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Instanton sheaves on the blow-up of $\mathbb{P}^n$ at a point.

Florianópolis
2025

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Doctoral thesis presented for the Postgraduate Program in Pure and Applied Mathematics at the Federal University of Santa Catarina, to obtain the degree of Doctor in Pure and Applied Mathematics.
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ABSTRACT

This doctoral thesis studies instantons, bundles with specific conditions, in the blow-up at a point of projective spaces. We propose an extension of the instantons definition to odd-dimensional projective spaces presented by Henni [2]. Thus, this definition was extended to even-dimensional projective spaces. Furthermore, we present two examples that support the validity of the generalized definition. We also present some properties of these examples, namely: restriction of the bundle to a line, analysis of whether they are Ulrich, among others.

Keywords: Instantons, blow-up, projective spaces.

RESUMO

Esta tese de doutorado tem como objeto de estudo os instantons, feixes com condições específicas, no *blow-up* em um ponto de espaços projetivos. Nos propomos uma extensão da definição de instantons para dimensões ímpares dos espaços projetivos apresentada por Henni [2]. Desta forma, esta definição foi estendida para dimensões pares dos espaços projetivos. Além disso, mostramos dois exemplos que sustentam a validade da definição generalizada. Apresentamos também algumas propriedades desses exemplos, quais sejam: restrição do feixe a uma linha, analisar se são Ulrich, entre outras.

Palavras-chave: Instantons, *blow-up*, espaços projetivos.

RESUMO EXPANDIDO

Introdução

Os instantons foram introduzidos na geometria algébrica na década de 1970, a partir dos trabalhos pioneiros de [7] e [8]. Devido à sua grande relevância na classificação de espaços de módulos de feixes vetoriais estáveis, os instantons têm sido amplamente estudados. Eles exercem influência em diversas áreas, como topologia diferencial, geometria diferencial, física matemática, teoria da representação e geometria algébrica.

Os exemplos mais simples são os instantons sobre o espaço projetivo $\mathbb{P}^3$, e, ao longo dos anos, diferentes estudos estenderam esse conceito para outras dimensões e variedades. Um exemplo recente é a definição proposta por Henni, em 2021, para instantons no *blow-up* de espaços projetivos em um ponto. Contudo, essa definição contempla apenas espaços projetivos de dimensão ímpar. Até o momento, não há na literatura uma definição ou exemplos conhecidos de instantons no caso de dimensões pares. Sendo assim, os objetivos deste trabalho são os seguintes.

Objetivos

Esta tese de doutorado tem os seguintes objetivos:

- **Objetivo 1:** Completar a definição de instanton no *blow-up* de espaços projetivos em um ponto. Generalizando para dimensões pares do espaço projetivo.
- **Objetivo 2:** Fornecer exemplos que verificam essa definição generalizada.
- **Objetivo 3:** Estudar algumas das propriedades que esses exemplos possuem.

Metodologia

- Revisão bibliográfica e fundamentação teórica.

- Estudo e análise crítica da definição existente.
- Generalização da definição para dimensões pares.
- Construção e verificação de exemplos.
- Análise dos resultados e perspectivas futuras.

Resultados e discussão

A partir da Definição 3.1, introduzida por Henni em [2], e da Definição 3.3, proposta neste trabalho, obtemos a definição geral 5.1. A mesma, permite definir feixes de instantons no *blow-up* em um ponto de espaços projetivos de qualquer dimensão n . Cada um desses instantons é obtido como a cohomologia de uma mônada descrita no Lema 3.2. O caso em que n é ímpar foi estudado por Henni em [2], onde é apresentada uma família de instantons localmente livres.

Neste trabalho, analisamos o caso em que n é par, obtendo duas famílias de feixes de instantons, todos de posto 2. Entretanto, esses feixes não são localmente livres, isto é, não são fibrados vetoriais. Ainda assim, eles surgem como coomologias de mônadas e exibem propriedades interessantes, conforme discutido no Capítulo 3.

Considerações finais

Como os instantons apresentados não são localmente livres, permanece como uma perspectiva futura a construção de exemplos, no caso n par, em que os instantons sejam localmente livres. Ademais, é desejável encontrar exemplos de instantons de posto superior, uma vez que, até o momento, dispomos apenas de exemplos de posto 2. Também é natural investigar a existência, ou não, de uma deformação do feixe de instantons $\mathcal{E}$ para $n = 4$, conforme apresentado na Seção 3.2.4.

No caso em que n é ímpar, como a restrição da mônada a uma linha assume uma forma análoga àquela considerada em [24], é possível prosseguir com a sua análise visando à obtenção de resultados semelhantes aos discutidos na Seção 3.4 de [24].

Palavras-chave: Instantons, *blow-up*, espaços projetivos.

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1 Introduction

Instantons entered algebraic geometry concepts in the 1970s introduced by [7] and [8]. Due to their great importance in classifying moduli spaces of stable vector bundles, instantons have been widely studied. They have influence over several areas, namely differential topology, differential geometry, mathematical physics, representation theory, and algebraic geometry. It should be noted that the simplest case of instantons found in the literature is the rank 2 bundles in $\mathbb{P}^3$, [28]. It was initially known as mathematical instantons and was later extended to projective spaces of dimension $2n + 1$ [27]. In summary, Figure 1.0.1 shows a timeline of instantons from their origin to the present day. This figure highlights only the most important advances related to the objective of this thesis.

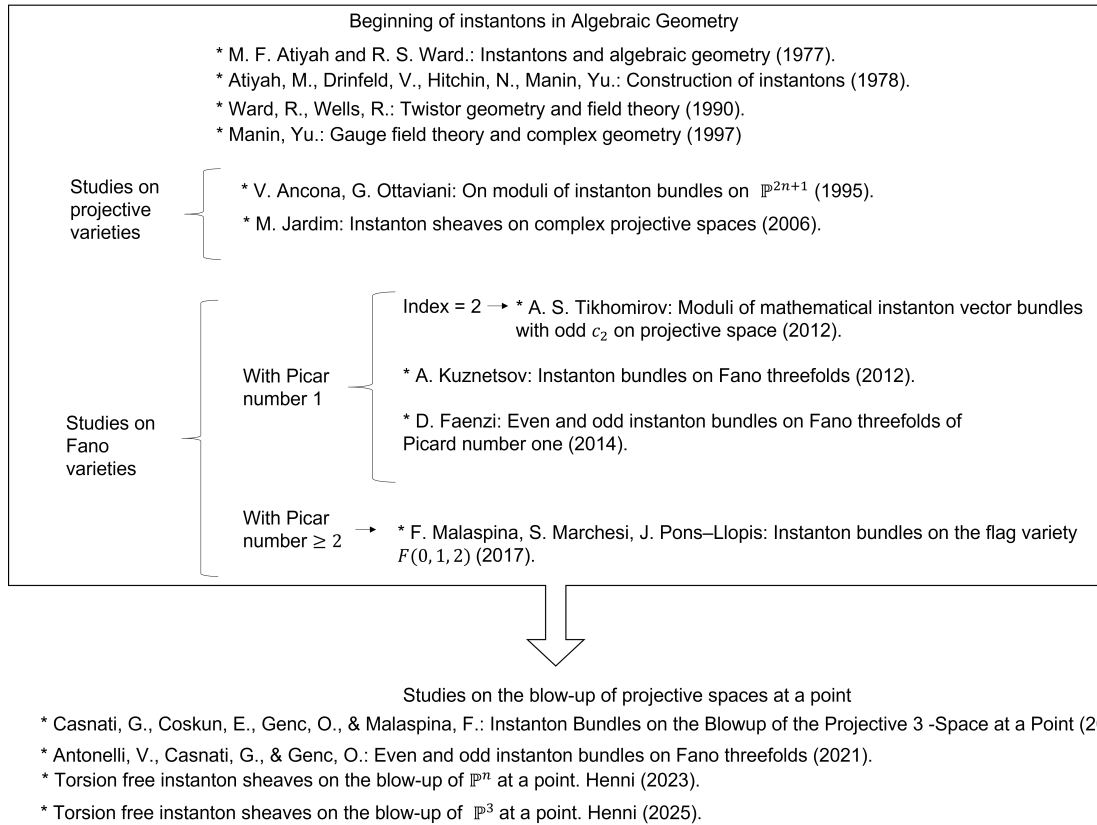


Figure 1.0.1: Instanton timeline. Font: Created by the author in Power Point.

Figure 1.0.1 shows that most studies have been published in the last two decades. Since then, important advances have been made in the instanton study. Among them is the study of its definition for several projective varieties (see [9], [25], [15]).

Studies on a specific class of torsion-free sheaves, admissible sheaves [1], allowed the construction of a complete definition of instanton sheaves in complex projective spaces [21]. These instantons are obtained as cohomologies of monads and are semi-stable if their rank is not too large.

Following this same logic, studies of instantons in the blow-up at a point of projective spaces began. First, Malaspina and Casnati [17] defined these instantons over the blow-up at a point of the 3-dimensional projective space [2].

Definition 1.1. [17] Let X be a Fano threefold. A vector bundle $\mathcal{E}$ of rank 2 on X is called an instanton bundle if the following properties hold:

- (1) $c_1(\mathcal{E}) = (2q_X - i_X)h$;
- (2) $\mathcal{E}$ is μ -semi-stable and $h^0(X, \mathcal{E}) = 0$;
- (3) $h^1(X, \mathcal{E}(-q_X h)) = 0$.

Later, Henni [2] extended this definition to odd-dimensional projective spaces. Again, these instantons are given by the cohomology of monads and semi-stables with respect to the corresponding polarization. In addition, it features a family of instanton bundles that verify these results.

Definition 1.2. On $(\widetilde{\mathbb{P}^n}, \mathcal{L}) = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, N_n)$ with

$$N_n = \begin{cases} 1 & n = 3; \\ \frac{n-3}{2} & n > 3 \text{ and odd} \end{cases}$$

a $\mu_{\mathcal{L}}$ -semi-stable rank r torsion free coherent sheaf $\mathcal{E}$, with Chern class

$$c_1(\mathcal{E}) = \begin{cases} -2\alpha & n > 3 \text{ and odd;} \\ 0 & n = 3 \end{cases}$$

will be called instanton if it satisfies

- (1) $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -2)) = 0$, $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 0)) = 0$;

- (2) $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, -1)) = 0$, $H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 3-n)) = 0$;
- (3) $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 2-n)) = 0$, $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 4-n)) = 0$;
- (4) for $n \geq 4$, $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -3)) = 0 = H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 5-n))$ and $H^i(\widetilde{\mathbb{P}^n}, \mathcal{E} \otimes \mathcal{L}') = 0$, for all $2 \leq i \leq n-2$ and all invertible sheaves $\mathcal{L}'$ in $\widetilde{\mathbb{P}^n}$.

However, no studies were found in the literature on the extension of this definition to even-dimensional projective spaces. No examples were found. We propose an extension of the definition of instantons to the odd-dimensional projective space presented by Henni [2]. Thus, this definition has been extended to even-dimensional projective space. Furthermore, we present two examples that support the validity of the generalized definition. In addition, we analyze some properties of these examples, namely their restriction to a line, charge, and Chern classes, among others.

In the following, we present a summary of the structure of the results of this work. In Chapter 2, we introduced the main topics related to the geometry of the blow-up at a point of projective spaces that we will use, as well as some results that will be used in the development of this work. In addition, two tools are presented. The first, called Hartshorne-Serre Correspondence, [19], will be used to construct the sheaf families. In turn, the second, called Hoppe's Criterion, [22], will allow demonstrating the semi-stability of these families.

On the other hand, in Chapter 3 is presented our definition of an instanton sheaf in the blow-up of even-dimensional projective spaces n at a point. First, we justify the choice of the hypotheses of the definition, and then, we will present examples that verify this new definition. Furthermore, we will explore some of the properties that these families of bundles present.

In Chapter 4 we will continue the study of instanton sheaves in the blow-up at a point of odd-dimensional projective spaces. We will focus mainly on the properties of the monad from the example presented by Henni [2] in $\widetilde{\mathbb{P}^5}$.

Finally, in Chapter 4, we will make some final considerations following the references used in this work.

2 Basic concepts

In this chapter, the main definitions and results that will be used in the development of the work are presented. For better understanding, it has been divided into five sections, which are described below. In Section 2.1, the properties of the blow-up at a point in projective spaces are reviewed. In addition, we present tools for constructing bundles and obtaining complexes in these spaces. In Section 2.2, we will present the stability definition and a result to find Chern classes embedded. Finally, in Section 2.3, we will present the results that will be used to analyze the fined sheaves.

2.1 Blow-up at a point of projective space.

Let X be a complex variety and $Y \subset X$ be a closed subvariety, the blow-up of X over Y is a complex variety $\tilde{X} = \text{Bl}_Y(X)$ with proper holomorphic map $\pi: \tilde{X} \rightarrow X$ such that $\pi: \tilde{X} \setminus \pi^{-1}(Y) \xrightarrow{\cong} X \setminus Y$ and $\pi: \pi^{-1}(Y) \rightarrow Y$ is isomorphic to $\mathbb{P}(\mathcal{N}_{Y/X}) \rightarrow Y$. We call $\pi^{-1}(Y) = \mathbb{P}(\mathcal{N}_{Y/X}) \subset \text{Bl}_Y(X)$ the exceptional divisor of the blow-up.

The case of interest to us is when $X = \mathbb{P}^n$ and Y is a point p . In this situation, the blow-up $\pi: \widetilde{\mathbb{P}^n} \rightarrow \mathbb{P}^n$ replaces the point $p \in \mathbb{P}^n$ with a copy of $\mathbb{P}^{n-1}$. The corresponding exceptional divisor is $\pi^{-1}(p)$ which we denote by E . It is well known that $\widetilde{\mathbb{P}^n}$ has the structure of the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$ on $\mathbb{P}^{n-1}$. The canonical projection is denoted by $pr: \widetilde{\mathbb{P}^n} \rightarrow \mathbb{P}^{n-1}$ [11, Example 2.11.4]. It follows that the Chow ring of a blow-up $\widetilde{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is given by

$$A^*(\widetilde{\mathbb{P}^n}) = \frac{\mathbb{Z}[\alpha, \xi]}{(\alpha^n, \xi^2 - \alpha\xi)}.$$

and the exceptional divisor class E is given by

$$[E] = \xi - \alpha$$

as presented in [12, Theorem 9.12]. Here, α is the pull-back of a hyperplane class in $\mathbb{P}^{n-1}$ and $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$, which is also identified with the pull-back of hyperplane class in $\mathbb{P}^n$. In particular, the Picard group of blow-up $\widetilde{\mathbb{P}^n}$ is isomorphic to $\xi\mathbb{Z} \oplus \alpha\mathbb{Z}$, [18, Exercise II.7.9] and

[11, Corollary 6.13]. Note that the elements c^* of the Chow ring,

$$c^* = r_0 \cdot 1 \oplus \left(\oplus_{k=1}^n r_k \xi^k \right) \oplus \left(\oplus_{l=1}^{n-1} s_l \alpha^l \right),$$

are polynomials in the variables $\xi \in \mathbb{P}^n$ and $\alpha \in \mathbb{P}^{n-1}$ with coefficients in $\mathbb{Z}$ taken modulo the equivalence relations given by

$$\begin{cases} \alpha^n &= 0 \\ \xi^2 - \alpha\xi &= 0. \end{cases}$$

Moreover, observe that

$$\begin{aligned} \text{if } k+l \leq n, k \leq n-1 &\Rightarrow \xi^k \cdot \alpha^l = \xi^k \cdot \xi^l = \xi^{k+l} \\ \text{if } k+l > n &\Rightarrow \xi^k \cdot \alpha^l = \xi^{k+l} = \alpha^{k+l} = \alpha^n \cdot \alpha^{k+l-n} = 0 \\ \text{if } k > n+1 &\Rightarrow \xi^k \cdot \alpha^l = \alpha^n \cdot \alpha^{k+l-n} = 0. \end{aligned}$$

Therefore

$$\xi^k \cdot \alpha^l = \begin{cases} \xi^{k+l} & k+l \leq n, k \leq n-1 \\ 0 & \text{otherwise.} \end{cases}$$

In this work, the following nomenclature is defined and will be adopted in the whole work. Denote by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q)$ the line bundle $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(p\xi + q\alpha)$ associated with the divisor $p\xi + q\alpha$, except when it is necessary to specify the generators explicitly. Thus, the canonical sheaf $\omega_{\widetilde{\mathbb{P}^n}}$ of $\widetilde{\mathbb{P}^n}$ is the line bundle $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 1-n)$ as shown in [2].

The following result allows the construction of bundles in $\widetilde{\mathbb{P}^n}$. This is important since these constructed bundles are part of exact sequences and will be used to find our examples.

Theorem 2.1. *Let X be a smooth algebraic variety and Y be a local complete intersection subscheme and codimension 2 in X . Let $\mathcal{N}$ be the normal bundle of Y in X and S be a line bundle in X such that $H^2(X, S^*) = 0$. Suppose $\wedge^2 \mathcal{N} \otimes S_Y^*$ is generated by $r-1$ global sections $s_1, \dots, s_{r-1}$. Then there is a vector bundle $\mathcal{E}$ of rank r in X such that*

$$(1) \det \mathcal{E} = \wedge^r \mathcal{E};$$

- (2) $\mathcal{E}$ has $r - 1$ global sections $\alpha_1, \dots, \alpha_{r-1}$ whose dependency locus is Y and such that $s_1 \alpha_1|_Y + \dots + s_{r-1} \alpha_{r-1}|_Y = 0$.

Furthermore, if $H^1(X, S^*) = 0$, the previous conditions determine $\mathcal{E}$ to less than isomorphism.

Proof. The demonstration can be found in [14, Theorem 1]. ■

According to [5], one of the main advantages of this correspondence is to construct bundles and express the properties of the subschemes of 2-codimension in terms of bundle properties. In addition, these bundles are in the exact sequence follows

$$0 \rightarrow \mathcal{O}_X^{\oplus(r-1)} \rightarrow \mathcal{E} \rightarrow I_Y \otimes S \rightarrow 0, \quad (2.1.1)$$

with I_Y in the restriction sequence

$$0 \rightarrow I_Y \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0. \quad (2.1.2)$$

In this way, information about $\mathcal{E}$ can be derived in terms of Y , thus facilitating the construction of explicit examples.

The following result is presented by Ancona and Ottaviani in [4]. This result is on Brieskorn varieties with projective description $\mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ direct sum of line bundles on $\mathbb{P}^n$.

Theorem 2.2. [4] *Let $\mathcal{E} = \bigoplus \mathcal{O}(a_i)$ be a bundle in $\mathbb{P}^n$ with $a_i \geq 0$ and $pr: X = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$. Let $\mathcal{U}_{rel}(-1)$ and $\mathcal{Q}$ be the universal relative and quotient bundles, respectively. Then any complex of coherent bundles $\mathcal{F}$ in X is obtained as the cohomology of the complex $\mathcal{C}_{\mathcal{F}}^\bullet$ where*

$$\mathcal{C}_{\mathcal{F}}^p = \bigoplus_{s-i=p} \bigoplus_{q-h=i} H^s(\pi^* \mathcal{O}_{\mathbb{P}^n}(-q) \otimes \mathcal{U}_{rel}(-h) \otimes \mathcal{F}^\bullet) \otimes pr^* \Omega_{\mathbb{P}^n}^q(q) \otimes \wedge^h \mathcal{Q}^*. \quad (2.1.3)$$

Proof. The demonstration can be found in [4, Theorem 8]. ■

Considering that $\widetilde{\mathbb{P}^n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$, we can apply Theorem 2.2 to find the spectral sequence of any coherent sheaf $\mathcal{F}$. In this case, the quotient bundle is $\mathcal{Q} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, 0)$ and the relative universal bundle is $\mathcal{U}_{rel}(-1) = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 1)$. Thus, the complex terms of Theorem

2.2 become

$$\mathcal{C}_{\mathcal{F}}^p = \bigoplus_{s-i=p} \bigoplus_{q-h=i} H^s(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(-h, h-q)) \otimes pr^* \Omega_{\mathbb{P}^{n-1}}^q \otimes \mathcal{O}_{\mathbb{P}^n}(-h, q). \quad (2.1.4)$$

with $h \in \{0, 1\}$ and $0 \leq q \leq n-1$, as shown in [2]. To calculate the cohomologies that appear in each term of the complex, we will use some formulae described in [2]. Considering a line bundle $\mathcal{O}_{\mathbb{P}^n}(p, q)$, for $p \geq 0$,

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p, q)) = \begin{cases} \sum_{k=0}^p \binom{n+q+k-1}{n-1} & \text{if } i = 0 \\ \sum_{k=0}^{-2-p} \binom{n+q-k-2}{n-1} & \text{if } i = 1 \\ \sum_{k=0}^p \binom{-q-k-1}{n-1} & \text{if } i = n-1 \\ \sum_{k=0}^{-2-p} \binom{k-q}{n-1} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \quad (2.1.5)$$

In the case $p < 0$, the formula can be obtained analogously by applying Serre's duality

$$h^i(X, \mathcal{O}_{\mathbb{P}^n}(p, q)) \cong h^{n-i}(X, \omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(p, q)^\vee) \quad (2.1.6)$$

to compute the cohomologies of the dual using the previously established formula.

In case of $\Omega^l(p, q)$, where $\Omega^l(p, q) = pr^* \Omega_{\mathbb{P}^{n-1}}^l \otimes \mathcal{O}_{\mathbb{P}^n}(p, q)$; these cohomologies can be derived using Bott's formulae, similar to the calculation for line bundles [2].

$$h^i(\mathbb{P}^n, \Omega^l(p, q)) = \begin{cases} \sum_{k=0}^p \binom{q+k+n-1-l}{n-1-l} \binom{q+k-1}{l} & \text{if } i = 0 \\ \sum_{k=0}^{-2-p} \binom{q-k+n-l-2}{n-1-l} \binom{q-k-2}{l} & \text{if } i = 1 \\ 1 & \text{if } 0 \leq l = i \leq n-1, k+q=0 \\ \sum_{k=0}^p \binom{-q-k+l}{l} \binom{-q-k-1}{n-1-l} & \text{if } i = n-1 \\ \sum_{k=0}^{-2-p} \binom{k-q+l+1}{l} \binom{k-q}{n-1-l} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \quad (2.1.7)$$

Note that this bundle is part of the following exact sequence

$$0 \rightarrow \Omega^l \rightarrow \mathcal{O}(-l)^{\binom{n+1}{l}} \rightarrow \Omega^{l-1} \rightarrow 0. \quad (2.1.8)$$

In addition, as $\Omega^{n-2}(n-1) = T\mathbb{P}^{n-1}(-1)$ it follows that

$$0 \rightarrow \mathcal{O}(-n) \rightarrow \mathcal{O}(1-n)^n \rightarrow \Omega^{n-2} \rightarrow 0. \quad (2.1.9)$$

Using the formula (2.1.5) we can show that $H^i(\widetilde{\mathbb{P}^n}, \mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q)) = 0$ for all $i = 0, \dots, n$ and $(p, q) = (0, 0), \dots, (0, 1-n)$ or $(p, q) = (-1, 1), \dots, (-1, 2-n)$. In particular, we obtain an exceptional full collection of line bundles given by

$$\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 2-n), \dots, \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 1), \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1-n), \dots, \mathcal{O}_{\widetilde{\mathbb{P}^n}}. \quad (2.1.10)$$

2.2 Stability.

In this work, we will use the definition of μ -(semi)-stability, or Mumford-Takemoto stability given below.

Definition 2.3. [22] Let $\mathcal{F}$ be a coherent bundle of rank r over $\mathbb{P}^n$ and $\mathcal{L}$ be an ample line bundle over X a non-singular projective variety over $\mathbb{C}$. We define

(1) $\mathcal{L}$ -degree of $\mathcal{F}$ by

$$\deg_{\mathcal{L}} \mathcal{F} = c_1(\mathcal{F}) \cdot c_1(\mathcal{L})^{n-1}$$

with $c_1(\mathcal{F})$ the first Chern class of $\mathcal{F}$.

(2) $\mathcal{L}$ -slope of $\mathcal{F}$ with respect to $\mathcal{L}$ by

$$\mu_{\mathcal{L}}(\mathcal{F}) = \frac{\deg_{\mathcal{L}} \mathcal{F}}{\text{rk} \mathcal{F}}.$$

We then define that $\mathcal{F}$ as before is semi-stable if for every coherent subsheaf $0 \neq \mathcal{F}' \subset \mathcal{F}$ we have

$$\mu_{\mathcal{L}}(\mathcal{F}') \leq \mu_{\mathcal{L}}(\mathcal{F}). \quad (2.2.1)$$

If for every coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$, with $0 < \text{rk} \mathcal{F}' < \text{rk} \mathcal{F}$, the inequality (2.2.1) is strict, we say that $\mathcal{F}$ is stable.

Verifying stability via definition is not always feasible. In this work, we will use a special case of the Generalized Hoppe Criterion [20, Theorem 3]. For this we will consider [26, Definition 1.1.14], where for torsion-free sheaves of rank r we have

$$\det \mathcal{F} = (\wedge^r \mathcal{F})^{**},$$

which is a line bundle. Being able to apply the theorem to torsion free sheaves. This theorem can be used to prove the μ -stability or μ -semi-stability of a sheaf of rank 2 with respect to a given polarization

Theorem 2.4. [22] *Let $\mathcal{E}$ be a torsion free sheaf of rank 2 on a polarized variety $(X, \mathcal{L})$ with polycycle Picar group. Then $\mathcal{E}$ is $\mu_{\mathcal{L}}$ -stable (semi-stable) if and only if*

$$H^0(X, \mathcal{E}_{\text{norm}} \otimes \mathcal{F}^{\vee}) = 0$$

for every line bundle $\mathcal{F}$ such that $\deg_{\mathcal{L}} \mathcal{F} \leq -\mu_{\mathcal{L}}(\mathcal{E})$, (respectively, $\deg_{\mathcal{L}} \mathcal{F} < -\mu_{\mathcal{L}}(\mathcal{E})$).

To calculate the Chern classes, we will use their properties, which can be found in [26, 18, 20]. In particular case, we obtain an equality for ch in embedded form as a consequence of the Hirzebruch-Riemann-Roch Theorem [18, 16].

Proposition 2.5. [16] *For a closed immersion $i: Y \rightarrow X$ of a smooth subvariety Y into a smooth projective variety X and a vector bundle $\mathcal{E}$ on Y , we have that*

$$\text{ch}(i_* \mathcal{E}) = i_*(\text{ch}(\mathcal{E}) \cdot \text{Td}(\mathcal{N}_{Y/X})^{-1}). \quad (2.2.2)$$

by the sequence $0 \rightarrow T_Y \rightarrow T_X|_Y \rightarrow \mathcal{N}_{Y/X} \rightarrow 0$.

2.3 Other results.

The objective of this section is to present several results and definitions concerning bundles that can provide further insight into their classification.

Definition 2.6. [26] An monad in a compact complex variety X is a complex

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \quad (2.3.1)$$

of vector bundles over X that is exact in A and C , such that $\text{Im}(\alpha)$ is a sub-bundle of B . The holomorphic vector bundle

$$\mathcal{E} = \text{Ker}\beta / \text{Im}\alpha$$

is the cohomology of the monad. Furthermore,

$$\text{rk}\mathcal{E} = \text{rk}B - \text{rk}A - \text{rk}C. \quad (2.3.2)$$

Since $\mathcal{E}$ is a holomorphic vector bundle, the previous definition is a very useful tool for finding bundles.

Furthermore, the monad (2.3.1) presents the following exact sequence display

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & A & \xlongequal{\quad} & A & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Ker}\beta & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \text{CoKer}\alpha & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Grothendieck established that every algebraic vector bundle over $\mathbb{P}^1$ admits a decomposition into a direct sum of line bundles. Consequently, any coherent sheaf on $\mathbb{P}^1$ can be expressed as a direct sum of line bundles together with torsion sheaves. Therefore, if we draw a line $L \cong \mathbb{P}^1$, then the restriction $\mathcal{E}|_L$ must be split. Furthermore, the restriction of a line bundle to any H_i is in the long exact sequence

$$0 \rightarrow \mathcal{O}(-i) \rightarrow \mathcal{O}(-i+1)^{\binom{i}{i-1}} \rightarrow \mathcal{O}(-i+2)^{\binom{i}{i-2}} \rightarrow \cdots \rightarrow \mathcal{O}(-i+(i-1))^{\binom{i}{i-(i-1)}} \rightarrow \mathcal{O} \rightarrow \mathcal{O}|_{H_i} \rightarrow 0 \quad (2.3.3)$$

where $L = H_{n-1} \subset H_{n-2} \subset \cdots \subset H_1 = H$ and H_i are embedded linear subspaces of codimension

i on $\mathbb{P}^n$, each is a hyperplane on the other, called a complete Flag.

In addition, Horrocks established that a vector bundle $\mathcal{E}$ of rank r on $\mathbb{P}^n$ split iff $H^i(\mathcal{E}(k)) = 0$ for every $k \in \mathbb{Z}$ and every $0 < i < n$.

An important concept to consider in the sheaves we want to construct is whether they should be Ulrich sheaves. For sheaves, we will consider the Definition 2.7 given for bundles.

Definition 2.7. [10] For a polarized variety $(X, \mathcal{L})$, an $\mathcal{L}$ -Ulrich bundle is a bundle $\mathcal{E}$ satisfying

- (1) $H^i(X, \mathcal{E} \otimes \mathcal{L}^{-i}) = 0$, for $i > 0$ and
- (2) $H^j(X, \mathcal{E} \otimes \mathcal{L}^{-(j+1)}) = 0$, for $j < n$.

Although less well known, we also have the definition for weakly Ulrich and almost Ulrich as presented in [10, 23, 13].

A bundle $\mathcal{E}$ in a projective variety X with polarization $\mathcal{L} = \mathcal{O}(1, 1)$ and dimension n is a weakly Ulrich bundle in X if

- $H^i(\mathcal{E}(-t, -t)) = 0$, for $1 \leq i \leq n$ and $t \leq i - 1 \leq n - 1$, and
- $H^i(\mathcal{E}(-t, -t)) = 0$, for $0 \leq i \leq n - 1$ and $t \geq i + 2$.

If, in addition, $\mathcal{E}$ is an Arithmetically Cohen-Macaulay (ACM) bundle, we say that $\mathcal{E}$ is almost Ulrich bundle. Recall that an ACM bundle satisfies $H^i(\mathcal{E}(t, t)) = 0$ for $1 \leq i \leq n - 1$ and all $t \in \mathbb{Z}$. As mentioned in [23], we have

$$\mathcal{E} \text{ is Ulrich} \Rightarrow \mathcal{E} \text{ is almost Ulrich} \Rightarrow \mathcal{E} \text{ is weakly Ulrich.}$$

3 Instantons sheaf on a blow-up at a point of projective space.

Henni [2] defines instanton sheaf on blow-up at a point of projective spaces, as follows.

Definition 3.1. On $(\widetilde{\mathbb{P}^n}, \mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, N_n))$ with

$$N_n = \begin{cases} 1 & \text{if } n = 3; \\ \frac{n-3}{2} & \text{if } n > 3 \text{ and odd} \end{cases}$$

a $\mu_{\mathcal{L}}$ -semi-stable rank r torsion-free coherent sheaf $\mathcal{E}$, with Chern class

$$c_1(\mathcal{E}) = \begin{cases} -2\alpha & \text{if } n > 3 \text{ and odd;} \\ 0 & \text{if } n = 3 \end{cases}$$

will be called instanton if it satisfies

- (1) $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -2)) = 0$, $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 0)) = 0$;
- (2) $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, -1)) = 0$, $H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 3-n)) = 0$;
- (3) $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 2-n)) = 0$, $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 4-n)) = 0$;
- (4) for $n \geq 4$, $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -3)) = 0 = H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 5-n))$ and $H^i(\widetilde{\mathbb{P}^n}, \mathcal{E} \otimes \mathcal{L}') = 0$, for all $2 \leq i \leq n-2$ and all invertible sheaves $\mathcal{L}'$ in $\widetilde{\mathbb{P}^n}$.

Note that the above definition requires n to be odd. This is because, as shown by Henni [2], the definition does not work for even n . Therefore, changes to assumptions must be made to achieve a complete definition. In addition, for the case where the dimension n is odd, in [2], the following lemma is stated.

Lemma 3.2. *An instanton sheaf as the Definition 3.1 is the cohomology of a monad*

$$\mathbf{M}: 0 \rightarrow \mathcal{C}^{-1} \rightarrow \mathcal{C}^0 \rightarrow \mathcal{C}^1 \rightarrow 0,$$

where

$$\begin{aligned} \mathcal{C}^1 = & H^1(\mathcal{F}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}} \oplus H^{n-1}(\mathcal{F}(0, 2-n)) \otimes \Omega^1(0, n-2) \oplus H^{n-1}(\mathcal{F}(-1, 4-n)) \otimes \Omega^{n-3}(-1, n-3) \\ & \oplus H^n(\mathcal{F}(0, 1-n)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1) \oplus H^n(\mathcal{F}(-1, 3-n)) \otimes \Omega^{n-2}(-1, n-2); \end{aligned} \quad (3.0.1)$$

$$\begin{aligned} \mathcal{C}^0 = & H^0(\mathcal{F}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}} \oplus H^1(\mathcal{F}(0, -1)) \otimes \Omega^1(0, 1) \oplus H^1(\mathcal{F}(-1, 1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0) \\ & \oplus H^{n-1}(\mathcal{F}(0, 1-n)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1) \oplus H^{n-1}(\mathcal{F}(-1, 3-n)) \otimes \Omega^{n-2}(-1, n-2) \\ & \oplus H^n(\mathcal{F}(-1, 2-n)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1) \end{aligned} \quad (3.0.2)$$

$$\begin{aligned} \mathcal{C}^{-1} = & H^0(\mathcal{F}(0, -1)) \otimes \Omega^1(0, 1) \oplus H^0(\mathcal{F}(-1, 1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0) \oplus H^1(\mathcal{F}(0, -2)) \otimes \Omega^2(0, -2) \\ & \oplus H^1(\mathcal{F}(-1, 0)) \otimes \Omega^1(-1, 1) \oplus H^{n-1}(\mathcal{F}(-1, 2-n)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1). \end{aligned} \quad (3.0.3)$$

Proof. The demonstration can be found in [2]. ■

We want the changes defined in the even case n to not interfere with the Lemma 3.2. Thus, the instantons given by the new definition continue to be monad cohomology, as described in the lemma.

The main objective of this chapter is to present a generalization of Definition 3.1, that is, to define sheaf instantons when the dimension of the projective space is even. For this aim, we will first analyze the proposed definition and then show examples that verify the extension of the definition to the even case. Furthermore, we will analyze some properties of the families found.

3.1 Analysis of the proposed definition.

In this section we will propose and analyze a definition for the case of the dimension n of the projective space being even.

Definition 3.3. On $(\widetilde{\mathbb{P}^n}, \mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, N_n))$ for n even with

$$N_n = \begin{cases} 1 & \text{if } n = 4; \\ \frac{n-2}{2} & \text{if } n > 4. \end{cases}$$

a $\mu_{\mathcal{L}}$ -semi-stable rank r torsion-free coherent sheaf $\mathcal{E}$, with Chern class

$$c_1(\mathcal{E}) = -\alpha$$

will be called instanton if it satisfies

- (1) $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -2)) = 0$, $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 0)) = 0$;
- (2) $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, -1)) = 0$, $H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 3-n)) = 0$;
- (3) $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 2-n)) = 0$, $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 4-n)) = 0$;
- (4) for $n \geq 4$, $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -3)) = 0 = H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 5-n))$ and $H^i(\widetilde{\mathbb{P}^n}, \mathcal{E} \otimes \mathcal{L}') = 0$, for all $2 \leq i \leq n-2$ and all invertible sheaves $\mathcal{L}'$ in $\widetilde{\mathbb{P}^n}$.

If we compare Definitions 3.1 and 3.3 we can notice that the cohomological conditions are the same. This follows from Lemma 3.2, since in complex (2.1.4) the terms C^{-2} and C^2 must be zero. Considering that $s = \{0, \dots, n\}$, $h = \{0, 1\}$, and $q = \{0, \dots, n-1\}$, and from the definition of each term of the complex $i = s - p$ and $q + h = i$, we have the following table.

		$s = 0$	$s = 1$	$s = 2$	$\dots$	$s = n-2$	$s = n-1$	$s = n$
C^{-2}	(h, q)	$(0, 2)$ $(1, 1)$	$(0, 3)$ $(1, 2)$	$(1, 3)$	$\dots$	$(1, n-1)$	---	---
C^2	(h, q)	---	---	$(0, 1)$ $(1, 0)$	$\dots$	$(0, n-4)$ $(1, n-5)$	$(0, n-3)$ $(1, n-4)$	$(0, n-2)$ $(1, n-3)$

Table 3.1.1: Possibilities of combining parameters s , h and q in the complex. Font: Created by the author.

Substituting the possible combinations from the table 3.1 into the equation (2.1.4) we get the terms C^{-2} and C^2 as follows.

$$\begin{aligned}
 C^{-2} = & H^0(\mathcal{E}(0, -2)) \otimes \Omega^2(0, 2) \oplus H^0(\mathcal{E}(-1, 0)) \otimes \Omega^1(-1, 1) \\
 & \oplus H^1(\mathcal{E}(0, -3)) \otimes \Omega^3(0, 3) \oplus H^1(\mathcal{E}(-1, -1)) \otimes \Omega^2(-1, 2) \\
 & \oplus H^2(\mathcal{E}(-1, -1)) \otimes \Omega^3(-1, 3) \\
 & \vdots \\
 & \oplus H^{n-2}(\mathcal{E}(-1, -n+2)) \otimes \Omega^{n-1}(-1, n-1);
 \end{aligned} \tag{3.1.1}$$

$$\begin{aligned}
C^2 = & H^2(\mathcal{E}) \otimes \Omega^0 \\
& \oplus H^3(\mathcal{E}(0, -1)) \otimes \Omega^1(0, 1) \oplus H^3(\mathcal{E}(-1, 1)) \otimes \Omega^0(-1, 0) \\
& \vdots \\
& \oplus H^n(\mathcal{E}(0, -n+2)) \otimes \Omega^{n-2}(0, n-2) \oplus H^n(\mathcal{E}(-1, -n+4)) \otimes \Omega^{n-3}(-1, n-3).
\end{aligned} \tag{3.1.2}$$

Comparing the cohomologies that appear in the sums and the cohomology conditions of Definition 3.1 we see that they do not change regardless of whether n is even or odd. Note also that the cohomological conditions seem to be independent of the polarization. However, the choice of polarization interferes with semi-stability. For the first Chern class, we have

$$\det(\mathcal{E}) = \mathcal{L}^{\otimes 2} \otimes \omega_{\widetilde{\mathbb{P}^n}} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1).$$

As a result, $c_1(\mathcal{E}) = -\alpha$. This relation between the first Chern class and the polarization implies that we can consider as cohomological conditions $H^i(\mathcal{E} \otimes \mathcal{L}^{-1}) = 0$ for all i . This holds when $\mathcal{E}$ is a rank 2 locally free sheaf, as mentioned in [2].

Recently, Antonelli and Casnati [6] defined a class of sheaves on a projective scheme X with respect to an ample and globally generated line bundle $\mathcal{O}_X(h)$ as follows.

Definition 3.4. [6, Definition 1.3 and Theorem 1.4] Let X be an irreducible projective scheme of dimension n ($n \geq 1$) endowed with an ample and globally generated line bundle $\mathcal{O}_X(h)$. If $\mathcal{E}$ is a coherent sheaf on X , k a non-negative integer and $\delta = \{0, 1\}$ then $\mathcal{E}$ is called an h-instanton sheaf if the following assertions hold:

- (1) $h^0(\mathcal{E}(-h)) = h^n(\mathcal{E}(\delta - n)h) = 0$;
- (2) $h^i(\mathcal{E}(-(i+1)h)) = h^{n-i}(\mathcal{E}((\delta - n + i)h)) = 0$ if $1 \leq i \leq n-2$;
- (3) $\delta h^i(\mathcal{E}(-ih)) = 0$ for $2 \leq i \leq n-2$;
- (4) $h^1(\mathcal{E}(-h)) = h^{n-1}(\mathcal{E}((\delta - n)h)) = k$;
- (5) $\delta(\chi(E) - (-1)^n \chi(\mathcal{E}(-nh))) = 0$.

Note that Definition 3.4 does not require stability for an h-instanton sheaf. Furthermore, when $k = 0 = \delta$ the h-instanton sheaf $\mathcal{E}$ is Ulrich. An h-instanton sheaf $\mathcal{E}$ with $\delta = 0$ is called an

ordinary instanton, and $\mathcal{E}$ is called non-ordinary if $\delta = 1$. Considering an instanton given by the generalized definition composed of 3.1 and 3.3, we then find that the parameter k of Definition 3.4 is null. Thus, the Ulrich instantons will then be semi-stable ordinary h-instantons.

Next, we will study families of sheaf instantons that verify this definition. We will first consider an example in the case $n = 4$, for better understanding, and then generalize it to a family of sheaf instantons for any even n . Later, we will present another example that focuses on generalization.

3.2 First example for $n = 4$

Consider in $\widetilde{\mathbb{P}^4}$ the subvarieties of codimensions 2 denoted by $\wp$ and κ . Here, $\wp$ is the pull-back of a blow-down π of a linear space that does not contain the blow-up point, and κ is the linear space contained in the exceptional divisor. Each of these subvarieties has resolution given by the sequences

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-2, 0) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)^{\oplus 2} \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}} \rightarrow \mathcal{O}_{\wp} \rightarrow 0. \quad (3.2.1)$$

and

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1) \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1) \rightarrow \mathcal{O}_{\kappa} \rightarrow 0. \quad (3.2.2)$$

From these two resolutions, we can obtain important information that we will use later. Among them, the second Chern class given by $c_2(\mathcal{O}_{\wp}) = -\xi^2$ and $c_2(\mathcal{O}_{\kappa}) = \alpha^2 - \xi^2$. The determinants of its normal bundle in $\wp$ and κ are $\det \mathcal{N}_{\wp} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and $\det \mathcal{N}_{\kappa} \cong \mathcal{O}_{\mathbb{P}^2}$, respectively. Define X by the disjoint union of $\wp$ and κ . Next, we will construct an instanton of rank 2 using the variety X and the tools presented in the previous sections. Note that

$$\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(p, q) \cong \mathcal{O}_{\mathbb{P}^2}(p + q) \oplus \mathcal{O}_{\mathbb{P}^2}(q) \quad (3.2.3)$$

because doing the intersections in each part of X we have

$$[\wp](p\xi + q\alpha) = \xi^2(p\xi + q\alpha) = (p + q)\xi^3;$$

$$[\kappa](p\xi + q\alpha) = (\xi^2 - \alpha^2)(p\xi + q\alpha) = q\xi^3.$$

This specific variety was chosen because it satisfies the Hartshorne-Serre Correspondence hypothesis and is a disjoint union.

3.2.1 Construction of the instanton via Hartshorne-Serre Correspondence.

We want to find the sheaf using the Hartshorne-Serre Correspondence. First, we need to find the line bundle S that fits all the required hypotheses. Afterwards, we must check if the constructed sheaf is an instanton.

In the initial studies, the choice of S was made through several attempts until the correct one was found. However, we later realized that analyzing the necessary hypotheses more generally considerably reduced the number of choices for S . This may seem unfeasible since Definition 3.3 presents several cohomological conditions. However, we managed to reduce these hypotheses to three. Because of this, the S found does not ensure the construction of an instanton, but those that do not fit the possibilities certainly will not. The following result shows which S we could choose for X on $\widetilde{\mathbb{P}^4}$.

Proposition 3.5. *In $\widetilde{\mathbb{P}^4}$ and for $X = \wp \cup \kappa$, there is one single possibility for S to construct instantons of rank 2 via Hartshorne-Serre Correspondence: $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(2, -1)$.*

Proof. Consider $S = \mathcal{O}_{\widetilde{\mathbb{P}^4}}(p, q)$ a line bundle on $\widetilde{\mathbb{P}^4}$. We now analyze the hypotheses of the correspondence and the properties of instantons to determine the possible values of p and q . First, note that $H^2(S^*)$ is always zero in this space. Furthermore, considering that the determinant of the normal in $\wp$ and κ is given by $\det \mathcal{N}_{\wp} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and $\det \mathcal{N}_{\kappa} \cong \mathcal{O}_{\mathbb{P}^2}$ it follows that

$$\det \mathcal{N}_X \otimes S^*|_X \cong \begin{cases} \mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathbb{P}^2}(-p-q) & \text{in } \wp; \\ \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-q) & \text{in } \kappa. \end{cases}$$

Since we want $\det \mathcal{N}_X \otimes S^*|_X$ to have sections, we need $H^0(\mathbb{P}^2, \det \mathcal{N}_X \otimes S^*|_X)$ to be non-

zero. Therefore, the line bundle must be the following form $\mathcal{O}_{\mathbb{P}^2}(a)$ with $a \geq 0$. We thus obtain

$$\begin{cases} -p - q \geq -2; \\ -q \geq 0. \end{cases} \quad (3.2.4)$$

With the above inequality satisfied, we obtain the existence of a sheaf $\mathcal{F}$ given by the sequence

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}} \rightarrow \mathcal{F} \rightarrow I_X(p, q) \rightarrow 0. \quad (3.2.5)$$

Previously, we only analyzed the correspondence. Let us now use the hypotheses of Definition 3.3. We know that, by definition of instantons for even n , the first Chern class must be $-\alpha$, so if we twist the sequence by $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(a, b)$ we obtain that

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(a, b) \rightarrow \mathcal{E} \rightarrow I_X(p + a, q + b) \rightarrow 0 \quad (3.2.6)$$

and

$$\begin{cases} p \text{ even, } q \text{ odd;} \\ a = -\frac{p}{2}; \\ b = -\frac{q+1}{2}. \end{cases} \quad (3.2.7)$$

Consequently, for an instanton to be in an exact sequence, the latter should be of the following form

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}\left(-\frac{p}{2}, -\frac{q+1}{2}\right) \rightarrow \mathcal{E} \rightarrow I_X\left(\frac{p}{2}, \frac{q-1}{2}\right) \rightarrow 0. \quad (3.2.8)$$

Furthermore, again by the definition of instantons, we need $H^3(\mathcal{E}(0, -1))$ to be zero. Note that,

$$\cdots \rightarrow 0 \rightarrow H^3\left(\mathcal{O}_{\widetilde{\mathbb{P}^4}}\left(\frac{-p}{2}, \frac{-q-1}{2} - 1\right)\right) \rightarrow H^3(\mathcal{E}(0, -1)) \rightarrow H^3\left(I_X\left(\frac{p}{2}, \frac{q-1}{2} - 1\right)\right) \rightarrow 0 \rightarrow \cdots (3.2.9)$$

in this way

- $H^3 \left(\mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{-p}{2}, \frac{-q-1}{2} - 1 \right) \right) = 0$: using Bott's formulae we have

$$h^3 \left(\mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{-p}{2}, \frac{-q-1}{2} - 1 \right) \right) = \begin{cases} \sum_{k=0}^{\frac{-p}{2}} \binom{\frac{q+1}{2} - k}{3} & \text{if } p \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and, for $p \leq 0$ the above binomial will be non-zero when $5 \leq q$. So, we obtain the inequality

$$\text{for } p \leq 0 \text{ need } q \leq 3. \quad (3.2.10)$$

- $H^3 \left(I_X \left(\frac{p}{2}, \frac{q-1}{2} - 1 \right) \right) = 0$: in the restriction sequence we have the cohomologies

$$\begin{aligned} \cdots \rightarrow 0 \rightarrow H^2 \left(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{p}{2}, \frac{q-1}{2} - 1 \right) \right) &\rightarrow H^3 \left(I_X \left(\frac{p}{2}, \frac{q-1}{2} - 1 \right) \right) \rightarrow \\ &\rightarrow H^3 \left(\mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{p}{2}, \frac{q-1}{2} - 1 \right) \right) \rightarrow 0 \rightarrow \cdots \end{aligned}$$

therefore both ends must have zero cohomologies. Similarly to the reasoning of the previous item, we obtain $h^3 \left(\mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{p}{2}, \frac{q-1}{2} - 1 \right) \right) = 0$ if $p < 0$ or $p \geq 0$ with $-q \leq 3$. On the other hand

$$\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{p}{2}, \frac{q-1}{2} - 1 \right) \cong \mathcal{O}_{\mathbb{P}^2} \left(\frac{p}{2} + \frac{q-1}{2} - 1 \right) \oplus \mathcal{O}_{\mathbb{P}^2} \left(\frac{q-1}{2} - 1 \right)$$

which will have zero cohomology in h^2 if each of the elements inside the parentheses is greater than or equal to -2 , so then we obtain other important inequalities

$$p + q \geq -1; \quad q \geq -1.$$

Finally, since we want the bundle to be semi-stable with respect to polarization, in particular, we need $H^0(\mathcal{E})$ to be zero. Therefore, $H^0 \left(\mathcal{O}_{\widetilde{\mathbb{P}^4}} \left(\frac{-p}{2}, \frac{-q-1}{2} \right) \right)$ must be zero as well.

Similarly, we obtain

$$p > 0 \text{ or } p \leq 0 \text{ with } 0 \leq p + q.$$

We obtain the following inequalities

$$\left\{ \begin{array}{l} p \leq 0 \\ -1 \leq q \leq 3 \\ 0 \leq p + q \end{array} \right. \cup \left\{ \begin{array}{l} p > 0 \\ p + q \geq -1 \\ q \geq -1. \end{array} \right. \quad (3.2.11)$$

Intersecting the inequalities (3.2.4) and (3.2.11), and recall that p is even and q is odd, we find that S can only be the line bundle $\mathcal{O}_{\mathbb{P}^4}^{\sim}(2, -1)$. ■

Substituting the values of p and q in the sequence (3.2.8) found, we obtain a sheaf $\mathcal{E}$ given by the following exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}^{\sim}(-1, 0) \rightarrow \mathcal{E} \rightarrow I_X(1, -1) \rightarrow 0. \quad (3.2.12)$$

Recall that this possibility of choosing S can still give us a sheaf $\mathcal{E}$ with a sequence that does not satisfy any of the other hypotheses of the instanton definition. However, now we only have this choice to analyze instead of testing randomly.

Note also that we do not say that $\mathcal{E}$ is a bundle but rather a sheaf, since the correspondence guarantees a bundle when the sequence is of the form (2.1.1) with $r - 1$ copies of $\mathcal{O}_{\mathbb{P}^4}^{\sim}$. However, we can ensure the existence of the sequence by taking only one copy of the line bundle to obtain

a sheaf of rank 2. With these two sequences, we can create the following diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} & \xlongequal{\quad} & \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}^{\oplus 6} & \longrightarrow & \mathcal{F}' & \longrightarrow & I_X(2, -1) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}^{\sim} & \longrightarrow & \mathcal{F} & \longrightarrow & I_X(2, -1) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

We then have that $\mathcal{F}$ is the cokernel of the exact sequence in the middle of the diagram. Although $\mathcal{F}'$ is locally free by the Hartshorne-Serre Correspondence, we cannot say the same for $\mathcal{F}$ since it is the cokernel. Furthermore, since $\mathcal{E}$ and $\mathcal{F}$ are cohomologically equal, this is also true for $\mathcal{E}$. Later, we will show that $\mathcal{E}$ is in fact not locally free.

3.2.2 Construction of the instanton via spectral sequence.

In the previous section, we found a sheaf using Hartshorne-Serre Correspondence. In this section, we will give another demonstration of the existence of the sequence (3.2.12) using a spectral sequence. For this aim, the steps done in [14] will be followed as shown below.

Let us consider

$$E_2^{p,q} = H^q(\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) = \text{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$$

and

$$E_n = \text{Ext}^n(\mathcal{F}, \mathcal{G}).$$

In particular, considering also $\mathcal{F} = I_X(2, -1)$, $\mathcal{G} = \mathcal{O}_{\mathbb{P}^4}^{\sim}$ and $S = \mathcal{O}_{\mathbb{P}^4}^{\sim}(2, -1)$, with the above sheaf, we can construct the following exact sequence

$$0 \rightarrow E_2^{1,0} \rightarrow E_1 \rightarrow E_2^{0,1} \rightarrow E_2^{2,0}.$$

Substituting each notation we obtain the following sequence

$$0 \rightarrow H^1 \left(\mathcal{E}xt^0(\mathcal{F}, \mathcal{G}) \right) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{G}) \rightarrow H^0 \left(\mathcal{E}xt^1(\mathcal{F}, \mathcal{G}) \right) \rightarrow H^2 \left(\mathcal{E}xt^0(\mathcal{F}, \mathcal{G}) \right) \quad (3.2.13)$$

Note that if we show that $\text{Ext}^1(\mathcal{F}, \mathcal{G})$ is non-zero, then the existence of the sequence is proven again. To do this, we will use the results for groups $\mathcal{E}xt$ in [18, Section III.6] and the restriction sequence (2.1.2). In this way, it will be possible to calculate the terms we need from the spectral sequence.

$$\begin{aligned} H^1 \left(\mathcal{E}xt^0(I_X(2, -1), \mathcal{O}_{\mathbb{P}^4}^\sim) \right) &\cong H^1 \left(\mathcal{H}om(I_X(2, -1), \mathcal{O}_{\mathbb{P}^4}^\sim) \right) \\ &= H^1(I_X(2, -1)^\vee) \\ &\cong H^1(\mathcal{O}_{\mathbb{P}^4}^\sim(-2, 1)) \\ &\cong H^3(\mathcal{O}_{\mathbb{P}^4}^\sim(0, -4)) \cong \mathbb{C}; \end{aligned} \quad (3.2.14)$$

and

$$\begin{aligned} H^0 \left(\mathcal{E}xt^1(I_X(2, -1), \mathcal{O}_{\mathbb{P}^4}^\sim) \right) &\cong H^0 \left(\mathcal{E}xt^2(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^4}^\sim) \right) \\ &\cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) \cong \mathbb{C}^{\oplus 6}. \end{aligned} \quad (3.2.15)$$

Consequently,

$$\text{Ext}^1(I_X(2, -1), \mathcal{O}_{\mathbb{P}^4}^\sim) \neq 0$$

which implies that there are sequences of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}^\sim \rightarrow \mathcal{E}^l \rightarrow I_X(2, -1) \rightarrow 0. \quad (3.2.16)$$

Twisting the above sequence by $\mathcal{O}_{\mathbb{P}^4}^\sim(-1, 0)$ we obtain again the sequence (3.2.12).

3.2.3 The sheaf $\mathcal{E}$ verifies Definition 3.3.

In this section, we will show that the assumptions of Definition 3.3 are satisfied by $\mathcal{E}$ for $n = 4$. It should be noted that, due to the construction itself, it is already ensured that $c_1(\mathcal{E}) = -\alpha$.

For cohomological conditions, we will calculate the cohomology table of the beam using the full exceptional sequence (2.1.10). Each of the cohomological conditions required in the definition will appear in the table.

Proposition 3.6. *The sheaf $\mathcal{E}$ presents the table 3.2.1 of cohomologies.*

$\mathcal{E}(-1, -2)$	$\mathcal{E}(-1, -1)$	$\mathcal{E}(-1, 0)$	$\mathcal{E}(-1, 1)$	$\mathcal{E}(0, -3)$	$\mathcal{E}(0, -2)$	$\mathcal{E}(0, -1)$	$\mathcal{E}$	
0	0	0	0	0	0	0	0	h^0
0	0	0	2	0	0	0	0	h^1
0	0	0	0	0	0	0	0	h^2
2	0	0	0	5	1	0	0	h^3
0	0	0	0	0	0	0	0	h^4

Table 3.2.1: Cohomological dimensions of $\mathcal{E}$ in $\widetilde{\mathbb{P}^4}$. Font: Created by the author.

Furthermore, $H^i(\widetilde{\mathbb{P}^n}, \mathcal{E} \otimes \mathcal{L}') = 0$, for $i = 2$ and all invertible sheaves $\mathcal{L}'$ in $\widetilde{\mathbb{P}^4}$.

Proof. To calculate each of the cohomologies in the table 3.2.1, we will use the sequence (3.2.12) twisted by each of the corresponding line bundles. In the case of cohomologies of $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(p, q)$, just apply the equation (2.1.5) and in the case of $I_X(1, -1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(p, q)$, we must consider the restriction sequence (2.1.2) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(p, q)$ and the equation (3.2.3). For example, for $\mathcal{E}(-1, -2)$ the sequence (3.2.12) is as follows

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-2, -2) \rightarrow \mathcal{E}(-1, -2) \rightarrow I_X(0, -3) \rightarrow 0. \quad (3.2.17)$$

By (2.1.5) and (2.1.6) it follows that

$$h^i(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-2, -2)) \cong h^{n-i}(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)) = 0; \forall i \in \{0, \dots, 4\}.$$

On the other hand, we have to calculate the cohomologies of the restriction sequence (2.1.2) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -3)$. In this way, the followed cohomology exact sequence is obtained

$$0 \rightarrow \overbrace{H^2(\mathcal{O}_{\mathbb{P}^2}(-3))^{\oplus 2}}^{\mathbb{C}^{\oplus 2}} \rightarrow H^3(I_X(0, -3)) \rightarrow \overbrace{H^3(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -3))}^0 \rightarrow 0,$$

because

$$h^i(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -3)) = h^i(\mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2}) = \begin{cases} 2 & \text{if } i = 2; \\ 0 & \text{otherwise.} \end{cases}$$

We then obtain the dimension cohomologies of $I_X(0, -3)$ given by

$$h^i(I_X(0, -3)) = \begin{cases} 2 & \text{if } i = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Considering that the cohomologies of $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-2, -2)$ are all zero, we find that the cohomological dimensions of $\mathcal{E}(-1, -2)$ and those related to $I_X(0, -3)$ are equal. Thus, we obtain the corresponding column. For the other columns, the calculations are done analogously and, therefore, are not presented here. ■

We can observe from the table that the cohomologies required in Definition 3.3 are satisfied. Furthermore, since the polarization for $n = 4$ is $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 1)$, we have $H^i(\mathcal{E} \otimes \mathcal{L}^{-1}) = 0$ for $0 \leq i \leq 4$. Let us now show whether the sheaf is semi-stable.

Proposition 3.7. *The sheaf $\mathcal{E}$ constructed before is semi-stable with respect to polarization $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 1)$ in $\widetilde{\mathbb{P}^4}$.*

Proof. To demonstrate the semi-stability, we will use the Hoppe Criterion 2.4 where

$$\mu_{\mathcal{L}}(\mathcal{E}) = \frac{-7}{2} \tag{3.2.18}$$

and

$$\deg(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(p, q)) = 8p + 7q. \tag{3.2.19}$$

Therefore, we need to prove that

$$H^0(\mathcal{E}(-p, -q)) = 0, \quad \forall \quad -q < \frac{1}{2} + \frac{8}{7}p. \tag{3.2.20}$$

Let us then consider the sequence (3.2.12) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-p, -q)$ with p and q satisfying

the inequality in (3.2.20). Obtaining the following sequence

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1-p, -q) \rightarrow \mathcal{E}(-p, -q) \rightarrow I_X(1-p, -1-q) \rightarrow 0. \quad (3.2.21)$$

We will start with the cohomologies of $I_X(1-p, -1-q)$ because the inclusion $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1-p, -q) \hookrightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1-p, -1-q)$ gives us

$$h^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1-p, -q)) \leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1-p, -1-q)). \quad (3.2.22)$$

For this aim and for better understanding, we divide the demonstration into two cases. It should be noted that, in each case, the coefficient of ξ , $1-p$, can be negative or not. For each of these situations, there are different ways to analyze cohomologies, which will be discussed below.

- For $1-p < 0$, we have

$$\begin{aligned} h^0(I_X(1-p, -1-q)) &\leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1-p, -1-q)) \cong h^4(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-3+p, -2+q)) \\ &= \sum_{k=0}^{1-p} \binom{2-q+k}{3} = 0 \end{aligned}$$

because the sum above goes from $k = 0$ to $1-p < 0$. This "upper" limit is a negative number, so the sum is zero. As result, we have $H^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1-p, -q)) = 0$. Hence $H^0(\mathcal{E}(-p, -q)) = 0$ in this case.

- For $1-p \geq 0$, we have $1-p-1-q < \frac{1}{2} + \frac{1}{7}p$ and the latter is less than zero when $p \leq -4$. So, for these cases, $p \leq -4$, $H^0(I_X(1-p, -1-q)) = 0$. Now, for the others $-3 \leq p \leq 1$, we have

$$h^0(I_X(1-p, -1-q)) \leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1-p, -1-q)) = \sum_{k=0}^{1-p} \binom{2-q+k}{3}.$$

The only cases it is worth noting that the binomial is not zero are when $-q = p$. But in these cases, we have that the cohomologies of $I_X(1-p, -(1-p))$ are null since X is not contained in the exceptional divisor. In these cases, we cannot use the inequality (3.2.22) to calculate the cohomological dimension of the other side of the sequence. Thus, we

must calculate it directly for $1 - p \leq 0$. Note that, in this case, $-1 - p \geq -2$.

$$h^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1-p, -q)) \begin{cases} \cong h^4(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1+p, -3+q)) = 0 & \text{if } -2 \leq -1-p < 0 \\ = \sum_{k=0}^{-1-p} \binom{3-q+k}{3} = 0 & \text{if } -1-p \geq 0. \end{cases}$$

The second summary is in fact null because

$$3 \leq 3 - q + k \leq 3 - q - 1 - p < \frac{5}{2} + \frac{1}{7}p \Rightarrow \frac{5}{2} + \frac{1}{7}p > 3 \Rightarrow p \geq 4$$

and in this case, $p \leq 1$. Consequently, we obtain $H^0(\mathcal{E}(-p, -q)) = 0$.

■

Since the sheaf $\mathcal{E}$ is part of an extension where the other terms are coherent and torsion-free, it follows that it also presents these properties. However, it will not be locally free as in the example found by Henni in [2] for $n = 5$. Unfortunately, for the case of even n , it is difficult to find locally free instanton sheaves. For each constructed example, we will show that they are not locally free.

Recall that in Definition 3.3, we made it as a condition that each instanton be the cohomology of a monad given by Lemma 3.2. This condition is satisfied, as shown below.

Corollary 3.8. *The instantons $\mathcal{E}$ is the cohomology of a monad*

$$\begin{array}{c} \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)^{\oplus 2} \\ \text{M: } 0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)^{\oplus 2} \xrightarrow{\alpha} \oplus \xrightarrow{\beta} \Omega^2(0, 2) \rightarrow 0. \quad (3.2.23) \\ \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 5} \end{array}$$

Proof. The verification that the monad is given in this form follows from Lemma 3.2 and the

cohomologies shown in table 3.2.1 referring to Proposition 3.6.

$$\begin{aligned}
 C^{-1} &= \overbrace{H^3(\mathcal{E}(-1, -2))}^{\mathbb{C}^{\oplus 2}} \otimes \Omega_{\mathbb{P}^3}^3(-1, 3) \cong \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)^{\oplus 2} \\
 C^0 &= \overbrace{H^1(\mathcal{E}(-1, 1))}^{\mathbb{C}^{\oplus 2}} \otimes \Omega_{\mathbb{P}^3}^0(-1, 0) \oplus \overbrace{H^3(\mathcal{E}(0, -3))}^{\mathbb{C}^{\oplus 5}} \otimes \Omega_{\mathbb{P}^3}^3(0, -3) \\
 &\cong \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 5} \\
 C^1 &= \overbrace{H^3(\mathcal{E}(0, -2))}^{\mathbb{C}} \otimes \Omega_{\mathbb{P}^3}^2(0, 2)
 \end{aligned}$$

Note that properties of $\mathcal{E}$ are maintained. For example, the rank

$$\text{rk}(\mathcal{E}) = \text{rk}(\mathcal{O}(-1, 0)^{\oplus 2} \oplus \mathcal{O}(0, -1)^{\oplus 5}) - \text{rk}(\mathcal{O}(-1, -1)^{\oplus 2}) - \text{rk}(\Omega_{\mathbb{P}^3}^2(0, 2)) = 7 - 2 - 3 = 2$$

and the Chern classes

$$c(\mathcal{E}) = \frac{c(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 5})}{c(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)^{\oplus 2})c(\Omega_{\mathbb{P}^3}^2(0, 2))} = 1 - \alpha + 2\xi^2 - \alpha^2 + \dots$$

To find the rank and the Chern classes of the $\Omega^i(p, q)$ just use the sequence (2.1.8) twisted by the corresponding line bundle.

■

This example verifies the proposed definition for the case $n = 4$. In the following sections, we will study the properties of this specific example and then generalize it to any even n . We recommend doing at least one example for $n = 4$ for a better understanding of the resented results. However, for more general examples, the results are similar. Thus, the second family of instanton sheaves will present more generic results.

3.2.4 Properties of instanton sheaf.

In this section, we will analyze some properties of the instanton sheaf $\mathcal{E}$ in $\widetilde{\mathbb{P}^4}$.

Proposition 3.9. *The second Chern class is $c_2(\mathcal{E}) = 2\xi^2 - \alpha^2$ and its charge $c = 5$.*

Proof. To calculate the Chern classes it is necessary to use the $\mathcal{E}$ sequence. In that manner, we

obtain

$$c(\mathcal{E}) = c(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0))c(I_X(1, -1)) = 1 - \alpha + 2\xi^2 - \alpha^2 + \dots$$

Using the Chern classes of $\mathcal{E}$ and the polarization $\mathcal{L}$ we can calculate the charge as follows

$$c_2(\mathcal{E})c_1(\mathcal{L})^2 = (2\xi^2 - \alpha^2)(\xi + \alpha)^2 = 5\xi^4.$$

■

Although the ideal would be for the instanton sheaf $\mathcal{E}$ to be locally free, unfortunately this does not happen because $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}})$ is not null.

Proposition 3.10. *The instanton sheaf $\mathcal{E}$ is not locally free.*

Proof. To verify this, let us consider the $\mathcal{H}om(-, \mathcal{O}_{\widetilde{\mathbb{P}^4}})$ sequence of (3.2.12).

$$\begin{aligned} \text{I: } 0 &\rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1) \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 0) \\ &\rightarrow \mathcal{E}xt^1(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^1(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\rightarrow \mathcal{E}xt^2(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^2(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^2(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\rightarrow \dots \end{aligned}$$

Now we will use the properties of the Ext group presented in [18, Section III.6] to calculate the dimension of each element of the previous sequence. First, we have to

$$\mathcal{E}xt^i(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \cong \mathcal{E}xt^i(\mathcal{O}_{\widetilde{\mathbb{P}^4}}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 0)) = 0, \forall i > 0. \quad (3.2.24)$$

For the others $\mathcal{E}xt$ groups, we need to take the restriction sequence (2.1.2) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1)$ and do $\mathcal{H}om(-, \mathcal{O}_{\widetilde{\mathbb{P}^4}})$. We then obtain the following sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{H}om(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1) \rightarrow \mathcal{H}om(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\rightarrow \mathcal{E}xt^1(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^1(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^1(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\rightarrow \mathcal{E}xt^2(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^2(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^2(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\rightarrow \mathcal{E}xt^3(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^3(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow \mathcal{E}xt^3(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\rightarrow \dots \end{aligned}$$

Since X is of codimension 2 it follows that $\mathcal{E}xt^i(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) = 0$ for $i \neq 2$ and $\mathcal{E}xt^i(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) = 0$ for $i > 0$. Therefore,

$$\begin{aligned} \mathcal{E}xt^1(I_X(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) &\cong \mathcal{E}xt^2(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \\ &\cong \mathcal{E}xt^2(\mathcal{O}_X, \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1) \\ &\cong \omega_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1) \otimes \omega_{\widetilde{\mathbb{P}^4}}^{-1} \\ &\cong \omega_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 4). \end{aligned}$$

Substituting the results obtained above into sequence I, we have

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 1) \rightarrow \mathcal{E}^\vee \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 0) \xrightarrow{f} \omega_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 4) \rightarrow \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \rightarrow 0. \quad (3.2.25)$$

If f is not a surjective map, we find that $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}})$ is not null, as we had claimed before. However,

$$\begin{aligned} f \in \text{Hom}(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 0), \omega_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 4)) &\cong H^0(\omega_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, 4)) \\ &\cong H^0((\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(4))^{\oplus 2}) = H^0((\mathcal{O}_{\mathbb{P}^2}(1))^{\oplus 2}) \\ &\cong \mathbb{C}^{\oplus 6} \end{aligned}$$

and in the diagram

$$\begin{array}{ccc} \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 0) & \xrightarrow{f} & \omega_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 4) \cong \mathcal{O}_{\wp}(2) \oplus \mathcal{O}_{\mathbf{K}}(1) \\ r_X \downarrow & \nearrow \bar{f} & \\ \mathcal{O}_{\wp}(1) \oplus \mathcal{O}_{\mathbf{K}} & & \end{array}$$

we find that $\bar{f}$ is not surjective. Therefore, f can also not be surjective, and consequently $\mathcal{E}$ is not locally free. ■

Now we want to know if the instanton restrictions on the pull-back of the flag elements $L = H_3 \subset H_2 \subset H_1 = H$ in $\mathbb{P}^4$ are split. To do this, we will use the sequence (2.3.3) with $i = 1, 2, 3$.

Note that, $c_1(\mathcal{E}|_{H_i}) = c_1(\mathcal{E}|_H)$. Then it is sufficient to calculate the first Chern class

for $\mathcal{E}|_H$ in the sequence $0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}} \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}|_H \rightarrow 0$ twisted by $\mathcal{E}$. Using $c(\mathcal{E}) = 1 - \alpha + 2\xi^2 - \alpha^2 + \dots$ and the properties of Chern classes, we obtain that

$$c(\mathcal{E} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)) = 1 - 2\xi - \alpha + 4\xi^2 - \alpha^2 + \dots$$

and so on

$$\begin{aligned} c(\mathcal{E}|_H) &= \frac{c(\mathcal{E})}{c(\mathcal{E}(-1, 0))} = \frac{1 - \alpha + 2\xi^2 - \alpha^2 + \dots}{1 - 2\xi - \alpha + 4\xi^2 - \alpha^2 + \dots} \\ &= \left(1 - \alpha + 2\xi^2 - \alpha^2 + \dots\right) \left(1 + 2\xi + \alpha - 4\xi^2 + \alpha^2 + 8\xi^2 + \alpha^4 + \dots\right) \\ &= 1 + 2\xi + 4\xi^2 + \dots \end{aligned}$$

With this result, we find the Chern classes of the restriction on $\widetilde{\mathbb{P}^4}$. However, it must be on $\mathbb{P}^3$. For the aim, we will use the Proposition 2.5 where $i: \mathbb{P}^3 \rightarrow \widetilde{\mathbb{P}^4}$ and the $\mathcal{E}$ rank is

$$r = \text{rk} \mathcal{E}|_H = \begin{cases} 0 & \text{in } \widetilde{\mathbb{P}^4}; \\ 2 & \text{in } \mathbb{P}^3. \end{cases}$$

In addition, from the sequence $T_{\mathbb{P}^3} \rightarrow T_{\widetilde{\mathbb{P}^4}/\mathbb{P}^3} \rightarrow \mathcal{N} \rightarrow 0$ we obtain $\mathcal{N} \cong \mathcal{O}_{\mathbb{P}^3}(1)$. Therefore, the Todd classes that appear in the formula are given as follows

$$(\text{Td}(\mathcal{O}_{\mathbb{P}^3}(\mathbf{h})))^{-1} = \left(1 + \frac{\mathbf{h}}{2} + \frac{\mathbf{h}^2}{12} + \dots\right)^{-1} = 1 - \frac{\mathbf{h}}{2} + \frac{\mathbf{h}^2}{3} + \dots$$

where $\mathbf{h}$ is the generator of the Chow ring $\mathbb{P}^3$. Consequently, the equation (2.2.2) is as follows

$$\begin{aligned} r + \text{ch}_1(i_* \mathcal{E}|_H) + \text{ch}_2(i_* \mathcal{E}|_H) + \dots &= i_* \left[(r + \text{ch}_1(\mathcal{E}|_H) + \text{ch}_2(\mathcal{E}|_H) + \dots) \left(1 - \frac{\mathbf{h}}{2} + \frac{\mathbf{h}^2}{3} + \dots\right) \right] \\ 2\xi - 2\xi^2 + \dots &= i_* \left[2 + (\text{ch}_1(\mathcal{E}|_H) - \mathbf{h}) + \left(\text{ch}_2(\mathcal{E}|_H) + \frac{2}{3}\mathbf{h}^2 - \frac{\text{ch}_1(\mathcal{E}|_H)\mathbf{h}}{2}\right) + \dots \right] \\ 2\xi - 2\xi^2 + \dots &= 2\xi + (\text{ch}_1(\mathcal{E}|_H)\xi - \xi^2) + \left(\xi \text{ch}_2(\mathcal{E}|_H) + \frac{2}{3}\xi^3 - \frac{\text{ch}_1(\mathcal{E}|_H)\xi^2}{2}\right) + \dots \end{aligned}$$

and

$$-2\xi^2 = \text{ch}_1(\mathcal{E}|_H)\xi - \xi^2 \Rightarrow \text{ch}_1(\mathcal{E}|_H) = -\mathbf{h}.$$

As a result, we have

$$c_1(\mathcal{E}|_{H_i}) = -1.$$

For the case of $i = 1$, we obtain a hyperplane H and $H^2(\mathcal{E}|_H \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(0, -2)) \neq 0$. This happens because in the sequence (2.3.3) with $i = 1$ twisted by $\mathcal{E}$:

$$0 \rightarrow \mathcal{E}(-1, -2) \rightarrow \mathcal{E}(0, -2) \rightarrow \mathcal{E}|_H \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(0, -2) \rightarrow 0 \quad (3.2.26)$$

we have $h^2(\mathcal{E}|_H \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(0, -2)) - h^3(\mathcal{E}|_H \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(0, -2)) = 1$. Therefore, by Horrocks it follows that $\mathcal{E}|_H$ is not split. Similarly, we can show that for $i = 2$ the restriction $\mathcal{E}|_{H_2}$ is also not split because $H^1(\mathcal{E}|_{H_2} \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(0, -1)) \neq 0$.

For $i = 3$, we get a line L and the following result.

Proposition 3.11. *Let L be the pull-back of a line in $\mathbb{P}^4$, the restriction of the instanton sheaf $\mathcal{E}$ to this line is the form*

$$\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

Proof. We know that $\mathcal{E}|_L$ must be split, by Grothendieck Theorem, and that its rank is 2. So, we can write that

$$\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

for a and b integers. Next, we will obtain its first Chern class and cohomologies to find the values of a and b . For this aim, we will use the sequence (2.3.3) with $L = H_{n-1} = H_3$ twisted by $\mathcal{E}$

$$0 \rightarrow \mathcal{E}(-3, 0) \rightarrow \mathcal{E}(-2, 0)^{\oplus 3} \rightarrow \mathcal{E}(-1, 0)^{\oplus 3} \rightarrow \mathcal{E} \rightarrow \mathcal{E}|_L \rightarrow 0 \quad (3.2.27)$$

To calculate the cohomologies we must break the long exact sequence into three short exact sequences, obtaining the following ones

$$\begin{array}{ccccccccc} 0 & \rightarrow & K_1 & \rightarrow & \mathcal{E} & \rightarrow & \mathcal{E}|_L & \rightarrow & 0 \\ 0 & \rightarrow & K_2 & \rightarrow & \mathcal{E}(-1, 0)^{\oplus 3} & \rightarrow & K_1 & \rightarrow & 0 \\ 0 & \rightarrow & \mathcal{E}(-3, 0) & \rightarrow & \mathcal{E}(-2, 0)^{\oplus 3} & \rightarrow & K_2 & \rightarrow & 0. \end{array} \quad (3.2.28)$$

We will calculate the cohomological dimensions starting with the last sequence. By first calculating those of K_2 and then of K_1 , we will be able to know those of $\mathcal{E}|_L$. To calculate the cohomologies of the terms $\mathcal{E}(p, q)$ we will use the sequence (3.2.12) twisted by the corresponding line bundle. We then obtain that

$$h^i(K_2) = \begin{cases} 1 & \text{if } i = 2; \\ 0 & \text{otherwise;} \end{cases} \quad h^i(K_1) = \begin{cases} 1 & \text{if } i = 1; \\ 0 & \text{otherwise,} \end{cases} \quad (3.2.29)$$

and, consequently

$$h^i(\mathcal{E}|_L) = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{if } i = 1. \end{cases} \quad (3.2.30)$$

As a result, $a \geq -1$ and $b \geq -1$ so that the cohomologies of $\mathcal{O}_{\mathbb{P}^1}(a)$ and $\mathcal{O}_{\mathbb{P}^1}(b)$ vanish in dimension 1. As $c_1(\mathcal{E}|_L) = -1$, it follows that $a = -1$ and $b = 0$, and therefore

$$\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

■

Another property that could be verified is whether the instanton sheaf $\mathcal{E}$ is Ulrich or not, Definition 2.7. Let us now calculate $H^4(\mathcal{E}(-4, -4))$ using the sequence (3.2.12) twisted by $\mathcal{O}_{\mathbb{P}^4}(-4, -4)$. We then obtain the following cohomology sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & H^3(\mathcal{E}(-4, -4)) & \rightarrow & \overbrace{H^3(I_X(-3, -5))}^{\mathbb{C}^{\oplus 27}} \\ \rightarrow & \underbrace{H^4(\mathcal{O}_{\mathbb{P}^4}(-5, -4))}_{\mathbb{C}^{\oplus 69}} & \rightarrow & H^4(\mathcal{E}(-4, -4)) & \rightarrow & \underbrace{H^4(I_X(-3, -5))}_{\mathbb{C}^{\oplus 30}} & \rightarrow 0 \end{array}$$

because

$$h^i(\mathcal{O}_{\mathbb{P}^4}(-5, -4)) \cong h^{n-i}(\mathcal{O}_{\mathbb{P}^4}(3, 1)) = \begin{cases} 69 & \text{if } i = 4; \\ 0 & \text{otherwise} \end{cases}$$

and the cohomologies of $I_X(-3, -5)$ come from

$$0 \rightarrow I_X(-3, -5) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-3, -5) \rightarrow \mathcal{O}_{\mathbb{P}^2}(-8) \oplus \mathcal{O}_{\mathbb{P}^2}(-5).$$

As a result, we have $h^4(\mathcal{E}(-4, -4)) = 72 + h^3(\mathcal{E}(-4, -4))$.

Similarly, we can show that $h^4(\mathcal{E}(-3, -3)) \geq 6$. Thus, showing that $\mathcal{E}$ is not weakly Ulrich and, as a result, it will also not be almost Ulrich or Ulrich.

Proposition 3.12. *There is an unobstructed component*

$$\mathcal{I}_{\widetilde{\mathbb{P}^4}}(\mathcal{L}, -\alpha, 2\xi^2 - \alpha^2) \subset \mathcal{M}_{\widetilde{\mathbb{P}^4}}^{ss}(\mathcal{L}, -\alpha, 2\xi^2 - \alpha^2),$$

of dimension

$$12 \leq \dim \mathcal{I}_{\widetilde{\mathbb{P}^4}}(\mathcal{L}, -\alpha, 2\xi^2 - \alpha^2) \leq 31$$

whose points are instantons sheaf.

Proof. Next, the dimension of $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ or at least an interval where it is located must be calculated. For this purpose, we will consider the sequence of $\mathcal{E}$ and apply $\text{Hom}(\mathcal{E}, _)$

$$\begin{aligned} \text{I}_{\mathcal{E}} : 0 &\rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Hom}(\mathcal{E}, I_X(1, -1)) \\ &\rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^1(\mathcal{E}, I_X(1, -1)) \\ &\rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)) \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^2(\mathcal{E}, I_X(1, -1)) \rightarrow \dots \end{aligned}$$

Now, let us calculate the dimensions of each of the terms in this sequence. But first, consider

$$\Theta_{\mathcal{F}}^i = \text{Ext}^i(\mathcal{E}, \mathcal{F}),$$

$$\theta_{\mathcal{F}}^{i,j} = (\dim(\Theta_{(\mathcal{F})}^i), \dim(\Theta_{(\mathcal{F})}^j))$$

and

$$\theta_{\mathcal{F}}^{i,j,k} = (\dim(\Theta_{(\mathcal{F})}^i), \dim(\Theta_{(\mathcal{F})}^j), \dim(\Theta_{(\mathcal{F})}^k)).$$

This notation will be used to facilitate writing, since the number of possibilities is significant.

The following diagram briefly shows the steps of the demonstration.

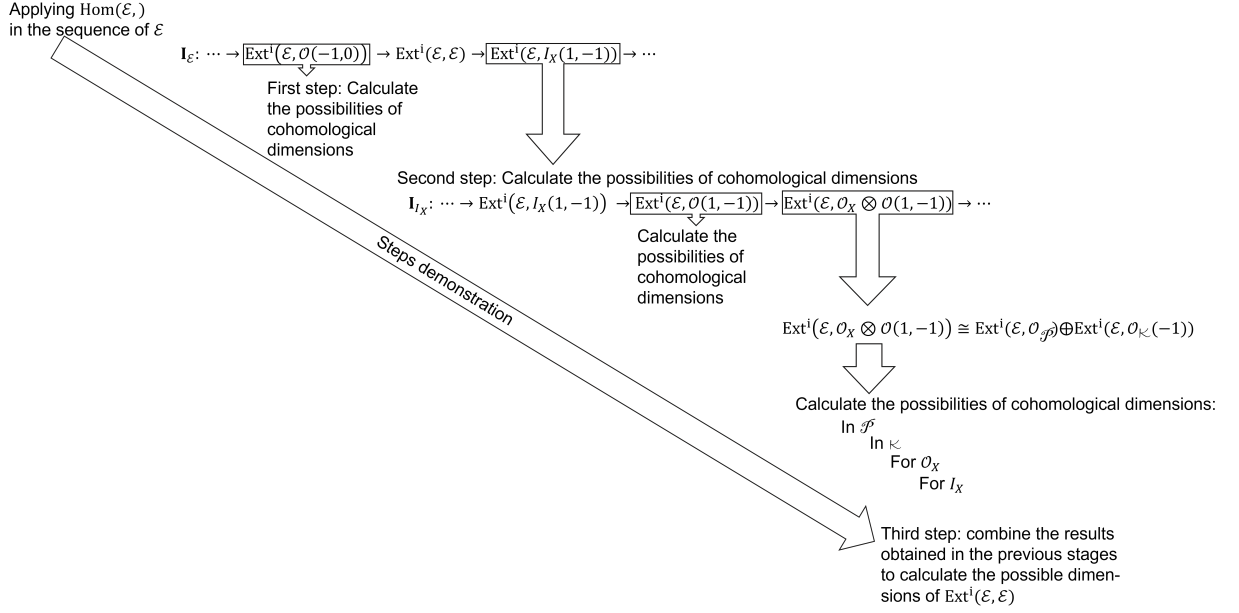


Figure 3.2.1: Steps for demonstrating Proposition 3.12. Font: Created by the author in Power Point.

First step: Let us first calculate $\text{Ext}^i(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}(-1,0))$ using the properties of the groups Ext . It follows that

$$\begin{aligned} \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}(-1,0)) &\cong \text{Ext}^{4-i}(\mathcal{O}_{\mathbb{P}^4}(-1,0), \mathcal{E} \otimes \omega_{\mathbb{P}^4}) \\ &\cong H^{4-i}(\mathcal{E}(-1, -3)). \end{aligned}$$

Considering the sequence of cohomologies of

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-2, -3) \rightarrow \mathcal{E}(-1, -3) \rightarrow I_X(0, -4) \rightarrow 0 \quad (3.2.31)$$

we have

$$0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(-1,0)}^0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^{\oplus 7} \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(-1,0)}^1 \rightarrow 0. \quad (3.2.32)$$

As a result, the possibilities of the cohomological dimensions are

$$\theta_{(-1,0)}^{0,1} = \theta_{\mathcal{O}_{\mathbb{P}^4}(-1,0)}^{0,1} = \{(0,6); (1,7)\}. \quad (3.2.33)$$

Second step: To find the possible cohomological dimensions of $\text{Ext}^i(\mathcal{E}, I_X(1, -1))$ we will use the restriction sequence twisted by $\mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)$ and then applying $\text{Hom}(\mathcal{E}, _)$.

$$\begin{aligned} I_X : 0 &\rightarrow \text{Hom}(\mathcal{E}, I_X(1, -1)) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \\ &\rightarrow \text{Ext}^1(\mathcal{E}, I_X(1, -1)) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \\ &\rightarrow \text{Ext}^2(\mathcal{E}, I_X(1, -1)) \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \rightarrow \dots \end{aligned}$$

Analogously to the previous step, we have that

$$\begin{aligned} \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) &\cong \text{Ext}^{4-i}(\mathcal{O}_{\mathbb{P}^4}^\sim(1, -1), \mathcal{E} \otimes \omega_{\mathbb{P}^4}) \\ &\cong H^{4-i}(\mathcal{E}(-3, -2)) \end{aligned}$$

Considering the sequence of cohomologies of (3.2.12) twisted by $\mathcal{O}_{\mathbb{P}^4}^\sim(-3, -2)$ we obtain

$$0 \rightarrow \mathbb{C} \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)}^0 \rightarrow \mathbb{C}^{\oplus 5} \rightarrow \mathbb{C}^{\oplus 7} \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)}^1 \rightarrow 0$$

and the definition possibilities

$$\theta_{(1, -1)}^{0,1} = \theta_{\mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)}^{0,1} = \{(0, 1), (1, 2), \dots, (5, 6)\}. \quad (3.2.34)$$

To calculate the definitions of

$$\text{Ext}^i(\mathcal{E}, \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)) \cong \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\mathcal{P}}) \otimes \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\mathbf{K}}(-1))$$

we must consider the resolutions of each component to find the corresponding dimensions and then add the results obtained in both cases.

- In $\mathcal{P}$: the resolution sequence (3.2.1) twisted by $\mathcal{O}_{\mathbb{P}^4}^\sim(1, -1)$ gives us the following short exact sequences

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}^\sim(-1, -1) \rightarrow \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}(0, -1) \rightarrow \mathcal{Q} \rightarrow 0 \quad (3.2.35)$$

and

$$0 \rightarrow Q \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1) \rightarrow \mathcal{O}_{\mathcal{P}} \rightarrow 0. \quad (3.2.36)$$

Applying $\text{Hom}(\mathcal{E}, _)$ to (3.2.35) we obtain

$$\begin{aligned} 0 &\rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}^{\oplus 2}(0, -1)) \rightarrow \text{Hom}(\mathcal{E}, Q) \\ &\rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}^{\oplus 2}(0, -1)) \rightarrow \text{Ext}^1(\mathcal{E}, Q) \\ &\rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)) \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}^{\oplus 2}(0, -1)) \rightarrow \text{Ext}^2(\mathcal{E}, Q) \rightarrow \dots \end{aligned}$$

Using the same reasoning as before, we have

$$\text{Ext}^i(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)) \cong H^{4-i}(\mathcal{E}(-1, -2)) \cong \begin{cases} \mathbb{C}^{\oplus 2} & \text{if } i = 1 \\ 0 & \text{otherwise;} \end{cases} \quad (3.2.37)$$

and

$$\text{Ext}^i(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)) \cong H^{4-i}(\mathcal{E}(-2, -2)^{\oplus 2}).$$

Considering then the cohomologies of the sequence (3.2.12) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-2, -2)$ we have

$$0 \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)}^0 \rightarrow \mathbb{C}^{\oplus 1} \rightarrow \mathbb{C}^{\oplus 4} \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)}^1 \rightarrow 0.$$

and the possibilities for the dimensions

$$\theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)}^{0,1} = \{(0, 3), (1, 4)\} \quad (3.2.38)$$

Since we have two copies of the yarn fiber, we need to add the possibilities, thus obtaining the following table.

$$\theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 2}}^{0,1} = \{(0, 6), (1, 7), (2, 8)\} \quad (3.2.39)$$

Consequently for $\text{Ext}^i(\mathcal{E}, Q)$ we have the sequence

$$0 \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 2}}^0 \rightarrow \Theta_Q^0 \rightarrow \mathbb{C}^{\oplus 2} \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 2}}^1 \rightarrow \Theta_Q^1 \rightarrow 0$$

and the table 3.2.2 shows the possibilities for $\theta_Q^{0,1}$.

$\theta_{\mathcal{O}_{\mathbb{P}^4}(0,-1)^{\oplus 2}}^{0,1}$		
(0,6)	(1,7)	(2,8)
$\theta_Q^{0,1} = \{(0,4), (1,5), (2,6)\}$	$\theta_Q^{0,1} = \{(1,5), (2,6)\}$	$\theta_Q^{0,1} = \{(2,6)\}$

Table 3.2.2: Possible dimensions of $\text{Ext}^i(\mathcal{E}, Q)$ with $i = 0, 1$. Font: Created by the author.

This result is obtained because these possibilities depend on the possibilities with respect to $\mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 2}$. As a result, we get that

$$\theta_Q^{0,1} = \{(0,4), (1,5), (2,6)\} \quad (3.2.40)$$

Now considering the $\text{Hom}(\mathcal{E}, _)$ of the sequence (3.2.36) and equations (3.2.34) and (3.2.40) we obtain the sequence

$$0 \rightarrow \Theta_Q^0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^0 \rightarrow \Theta_{\mathcal{O}_{\wp}}^0 \rightarrow \Theta_Q^1 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^1 \rightarrow \Theta_{\mathcal{O}_{\wp}}^1 \rightarrow 0. \quad (3.2.41)$$

It is worth noting that the possibilities for $\theta_{\mathcal{O}_{\wp}}^{0,1}$ depend on those related of $\theta_Q^{0,1}$ at the top and those of $\theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^{0,1}$. The table 3.2.3 shows the possible dimensions obtained.

		$\theta_Q^{0,1}$		
		(0,4)	(1,5)	(2,6)
$\theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^{0,1}$	(0,1)	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(4,1), (3,0)\}$	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(4,1), (3,0)\}$	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(4,1), (3,0)\}$
	(1,2)	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(5,2)(4,1), (3,0)\}$	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(5,2)(4,1), (3,0)\}$	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(5,2)(4,1), (3,0)\}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	(5,6)	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(9,6), \dots, (3,0)\}$	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(9,6), \dots, (3,0)\}$	$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(9,6), \dots, (3,0)\}$

Table 3.2.3: Possible dimensions of $\text{Ext}^i(\mathcal{E}, \mathcal{O}_{\wp})$ with $i = 0, 1$. Font: Created by the author.

As a result, we have

$$\theta_{\mathcal{O}_{\wp}}^{0,1} = \{(9,6), \dots, (4,1), (3,0)\}. \quad (3.2.42)$$

- In κ : The process to find the possible dimensions is analogous to that done for $\wp$.
Let us get it the resolution sequence (3.2.2) twisted by $\mathcal{O}_{\mathbb{P}^4}(1, -1)$ and break it into

two short exact sequences

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2) \rightarrow R \rightarrow 0 \quad (3.2.43)$$

and

$$0 \rightarrow R \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -1) \rightarrow \mathcal{O}_{\kappa}(-1) \rightarrow 0. \quad (3.2.44)$$

Applying $\text{Hom}(\mathcal{E}, _)$ to (3.2.43) we obtain

$$\begin{aligned} 0 &\rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)) \rightarrow \text{Hom}(\mathcal{E}, R) \\ &\rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)) \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)) \rightarrow \text{Ext}^1(\mathcal{E}, R) \\ &\rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)) \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)) \rightarrow \text{Ext}^2(\mathcal{E}, R) \rightarrow \dots \end{aligned}$$

For $\text{Ext}^1(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2))$, note that

$$\begin{aligned} \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)) &\cong \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}) \oplus \text{Ext}^i(\mathcal{E}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)) \\ &\cong H^{4-i}(\mathcal{E}(-2, -3)) \oplus H^{4-i}(\mathcal{E}(-3, -1)). \end{aligned}$$

Let us now consider the cohomology sequences of $\mathcal{E}(-2, -3)$ and $\mathcal{E}(-3, -1)$ to calculate the possible dimensions of the previous sum. First, we have

$$0 \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}}^0 \rightarrow \mathbb{C}^{\oplus 5} \rightarrow \mathbb{C}^{\oplus 9} \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}}^1 \rightarrow 0$$

then

$$\theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}}^{0,1} = \{(0, 4), (1, 5), \dots, (5, 9)\} \quad (3.2.45)$$

and

$$0 \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)}^0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^{\oplus 3} \rightarrow \Theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)}^1 \rightarrow 0$$

then

$$\theta_{\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, -2)}^{0,1} = \{(0, 2), (1, 3), \dots, (5, 9)\}. \quad (3.2.46)$$

For this reason, we have

$$\theta_{\oplus} = \theta_{\mathcal{O}_{\mathbb{P}^4}(1,-2) \oplus \mathcal{O}_{\mathbb{P}^4}(1,-2)}^{0,1} = \{(0,6), (1,7), \dots, (6,12)\}. \quad (3.2.47)$$

As results, from the sequence (3.2.43) and using the possibilities (3.2.39) we have

$$0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(0,-1)}^0 \rightarrow \Theta_{\oplus}^0 \rightarrow \Theta_R^0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(0,-1)}^1 \rightarrow \Theta_{\oplus}^1 \rightarrow \Theta_R^1 \rightarrow 0.$$

Table 3.2.4 shows the possible dimensions of $\theta_R^{0,1}$ obtained from the possible dimensions of $\theta_{\oplus}^{0,1}$ and $\theta_{\mathcal{O}_{\mathbb{P}^4}(0,-1)}^{0,1}$.

		$\theta_{\mathcal{O}_{\mathbb{P}^4}(0,-1)}^{0,1}$	
		(0,3)	(1,4)
$\theta_{\oplus}^{0,1}$	(0,6)	$\theta_R^{0,1} = \{(3,6), (2,5), \dots, (0,3)\}$	$\theta_R^{0,1} = \{(3,6), (2,5), \dots, (0,3)\}$
	(1,7)	$\theta_R^{0,1} = \{(4,7), (3,6), \dots, (0,3)\}$	$\theta_R^{0,1} = \{(4,7), (3,6), \dots, (0,3)\}$
	$\vdots$	$\vdots$	$\vdots$
	(6,12)	$\theta_R^{0,1} = \{(9,12), (8,11), \dots, (0,3)\}$	$\theta_R^{0,1} = \{(9,12), (8,11), \dots, (0,3)\}$

Table 3.2.4: Possible dimensions of $\text{Ext}^i(\mathcal{E}, R)$ with $i = 0, 1$. Font: Created by the author.

Therefore

$$\theta_R^{0,1} = \{(9,12), (8,11), \dots, (0,3)\}. \quad (3.2.48)$$

Considering the other sequence of resolution (3.2.44) we obtain

$$0 \rightarrow \Theta_R^0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^0 \rightarrow \Theta_R^1 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^1 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^1 \rightarrow 0.$$

From the possibilities (3.2.34) and ((3.2.48)), the table 3.2.5 is obtained.

		$\theta_R^{0,1}$		
		(0,3)	(1,4)	...
$\theta_{\mathcal{O}_{\mathbb{P}^4}(1,-1)}^{0,1}$	(0,1)	$\theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^{0,1} = \{(3,1), (2,0)\}$	$\theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^{0,1} = \{(3,1), (2,0)\}$	...
	(1,2)	$\theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^{0,1} = \{(4,2), (3,1), (2,0)\}$	$\theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^{0,1} = \{(4,2), (3,1), (2,0)\}$	...
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	(5,6)	$\theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^{0,1} = \{(8,6), \dots, (2,0)\}$	$\theta_{\mathcal{O}_{\mathbb{P}^4}(-1)}^{0,1} = \{(8,6), \dots, (2,0)\}$	...

Table 3.2.5: Possible dimensions of $\text{Ext}^i(\mathcal{E}, \mathcal{O}_{\mathbb{P}^4}(-1))$ with $i = 0, 1$. Font: Created by the author.

For the other columns the values are the same and for this reason they are not presented here.

Add the results obtained in both cases we can find the dimension possibilities of $\text{Ext}^i(\mathcal{E}, \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}(1, -1))$ and the table 3.2.6.

		$\theta_{\mathcal{O}_{\mathbb{P}^4}}^{0,1}$						
		(3,0)	(4,1)	(5,2)	(6,3)	(7,4)	(8,5)	(9,6)
$\theta_{\mathcal{O}_X(-1)}^{0,1}$	(2,0)	$\theta_{\mathcal{O}_X}^{0,1} = (5,0)$	$\theta_{\mathcal{O}_X}^{0,1} = (6,1)$	$\theta_{\mathcal{O}_X}^{0,1} = (7,2)$	$\theta_{\mathcal{O}_X}^{0,1} = (8,3)$	$\theta_{\mathcal{O}_X}^{0,1} = (9,4)$	$\theta_{\mathcal{O}_X}^{0,1} = (10,5)$	$\theta_{\mathcal{O}_X}^{0,1} = (11,6)$
	(3,1)	$\theta_{\mathcal{O}_X}^{0,1} = (6,1)$	$\theta_{\mathcal{O}_X}^{0,1} = (7,2)$	$\theta_{\mathcal{O}_X}^{0,1} = (8,3)$	$\theta_{\mathcal{O}_X}^{0,1} = (9,4)$	$\theta_{\mathcal{O}_X}^{0,1} = (10,5)$	$\theta_{\mathcal{O}_X}^{0,1} = (11,6)$	$\theta_{\mathcal{O}_X}^{0,1} = (12,7)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	(8,6)	$\theta_{\mathcal{O}_X}^{0,1} = (11,6)$	$\theta_{\mathcal{O}_X}^{0,1} = (12,7)$	$\theta_{\mathcal{O}_X}^{0,1} = (13,8)$	$\theta_{\mathcal{O}_X}^{0,1} = (14,9)$	$\theta_{\mathcal{O}_X}^{0,1} = (15,10)$	$\theta_{\mathcal{O}_X}^{0,1} = (16,11)$	$\theta_{\mathcal{O}_X}^{0,1} = (17,12)$

Table 3.2.6: Possible dimensions of $\text{Ext}^i(\mathcal{E}, \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}(1, -1))$ with $i = 0, 1$. Font: Created by the author.

In this way

$$\theta_{\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^4}(1, -1)}^{0,1} = \theta_{\mathcal{O}_X}^{0,1} = \{(5,0), (6,1), \dots, (16,11), (17,12)\}. \quad (3.2.49)$$

Now we can find the possibilities of $\text{Ext}^i(\mathcal{E}, I_X(1, -1))$ using the following sequence

$$0 \rightarrow \Theta_{I_X(1, -1)}^0 \rightarrow \Theta_{\mathcal{O}_{\mathbb{P}^4}(1, -1)}^0 \rightarrow \Theta_{\mathcal{O}_X}^0 \rightarrow \Theta_{I_X(1, -1)}^1 \rightarrow 0 \rightarrow \Theta_{\mathcal{O}_X}^1 \rightarrow \Theta_{I_X(1, -1)}^2 \rightarrow 0.$$

As results, the table 3.2.7 of dimension possibilities is obtained.

		$\theta_{\mathcal{O}_{\mathbb{P}^4}(1, -1)}^{0,1}$			
		(0,1)	(1,2)	...	(5,6)
$\theta_{\mathcal{O}_X}^{0,1}$	(5,0)	$\theta_{I_X(1, -1)}^{0,1,2} = (0,6,0)$	$\theta_{I_X(1, -1)}^{0,1,2} = \{(0,6,0), (1,7,0)\}$		$\theta_{I_X(1, -1)}^{0,1,2} = \{(0,6,0), (1,7,0), \dots, (5,11,0)\}$
	(6,1)	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), \\ (0,7,1) \end{cases}$	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), (1,7,0), \\ (0,7,1), (1,8,1) \end{cases}$	...	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), \dots, (5,11,0), \\ (0,7,1), \dots, (5,12,1) \end{cases}$
	(7,2)	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), \\ (0,7,1), \\ (0,8,2) \end{cases}$	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), (1,7,0), \\ (0,7,1), (1,8,1), \\ (0,8,2), (1,9,2) \end{cases}$	...	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), \dots, (5,11,0), \\ (0,7,1), \dots, (5,12,1), \\ (0,8,2), \dots, (5,13,2) \end{cases}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	(17,12)	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), \\ (0,7,1), \\ \vdots \\ (0,19,13) \end{cases}$	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), (1,7,0), \\ (0,7,1), (1,8,1), \\ \vdots \\ (0,19,13), (1,20,13) \end{cases}$	...	$\theta_{I_X(1, -1)}^{0,1,2} = \begin{cases} (0,6,0), \dots, (5,11,0), \\ (0,7,1), \dots, (5,12,1), \\ \vdots \\ (0,19,13), \dots, (5,24,13) \end{cases}$

Table 3.2.7: Possible dimensions of $\text{Ext}^i(\mathcal{E}, I_X(1, -1))$ with $i = 0, 1, 2$. Font: Created by the author.

Consequently

$$\theta_{I_X(1,-1)}^{0,1,2} = \{(0,6,0), \dots, (5,11,0), (0,7,1), \dots, (5,12,1), (0,19,13), \dots, (5,24,13)\} \quad (3.2.50)$$

Third step: Calculating the possibilities of $\text{Ext}^1(\mathcal{E}, \mathcal{E})$.

Finally, we can use the possibilities (3.2.33) and (3.2.50) in the sequence $I_{\mathcal{E}}$ to obtain the following sequence

$$0 \rightarrow \Theta_{(-1,0)}^0 \rightarrow \Theta_{\mathcal{E}}^0 \rightarrow \Theta_{I_X(1,-1)}^0 \rightarrow \Theta_{(-1,0)}^1 \rightarrow \Theta_{\mathcal{E}}^1 \rightarrow \Theta_{I_X(1,-1)}^1 \rightarrow 0 \rightarrow \Theta_{\mathcal{E}}^2 \rightarrow \Theta_{I_X(1,-1)}^2 \rightarrow 0$$

and the table 3.2.8.

		$\theta_{\mathcal{E}(-1,0)}^{0,1}$	
		(0,6)	(1,7)
$\theta_{I_X(1,-1)}^{0,1}$	(0,6,0)	$\theta_{\mathcal{E}}^{0,1,2} = (0,12,0)$	$\theta_{\mathcal{E}}^{0,1,2} = (1,13,0)$
	(0,7,1)	$\theta_{\mathcal{E}}^{0,1,2} = (0,13,1)$	$\theta_{\mathcal{E}}^{0,1,2} = (1,14,1)$
	(0,8,2)	$\theta_{\mathcal{E}}^{0,1,2} = (0,14,2)$	$\theta_{\mathcal{E}}^{0,1,2} = (1,15,2)$
	$\vdots$	$\vdots$	$\vdots$
	(0,19,13)	$\theta_{\mathcal{E}}^{0,1,2} = (0,25,13)$	$\theta_{\mathcal{E}}^{0,1,2} = (1,26,13)$
	(1,7,0)	$\theta_{\mathcal{E}}^{0,1,2} = \{(0,12,0), (1,13,0)\}$	$\theta_{\mathcal{E}}^{0,1,2} = (2,14,0)$
	(1,8,1)	$\theta_{\mathcal{E}}^{0,1,2} = \{(0,13,1), (1,14,1)\}$	$\theta_{\mathcal{E}}^{0,1,2} = (2,15,1)$
	$\vdots$	$\vdots$	$\vdots$
	(1,20,13)	$\theta_{\mathcal{E}}^{0,1,2} = \{(0,25,13), (1,26,13)\}$	$\theta_{\mathcal{E}}^{0,1,2} = (2,27,13)$
	$\vdots$	$\vdots$	$\vdots$

Table 3.2.8: Possible dimensions of $\text{Ext}^i(\mathcal{E}, \mathcal{E})$ with $i = 0, 1, 2$. Font: Created by the author.

The table 3.2.8 is divided into 6 blocks with 14 rows each. But we only present two blocks because each resulting column follows a pattern from block to block. These rows depend on the possible dimensions with respect to $I_X(1, -1)$. Note that each of these possibilities is a triple of the form $(j, i+j, i-6)$ with $j = 0, \dots, 5$ and $i = 6, \dots, 19$. Fixing j and varying i , we obtain the elements of each block. Also note that the larger i and j , the larger the second entry of the possible dimension triples for $\theta_{\mathcal{E}}^{0,1,2}$. In other

words, the larger the possible value of $\theta_{\mathcal{E}}^1$. Furthermore, for the second column of $\theta_{\mathcal{E}(-1,0)}^{0,1}$ possibilities, we also obtain triples with a larger second entry compared to the possibilities in the same row in the first column.

In Table 3.2.8, we observe that the possibilities from the first block of the resulting first column are contained in the second block of this column by adding a triple in each row. The triple added in each row is the result of adding 1 to the first and second entries of the triple to the possibility of the previous block. For example, in the first row of the first block, we have $(0, 12, 0)$ while in the first row of the second block we have $(0, 12, 0)$ and $(1, 13, 0)$. The elements of the remaining blocks are obtained in a similar manner. The elements of the remaining blocks are obtained in a similar manner.

In the second resulting column, there will be only 14 triples per block. Each block repeats the rows of the previous block with the first and second entries added in 1. For example, $(1, 13, 0)$ in the first row of the first block and $(2, 14, 0)$ in the first row of the second block. The other blocks follow this same pattern from the previous block. Note further that for each triple (a, b, c) of possibilities relative to $I_X(1, -1)$, we obtain in this column the triple $(a + 1, b + 7, c)$.

To complete the table, simply follow these patterns. Furthermore, based on the above description of the table behavior, and remembering that our objective is to find the smallest and largest value that the second entry of all the triples found has, we can go directly to calculating the dimension of $\text{Ext}^1(\mathcal{E}, \mathcal{E})$. For the lowest value we have the triple $(0, 12, 0)$. For the largest value, simply calculate the triple obtained from the combination of $(5, 24, 13)$ with $(1, 7)$. Obtaining then that the possibility with the largest $\text{Ext}^1(\mathcal{E}, \mathcal{E})$ is

$$\theta_{\mathcal{E}}^{0,1,2} = (6, 31, 13). \quad (3.2.51)$$

Consequently

$$12 \leq \dim \text{Ext}^1(\mathcal{E}, \mathcal{E}) \leq 31.$$

■

Let us analyze some results of the monad that its cohomology represents the instan-

ton. Let us consider now $U = V \oplus W$ with $\alpha \in V = \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5})$ and $\beta \in W = \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}, \Omega^2(0, 2))$.

Proposition 3.13. *For the monad of $\mathcal{E}$ in (3.2.23) we have that*

$$\text{End}(M) \subset \text{End}(\mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}) \oplus \text{End}(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}) \oplus \text{End}(\Omega^2(0, 2)) \quad (3.2.52)$$

then

$$\dim U - \dim \text{End}(M) \geq 50.$$

Proof. The idea here is to take the homomorphisms of the monad itself to obtain the endomorphisms of each element. With this we obtain the containment (3.2.52) of the endomorphisms of M that appears in the statement. Let us calculate each of the endomorphisms of the right side of the containment

$$\text{End}(\mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}) \cong \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}) \cong H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \cong \mathbb{C}^{\oplus 4}; \quad (3.2.53)$$

$$\begin{aligned} \text{End}(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}) &\cong \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2}) \quad (3.2.54) \\ &\oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}, \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}) \\ &\oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}) \\ &\oplus \text{Hom}(\mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}, \mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2}) \\ &\cong H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \oplus H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 25}) \oplus H^0(\mathcal{O}_{\mathbb{P}^4}(1, -1)^{\oplus 10}) \\ &\oplus H^0(\mathcal{O}_{\mathbb{P}^4}(-1, 1)^{\oplus 10}) \\ &\cong \mathbb{C}^{\oplus 39}; \end{aligned}$$

and

$$\text{End}(\Omega^2(0, 2)) \cong H^0(\Omega^2) \cong \mathbb{C}. \quad (3.2.55)$$

Furthermore, there are no homotopies since

$$\text{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}, \mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}) = 0;$$

$$\mathrm{Hom}(\Omega^2(0, 2), \mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}) = 0;$$

$$\mathrm{Hom}(\Omega^2(0, 2), \mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}) = 0.$$

The last two follows from the sequence (2.3.3). As results, we have

$$\mathrm{End}(\mathbf{M}) \subset \mathbb{C}^{\oplus 44} \quad (3.2.56)$$

and consequently the largest dimension that $\mathrm{End}(\mathbf{M})$ can reach is 44.

On the other hand

$$\begin{aligned} \alpha \in V &\cong \mathrm{H}^0((\mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}, (\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2}) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(-1, -1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5})) \\ &\cong \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(0, 1)^{\oplus 4}) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(1, 0)^{\oplus 10}) \\ &\cong \mathbb{C}^{\oplus 16} \oplus \mathbb{C}^{\oplus 50} \end{aligned}$$

and, by the formula (2.1.7),

$$\begin{aligned} \beta \in W &\cong \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(-1, 0)^{\oplus 2}, \Omega^2(0, 2)) \oplus \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^4}(0, -1)^{\oplus 5}, \Omega^2(0, 2)) \\ &\cong \mathrm{H}^0(\Omega^2(1, 2)^{\oplus 2}) \oplus \mathrm{H}^0(\Omega^2(0, 3)^{\oplus 5}) \\ &\cong \mathbb{C}^{\oplus 8} \oplus \mathbb{C}^{\oplus 20} \end{aligned}$$

In this way $\dim V = 66$ and $\dim W = 28$, then $\dim U = \dim(V + W) = 94$. Consequently

$$\dim U - \dim \mathrm{End}(\mathbf{M}) \geq 94 - 44 = 50.$$

■

Considering that the maximum dimension of $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ is 31, we have

$$\dim U - \dim \mathrm{End}(\mathbf{M}) \geq 50 > 31 = \max\{\dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})\}.$$

Therefore, there is the possibility of having a locally free sheaf. Recall from standard sheaf theory that the singular locus of a monad is $\mathrm{Supp}(\mathrm{Sing}(\mathcal{E})) = \{x \in \widetilde{\mathbb{P}^4}; \alpha(x) \text{ not injective}\}$ see

[3], where $\alpha(x)$ denotes the fiber map at the point $x \in \widetilde{\mathbb{P}^4}$ of the morphism α appearing in M. As $\mathcal{E}$ is locally free if and only if $\text{Supp}(\text{Sing}(\mathcal{E})) = \emptyset$, we will analyze the monad maps to show whether this bundle exists.

Let us first analyze α . Note that

$$\begin{aligned} \alpha &\in \text{Hom}(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)^{\oplus 2}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0)^{\oplus 2}) \oplus \text{Hom}(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, -1)^{\oplus 2}, \mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1)^{\oplus 5}) \\ &\cong H^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, 1))^{\oplus 4} \oplus H^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(1, 0))^{\oplus 10} \cong \langle y_0, \dots, y_3 \rangle^{\oplus 4} \oplus \langle z_0, \dots, z_4 \rangle^{\oplus 10} \end{aligned} \quad (3.2.57)$$

where y_i and z_i are coordinates in $\mathbb{P}^3$ and in $\mathbb{P}^4$, respectively, given by projection maps of $\widetilde{\mathbb{P}^4}$. For these coordinates, there is a $\lambda \in H^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(E))$ such that $\lambda y_i = z_i$. This λ is unique because $h^0(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(E)) = 1$. Therefore, α can be represented by a matrix 7×2 with the first two rows generated by y_i and the remaining ones by z_i . We can represent it in general form as follows:

$$A = \begin{pmatrix} \sum_{i=0}^4 A_i^1 y_i \\ \sum_{j=0}^5 A_j^2 z_j \end{pmatrix}, \quad (3.2.58)$$

where A_i^1 are matrices of order 2×2 and A_j^2 of order 5×2 , both with constant entries.

Considering the blow-up point, for example, as $p = [0 : 0 : 0 : 0 : 1]$ we have the matrices

$$A = \begin{pmatrix} y_0 & y_1 \\ y_2 & y_3 \\ 0 & z_0 \\ z_1 & 0 \\ z_4 & z_2 \\ z_3 & 0 \\ 0 & z_4 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} y_0 & 0 \\ y_1 & y_0 \\ z_2 & z_1 \\ z_3 & z_2 \\ 0 & z_3 \\ z_4 & 0 \\ 0 & z_4 \end{pmatrix}$$

are examples of matrices for injective α .

Let us now analyze the map β . Note that

$$\begin{aligned} \beta &\in \text{Hom}\left(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(-1, 0), \Omega^2(0, 2)\right)^{\oplus 2} \oplus \text{Hom}\left(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(0, -1), \Omega^2(0, 2)\right)^{\oplus 5} \\ &\cong H^0\left(\Omega^2(1, 2)\right) \oplus H^0\left(\Omega^2(0, 3)\right) \cong \langle \lambda \rangle^{\oplus 8} \oplus \mathbb{C}^{\oplus 20} \end{aligned} \quad (3.2.59)$$

Therefore, β is given by a matrix 4×7 where the first two columns are generated by λ and the others give constants. We can write it in general as follows.

$$B = \begin{pmatrix} \lambda B' & M \end{pmatrix} \quad (3.2.60)$$

where B' and M are matrices of order 4×2 and 4×5 , respectively, with constant entries. Again, we can construct matrices, in this case B surjective, for example, B given by

$$B = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We then show that there exist injective and surjective maps α and β , respectively. If we find a combination of matrices A and B such that A is injective, B is surjective, and $BA \cong 0$, then we can say that there exists an instanton bundle that is the cohomology of monad M . However, despite many attempts, we were unable to find this matrix pair. We could not even prove that it does not exist. The possibility remains that we could deform the instanton $\mathcal{E}$ into a locally free one. In the case of $n = 5$, we will see in the next chapter that we can give a specific example of a matrix pair A and B .

3.3 Generalization of the example for any even n .

In the previous sections, we showed the existence of an instanton sheaf for $n = 4$. Following Henni's idea in [2] for the odd n case, the next step will be to extend this example to any even n . Thus, we obtain a family of instanton sheaves.

Proposition 3.14. *The instanton $\mathcal{E}$ has the following table of cohomologies in $\widetilde{\mathbb{P}^n}$.*

$\mathcal{E}(-1, -n+2)$	$\mathcal{E}(-1, -n+3)$	$\dots$	$\mathcal{E}(-1, 0)$	$\mathcal{E}(-1, 1)$	$\mathcal{E}(0, -n+1)$	$\mathcal{E}(0, -n+2)$	$\mathcal{E}(0, -n+3)$	$\dots$	$\mathcal{E}$	
0	0	$\dots$	0	0	0	0	0	$\dots$	0	h^0
0	0	$\dots$	0	2	0	0	0	$\dots$	0	h^1
0	0	$\dots$	0	0	0	0	0	$\dots$	0	h^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	$\dots$	0	0	0	0	0	$\dots$	0	h^{n-2}
2	0	$\dots$	0	0	$n+1$	1	0	$\dots$	0	h^{n-1}
0	0	$\dots$	0	0	0	0	0	$\dots$	0	h^n

Table 3.3.1: Cohomologies of $\mathcal{E}$ in $\widetilde{\mathbb{P}^n}$. Font: Created by the author.

Proof. The demonstration will be divided into several cases as follows.

- $1 \leq t \leq n-1$ in $\mathcal{E}(0, -t)$: Let us consider

$$0 \rightarrow I_X(1, 1-t) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, -1-t) \rightarrow \begin{array}{c} \mathcal{O}_{\mathbb{P}^{n-2}}(-t) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^{n-2}}(-1-t) \end{array} \rightarrow 0.$$

In this case,

$$h^i(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, -1-t)) = \begin{cases} 1 & \text{if } i = n-1 = t \\ 0 & \text{otherwise.} \end{cases}$$

and

$$h^i(\mathcal{O}_{\mathbb{P}^{n-2}}(-t) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(-1-t)) = \begin{cases} \binom{t-1}{t+1-n} + \binom{t}{t-n+2} & \text{if } i = n-2, t \geq n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$h^i(I_X(1, -1-t)) = \begin{cases} n+1 & \text{if } i = n-1 = t \\ 1 & \text{if } i = n-1, t = n-2 \\ 0 & \text{otherwise} \end{cases}$$

Considering the sequence 3.3.1

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -t) \rightarrow \mathcal{E}(0, -t) \rightarrow I_X(1, -1-t) \rightarrow 0 \quad (3.3.1)$$

and that $h^i(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -t)) = 0$ for all i , we can deduce that the cohomologies of $\mathcal{E}(0, -t)$ are the same as those of $I_X(1, -1-t)$.

- $0 \leq t \leq n-2$ in $\mathcal{E}(-1, -t)$: Let us consider

$$0 \rightarrow I_X(0, -1-t) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1-t) \rightarrow \mathcal{O}_{\mathbb{P}^{n-2}}(-1-t)^{\oplus 2} \rightarrow 0.$$

In this case, $h^i(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1-t)) = 0$ and

$$h^i(\mathcal{O}_{\mathbb{P}^{n-2}}(-1-t)^{\oplus 2}) = \begin{cases} 2 \binom{t}{t-n+2} & \text{if } i = n-2, t \geq n-2; \\ 0 & \text{otherwise.} \end{cases}$$

As result, we have

$$h^i(I_X(0, -1-t)) = \begin{cases} 2 & \text{if } i = n-1, t = n-2 \\ 0 & \text{otherwise} \end{cases}.$$

Considering the sequence $0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, -t) \rightarrow \mathcal{E}(-1, -t) \rightarrow I_X(0, -1-t) \rightarrow 0$ and that $h^i(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, -t)) = 0$ the cohomologies of $\mathcal{E}(1, -t)$ are the same as $I_X(0, -1-t)$.

■

Theorem 3.15. *The sheaf $\mathcal{E}$ is an instanton on $\widetilde{\mathbb{P}^n}$ with n even, $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}\left(1, \frac{n-2}{2}\right)$ and the Chern class $c_1(\mathcal{E}) = -\alpha$, $c_2(\mathcal{E}) = 2\xi^2 - \alpha^2$ with charge $\frac{n^{n-2} + (n-2)^{n-2}}{2^{n-2}}$.*

Proof. We showed in previous sections that for $n = 4$ the result holds. Let us now show that for any n even. The cohomology conditions follow from Proposition 3.14. Let us verify that $\mathcal{E}$ is

semi-stable with respect to polarization $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n} \left(1, \frac{n-2}{2}\right)$. Note that,

$$\begin{aligned}
 \deg(\mathcal{O}_{\mathbb{P}^n}(p, q)) &= c_1(\mathcal{O}_{\mathbb{P}^n}(p, q))c_1(\mathcal{L}) \\
 &= (p\xi + q\alpha) \left[\frac{1}{2^{n-1}} \left[(\alpha(n-2))^{n-1} + (n^{n-1} - (n-2)^{n-1}) \xi^{n-1} \right] \right] \\
 &= \frac{1}{2^{n-1}} \left[((n-2)^{n-1} + n^{n-1} - (n-2)^{n-1}) p + (n^{n-1} - (n-2)^{n-1}) q \right] \\
 &= \frac{1}{2^{n-1}} \left[(n^{n-1}) p + (n^{n-1} - (n-2)^{n-1}) q \right]
 \end{aligned}$$

and

$$\begin{aligned}
 \mu_{\mathcal{L}}(\mathcal{E}) &= \frac{c_1(\mathcal{E})c_1(\mathcal{L})^{n-1}}{rk(\mathcal{E})} = \frac{-\alpha(\xi + (\frac{n-2}{2}\alpha)^{n-1})}{2} \\
 &= \frac{1}{2}(-\alpha) \left[\frac{1}{2^{n-1}} (\alpha(n-2))^{n-1} + \frac{1}{2^{n-2}} (n-1)\xi(\alpha(n-2))^{n-2} + \right. \\
 &\quad \left. + \frac{1}{2^{n-2}} (n-2)(n-1)\xi^2(\alpha(n-2))^{n-3} + \dots \right] \\
 &= \frac{1}{2^n}(-\alpha) \left[(\alpha(n-2))^{n-1} + 2 \binom{n-1}{1} (n-2)^{n-1-1} \xi^{n-1} + 2^2 \binom{n-1}{2} (n-2)^{n-1-2} \xi^{n-1} + \dots \right] \\
 &= \frac{1}{2^n}(-\alpha) \left[(\alpha(n-2))^{n-1} + \sum_{k=1}^{n-1} 2^k \binom{n-1}{k} (n-2)^{n-1-k} \right] \\
 &= \frac{1}{2^n}(-\alpha) \left[(\alpha(n-2))^{n-1} + (n^{n-1} - (n-2)^{n-1}) \xi^{n-1} \right] \\
 &= \frac{1}{2^n} \left[((n-2)^{n-1} - n^{n-1}) \xi^n \right].
 \end{aligned}$$

Following Hoppe's Criterion, we must then show $H^0(\mathcal{E}(-p, -q)) = 0$ for any p and q so that $\deg(\mathcal{O}_{\mathbb{P}^n}(p, q)) > \mu_{\mathcal{L}}(\mathcal{E})$. The last inequality gives us that

$$q > -\frac{1}{2} - \frac{p}{2 \left(1 - (\frac{n-2}{n})^{n-1}\right)} \quad (3.3.2)$$

Twisting the sequence (3.2.12) by $\mathcal{O}_{\mathbb{P}^n}(-p, -q)$ we obtain

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1-p, -q) \rightarrow \mathcal{E}(-p, -q) \rightarrow I_X(1-p, -1-q) \rightarrow 0. \quad (3.3.3)$$

To demonstrate that $H^0(\mathcal{E}(-p, -q))$, we need to verify that the ends in the sequence

(3.3.3) have cohomologies in the null dimension equal to 0. We will start with the cohomologies of $I_X(1-p, -1-q)$ because I_X is in the sequence (2.1.2) and $\mathcal{O}_{\mathbb{P}^n}(-1-p, -q) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(1-p, -1-q)$. To do this, we divide it into two cases, namely:

- $1-p \geq 0$: In this case $p \leq 1$ and

$$h^0(I_X(1-p, -1-q)) \leq h^0(\mathcal{O}_{\mathbb{P}^n}(1-p, -1-q)) = \sum_{k=0}^{1-p} \binom{n-2-q+k}{n-1}.$$

Therefore, to find the only p that will contribute to this sum, we need to consider the following inequality

$$\begin{aligned} n-1 \leq n-2-q+k \leq n-2+1-p < n-1 + \frac{1}{2} + \frac{p}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)} - p \\ = n - \frac{1}{2} + \left(\frac{1 - 2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)} \right) p. \end{aligned} \quad (3.3.4)$$

As the term $\frac{1 - 2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)}$ converges at infinity to $\frac{-2+e^2}{2(-1+e^2)} \cong 0.42174$. Without loss of generality, we can implement that on the right side of the inequality (3.3.4) is equal to $n - \frac{1}{2} + p$. Consequently, the values of p are

$$n-1 \leq n - \frac{1}{2} + p \Rightarrow p > -\frac{1}{2} \Rightarrow p \geq 0 \Rightarrow p = 0, 1.$$

Therefore, for any $p \leq -1$ the above binomial is null and consequently, the two extremes of the sequence (3.3.3) have zero cohomologies, in dimension zero. Now, for each object equal to $p = \{0, 1\}$, find q such that the pair (p, q) contributes to the sum. For $p = 0$ in the inequality (3.3.2) we obtain $q \geq 0$. On the other hand

$$n-1 \leq n-2-q+1 = n-1-q$$

and $0 \geq q$. Thus, $q = 0$, that is, for $p = 0$ only when $q = 0$ the above binomial is nonzero. In this specific case, we must calculate the cohomology using another tool. But $h^0(I_X(1, -1))$ is 0 because X is not contained in the exceptional divisor. Furthermore, at the other extreme we get $h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)) = 0$. Therefore, for this pair, we also have zero cohomologies at the sequence end. Lastly, for $p = 1$, in the inequality (3.3.2) we obtain

$$q > -\frac{1}{2} - \frac{1}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)} = \frac{-1 + \left(\frac{n-2}{n}\right)^{n-1} - 1}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)} \xrightarrow{n \rightarrow \infty} \frac{1 - 2e^2}{2(-1 + e^2)} \cong -1.07825,$$

so that $q \geq -1$. On the other hand $n - 1 \leq n - 2 - q$, and $-1 \geq q$. Similarly to before, we have to calculate $h^0(I_X)$ and $h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 1))$. But these two cohomologies are also null too.

- $1 - p < 0$: In this case $p > 1$ and

$$\begin{aligned} h^0(I_X(1 - p, -1 - q)) &\leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(1 - p, -1 - q)) \\ &\cong h^n(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2 - 1 + p, 1 - n + 1 + q)) \\ &= h^n(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-3 + p, 2 - n + q)) = \sum_{k=0}^{1-p} \binom{n-2-q+k}{n-1} \\ &= 0 \end{aligned}$$

because $1 - p < 0$. Therefore, the other extreme also has zero cohomology.

Based on the above it follows that $H^0(\mathcal{E}(-p, -q)) = 0$ for the pairs p and q in the inequality (3.3.2), as we wanted. As result, the sheaf $\mathcal{E}$ is semi-stable for any even n .

The Chern classes can be found in a similar way to what was done in the case $n = 4$. For

the charge we have

$$\begin{aligned}
c &= c_2(\mathcal{E})c_1(\mathcal{L})^{n-2} = (2\xi^2 - \alpha^2) \left(\xi + \left(\frac{n-2}{2} \right) \alpha \right)^{n-2} \\
&= 2\xi^2 \left(\xi + \left(\frac{n-2}{2} \right) \alpha \right)^{n-2} - \alpha^2 \left(\xi + \left(\frac{n-2}{2} \right) \alpha \right)^{n-2} \\
&= 2 \left(\xi^{\frac{2}{n-2}} \xi + \xi^{\frac{2}{n-2}} \left(\frac{n-2}{2} \right) \alpha \right)^{n-2} - \left(\alpha^{\frac{2}{n-2}} \xi + \alpha^{\frac{2}{n-2}} \left(\frac{n-2}{2} \right) \alpha \right)^{n-2} \\
&\stackrel{*}{=} 2 \left(\xi^{\frac{n}{n-2}} + \xi^{\frac{n}{n-2}} \left(\frac{n-2}{2} \right) \right)^{n-2} - \left(\xi^{\frac{n}{n-2}} + \xi^{\frac{n}{n-2}} \left(\frac{n-2}{2} \right) \right)^{n-2} + \left(\frac{n-2}{2} \right)^{n-2} \xi^n \\
&= \left(\left(\frac{n}{2} \right)^{n-2} + \left(\frac{n-2}{2} \right)^{n-2} \right) \xi^n = \frac{n^{n-2} + (n-2)^{n-2}}{2^{n-2}} \xi^n.
\end{aligned}$$

In the above equality (*) the term at the end appears because it is the coefficient of α^n which in turn is zero. But to be able to exchange α for ξ we must disregard this coefficient because in the exchange it would be counted. ■

We then show that there exists a family of sheaf instantons that verify Definition 3.3 for any even $n \geq 4$. Next, we will show that these instantons are cohomologies of monads as in Lemma 3.2.

Proposition 3.16. *The instanton $\mathcal{E}$ in $\widetilde{\mathbb{P}^n}$ is the cohomology of a monad*

$$\begin{aligned}
&\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 2} \\
M_n : 0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus 2} \rightarrow &\quad \oplus \quad \rightarrow \Omega^{n-2}(0, n-2) \rightarrow 0. \quad (3.3.5) \\
&\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus (n+1)}
\end{aligned}$$

Proof. The verification that the monad is given in this form follows from Lemma 3.2 and the cohomologies of the table 3.3.1 in Proposition 3.14.

Note that, the rank of $\Omega^{n-2}(0, n-2)$ follows from the exact sequence

$$0 \rightarrow \Omega^{n-1}(n-2) \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\binom{n}{n-1}} \rightarrow \Omega^{n-2}(n-2) \rightarrow 0.$$

Thus, the rank and Chern classes of $\Omega^{n-2}(0, n-2)$ are calculated using this sequence. Therefore

$$\begin{aligned} \text{rk}(\Omega^{n-2}(n-2)) &= \text{rk}\left(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\binom{n}{n-1}}\right) - \text{rk}(\Omega^{n-1}(n-2)) \\ &= \text{rk}\left(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\binom{n}{n-1}}\right) - \text{rk}(\mathcal{O}_{\mathbb{P}^{n-1}}(-2)) \\ &= \binom{n}{n-1} - 1 = n-1 \end{aligned}$$

and

$$\begin{aligned} c(\Omega^{n-2}(n-2)) &= \frac{c\left(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\binom{n}{n-1}}\right)}{c(\Omega^{n-1}(n-2))} = \frac{c\left(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\binom{n}{n-1}}\right)}{c(\mathcal{O}_{\mathbb{P}^{n-1}}(-2))} \\ &= \frac{(1-\alpha)^n}{1-2\alpha} \end{aligned}$$

As a result, the monad has rank and Chern classes given by

$$\begin{aligned} \text{rk}(\mathcal{E}) &= \text{rk}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus (n+1)}) - \text{rk}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus 2}) - \text{rk}(\Omega^{n-2}(0, n-2)) \\ &= (n+1+2) - 2 - (n-1) = 2 \end{aligned}$$

and

$$\begin{aligned} c(\mathcal{E}) &= \frac{c(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 2} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus (n+1)})}{c(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus 2})c(\Omega^{n-2}(0, n-2))} = \frac{(1-\xi)^2(1-\alpha)^{n+1}(1-2\alpha)}{(1-\xi-\alpha)^2(1-\alpha)^n} \\ &= 1 - \alpha + 2\xi^2 - \alpha^2. \end{aligned}$$

■

Theorem 3.17. For any $\mu_{\mathcal{L}}$ -semi-stable sheaf $\mathcal{F}$ that is the cohomology of the monad M_n , we have that $\mathcal{F}$ is an instanton on $\widetilde{\mathbb{P}^n}$ with even $n \geq 4$.

Proof. Since $\mathcal{F}$ is cohomology of the monad M_n , it is part of its display and therefore we obtain the following exact sequences

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus 2} \rightarrow K \rightarrow \mathcal{F} \rightarrow 0 \quad (3.3.6)$$

and

$$\begin{aligned}
& \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 2} \\
0 \rightarrow K \rightarrow & \oplus \rightarrow \Omega^{n-2}(0, n-2) \rightarrow 0 \\
& \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus(n+1)}
\end{aligned} \tag{3.3.7}$$

where K is the kernel of the second map of the monad. Let us now show that the cohomologies in Definition 3.3 are satisfied. To do this, in each case, we will calculate the cohomologies of $K \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q)$ by the sequence (3.3.7) and then we will calculate those of $\mathcal{F} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q)$ using the sequence (3.3.6). In the case of $\Omega^{n-2}(0, n-2) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q)$ cohomologies we will use the sequence

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -2) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1) \rightarrow \Omega^{n-2}(0, n-2) \rightarrow 0. \tag{3.3.8}$$

- $H^0(\mathcal{F}(0, -2))$: Note that

$$\dots \rightarrow \overbrace{H^0(K(0, -2))}^0 \rightarrow H^0(\mathcal{F}(0, -2)) \rightarrow \overbrace{H^1(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -3)^{\oplus 2})}^0 \rightarrow \dots$$

by Bott's formulae and

$$H^0(K(0, -2)) \leq H^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -2)^{\oplus 2}) \oplus H^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -3)^{\oplus(n+1)}) = 0.$$

Then, $H^0(\mathcal{F}(0, -2)) = 0$.

- $H^0(\mathcal{F}(-1, 0))$: Note that

$$\dots \rightarrow \overbrace{H^0(K(-1, 0))}^0 \rightarrow H^0(\mathcal{F}(-1, 0)) \rightarrow \overbrace{H^1(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, -1)^{\oplus 2})}^0 \rightarrow \dots$$

because

$$\begin{aligned}
H^0(K(-1, 0)) & \leq H^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 0)^{\oplus 2}) \oplus H^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus(n+1)}) \\
& \cong H^n(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1-n)^{\oplus 2}) \oplus H^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus(n+1)}) = 0
\end{aligned}$$

and

$$H^1(\mathcal{O}_{\mathbb{P}^n}(-2, -1)^{\oplus 2}) \cong H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(0, -n+2)^{\oplus 2}) = 0.$$

Therefore $H^0(\mathcal{F}(-1, 0)) = 0$.

- $H^1(\mathcal{F}(-1, -1))$: Note that

$$\cdots \rightarrow \overbrace{H^1(K(-1, -1))}^0 \rightarrow H^1(\mathcal{F}(-1, -1)) \rightarrow \overbrace{H^2(\mathcal{O}_{\mathbb{P}^n}(-2, -2)^{\oplus 2})}^0 \rightarrow \cdots$$

Let's first calculate the necessary cohomology of $K(-1, -1)$. In addition to using Bott's formulae, we have

$$\cdots \rightarrow \overbrace{H^0(\Omega^{n-2}(1, n-3))}^0 \rightarrow H^1(K(-1, -1)) \rightarrow \overbrace{H^1(\mathcal{O}_{\mathbb{P}^n}(-2, -1)^{\oplus 2}) \oplus H^1(\mathcal{O}_{\mathbb{P}^n}(-1, -2)^{\oplus (n+1)})}^0 \rightarrow \cdots$$

because

$$\cdots \rightarrow \overbrace{H^0(\mathcal{O}_{\mathbb{P}^n}(-1, -1)^{\binom{n}{n-1}})}^0 \rightarrow H^0(\Omega^{n-2}(1, n-3)) \rightarrow \overbrace{H^1(\mathcal{O}_{\mathbb{P}^n}(-1, -3))}^0 \rightarrow \cdots$$

Consequently $H^1(\mathcal{F}(-1, -1)) = 0$.

- $H^{n-1}(\mathcal{F}(0, 3-n))$: Note that

$$\cdots \rightarrow \overbrace{H^{n-1}(K(0, 3-n))}^0 \rightarrow H^{n-1}(\mathcal{F}(0, 3-n)) \rightarrow \overbrace{H^n(\mathcal{O}_{\mathbb{P}^n}(-1, 2-n)^{\oplus 2})}^0 \rightarrow \cdots$$

Again we have

$$\cdots \rightarrow \overbrace{H^{n-2}(\Omega^{n-2}(0, 1))}^0 \rightarrow H^{n-1}(K(0, 3-n)) \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-1, 3-n)^{\oplus 2}) \oplus H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(0, 2-n)^{\oplus (n+1)})}^0 \rightarrow \cdots$$

because

$$\cdots \rightarrow \overbrace{H^{n-2}(\mathcal{O}_{\mathbb{P}^n}(0, 2-n)^{\binom{n}{n-1}})}^0 \rightarrow H^{n-2}(\Omega^{n-2}(0, 1)) \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(0, 1-n))}^0 \rightarrow \cdots$$

Then $H^{n-1}(\mathcal{F}(0, 3-n)) = 0$.

- $H^n(\mathcal{F}(0, 2-n))$: Note that

$$\cdots \rightarrow \overbrace{H^n(K(0, 2-n))}^0 \rightarrow H^n(\mathcal{F}(0, 2-n)) \rightarrow 0.$$

Once more we have

$$\cdots \rightarrow \overbrace{H^{n-1}(\Omega^{n-2})}^0 \rightarrow H^n(K(0, 2-n)) \rightarrow \overbrace{H^n(\mathcal{O}_{\mathbb{P}^n}(-1, 2-n)^{\oplus 2}) \oplus H^n(\mathcal{O}_{\mathbb{P}^n}(0, 1-n)^{\oplus (n+1)})}^0 \rightarrow \cdots$$

because

$$\cdots \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(0, 1-n)^{\binom{n}{n-1}})}^0 \rightarrow H^{n-1}(\Omega^{n-2}) \rightarrow \overbrace{H^n(\mathcal{O}_{\mathbb{P}^n}(0, n))}^0 \rightarrow \cdots$$

Hence $H^n(\mathcal{F}(0, 2-n)) = 0$.

- $H^n(\mathcal{F}(-1, 4-n))$: Note that

$$\cdots \rightarrow \overbrace{H^n(K(-1, 4-n))}^0 \rightarrow H^n(\mathcal{F}(-1, 4-n)) \rightarrow 0.$$

Once again we have

$$\cdots \rightarrow \overbrace{H^{n-1}(\Omega^{n-2}(-1, 2))}^0 \rightarrow H^n(K(-1, 4-n)) \rightarrow \overbrace{H^n(\mathcal{O}_{\mathbb{P}^n}(-2, 4-n)^{\oplus 2}) \oplus H^n(\mathcal{O}_{\mathbb{P}^n}(-1, 3-n)^{\oplus (n+1)})}^0 \rightarrow \cdots$$

because

$$\cdots \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-1, 3-n)^{\binom{n}{n-1}})}^0 \rightarrow H^{n-1}(\Omega^{n-2}(-1, 2)) \rightarrow \overbrace{H^n(\mathcal{O}_{\mathbb{P}^n}(-1, 2-n))}^0 \rightarrow \cdots$$

Then $H^n(\mathcal{F}(-1, 4-n)) = 0$.

- $H^1(\mathcal{F}(0, -3))$: Note that

$$\cdots \rightarrow \overbrace{H^1(K(0, -3))}^0 \rightarrow H^1(\mathcal{F}(0, -3)) \rightarrow \overbrace{H^2(\mathcal{O}_{\mathbb{P}^n}(-1, -4)^{\oplus 2})}^0 \rightarrow \cdots$$

Once more,

$$\cdots \rightarrow \overbrace{H^0(\Omega^{n-2}(0, n-5))}^0 \rightarrow H^1(K(0, -3)) \rightarrow \overbrace{H^1(\mathcal{O}_{\mathbb{P}^n}(-1, -3)^{\oplus 2}) \oplus H^1(\mathcal{O}_{\mathbb{P}^n}(0, -4)^{\oplus (n+1)})}^0 \rightarrow \cdots$$

because

$$\cdots \rightarrow \overbrace{H^0(\mathcal{O}_{\mathbb{P}^n}(0, -4)^{\binom{n}{n-1}})}^0 \rightarrow H^0(\Omega^{n-2}(0, n-5)) \rightarrow \overbrace{H^1(\mathcal{O}_{\mathbb{P}^n}(0, n-5))}^0 \rightarrow \cdots$$

Consequently $H^1(\mathcal{F}(0, -3)) = 0$.

- $H^{n-1}(\mathcal{F}(-1, 5-n))$: Note that

$$\cdots \rightarrow \overbrace{H^{n-1}(K(-1, 5-n))}^0 \rightarrow H^{n-1}(\mathcal{F}(0-1, 5-n)) \rightarrow \overbrace{H^n(\mathcal{O}_{\mathbb{P}^n}(-2, 4-n)^{\oplus 2})}^0 \rightarrow \cdots$$

Again we have

$$\cdots \rightarrow \overbrace{H^{n-2}(\Omega^{n-2}(-1, 3))}^0 \rightarrow H^{n-1}(K(-1, 5-n)) \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-2, 5-n)^{\oplus 2}) \oplus H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-1, 4-n)^{\oplus (n+1)})}^0 \rightarrow \cdots$$

because

$$\cdots \rightarrow \overbrace{H^{n-2}(\mathcal{O}_{\mathbb{P}^n}(-1, 4-n)^{\binom{n}{n-1}})}^0 \rightarrow H^{n-2}(\Omega^{n-2}(-1, 3)) \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-1, 3-n))}^0 \rightarrow \cdots$$

Then $H^{n-1}(\mathcal{F}(-1, 5-n)) = 0$.

Since the cohomologies required in the definition of instanton are satisfied and the other properties of $\mathcal{F}$ in the definition are given by hypothesis, it follows that any sheaf $\mathcal{F}$ given in this way is an instanton. ■

3.4 Another example for even n with different X .

In the previous sections, we always used the same space $X = \wp \cup \kappa$; in this section, we will construct examples using another space. We will consider $X' = \wp \cup \kappa'$ with $\wp$ as before,

but $[\kappa'] = 2(\xi^2 - \alpha^2)$. The sheaf of structures κ' has the resolution

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -2) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}} \rightarrow \mathcal{O}_{\kappa'} \rightarrow 0. \quad (3.4.1)$$

Its second Chern class is $c_2(\mathcal{O}_{\kappa'}) = 2(\alpha^2 - \xi^2)$ and the determinant of its normal bundle is $\det \mathcal{N}_{\kappa'} \cong \mathcal{O}_{\mathbb{P}^{n-2}}(2)$. Furthermore,

$$\mathcal{O}_{X'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q) \cong \mathcal{O}_{\mathbb{P}^{n-2}}(p + q) \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(2q) \quad (3.4.2)$$

changing only the restriction part to κ' because, for example with $n = 4$,

$$[\kappa'](p\xi + q\alpha) = 2(\xi^2 - \alpha^2)(p\xi + q\alpha) = 2q\xi^3.$$

Considering $S = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1)$ on $\widetilde{\mathbb{P}^n}$, by Bott's formulae we have $H^1(S^*) = 0$ and $H^2(S^*) = 0$ and $\det \mathcal{N}_{X'} \otimes S^*|_{X'} \cong \begin{cases} \mathcal{O}_{\mathbb{P}^{n-2}}(1) & \text{in } \mathcal{P} \\ \mathcal{O}_{\mathbb{P}^{n-2}} & \text{in } \kappa' \end{cases}$. Thus, by the Hartshorne-Serre Correspondence 2.1, there is a sheaf $\mathcal{F}$ given by the sequence $0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}} \rightarrow \mathcal{F} \rightarrow I_{X'}(0, 1) \rightarrow 0$. Twisting by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)$ we obtain a sheaf $\mathcal{E}'$ given by the exact sequence

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1) \rightarrow \mathcal{E}' \rightarrow I_{X'} \rightarrow 0 \quad (3.4.3)$$

Unfortunately, we again find a non-locally free candidate. This is because $\mathcal{E}xt^1(\mathcal{E}', \mathcal{O}_{\widetilde{\mathbb{P}^n}})$ is non-zero, as we will show below.

Proposition 3.18. *The sheaf $\mathcal{E}'$ defined as before is not locally free.*

Proof. To verify this, let us consider the $\mathcal{H}om(-, \mathcal{O}_{\widetilde{\mathbb{P}^4}})$ sequence of (3.4.3).

$$\begin{aligned} \Pi : 0 &\rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}} &\rightarrow \mathcal{E}'^\vee &\rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1) \\ &\rightarrow \mathcal{E}xt^1(I_{X'}, \mathcal{O}_{\widetilde{\mathbb{P}^n}}) &\rightarrow \mathcal{E}xt^1(\mathcal{E}', \mathcal{O}_{\widetilde{\mathbb{P}^n}}) &\rightarrow \mathcal{E}xt^1(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1), \mathcal{O}_{\widetilde{\mathbb{P}^n}}) \\ &\rightarrow \mathcal{E}xt^2(I_{X'}, \mathcal{O}_{\widetilde{\mathbb{P}^n}}) &\rightarrow \mathcal{E}xt^2(\mathcal{E}', \mathcal{O}_{\widetilde{\mathbb{P}^n}}) &\rightarrow \mathcal{E}xt^2(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1), \mathcal{O}_{\widetilde{\mathbb{P}^n}}) \rightarrow \dots \end{aligned}$$

Using the properties of the groups $\mathcal{E}xt$ as done in previous sections we obtain

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}} \rightarrow \mathcal{E}'^\vee \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1) \xrightarrow{f} \omega_{X'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2, -(1-n)) \rightarrow \mathcal{E}xt^1(\mathcal{E}', \mathcal{O}_{\widetilde{\mathbb{P}^n}}) \rightarrow 0. \quad (3.4.4)$$

If f is not a surjective map, we obtain that $\mathcal{E}xt^1(\mathcal{E}', \mathcal{O}_{\widetilde{\mathbb{P}^n}})$ is not null as we claimed before. But,

$$\begin{aligned} f \in \text{Hom}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1), \omega_{X'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2, n-1)) &\cong H^0(\omega_{X'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2, n-2)) \\ &\cong H^0(\omega_{\mathbb{P}^{n-2}} \otimes (\mathcal{O}_{\mathbb{P}^{n-2}}(n) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(2n-4))) \\ &\cong \mathcal{O}_{\mathbb{P}^{n-2}}(1) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(n-3) \end{aligned}$$

and in the diagram

$$\begin{array}{ccc} \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, 1) & \xrightarrow{f} & \omega_{X'} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2, n-1) \cong \mathcal{O}_{\mathcal{P}}(2) \oplus \mathcal{O}_{\mathcal{K}}(n-1) \\ \downarrow r_{X'} & \nearrow \bar{f} & \\ \mathcal{O}_{\mathcal{P}}(1) \oplus \mathcal{O}_{\mathcal{K}}(2) & & \end{array}$$

we find that $\bar{f}$ is not surjective. Therefore, f can not be surjective and, consequently, $\mathcal{E}'$ is not locally free. ■

Although they are not locally free, these sheaves satisfy the Definition 3.3. We will perform the following proof, first calculating the cohomologies.

Proposition 3.19. *The sheaf $\mathcal{E}'$ has the following cohomology table on $\widetilde{\mathbb{P}^n}$*

$\mathcal{E}'(-1, -n+2)$	$\mathcal{E}'(-1, -n+3)$	...	$\mathcal{E}'(-1, 0)$	$\mathcal{E}'(-1, 1)$	$\mathcal{E}'(0, -n+1)$	$\mathcal{E}'(0, -n+2)$	$\mathcal{E}'(0, -n+3)$	...	$\mathcal{E}'(0, -1)$	$\mathcal{E}'$	
0	0	...	0	0	0	0	0	...	0	0	h^0
0	0	...	1	7	0	0	0	...	0	1	h^1
0	0	...	0	0	0	0	0	...	0	0	h^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	...	0	0	0	0	0	...	0	0	h^{n-2}
$\binom{2n-5}{n-3} + 1$	$\binom{2n-7}{n-5}$	...	0	0	$\binom{2n-3}{n-1} + 2$	$\binom{2n-5}{n-3}$	$\binom{2n-7}{n-5}$	...	0	0	h^{n-1}
0	0	...	0	0	0	0	0	...	0	0	h^n

Table 3.4.1: Cohomologies of $\mathcal{E}'$ in $\widetilde{\mathbb{P}^n}$. Font: Created by the author.

Proof. We will do the proof by twisting the sequence (3.4.3) by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -t)$ and $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -t)$ considering specific values of t as done in the previous section.

- Calculating the cohomologies of $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -t)$ with $1 \leq t \leq n-1$: Using the sequences (3.4.3) and (2.1.2) we obtain

$$0 \rightarrow \overbrace{\mathbf{h}^{n-1}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1-t))}^{\binom{t}{n-1}} \rightarrow \mathbf{h}^{n-1}(\mathcal{E}'(0, -t)) \rightarrow \overbrace{\mathbf{h}^{n-1}(I_{X'}(0, -t))}^{\binom{t-1}{t+1-n} + \binom{2t-1}{2t+1-n}} \rightarrow 0. \quad (3.4.5)$$

Be careful with the values of t for which the binomials appearing in each cohomological dimension are well defined. For example, the binomial $\binom{2t-1}{2t+1-n}$ is only well defined for $\frac{n-1}{2} \leq t$ and only for these and for $t \leq n-1$ will it be counted in the dimension.

- Calculating the cohomologies of $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -t)$ with $1 \leq t \leq n$: Using the sequences (3.4.3) and (2.1.2) we obtain

$$0 \rightarrow \mathbf{h}^{n-1}(\mathcal{E}'(-1, -t)) \rightarrow \overbrace{\mathbf{h}^{n-1}(I_{X'}(-1, -t))}^{\binom{t}{t+2-n} + \binom{2t-1}{2t+1-n}} \rightarrow 0. \quad (3.4.6)$$

Again, be careful with the values of t .

- To calculate the cohomologies of $\mathcal{E}'$ and $\mathcal{E}'(-1, 0)$ simply use the sequences (3.4.3) and (2.1.2).

■

Theorem 3.20. *The sheaf $\mathcal{E}'$ is an instanton in $\widetilde{\mathbb{P}^n}$ with $n \geq 4$ and even and polarization $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}\left(1, \frac{n-2}{2}\right)$. In addition, $c_2(\mathcal{E}') = 3\xi^2 - 2\alpha^2$ and charge $c = \frac{n^{n-2} + 2(n-2)^{n-2}}{2^{n-2}}$.*

Proof. The cohomologies follow Proposition 3.19. For Chern classes and charge, use the sequence (3.4.3) and follow the same reasoning as in the example shown in the previous section. Now, let us show the semi-stability with respect to polarization. To do this, we will use Hoppe's Criterion 2.4 again and show that $H^0(\mathcal{E}(-p, -q)) = 0$ for any p and q so that 3.3.2. Recall that where $\mathbf{h}^0(I_{X'}(-p, -q))$ is null, $\mathbf{h}^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-p, -1-q))$ will also be null. Thus, we will start by calculating the cohomological dimensions of $I_{X'}(-p, -q)$ by dividing into three cases that depend on the values of p .

$-p > 0$: Since $-p$ is positive, it follows from Bott's formulae that

$$h^0(I_{X'}(-p, -q)) \leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-p, -q)) = \sum_{k=0}^{-p} \binom{n-1-q+k}{n-1}.$$

We will analyze for which values of p this binomial is defined.

$$\begin{aligned} n-1 \leq n-1-q+k &\leq n-1-q-p < n-1 + \frac{1}{2} + \frac{p}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)} - p \\ &= n - \frac{1}{2} + \left(\frac{1 - 2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)}{2 \left(1 - \left(\frac{n-2}{n}\right)^{n-1}\right)} \right) p. \end{aligned}$$

From the previous section we know that this last term is smaller than $n - \frac{1}{2} + p$, which in turn will be larger than $n-1$ only when $p \geq 0$. Therefore, in this case, we analyze when $p < 0$, so this binomial here will be zero. Consequently, $h^0(I_{X'}(-p, -q)) = 0$ and $h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-p, -1-q))$.

$p = 0$: In this case, we still have positive p , so we can apply the same idea as before using the binomial with $k = 0$. Thus, $n-1 \leq n-1-q$ implies that $0 \leq -q$. On the other hand, by the inequality (3.3.2) with $p = 0$ we have $-q \leq 0$. Therefore, for $p = 0$ and $q = 0$ it is the only pair where the binomial does not vanish, and thus only in this case we cannot conclude that the dimension of $I_{X'}(-p, -q)$ is vanishing. However, we know that $h^0(I_{X'}) = 0$. In addition, $h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)) = 0$. As a result, we find that the desired cohomologies also vanish for this case.

$-p < 0$: Since p is now negative, we need to use Serre's duality. We then obtain that

$$h^0(I_{X'}(-p, -q)) \leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-p, -q)) \cong h^n(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2+p, 1-n+q)) = 0$$

because $-p < 0$. Again, $h^0(I_{X'}(-p, -q)) = 0$ and $h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-p, -1-q))$.

■

As the previous results show, considering the space X' determined at the beginning of

the section, we obtain a family of instanton sheaves. Each of these sheaves is given as the cohomology of a monad as shown below.

Proposition 3.21. *The sheaf $\mathcal{E}'$ on $\widetilde{\mathbb{P}^n}$ is the cohomology of a monad M'_n .*

$$\begin{array}{ccccccc}
 \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus(1+\binom{2n-5}{n-3})} & & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 7} & & \mathcal{O}_{\widetilde{\mathbb{P}^n}} & & \\
 M'_n : 0 \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow 0 \quad (3.4.7) \\
 & \Omega(-1, 1) & & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus(2+\binom{2n-3}{n-1})} & & \Omega^{n-2}(0, n-2)^{\oplus(2+\binom{2n-5}{n-3})} &
 \end{array}$$

Proof. The proof is analogous to that given in the previous sections. It suffices to use the cohomology table 3.4.1 of $\mathcal{E}'$ and the complex given by the Lemma 3.2.

$$\begin{aligned}
 C^{-1} &= \overbrace{H^{n-1}(\mathcal{E}'(-1, -n+2))}^{\mathbb{C}^{\oplus(1+\binom{2n-5}{n-3})}} \otimes \Omega^{n-1}(-1, n-1) \oplus \overbrace{H^1(\mathcal{E}'(-1, 0))}^{\mathbb{C}} \otimes \Omega(-1, 1) \\
 &\cong \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus(1+\binom{2n-5}{n-3})} \oplus \Omega(-1, 1) \\
 C^0 &= \overbrace{H^1(\mathcal{E}'(-1, 1))}^{\mathbb{C}^{\oplus 7}} \otimes \Omega_{\mathbb{P}^3}^0(-1, 0) \oplus \overbrace{H^3(\mathcal{E}'(0, 1-n))}^{\mathbb{C}^{\oplus(2+\binom{2n-3}{n-1})}} \otimes \Omega^{n-1}(0, 1-n) \\
 &\cong \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 7} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus(2+\binom{2n-3}{n-1})} \\
 C^1 &= \overbrace{H^1(\mathcal{E}')}^{\mathbb{C}} \otimes \Omega^0 \oplus \overbrace{H^{n-1}(\mathcal{E}'(0, 2-n))}^{\mathbb{C}^{\oplus(2n-5)}} \otimes \Omega^{n-2}(0, n-2) \cong \mathcal{O}_{\widetilde{\mathbb{P}^n}} \oplus \Omega^{n-2}(0, n-2)^{\oplus(2n-5)}.
 \end{aligned}$$

■

Theorem 3.22. *For any $\mu_{\mathcal{L}}$ -semi-stable sheaf $\mathcal{F}$ that is the cohomology of the monad M'_n , we find that $\mathcal{F}$ is an instanton sheaf on $\widetilde{\mathbb{P}^n}$ with even $n \geq 4$.*

Proof. To demonstrate this proposition, we will use exact sequences given by the monad display. These sequences will be twisted by the corresponding line bundle for each cohomological condition. We then consider a coherent and torsion-free sheaf $\mathcal{F}$ that is semi-stable with respect to the polarization $\mathcal{L}$ on $\widetilde{\mathbb{P}^n}$ ($n \geq 4$ even). To calculate the cohomological conditions, follow the

process described in Proposition 3.17. Considering that the display sequences are as follows

$$\begin{array}{ccccccc}
 & & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)^{\oplus(1+\binom{2n-5}{n-3})} & & & & \\
 0 \rightarrow & & \oplus & \rightarrow & K \rightarrow & \mathcal{F} \rightarrow & 0 \\
 & & \Omega(-1, 1) & & & &
 \end{array} \quad (3.4.8)$$

and

$$\begin{array}{ccccccc}
 & & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0)^{\oplus 7} & & & \mathcal{O}_{\widetilde{\mathbb{P}^n}} & \\
 0 \rightarrow K \rightarrow & & \oplus & \rightarrow & \oplus & \rightarrow & 0 \\
 & & \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -1)^{\oplus(2+\binom{2n-3}{n-1})} & & \Omega^{n-2}(0, n-2)^{\oplus(2+\binom{2n-5}{n-3})} & &
 \end{array} \quad (3.4.9)$$

where K is the kernel of the second map of the monad. Let us calculate, for example, the cohomological condition $H^{n-1}(\mathcal{F}(-1, 5-n)) = 0$. Note that

$$\cdots \rightarrow \overbrace{H^{n-1}(K(-1, 5-n))}^0 \rightarrow H^{n-1}(\mathcal{F}(0-1, 5-n)) \rightarrow \overbrace{H^n(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 4-n)^{\oplus(1+\binom{2n-5}{n-3})}) \oplus H^n(\Omega(-2, 6-n))}^0 \rightarrow \cdots$$

The cohomology of $K(-1, 5-n)$ is null since

$$\begin{array}{c}
 \overbrace{\cdots \rightarrow H^{n-2}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 5-n)) \oplus H^{n-2}(\Omega^{n-2}(-1, 3)^{\oplus(2+\binom{2n-5}{n-3})})}^0 \rightarrow H^{n-1}(K(-1, 5-n)) \rightarrow \\
 \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 5-n)^{\oplus 7}) \oplus H^{n-1}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 4-n)^{\oplus(2+\binom{2n-3}{n-1})})}^0 \rightarrow \cdots
 \end{array}$$

To calculate the cohomology of $H^{n-2}(\Omega^{n-2}(-1, 3))$ we use the exact sequence $0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 3-n) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 4-n)^{\oplus \binom{n}{n-1}} \rightarrow \Omega^{n-2}(-1, 3) \rightarrow 0$, and in this way, we obtain

$$\cdots \rightarrow \overbrace{H^{n-2}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 4-n)^{\oplus \binom{n}{n-1}})}^0 \rightarrow H^{n-2}(\Omega^{n-2}(-1, 3)) \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 3-n))}^0 \rightarrow \cdots$$

On the other hand, considering the sequence (2.1.8) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 6-n)$, we have

$$\cdots \rightarrow \overbrace{H^{n-1}(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 6-n))}^0 \rightarrow H^n(\Omega(-2, 6-n)) \rightarrow \overbrace{H^n(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-2, 5-n)^{\oplus \binom{n}{n-1}})}^0 \rightarrow \cdots$$

As result, we have $H^{n-1}(\mathcal{F}(-1, 5-n)) = 0$. ■

We can also analyze some properties of this new family in $\widetilde{\mathbb{P}^4}$ as follows.

Proposition 3.23. *Let L be a line, the restriction of the instanton sheaf $\mathcal{E}'$ on $\widetilde{\mathbb{P}^4}$ to this line is the form*

$$\mathcal{E}'|_L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$$

or

$$\mathcal{E}'|_L = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$

depending on the value of their cohomologies.

Proof. Following the proof given for Proposition 3.11, we see that the Chern class in $\mathbb{P}^1$ is -1 . For cohomologies, using the sequences (3.2.28) we obtain for K_2 two possible cohomologies

$$h^i(K_2) = \begin{cases} 2 & \text{if } i = 1; \\ 1 & \text{if } i = 2; \\ 0 & \text{otherwise;} \end{cases} \quad \text{or} \quad h^i(K_2) = \begin{cases} 1 & \text{if } i = 0; \\ 3 & \text{if } i = 1; \\ 1 & \text{if } i = 2; \\ 0 & \text{otherwise;} \end{cases} \quad (3.4.10)$$

However, since $H^0(\mathcal{E}(-1, 0)) = 0$, the second cohomology option of K_2 is not valid. Consequently, we obtain three options for K_1

$$h^i(K_1) = \begin{cases} 2 & \text{if } i = 1; \\ 0 & \text{otherwise;} \end{cases} \quad \text{or} \quad h^i(K_1) = \begin{cases} 1 & \text{if } i = 0; \\ 3 & \text{if } i = 1; \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad h^i(K_1) = \begin{cases} 2 & \text{if } i = 0; \\ 4 & \text{if } i = 1; \\ 0 & \text{otherwise} \end{cases} \quad (3.4.11)$$

Again, since $H^0(\mathcal{E}) = 0$, the last two cohomologies options for K_1 are not valid. As a result, we obtain

$$h^i(\mathcal{E}|_L) = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{if } i = 1. \end{cases} \quad \text{or} \quad h^i(\mathcal{E}|_L) = \begin{cases} 2 & \text{if } i = 0; \\ 1 & \text{if } i = 1. \end{cases} \quad (3.4.12)$$

Each of these cohomology options together with the value of the Chern class gives us the options presented in the statement. ■

Similarly to the previous sections, we can show that $H^4(\mathcal{E}'(-4, -4)) \geq 57$. Thus, the instanton sheaf $\mathcal{E}'$ is not Ulrich. Furthermore, since $h^1(\mathcal{E}') = 1$, it follows that $\mathcal{E}'$ is not weakly Ulrich and as a consequence is neither almost Ulrich nor Ulrich.

Proposition 3.24. *There is an unobstructed component*

$$\mathcal{I}_{\mathbb{P}^4}^{\sim}(\mathcal{L}, -\alpha, 3\xi^2 - 2\alpha^2) \subset \mathcal{M}_{\mathbb{P}^4}^{ss}(\mathcal{L}, -\alpha, 3\xi^2 - 2\alpha^2),$$

of dimension

$$36 \leq \mathcal{I}_{\mathbb{P}^4}^{\sim}(\mathcal{L}, -\alpha, 3\xi^2 - 2\alpha^2) \leq 55$$

whose points are instantons sheaf.

Proof. The idea of the demonstration is analogous to that made in Proposition 3.12. Setting tables with the possible dimensions of each group Ext until the table (3.4.2).

The table 3.4.2 is divided into 5 blocks with 16 rows each. These rows depend on the possible dimensions with respect to $I_{X'}$ which are triples of the form $(j, i + j, i - 31)$ with $j = 0, \dots, 4$ and $i = 31, \dots, 46$. Fixing j and varying i , we obtain the elements of each block. Also note that the larger i and j , the larger the second entry of the possible dimension triples for $\theta_{\mathcal{E}'}^{0,1,2}$. In other words, the larger the possible value of $\theta_{\mathcal{E}'}^1$. Furthermore, for the second column of possibilities $\theta_{\mathcal{O}_{(0,-1)}}^{0,1}$, we also obtain triples with a larger second entry compared to the possibilities in the same row of the first column.

We can notice a pattern within the 5 blocks. For the blocks in the first resulting column, the possibilities shown in the first block are repeated in the second, and a triple is added in each row. This added triple is the triple of the possibility from the previous block, adding 1 to both the first and second entries. For example, in the first row of the second block, the triple $(0, 36, 0)$ is repeated and $(1, 37, 0)$ is added. The elements of the remaining blocks are obtained in a similar manner.

In the second resulting column, only 16 triples will be repeated in the first two blocks. Starting with the third, the previous block is repeated and a triple is added in each row. This

		$\theta_{\widetilde{\mathcal{C}}^4(0,-1)}^{0,1}$	
		(0,5)	(1,6)
$\theta_{I_{X'}}^{0,1}$	(0,31,0)	$\theta_{\mathcal{C}'}^{0,1,2} = (0,36,0)$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,37,0)$
	(0,32,1)	$\theta_{\mathcal{C}'}^{0,1,2} = (0,37,1)$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,38,1)$
	(0,33,2)	$\theta_{\mathcal{C}'}^{0,1,2} = (0,38,2)$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,39,2)$
	$\vdots$	$\vdots$	$\vdots$
	(0,46,15)	$\theta_{\mathcal{C}'}^{0,1,2} = (0,51,15)$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,52,15)$
	(1,32,0)	$\theta_{\mathcal{C}'}^{0,1,2} = \{(0,36,0), (1,37,0)\}$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,37,0)$
	(1,33,1)	$\theta_{\mathcal{C}'}^{0,1,2} = \{(0,37,1), (1,38,1)\}$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,38,1)$
	$\vdots$	$\vdots$	$\vdots$
	(1,47,15)	$\theta_{\mathcal{C}'}^{0,1,2} = \{(0,51,15), (1,52,15)\}$	$\theta_{\mathcal{C}'}^{0,1,2} = (1,52,15)$
	(2,33,0)	$\theta_{\mathcal{C}'}^{0,1,2} = \{(0,36,0), (1,37,0), (2,38,0)\}$	$\theta_{\mathcal{C}'}^{0,1,2} = \{(1,37,0), (2,38,0)\}$
	(2,34,1)	$\theta_{\mathcal{C}'}^{0,1,2} = \{(0,37,1), (1,38,1), (2,39,1)\}$	$\theta_{\mathcal{C}'}^{0,1,2} = \{(1,38,1), (2,39,1)\}$
	$\vdots$	$\vdots$	$\vdots$
	(2,48,15)	$\theta_{\mathcal{C}'}^{0,1,2} = \{(0,51,15), (1,52,15), (2,53,15)\}$	$\theta_{\mathcal{C}'}^{0,1,2} = \{(1,52,15), (2,53,15)\}$
	$\vdots$	$\vdots$	$\vdots$

Table 3.4.2: Possible dimensions of $\text{Ext}^i(\mathcal{C}', \mathcal{C}^I)$ with $i = 0, 1, 2$. Font: Created by the author.

added triple is the triple of the previous block's possibilities, adding 1 to the first and second entries. Note that for each triple (a, b, c) , starting from the second block, of possibilities relative to $I_{X'}$, the largest triple we obtain in this column is $(a, b + 5, c)$.

To complete the table, simply follow these patterns. In this way, we can find the smallest and largest dimension values that $\text{Ext}^1(\mathcal{E}', \mathcal{E}')$ can achieve. Furthermore, based on the analysis of table 3.4.2, the largest value is given by the combination of $(4, 50, 15)$ with $(1, 6)$.

Obtaining then that the possibility with the largest $\text{Ext}^1(\mathcal{E}', \mathcal{E}')$ is

$$\theta_{\mathcal{E}'}^{0,1,2} = (4, 55, 15). \quad (3.4.13)$$

Consequently

$$36 \leq \dim \text{Ext}^1(\mathcal{E}', \mathcal{E}') \leq 55.$$

■

4 Continuing with the odd n case.

Henni [2], together with Definition 3.1 presents the following example that verifies this definition.

Example 4.1. [2] Let us consider the following codimension 2 sub-varieties; $\mathcal{P}$ is the pull-back via the blow-down map $\pi: \widetilde{\mathbb{P}^5} \rightarrow \mathbb{P}^5$, of a codimension 2 linear space that does not contain the blow-up point, and κ is the general codimension 2 linear space contained in the exceptional divisor. By the Hartshorne-Serre Correspondence 2.1, with $S = \mathcal{O}_{\widetilde{\mathbb{P}^5}}(2, 0)$, there is a bundle $\mathcal{F}$ given by the following exact sequence $0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^5}} \rightarrow \mathcal{F} \rightarrow I_X(2, 0) \rightarrow 0$. We set $\mathcal{E} := \mathcal{F} \otimes \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, -1)$. This bundle $\mathcal{E}$ is an instanton sheaf on $(\widetilde{\mathbb{P}^5}, \mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^5}}(1, 1))$ with exact sequence 4.0.1

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1) \rightarrow \mathcal{E} \rightarrow I_X(1, -1) \rightarrow 0. \quad (4.0.1)$$

More generally, considering $\mathcal{E}$ in $\widetilde{\mathbb{P}^n}$, it will remain an instanton with polarization given by $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}\left(1, \frac{n-3}{2}\right)$. In a fact,

- the Chern class of $\mathcal{E}$ are $c_1(\mathcal{E}) = -2\alpha$ and $c_2(\mathcal{E}) = \xi^2$. Using the sequence of $\mathcal{E}$ (4.0.1) and the restriction sequence (2.1.2) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, -1)$ we have,

$$\begin{aligned} c(\mathcal{E}) &= c(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -1)) \cdot c(I_X(1, -1)) \\ &= (1 - \xi - \alpha) \left(1 + \xi - \alpha + 2\xi^2 - \alpha^2 + \dots\right) \\ &= 1 - 2\alpha + \xi^2 + \dots \end{aligned} \quad (4.0.2)$$

- the cohomologies we need null. To do this, consider the sequence of cohomologies of (4.0.1) and (2.1.2) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(p, q)$. For example, for $\mathcal{E}(0, -2)$ we have

$$h^0(I_X(1, -1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(0, -2)) = h^0(I_X(1, -3)) \leq h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, -3)) = 0$$

and

$$h^0(\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, -3)) = 0.$$

Consequently $h^0(\mathcal{E}(0, -2)) = 0$.

- is $\mu_{\mathcal{L}}$ -semi-stable. Here

$$\mu_{\mathcal{L}}(\mathcal{E}) = \frac{1}{2^{n-1}} \left[\left((n-3)^{n-1} - (n-1)^{n-1} \right) \xi^{n-1} \right]. \quad (4.0.3)$$

By the Hoppe Criterion (2.4) we must show that

$$H^0(\mathcal{E}(-p, -q)) = 0 \quad \forall q > -1 - \frac{p}{1 - \left(1 - \frac{2}{n-1}\right)^{n-1}}.$$

For this, we used the cohomologies sequence of (4.0.1) twisted by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(-p, -q)$.

For more details, see [2].

We can do the same analysis of Proposition 3.5 as we did for the choice of S . In addition, to find the S chosen by Henni [2] to construct the families in the previous example, we will find $S = \mathcal{O}_{\widetilde{\mathbb{P}^5}}$.

Proposition 4.2. *In the blow-up $\widetilde{\mathbb{P}^5}$ and for $X = \wp \cup \kappa$, there are only two possibilities for S to construct instantons via Hartshorne-Serre Correspondence. There are $\mathcal{O}_{\widetilde{\mathbb{P}^5}}(2, 0)$ and $\mathcal{O}_{\widetilde{\mathbb{P}^5}}$.*

Proof. Let us consider $S = \mathcal{O}_{\widetilde{\mathbb{P}^5}}(p, q)$ a line bundle on $\widetilde{\mathbb{P}^5}$. We now analyze the hypotheses of the correspondence and the properties of instantons to determine the possible values of p and q . First, note that $h^2(S^*)$ is always zero in this space. Furthermore, considering that the determinant of the normal in $\wp$ and κ is given by $\det N_{\wp} \cong \mathcal{O}_{\mathbb{P}^3}(2)$ and $\det N_{\kappa} \cong \mathcal{O}_{\mathbb{P}^3}$ it follows that

$$\det N_X \otimes S^*|_X \cong \begin{cases} \mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{\mathbb{P}^3}(-p-q) \text{ in } \wp \\ \mathcal{O}_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-q) \text{ in } \kappa \end{cases} \cong \begin{cases} \mathcal{O}_{\mathbb{P}^3}(2-p-q) \text{ in } \wp \\ \mathcal{O}_{\mathbb{P}^3}(-q) \text{ in } \kappa. \end{cases}$$

Since we want this tensor product to have sections, we need h^0 to be non-zero. Then obtaining the following inequalities

$$-p-q \geq -2; \quad -q \geq 0. \quad (4.0.4)$$

Satisfying these hypotheses, we then obtain a sequence given by

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}^\sim \rightarrow \mathcal{F} \rightarrow I_X(p, q) \rightarrow 0. \quad (4.0.5)$$

However, we know that, by definition, the first Chern class must be -2α , so if we twist the sequence by $\mathcal{O}_{\mathbb{P}^5}^\sim(a, b)$ we obtain that

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}^\sim(a, b) \rightarrow \mathcal{E} \rightarrow I_X(p + a, q + b) \rightarrow 0. \quad (4.0.6)$$

and

$$\begin{cases} p, q \text{ even numbers} \\ a = \frac{-p}{2} \\ b = \frac{-q}{2} - 1. \end{cases} \quad (4.0.7)$$

Consequently, for the instanton to belong to a sequence given by the correspondence, the sequence in question must be of the following form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}^\sim \left(\frac{-p}{2}, \frac{-q}{2} - 1 \right) \rightarrow \mathcal{E} \rightarrow I_X \left(\frac{p}{2}, \frac{q}{2} - 1 \right) \rightarrow 0. \quad (4.0.8)$$

Furthermore, again by the definition of instantons, we need $H^4(\mathcal{E}(0, -2))$ to be zero. Note that,

$$\cdots \rightarrow 0 \rightarrow H^4 \left(\mathcal{O}_{\mathbb{P}^5}^\sim \left(\frac{-p}{2}, \frac{-q}{2} - 3 \right) \right) \rightarrow H^4(\mathcal{E}(0, -2)) \rightarrow H^4 \left(I_X \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \right) \rightarrow 0 \rightarrow \cdots \quad (4.0.9)$$

in this way

- $H^4 \left(\mathcal{O}_{\mathbb{P}^5}^\sim \left(\frac{-p}{2}, \frac{-q}{2} - 3 \right) \right) = 0$: using Bott's formulae we have

$$h^4 \left(\mathcal{O}_{\mathbb{P}^5}^\sim \left(\frac{-p}{2}, \frac{-q}{2} - 3 \right) \right) = \begin{cases} \sum_{k=0}^{\frac{-p}{2}} \binom{\frac{p}{2} + 2 - k}{4} & \text{if } p \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and, for $p \leq 0$ the above binomial will be non-zero when $4 \leq q$. So, we obtain the third

important inequality

$$\text{for } p \leq 0 \text{ need } q \leq 2. \quad (4.0.10)$$

- $H^4 \left(I_X \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \right) = 0$: in the restriction sequence we have

$$\cdots \rightarrow 0 \rightarrow H^3 \left(\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^5}} \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \right) \rightarrow H^4 \left(I_X \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \right) \rightarrow H^4 \left(\mathcal{O}_{\widetilde{\mathbb{P}^5}} \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \right) \rightarrow 0 \rightarrow \cdots$$

therefore both ends must have zero cohomologies. Analogously to the reasoning of the previous item we obtain $h^4 \left(\mathcal{O}_{\widetilde{\mathbb{P}^5}} \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \right) = 0$ if $p < 0$ or $p \geq 0$ need $-q \leq 2$. On the other hand

$$\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{\mathbb{P}^5}} \left(\frac{p}{2}, \frac{q}{2} - 3 \right) \cong \mathcal{O}_{\mathbb{P}^3} \left(\frac{p}{2} + \frac{q}{2} - 3 \right) \oplus \mathcal{O}_{\mathbb{P}^3} \left(\frac{q}{2} - 3 \right)$$

which will have zero cohomology in h^3 if each of the elements inside the parentheses is greater than or equal to -3 , so then we obtain other important inequalities

$$q \geq -p; \quad q \geq 0. \quad (4.0.11)$$

Finally, since we want the bundle to be semi-stable with respect to polarization, in particular, we need $H^0(\mathcal{E})$ to be zero. Therefore, $H^0 \left(\mathcal{O}_{\widetilde{\mathbb{P}^5}} \left(\frac{-p}{2}, \frac{-q}{2} - 1 \right) \right)$ must be zero as well. Similarly, we obtain

$$p \leq 0 \text{ or } p > 0 \text{ need } p + q \leq 4. \quad (4.0.12)$$

Combining the inequalities (4.0.4), (4.0.7), (4.0.10), (4.0.11) and (4.0.12) we obtain that

$$\left\{ \begin{array}{l} -p - q \geq -2 \\ -q \geq 0 \end{array} \right. \cap \left(\left\{ \begin{array}{l} p \leq 0 \\ q \geq -p \\ q \geq 0 \\ 2 > -q - p \end{array} \right\} \cup \left\{ \begin{array}{l} p > 0 \\ q \geq -p \\ q \geq 0 \\ q \geq -2 \end{array} \right\} \right) \quad (4.0.13)$$

With these conditions, only the possibilities for choosing S remain, given by $S = \mathcal{O}_{\widetilde{\mathbb{P}^5}}(2, 0)$

and $S = \mathcal{O}_{\widetilde{\mathbb{P}^5}}$. ■

Proposition 4.3. *Considering in $\widetilde{\mathbb{P}^5}$ the space $X = \wp \cup \kappa$ there is an instanton $\mathcal{E}$ given by the sequence*

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^5}}(0, -1) \rightarrow \mathcal{E} \rightarrow I_X(0, -1) \rightarrow 0.$$

Proof. The proof is analogous to the one made [2] changing S to $\mathcal{O}_{\widetilde{\mathbb{P}^5}}$. ■

It can be shown that this instanton has properties similar to the example presented in the article [2] by Henni. Thus, we leave it to the reader.

4.1 Properties of the monad in the example $n = 5$.

According to [2, Corollary 5.5], the instanton corresponding to case $n = 5$ is described by the cohomology of the monad

$$\begin{array}{c} \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, 0) \\ \mathbf{N}: 0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, -1) \xrightarrow{\alpha} \oplus \xrightarrow{\beta} \Omega^3(0, 3) \rightarrow 0. \\ \mathcal{O}_{\widetilde{\mathbb{P}^5}}(0, -1)^{\oplus 6} \end{array} \quad (4.1.1)$$

In the example for $n = 4$, section 3.2.4, we were unable to show the existence or not of matrices A and B to obtain a bundle as cohomology of this monad. For this case, $n = 5$, we can even present a specific pair of matrices A and B , as follows.

Proposition 4.4. *There are specific maps α and β in $\mathbf{N}$ such that its cohomology is an instanton bundle.*

Proof. The idea of the proof is to construct matrices A and B of the maps α and β , respectively such that A is injective, B is surjective and $BA \cong 0$. Note that

$$\begin{aligned} \alpha &\in \text{Hom}(\mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, -1), \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, 0) \oplus \mathcal{O}_{\widetilde{\mathbb{P}^5}}(0, -1)^{\oplus 6}) \cong H^0(\mathcal{O}_{\widetilde{\mathbb{P}^5}}(0, 1) \oplus H^0(\mathcal{O}_{\widetilde{\mathbb{P}^5}}(1, 0)^{\oplus 6}) \\ &\cong \langle y_0, \dots, y_4 \rangle \oplus \langle z_0, \dots, z_5 \rangle^{\oplus 6} \end{aligned}$$

with $\lambda y_i = z_i$ for $i = 0, \dots, 4$ and $\lambda \in H^0(\mathcal{O}(E))$ for E the exceptional divisor. Furthermore

$$\begin{aligned} \beta &\in \text{Hom}(\mathcal{O}_{\mathbb{P}^5}(-1, 0) \oplus \mathcal{O}_{\mathbb{P}^5}(0, -1)^{\oplus 6}, \Omega^3(0, 3)) \cong H^0(\Omega^3(1, 3) \oplus H^0(\Omega^3(0, 4)^{\oplus 6}) \\ &\cong \langle \lambda \rangle^{\oplus 5} \oplus (\mathbb{C}^{\oplus 5})^{\oplus 6} \end{aligned}$$

In this way, we can choose

$$A = \begin{pmatrix} y_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_0 \end{pmatrix}; \quad B = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is clear that A is injective, B is surjective and the product of

$$BA \cong \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & z_4 \end{pmatrix}^t \cong 0$$

as we wanted. ■

The previous result shows that there are matrices A and B , representing α and β , respectively. Let us consider the set formed by these matrices, that is, formed by matrices A of order 7×1 injective and B of order 5×1 surjective such that $BA \cong 0$ and

$$A = \begin{pmatrix} \sum_{i=0}^4 a_i y_i \\ \sum_{j=0}^5 A_j z_j \end{pmatrix}; \quad B = (\lambda B' \quad M) \tag{4.1.2}$$

with a_i constants, A_j , M and B' constant matrices of order 6×1 , 5×6 and 5×1 , respectively.

Since $BA \cong 0$, we have that

$$\lambda B' \sum_{i=0}^4 a_i y_i + \sum_{j=0}^5 MA_j z_j \cong 0 \implies \sum_{i=0}^4 (a_i B' + MA_i) + MA_5 z_5 \cong 0 \quad (4.1.3)$$

Recall that in these spaces being congruent to zero is the same as being of the form $\sum_{j=0}^4 z_j e_j$ where e_j are matrices of order 6×1 with $a_{j1} = 1$ and the others are all zero. Considering t as fixed, it follows that

$$\begin{cases} MA_5 = 0 \\ a_i B' + MA_i \cong t e_i \quad \forall \quad 0 \leq i \leq 4. \end{cases} \quad (4.1.4)$$

Thus, we obtain the following representation of quivers.

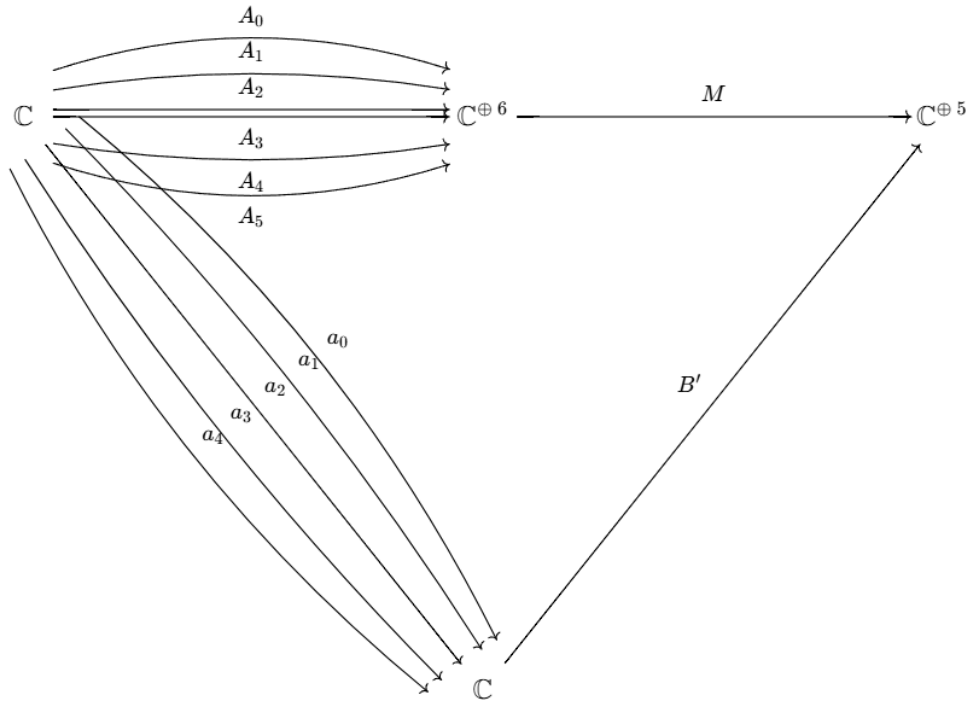


Figure 4.1.1: Quiver representation. Font: Created by the author.

We also observe that the automorphisms of the monad are given by:

$$\begin{aligned} \text{Aut}(\mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, -1)) &\cong \mathbb{C}^* \\ \text{Aut}(\mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, 0) \oplus \mathcal{O}_{\widetilde{\mathbb{P}^5}}(0, -1)^{\oplus 6}) &\cong \begin{pmatrix} \mathbb{C}^* & <\lambda>_{1 \times 6} \\ 0_{6 \times 1} & \text{Gl}(\mathbb{C})_{6 \times 6} \end{pmatrix} \\ \text{Aut}(\Omega^3(0, 3)) &\cong \mathbb{C}^* \end{aligned} \quad (4.1.5)$$

Proposition 4.5. *The restriction of the monad $\mathbf{N}$ to any line L is given by*

$$\begin{array}{ccccccc} & & \mathcal{O}_{\mathbb{P}^1}(-1) & & \mathcal{O}_{\mathbb{P}^1} & & \\ & & \downarrow & & \downarrow & & \\ \mathbf{N}|_L : 0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) & \xrightarrow{\alpha|_L} & \oplus & \xrightarrow{\beta|_L} & \oplus & \rightarrow 0. & (4.1.6) \\ & & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6} & & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} & & \end{array}$$

Proof. Considering the sequence $0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-1, 0) \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}} \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}^n}}|_H$, with H hyperplane, we obtain the following diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-2, -1) & \longrightarrow & \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-2, 0) \oplus \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, -1)^{\oplus 6} & \longrightarrow & \Omega^3(-1, 3) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, -1) & \longrightarrow & \mathcal{O}_{\widetilde{\mathbb{P}^5}}(-1, 0) \oplus \mathcal{O}_{\widetilde{\mathbb{P}^5}}(0, -1)^{\oplus 6} & \longrightarrow & \Omega^3(0, 3) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}|_H(-2) & \longrightarrow & \mathcal{O}|_H(-1) \oplus \mathcal{O}|_H(-1)^{\oplus 6} & \longrightarrow & \Omega^3|_H(3) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Consequently, the restriction to a line is given by the sequence

$$\begin{array}{ccccccc}
 & & \mathcal{O}|_L(-1) & & & & \\
 0 \rightarrow \mathcal{O}|_L(-2) \rightarrow & & \oplus & \rightarrow \Omega^3|_L(3) \rightarrow 0. \\
 & & \mathcal{O}|_L(-1)^{\oplus 6} & & & &
 \end{array}$$

The restriction $\Omega^3(3)$ remains to be calculated. The restriction on L of the sequence (2.1.8) is of the form $0 \rightarrow \Omega|_L \rightarrow \mathcal{O}|_L^5(-1) \rightarrow \mathcal{O}|_L \rightarrow 0$. We then obtain the

$$\left\{ \begin{array}{l} c_1(\Omega|_L) = -5 \\ h^0(\Omega|_L) = 0 \\ h^1(\Omega|_L) = 1 \\ \text{rk} \Omega|_L = 4 \end{array} \right. \implies \left\{ \begin{array}{l} \Omega|_L \cong \mathcal{O}|_L(a) \oplus \mathcal{O}|_L(b) \oplus \mathcal{O}|_L(c) \oplus \mathcal{O}|_L(d) \\ a + b + c + d = -5 \\ a, b, c, d \text{ negatives} \end{array} \right. \quad (4.1.7)$$

$$\implies \Omega|_L \cong \mathcal{O}|_L(-2) \oplus \mathcal{O}|_L(-1)^3. \quad (4.1.8)$$

Since $\Omega^3 \cong \wedge^3 \Omega$, it follows that

$$\Omega^3|_L \cong \wedge^3 \Omega|_L = \wedge^3 (\mathcal{O}|_L(-2) \oplus \mathcal{O}|_L(-1)^3) \cong \mathcal{O}|_L(-3) \oplus \mathcal{O}|_L(-4)^3.$$

As a result, we obtain $\Omega^3|_L(3) \cong \mathcal{O}|_L \oplus \mathcal{O}|_L(-1)^3$ and the sequence $\mathbf{N}|_L$. ■

By Corollary 5.3 in [2] we have $\mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Thus, $\mathcal{E}|_L(1)$ is trivial. The following result is an adaptation of Proposition 2.3.2 in [24], which we can use in the monad $\mathbf{N}|_L$ twisted by $\mathcal{O}_{\mathbb{P}^1}(1)$ using this triviality.

Proposition 4.6. *Suppose $\mathcal{F}$ is a vector bundle in $\mathbb{P}^1$ that is cohomology of a monad $0 \rightarrow \mathcal{U} \rightarrow \mathcal{V} \rightarrow \mathcal{W} \rightarrow 0$ with $\mathcal{V}$ trivial and $\mathcal{W}$ direct sum of two bundles with one of them trivial. Then there is a naturally induced map $\phi: H^1(\mathcal{U}(-1)) \rightarrow H^0(\mathcal{W}(-1))$ which is an isomorphism if and only if $\mathcal{E}$ is trivial.*

Proof. The proof is analogous to the proof for Proposition 2.3.2 in [24]. ■

5 Final considerations.

With Definition 3.1, defined by Henni in [2], and Definition 3.3, defined in this work, we obtain the following general definition.

Definition 5.1. On $(\widetilde{\mathbb{P}^n}, \mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(1, N_n))$ with

$$N_n = \begin{cases} 1 & \text{if } n = 3; \\ \frac{n-2}{2} & \text{if } n > 3 \text{ and even;} \\ \frac{n-3}{2} & \text{if } n > 3 \text{ and odd} \end{cases}$$

a $\mu_{\mathcal{L}}$ -semi-stable rank r torsion-free coherent sheaf $\mathcal{E}$, with Chern class

$$c_1(\mathcal{E}) = \begin{cases} 0 & \text{if } n = 3; \\ -\alpha & \text{if } n > 3 \text{ and even;} \\ -2\alpha & \text{if } n > 3 \text{ and odd} \end{cases}$$

will be called instanton if it satisfies

- (1) $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -2)) = 0$, $H^0(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 0)) = 0$;
- (2) $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, -1)) = 0$, $H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 3-n)) = 0$;
- (3) $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, 2-n)) = 0$, $H^n(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 4-n)) = 0$;
- (4) for $n \geq 4$, $H^1(\widetilde{\mathbb{P}^n}, \mathcal{E}(0, -3)) = 0 = H^{n-1}(\widetilde{\mathbb{P}^n}, \mathcal{E}(-1, 5-n))$ and $H^i(\widetilde{\mathbb{P}^n}, \mathcal{E} \otimes \mathcal{L}') = 0$, for all $2 \leq i \leq n-2$ and all invertible sheaves $\mathcal{L}'$ in $\widetilde{\mathbb{P}^n}$.

We can then define instanton sheaves on blow-up at a point of projective spaces of any dimension n . Each of these instantons is the cohomology of a monad given by the Lemma 3.2. The odd n case was verified by Henni in [2], presenting a family of instanton bundles.

In this work, the even n case was verified, presenting two families of instanton sheaves. Unfortunately, these sheaves are not locally free, that is, they are not bundles. However, they are cohomologies of monads and present interesting properties, as shown in Chapter 3. Therefore, we would like to find examples in the future for even n where the instantons are locally free.

In addition, we would like to find examples of higher-rank instantons. So far, we only have examples of rank 2. Furthermore, it is possible to demonstrate the existence or non existence of a deformation of the instanton sheaf $\mathcal{E}$ for $n = 4$ of the section 3.2.4.

In the case where n is odd, since the restriction of the monad to a line assumes a format similar to that presented in [24], we can proceed with its analysis, with the aim of obtaining results analogous to those discussed in Section 3.4 of [24].

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