. First, observe that we can decompose E in this way due to Proposition 2.24. The well definition, linearity and the property of being isometric are easily obtained: it suffices to write $E = E_0 S$. Furthermore, to note that the map is surjective, observe that the image of $\mathcal{J}$ is closed in $\mathcal{H}(E)$ because the map is an isometry. As this image contains $K_E(w,z)$ for each w because[3]

$$K_E(w,z) = S(z)K_{E_0}(w,z)S(\bar{w}) = \mathcal{J}(K_{E_0}(w,\cdot)S(\bar{w}))(z), \tag{3.10}$$

if an element is orthogonal to the image of $\mathcal{J}$, it must be orthogonal to $K_E(w,z)$ for all $w \in \mathbb{C}$, and therefore, it is identically zero.

$\square$

Proposition 3.12. Let $\mathcal{H}(E)$ be a given space. If w is a nonreal number, then $F(z)/(z-w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and vanishes at w. The same conclusion holds for a real number w if, and only if, $E(w) \neq 0$.

Proof. Let $U_w(z) = (z-w)$. First, let us assume that w is not a real number. It is important to note that $G(z) = F(z)/U(z)$ is an entire function since the singularity is canceled out. Additionally, it meets the integrability condition of $\mathcal{H}(E)$ because $1/|t-w|^2 \leq C$ for some constant for any real t. Furthermore, it can be observed that G/E has a mean type that is nonpositive

$$0 \geq v(FE^{-1}) = v(U_w GE^{-1}) = v(GE^{-1}) + v\left(U_w\right) = v(GE^{-1})$$

and the same holds for G^*/E.

For the second part, note that the forward direction proceeds by contradiction: assuming the validity of the result and the vanishing of E at a point w, we can use induction on the order of the real zero w of F. Subsequently, we can see that the integral (3.3) diverges for this modified function at $t = w$ due to the zero of E at that point. The remaining steps entail splitting the integral and making use of continuity.

$\square$

Corollary 3.13. Let $\mathcal{H}(E)$ be a given de Branges space. Then $K(w,w) = 0$ if and only if $w \in \mathbb{R}$ and $E(w) = 0$.

Proof. By definition of K in terms of E in (3.6), the converse implication is obvious. Suppose now that $K(w,w) = 0$. But

$$0 = K(w,w) = \langle K(w,\cdot), K(w,\cdot)\rangle_{\mathcal{H}(E)} = \|K(w,\cdot)\|^2_{\mathcal{H}(E)}$$

[3] To prove the first equality in (3.10) we just need to use the fact that S is real so $S^* = S$.

and we obtain that $K(w,t)=0$ for all $t\in\mathbb{R}$. By the reproducing kernel property, for any $F\in\mathcal{H}(E)$ we have

$$F(w)=\langle K(w,\cdot),F\rangle_{\mathcal{H}(E)}=0. \tag{3.11}$$

This is enough to conclude the result. In fact, if we suppose, by contradiction, that $w\notin\mathbb{R}$, it follows from Proposition 3.12 that $F(z)/(z-w)\in\mathcal{H}(E)$ for a nonidentically null $F\in\mathcal{H}(E)$. Applying this by induction over the order r of a zero w of F (which must be finite), we conclude that $F(z)/(z-w)^r\in\mathcal{H}(E)$. However, this function does not satisfy (3.11), a contradiction. Similarly, if $w\in\mathbb{R}$ and $E(w)\neq 0$, then $F(z)/(z-w)^r\in\mathcal{H}(E)$ by Proposition 3.12, and we encounter a contradiction again with (3.11). □

Lemma 3.14. Suppose that $E(z)$ is a Hermite-Biehler function. So $A(z)$ and $B(z)$ have only real zeros. If E has no zeros on the real line, then $A(z)$ and $B(z)$ have only zeros of order one and cannot be the same.

Proof. By the Hermite-Biehle and (3.4), all zeros of $A(z)$ and $B(z)$ are real. Moreover, the formula for $K(w,\bar{w})$ obtained from (3.5) using L'Hôpital's rule

$$K(w,\bar{w})=\frac{B'(\bar{w})A(\bar{w})-A'(\bar{w})B(\bar{w})}{\pi} \tag{3.12}$$

is valid. In particular

$$K(x,x)=\frac{B'(x)A(x)-A'(x)B(x)}{\pi}. \tag{3.13}$$

So, if $A(z)$ has a zero of order 2 at a point x, it follows that $K(x,x)=0$. By Corollary 3.13, we have $E(x)=0$, which is impossible because we assumed that E has no real zeros. The same argument can be applied to the other cases.

□

3.3 Alternative definition and completeness

An alternative definition of the space $\mathcal{H}(E)$ involves an explicit estimate in the complex plane.

Theorem 3.15. A necessary and sufficient condition for an entire function F to belong to $\mathcal{H}(E)$ is that F satisfies (3.3) and that for all complex z

$$|F(z)|^2\leq\|F\|^2_{\mathcal{H}(E)}\,K(z,z). \tag{3.14}$$

Proof. The necessity follows from applying the Cauchy-Schwarz inequality in Theorem 3.8. For the sufficiency, we only need to show that $F(z)/E(z)$ and $F^*(z)/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. Bounded type is obtained by Theorem 2.55, and for this, we will check its hypotheses. The condition (2.44) is satisfied with the same argument as the proof of Theorem 3.2, because $F(x)/E(x)$ is square summable over the real line. The growth hypothesis (2.45) in the upper half-plane follows from the fact that $G(z) \stackrel{def}{=} F(z)/E(z)$ satisfies

$$|G(z)|^2 \leq C\frac{|K(z,z)|}{|E(z)|^2}$$

by (3.14) and

$$\lim_{r\to\infty} r^{-1}\int_0^\pi \log^+ |K(re^{i\theta}, re^{i\theta})E(re^{i\theta})^{-2}| \sin\theta \, d\theta = 0. \tag{3.15}$$

In fact, (3.15) is true because from the definition of K in terms of E, we have

$$|K(w,z)| = \left|\frac{E(z)\bar{E}(w) - E^*(z)E(\bar{w})}{2\pi i(\bar{w} - z)}\right| \leq \frac{2|E(z)||E(w)|}{2\pi|z - \bar{w}|},$$

given that in the last inequality we used the Hermite-Biehler condition. In particular

$$K(re^{i\theta}, re^{i\theta}) \leq C|E(re^{i\theta})|^2/(r\sin\theta)$$

for $0 < \theta < \pi$, so that

$$\left|G(re^{i\theta})\right|^2 \leq \frac{C}{r\sin\theta}. \tag{3.16}$$

As

$$\lim_{r\to\infty} r^{-2}\int_0^\pi \log^+ |1/(r\sin\theta)| \sin\theta \, d\theta = 0,$$

the condition (2.45) follows from (3.16). Finally, the condition (2.46) is obtained using (3.16) over the line iy, with $\theta = \pi/2$ and $r = y$.

□

The explicit estimate of Theorem 3.15 is used to prove the completeness of the space.

Theorem 3.16. The space $\mathcal{H}(E)$ is a Hilbert space.

Proof. Consider any Cauchy sequence $\{F_n(z)\}$ in the metric of $\mathcal{H}(E)$. Since

$$|F_k(w) - F_n(w)|^2 \leq \|F_k - F_n\|^2_{\mathcal{H}(E)} K(w,w) \tag{3.17}$$

for all complex w, the sequence $\{F_n(w)\}$ is a Cauchy sequence for any fixed w. By the completeness of $\mathbb{C}$, $F(w) = \lim F_n(w)$ exists. Since $K(w,w)$ is continuous by Remark 3.7, it remains bounded on any bounded set. The convergence is therefore uniform on bounded sets, and the limit is entire. If (a,b) is any finite interval,

$$\begin{aligned}\int_a^b |(F(t) - F_n(t))/E(t)|^2 \, dt &= \lim_{k\to\infty}\int_a^b |(F_k(t) - F_n(t))/E(t)|^2 \, dt \\ &\leq \lim_{k\to\infty} ||F_k - F_n||^2_{\mathcal{H}(E)}\end{aligned}$$

where the limit on the right exists because the sequence is Cauchy. Thus, it follows that

$$||F - F_n||^2_{\mathcal{H}(E)} \leq \lim_{k\to\infty} ||F_k - F_n||^2_{\mathcal{H}(E)}. \tag{3.18}$$

By (3.17), we have

$$\begin{aligned} |F(w) - F_n(w)|^2 &= \lim_{k\to\infty} |F_k(w) - F_n(w)|^2 \\ &\leq \lim_{k\to\infty} \|F_k - F_n\|^2_{\mathcal{H}(E)} K(w,w) \\ &\leq \lim_{k\to\infty} \Big(\|F_k - F\|_{\mathcal{H}(E)} + \|F - F_n\|\Big)^2_{\mathcal{H}(E)} K(w,w) \\ &= \|F - F_n\|^2_{\mathcal{H}(E)} K(w,w) \end{aligned}$$

for all complex w. For the last equality, we take the limit in n in (3.18) using the fact that the sequence is Cauchy in the norms of the space. By Theorem 3.15, $F - F_n \in \mathcal{H}(E)$. Since $F_n \in \mathcal{H}(E)$ and $\mathcal{H}(E)$ is a vector space, $F \in \mathcal{H}(E)$. Using (3.18) and the fact that the given sequence is Cauchy, we conclude that it converges to $F(z)$ in the metric of $\mathcal{H}(E)$. This completes the proof. □

3.4 Orthogonal sets and characterization of $\mathcal{H}(E)$

Definition 3.17 (Phase function). Let $E(z)$ be a given entire function that satisfies the Hermite-Biehler condition. If there exists a continuous function $\varphi(x)$ of real x such that $E(x)\exp[i\varphi(x)]$ is real for all values of x, we call this function the phase function associated with $E(z)$.

Proposition 3.18. Let $E(z)$ be a given entire function that satisfies the Hermite-Biehler condition. Then there exists some phase function associated with $E(z)$. Moreover, if $\varphi(x)$ is any such function

$$\varphi'(x) = \pi K(x,x)|E(x)|^{-2} > 0$$

for all real x.

Proof. The Hermite-Biehler condition implies that $G(z) = E^*(z)/E(z)$ is analytic in an open set $V \subset \mathbb{C}$ such that $\overline{\mathbb{C}^+} \subset V$. Since $|G(x)| = 1$ on the real line, we have $G(z) \neq 0$ for z in an open set $W \subset V$ with $\overline{\mathbb{C}^+} \subset W$. Therefore, we can write

$$G(z) = \exp(2i\varphi(z)),$$

with $z \in W$ and φ analytical in W. Note that for $x \in \mathbb{R}$, we have $\varphi(x) \in \mathbb{R}$, because $|G(x)| = 1$. In particular, $\varphi(z) = \varphi^*(z)$ for $z, \bar{z} \in W$. In this case, by the definition of $G(z)$ we have

$$E^*(z)\exp(-i\varphi(z)) = E(z)\exp(-i\varphi(z))$$

and defining $F(z) = E(z)\exp(-i\varphi(z))$, the last equality becomes $F^*(z) = F(z)$. Because of this, $F(x)$ is real for $x \in \mathbb{R}$ and the first part is proved. For the second part, note that

$$\frac{(E^*)'(z)E(z) - E^*(z)E'(z)}{E(z)^2} = 2i\varphi'(z)\frac{E^*(z)}{E(z)}.$$

Taking $z = x$ real, we get that

$$\varphi'(x) = \frac{(E^*)'(x)E(x) - E^*(x)E'(x)}{2i|E(x)|^2} = \frac{\pi K(x,x)}{|E(x)|^2}, \tag{3.19}$$

because by L'Hôpital's rule

$$K(w,\bar{w}) = \frac{E'(\bar{w})E^*(\bar{w}) - (E^*)'(\bar{w})E(\bar{w})}{-2\pi i}.$$

By Corollary 3.13, we conclude that $\varphi'(x) > 0$. The only potentially problematic case would be when $E(x) = 0$, but the result still holds true. In fact, in this case, the zeros of the numerator and denominator cancel each other out in equation (3.19). To see this, we can use Proposition 3.11 to express $E(z) = S(z)E_1(z)$, where E_1 has no real zeros.

$\square$

Proposition 3.19. Let $\mathcal{H}(E)$ be a given space and let $\varphi(x)$ be a phase function associated with $E(z)$. Let a, b, and c be real numbers, with a and b distinct.

(i) The function $\bar{E}(c)^{-1}K(c,z)$ belongs to $\mathcal{H}(E)$ and

$$\frac{K(c,z)}{\bar{E}(c)} = \frac{E(z) - E^*(z)\exp(-2i\varphi(c))}{2\pi i(c-z)};$$

(ii) For every $F(z)$ in $\mathcal{H}(E)$,

$$E(c)^{-1}F(c) = \langle F, \bar{E}(c)^{-1}K(c,\cdot)\rangle;$$

(iii)

$$\langle \bar{E}(a)^{-1}K(a,\cdot), \bar{E}(b)^{-1}K(b,\cdot)\rangle = \frac{1 - e^{2i(\varphi(b)-\varphi(a))}}{2\pi i(a-b)}. \tag{3.20}$$

Proof. The items (i) and (ii) are trivial if $E(c)$ is nonzero; it suffices to use the definition of the phase function and the conjugation property of the inner product, as well as the definition of the reproducing kernel. In case $E(c)$ is 0, we must proceed with more caution and employ Proposition 3.11 to express $E(z) = S(z)E_1(z)$ with E_1 having no real zeros. We also utilize the fact that $K_E(z) = S(z)K_{E_1}(w,z)S(\bar{w})$

to eliminate the singularities and the result remains valid. For (iii), we can use the item (ii) and conclude that

$$\begin{aligned}\langle \bar{E}(a)^{-1}K(a,\cdot), \bar{E}(b)^{-1}K(b,\cdot)\rangle_{\mathcal{H}(E)} &= E(b)^{-1}\bar{E}(a)^{-1}K(a,b) \\ &= \frac{E(b)\bar{E}(a) - E^*(b)E(\bar{a})}{E(b)\bar{E}(a)2\pi i(\bar{a}-b)} = \frac{1-e^{2i(\varphi(b)-\varphi(a))}}{2\pi i(a-b)}.\end{aligned}$$

□

Phase functions are used to construct orthogonal sets in $\mathcal{H}(E)$, which yield a remarkable formula for norms in the space. The motivation for this construction comes from (3.20), where we consider a and b as two distinct real numbers with $\varphi(a) \equiv \varphi(b) \bmod \pi$.

Theorem 3.20 (Orthogonal sets). Let $\mathcal{H}(E)$ be a given space and let $\varphi(x)$ be a phase function associated with $E(z)$. If α is a given real number, then

$$\Lambda_\alpha = \left\{\bar{E}(t_n)^{-1} \times K(t_n, z) \,:\, t_n \in \mathbb{R} \textit{ with } \varphi(t_n) \equiv \alpha \bmod \pi\right\} \tag{3.21}$$

is an orthogonal set in $\mathcal{H}(E)$. The only elements of $\mathcal{H}(E)$ that are orthogonal to Λ_α are constant multiples of

$$e^{i\alpha}E(z) - e^{-i\alpha}E^*(z).$$

If this function does not belong to space, then Λ_α is a basis for $\mathcal{H}(E)$ and we have the representation formulas

$$F(z) = \sum_n F(t_n)\frac{K(t_n,z)}{K(t_n,t_n)} \tag{3.22}$$

and

$$\|F\|^2_{\mathcal{H}(E)} = \sum_n \left|\frac{F(t_n)}{E(t_n)}\right|^2 \frac{\pi}{\varphi'(t_n)} = \sum_n \frac{|F(t_n)|^2}{K(t_n,t_n)} \tag{3.23}$$

for every F in $\mathcal{H}(E)$.

Corollary 3.21. Let $\alpha = 0$ in the definition of Λ_α. Then (3.21) becomes, up to a constant, equal to

$$\Lambda_0 = \left\{\frac{B(z)}{(z-\xi)} \,:\, \xi \in \mathbb{R} \textit{ with } B(\xi) = 0\right\}.$$

So

$$F(z) = \sum_{B(\xi)=0} \frac{F(\xi)}{B'(\xi)}\frac{B(z)}{(z-\xi)}$$

and

$$\|F\|^2_{\mathcal{H}(E)} = \sum_{B(\xi)=0} \frac{\pi|F(\xi)|^2}{B'(\xi)A(\xi)}.$$

In the same way, if $\alpha = \pi/2$, we have

$$\Lambda_{\pi/2} = \left\{ \frac{A(z)}{(z-\xi)} \;:\; \xi \in \mathbb{R} \textit{ with } A(\xi) = 0 \right\}.$$

It follows that

$$F(z) = \sum_{A(\xi)=0} \frac{F(\xi)}{A'(\xi)} \frac{A(z)}{(z-\xi)} \tag{3.24}$$

and

$$\|F\|^2_{\mathcal{H}(E)} = \sum_{A(\xi)=0} \frac{\pi |F(\xi)|^2}{-A'(\xi)B(\xi)}. \tag{3.25}$$

Proof. By the definition of the phase function

$$E^*(x) = e^{2i\varphi(x)} E(x),$$

it follows that each t where $\varphi(t) \equiv 0 \bmod \pi$ is a zero of $B(z)$. For the same reason, each t where $\varphi(t) \equiv \pi/2 \bmod \pi$ is a zero of A. If $A(\xi) = 0$, we have by (3.5) and (3.13) that

$$K(\xi, z) = -\frac{A(z)B(\xi)}{\pi(z-\xi)} \qquad \text{and} \qquad K(\xi,\xi) = -\frac{A'(\xi)B(\xi)}{\pi}. \tag{3.26}$$

Therefore, the formulas (3.24) and (3.25) follow from the replacement of (3.26) in (3.22) and (3.23). The same occurs with formulas with zeros in B. □

Proof of Theorem 3.20. Orthogonality follows from Proposition 3.19. An explicit proof of the theorem is restricted to the case $\alpha = 0$. The general case then follows because $E(z)$ can be replaced by $e^{i\alpha}E(z)$ without a change in the corresponding space. Let

$$\Lambda_0 = \left\{ \bar{E}(t_n)^{-1} \times K(t_n, z) \;:\; t_n \in \mathbb{R} \textit{ with } \varphi(t_n) \equiv 0 \bmod \pi \right\}$$

and consider $f(z) \overset{def}{=} -A(z)/B(z)$. Since $A(z)$ and $B(z)$ are real entire functions, $f^*(z) = f(z)$. Additionally, $K(z,z) > 0$ and

$$\begin{aligned} K(z,z) = \frac{B(z)A(\bar{z}) - A(z)B(\bar{z})}{2\pi i y} &= \frac{|B(z)|^2}{2\pi i y}\left(\frac{A(\bar{z})}{B(\bar{z})} - \frac{A(z)}{B(z)}\right) \\ &= \frac{|B(z)|^2}{2\pi i y}\left(f(z) - f(\bar{z})\right) \\ &= \frac{|B(z)|^2}{2\pi y}\left(2\,\mathrm{Im}\, f(z)\right) \end{aligned}$$

when z is not real, it follows that $\mathrm{Re}(-if(z)) > 0$ when $z \in \mathbb{C}^+$. The idea here is to apply Proposition 2.17, and for that, we need to check its hypotheses. The singularities of $f(z)$ are the real points where $B(z)$ has a zero of higher multiplicity than $A(z)$. Remember that Lemma 3.14 implies that all the zeros of A and B are real. If E has no real zeros, the same lemma implies that the sets of zeros of A and B are disjoint, and these zeros are all simple. So, if E has no real zeros, $f(z)$ has only simple zeros and simple poles. In the general case where $E(x) = 0$ for some real x, we can write $E(z) = E_0(z)S(z)$ as in Proposition 3.11. This implies that the poles of $f(z)$ are simple due to the cancellation of this zero[4]. By Corollary 3.21, the points $t_n \in \mathbb{R}$ with $\varphi(t_n) \equiv 0 \mod \pi$ are poles of $f(z)$, as they correspond to the zeros of $B(z)$.

In summary, the function $f(z) = A(z)/B(z)$ is analytic with a positive real part in the upper half-plane. It has simple poles only on the real line at points where $\varphi(t_n) \equiv 0 \mod \pi$ and simple zeros only on the real line at points where $\varphi(s_n) \equiv \pi/2 \mod \pi$. Therefore, by Proposition 2.17, there exist positive numbers $\{p_n\}$ and a nonnegative number p such that

$$[f(z) - \bar{f}(w)]/(z - \bar{w}) = p + \sum_n p_n(t_n - z)^{-1}(t_n - \bar{w})^{-1}$$

when z and w are not real. The numbers p_n are given by

$$p_n = \lim_{z \to t_n} (z - t_n)A(z)/B(z) = A(t_n)/B'(t_n).$$

As

$$K(w, z) = \frac{B(z)B(\bar{w})}{\pi} \left(\frac{f(z) - f(\bar{w})}{z - \bar{w}} \right),$$

explicitly, the formula reads

$$K(w, z) = \frac{p}{\pi} B(z)\bar{B}(w) + \sum_n \frac{A(t_n)}{\pi B'(t_n)} \frac{B(z)}{(z - t_n)} \frac{B(\bar{w})}{(\bar{w} - t_n)}. \tag{3.27}$$

Written in this way, the formula is valid for all complex z and w. We will now show that convergence takes place in the metric of $\mathcal{H}(E)$. Since t_n is a zero of $B(z)$, it follows that $E^*(t_n) = E(t_n)$, so $A(t_n) = E(t_n)$ and

$$\pi \frac{K(t_n, z)}{\bar{E}(t_n)} = \pi \frac{B(z)A(t_n)}{\pi(z - t_n)\bar{E}(t_n)} = \frac{B(z)}{(z - t_n)}.$$

This implies that the functions on the right-hand side form an orthogonal sequence in $\mathcal{H}(E)$. Moreover, note that

$$\begin{aligned} &\left\| \sum_{n=1}^{N} \frac{A(t_n)}{\pi B'(t_n)} \frac{B(t)}{(t - t_n)} \frac{B(\bar{w})}{(\bar{w} - t_n)} \right\|_{\mathcal{H}(E)} \\ &\qquad = \sum_{n=1}^{N} \left(\frac{A(t_n)}{\pi B'(t_n)} \right)^2 \left(\frac{B(\bar{w})}{\bar{w} - t_n} \right) \left(\frac{B(w)}{w - t_n} \right) \left\| \frac{B(t)}{t - t_n} \right\|_{\mathcal{H}(E)}^2 \end{aligned} \tag{3.28}$$

[4] If $B(z)$ has a zero at a point x, then $A(z)$ has the same zero but with one lower order.

and

$$\begin{aligned}\|B(t)/(t-t_n)\|^2_{\mathcal{H}(E)} &= \pi^2\langle \bar{E}(t_n)^{-1}K(t_n,t), \bar{E}(t_n)^{-1}K(t_n,t)\rangle \\ &= \pi^2 K(t_n,t_n)|E(t_n)|^{-2} \\ &= \pi B'(t_n)/A(t_n),\end{aligned}$$

by the formula of $K(x,x)$ in (3.13). So, it follows from (3.28) and (3.27) that

$$\begin{aligned}&\left\|\sum_{n=1}^{N}\frac{A(t_n)}{\pi B'(t_n)}\frac{B(t)}{(t-t_n)}\frac{B(\bar{w})}{(\bar{w}-t_n)}\right\|_{\mathcal{H}(E)} \\ &\quad = \sum_{n=1}^{N}\left(\frac{A(t_n)}{\pi B'(t_n)}\right)\left(\frac{B(\bar{w})}{\bar{w}-t_n}\right)\left(\frac{B(w)}{w-t_n}\right) = K(w,w) - \frac{p}{\pi}|B(w)|^2 \leq K(w,w).\end{aligned}$$

This means that the partial sums of the series are uniformly bounded in the metric of $\mathcal{H}(E)$. Since $\mathcal{H}(E)$ is a Hilbert space by Theorem 3.16, the convergence of the orthogonal series

$$\sum_n \frac{A(t_n)}{\pi B'(t_n)}\frac{B(t)}{t-t_n}\frac{\bar{B}(w)}{\bar{w}-t_n}$$

occurs in the metric of $\mathcal{H}(E)$. In particular, the sum of the orthogonal series belongs to $\mathcal{H}(E)$. Since $K(w,z)$ belongs to $\mathcal{H}(E)$, it follows from (3.27) that $pB(z)\bar{B}(w)$ belongs to $\mathcal{H}(E)$. If $F(z)$ is an element of $\mathcal{H}(E)$ that is orthogonal to Λ_0, then by (3.27)

$$F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}(E)} = \langle F, (p/\pi)\bar{B}(w)B\rangle_{\mathcal{H}(E)} = B(w)\langle F, (p/\pi)B\rangle_{\mathcal{H}(E)}.$$

Due to the arbitrariness of w, it follows that $F(w) = cB(w)$ for all w, for some constant c that does not depend on w. If $B(z)$ does not belong to $\mathcal{H}(E)$, then $c = 0$, and $F(z)$ vanishes identically, making the orthogonal set complete. In this case, we can divide by the norm to obtain an orthonormal basis for $\mathcal{H}(E)$. In particular, if $F \in \mathcal{H}(E)$, we have by the properties of orthogonal sets

$$\begin{aligned}\|F\|^2_{\mathcal{H}(E)} &= \sum_n \frac{\left|\langle F, \bar{E}(t_n)^{-1}K(t_n,\cdot)\rangle_{\mathcal{H}(E)}\right|^2}{\left\|\bar{E}(t_n)^{-1}K(t_n,t)\right\|^2_{\mathcal{H}(E)}} \\ &= \sum_n \frac{|F(t_n)|^2}{K(t_n,t_n)} = \sum_n \left|\frac{F(t_n)}{E(t_n)}\right|^2 \frac{\pi}{\varphi'(t_n)}\end{aligned}$$

and the last equality is valid due to (3.19). For similar reasons,

$$F(z) = \sum_{\psi\in\Lambda_\alpha}\frac{\langle F,\psi\rangle_{\mathcal{H}(E)}}{\|\psi\|_{\mathcal{H}(E)}}\psi(z) = \sum_n F(t_n)\frac{K(t_n,z)}{K(t_n,t_n)}.$$

□

Proposition 3.22. If $\mathcal{H}(E)$ is a given space, there is at most one real number α, modulo π, such that $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$.

Proof. Assume that there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 \not\equiv \alpha_2 \pmod{\pi}$ such that

$$F_{\alpha_1}(z) = e^{i\alpha_1}E(z) - e^{-i\alpha_1}E^*(z) \in \mathcal{H}(E)$$

and

$$F_{\alpha_2}(z) = e^{i\alpha_2}E(z) - e^{-i\alpha_2}E^*(z) \in \mathcal{H}(E).$$

Because $\mathcal{H}(E)$ is a Hilbert space, it follows that $G(z) = e^{i\alpha_1}F_{\alpha_1}(z) - e^{i\alpha_2}F_{\alpha_2}(z)$ belongs to the space. But

$$G(z) = (e^{2i\alpha_1} - e^{2i\alpha_2})E(z) \tag{3.29}$$

and since $\alpha_1 \not\equiv \alpha_2 \bmod \pi$, it follows that $|e^{2i\alpha_1} - e^{2i\alpha_2}| > 0$. This, along with (3.29), implies that $E \in \mathcal{H}(E)$, which contradicts the integrability condition in (3.3) because

$$\|E\|^2_{\mathcal{H}(E)} = \int_{\mathbb{R}} |E(t)|^2|E(t)|^{-2}dt = \int_{\mathbb{R}} dt,$$

and the last integral diverges.

$\square$

Remark 3.23. In Theorem 3.20, the set Λ_α is not a basis for $\mathcal{H}(E)$ unless $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z) \in \mathcal{H}(E)$. However, Proposition 3.22 guarantees that there is at most one real α for which this condition holds. For all other real values of α, Λ_α becomes a basis for $\mathcal{H}(E)$.

Proposition 3.24. Any space $\mathcal{H}(E)$ has the following properties:

($\mathcal{H}_1$) Whenever $F(z)$ is in the space and has a nonreal zero w, the function

$$G(z) = F(z)(z - \overline{w})/(z - w)$$

is in the space and has the same norm as $F(z)$.

($\mathcal{H}_2$) For every nonreal number w, the linear functional of evaluation defined on the space by $F(z) \to F(w)$ is continuous.

($\mathcal{H}_3$) The function $F^*(z)$ belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

Proof.
($\mathcal{H}_1$) Since w is nonreal, Proposition 3.12 implies that $J(z) = F(z)/(z - w)$ belongs to $\mathcal{H}(E)$. In particular, J is an entire function such that $J(z)/E(z)$ and $J^*(z)/E(z)$ have mean type nonpositive; the same occurs with G because polynomials have mean type zero, as shown in Corollary 2.48. Moreover, the equality $\|F\|_{\mathcal{H}(E)} = \|G\|_{\mathcal{H}(E)}$ is obvious.

($\mathcal{H}_2$) This is clear from the Cauchy–Schwarz inequality (or see Theorem 3.15).

($\mathcal{H}_3$) This is trivial by the definition of $\mathcal{H}(E)$.

□

Remark 3.25. If $\mathcal{H}(E)$ is a space, the function $L(w,z) = 2\pi i(\bar{w} - z)K(w,z)$ satisfies the identity

$$L(w,z) = L(\alpha,z)L(\alpha,\alpha)^{-1}L(w,\alpha) + L(\bar{\alpha},z)L(\bar{\alpha},\bar{\alpha})^{-1}L(w,\bar{\alpha}) \tag{3.30}$$

for every nonreal number α.

Lemma 3.26. Let $z \mapsto K(w,z)$ be an entire function of z defined for every nonreal number w, such that

(i) $K(z,w) = \overline{K(w,z)}$ for all $z, w \in \mathbb{C} \setminus \mathbb{R}$.

(ii) $K(w,w) > 0$ for all $w \in \mathbb{C} \setminus \mathbb{R}$.

(iii) $K(\bar{w},z) = \overline{K(w,\bar{z})}$ for all nonreal w.

(iv) $L(w,z) = 2\pi i(\bar{w} - z)K(w,z)$ satisfies (3.30) for all $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

Then there exist entire real functions A and B such that

$$K(w,z) = \frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{\pi(z - \bar{w})}$$

and $E(z) = A(z) - iB(z)$ is a Hermite-Biehler function.

Proof. In fact, let α be some nonreal number, and

$$E(z) \stackrel{def}{=} \frac{L(\alpha,z)}{L(\alpha,\alpha)^{1/2}}. \tag{3.31}$$

We assert that E is the sought Hermite-Biehler function. If this is the case, we can choose as usual

$$A(z) = \frac{1}{2L(\alpha,\alpha)^{1/2}} \left(L(\alpha,z) + \overline{L(\alpha,\bar{z})}\right)$$

and

$$B(z) = \frac{i}{2L(\alpha,\alpha)^{1/2}} \left(L(\alpha,z) - \overline{L(\alpha,\bar{z})}\right).$$

We will now prove that this choice of E is correct. From property (iii) and the fact that $K(\alpha,\alpha)$ is real by (i), we get that $L(\alpha,\alpha)$ is real. By (iv), we have

$$L(\alpha,\alpha) = -L(\bar{\alpha},\bar{\alpha}). \tag{3.32}$$

Taking $w = z$ in (3.30), using equation (3.32) and properties (i) and (iii), we conclude that

$$L(z,z) = |L(\alpha,z)|^2 \frac{1}{L(\alpha,\alpha)} - |L(\alpha,z)|^2 \frac{1}{L(\alpha,\alpha)}. \tag{3.33}$$

If we consider any fixed $\alpha \in \mathbb{C}^+$, we have $L(\alpha,\alpha) > 0$ by (ii), and we can define the entire function of the variable z as in (3.31). From equation (3.33) and (ii), we find that

$$|E(\alpha,z)|^2 - |E(\alpha,\bar{z})|^2 = L(z,z) = 4\pi \operatorname{Im}(z) K(z,z) > 0$$

for all $z \in \mathbb{C}^+$. This is the Hermite-Biehler property and the identity

$$L(w,z) = E(\alpha,z)\overline{E(\alpha,w)} - \overline{E(\alpha,\bar{z})}E(\alpha,\bar{w})$$

is equivalent to equation (3.30). $\square$

The axioms $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ of Proposition 3.24 completely characterize $\mathcal{H}(E)$.

Theorem 3.27. A Hilbert space $\mathcal{H}$ whose elements are entire functions, which contains a nonzero element, and which satisfies $(\mathscr{H}_1)$, $(\mathscr{H}_2)$, and $(\mathscr{H}_3)$ is isometrically equal to some de Branges space $\mathcal{H}(E)$.

Proof. Due to $(\mathscr{H}_2)$, it follows from the Riesz Representation Theorem that for every $w \in \mathbb{C} \setminus \mathbb{R}$, there exists a unique element $z \mapsto K(w,z) \in \mathcal{H}$ such that

$$F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}}$$

for every F in $\mathcal{H}$. The idea here is simply to prove that the function $K(w,z)$ satisfies the hypotheses (i)-(iv) of Lemma 3.26. Thus, we obtain our function E to construct $\mathcal{H}(E)$.

(i) Let w, z be nonreal numbers. So

$$\overline{K(w,z)} = \overline{\langle K(w,\cdot), K(z,\cdot)\rangle_{\mathcal{H}}} = \langle K(z,\cdot), K(w,\cdot)\rangle_{\mathcal{H}} = K(z,w).$$

(ii) Let $w \in \mathbb{C} \setminus \mathbb{R}$. The inequality $K(w,w) = \langle K(w,\cdot), K(w,\cdot)\rangle_{\mathcal{H}} \geq 0$ holds due to the positivity of an inner product. Therefore, if $K(w,w) = 0$, then $K(w,\cdot) = 0$ and

$$F(w) = \langle F, K(w,\cdot)\rangle_{\mathcal{H}} = 0$$

for all $F \in \mathcal{H}$. Since

$$F(z)\frac{(z-\bar{w})}{(z-w)} = F(z) + (w-\bar{w})\frac{F(z)}{(z-w)},$$

the axiom $(\mathscr{H}_1)$ implies that $F(z)/(z-w)$ belongs to $\mathcal{H}$ whenever $F(z)$ belongs to $\mathcal{H}$ and has a zero at w. If every element of $\mathcal{H}$ vanishes at w, it follows inductively that $F(z)/(z-w)^n$ belongs to $\mathcal{H}$ and vanishes at w for every $n \in \mathbb{N}$. Then F has a root in w of infinite order and, being entire, it follows by Taylor expansion that F is identically null. The hypothesis that $\mathcal{H}$ contains a nonzero element therefore implies that $K(w,w) > 0$.

(iii) Here we will apply the axiom $(\mathscr{H}_3)$. It implies that if $F \in \mathcal{H}$ then $F^* \in \mathcal{H}$ and $\|F\|_{\mathcal{H}} = \|F^*\|_{\mathcal{H}}$. In particular, using polarization we conclude that

$$\langle F^*, G^* \rangle_{\mathcal{H}} = \langle G, F \rangle_{\mathcal{H}} = \overline{\langle F, G \rangle}_{\mathcal{H}}$$

for every $F, G \in \mathcal{H}$. Let w be a nonreal number. Then $(K(w,z))^* = \overline{K(w,\bar{z})}$ belongs to $\mathcal{H}$ for every nonreal number w, with

$$\langle F, \overline{K(w,\cdot)} \rangle_{\mathcal{H}} = \overline{\langle F^*, K(w,\cdot) \rangle}_{\mathcal{H}} = F(\bar{w}) = \langle F, K(\bar{w},\cdot) \rangle_{\mathcal{H}}.$$

By the arbitrariness of F, we have $\overline{K(\bar{w},z)} = K(\bar{w},z)$.

(iv) Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. The function

$$z \mapsto K(w,z) - K(\alpha,z)K(\alpha,\alpha)^{-1}K(w,\alpha)$$

belongs to $\mathcal{H}$ as a function of z for every nonreal number w. By $(\mathscr{H}_1)$, we have

$$G(z) \stackrel{\text{def}}{=} \left[K(w,z) - \frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)}\right] \frac{(z-\bar{\alpha})}{(z-\alpha)} \in \mathcal{H}.$$

Note that $G(\bar{\alpha}) = 0$. Moreover, note that if F and G are in $\mathcal{H}$ and both vanish at $\bar{\alpha}$, then because of $(\mathscr{H}_1)$, we can use polarization again to conclude that

$$\langle F(t)(t-\alpha)(t-\bar{\alpha})^{-1}, G \rangle_{\mathcal{H}} = \langle F, (t-\alpha)(t-\bar{\alpha})^{-1}G(t) \rangle_{\mathcal{H}}. \tag{3.34}$$

Let $F \in \mathcal{H}$ such that $F(\bar{\alpha}) = 0$. Because of (3.34), we have

$$\begin{aligned}
\langle F, G \rangle_{\mathcal{H}} &= \langle F(t)(t-\alpha)/(t-\bar{\alpha}), K(w,t) - K(\alpha,t)K(\alpha,\alpha)^{-1}K(w,\alpha) \rangle_{\mathcal{H}} \\
&= F(w)(w-\alpha)/(w-\bar{\alpha}) \\
&= \langle F(t), [K(w,t) - K(\bar{\alpha},t)K(\bar{\alpha},\bar{\alpha})^{-1}K(w,\bar{\alpha})](\bar{w}-\bar{\alpha})/(\bar{w}-\alpha) \rangle_{\mathcal{H}}.
\end{aligned}$$

Since $[K(w,z) - K(\bar{\alpha},z)K(\bar{\alpha},\bar{\alpha})^{-1}K(w,\bar{\alpha})](\bar{w}-\bar{\alpha})/(\bar{w}-\alpha)$ vanishes at $\bar{\alpha}$ and since F is an arbitrary element of $\mathcal{H}$ which vanishes at $\bar{\alpha}$, we can conclude that

$$G(z) = [K(w,z) - K(\bar{\alpha},z)K(\bar{\alpha},\bar{\alpha})^{-1}K(w,\bar{\alpha})](\bar{w}-\bar{\alpha})/(\bar{w}-\alpha).$$

This identity is equivalent to the identity in (3.30) from Remark 3.25.

By Lemma 3.26, we obtain a space $\mathcal{H}(E)$ associated with E, with a reproducing kernel $K(w,z)$. We show that $\mathcal{H}(E)$ is isometrically equal to $\mathcal{H}$. The function $K(w,z)$ belongs to both spaces for every non-real number w. The inner product of two such functions is the same in $\mathcal{H}$ as it is in $\mathcal{H}(E)$. A finite linear combination of such functions, therefore, has the same norm in $\mathcal{H}$ as it does in $\mathcal{H}(E)$. In fact, if $F(z) = \sum_{i=1}^{n} a_i K(w_i, z) \in \mathcal{H} \cap \mathcal{H}(E)$ for $w_i \in \mathbb{C} \setminus \mathbb{R}$, then

$$\|F\|_{\mathcal{H}}^2 = \sum_{i,j=1}^{n} a_i \overline{a_j} K(w_i, w_j) = \|F\|_{\mathcal{H}(E)}^2 .$$

Moreover, note that $S \stackrel{\mathrm{def}}{=} \operatorname{span}\{K(w, z) \; : \; w \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in $\mathcal{H}$ and $\mathcal{H}(E)$. Otherwise, there exists a nonzero $F \in \mathcal{H}$ such that F is orthogonal to S in $\mathcal{H}$, then

$$F(w) = \langle F, K(w, \cdot)\rangle_{\mathcal{H}} = 0$$

for all w nonreal, and so F is identically null because F is entire. The same applies to $\mathcal{H}(E)$. If $F(z)$ is in $\mathcal{H}$, there exists a sequence $\{F_n\} \subset S$ such that $F = \lim F_n$ in the metric of $\mathcal{H}$. Since the sequence is Cauchy in the metric of $\mathcal{H}$ and the approximating functions have the same norms and inner products in $\mathcal{H}(E)$ as in $\mathcal{H}$, the sequence is also Cauchy in the metric of $\mathcal{H}(E)$. As $\mathcal{H}(E)$ is a Hilbert space, $G = \lim F_n$ exists in the metric of $\mathcal{H}(E)$. For every nonreal number w,

$$\begin{aligned} G(w) = \langle G, K(w, \cdot)\rangle_{\mathcal{H}(E)} &= \lim_{n\to\infty} \langle F_n, K(w, \cdot)\rangle_{\mathcal{H}(E)} \\ &= \lim_{n\to\infty} \langle F_n, K(w, \cdot)\rangle_{\mathcal{H}} = \langle F, K(w, \cdot)\rangle_{\mathcal{H}} = F(w). \end{aligned}$$

By the arbitrariness of w, it follows that $G(z) = F(z)$ for every z by analytical continuation. Also,

$$\|G\|_{\mathcal{H}(E)} = \lim_{n\to\infty} \|F_n\|_{\mathcal{H}(E)} = \lim_{n\to\infty} \|F_n\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}.$$

So $\mathcal{H}$ is contained isometrically in $\mathcal{H}(E)$. A similar argument will show that $\mathcal{H}(E)$ is contained isometrically in $\mathcal{H}$.

$\square$

Chapter 4

Some extremal problems in Fourier analysis

This chapter addresses some extremal problems in Hilbert spaces of entire functions, exploring weighted Paley-Wiener spaces and their connections with de Branges spaces to understand the nature of some sharp constants in the context of Fourier analysis. To study this type of problem, we need two definitions.

Definition 4.1 (Fourier optimization problem). A Fourier optimization problem is a problem that seeks to optimize a functional or quantity over a set of functions, with constraints imposed on their Fourier transforms to ensure the significance and nontriviality of the problem. These constraints may be related to decay rates, support restrictions or pointwise requirements, among others.

Definition 4.2 (Extremizer). In a Fourier optimization problem, an extremizer is a function that represents the ideal solution that achieves the optimal value of the objective functional while meeting the specified Fourier transform constraints.

The existence of an extremizer function depends on the specific problem formulation. The quest for an optimizer involves employing mathematical techniques such as Fourier analysis techniques, variational methods or optimization algorithms.

In the first section, we define Paley-Wiener spaces with weight and prove that they are Hilbert spaces. In the second section, we introduce the principal extremal problem ($\mathbb{EP}$). Moving on to Section 4.3, we present an extremal problem related to the norm of an inclusion operator and analyze the characteristics of this problem in connection with the previous one to discover its properties.

In Section 4.4, we revisit the original problem with a sense of "déjà vu", providing a new perspective. Finally, in Section 4.5, we delve into determining extremizers and sharp constants in a more general setting within the theory of de Branges spaces introduced and studied in Chapter 3.

4.1 Weighted Paley-Wiener spaces

Definition 4.3 (Weighted Paley-Wiener spaces). For $\alpha > -1$ and $\delta > 0$, we define $\mathcal{H}_\alpha(d;\delta)$ as the multidimensional weighted Paley-Wiener space. This space consists of entire functions $F : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most δ such that

$$\|F\|_{\mathcal{H}_\alpha(d;\delta)} = \left(\int_{\mathbb{R}^d} |F(x)|^2 |x|^{2\alpha+2-d}\, dx\right)^{1/2} < \infty. \tag{4.1}$$

Remark 4.4. We need the condition $\alpha > -1$ to guarantee that the integral converges in a neighborhood of 0. In fact, since $F : \mathbb{C}^d \to \mathbb{C}$ is entire, it follows that F is bounded in $B_1 \overset{def}{=} \left\{x \in \mathbb{R}^d : |x| < 1\right\}$. Also, since $2\alpha + 2 - d > -d$, it follows that

$$\int_{B_1} |F(x)|^2 |x|^{2\alpha+2-d}\, dx \le C \int_{B_1} |x|^{2\alpha+2-d}\, dx < \infty.$$

On the other hand, the integral over the ball B_1 diverges if $\alpha \le -1$ and $F(0) \neq 0$.

Proposition 4.5. For $\alpha \ge \beta > -1$, there is an inclusion $\mathcal{H}_\alpha(d;\delta) \subseteq \mathcal{H}_\beta(d;\delta)$ as sets.

Proof. Let $\alpha > \beta > -1$ and $F \in \mathcal{H}_\alpha(d;\delta)$. Since $|x|^{2\alpha+2-d} \ge |x|^{2\beta+2-d}$ for $x \in B_1^c$, we have

$$\int_{B_1^c} |F(x)|^2 |x|^{2\beta+2-d}\, dx \le \int_{B_1^c} |F(x)|^2 |x|^{2\alpha+2-d}\, dx < \infty. \tag{4.2}$$

Moreover, the integral over the ball B_1 converges, as shown in Remark 4.4.

□

Proposition 4.6. Let $\alpha > -1$ and $\delta > 0$. Suppose that $F : \mathbb{C}^d \to \mathbb{C}$ satisfies the condition (4.1). Then $F \in \mathcal{H}_\alpha(d;\delta)$ if and only if $\widehat{F}$ (in distributional sense) has support in the ball $B_{\delta/2\pi}$.

Proof. Let us split the proof into two cases.
Case 1. ($\alpha \ge \frac{d}{2} - 1$).
Suppose that $F \in \mathcal{H}_\alpha(d;\delta)$. By Proposition 4.5, it follows that $F \in L^2(\mathbb{R}^d)$ and by the classical Paley-Wiener Theorem 1.14, F being entire with exponential type equal to δ implies that $\widehat{F}$ has compact support in the ball $B_{\delta/2\pi}$.

If $\widehat{F}$ has compact support in the ball $B_{\delta/2\pi}$, then by (1.11) from Lemma 1.15, it follows that F is entire. This, combined with the proof of Proposition 4.5, leads to $F \in L^2(\mathbb{R}^d)$. Therefore, according to Theorem 1.14, F is an entire function with exponential type δ.

Case 2. ($\alpha < \frac{d}{2} - 1$).
Suppose $F \in \mathcal{H}_\alpha(d;\delta)$, meaning F is an entire function of exponential type δ. We will use approximation techniques for this proof. Let $\varepsilon > 0$ and $\varphi \in C^\infty(\mathbb{R}^d)$ be a

function such that $\operatorname{supp}(\varphi) \subset B_\varepsilon$. Then the Fourier transform $\widehat{\varphi}$ is an entire function with $\tau(\widehat{\varphi}) = \varepsilon$. It is worth noting that $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$, due to the Fourier transform being an isomorphism in Schwartz space (as shown in Proposition 1.5).

Let $F_\varepsilon(x) \overset{def}{=} F(x)\widehat{\varphi}(x)$. By the definition of exponential type, we have $\tau(F_\varepsilon) \leq \delta + \varepsilon$. Additionally, since $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$, it follows that $F_\varepsilon \in L^2(\mathbb{R}^d)$ and by Theorem 1.14

$$\operatorname{supp}(\widehat{F_\varepsilon}) \subset B_{(\delta+\varepsilon)/2\pi}. \tag{4.3}$$

Note that $\widehat{F_\varepsilon} = \widehat{F} * \breve{\varphi}$, where $\breve{\varphi}(x) = \varphi(-x)$. Although $\widehat{F}$ is initially only a tempered distribution, $\widehat{F} * \breve{\varphi}$ is a function (as discussed in (1.3)), given by $F_\varepsilon(x) = \widehat{F}(\tau_x\varphi)$. The formula for $\widehat{F_\varepsilon}$ and (4.3) imply that $\operatorname{supp} \widehat{F} \subset B_{(\delta+2\varepsilon)/2\pi}$ in the distributional sense. This is because for any x with $|x| > (\delta + 2\varepsilon)/2\pi$, there exists a neighborhood of x where $\widehat{F}$ and 0 are equal as distributions. Since ε is arbitrary, we conclude that

$$\operatorname{supp}(\widehat{F}) \subset B_{\delta/2\pi}.$$

If $\widehat{F}$ (in a distributional sense) has support in the ball $B_{\delta/2\pi}$, it follows from Lemma 1.15 that F is entire. According to the Paley-Wiener-Schwartz Theorem 1.16, there exist positive constants C and N such that

$$|F(z)| \leq C(1+|z|)^N e^{\delta|\operatorname{Im}(z)|} = C(1+|z|)^N e^{\delta||\operatorname{Im}(z)||} \leq C(1+|z|)^N e^{\delta||z||}$$

and in the last equality we use the fact that $|z| = ||z||$ for z purely complex or real. In the language of lim sup, we can see that $\tau(F) \leq \delta$, and the result follows.

$\square$

One important fact about these spaces is the following.

Theorem 4.7. Let $\alpha > -1$ and $\delta > 0$. Then $\mathcal{H}_\alpha(d;\delta)$ with norm given by (4.1) is a Hilbert space.

Proof. Without loss of generality we can take $\delta = 2\pi$, because we have that $F \in \mathcal{H}_\alpha(d;\delta)$ if and only if $G(z) \overset{def}{=} F(2\pi z/\delta)$ is such that $G \in \mathcal{H}_\alpha(d;2\pi)$.

Let $\{F_n\} \subset \mathcal{H}_\alpha(d;2\pi)$ be a Cauchy sequence. We can identify the norm of this Paley-Wiener space with the norm of the weighted space $L^2(|x|^{2\alpha+2-d}\,dx, \mathbb{R}^d)$, which is a Hilbert space. In particular, there exists $F \in L^2(|x|^{2\alpha+2-d}\,dx, \mathbb{R}^d)$ such that $F_n \to F$ in this norm. We need to prove that F is a restriction to $\mathbb{R}^d$ of an entire function of exponential type 2π, and for this, we will split the proof of this theorem into two parts.
Case 1. ($\alpha \geq \frac{d}{2} - 1$).

By Proposition 4.5, it follows that $\mathcal{H}_\alpha(d;2\pi) \subseteq L^2(\mathbb{R}^d)$ as sets. According to Nazarov's Theorem (see [Jam07]), it follows from Fatou's Lemma that

$$\begin{aligned}
\int_{\mathbb{R}^d} |(F-F_n)(x)|^2\,dx &\le \liminf_{m\to\infty} \int_{\mathbb{R}^d} |(F_m-F_n)(x)|^2\,dx \\
&\le C \liminf_{m\to\infty} \left[\int_{B_1^c} |(F_m-F_n)(x)|^2\,dx + \int_{B_1^c} |\mathcal{F}(F_m-F_n)(x)|^2\,dx \right] \\
&= C \liminf_{m\to\infty} \int_{B_1^c} |(F_m-F_n)(x)|^2\,dx \\
&\le C \liminf_{m\to\infty} \int_{B_1^c} |(F_m-F_n)(x)|^2 |x|^{2\alpha+2-d}\,dx
\end{aligned}$$

and the equality above follows because every $\widehat{F_n}$ has compact support in the ball B_1 by the Paley-Wiener Theorem 1.14. Taking $n \to \infty$, we obtain that $F_n \to F$ in the $L^2(\mathbb{R}^d)$ norm and in particular $F \in L^2(\mathbb{R}^d)$. As the Fourier transform is an isometry in $L^2(\mathbb{R}^d)$, it follows from $\widehat{F_n}$ having compact support that

$$\begin{aligned}
&\int_{B_1^c} |\widehat{F}(x)|^2\,dx + \lim_{n\to\infty} \int_{B_1^c} |\mathcal{F}(F-F_n)(x)|^2\,dx \\
&\qquad = \lim_{n\to\infty} \int_{\mathbb{R}^d} |\mathcal{F}(F-F_n)(x)|^2\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} |(F-F_n)(x)|^2\,dx = 0.
\end{aligned}$$

So $\widehat{F}$ has compact support in the ball B_1. By Theorem 1.14, we conclude that F is a restriction of an entire function of exponential type equal to 2π.

Case 2. ($\alpha < \frac{d}{2} - 1$).

Here we will use approximation arguments. Let $\varepsilon > 0$ and $\varphi \in C^\infty(\mathbb{R}^d)$ be a function such that $\operatorname{supp}(\varphi) \subset B_\varepsilon$. Then $\widehat{\varphi}$ is an entire function with $\tau(\widehat{\varphi}) = \varepsilon$.

Define for each $n \in \mathbb{N}$ the functions $F_n(x) \overset{\text{def}}{=} F_{n,\varepsilon}(x)\widehat{\varphi}(x)$ and $F_\varepsilon(x) \overset{def}{=} F(x)\widehat{\varphi}(x)$. Note that, by the definition of exponential type, we have $\tau(F_n(x)\widehat{\varphi}) \le \delta+\varepsilon$. Moreover, $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ implies $F_{(n,\varepsilon)}(x) \to F_\varepsilon$ in the $L^2(\mathbb{R}^d)$ norm when $n \to \infty$, because

$$\begin{aligned}
\left\| F_{(n,\varepsilon)}(x) - F_\varepsilon \right\|_2 &\le \sup_{x\in\mathbb{R}^d} \left(|x|^{d-2-2\alpha} |\widehat{\varphi}(x)| \right) \|F_n - F\|_{L^2(|x|^{2\alpha+2-d}\,dx,\mathbb{R}^d)} \\
&= C \|F_n - F\|_{L^2(|x|^{2\alpha+2-d}\,dx,\mathbb{R}^d)}
\end{aligned}$$

Therefore, due to the first case, $F_\varepsilon \in L^2(\mathbb{R}^d)$ is an entire function of exponential type $\delta+\varepsilon$. By the proof of Proposition 4.6 (case 2), we observe that $\operatorname{supp} \widehat{F} \subset B_{\delta/2\pi}$. According to the same proposition, $F \in \mathcal{H}_\alpha(d;\delta)$, and the result follows.

□

Now that we have covered some fundamental properties of weighted Paley-Wiener spaces, we can move on to discussing extremal problems. But first, let us delve into one of the most classic problems in Fourier optimization: the one-delta problem.

Example 4.8 (One-delta problem - dimension $d = 1$). The classical one-delta problem seeks to determine the infimum value denoted by $\mathcal{A}_1$ given by

$$\mathcal{A}_1 = \inf_{f \in \mathcal{C}_1} \int_{\mathbb{R}} f(x)\,dx,$$

where the class $\mathcal{C}_1$ comprises real entire functions $f : \mathbb{C} \to \mathbb{C}$ of exponential type at most 2π that act as majorants for the delta function at the origin, satisfying $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $f(0) \geq 1$. It is known that $\mathcal{A}_1 = 1$ and is achieved by the function $f(z) = (\sin(\pi z)/\pi z)^2$. This result can be established by leveraging two fundamental tools in harmonic analysis: Plancherel's Theorem (Theorem 1.4) and the Poisson summation formula (Theorem 1.11).

Several of its variations are named after Carathéodory, Fejér, and Turán. After this problem was studied and solved, an interesting question arose about what would happen in higher dimensions. The answer to this question was provided by Holt and Vaaler, as we will see in the next example.

Example 4.9 (One-delta problem - multidimensional formulation). Find

$$\mathcal{A}_d = \inf_{F \in \mathcal{C}_d} \int_{\mathbb{R}^d} F(x)\,dx,$$

where the class $\mathcal{C}_d$ consists of real entire functions $F : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most 1 which are majorants of a delta function at the origin, i.e., $F(x) \geq 0$ for all $x \in \mathbb{R}^d$ and $F(0) \geq 1$. This question was solved by Holt and Vaaler in 1996 (see the paper [HV96]) and the proof of this result requires the use of weighted Paley-Wiener spaces and connections with the theory of de Branges spaces. We know that $\mathcal{A}_d = d2^{d-1}\pi^{d/2}\Gamma(d/2)$.

4.2 The critical extremal problem

In this section, we will introduce the central extremal problem, which is an analogous problem mentioned in the previous section with an additional monotonicity restriction.

Extremal Problem $\mathbb{EP}$ (Radial nonincreasing delta majorant): Let $\mathcal{R}^+(d; 2\delta)$ be the class of real entire functions $M : \mathbb{C}^d \to \mathbb{C}$ that satisfy the following properties:

(i) M has exponential type at most 2δ;

(ii) M is nonnegative and radial nonincreasing on $\mathbb{R}^d$;

(iii) $M(0) \geq 1$.

Find the value of

$$(\mathbb{EP})(d;\delta) \stackrel{def}{=} \inf_{M\in\mathcal{R}^+(d;2\delta)} \int_{\mathbb{R}^d} M(x)\,dx. \tag{4.4}$$

This problem is not trivial, particularly the sharp value of (4.4) when d is odd is unknown. In dimension 1 and $\delta = \pi$, with this additional restriction, the sharp constant satisfies $1.2750 < (\mathbb{EP})(1;\pi) < 1.27714$ and the proof of this fact uses the specific decomposition of the Paley-Wiener space into an orthogonal basis (for more details, see the paper [Chi+23]).

Monotone problems like these are important in analysis, not just for the theory of Fourier optimization but also for other fields like analytical number theory. A prime example of this is the application of these techniques to establish the Weighted Hilbert-Montgomery-Vaughan inequality, as detailed in the paper [CL24].

The purpose of this chapter is to prove the existence of extremizers for all dimensions and compute explicitly the values of the constant $(\mathbb{EP})(d;\delta)$ in even dimensions. To achieve this, we need to introduce another interesting extremal problem in Fourier analysis that has a direct connection with this one and the theory of Paley-Wiener spaces.

4.3 Embeddings between Paley-Wiener spaces

Proposition 4.10. For any $\alpha \geq \beta > -1$, the inclusion map $I : \mathcal{H}_\alpha(d;\delta) \to \mathcal{H}_\beta(d;\delta)$ is a bounded operator.

Proof. Let $\alpha > \beta > -1$. To prove the continuity of the inclusion operator, we will use the fact that $\mathcal{H}(d,\alpha)$ is a Hilbert space and apply the Closed Graph Theorem. Assume $F_n \to F$ in $\mathcal{H}(d,\alpha)$ and $F_n \to G$ in $\mathcal{H}(d,\beta)$. Using (4.2), we have

$$\begin{aligned}
\left\|(F-G)\chi_{B_1^c}\right\|_{\mathcal{H}(d,\beta)} &\leq \left\|(F_n-G)\chi_{B_1^c}\right\|_{\mathcal{H}(d,\beta)} + \left\|(F_n-F)\chi_{B_1^c}\right\|_{\mathcal{H}(d,\beta)} \\
&\leq \left\|(F_n-G)\chi_{B_1^c}\right\|_{\mathcal{H}(d,\beta)} + \left\|(F_n-F)\chi_{B_1^c}\right\|_{\mathcal{H}(d,\alpha)} \\
&\leq \|F_n-G\|_{\mathcal{H}(d,\beta)} + \|F_n-F\|_{\mathcal{H}(d,\alpha)}
\end{aligned}$$

and taking the limit as $n \to \infty$, we conclude that $\|(F-G)\chi_{B_1}\|_{\mathcal{H}(d,\beta)} = 0$.

Since F and G are entire functions, they are equal in B_1^c. By analytical continuation, we have $F(z) = G(z)$ for all $z \in \mathbb{C}^d$.

□

Given the previous proposition, we turn our attention to examining the operator norm associated with the embedding $I : \mathcal{H}_\alpha(d;\delta) \to \mathcal{H}_\beta(d;\delta)$.

Extremal Problem $\mathbb{EP}1$: For real parameters $\alpha \geq \beta > -1$, $\delta > 0$, and a natural number d, determine the value of

$$(\mathbb{EP}1)(\alpha, \beta; d; \delta) \stackrel{\text{def}}{=} \inf_{\substack{F \in \mathcal{H}_\alpha(d;\delta) \\ F \neq 0}} \frac{\int_{\mathbb{R}^d} |F(x)|^2 |x|^{2\alpha+2-d} \, dx}{\int_{\mathbb{R}^d} |F(x)|^2 |x|^{2\beta+2-d} \, dx}.$$

Note that the expression

$$(\mathbb{EP}1)(\alpha, \beta; d; \delta) = \left(\|I\|_{\mathcal{H}_\alpha(d;\delta) \to \mathcal{H}_\beta(d;\delta)} \right)^{-2}$$

is always greater than zero. This inequality can be interpreted as a Fourier uncertainty principle, as shown in Proposition 4.6. According to the Fourier uncertainty paradigm, the mass of F cannot be too concentrated around the origin. The $(\mathbb{EP}1)$ problem is not only interesting on its own but also helps in solving the problem discussed in the previous section.

Theorem 4.11. There exist extremizers for $\mathbb{EP}(d; \delta)$ and

$$(\mathbb{EP})(d; \delta) = \frac{\omega_{d-1}}{d} (\mathbb{EP}1)(\tfrac{d}{2}, 0; 1; \delta), \tag{4.5}$$

where $\omega_{d-1} = 2\pi^{d/2} \Gamma\left(d/2\right)^{-1}$ is the surface area of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$.

To prove this, we can first study the properties of $(\mathbb{EP}1)$ and then come back to $(\mathbb{EP})$.

Proposition 4.12. By dilation

$$(\mathbb{EP}1)(\alpha, \beta; d; \delta) = \delta^{2\beta - 2\alpha} (\mathbb{EP}1)(\alpha, \beta; d; 1). \tag{4.6}$$

Proof. Observe that $F(\cdot) \in \mathcal{H}_\alpha(d; \delta)$ if and only if $F(\cdot/\delta) \in \mathcal{H}_\alpha(d; 1)$. This change of variables proves (4.6).

□

The first nontrivial observation about the extremal problem $(\mathbb{EP}1)$ addresses the existence of extremizers in dimension $d = 1$. This is indeed the most important case because, as we will see, the solution in higher dimensions can be derived from this one-dimensional case.

Theorem 4.13 (Existence of even extremizers in dimension $d = 1$). Let $\alpha > \beta > -1$ and $\delta > 0$ be real parameters. There is an extremizer for the problem $(\mathbb{EP}1)(\alpha, \beta; 1; \delta)$ and it must be an even function.

Proof. We will divide the proof into two steps:
Step 1. **(Existence of extremizers):** By Proposition 4.12, we only need to consider the case where $\delta = 1$. For brevity, we will denote $(\mathbb{EP}1)(\alpha, \beta; 1; 1)$ as $(\mathbb{EP}1)$ in this proof. Additionally, assume $\alpha > \beta$.

Let $\{f_n\}_{n\geq 1} \subset \mathcal{H}_\alpha(1;1)$ be an extremizing sequence, normalized so that $||f_n||_{\mathcal{H}_\alpha(1;1)} = 1$. This means that

$$||f_n||_{\mathcal{H}_\beta(1;1)} \to (\mathbb{EP}1)^{-1}.$$

Because $\mathcal{H}_\alpha(1;1)$ is a Hilbert space and the sequence of norms is bounded, it follows by reflexivity (up to a subsequence) that $f_n \rightharpoonup g$ for some $g \in \mathcal{H}_\alpha(1;1)$. In particular,

$$||g||_{\mathcal{H}_\alpha(1;1)} \leq \liminf_{n\to\infty} ||f_n||_{\mathcal{H}_\beta(1;1)} = 1.$$

Since $\mathcal{H}_\alpha(1;1) = \mathcal{H}(E_\alpha)$ as sets, with norms differing by a multiplicative constant (Lemma A9), this is a reproducing kernel Hilbert space. Then, for any $w \in \mathbb{C}$, we have the pointwise convergence

$$f_n(w) = \langle f_n, K_\alpha(w,\cdot)\rangle_{\mathcal{H}(E_\alpha)} \to \langle g, K_\alpha(w,\cdot)\rangle_{\mathcal{H}(E_\alpha)} = g(w).$$

By the Cauchy-Schwarz inequality, we also get that

$$|f_n(w)| = |\langle f_n, K_\alpha(w,\cdot)\rangle_{\mathcal{H}(E_\alpha)}| \leq ||f_n||_{\mathcal{H}(E_\alpha)} ||K_\alpha(w,\cdot)||_{\mathcal{H}(E_\alpha)} = c_\alpha^{1/2} K_\alpha(w,w)^{1/2}$$

and since $K_\alpha(w,w)$ is a continuous function by Remark 3.7, we get that f_n is uniformly bounded in compact subsets of $\mathbb{C}$. In particular, we have that the convergence $f_n \to g$ is uniform on every compact subset $K \subset \mathbb{C}$. We will use this fact in the following way.

For any $0 < a < (\mathbb{EP}1)^{-1}$, there exists an $N_0 = N_0(a)$ such that for $n \geq N_0$ we have

$$\begin{aligned} a &\leq \int_{\mathbb{R}} |f_n(x)|^2 |x|^{2\beta+1}\, dx \\ &= \int_{|x|\leq R} |f_n(x)|^2 |x|^{2\beta+1}\, dx + \int_{|x|>R} |f_n(x)|^2 |x|^{2\beta+1}\, dx \\ &\leq \int_{|x|\leq R} |f_n(x)|^2 |x|^{2\beta+1}\, dx + R^{2\beta-2\alpha} \int_{|x|>R} |f_n(x)|^2 |x|^{2\alpha+1}\, dx \\ &\leq \int_{|x|\leq R} |f_n(x)|^2 |x|^{2\beta+1}\, dx + R^{2\beta-2\alpha} \end{aligned} \tag{4.7}$$

for any fixed R, where we have used that $\alpha > \beta$ and $||f_n||_{\mathcal{H}_\alpha(1;1)} = 1$. Letting $n \to \infty$ and applying the Dominated Convergence Theorem in (4.7), we get

$$a - R^{2\beta-2\alpha} \leq \int_{|x|\leq R} |g(x)|^2 |x|^{2\beta+1}\, dx \leq \int_{\mathbb{R}} |g(x)|^2 |x|^{2\beta+1}\, dx. \tag{4.8}$$

Letting $R \to \infty$ in (4.8), we arrive at

$$a \leq \int_{\mathbb{R}} |g(x)|^2 |x|^{2\beta+1}\, dx.$$

This shows that $g \neq 0$ and, since $a < (\mathbb{EP}1)^{-1}$ is arbitrary, we conclude by the limit that

$$(\mathbb{EP}1)^{-1} \leq \int_{\mathbb{R}} |g(x)|^2 |x|^{2\beta+1}\, dx. \tag{4.9}$$

Step 2. **(The extremizer is even):** Let $g \in \mathcal{H}_\alpha(1;\delta)$ be an extremizer of $(\mathbb{EP}1)(\alpha,\beta;1;\delta)$. We can express

$$g(z) = g_e(z) + g_o(z),$$

where $g_e(z) \overset{def}{=} \frac{1}{2}(g(z)+g(-z))$ and $g_o(z) \overset{def}{=} \frac{1}{2}(g(z)-g(-z))$ represent the even and odd parts of g, respectively. Due to the orthogonality between g_e and g_o observe that

$$\begin{aligned}
(\mathbb{EP}1)(\alpha,\beta;1;\delta) &= \frac{\int_{\mathbb{R}} |g(x)|^2|x|^{2\alpha+1}dx}{\int_{\mathbb{R}} |g(x)|^2|x|^{2\beta+1}dx} \\
&= \frac{\langle g_e, g_e\rangle_{\mathcal{H}_\alpha(1;\delta)} + 2\operatorname{Re}\langle g_e, g_o\rangle_{\mathcal{H}_\alpha(1;\delta)} + \langle g_o, g_o\rangle_{\mathcal{H}_\alpha(1;\delta)}}{\langle g_e, g_e\rangle_{\mathcal{H}_\beta(1;\delta)} + 2\operatorname{Re}\langle g_e, g_o\rangle_{\mathcal{H}_\beta(1;\delta)} + \langle g_o, g_o\rangle_{\mathcal{H}_\beta(1;\delta)}} \\
&= \frac{\int_{\mathbb{R}} |g_e(x)|^2|x|^{2\alpha+1}dx + \int_{\mathbb{R}} |g_o(x)|^2|x|^{2\alpha+1}dx}{\int_{\mathbb{R}} |g_e(x)|^2|x|^{2\beta+1}dx + \int_{\mathbb{R}} |g_o(x)|^2|x|^{2\beta+1}dx} \overset{def}{=} \frac{a+b}{c+d}.
\end{aligned} \tag{4.10}$$

Assume g_o is not identically zero. Since g_e and g_o are both in $\mathcal{H}_\alpha(1;\delta)$, by (4.10), we observe that g_o must also be an extremizer.

In fact, if $g_e = 0$, this result is obvious. If $g_e \neq 0$ and $b/d > (\mathbb{EP}1)(\alpha,\beta;1;\delta)$, then $a/c < (\mathbb{EP}1)(\alpha,\beta;1;\delta)$, and this is a contradiction.

We may write $g_o(z) = zh(z)$ with $h \in \mathcal{H}_{\alpha+1}(1;\delta) \subset \mathcal{H}_\alpha(1;\delta)$. Then, by the setup of our problem,

$$\frac{\int_{\mathbb{R}} |h(x)|^2|x|^{2\alpha+3}dx}{\int_{\mathbb{R}} |h(x)|^2|x|^{2\beta+3}dx} = \frac{\int_{\mathbb{R}} |g_o(x)|^2|x|^{2\alpha+1}dx}{\int_{\mathbb{R}} |g_o(x)|^2|x|^{2\beta+1}dx} = (\mathbb{EP}1)(\alpha,\beta;1;\delta) \leq \frac{\int_{\mathbb{R}} |h(x)|^2|x|^{2\alpha+1}dx}{\int_{\mathbb{R}} |h(x)|^2|x|^{2\beta+1}dx}. \tag{4.11}$$

Let $d\mu(x) = |h(x)|^2|x|^{2\beta+1}\, dx$ and assume, without loss of generality, that $d\mu$ is normalized such that $\int_{\mathbb{R}} d\mu(x) = 1$. The inequality in (4.11) can be expressed as

$$\int_{\mathbb{R}} |x|^{2\gamma+2}\, d\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2\gamma}\, d\mu(x)\right)\left(\int_{\mathbb{R}} |x|^2\, d\mu(x)\right), \tag{4.12}$$

with $\gamma = \alpha - \beta > 0$. On the other hand, by Hölder's inequality we have

$$\int_{\mathbb{R}} |x|^{2\gamma}\, d\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2\gamma+2}\, d\mu(x)\right)^{\frac{2\gamma}{2\gamma+2}} \left(\int_{\mathbb{R}} d\mu(x)\right)^{\frac{2}{2\gamma+2}} \tag{4.13}$$

and

$$\int_{\mathbb{R}} |x|^{2}\, d\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2\gamma+2}\, d\mu(x)\right)^{\frac{2}{2\gamma+2}} \left(\int_{\mathbb{R}} d\mu(x)\right)^{\frac{2\gamma}{2\gamma+2}}. \tag{4.14}$$

Multiplying (4.13) by (4.14) and using the fact that $d\mu$ is normalized, we get

$$\left(\int_{\mathbb{R}} |x|^{2\gamma}\, d\mu(x)\right)\left(\int_{\mathbb{R}} |x|^2\, d\mu(x)\right) \leq \int_{\mathbb{R}} |x|^{2\gamma+2}\, d\mu(x). \tag{4.15}$$

By (4.12) and (4.15), the inequalities (4.13) and (4.14) become equalities. Using the case of equality in Hölder's inequality, this implies that $|x|^{2\gamma+2}$ is constant in the support of $d\mu$, which leads to a contradiction. Therefore, we can conclude that our original extremizer must be an even function.

$\square$

Theorem 4.14 (Dimension shifts). We have

$$(\mathbb{EP}1)(\alpha,\beta;d;\delta) = (\mathbb{EP}1)(\alpha,\beta;1;\delta). \tag{4.16}$$

To prove this theorem, we introduce Extremal Problem $\mathbb{EP}2$, which is based on suitable radial symmetrization mechanisms. Furthermore, it is an extremely interesting auxiliary problem in itself.

Extremal Problem $\mathbb{EP}$2: Let $\alpha \geq \beta > -1$ and $\delta > 0$ be real parameters, and $d \in \mathbb{N}$. Define $\mathcal{W}_\alpha^+(d;2\delta)$ as the set of real entire functions $M : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most 2δ that are nonnegative on $\mathbb{R}^d$ and such that

$$\int_{\mathbb{R}^d} M(x)|x|^{2\alpha+2-d}\,dx < \infty.$$

Find

$$(\mathbb{EP}2)(\alpha,\beta;d;\delta) \overset{def}{=} \inf_{M\in\mathcal{W}_\alpha^+(d;2\delta)} \frac{\int_{\mathbb{R}^d} M(x)|x|^{2\alpha+2-d}\,dx}{\int_{\mathbb{R}^d} M(x)|x|^{2\beta+2-d}\,dx}. \tag{4.17}$$

Proof of Theorem 4.14. The idea here is to prove that ($\mathbb{EP}$2) satisfies the dimension equality (4.16) and pass this property to ($\mathbb{EP}$1) via inequalities. First, observe that if $M \in \mathcal{W}_\alpha^+(d;2\delta)$, by Theorem A7 (Appendix A.2), we have that $\widetilde{M} \in \mathcal{W}_\alpha^+(d;2\delta)$ and

$$\int_{\mathbb{R}^d} M(x)|x|^{2\nu+2-d}\,dx = \int_{\mathbb{R}^d} \widetilde{M}(x)|x|^{2\nu+2-d}\,dx$$

for any $\nu > -1$. In particular, we can restrict our search for the infimum in (4.17) to functions $M \in \mathcal{W}_\alpha^+(d;2\delta)$ that are radial on $\mathbb{R}^d$.

According to Corollary A8 (Appendix A.2), such M corresponds to the lift $\mathcal{L}_d(f)$ of an even entire function $f \in \mathcal{W}_\alpha^+(1;2\delta)$ and vice versa. Utilizing the equality (A5) from Theorem A5 (Appendix A.2), we have

$$\frac{\int_{\mathbb{R}^d} M(x)|x|^{2\alpha+2-d}\,dx}{\int_{\mathbb{R}^d} M(x)|x|^{2\beta+2-d}\,dx} = \frac{(\omega_{d-1}/2)^{-1}\int_{\mathbb{R}^d} \mathcal{L}_d(f)(x)|x|^{2\alpha+2-d}\,dx}{(\omega_{d-1}/2)^{-1}\int_{\mathbb{R}^d} \mathcal{L}_d(f)(x)|x|^{2\beta+2-d}\,dx} = \frac{\int_{\mathbb{R}} f(x)|x|^{2\alpha+1}\,dx}{\int_{\mathbb{R}} f(x)|x|^{2\beta+1}\,dx}$$

and we conclude that

$$(\mathbb{EP}2)(\alpha,\beta;d;\delta) = (\mathbb{EP}2)(\alpha,\beta;1;\delta). \tag{4.18}$$

Now, if $F \in \mathcal{H}_\alpha(d;\delta)$ then $M(z) = F(z)F^*(z) \in \mathcal{W}_\alpha^+(d;2\delta)$. Hence

$$(\mathbb{EP}1)(\alpha,\beta;d;\delta) \geq (\mathbb{EP}2)(\alpha,\beta;d;\delta). \tag{4.19}$$

By Krein's decomposition (see Lemmas A9 and A10), every $f \in \mathcal{W}_\alpha(1; 2\delta)$ can be written as $f(z) = g(z)g^*(z)$ with $g \in \mathcal{H}_\alpha(1; \delta)$ and conversely. This implies that

$$(\mathbb{EP}2)(\alpha, \beta; 1; \delta) = (\mathbb{EP}1)(\alpha, \beta; 1; \delta). \tag{4.20}$$

By Theorem 4.13, there exists an even extremizer $g \in \mathcal{H}_\alpha(1; \delta)$ for $(\mathbb{EP}1)(\alpha, \beta; 1; \delta)$. According to Theorem A5 (see Appendix A.2), we have $\mathcal{L}_d(g) \in \mathcal{H}_\alpha(d; \delta)$ and we find that

$$(\mathbb{EP}1)(\alpha, \beta; 1; \delta) = \frac{\int_{\mathbb{R}} |g(x)|^2 |x|^{2\alpha+1}\, dx}{\int_{\mathbb{R}} |g(x)|^2 |x|^{2\beta+1}\, dx} = \frac{\int_{\mathbb{R}^d} |\mathcal{L}_d(g)(x)|^2 |x|^{2\alpha+2-d}\, dx}{\int_{\mathbb{R}^d} |\mathcal{L}_d(g)(x)|^2 |x|^{2\beta+2-d}\, dx} \geq (\mathbb{EP}2)(\alpha, \beta; d; \delta). \tag{4.21}$$

Combining (4.19), (4.18), (4.20), and (4.21), we arrive at

$$\begin{aligned}&(\mathbb{EP}1)(\alpha, \beta; d; \delta)\\ &\quad \geq (\mathbb{EP}2)(\alpha, \beta; d; \delta) = (\mathbb{EP}2)(\alpha, \beta; 1; \delta) = (\mathbb{EP}1)(\alpha, \beta; 1; \delta) \geq (\mathbb{EP}1)(\alpha, \beta; d; \delta),\end{aligned}$$

hence we must have

$$(\mathbb{EP}1)(\alpha, \beta; d; \delta) = (\mathbb{EP}2)(\alpha, \beta; d; \delta) = (\mathbb{EP}2)(\alpha, \beta; 1; \delta) = (\mathbb{EP}1)(\alpha, \beta; 1; \delta). \tag{4.22}$$

□

Theorem 4.15 (Radial extremizers in higher dimensions). Let $\alpha > \beta > -1$ and $\delta > 0$ be real parameters. There exists a radial extremizer for $(\mathbb{EP}1)(\alpha, \beta; d; \delta)$.

Proof. Let g be an extremizer for $(\mathbb{EP}1)(\alpha, \beta; 1; \delta)$. By Theorem 4.13, g is an even function. From Theorems 4.14 and A5, one can plainly verify that the lift $\mathcal{L}_d(g)$ is a radial extremizer for $(\mathbb{EP}1)(\alpha, \beta; d; \delta)$. □

Remark 4.16. Based on the proof of Theorem 4.14 and equation (4.22), we can deduce the existence of extremizers for $(\mathbb{EP}2)$.

The idea now is to compute the sharp value of $(\mathbb{EP}1)$ for the cases where $\alpha = \beta + k$ for $k \in \mathbb{N}$. The solution is given in terms of the smallest positive solution to a specific determinant equation involving Bessel functions. Let

$$0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \dots$$

denote the sequence of positive zeros of the Bessel function J_ν (see Appendix A.1), and define the meromorphic function[1]

$$C_\nu(z) \stackrel{def}{=} \frac{B_\nu(z)}{A_\nu(z)}.$$

[1] See Appendix A.1, Definition A3.

Theorem 4.17 (Sharp constants for $(\mathbb{EP}1)$**).** For $\beta > -1$ and $k \in \mathbb{N}$, let λ_0 be defined as $\lambda_0 \stackrel{\text{def}}{=} \left((\mathbb{EP}1)(\beta+k, \beta; 1; 1)\right)^{1/2k}$.

(i) If $k = 1$, we have $\lambda_0 = j_{\beta,1}$.

(ii) If $k \geq 2$, set $\ell \stackrel{\text{def}}{=} \lfloor k/2 \rfloor$. Then λ_0 is the smallest positive solution of the equation

$$A_\beta(\lambda) \det \mathcal{V}_\beta(\lambda) = 0$$

where $\mathcal{V}_\beta(\lambda)$ is the $\ell \times \ell$ matrix with entries

$$\left(\mathcal{V}_\beta(\lambda)\right)_{mj} = \sum_{r=0}^{k-1} \omega^{r(4\ell-2m-2j+3)} C_\beta\left(\omega^r \lambda\right),$$

for $1 \leq m, j \leq \ell$ and $\omega \stackrel{\text{def}}{=} e^{\pi i/k}$.

Moreover, the extremizers for $(\mathbb{EP}1)(\beta+k, \beta; 1; 1)$ are, when $k = 1$, complex multiples of

$$f(z) = \frac{A_\beta(z)}{(z^2 - j_{\beta,1}^2)},$$

and, when $k \geq 2$, have the form

$$f(z) = \sum_{n=1}^{\infty} a_n j_{\beta,n} \frac{A_\beta(z)}{(z^2 - j_{\beta,n}^2)},$$

where $(a_1, a_2, \ldots, a_\ell)$ belongs to the kernel of a certain $\ell \times \ell$ matrix (here $\ell = \lfloor k/2 \rfloor$), and each a_n, for $n > \ell$, is given in terms of $a_1, \ldots, a_\ell$.

The approach to prove this theorem involves connecting the Extremal Problem $(\mathbb{EP}1)$ with the powerful theory of de Branges spaces of entire functions. In fact, Theorem 4.17 turns out to be a special case of the much more general Theorems 4.20 and 4.21, which will be proved in Section 4.5. These last two theorems address an extremal problem related to the operator of multiplication by z^k in a de Branges space. Refer to Remark 4.22 for more details.

4.4 Déjà vu: revisiting the original problem

Remember the original problem $(\mathbb{EP})(d; \delta)$ from Section 4.2. Now that we have the necessary tools, we can prove the Theorem 4.11.

Proof of Theorem 4.11. Let $M \in \mathcal{R}^+(d; 2\delta)$ and assume that $\int_{\mathbb{R}^d} M(x)\, dx < \infty$. By Corollary A8, we know that M is the lift $\mathcal{L}_d(f)$ of a real entire function $f : \mathbb{C} \to \mathbb{C}$ of exponential type at most 2δ that is also radial (even) and nonincreasing with

$$\int_{\mathbb{R}^d} M(x)\, dx = \tfrac{1}{2}\omega_{d-1} \int_{\mathbb{R}} f(x)|x|^{d-1}\, dx. \tag{4.23}$$

The fact that $M(0) \geq 1$ implies that $f(0) \geq 1$. We can assume that $f(0) = 1$. Let $g = f'$. Since f has exponential type at most 2δ and belongs to $L^1(\mathbb{R})$ by (4.23), by a classical result of Plancherel and Pólya [PP36], so does g. We can then write

$$f(x) = \int_{-\infty}^{x} g(u)\, du \tag{4.24}$$

Using (4.24) and Fubini's theorem, we arrive at

$$\int_{\mathbb{R}} f(x)|x|^{d-1}\, dx = \frac{1}{d}\int_{\mathbb{R}} |g(u)||u|^d\, du. \tag{4.25}$$

Note that $H(z) = -zg(z)$ is a function of exponential type at most 2δ, nonnegative on $\mathbb{R}$ and belongs to $L^1(\mathbb{R})$. Using Krein's decomposition from Lemma A10, we can express $H(z) = z^2 h(z)h^*(z)$, where h has exponential type at most δ. This implies that

$$g(z) = -zh(z)h^*(z).$$

Since g is odd, the condition $f(0) = 1$ implies $\|g\|_1 = 2$. Therefore, by definition of h

$$\|h\|_{\mathcal{H}_0(1;\delta)} = \int_{\mathbb{R}} |h(u)|^2|u|\, du = \int_{\mathbb{R}} |g(u)|\, du = 2. \tag{4.26}$$

In particular, $h \in \mathcal{H}_0(1;\delta)$. The quantity we want to minimize now is by[2] (4.23) and (4.25)

$$\frac{\omega_{d-1}}{2d}\int_{\mathbb{R}} |h(u)|^2|u|^{d+1}\, du = \frac{\omega_{d-1}}{2d}\|h\|_{\mathcal{H}_{d/2}(1;\delta)} = \frac{\omega_{d-1}}{d}\left(\frac{\|h\|_{\mathcal{H}_{d/2}(1;\delta)}}{\|h\|_{\mathcal{H}_0(1;\delta)}}\right)$$

and we can conclude that

$$(\mathbb{EP})(d;\delta) \geq \frac{\omega_{d-1}}{d}(\mathbb{EP}1)\left(\tfrac{d}{2}, 0; 1; \delta\right).$$

Conversely, by Theorem 4.13, the extremal problem $(\mathbb{EP}1)\left(\tfrac{d}{2}, 0; 1; \delta\right)$ has an extremizer $h \in \mathcal{H}_{d/2}(1;\delta)$ that is even. Using the normalization (4.26), we can reverse the steps by defining g as before, then f as in (4.24), and finally M as $\mathcal{L}_d(f)$. So M belongs to $\mathcal{R}^+(d; 2\delta)$ and by Theorem A5,

$$(\mathbb{EP})(d;\delta) \leq \frac{\omega_{d-1}}{d}(\mathbb{EP}1)\left(\tfrac{d}{2}, 0; 1; \delta\right).$$

We therefore conclude (4.5)

$$(\mathbb{EP})(d;\delta) = \frac{\omega_{d-1}}{d}(\mathbb{EP}1)\left(\tfrac{d}{2}, 0; 1; \delta\right),$$

and that extremizers exist for $(\mathbb{EP})(d;\delta)$ by Theorem 4.15.

□

[2] Remember that $h \in \mathcal{H}_{d/2}(1;\delta)$ by Proposition 4.5.

From Theorems 4.11 and 4.17, we obtain the exact solution to the extremal problem ($\mathbb{EP}$) when the dimension d is even. The characterization of the extremizers is also obtained through Theorem 4.17 and radial symmetrization considerations from Theorem 4.15. Below is a table with the initial values for $\delta = 1$:

	$d=2$	$d=4$	$d=6$	$d=8$	$d=10$	$d=12$	$d=14$
$((\mathbb{EP})(d;1))^{1/d}$	$4.26\ldots$	$4.76\ldots$	$5.23\ldots$	$5.66\ldots$	$6.07\ldots$	$6.45\ldots$	$6.81\ldots$

4.5 Sharp constants: settings in de Branges spaces

Let $E(z) = A(z) - iB(z)$ be a Hermite-Biehler function, with A and B real entire functions, and $\mathcal{H}(E)$ be the associated de Branges space.

($\mathcal{C}_1$) E has no real zeros.

($\mathcal{C}_2$) The function $z \mapsto E(iz)$ is real entire (or, equivalently, A is even and B is odd).

($\mathcal{C}_3$) $A \notin \mathcal{H}(E)$.

Extremal Problem $\mathbb{EP}$3 (de Branges Spaces setting): Let $k \in \mathbb{N}$ and let E be a Hermite-Biehler function satisfying conditions ($\mathcal{C}_1$), ($\mathcal{C}_2$), and ($\mathcal{C}_3$). Suppose that A has at least $k+1$ zeros. Determine

$$(\mathbb{EP}3)(E;k) \stackrel{\text{def}}{=} \inf_{0 \neq f \in \mathcal{H}(E)} \frac{\|z^k f\|^2_{\mathcal{H}(E)}}{\|f\|^2_{\mathcal{H}(E)}}. \tag{4.27}$$

Remember that Lemma 3.14 implies that all the zeros of A and B are real. Since E has no real zeros, the same lemma implies that the sets of zeros of A and B are disjoint (in particular $A(0) \neq 0$) and these zeros are all simple. Condition ($\mathcal{C}_3$) is generic and allows for the use of the interpolation formulas in Corollary 3.21. The minor technical assumption that A has at least $k+1$ zeros guarantees that the subspace

$$\mathcal{X}_k(E) \stackrel{\text{def}}{=} \{f \in \mathcal{H}(E) \,:\, z^k f \in \mathcal{H}(E)\}$$

is nonempty. In fact, if $\xi_1, \ldots, \xi_{k+1}$ are zeros of A, then for each $j \in \{1, \ldots, k+1\}$ the functions $K(\xi_j, z) = A(z)/\pi(z - \xi_j)$ belong to $\mathcal{H}(E)$. A proper linear combination of these functions will decay as $|A(x)/x^{k+1}|$ as $|x| \to \infty$, and therefore, it will be in $\mathcal{X}_k(E)$.

We will first establish a few qualitative properties about the extremal problem ($\mathbb{EP}$3).

Proposition 4.18. The following statements are true:

(i) $(\mathbb{EP}3)(E;k) > 0$.

(ii) There exist extremizers for $(\mathbb{EP}3)(E;k)$.

(iii) Any extremizer for $(\mathbb{EP}3)(E;k)$ must be an even function.

Proof. (i) Let $f \in \mathcal{X}_k(E)$ be such that $||f||^2_{\mathcal{H}(E)} = 1$. By Theorem 3.15 we have

$$|f(z)|^2 \leq ||f||^2_{\mathcal{H}(E)} K(z,z) \tag{4.28}$$

for all $z \in \mathbb{C}$. Since $K(x,x)$ is a continuous function, as stated in Remark 3.7, we can set $M \stackrel{def}{=} \max\{K(x,x) : -1 \leq x \leq 1\}$. Define $d\mu(x) \stackrel{def}{=} |E(x)|^{-2}dx$ and choose $\eta \leq 1$ such that $M \int_{[-\eta,\eta]} d\mu(x) \leq \frac{1}{2}$ (this η exists because $1/E(x)$ is continuous for real x). By (4.28)

$$\int_{[-\eta,\eta]} |f(x)|^2 d\mu(x) \leq \int_{[-\eta,\eta]} K(x,x)d\mu(x) \leq M \int_{[-\eta,\eta]} d\mu(x) \leq \frac{1}{2}$$

and we obtain a positive lower bound that does not depend on f but only on K and E

$$||z^k f||^2_{\mathcal{H}(E)} \geq \int_{[-\eta,\eta]^c} |f(x)|^2 |x|^{2k} d\mu(x) \geq \eta^{2k} \int_{[-\eta,\eta]^c} |f(x)|^2 d\mu(x) \geq \frac{\eta^{2k}}{2} > 0.$$

(ii) Let $\{f_n\}_{n\geq 1} \subset \mathcal{X}_k(E)$ be an extremizing sequence, normalized so that $||z^k f_n||_{\mathcal{H}(E)} = 1$. This normalization implies that $||f_n||^2_{\mathcal{H}(E)} \to (\mathbb{EP}3)(E;k)^{-1}$. By the reflexivity of the space, we can extract a subsequence such that $f_n \rightharpoonup g$ for some $g \in \mathcal{H}(E)$. The reproducing kernel identity ensures that $f_n \to g$ pointwise everywhere, and Fatou's lemma implies $||z^k g||_{\mathcal{H}(E)} \leq 1$. Thus, $g \in \mathcal{X}_k(E)$. Additionally, from (4.28), we observe that f_n is uniformly bounded in compact subsets of $\mathbb{C}$. The proof that $||g||^2_{\mathcal{H}(E)} \geq (\mathbb{EP}3)(E;k)^{-1}$ (and hence g is the desired extremizer) follows a similar argument as in (4.7), (4.8), and (4.9).

(iii) The proof follows the same reasoning as the proof of step 2 of Theorem 4.13. Note that in our case, $x \mapsto |E(x)|^{-2}$ is an even function. □

The last proposition implies that any extremizer for $(\mathbb{EP}3)(E;k)$ in (4.27) must be an even function. If f is an extremizer, we can write $f(z) = g(z) - ih(z)$, where g and h are real entire functions. This means that $g = (f + f^*)/2$ and $h = i(f - f^*)/2$. For $x \in \mathbb{R}$, we have $|f(x)|^2 = |g(x)|^2 + |h(x)|^2$. Similar to (4.10), we can see that g and h must also be extremizers (unless they are identically zero). The proof below will show that the set of real entire extremizers forms a finite-dimensional vector space over $\mathbb{R}$. Hence, the full space of extremizers is the span over $\mathbb{C}$ of the real entire extremizers.

Using the fact that A is even by condition $(\mathcal{C}_2)$, we can group the zeros $+\xi$ and $-\xi$ in the interpolation formula (3.24) of Corollary 3.21. If $f \in \mathcal{H}(E)$ is an even function, we get

$$f(z) = \sum_{n=1}^{\infty} \frac{2\xi_n f(\xi_n)}{A'(\xi_n)} \frac{A(z)}{(z^2 - \xi_n^2)} \tag{4.29}$$

and if $f \in \mathcal{H}(E)$ is odd, we obtain

$$f(z) = \sum_{n=1}^{\infty} \frac{2f(\xi_n)}{A'(\xi_n)} \frac{zA(z)}{(z^2 - \xi_n^2)}. \tag{4.30}$$

Both representations in (4.29) and (4.30) are uniformly convergent on compact subsets of the complex plane, as proved in Theorem 3.20.

Proposition 4.19. If $f \in \mathcal{X}_k(E)$, then $g(z) \stackrel{\text{def}}{=} z^k f(z) \in \mathcal{H}(E)$, and we have the following constraints

$$g(0) = g'(0) = \ldots = g^{(k-1)}(0) = 0. \tag{4.31}$$

Reciprocally, if $g \in \mathcal{H}(E)$ satisfies (4.31), then $g(z) = z^k f(z)$ with $f \in \mathcal{X}_k(E)$.

Proof. The first part is obvious because f is entire, so 0 is a zero of order greater than or equal to k of g. For the other direction, we use (4.31) and the power series of g to see that 0 is a zero of order greater than or equal to k. The result follows by induction in n for $1 \leq n \leq k$ and by Proposition 3.12 because E has no real zeros. $\square$

Theorem 4.20 (Sharp constants for (𝔼ℙ3)). Let $k \in \mathbb{N}$ and $\lambda_0 \stackrel{\text{def}}{=} ((\mathbb{EP}3)(E;k))^{1/2k}$.

(i) If $k = 1$, we have $\lambda_0 = \xi_1$.

(ii) If $k \geq 2$, set $\ell \stackrel{\text{def}}{=} \lfloor k/2 \rfloor$. Then λ_0 is the smallest positive solution of the equation

$$A_\beta(\lambda) \det \mathcal{V}(\lambda) = 0$$

where $\mathcal{V}(\lambda)$ is the $\ell \times \ell$ matrix with entries

$$(\mathcal{V}(\lambda))_{mj} = \sum_{r=0}^{k-1} \omega^{r(4\ell-2m-2j+3)} C(\omega^r \lambda)$$

for $1 \leq m, j \leq \ell$ and $\omega \stackrel{\text{def}}{=} e^{\pi i/k}$.

We are also able to classify the extremizers for $(\mathbb{EP}3)(E;k)$. In what follows, let

$$c_n \stackrel{\text{def}}{=} -\frac{A'(\xi_n)}{B(\xi_n)} \tag{4.32}$$

From (3.12), note that $c_n > 0$. Also, for $\ell = \lfloor k/2 \rfloor$ (when $k \geq 2$), let $\mathcal{T}$ be the $\ell \times \ell$ Vandermonde matrix with entries

$$\mathcal{T}_{mj} = \xi_m^{2\ell-2j+1} \quad \text{for} \quad 1 \leq m, j \leq \ell, \tag{4.33}$$

with $\mathcal{T}^{-1}$ denoting its inverse, and let $Q(\lambda)$ be the $\ell \times \ell$ matrix with entries

$$Q(\lambda)_{mj} = \left\{ \frac{(\mathcal{V}(\lambda))_{mj}}{2k\lambda^{2k-4\ell+2m+2j-3}} \quad \text{if} \quad 1 \leq m, j \leq \ell. \right.$$

Theorem 4.21 (Classification of extremizers). Let $k \in \mathbb{N}$ and $\lambda_0 = ((\mathbb{EP}3)(E;k))^{1/2k}$.

(i) If $k = 1$, the extremizers for $(\mathbb{EP}3)(E;k)$ are spanned over $\mathbb{C}$ by the real entire function

$$f(z) = \frac{A(z)}{(z^2 - \xi_1^2)};$$

(ii) If $k \geq 2$, set $\ell \stackrel{def}{=} \lfloor k/2 \rfloor$. The extremizers for $(\mathbb{EP}3)(E;k)$ are spanned over $\mathbb{C}$ by the real entire functions

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{\xi_n A(z)}{(z^2 - \xi_n^2)}$$

where

$$a_n = \frac{\sum_{i=1}^{\ell} c_i a_i \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right)}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)} \quad \text{for } n > \ell,$$

and $(a_1, a_2, \ldots, a_\ell) \in \mathbb{R}^d \setminus \{0\}$ belongs to ker $\mathcal{W}(\lambda_0)$, where $\mathcal{W}(\lambda_0)$ is the $\ell \times \ell$ matrix with entries

$$(\mathcal{W}(\lambda_0))_{ij} = c_i(\xi_i^{2k} - \lambda_0^{2k}) \left((\mathcal{T}^{-1})^t \mathcal{Q}(\lambda_0) \right)_{ij} \quad \text{if} \quad 1 \leq m, j \leq \ell.$$

Remark 4.22. Note that Theorem 4.17 is a specialization of Theorem 4.20 in the case where $E(z) = E_\beta(z) = A_\beta(z) - iB_\beta(z)$, with A_β and B_β introduced in Appendix A3. In fact, if $\alpha = \beta + k$, note that $f \in \mathcal{H}_\alpha(1;1)$ if and only if $f \in \mathcal{X}_k(E_\beta)$, and from Lemma A9, we plainly have

$$(\mathbb{EP}1)(\beta + k, \beta; 1; 1) = (\mathbb{EP}3)(E_\beta; k).$$

The extremizers in Theorem 4.17 can be obtained from Theorem 4.21 with $E = E_\beta$, and hence $\xi_n = j_{\beta,n}$. In this case, due to (A4), we have that $c_n = 1$ for all $n \geq 1$.

Proof of Theorems 4.20 and 4.21. Let us split the proof into two parts.
Case 1. Let $k = 2\ell$ with $\ell \in \mathbb{N}$. Let $f \in \mathcal{X}_k(E)$ be an even and real entire function. In this case, since $g(z) \stackrel{def}{=} z^k f(z)$ is even, half of the conditions in (4.31) are already taken care of, and we only need to look for

$$g(0) = g^{(2)}(0) = \ldots = g^{2(\ell-1)}(0) = 0. \tag{4.34}$$

From the representation (4.29), we have

$$g(z) = \sum_{n=1}^{\infty} 2 \frac{\xi_n^{k+1} f(\xi_n)}{A'(\xi_n)} \frac{A(z)}{(z^2 - \xi_n^2)}. \tag{4.35}$$

Then $g(0) = 0$ if and only if we have

$$\sum_{n=1}^{\infty} \frac{\xi_n^{k-1} f(\xi_n)}{A'(\xi_n)} = 0. \tag{4.36}$$

Since the series in (4.35) converges uniformly, we can proceed inductively by differentiating term by term and plugging in $z = 0$ (the initial case being (4.36)). This leads us to the conclusion that (4.34) is equivalent to

$$\sum_{n=1}^{\infty} \frac{\xi_n^{2\ell-2j+1} f(\xi_n)}{A'(\xi_n)} = 0 \quad \text{for all} \quad j \in \{1, 2, \dots, \ell\}. \tag{4.37}$$

Let $a_n \stackrel{\text{def}}{=} f(\xi_n)/A'(\xi_n)$. Note that a_n is a real number since we are assuming that f is real entire. Using the definition of the positive constant c_n in (4.32) and the interpolation formulas (3.24) - (3.25) of Corollary 3.21, we obtain from f being even that

$$\|f\|^2_{\mathcal{H}(E)} = 2\pi \sum_{n=1}^{\infty} c_n a_n^2 \quad \text{and} \quad \|z^k f\|^2_{\mathcal{H}(E)} = 2\pi \sum_{n=1}^{\infty} c_n a_n^2 \xi_n^{2k}. \tag{4.38}$$

Letting $\lambda_0 \stackrel{\text{def}}{=} ((\mathbb{EP}3)(E;k))^{1/2k}$, an equivalent formulation of ($\mathbb{EP}$3) is given by (4.38):

$$\lambda_0^{2k} = \inf_{\{a_n\} \in \mathcal{A}} \frac{\sum_{n=1}^{\infty} c_n a_n^2 \xi_n^{2k}}{\sum_{n=1}^{\infty} c_n a_n^2}, \tag{4.39}$$

where the infimum is taken over the class $\mathcal{A}$ of real-valued sequences $\{a_n\}_{n\geq 1}$, nonidentically zero, such that[3]

(i) $\sum_{n=1}^{\infty} c_n a_n^2 \xi_n^{2k} < \infty$;

(ii) the sequence verifies the ℓ conditions given by (4.37), i.e.,

$$\sum_{n=1}^{\infty} a_n \xi_n^{2\ell-2j+1} = 0 \quad \text{for all} \quad j \in \{1, 2, \dots, \ell\}. \tag{4.40}$$

Recall the definition of the $\ell \times \ell$ Vandermonde matrix $\mathcal{T}$ in (4.33). This system of equations (4.40) states that we can express the variables $a_1, a_2, \dots, a_\ell$ in terms of the variables $\{a_n\}_{n>\ell}$. In fact, defining for each $j \in \{1, 2, \dots, \ell\}$

$$S_j \stackrel{\text{def}}{=} \sum_{n>\ell} a_n \xi_n^{2\ell-2j+1}, \tag{4.41}$$

we have by (4.40) that

$$(a_1, \dots, a_\ell)\mathcal{T} = (-S_1, -S_2, \dots, -S_\ell). \tag{4.42}$$

[3] The function $K(0,z) = A(0)B(z)/(\pi z)$ belongs to $\mathcal{H}(E)$, and by (3.23) we have $\sum_{n=1}^{\infty} 1/\left(c_n \xi_n^2\right) < \infty$. From this and the condition $\sum_{n=1}^{\infty} c_n a_n^2 \xi_n^{2k} < \infty$, we have an alternative way to see that (4.40) is absolutely summable by applying the Cauchy-Schwarz inequality.

and using (4.41) we can deduce that for each $i \in \{1, 2, \dots, \ell\}$ the following holds

$$a_i = -\sum_{r=1}^{\ell} S_r (\mathcal{T}^{-1})_{ri} = -\sum_{r=1}^{\ell} \left(\sum_{n>\ell} a_n \xi_n^{2\ell-2j+1} \right) (\mathcal{T}^{-1})_{ri}. \tag{4.43}$$

An easy choice for $\{a_n\}_{n>\ell}$ is $a_{\ell+1} = 1$ and $a_n = 0$ for $n > \ell + 1$. Then, $\{a_i\}_{i=1}^{\ell}$ are given by (4.43) and they are not all zero since $\mathcal{T}$ is invertible. Therefore, using the test sequence, we find that

$$\lambda_0^{2k} \le \frac{\sum_{n=1}^{\ell+1} c_n a_n^2 \xi_n^{2k}}{\sum_{n=1}^{\ell+1} c_n a_n^2} \le \frac{c_{\ell+1}\xi_{\ell+1}^{2k} + \sum_{n=1}^{\ell} c_n a_n^2 \xi_n^{2k}}{c_{\ell+1}} < \frac{c_{\ell+1}\xi_{\ell+1}^{2k}}{c_{\ell+1}} = \xi_{\ell+1}^{2k}$$

and hence $\lambda_0 < \xi_{\ell+1}$. From (4.43), we can consider

$$F\left(\{a_n\}_{n>\ell}\right) \stackrel{\text{def}}{=} \sum_{i=1}^{\ell} c_i \left(\sum_{r=1}^{\ell} \left(\sum_{n>\ell} a_n \xi_n^{2\ell-2r+1} \right) (\mathcal{T}^{-1})_{ri} \right)^2 \xi_i^{2k} + \sum_{n>\ell} c_n a_n^2 \xi_n^{2k},$$

$$G\left(\{a_n\}_{n>\ell}\right) \stackrel{\text{def}}{=} \sum_{i=1}^{\ell} c_i \left(\sum_{r=1}^{\ell} \left(\sum_{n>\ell} a_n \xi_n^{2\ell-2r+1} \right) (\mathcal{T}^{-1})_{ri} \right)^2 + \sum_{n>\ell} c_n a_n^2,$$

and now the problem (4.39) is to minimize $F\left(\{a_n\}_{n>\ell}\right) / G\left(\{a_n\}_{n>\ell}\right)$ over all the class $\mathcal{A}_1$ of real-valued sequences $\{a_n\}_{n>\ell}$, nonidentically zero, that verify the condition

$$\sum_{n>\ell} c_n a_n^2 \xi_n^{2k} < \infty.$$

From Proposition 4.18, we know that there exists an extremal sequence denoted by $\{a_n^*\}_{n>\ell}$. This means that

$$\lambda_0^{2k} = \frac{F\left(\{a_n^*\}_{n>\ell}\right)}{G\left(\{a_n^*\}_{n>\ell}\right)}, \tag{4.44}$$

and if we perturb this sequence by considering $\{a_n^* + \varepsilon b_n\}_{n>\ell}$, where $\{b_n\}_{n>\ell} \in \mathcal{A}_1$ and ε is small, we find that the function

$$\varphi(\varepsilon) = \frac{F\left(\{a_n^* + \varepsilon b_n\}_{n>\ell}\right)}{G\left(\{a_n^* + \varepsilon b_n\}_{n>\ell}\right)}$$

is differentiable. Moreover, as φ has a minimum at zero by (4.44), it follows that $\varphi'(0) = 0$. In particular, for any $m \in \mathbb{N}$, with the perturbation $\{b_n\}_{n>\ell}$ given by $b_n = 1$ if $n = \ell + m$ and $b_n = 0$ otherwise, we arrive at the Lagrange multipliers

$$\frac{\partial F}{\partial a_n}(\{a_n^*\}_{n>\ell}) = \lambda_0^{2k} \frac{\partial G}{\partial a_n}\left(\{a_n^*\}_{n>\ell}\right) \tag{4.45}$$

for all $n > \ell$. Using (4.43), we have

$$\frac{\partial F}{\partial a_n}(\{a_n\}_{n>\ell}) = \sum_{i=1}^{\ell} (-2c_i a_i) \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \xi_i^{2k} + 2c_n a_n \xi_n^{2k},$$

$$\frac{\partial G}{\partial a_n}\left(\{a_n\}_{n>\ell}\right) = \sum_{i=1}^{\ell} (-2c_i a_i) \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) + 2c_n a_n.$$

Dividing the last identities by 2, we find that (4.45) is equivalent to

$$c_n a_n^*(\xi_n^{2k} - \lambda_0^{2k}) = \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) (\xi_i^{2k} - \lambda_0^{2k}) \tag{4.46}$$

for all $n > \ell$. Since we have already established that $\lambda_0 < \xi_{\ell+1}$, we find that

$$a_n^* = \sum_{i=1}^{\ell} c_i a_i^* \frac{\left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right)}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)} \tag{4.47}$$

for all $n > \ell$. In particular, note that $(a_1^*, a_2^*, \dots, a_\ell^*) \neq (0, 0, \dots, 0)$, as $\mathcal{T}$ is invertible.

If we multiply both sides of (4.47) by $\xi_n^{2\ell-2j+1}$ and sum over $n > \ell$, using (4.41), we get[4]

$$S_j^* = \sum_{n>\ell} a_n^* \xi_n^{2\ell-2j+1} = \sum_{i=1}^{\ell} c_i a_i^* \sum_{n>\ell} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right) \xi_n^{2\ell-2j+1}}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)}.$$

Recall from (4.42) that for each $j \in \{1, 2, \dots, \ell\}$ we have

$$S_j^* = -\sum_{i=1}^{\ell} a_i^* \mathcal{T}_{ij} = -\sum_{i=1}^{\ell} a_i^* \xi_i^{2\ell-2j+1}.$$

If we subtract these two equations, we get

$$0 = \sum_{n \geq 1} a_n^* \xi_n^{2\ell-2j+1} = \sum_{i=1}^{\ell} a_i^* \left[\sum_{n>\ell} \frac{c_i \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) (\xi_i^{2k} - \lambda_0^{2k}) \xi_n^{2\ell-2j+1}}{c_n (\xi_n^{2k} - \lambda_0^{2k})} + \xi_i^{2\ell-2j+1} \right] \tag{4.48}$$

for each $j \in \{1, 2, \dots, \ell\}$. Denote by $\mathcal{W}(\lambda)$ the $\ell \times \ell$ matrix with entries

$$(\mathcal{W}(\lambda))_{ij} = \left(\sum_{n>\ell} \frac{c_i \left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2} (\mathcal{T}^{-1})_{ri} \right)}{c_n (\xi_n^{2k} - \lambda^{2k})} \right) (\xi_i^{2k} - \lambda^{2k}) + \xi_i^{2\ell-2j+1}, \tag{4.49}$$

what we have seen from (4.48) is that we must have $\det \mathcal{W}(\lambda_0) = 0$, because the vector $(a_1^*, a_2^*, \dots, a_\ell^*) \in \mathbb{R}^\ell \setminus \{0\}$ belongs to $\ker \mathcal{W}(\lambda_0)$.

Conversely, if $0 < \lambda_0 < \xi_{\ell+1}$ is such that $\det \mathcal{W}(\lambda_0) = 0$, and we let $(a_1^*, a_2^*, \dots, a_\ell^*)$ be a nonzero element in $\ker \mathcal{W}(\lambda_0)$, we may define a_n^* for $n > \ell$ using (4.47). Then, the constraints (4.40) are reduced to (4.48). Multiplying (4.47) by $c_n a_n^* \xi_n^{2k}$ and summing

[4] S_j^* will be used in place of S_j for the sequence $\{a_n^*\}_{n>\ell}$.

over $n > \ell$, we can check that $\sum_{n\geq 1} c_n(a_n^*)^2\xi_n^{2k} < \infty$, because[5]

$$\begin{aligned}
\sum_{n>\ell} c_n(a_n^*)^2\xi_n^{2k} &= \sum_{n>\ell}\left(\frac{\xi_n^{2k}}{\xi_n^{2k}-\lambda_0^{2k}}\right) a_n^* \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1}(\mathcal{T}^{-1})_{ri}\right)(\xi_i^{2k}-\lambda_0^{2k})\\
&\leq C\sum_{n>\ell} a_n^* \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1}(\mathcal{T}^{-1})_{ri}\right)(\xi_i^{2k}-\lambda_0^{2k})\\
&\leq C\sum_{i=1}^{\ell} c_i a_i^* \sum_{r=1}^{\ell}\left(\sum_{n>\ell} a_n^*\xi_n^{2\ell-2r+1}\right)(\mathcal{T}^{-1})_{ri}(\xi_i^{2k}-\lambda_0^{2k})\\
&\leq C\sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} S_r^*(\mathcal{T}^{-1})_{ri}\right)(\xi_i^{2k}-\lambda_0^{2k})\\
&\leq C\sum_{i=1}^{\ell} c_i (a_i^*)^2(\lambda_0^{2k}-\xi_i^{2k}) \qquad (4.50)
\end{aligned}$$

Working backward, (4.46) follows from (4.47). Using the same argument as in (4.50), we obtain by (4.46)

$$\begin{aligned}
\sum_{n>\ell} c_n(a_n^*)^2(\xi_n^{2k}-\lambda_0^{2k}) &= \sum_{n>\ell} a_n^* \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1}(\mathcal{T}^{-1})_{ri}\right)(\xi_i^{2k}-\lambda_0^{2k})\\
&= \sum_{i=1}^{\ell} c_i(a_i^*)^2(\lambda_0^{2k}-\xi_i^{2k}).
\end{aligned}$$

So, we can rewrite the last equality as

$$\frac{\sum_{n\geq 1} c_n(a_n^*)^2\xi_n^{2k}}{\sum_{n\geq 1} c_n(a_n^*)^2} = \lambda_0^{2k}.$$

The conclusion is that the desired value λ_0 is the smallest positive solution of the equation $\det \mathcal{W}(\lambda) = 0$, and all the real-valued extremal sequences are given by (4.47) with

$$\{0\} \neq (a_1^*, a_2^*, \dots, a_\ell^*) \in \ker \mathcal{W}(\lambda_0).$$

The idea now is to work with the determinant of the matrix $\mathcal{W}(\lambda)$, as given by (4.49), in order to describe it in terms of the functions A and B that define the de Branges space $\mathcal{H}(E)$. Note that $\lambda \mapsto \det \mathcal{W}(\lambda)$ is a continuous function on the interval $(0, \xi_{\ell+1})$.

Observe first that, by the definition of (4.49) and using the identity

$$\sum_{r=1}^{\ell} \xi^{2\ell-2r+1} n(\mathcal{T}^{-1})ri = \delta_{ni}$$

[5] We make use of (4.41) and (4.43), as well as the fact that there exists a positive constant C such that $\xi_n^{2k} \leq C(\xi_n^{2k}-\lambda_0^{2k})$ for all $n > \ell$, because $\lambda_0 < \xi_{\ell+1}$.

for $1 \leq i \leq \ell$ and $1 \leq n \leq \ell$, we obtain

$$
\begin{aligned}
(\mathcal{W}(\lambda))_{ij} &= c_i(\xi_i^{2k} - \lambda^{2k}) \left[\frac{\xi_i^{2\ell-2j+1}}{c_i(\xi_i^{2k} - \lambda^{2k})} + \sum_{n>\ell} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2} (\mathcal{T}^{-1})_{ri}\right)}{c_n(\xi_n^{2k} - \lambda^{2k})} \right] \\
&= c_i(\xi_i^{2k} - \lambda^{2k}) \left[\frac{\xi_n^{2\ell-2j+1}\delta_{\{n,i\}}}{c_n(\xi_n^{2k} - \lambda^{2k})} + \sum_{n>\ell} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2} (\mathcal{T}^{-1})_{ri}\right)}{c_n(\xi_n^{2k} - \lambda^{2k})} \right] \\
&= c_i\left(\xi_i^{2k} - \lambda^{2k}\right) \sum_{n=1}^{\infty} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2} (\mathcal{T}^{-1})_{ri}\right)}{c_n(\xi_n^{2k} - \lambda^{2k})} \\
&= c_i(\xi_i^{2k} - \lambda^{2k}) \left((\mathcal{T}^{-1})^t \mathcal{Q}(\lambda) \right)_{ij}.
\end{aligned} \tag{4.51}
$$

Let $\mathcal{Q}(\lambda)$ be the $\ell \times \ell$ matrix with entries given by

$$
(\mathcal{Q}(\lambda))_{mj} = \sum_{n=1}^{\infty} \frac{\xi_n^{4\ell-2m-2j+2}}{c_n(\xi_n^{2k} - \lambda^{2k})}
$$

for $1 \leq m, j \leq \ell$. From (4.51), we arrive at

$$
\det(\mathcal{W}(\lambda)) = \left(\prod_{i=1}^{\ell} c_i(\xi_i^{2k} - \lambda^{2k}) \right) (\det \mathcal{T})^{-1} \det \mathcal{Q}(\lambda), \tag{4.52}
$$

and now we need to work on this last determinant. The next lemma is helpful to rewrite each entry $(\mathcal{Q}(\lambda))_{mj}$ in terms of the companion functions A and B.

Lemma 4.23. Let $k \in \mathbb{N}$ and set $\zeta \stackrel{def}{=} e^{2\pi i/k}$. For $s \in \mathbb{Z}$ with $0 \leq s \leq k-1$, we have

$$
\sum_{r=0}^{k-1} \frac{\zeta^{-rs}}{x - \zeta^r y} = \frac{k x^{k-s-1} y^s}{x^k - y^k},
$$

as rational functions of the variables x and y.

Proof. We briefly argue via series expansions. Observe that

$$
\begin{aligned}
\sum_{r=0}^{k-1} \frac{\zeta^{-rs}}{x - \zeta^r y} &= \frac{1}{x} \sum_{r=0}^{k-1} \frac{\zeta^{-rs}}{1 - \zeta^r \left(\frac{y}{x}\right)} = \frac{1}{x} \sum_{r=0}^{k-1} \zeta^{-rs} \sum_{m=0}^{\infty} \zeta^{rm} \left(\frac{y}{x}\right)^m = \frac{1}{x} \sum_{m=0}^{\infty} \left(\frac{y}{x}\right)^m \sum_{r=0}^{k-1} \zeta^{r(m-s)} \\
&= \frac{k}{x} \sum_{\substack{m \geq 0 \\ m \equiv s \pmod{k}}} \left(\frac{y}{x}\right)^m = \frac{k}{x} \frac{y^s}{x^s} \left(\frac{1}{1 - \left(\frac{y}{x}\right)^k} \right) = \frac{k x^{k-s-1} y^s}{x^k - y^k}
\end{aligned}
$$

and the last equality follows from the geometric series formula. $\square$

The even function $K(0,z) = A(0)B(z)/(\pi z)$ belongs to $\mathcal{H}(E)$. From (4.29) and (4.32), we have

$$
\frac{B(z)}{2z} = \sum_{n=1}^{\infty} \frac{1}{c_n} \frac{A(z)}{(\xi_n^2 - z^2)}.
$$

Then, with $C(z) = B(z)/A(z)$,

$$\frac{C(z)}{2z} = \sum_{n=1}^{\infty} \frac{1}{c_n(\xi_n^2 - z^2)}.$$

By setting $\zeta = e^{2\pi i/k}$, we can utilize Lemma 4.23 with $x = \xi_n^2$ and $y = \lambda^2$. Then, we can apply the last equation to obtain

$$\begin{aligned}
(\mathcal{Q}(\lambda))_{mj} &= \sum_{n=1}^{\infty} \frac{\xi_n^{4\ell-2m-2j+2}}{c_n\left(\xi_n^{2k} - \lambda^{2k}\right)} \\
&= \sum_{n=1}^{\infty} \frac{1}{k\lambda^{2k-4\ell+2m+2j-4}} \frac{k\xi_n^{4\ell-2m-2j+2}\lambda^{2k-4\ell+2m+2j-4}}{c_n\left(\xi_n^{2k} - \lambda^{2k}\right)} \\
&= \sum_{n=1}^{\infty} \frac{1}{k\lambda^{2k-4\ell+2m+2j-4}} \sum_{r=0}^{k-1} \frac{\zeta^{-r(k-2\ell+m+j-2)}}{c_n\left(\xi_n^2 - \zeta^r\lambda^2\right)} \\
&= \sum_{r=0}^{k-1} \frac{\zeta^{-r(k-2\ell+m+j-2)}}{k\lambda^{2k-4\ell+2m+2j-4}} \sum_{n=1}^{\infty} \frac{1}{c_n\left(\xi_n^2 - \zeta^r\lambda^2\right)} \\
&= \sum_{r=0}^{k-1} \frac{\zeta^{-r(k-2\ell+m+j-2)}}{k\,\lambda^{2k-4\ell+2m+2j-4}} \frac{C\left(\zeta^{r/2}\lambda\right)}{2\zeta^{r/2}\lambda} \\
&= \frac{(\mathcal{V}(\lambda))_{mj}}{(2k)\lambda^{2k-4\ell+2m+2j-3}}
\end{aligned} \tag{4.53}$$

where $\mathcal{V}(\lambda)$ is the $\ell \times \ell$ matrix with entries

$$(\mathcal{V}(\lambda))_{mj}(\lambda) = \sum_{r=0}^{k-1} \omega^{r(4\ell-2m-2j+3)} C(\omega^r\lambda)$$

for $1 \leq m, j \leq \ell$ and $\omega \stackrel{def}{=} e^{\pi i/k}$.

Recall that we are only interested in the roots of our original determinant in the open interval $(0, \xi_{\ell+1})$. Since the power of λ in the denominator of $(\mathcal{Q}(\lambda))_{mj}$ in (4.53) depends only on $m + j$, when the determinant is multiplied out, there will be a common power of λ that can be factored out. Also, the expression $\prod_{i=1}^{\ell} c_i(\xi_i^{2k} - \lambda^{2k})$ appearing in (4.52) can be replaced by $A(\lambda)$, which shares the same zeros in the interval $(0, \xi_{\ell+1})$. Therefore, in the range $(0, \xi_{\ell+1})$, $\det \mathcal{W}(\lambda) = 0$ if and only if $A(\lambda) \det \mathcal{V}(\lambda) = 0$.

Case 2. Let $k = 2\ell + 1$ with $\ell \in \mathbb{Z}_+$. Let $f \in \mathcal{X}_k(E)$ be an even and real entire function. In this case, note that $g(z) \stackrel{def}{=} z^k f(z)$ is odd.

If $k = 1$, there are no nontrivial conditions in (4.31), and we get

$$\lambda_0^2 = \inf_{\{a_n\} \in A} \frac{\sum_{n=1}^{\infty} c_n a_n^2 \xi_n^2}{\sum_{n=1}^{\infty} c_n a_n^2} = \xi_1^2,$$

with the only extremal sequences are those where the first term a_1 is nonzero, and all subsequent terms a_n are zero for $n \geq 2$.

If $k \geq 3$, i.e. $\ell \geq 1$, then more than half of the conditions in (4.31) are already taken care of, and we only need to look for

$$g'(0) = g^{(3)}(0) = \ldots = g^{(2\ell-1)}(0) = 0. \tag{4.54}$$

From the representation (4.30), we have

$$g(z) = \sum_{n=1}^{\infty} \frac{2\xi_n^k f(\xi_n)}{A'(\xi_n)} \frac{zA(z)}{(z^2 - \xi_n^2)}, \tag{4.55}$$

and then $g'(0) = 0$ if and only if

$$\sum_{n=1}^{\infty} \frac{\xi_n^{k-2} f(\xi_n)}{A'(\xi_n)} = 0. \tag{4.56}$$

Proceeding inductively by differentiating (4.55) and plugging in $z = 0$ (the initial case being (4.56)), we come to the conclusion that (4.54) is equivalent to the set of identities

$$\sum_{n=1}^{\infty} \frac{\xi_n^{2\ell-2j+1} f(\xi_n)}{A'(\xi_n)} = 0 \quad \text{for all} \quad j \in \{1, 2, \ldots, \ell\}.$$

The rest of the proof follows the same steps as in *Case 1* for even k. The variables k and ℓ were controlled independently in the previous argument to allow for a direct application of the same reasoning to this case.

$\square$

Appendix

A.1 Properties of gamma and Bessel functions

Given a real and positive number x, we define the gamma function as

$$\Gamma(x) \stackrel{def}{=} \int_0^\infty t^{x-1} e^{-t}\, dt.$$

It can be shown that $\Gamma(x)$ is absolutely convergent for each positive x. This real function Γ extends to a holomorphic function in the half-plane $\mathrm{Re}(z) > 0$, and this extension further has an analytic continuation to a meromorphic function in $\mathbb{C}$ whose only singularities are simple poles at negative integers (see [SS10, 6§1]). Therefore, whenever we refer to the gamma function, we are referring to this meromorphic function. From these definitions, we obtain interesting properties condensed in the following result.

Theorem A1. For $z, w \in \mathbb{C}$ with $\mathrm{Re}\,(z) > 0$, the following equalities hold:

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin(\pi w)}. \tag{A1}$$

Finally, for $r > 0$, the d-dimensional volume of the ball $B_r \stackrel{def}{=} \{x \in \mathbb{R}^d : |x| < r\}$ is

$$m(B_r) = \frac{\pi^{d/2} r^d}{\Gamma(d/2+1)} \tag{A2}$$

and the surface area is

$$\omega(B_r) = \frac{2\pi^{d/2} r^{d-1}}{\Gamma\left(d/2\right)}. \tag{A3}$$

Proof. The proof of (A1) can be found in [SS10], Chapter 6§1, Lemma 1.2 and Theorem 1.4, respectively. The proof of (A2) and (A3) can be found in [Fol99], Chapter 2§7, Proposition 2.54 and Corollary 2.55.

□

For simplicity, we use the notation $\omega_{d-1} \stackrel{def}{=} \omega(B_1)$ when $B_1 \subset \mathbb{R}^d$.

Definition A2 (Bessel function). Let $\nu \in \mathbb{R} \setminus \{-1, -2, \ldots\}$. The Bessel function of the first kind, $J_\nu(x)$, is defined by

$$J_\nu(x) \stackrel{def}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{\nu+2n},$$

where $\Gamma(x)$ is the gamma function. If ν is a negative integer, we define

$$J_\nu(x) = (-1)^\nu J_{-\nu}(x).$$

If ν is an integer, then $J_\nu(x)$ defines an entire analytic function. For non-integer ν, it has a branch point at the origin and then we choose the principal branch of the logarithm when defining x^ν.

Definition A3. For $\nu > -1$, let $A_\nu : \mathbb{C} \to \mathbb{C}$ and $B_\nu : \mathbb{C} \to \mathbb{C}$ be the real entire functions defined by

$$A_\nu(z) \stackrel{\mathrm{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \, (z/2)^{2n}}{n!(\nu+1)(\nu+2)\cdots(\nu+n)};$$
$$B_\nu(z) \stackrel{\mathrm{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n \, (z/2)^{2n+1}}{n!(\nu+1)(\nu+2)\cdots(\nu+n+1)}.$$

These functions are related to the classical Bessel functions of the first kind by the following identities:

$$A_\nu(z) = \Gamma(\nu+1)\,(z/2)^{-\nu}\, J_\nu(z);$$
$$B_\nu(z) = \Gamma(\nu+1)\,(z/2)^{-\nu}\, J_{\nu+1}(z).$$

Both A_ν and B_ν have only real, simple zeros and do not share any common zeros. For example, when $\nu = -1/2$, $A_{-1/2}(z) = \cos(z)$ and $B_{-1/2}(z) = \sin(z)$. A_ν is even, B_ν is odd, and both have $\tau(A_\nu) = \tau(B_\nu) = 1$. Moreover, they satisfy the system of differential equations

$$\begin{aligned} A_\nu'(z) &= -B_\nu(z); \\ B_\nu'(z) &= A_\nu(z) - (2\nu+1)z^{-1}B_\nu(z). \end{aligned} \tag{A4}$$

A.2 Lifts and radial symmetrizations

Definition A4 (Lift). Let $F : \mathbb{C} \to \mathbb{C}$ be an even entire function such that

$$F(z) = \sum_{k=0}^{\infty} c_k z^{2k}.$$

The lift $\mathcal{L}_d(F)\colon \mathbb{C}^d \to \mathbb{C}$ is defined as the entire function given by

$$\mathcal{L}_d(F)(z) = \sum_{k=0}^{\infty} c_k (z_1^2 + z_2^2 + \cdots + z_d^2)^k.$$

Theorem A5. Let $F : \mathbb{C} \to \mathbb{C}$ be an even entire function. Then F has exponential type if and only if $\mathcal{L}_d(F)$ has exponential type, and $\tau(F) = \tau(\mathcal{L}_d(F))$. Moreover, we have

$$\frac{1}{2}\omega_{d-1} \int_{\mathbb{R}} |F(x)||x|^{2\nu+1}\, dx = \int_{\mathbb{R}^d} |\mathcal{L}_d(F)(x)||x|^{2\nu+2-d}\, dx$$

where $-1 < \nu$. If the integrals in the above inequalities are finite, then

$$\frac{1}{2}\omega_{d-1} \int_{\mathbb{R}} F(x)|x|^{2\nu+1}\, dx = \int_{\mathbb{R}^d} \mathcal{L}_d(F)(x)|x|^{2\nu+2-d}\, dx. \tag{A5}$$

The proof of Theorem A5 is provided in [HV96, Lemma 18]. For $N \geq 2$, $SO(d)$ is the topological group of real orthogonal $d \times d$ matrices M with $\det M = 1$. Let σ be the unique left-invariant Haar measure on the Borel subsets of $SO(d)$, such that $\sigma(SO(d)) = 1$. Since $SO(d)$ is compact, σ is also a right-invariant Haar measure.

For a $d \times d$ matrix A with d^2 complex variables and a column vector z with d complex variables, $(A, z) \mapsto Az$ is a polynomial map from $\mathbb{C}^{d^2+d}$ to $\mathbb{C}^d$, making it an analytic function. If $F : \mathbb{C}^d \to \mathbb{C}$ is entire, then $(A, z) \mapsto F(Az)$ is an entire function of $d^2 + d$ complex variables. Since $SO(d)$ is compact in $\mathbb{R}^{d^2}$, the map $A \mapsto F(Az)$ is integrable with respect to σ for each z in $\mathbb{C}^d$.

Definition A6 (Radial symmetrization). Let $F : \mathbb{C}^d \to \mathbb{C}$ be an entire function. We define its radial symmetrization $\widetilde{F} : \mathbb{C}^d \to \mathbb{C}$ by

$$\widetilde{F}(z) = \int_{SO(d)} F(Mz)\, d\sigma(M). \tag{A6}$$

If $d = 1$, it will be convenient to write $\widetilde{F}(z) = (F(z) + F(-z))/2$ for the even part of F. The principal theorem about radial symmetrization is the following and its proof can be found in [HV96, Lemma 19].

Theorem A7. Let $F : \mathbb{C}^d \to \mathbb{C}$ be an entire function. Then $\widetilde{F} : \mathbb{C}^d \to \mathbb{C}$ is an entire function that satisfies the following conditions:

(i) $\widetilde{F}$ has a power series expansion of the form

$$\widetilde{F}(z) = \sum_{k=0}^{\infty} c_k (z_1^2 + z_2^2 + \ldots + z_d^2)^k;$$

(ii) If the function F satisfies the property $F(x) \geq 0$ for all x in $\mathbb{R}^d$, then by (A6), the corresponding property also holds for $\widetilde{F}$.

(iii) If F has exponential type, then $\widetilde{F}$ also has exponential type and $\tau(\widetilde{F}) \leq \tau(F)$;

(iv) If $-1 < \nu$, then

$$\int_{\mathbb{R}^d} |\widetilde{F}(x)| |x|^{2\nu+2-d}\, dx \leq \int_{\mathbb{R}^d} |F(x)| |x|^{2\nu+2-d}\, dx;$$

(v) If the integral on the right of (iv) is finite, then

$$\int_{\mathbb{R}^d} F(x) |x|^{2\nu+2-d}\, dx = \int_{\mathbb{R}^d} \widetilde{F}(x) |x|^{2\nu+2-d}\, dx.$$

Using the last two theorems, we have the following result (see [HV96, Corollary 20]).

Corollary A8. Let $F : \mathbb{C}^d \to \mathbb{C}$ be an entire function, and suppose that $d \geq 2$. Then the following conditions are equivalent:

(i) $F(z) = F(Mz)$ for all $M \in SO(d)$ and for all $z \in \mathbb{C}^d$;

(ii) $F(z) = \widetilde{F}(z)$ for all z in $\mathbb{C}^d$;

(iii) F has a power series expansion of the form

$$F(z) = \sum_{k=0}^{\infty} c_k (z_1^2 + z_2^2 + \ldots + z_d^2)^k; \tag{A7}$$

(iv) $F(z) = \mathcal{L}_d(f)(z)$, where $f : \mathbb{C} \to \mathbb{C}$ is an entire even function, expressed in the form

$$f(z) = \sum_{k=0}^{\infty} c_k z^{2k},$$

using the same notation as in (A7).

A.3 Homogeneous spaces and Krein decomposition

If $-1 < \nu$, a de Branges space $\mathcal{H}(E)$ is considered homogeneous of order ν if, for all $0 < a < 1$ and all $f \in \mathcal{H}(E)$, the function $z \mapsto a^{\nu+1} f(az)$ is in $\mathcal{H}(E)$ and has the same norm as f. Louis de Branges provided a characterization of such spaces in [Bra62] and in [Bra68, Section 50].

Consider the real entire functions A_ν and B_ν defined in Appendix A.1. Define the function $E_\nu(z) \overset{def}{=} A_\nu(z) - iB_\nu(z)$, which happens to be a Hermite-Biehler function with no real zeros. The space $\mathcal{H}(E_\nu)$ is a homogeneous space of order ν, with $\tau(A_\nu) = \tau(B_\nu) = \tau(E_\nu) = 1$.

Additionally, A_ν and B_ν belong to $\mathcal{H}(E_\nu)$ due to the behavior of the Bessel function J_ν at infinity. The formulas in Corollary 3.21 are applicable in this context. Further details can be found in [Bra68, Section 50].

Lemma A9. Let $\nu > -1$. The following properties hold:

(i) There exist positive constants a_ν and b_ν such that

$$a_\nu |x|^{2\nu+1} \leq |E_\nu(x)|^{-2} \leq b_\nu |x|^{2\nu+1}$$

for all $x \in \mathbb{R}$ with $|x| \geq 1$.

(ii) For $f \in \mathcal{H}(E_\nu)$, we have the identity

$$\int_{\mathbb{R}} |f(x)|^2 |E_\nu(x)|^{-2}\, dx = c_\nu \int_{\mathbb{R}} |f(x)|^2 |x|^{2\nu+1}\, dx,$$

with $c_\nu = \pi 2^{-2\nu-1} \Gamma(\nu+1)^{-2}$.

(iii) An entire function f belongs to $\mathcal{H}(E_\nu)$ if and only if f has exponential type at most 1 and
$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\nu+1}\, dx < \infty.$$

The proof of Lemma A9 can be found in [HV96, Lemma 16]. Let us revisit a useful version of a classical result attributed to Krein concerning the decomposition of entire functions of exponential type that are nonnegative on $\mathbb{R}$. The version stated below is a special case derived from a more general version applicable to de Branges spaces. The proof is available in [CL14, Lemma 14].

Lemma A10 (Krein's decomposition). Let $\nu > -1$ and $\delta > 0$. Let $f : \mathbb{C} \to \mathbb{C}$ be a real entire function of exponential type at most 2δ, that is nonnegative on $\mathbb{R}$ and belongs to $L^1(\mathbb{R}, |x|^{2\nu+1}\, dx)$. Then there exists $g \in \mathcal{H}(E_\nu)$ such that

$$f(z) = g(\delta z) g^*(\delta z).$$

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