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A Fractional Order Approach for the Unsteady Stokes Equations

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Juan Carlos Oyola Ballesteros

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À Meu Orientador!

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“El problema de un profesor no es cometer errores, sino no corregirlos.”

Resumo

Neste trabalho, apresentamos uma nova formulação variacional para o estudo das equações de Stokes não estacionárias em um domínio de Lipschitz aberto e limitado $\Omega \subset \mathbb{R}^n$, envolvendo uma derivada fracionária de Caputo no tempo de ordem $\alpha \in (1/2, 1)$. A natureza não local da derivada fracionária impõe desafios analíticos significativos, tornando os métodos clássicos, como a abordagem de Faedo–Galerkin, inadequados em sua forma padrão para uma análise abrangente do problema. Para superar essas dificuldades, desenvolvemos e analisamos rigorosamente novos espaços funcionais, denotados por $L^p_\alpha(0, T; X)$, especificamente concebidos para o contexto fracionário. Esses espaços permitem reformular as soluções fracas de modo consistente com a dinâmica fracionária, possibilitando, assim, a implementação bem-sucedida de um esquema de Galerkin generalizado que garante a existência e unicidade das soluções da formulação proposta.

Palavras-chave: Cálculo Fracionário. Derivada Fracionária de Caputo. Equações de Stokes. Equações Diferenciais Parciais Fracionárias. Método de Faedo–Galerkin.

Abstract

In this work, we present a new variational framework for studying the evolution Stokes equations in an open, bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, involving a Caputo fractional derivative in time of order $\alpha \in (1/2, 1)$. The nonlocal nature of the fractional derivative poses significant analytical challenges, rendering classical methods, such as the Faedo–Galerkin approach, inadequate in their standard form for a comprehensive analysis of the problem. To overcome these difficulties, we develop and rigorously analyze new functional spaces, denoted by $L^p_\alpha(0, T; X)$, specifically designed for the fractional setting. These spaces allow us to reformulate weak solutions in a manner consistent with the fractional dynamics, thereby enabling the successful implementation of a generalized Galerkin scheme that ensures the existence and uniqueness of solutions to the proposed formulation.

Keywords: Fractional Calculus. Caputo Fractional Derivative. Stokes Equations. Fractional Partial Differential Equations. Faedo-Galerkin Method.

Resumo Expandido

Introdução

Um dos modelos matemáticos mais renomados na dinâmica dos fluidos são as equações de Navier–Stokes. Essas equações têm como objetivo determinar os campos de velocidade e pressão no interior de um fluido. Elas estão entre as equações mais amplamente utilizadas, descrevendo a física de inúmeros fenômenos de interesse tanto econômico quanto acadêmico. As equações de Navier–Stokes encontram aplicações na modelagem de padrões climáticos, correntes oceânicas, escoamento de água em dutos e em muitas outras áreas.

A motivação para o estudo dessas equações decorre do crescente esforço que os matemáticos têm dedicado a tais problemas; ver, por exemplo, [19, 27, 32]. Essas equações podem ser derivadas diretamente das leis de Newton sob a hipótese de incompressibilidade. Para mais detalhes, remetemos o leitor a [14, Capítulo 1].

Um avanço importante na teoria das EDPs foi a introdução do conceito de soluções fracas por J. Leray, conforme detalhado em [28], particularmente no contexto das equações de Navier–Stokes. A teoria de Leray estabelece a existência de soluções em um novo sentido variacional, que podem ser potencialmente irregulares. Essa abordagem baseia-se em estimativas de energia e em processos limite específicos nas topologias fraca e fraca-$*$ de alguns espaços de Bochner–Lebesgue. Para uma visão mais abrangente, remetemos aos trabalhos de [15, 16, 17].

Por outro lado, o cálculo fracionário e suas aplicações a métodos analíticos para a resolução de equações diferenciais de ordem fracionária, como demonstrado em [25, 33, 34, 36], vêm se tornando ferramentas essenciais para o tratamento de uma ampla gama de problemas relacionados à dinâmica dos fluidos, estruturas porosas, teoria de controle e sistemas dinâmicos, conforme discutido nas referências clássicas [39, 40, 41, 47]. Diferentemente do cálculo clássico, que lida principalmente com derivadas e integrais de ordem inteira, o cálculo fracionário estende essas operações para ordens reais e até complexas. Existem dois tipos de derivadas fracionárias comumente utilizadas: as derivadas de Riemann–Liouville e as derivadas de Caputo (ver [25]). A definição de derivadas fracionárias de Caputo foi introduzida pela primeira vez em [7] para estudar o efeito de memória da dissipação de energia em certos materiais anelásticos. Em comparação com a derivada de Riemann–Liouville [39], as derivadas de Caputo eliminam singularidades na origem e compartilham muitas semelhanças com a derivada ordinária, o que as torna mais adequadas para problemas de valor inicial.

Consequentemente, a relevância dos temas combinados acima enfatiza o foco de estudos recentes em equações de Navier–Stokes com derivadas fracionárias de Caputo de ordem $\alpha \in (0,1)$ no tempo, com o objetivo de generalizar o problema clássico e explorar o caso limite ($\alpha = 1$); ver [10, 12, 31, 43] para alguns exemplos.

Entretanto, a maioria dos estudos disponíveis aos autores concentra-se no estabelecimento da existência e unicidade de soluções brandas para o problema de Cauchy abstrato associado às equações de Navier–Stokes fracionárias. Em contraste, relativamente poucos trabalhos abordam a existência e unicidade de soluções fracas, principalmente devido aos desafios técnicos do método de Faedo–Galerkin. Por exemplo, [48] define e demonstra a existência de uma solução fraca para as equações de Navier–Stokes com derivada fracionária de Caputo de ordem $\alpha \in (0,1)$, utilizando a mesma definição de solução fraca do caso clássico quando $\alpha = 1$. De forma semelhante, [31] introduz um novo espaço funcional (ver [31, Item (4.5)]) para o estudo de soluções fracas das equações de Burgers fracionárias multidimensionais, um caso particular das equações de Navier–Stokes fracionárias compressíveis, mantendo a definição clássica de solução fraca (ver [31, Definição 5.1]).

Objetivos

Inspirados pelos trabalhos citados anteriormente, o objetivo principal deste trabalho é ir além das abordagens tradicionais e propor uma nova formulação para soluções fracas de EDPs com derivada fracionária de Caputo na variável temporal, no contexto das equações de Navier–Stokes linearizadas, também conhecidas como equações de Stokes não estacionárias. Em particular, não apenas introduzimos uma nova formulação fraca, como também realizamos uma análise detalhada de uma variação dos espaços funcionais apresentados em [31], justificando por que eles são mais adequados a esse arcabouço variacional.

Metodologia

A metodologia empregada nesta pesquisa é a usual na área de Matemática: realização de reuniões periódicas entre o doutorando e o orientador, com o objetivo de estudar novas abordagens e problemas da área de concentração. A partir da leitura e do estudo intensivos de trabalhos e artigos científicos, propõem-se novos caminhos para garantir a obtenção de soluções para os problemas investigados.

Resultados e Discussão

Em conclusão, a elaboração desta tese envolveu diversos aspectos técnicos que impuseram desafios teóricos e matemáticos significativos, muitos dos quais foram abordados com sucesso. Destacamos a seguir as três contribuições que consideramos mais relevantes para a estrutura e a conclusão deste trabalho.

(i) A Seção 3.1 introduz o problema de Cauchy fracionário (equação (3.1)) sob a hipótese de que a aplicação $G : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ seja uma função de Carathéodory. Assumindo adicionalmente que G é localmente Lipschitz contínua, o Teorema 3.12 estabelece a existência e a unicidade de solução global quando o operador de Nemytskii associado satisfaz a condição $p = q$.

(ii) A Seção 3.2 apresenta o espaço $L^p_\alpha(0, T; X)$ e uma série de resultados voltados a estabelecer relações entre os espaços $L^p_\alpha(0, T; X)$ para diferentes valores de $\alpha \in [0, 1]$ e $p \in [1, \infty)$, bem como sua conexão com o espaço clássico $L^p(0, T; X)$. Mostramos que $L^p_\alpha(0, T; X)$ é um espaço de Banach e que ele é reflexivo sempre que X é um espaço de Banach reflexivo, conforme afirmado no Teorema 3.25. Por fim, o Teorema 3.27 e o Corolário 3.28 fornecem uma caracterização do espaço dual de $L^p_\alpha(0, T; X)$.

(iii) A Seção 4 apresenta uma nova formulação para a solução fraca das equações de Stokes não estacionárias quando se consideram derivadas fracionárias na variável temporal, conforme descrito na Definição 4.4. Aplicando o método de Faedo–Galerkin em um arcabouço especificamente adaptado às derivadas fracionárias, provamos a existência e a unicidade da solução fraca.

É importante destacar que, ao impor a condição $\alpha \in (1/2, 1)$, o problema de Cauchy fracionário introduzido na Seção 3 garante a existência de soluções aproximadas e, consequentemente, de uma sequência de soluções aproximadas correspondentes à equação 4.3. Ademais, as estimativas de energia obtidas para essa sequência por meio do método de Faedo–Galerkin demonstram a relevância do espaço $L^p_\alpha(0, T; X)$.

Considerações Finais

A pesquisa desenvolvida nesta tese já resultou em contribuições científicas concretas. Em particular, parte dos resultados obtidos ao longo deste trabalho foi consolidada nos seguintes manuscritos:

- P. M. Carvalho-Neto, C. L. Frota, J. C. Oyola Ballesteros, P. G. P. Torelli, *Energy Estimates for Fractional Evolution Equations*, arXiv:2508.05780.
- P. M. Carvalho-Neto, J. C. Oyola Ballesteros, *A New Approach for the Unsteady Stokes Equations with Time Fractional Derivative in Bounded Domains*, arXiv:2511-.13896.

Esses trabalhos refletem os principais desenvolvimentos teóricos alcançados durante este projeto de doutorado, particularmente no que diz respeito a problemas de Cauchy fracionários, espaços de Bochner ponderados e formulações fracionárias de modelos da dinâmica dos fluidos.

Palavras-chave: Cálculo Fracionário. Derivada Fracionária de Caputo. Equações de Stokes. Equações Diferenciais Parciais Fracionárias. Método de Faedo–Galerkin.

Contents

Chapter 1

Introduction

One of the most renowned mathematical models in fluid dynamics is the Navier-Stokes equations. These equations aim to determine the velocity and pressure fields within a fluid. They are among the most widely used equations, describing the physics of numerous phenomena of both economic and academic interest. The Navier-Stokes equations find applications in modeling weather patterns, ocean currents, water flow in dutos, and many other areas. The motivation for studying these equations cames from the growing effort mathematicians have dedicated to such problems; see [19, 27, 32]. These equations can be derived directly from Newton's laws under the assumption of incompressibility. For more details, we refer to [14, Chapter 1].

An important breakthrough in the theory of PDEs was the introduction of the concept of weak solutions by J. Leray, as detailed in [28], particularly in the context of the Navier-Stokes equations. Leray's theory establishes the existence of solutions in a new variational sense, which can be potentially irregular. This approach is based on energy estimates, and on specific limit processes on the weak and weak-$*$ topologies of some Bochner-Lebesgue spaces. For a more comprehensive survey, we refer to the works of [15, 16, 17].

On the other hand, fractional calculus and its application to analytical methods for solving differential equations of fractional orders, as demonstrated in [25, 33, 34, 36], are becoming essential tools for addressing a wide range of problems related to fluid dynamics, porous structures, control theory, and dynamical systems, as discussed in the classical references [39, 40, 41, 47]. Unlike classical calculus, which primarily deals with derivatives and integrals of integer order, fractional calculus extends these operations to real and even complex orders. There are two types of fractional derivatives that are commonly used: the Riemann–Liouville derivatives and the Caputo derivatives (see [25]). Caputo's definition of fractional derivatives was first introduced in [7] to study the memory effect of energy dissipation in certain anelastic materials. Compared with the Riemann–Liouville derivative [39], Caputo derivatives eliminate singularities at the origin and share many similarities with the ordinary derivative, making them more suitable for initial value problems.

Consequently, the importance of the combined topics addressed above emphasize the focus in recent studies on Navier-Stokes equations with Caputo fractional derivatives of order $\alpha \in (0, 1)$ in time, aiming to generalize the classical problem and explore the limiting case ($\alpha = 1$); see [10, 12, 31, 43] for some examples.

However, most studies available to the authors focus on establishing the existence and uniqueness of mild solutions for the abstract Cauchy problem associated with fractional Navier-Stokes equations. In contrast, relatively few works address the existence and uniqueness of weak solutions, primarily due to technical challenges of the Faedo-Galerkin method. For example, [48] defines and proves the existence of a weak solution to the Navier-Stokes equations with a Caputo fractional derivative of order $\alpha \in (0, 1)$, using the same definition of weak solution as in the classical case when $\alpha = 1$. Likewise, [31] introduces a novel function space (see [31, Item (4.5)]) for studying weak solutions of the fractional multidimensional Burgers equations, a specific case of the fractional compressible Navier–Stokes equations, while retaining the classical weak solution definition (see [31, Definition 5.1]).

In conclusion, the primary aim of this work is to move beyond traditional approaches and to propose a novel formulation for weak solutions of PDEs with a Caputo fractional derivative in the time variable, within the framework of the linearized Navier–Stokes equations, also known as the unsteady Stokes equations. Specifically, we not only introduce a new weak formulation, but also carry out a detailed analysis of a variation of the functional spaces presented in [31], justifying why they are better suited to this variational framework.

1.1 A Novel Weak Formulation

Let us first recall the standard approach to the weak formulation of the unsteady Stokes equations. Suppose that H and V are Hilbert spaces satisfying the following conditions:

(i) V is dense and continuously embedded in H;

(ii) If H' denotes the dual of H, then by the Riesz representation theorem, we identify $H \equiv H'$.

Under these assumptions, if V' denotes the dual of V, we obtain the continuous embeddings

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

In this setting, following the standard procedure for deriving a variational formulation of the classical evolution Navier–Stokes equations (see [46, Chapter III] for details), we arrive at the following weak formulation.

Definition 1.1. Given $f \in L^2(0,T;V')$ and $u_0 \in H$, find $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that

$$\frac{\partial}{\partial t}(u(t),v)_H + \nu(u(t),v)_V + b(u(t),u(t),v) = \langle f(t),v\rangle_{V',V},$$

almost everywhere in $[0,T]$, for all $v \in V$, with initial condition $u(0) = u_0$. Here, $\langle\cdot,\cdot\rangle_{V',V}$ denotes the duality pairing between V' and V.

The nonlinear term in the weak formulation of the Navier–Stokes equations is represented by the trilinear form $b(\cdot,\cdot,\cdot)$. Therefore, to obtain the weak formulation of the unsteady Stokes equations, this term is omitted.

Definition 1.2. Given $f \in L^2(0,T;V')$ and $u_0 \in H$, find $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that

$$\frac{\partial}{\partial t}(u(t),v)_H + \nu(u(t),v)_V = \langle f(t),v\rangle_{V',V},$$

almost everywhere in $[0,T]$, for all $v \in V$, with initial condition $u(0) = u_0$.

Next, by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$, we obtain the following weak formulation for the fractional unsteady Stokes equations, where cD_t^α denotes the Caputo fractional derivative of order α.

Definition 1.3. Let $\alpha \in (0,1)$. Given $f \in L^2(0,T;V')$ and $u_0 \in H$, find u in suitable function spaces such that

$$cD_t^\alpha(u(t),v)_H + \nu(u(t),v)_V = \langle f(t),v\rangle_{V',V}, \tag{1.1}$$

almost everywhere in $[0,T]$, for all $v \in V$, with $u(0) = u_0$.

The above definition naturally raises a question of which function spaces are appropriate for the solution u. To address this, observe that if u satisfies (1.1), then, by taking $v = u(t)$ and assuming that

$$cD_t^\alpha\|u(t)\|_H^2 \le 2(cD_t^\alpha u(t),u(t))_H, \tag{1.2}$$

for almost every $t \in [0,T]$ (cf. [10, Theorem 4.16]), we derive the following energy inequality:

$$\frac{1}{2}cD_t^\alpha \|u(t)\|_H^2 + \nu\|u(t)\|_V^2 \leq \langle f(t), u(t)\rangle_{V',V}, \text{ for a.e. } t \in [0,T], \tag{1.3}$$

which becomes an equality when $\alpha = 1$.

Following the classical ideas introduced by Leray in [28, 29, 30], below we present a series of formal arguments to explore the regularity that can be expected from the weak solution of the unsteady Stokes equations, which is derived from (1.3).

(i) Recall that when $\alpha = 1$, as is classically justified in the literature, integrating both sides from 0 to t and applying Young's inequality allows us to deduce from (1.3) that

$$\|u(t)\|_H^2 + \nu \int_0^t \|u(s)\|_V^2\, ds \;\leq\; \|u_0\|_H^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_{V'}^2\, ds, \quad \text{for a.e. } t \;\in\; [0,T].$$

The above inequality suggests that requiring $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ is a reasonable regularity condition for the classical weak solution.

(ii) However, when $\alpha \in (0,1)$, these function spaces may no longer be ideal. Mimicking the classical formulation, we integrate both sides of (1.3) from 0 to t, use Proposition 2.42, and apply Young's inequality to obtain

$$\begin{aligned} J_t^{1-\alpha}\|u(t)\|_H^2 &+ \nu \int_0^t \|u(s)\|_V^2\, ds \\ &\leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\|u_0\|_H^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_{V'}^2\, ds, \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

From this, we can derive the estimate

$$J_t^{1-\alpha}\|u(t)\|_H^2\Big|_{t=T} \;+\; \nu\|u\|_{L^2(0,T;V)}^2 \;\leq\; \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|u_0\|_H^2 \;+\; \frac{1}{\nu}\|f\|_{L^2(0,T;V')}^2. \tag{1.4}$$

It follows from (1.4) that the solution u should belong to $L^2(0,T;V)$ and that $J_t^{1-\alpha}\|u(t)\|_H^2$, evaluated at $t = T$, must be finite.

The discussion above indicates the need to consider a new subspace of $L^p(0,T;X)$, denoted as $L_\alpha^p(0,T;X)$ for any Banach space X and $1 \leq p < \infty$. This subspace consists of measurable functions $f : [0, T] \to X$ for which

$$\int_0^T (T-s)^{\alpha-1} \|f(s)\|_X^p \, ds < \infty.$$

The study of this space, along with the establishment of existence and uniqueness of solutions to this weak formulation of the fractional unsteady Stokes equations using the Faedo-Galerkin method, becomes our main focus from this point onward. In carrying out this procedure, we encountered several challenges and technical difficulties, which we successfully overcame and present in this document.

1.2 Organization of the Manuscript

This manuscript is organized as follows. Chapter 2 reviews the main technical concepts relevant to this work, including the classical Bochner–Lebesgue spaces, the variational formulation of the unsteady Stokes equations, the Riemann–Liouville fractional integral, and the Caputo fractional derivative.

The fractional Cauchy problem, specifically designed to the requirements of the fractional unsteady Stokes equations, is addressed in Chapter 3. Due to the absence of references covering these specific results in the fractional setting, the necessary theoretical foundations are developed within this work. Also, in Section 3.2 we address the spaces $L^p_\alpha(0, T; X)$, their relationship with the classical spaces $L^p(0, T; X)$, and the conditions on X ensuring that they are reflexive Banach spaces.

A new formulation for weak solutions of the unsteady Stokes equations with a fractional time derivative is introduced in Chapter 4. By imposing the condition $\alpha \in (1/2, 1)$, the fractional Cauchy problem studied previously ensures the existence of approximate solutions. This allows the application of the Faedo–Galerkin method, leading to suitable energy estimates and, to the construction of a weak solution via a limiting process.

The recovery of the pressure associated with a given weak velocity solution is the focus of Chapter 5. By relying on classical results concerning gradient distributions and regularity properties of distributions with values in dual spaces, we establish the existence of a pressure function with L^2-regularity in space and continuous dependence on time.

Finally, Chapter 6 provides concluding remarks, including an overview of submitted works and a discussion of future research directions naturally suggested by the results of this thesis.

Chapter 2

Preliminary Knowledge

The first chapter introduces fundamental concepts and results that will be used throughout this thesis, aiming to provide a foundation for the development of subsequent material. We begin by presenting some important results from the classical theory of Bochner–Lebesgue spaces. Next, we explore the classical variational formulation of the unsteady Stokes equations. Finally, we provide an overview of the essentials of fractional calculus, focusing on fractional operators of integration and differentiation.

2.1 Bochner-Lebesgue spaces $L^p(I;X)$

This section discusses several properties of the Bochner integral for vector-valued functions on a Banach space. Given a subset $I \subset \mathbb{R}$, equipped with the Lebesgue measure induced from $\mathbb{R}$, we introduce the classical Bochner-Lebesgue spaces $L^p(I;X)$ and establish their reflexivity under a suitable condition on X, namely that the dual space X' has the Radon–Nikodym property.

It is worth emphasizing that throughout this section, X always denotes a Banach space, X' its dual space, and $\langle \cdot, \cdot \rangle_{X',X}$ denotes the duality pairing between X' and X.

Definition 2.1. Let $n \in \mathbb{N}^* := \{1, 2, \ldots\}$, S a subset of $\mathbb{R}^n$, Σ a $\sigma-$algebra on the set S, and μ a measure in (S, Σ). We call the triplet (S, Σ, μ) a measure space .

(i) A function $l : S \to X$ is said to be a simple Bochner function, if it can be written as

$$l(s) = \sum_{i=1}^{n} x_i \cdot 1_{E_i}(s) \quad \forall s \in S,$$

where $x_i \in X$, the sets $E_i \in \Sigma$ are pairwise disjoint with $\mu(E_i) < \infty$ for $i = 1, \ldots, n$, and $1_{E_i}(s)$ denotes the characteristic function of the set E_i.

(ii) A function $g : S \to X$ is said to be Bochner measurable if there exists a sequence (l_n) of simple Bochner functions that converges to g in norm of X, for almost every $s \in S$. Furthermore, g is said to be weakly measurable if the function $\langle x', g(\cdot)\rangle$ is measurable for all $x' \in X'$.

(iii) We say that g is almost separably valued if there exists a set N with $\mu(N) = 0$ such that $g(S \setminus N)$ is separable.

Remark 2.2. We highlight two important results that arise from Definition 2.1.

(i) The most common way to check the measurability of vector-valued functions is Pettis' theorem, which links the notions of measurability and weak measurability. This result can be stated as follows: a function $g : S \to X$ is Bochner measurable if and only if g is weakly measurable and almost separably valued (see [2, Theorem 1.1.1]).

(ii) As a corollary of the previous item, we have that $g : S \to X$ is Bochner measurable if and only if $\langle x', g(\cdot)\rangle$ is measurable for every $x' \in M$, and g is almost separably valued, where M is a normed subspace of X'.

Definition 2.3. (i) For a simple Bochner function $l : S \to X$ the integral can be defined as

$$\int_S l(s)\ d\mu = \sum_{i=1}^{n} x_i \mu(E_i),$$

where $l(s) = \sum_{i=1}^{n} x_i 1_{E_i}(s)$.

(ii) A measurable function $g : S \to X$ is Bochner-integrable if there exists a sequence (l_n) of simple functions such that

$$\lim_{n\to\infty} \int_S \|l_n(s) - g(s)\|_X \, d\mu = 0,$$

and each integrand belongs to $L^1(S; \mu)$. In this case, $\int_S l_n(s)\, d\mu$ converges in X to a limit that is independent of the sequence of simple functions, which we denote by $\int_S g(s)\, d\mu$.

Remark 2.4. The second item in the above definition allows us to relate the Bochner integral with the Lebesgue integral (see [2, Theorem 1.1.4]). In fact, if $g : S \to X$

is a measurable function, then g is Bochner-integrable if and only if $\|g\|_X$ is Lebesgue-integrable. Furthermore, we have the fundamental estimate

$$\left\| \int_S g(s)\ d\mu \right\|_X \leq \int_S \|g(s)\|_X\ d\mu.$$

Proposition 2.5. *Let $x' \in X'$ and g be Bochner-integrable, then*

$$\int_S \langle x', g(s) \rangle_{X',X}\ d\mu = \left\langle x', \int_S g(s)\ d\mu \right\rangle_{X',X}, \quad \forall\, x' \in X'.$$

Proof. By the definition of the integral, it holds that

$$\int_S \langle x', l(s) \rangle_{X',X}\ d\mu = \left\langle x', \int_S l(s)\ d\mu \right\rangle_{X',X}$$

for any simple functions l. Now, let (l_n) be a sequence of simple functions as in the definition of $\int g(s)\ d\mu$. Observe that we can define a new sequence of simple functions as follows:

$$\tilde{l}_n(s) := \begin{cases} l_n(s), & if\ \ \|l_n(s)\|_X \leq 2\|g(s)\|_X, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\langle x', \tilde{l}_n(s) \rangle \to \langle x', g(s) \rangle$ for almost every $s \in S$, and $|\langle x', \tilde{l}_n(s) \rangle| \leq 2\|x'\|_{X'}\|g(s)\|_X$. Thus by the Dominated Convergence Theorem, we have that

$$\begin{aligned} \int_S \langle x', g(s) \rangle_{X',X}\ d\mu &= \lim_{n\to\infty} \int_S \langle x', \tilde{l}_n(s) \rangle_{X',X}\ d\mu \\ &= \lim_{n\to\infty} \left\langle x', \int_S \tilde{l}_n(s)\ d\mu \right\rangle_{X',X} = \left\langle x', \int_S g(s)\ d\mu \right\rangle_{X',X}, \end{aligned}$$

where the last equality follows from the continuity of x' and the definition of the Bochner-integral. □

Definition 2.6. Let I be an interval of $\mathbb{R}$, equipped with the Lebesgue measure inherited from $\mathbb{R}$.

(i) Let $I = [a, b]$ with $a < b$ in $\mathbb{R}$. If $g : I \to X$ is Bochner integrable, we say that a function $G : I \to X$ is a primitive of g if

$$G(t) = G(a) + \int_a^t g(s)\ ds$$

for every $t \in I$.

(ii) A function $G : I \to X$ is said to be absolutely continuous on I if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every finite collection $\{(a_i, b_i)\}$ of pairwise disjoint intervals in I, if $\sum_i (b_i - a_i) < \delta$, then $\sum_i \|G(b_i) - G(a_i)\|_X < \epsilon$.

(iii) A function $G : I \to X$ is said to be Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|G(t) - G(s)\|_X \leq L\,|t - s|$$

for all $s, t \in I$.

Remark 2.7. From the above definitions we can deduce:

(i) Every Lipschitz continuous function is absolutely continuous.

(ii) Let $G : [a, b] \to X$ be absolutely continuous, and suppose that derivative $G'(t)$ exists for almost every $t \in [a, b]$. Then, G' is Bochner integrable and

$$G(t) = G(a) + \int_a^t G'(s)\, ds$$

for every $t \in [a, b]$. See Proposition 1.2.3 in [2].

Theorem 2.8. *Let X be a Banach space and let $\mathbb{R}_+ = [0, \infty)$. The following statements are equivalent:*

(i) Every absolutely continuous function $G : \mathbb{R}_+ \to X$ is differentiable almost everywhere.

(ii) Every Lipschitz continuous function $G : \mathbb{R}_+ \to X$ is differentiable almost everywhere.

Proof. See, for instance, [2, Proposition 1.2.4]. ☐

Remark 2.9. Let $a < b$ in $\mathbb{R}$. The Theorem 2.8 does not depend on the choice of interval. Indeed:

(i) Suppose that every absolutely continuous function $G : [0, 1] \to X$ is differentiable almost everywhere. We show that item (i) of the theorem holds.

Let $G : \mathbb{R}_+ \to X$ be absolutely continuous. For each integer $n \geq 0$, define the translated function

$$G_n(t) := G(n + t), \quad \forall t \in [0, 1].$$

Each G_n is absolutely continuous. By assumption, each G_n is differentiable almost everywhere on $[0, 1]$. Hence, G is differentiable almost everywhere on each interval $[n, n + 1]$. Since

$$\mathbb{R}_+ = \bigcup_{n=0}^{\infty} [n, n + 1]$$

and a countable union of null sets is still a null set, it follows that G is differentiable almost everywhere on $\mathbb{R}_+$.

Conversely, any absolutely continuous function G defined on $[0, 1]$ can be extended to an absolutely continuous function on $\mathbb{R}_+$ by defining

$$\tilde{G} := \begin{cases} G(t), & if\ \ t \in [0, 1], \\ G(1), & if\ \ t > 1. \end{cases}$$

If absolutely continuous functions on $\mathbb{R}_+$ are differentiable almost everywhere, then their restrictions to $[0, 1]$ are also differentiable almost everywhere.

Therefore, item (i) in the theorem is equivalent to the statement that every absolutely continuous function $G : [0, 1] \to X$ is differentiable almost everywhere.

(ii) Let $F : [a, b] \to X$ and defined

$$G(t) := F(a + (b - a)t), \quad t \in [0, 1].$$

The map

$$\phi : [0, 1] \to [a, b], \qquad \phi(t) = a + (b - a)t,$$

is a Lipschitz bijection with Lipschitz inverse. Hence, F is absolutely continuous on $[a, b]$ if and only if G is absolutely continuous on $[0, 1]$.

(iii) Items (i) and (ii) together imply that absolutely continuous functions on $[a, b]$ are differentiable almost everywhere if and only if absolutely continuous functions on $\mathbb{R}_+$ are differentiable almost everywhere.

(iv) By analogous arguments, Lipschitz continuous functions on $[a, b]$ are differentiable almost everywhere if and only if Lipschitz continuous functions on $\mathbb{R}_+$ are differentiable almost everywhere.

Corollary 2.10. *Let $I = [a, b]$ with $a < b$ in $\mathbb{R}$ and let X be a Banach space. The following statements are equivalent:*

(i) Every absolutely continuous function $G : I \to X$ is differentiable almost everywhere.

(ii) Every Lipschitz continuous function $G : I \to X$ is differentiable almost everywhere.

A Banach space X that satisfies either (and hence both) of the equivalent conditions in Corollary 2.10 is said to have the Radon-Nikodym.

Based on [2, Theorem 1.2.6], we show that the Radon-Nikodym property is not a typical among Banach spaces. Nevertheless, we present below two broad classes of spaces that do possess this property.

Proposition 2.11. *(Dunford-Pettis). Let $I = [a, b]$ with $a < b$ in $\mathbb{R}$. Let X be a Banach space whose dual space X' is separable. Then X' has the Radon-Nikodym property.*

Proof. We use the characterization (i) from Theorem 2.8 to prove the claim. Let $F : I \to X'$ be a Lipschitz continuous function with Lipschitz constant L. Consider the function $G(s) := [F(s) - F(a)]/L$. This transformation allows us to assume, without loss of generality, that $F(a) = 0$ and that $L = 1$.

For any $x \in X$, the function $\langle F(\cdot), x\rangle_{X',X}$ is Lipschitz continuous with Lipschitz constant $\|x\|_X$. By the Fundamental Theorem of Calculus (see [38, Theorem 7.20]), every absolutely continuous function is differentiable almost everywhere. Therefore, for each $x \in X$, there exists a measurable function g_x, unique up to sets of measure zero, such that

$$\langle F(t), x\rangle_{X',X} = \int_a^t g_x(s)ds,$$

for almost every $t \in I$. Moreover, since

$$g_x(t) = \lim_{h\to 0} \frac{\langle F(t+h), x\rangle - \langle F(t), x\rangle}{h},$$

we have

$$\|g_x\|_{L^\infty(I;\mathbb{R})} \le \|x\|_X.$$

Now, X' is separable, so is X. Let $D \subset X$ be a countable dense subset. Consider all finite linear combinations $x = \sum_{i=1}^n \alpha_i x_i$, where $x_i \in D$ and $\alpha_i \in \mathbb{Q}$ (for complex banach space, $\alpha_i \in \mathbb{Q} + i\mathbb{Q}$). For such x, we obtain

$$\langle F(t), x\rangle = \left\langle F(t), \sum_{i=1}^n \alpha_i x_i \right\rangle = \sum_{i=1}^n \alpha_i \langle F(t), x_i\rangle$$
$$= \sum_{i=1}^n \alpha_i \int_a^t g_{x_i}(s)ds = \int_a^t \sum_{i=1}^n \alpha_i g_{x_i}(s)ds.$$

This implies that $g_x = \sum_{i=1}^{n} \alpha_i g_{x_i}$ almost everywhere. Consequently, for almost every $t \in I$,

$$\left| \sum_{i=1}^{n} \alpha_i g_{x_i}(t) \right| = |g_x(t)| \leq \|g_x\|_{L^\infty(I;\mathbb{R})} \leq \|x\|_X = \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|_X .$$

Since the choices of n, α_i, and x_i are countable, we can select a null set E such that the above estimate holds for all $t \in I \setminus E$ and all such $x(n, \alpha_i, x_i)$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, the estimate extends to all $\alpha_i \in \mathbb{R}$. Thus, for almost every $t \in I$, the map $x \to g_x(t)$ defines a linear functional on span(D), whose bounded in norm by 1, and hence extend uniquely to all of X. We therefore obtain a function $f(t) \in X'$ for which $\|f(t)\|_{X'} \leq 1$ almost everywhere.

For each $x \in D$, we have $\langle f(t), x \rangle = g_x(t)$, which is measurable and bounded. Now, if $x \in X$ and $(x_n) \subset D$ be a sequence converging to x, then $\langle f(t), x_n \rangle \to \langle f(t), x \rangle$ almost everywhere. Since each $\langle f(\cdot), x_n \rangle$ is bounded and measurable, the Dominated Convergence Theorem implies that $\langle f(\cdot), x \rangle$ is measurable, and

$$\begin{aligned} \langle F(t), x \rangle_{X',X} &= \lim_{n\to\infty} \langle F(t), x_n \rangle_{X',X} = \lim_{n\to\infty} \int_a^t g_{x_n}(s)\, ds \\ &= \lim_{n\to\infty} \int_a^t \langle f(s), x_n \rangle_{X',X}\, ds = \int_a^t \langle f(s), x \rangle_{X',X}\, ds. \end{aligned}$$

Given that $f : I \to X'$, its image $f(I)$ is separable. Viewing $X \subset X''$, item (ii) from Remark 2.2 allows us to deduce that f is measurable. From the previous computation and Proposition 2.5, it follows that

$$\langle F(t), x \rangle_{X',X} = \int_a^t \langle f(s), x \rangle_{X',X}\, ds = \left\langle \int_a^t f(s)\, ds, x \right\rangle_{X',X}$$

for almost every $t \in I$ and $\forall x \in X$. This allows us to conclude that

$$F(t) = \int_a^t f(s)\, ds, \qquad \text{a.e. in } I.$$

Finally, Lebesgue's Differentiation Theorem (see [2, Proposition 1.2.2 (a)]) implies that F is differentiable almost everywhere. Hence, X' has the Radon-Nikodym property. ☐

Remark 2.12. Theorem 2 in [13, Section 3, Chapte III] ensures that X has the Radon-Nikodym property if and only if every closed separable subspace of X has it.

Corollary 2.13. *Every reflexive space has the Radon-Nikodym property.*

Proof. Let D a closed separable subspace of a reflexive banach space X. Since every closed subspace of a reflexive space is reflexive, D is reflexive. In particular, D is the dual of a Banach space. Thus, Proposition 2.11 ensures that D has the Radon-Nikodym property. Since D was an arbitrary closed separable subset of X, we conclude that X has the Radon-Nikodym property. $\square$

Definition 2.14. Consider $I \subset \mathbb{R}$ with the induced Lebesgue measure from $\mathbb{R}$, and let $p \in [1, \infty]$. We use the symbol $L^p(I; X)$ to represent the set of all Bochner measurable functions $g : I \to X$ for which $\|g\|_X \in L^p(I; \mathbb{R})$, where $L^p(I; \mathbb{R})$ stands for the classical Lebesgue space. Moreover, $L^p(I; X)$ is a Banach space when considered with the norm

$$\|g\|_{L^p(I;X)} := \begin{cases} \left[\int_I \|g(s)\|_X ds\right]^{1/p} & , \ if \ \ p \in [1, \infty), \\ \operatorname{ess\,sup}_{s\in I} \|g(s)\|_X & , \ if \ \ p = \infty. \end{cases}$$

When $I = [a, b]$ or $I = [a, \infty)$, we may also use the notation $L^p(a, b; X)$ to refer to these spaces.

We are now in a position to discuss the reflexivity of Bochner–Lebesgue spaces.

Theorem 2.15. *Let X be a Banach space such that X' has the Radon-Nikodym property, and let $I = [a, b]$ with $a < b$ in $\mathbb{R}$. Then, for $p \in [0, \infty)$, we have an isometric isomorphism*

$$L^p(I; X)' \cong L^q(I; X'),$$

where $q = p/(p - 1)$ is the Hölder conjugate of p.

Proof. At the beginning of Chapter IV in [13], J. Diestel and J.J. Uhl show that the inclusion mapping $L^q(I; X') \hookrightarrow L^p(I; X)'$ is an isometry. Furthemore, in [13, Theorem 1, Section 1, Chapter IV], they prove that this isometry is surjective. $\square$

Corollary 2.16. *(Phillips). Let $I = [a, b]$ with $a < b$ in $\mathbb{R}$. If X is reflexive and $p \in (1, \infty)$, then $L^p(I, X)$ is reflexive.*

Proof. Let $q = p/(p - 1)$. Consider the following

(i) Since X is reflexive, we have $X'' \cong X$, and hence $L^p(I; X'') = L^p(I, X)$, in the sense that each element of one space corresponds naturally to an element of the other.

(ii) By Theorem 2.15, we have the isometric isomorphism $L^p(I; X)' \cong L^q(I; X')$. Taking the dual of both sides, it follows that $(L^p(I; X)')' \cong L^q(I; X')'$.

(iii) Again, by Theorem 2.15, we have that $(L^q(I;X'))' \cong L^p(I;(X')')$. Using the identification from (i), we deduce that $(L^q(I;X'))' \cong L^p(I;X)$. Thus, for every $T \in L^q(I;X')'$, there exists a unique $v \in L^p(I;X)$ such that

$$\langle T, g\rangle = \int_I \langle g(s), v(s)\rangle_{X',X}\, ds \quad \forall g \in L^q(I;X').$$

From the above, we can conclude that $(L^p(I;X)')' \cong L^p(I;X)$, completing the proof. □

2.2 The Unsteady Stokes Equations.

In this section, we briefly study the classical unsteady Stokes equations with the aim of presenting a variational formulation of the problem. To that end, we first introduce a sequence of function spaces, along with the definitions and notations of their norms and inner products, where applicable. Next, we present the strong formulation of the unsteady Stokes equations, followed by their weak formulation.

Let $n \in \mathbb{N}^*$ and let Ω be a bounded, open, and Lipschitz subset of $\mathbb{R}^n$. For $m \in \mathbb{N}^*$ and $p \in [1,\infty]$, denote by $W^{m,p}(\Omega)$ the Sobolev space of all functions $v : \Omega \to \mathbb{R}$ such that, for each multi-index α with $|\alpha| \le m$, the weak derivative $D^\alpha v$ belongs to $L^p(\Omega;\mathbb{R})$. These are Banach spaces with the norms

(i) $\|v\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha v(x)|^p\, dx\right)^{1/p}, \quad if \;\; p \in [1,\infty);$

(ii) $\|v\|_{W^{m,p}(\Omega)} := \sum_{|\alpha|\le m} \operatorname{ess\,sup}_{\Omega} |D^\alpha v(x)|, \quad if \;\; p = \infty.$

When $p = 2$, we write $W^{m,2}(\Omega)$ as $H^m(\Omega)$. This is a Hilbert space with the inner product

$$\big(v, g\big)_{H^m(\Omega)} := \sum_{|\alpha|\le m} (D^\alpha v, D^\alpha g)_{L^2(\Omega;\mathbb{R})} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha v \; D^\alpha g \; dx.$$

In this section, let $\mathcal{D}(\Omega)$ denote the set of infinitely differentiable functions with compact support in Ω. The closures of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ and $H^m(\Omega)$ are denoted by $W_0^{m,p}(\Omega)$ and $H_0^m(\Omega)$, respectively. In particular, due to the Poincaré Inequality (see [24, Theorem 2.3.4] for details), the space $H_0^1(\Omega)$, equipped with the inner product

$$((g, v))_{H_0^1(\Omega)} := \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i}, \frac{\partial v}{\partial x_i}\right)_{L^2(\Omega;\mathbb{R})},$$

is a Hilbert space.

Remark 2.17. An important relationship between Sobolev spaces $W^{m,p}(\Omega)$ and $L^p(\Omega)$ spaces is given by the Sobolev embeddings, which depend on the dimension of the domain. Specifically, for a bounded open set $\Omega \subset \mathbb{R}^n$ with C^1-class boundary and $p \in [1, \infty]$, the following continuous embeddings holds:

(i) If $p < n$,
$$W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega), \quad \text{where } \frac{1}{p'} = \frac{1}{p} - \frac{1}{n}.$$

(ii) If $p = n$,
$$W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \forall \; q \in [p, \infty).$$

(iii) If $p > n$,
$$W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega).$$

Starting from the previous spaces and with the purpose of discussing the weak formulation of the unsteady Stokes equations, we consider the following function spaces:

(i) For $p \in [1, \infty]$, define $\mathbb{L}^p(\Omega)$ as the space of vector functions $v = (v_1, ..., v_n) : \Omega \to \mathbb{R}^n$, such that $v_i \in L^p(\Omega; \mathbb{R})$, for $i = 1, ..., n$. With the norms
$$\|v\|_{\mathbb{L}^p(\Omega)} := \begin{cases} \left(\displaystyle\sum_{i=1}^{n} \|v_i\|^2_{L^p(\Omega;\mathbb{R})} \right)^{1/2}, & if \quad p \in [1, \infty), \\ \displaystyle\sum_{i=1}^{n} \|v_i\|_{L^\infty(\Omega;\mathbb{R})}, & if \quad p = \infty, \end{cases}$$
these are Banach spaces. In particular, $\mathbb{L}^2(\Omega)$ with the inner product
$$(v, g)_{\mathbb{L}^2(\Omega)} := \sum_{i=1}^{n} (v_i, g_i)_{L^2(\Omega;\mathbb{R})} = \sum_{i=1}^{n} \int_\Omega v_i \; g_i \; dx, \tag{2.1}$$
is a Hilbert space.

(ii) For $m \in \mathbb{N}^*$, we define $\mathbb{H}^m(\Omega)$ as the space of vector functions $v = (v_1, ..., v_n) : \Omega \to \mathbb{R}^n$, such that $v_i \in H^m(\Omega)$, for all $i = 1, ..., n$. When equipped with the inner product

$$(v,g)_{\mathbb{H}^m(\Omega)} := \sum_{i=1}^{n}(v_i,g_i)_{H^m(\Omega)} = \sum_{i=1}^{n}\sum_{|\alpha|\leq m}(D^\alpha v_i, D^\alpha g_i)_{L^2(\Omega;\mathbb{R})},$$

it becames a Hilbert space. In particular, define $\mathbb{H}_0^1(\Omega)$ as the space of vector functions $v=(v_1,...,v_n):\Omega\to\mathbb{R}^n$, such that $v_i\in H_0^1(\Omega)$, for all $i=1,...,n$. The inner product

$$((v,g))_{\mathbb{H}_0^1(\Omega)} := \sum_{i=1}^{n}((v_i,g_i))_{H_0^1(\Omega)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial v_i}{\partial x_j},\frac{\partial g_i}{\partial x_j}\right)_{L^2(\Omega;\mathbb{R})},$$

also makes this a Hilbert space.

Remark 2.18. Throughout this manuscript, we use $(\cdot,\cdot)_{\mathbb{L}^2(\Omega)}$ to denote the inner product in $\mathbb{L}^2(\Omega)$ and $((\cdot,\cdot))$ to represent the inner product in $\mathbb{H}_0^1(\Omega)$.

Consider the functions space $\mathcal{V}:=\{v\in[\mathcal{D}(\Omega)]^n : \text{div } v=0\}$. We define

$$H := \overline{\mathcal{V}}^{\mathbb{L}^2(\Omega)} = \left\{v\in\mathbb{L}^2(\Omega) : \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v\cdot\vec{n}=0 \text{ on } \partial\Omega\right\},$$

which becomes a Hilbert space when endowed with the topology induced by $\mathbb{L}^2(\Omega)$. We denote its inner product $(\cdot,\cdot)_H$ when referring specifically to H.

We also define the function space

$$V := \overline{\mathcal{V}}^{\mathbb{H}_0^1(\Omega)} = \{v\in\mathbb{H}_0^1(\Omega) : \operatorname{div} v = 0 \ in \ \Omega\},$$

which is also Hilbert space, equipped with the inner product inherited from $\mathbb{H}_0^1(\Omega)$. By classical arguments, we identify H with its dual space H', and write the inclusions

$$V\hookrightarrow H\equiv H'\hookrightarrow V', \tag{2.2}$$

where each space is densely embedded in the next, the injections are continuous, and the inclusion $V\hookrightarrow H$ is compact.

Remark 2.19. As a consequence of (2.2), there exists $K>0$ such that, for each fixed $g\in H$, the inner product in H defines a bounded linear functional on V, given by

$$\langle g,v\rangle_{V',V} := \big(g,v\big)_H, \qquad \forall v\in V,$$

where $\|g\|_{V'} \leq K\|g\|_{H}$. .

Remark 2.20. The space $C([0,T];C^2(\overline{\Omega};\mathbb{R}^n))$ consists of continuous functions from $[0,T]$ into $C^2(\overline{\Omega};\mathbb{R}^n)$, and becomes a Banach space when equipped with the norm

$$\sup_{t\in[0,T]} \|g(t)\|_{C^2(\overline{\Omega};\mathbb{R}^n)}.$$

Here, the norm in $C^2(\overline{\Omega};\mathbb{R}^n)$ is defined by

$$\|v\|_{C^2(\overline{\Omega};\mathbb{R}^n)} = \sup_{x\in\overline{\Omega}} \|v(x)\|_{\mathbb{R}^n} + \sup_{x\in\overline{\Omega}} \|\nabla v(x)\|_{\mathbb{R}^{n\times n}} + \sup_{x\in\overline{\Omega}} \left\|\Delta^2 v(x)\right\|_{\mathbb{R}^n}.$$

We are now led to the main objective of this section: the weak formulation of the unsteady Stokes equations. For this purpose, we first present the strong formulation.

Given a constant $\nu > 0$, a vector-valued function $f : [0,T]\times\Omega \to \mathbb{R}^n$, and an initial condition $u_0 : \Omega \to \mathbb{R}^n$, find functions $u : [0,T] \to C^2(\overline{\Omega};\mathbb{R}^n)$ and $p : [0,T] \to C^1(\overline{\Omega};\mathbb{R})$, representing the fluid's velocity and pressure, respectively, such that

$$u_t(t,x) - \nu\Delta u(t,x) + \nabla p(t,x) = f(t,x), \qquad \textit{in}\ \ [0,T]\times\Omega, \tag{2.3}$$

$$\operatorname{div} u(t,x) = 0, \qquad \textit{in}\ \ [0,T]\times\Omega, \tag{2.4}$$

$$u(t,x) = 0, \qquad \textit{on}\ \ [0,T]\times\partial\Omega, \tag{2.5}$$

$$u(0,x) = u_0(x), \qquad \textit{in}\ \ \Omega. \tag{2.6}$$

Assuming $u(t,x)$ and $p(t,x)$ are classical solutions of the system (2.3)-(2.6), we then have $u \in C([0,T];C^2(\overline{\Omega};\mathbb{R}^n))$ and $p \in C^1(\overline{\Omega};\mathbb{R})$, for all $t\in[0,T]$, and $\partial u/\partial t \in C^2(\overline{\Omega};\mathbb{R}^n)$ for all $t\in[0,T]$. It follows that $u \in L^2(0,T;V)$ and $\nabla p(t) \in \mathbb{L}^2(\Omega)$ for all $t\in[0,T]$.

Now, observe that if $v \in \mathcal{V}$ and $f \in L^2(0,T;H)$, we obtain

$$\begin{aligned}(f(t),v)_{\mathbb{L}^2(\Omega)} &= \left(\frac{\partial u(t)}{\partial t} - \nu\Delta u(t) + \nabla p(t), v\right)_{\mathbb{L}^2(\Omega)} \\ &= \left(\frac{\partial u(t)}{\partial t}, v\right)_{\mathbb{L}^2(\Omega)} - \nu(\Delta u(t),v)_{\mathbb{L}^2(\Omega)} + (\nabla p(t),v)_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{2.7}$$

Remark 2.21. By analyzing each term on the right-hand side of (2.7), we can derive the following conclusions:

(i) The term $(-\Delta u(t),v)_{\mathbb{L}^2(\Omega)}$ can be rewritten as $((u(t),v))$, given that u satisfies the boundary condition (2.5). Specifically,

$$\begin{aligned}(-\Delta u(t),v)_{\mathbb{L}^2(\Omega)} &= -\sum_{i=1}^{n}(\Delta u_i(t),v_i) = -\sum_{i=1}^{n}\int_\Omega \Delta u_i(t,x)\cdot v_i(x)dx \\ &= \sum_{i=1}^{n}\int_\Omega \nabla u_i(t,x)\cdot\nabla v_i(x)dx = \sum_{i=1}^{n}(\nabla u_i(t),\nabla v_i).\end{aligned}$$

(ii) The term $(\nabla p(t),v)_{\mathbb{L}^2(\Omega)}$ vanishes due to the fact that $v\in\mathcal{V}$, which implies $\operatorname{div} v=0$. Indeed,

$$\begin{aligned}(\nabla p,v)_{\mathbb{L}^2(\Omega)} &= \sum_{i=1}^{n}\left(\frac{\partial p}{\partial x_i}(t),v_i\right) = \sum_{i=1}^{n}\left[\int_\Omega \frac{\partial p}{\partial x_i}(t,x)\ v_i(x)\ dx\right] \\ &= -\sum_{i=n}^{n}\left[\int_\Omega p(t,x)\frac{\partial v_i(x)}{\partial x_i}dx\right] = -\int_\Omega p(t,x)\left[\sum_{i=1}^{n}\frac{\partial v_i(x)}{\partial x_i}\right]dx \\ &= -\int_\Omega p(t,x)\ \operatorname{div} v(x)\ dx = 0.\end{aligned}$$

(iii) For any small $h>0$ such that $0<t-h<t+h<T$, we observe that

$$\left(\frac{u(t\pm h)-u(t)}{h},v\right)_H = \frac{(u(t\pm h),v)_H-(u(t),v)_H}{h}.$$

Therefore, by the continuity of the inner product $(\cdot,\cdot)_H$, we conclude

$$\left(\frac{\partial u(t)}{\partial t},v\right)_H = \frac{\partial}{\partial t}(u(t),v)_H, \qquad \text{for all } t\in[0,T].$$

(iv) Since $f\in L^2(0,T,H)$, Remark 2.19 allows us to regard $(f(t),\cdot)_H$ as a duality pairing in V. That is

$$\langle f(t),v\rangle_{V',V} := \big(f(t),v\big)_H, \qquad \text{for all } v\in V.$$

Thanks to the above observations, we can rewrite the original equation (2.7) as

$$\frac{\partial}{\partial t}\big(u(t),v\big)_H + \nu((u(t),v)) = \langle f(t),v\rangle_{V',V}, \qquad \text{for all } t\in[0,T],$$

Since $\mathcal{V}$ is dense in V, this equations also holds for every $v\in V$.

WEAK FORMULATION: Given $f \in L^2(0,T;V')$ and $u_0 \in H$, find $u \in L^2(0,T;V)$ such that

$$\frac{\partial}{\partial t}\big(u(t),v\big)_H + \nu((u(t),v)) = \langle f(t),v\rangle_{V',V}, \tag{2.8}$$

almost everywhere in $[0,T]$, for all $v \in V$, and such that $u(0) = u_0$.

Since we seek $u \in L^2(0,T;V)$, the initial condition $u(0)$ may not be well-defined in the classical sense. However, the favorable properties of the operator $((u(t),\cdot))$, together with the hypothesis $f \in L^2(0,T;V')$, imply that u has a representation in $C([0,T];V')$, which allows us to give meaning to $u(0)$, see [46, Chapter III].

2.3 Fractional Integration and Derivation

Our initial goal in this section is extend the classical operations of integration and differentiation to fractional (non-integer) orders. To this end, we begin by introducing the standard notations and fundamental definitions. We then focus on the study of the fractional operators, both integral and differential, and present several properties that are used throughout this thesis.

We begin by presenting some special functions that are used throughout this section. First, we introduce the Gamma function, denoted by $\Gamma(z)$, which is undoubtedly one of the fundamental functions in fractional calculus. It generalizes de factorial $n!$, allowing n to take non-integer and even complex values.

Definition 2.22. The Gamma Function is defined for all complex numbers z with positive real part, i.e., for $\mathrm{Re}(z) > 0$, by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{2.9}$$

Remark 2.23. The Gamma function admits an analytic extension to the domain

$$D(\Gamma) = \{z \in \mathbb{C} \setminus \{0,-1,-2,...\}\},$$

for which the following properties hold:

(i) If $\mathrm{Re}(z) > 0$, then $\Gamma(z)$ is given by the integral (2.9).

(ii) The identity $z\Gamma(z) = \Gamma(z+1)$ holds for all $z \in D(\Gamma)$.

(iii) $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

For more details, see [36, Section 1.1].

Another useful mathematical function in fractional calculus is the Beta function, denoted by $\mathcal{B}$. Its importance lies in the fact that its form closely resembles that of fractional integral of many functions, particularly polynomials of the form t^α.

Definition 2.24. The Beta function is defined for complex numbers x and y with positive real parts, i.e., $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, by

$$\mathcal{B}(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds.$$

Theorem 2.25. *Let $x, y \in \mathbb{C}$ with $\mathrm{Re}(x), \mathrm{Re}(y) > 0$. Then the Beta and Gamma functions are related by the identity*

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. See Section 1.1.4 of the reference [36]. □

Next, we introduce the Mittag-Leffler functions, which are useful in formulating a fractional version of the Gronwall's inequality. The Mittag-Leffler function is a special function that generalizes the exponential function.

Definition 2.26. Let α and β be strictly positive real numbers. The generalized Mittag-Leffler is the function $E_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}$ defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Remark 2.27. There are many important special cases and identities associated with the general Mittag-leffler function.

1. If $\beta = 1$, we obtain the standard Mittag-leffler function, denoted by

$$E_\alpha(z) = E_{\alpha,1}(z).$$

2. If $\alpha = \beta = 1$, the function reduces to the exponential function

$$E_{\alpha,\beta}(z) = E_{1,1}(z) = e^z.$$

3. For $\alpha = 2$ and $\beta = 1$, we have the identity

$$E_{\alpha,\beta}(-z^2) = E_{2,1}(-z^2) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos(z).$$

4. As an additional observation, for $|z| < 1$, if we formally set $\alpha = 0$ and $\beta = 1$, the series becomes
$$E_{0,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(1)} = \frac{1}{1-z},$$
which is the standard geometry series.

For more information, see the literature [4, 22, 35].

Theorem 2.28. *(Fractional Gronwall inequality theorem) Given $b \geq 0$, $\alpha > 0$ and $l : [0,\infty) \to \mathbb{R}$ be a non-negative function that is locally integrable on $[0,T)$ for some $T \leq \infty$. Assume that $u : [0,\infty) \to \mathbb{R}$ is also a non-negative function, locally integrable on $[0,T)$, and satisfies the inequality*

$$u(t) \leq l(t) + b \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

for all $t \in [0,T)$. Then,

$$u(t) \leq l(t) + \theta \int_0^t \frac{\partial}{\partial s} E_\alpha(\theta(t-s)) l(s)\, ds, \quad t \in [0,T),$$

where $\theta = (b\Gamma(\alpha))^{1/\alpha}$. Moreove, if there exists a constant $l \geq 0$ such that $l(t) = l$ for all $t \in [0,T)$, then
$$u(t) \leq l E_\alpha(\theta t).$$

Proof. See for instance [23, Lemma 7.1.1] □

Now we can begin an introduction to fractional calculus. To do this, we first present the classical integral used in the Bochner-Lebesgue spaces. Henceforth, we assume that $I = [0,T]$ with $T > 0$. However, replacing $[0,T]$ with any other interval $[t_0,t_1]$ does not affect the results obtained in this section, except for slight changes in the constants involved in the estimates. The structure of this section is partially based on [36, Chapter 2].

Let X be a Banach space. We consider the operator:

$$I : L^1(0,T;X) \to C([0,T];X),$$

where, for almost every $t \in [0,T]$, (Iv) is defined by

$$(Iv)(t) = \int_0^t v(s)ds.$$

We observe that

$$(I^2 v)(t) = \int_0^t \int_0^s v(r)\ dr\ ds = \int_0^t \int_r^t v(r)\ ds\ dr = \int_0^t \frac{(t-r)}{1!} v(r)\ dr.$$

If we repeat this operation once more, we obtain

$$(I^3 v)(t) = \int_0^t \int_0^r \int_0^s v(\tau)\ d\tau\ ds\ dr = \int_0^t \frac{(t-r^2)}{2!} v(r)\ dr.$$

By induction, we conclude that $(I^n v)$, that is, the result of applying the integral operator n times to the function $v(t)$, is given by

$$(I^n v)(t) = \frac{1}{(n-1)!} \int_0^t (t-r)^{n-1} v(r)\ dr.$$

To introduce a concept that generalizes this pattern, we rewrite the formula above in more suggestive way

$$(I^n v)(t) = \frac{1}{\Gamma(n)} \int_0^t (t-r)^{n-1} v(r)\ dr.$$

This expression provides a natural clue for generalization. Informally, we may say that the fractional integration of order $\alpha > 0$ of the function v is given by:

$$(I^\alpha v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} v(r)\ dr \quad \text{for a.e. } t \in [0,T].$$

Definition 2.29. For $\alpha \in (0,\infty)$ and $v : [0,T] \to X$, the Riemann-Liouville fractional integral of order α at 0 of the function v, denoted by $J_t^\alpha v(t)$, is defined as

$$J_t^\alpha v(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds, \tag{2.10}$$

for every $t \in [0,T]$ such that integral (2.10) exists.

Proposition 2.30. *Let* $\alpha > 0$.

(i) $v \in L^1(0,t;X)$, *for every* $t \in [0,T]$, *if and only if,* $J_t^\alpha v(t)$ *exists for almost every* $t \in [0,T]$.

(ii) if $v \in L^1(0,T;X)$, *then* $J_t^\alpha v(t)$ *is Bochner integrable in* $[0,T]$.

Proof. See, for instance, [8, Theorem 2.5]. ☐

To better understand this definition, let us recall some additional definitions and results.

Definition 2.31. Let $\alpha > 0$. Consider the function $g_\alpha : \mathbb{R} \to [0,\infty)$ given by

$$g_\alpha(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Proposition 2.32. *Let* $\alpha, \beta > 0$ *and consider* $g_\alpha : \mathbb{R} \to [0,\infty)$ *given in Definition 2.31. Then*

$$g_\alpha * g_b = g_{\alpha+\beta}, \qquad t > 0.$$

Proof. See, for instance, [36, Section 2.4.2]. ☐

Remark 2.33. Note that for any $\alpha \in (0,\infty)$, item (i) of Proposition 2.30 implies that if $J_t^\alpha v(t)$ exists for almost every $t \in [0,T]$, then $v \in L^1(0,T;X)$. Thus, the Riemann-Liouville fractional integral and g_α functions are related by the identity

$$J_t^\alpha v(t) = (v * g_\alpha)(t) = \int_0^t g_\alpha(t-s) v(s)\, ds, \qquad \text{for a.e. } t \in [0,T].$$

Theorem 2.34. *Let* $\alpha > 0$ *and* $p \in [1,\infty]$. *Then the Riemann-Liouville fractional integral operator*

$$\begin{aligned} J_t^\alpha : L^p(0,T;X) &\to L^p(0,T;X) \\ v(t) &\to \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \end{aligned}$$

is a bounded linear operator. Furthermore, it holds that

$$\left[\int_0^T \|J_t^\alpha v(t)\|_X^p \, dt\right] \leq \left[\frac{T^\alpha}{\Gamma(\alpha+1)}\right] \|v\|_{L^p(0,T;X)}.$$

Proof. This theorem was proved by Carvalho-Neto and Fehlberg Júnior in Theorem 3.1 of [8]. ☐

Theorem 2.35. *If $\alpha > 0$ and $v \in L^\infty(0,T;X)$, then $J_t^\alpha v \in C([0,T];X)$.*

Proof. Since $v \in L^\infty(0,T;X)$, we have the estimate

$$\|J_t^\alpha v(t)\|_X \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|v(s)\|_X ds \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|v\|_{L^\infty(0,T;X)},$$

for any $t \in [0,T]$. This shows that $J_t^\alpha v(t)$ exists for every $t \in [0,T]$.

Now, for $t, w \in [0,T]$ with $w < t$, we consider

$$\begin{aligned}\|J_t^\alpha v(t) - J_w^\alpha v(w)\|_X \leq & \frac{1}{\Gamma(\alpha)} \int_w^t (t-s)^{\alpha-1}\|v(s)\|_X \, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^w \left[(w-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \|v(s)\|_X \, ds.\end{aligned}$$

But then, we obtain the bound

$$\begin{aligned}\|J_t^\alpha v(t) - J_w^\alpha v(w)\|_X & \leq \left[\frac{(t-w)^\alpha}{\Gamma(\alpha+1)} + \frac{w^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{(t-w)^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right] \|v\|_{L^\infty(0,T;X)} \\ & \leq \frac{2(t-w)^\alpha}{\Gamma(\alpha+1)}\|v\|_{L^\infty(0,T;X)}.\end{aligned}$$

Therefore, $J_t^\alpha v(t)$ is continuous on $[0,T]$, i.e., $J_t^\alpha v \in C([0,T];X)$. □

Definition 2.36. Let $\alpha \in (0,1)$, and $v \in L^1(0,T;X)$ such that $v * g_{1-\alpha} \in W^{1,1}(0,T;X)$. The Riemann-Liouville fractional derivative of order α, denoted by $D_t^\alpha v(t)$, is defined as

$$D_t^\alpha v(t) := D_t^1 J_t^{1-\alpha} v(t) = D_t^1 (g_{1-\alpha} * v)(t) \quad \text{for a.e. } t \in [0,T],$$

where $D_t^1 = \left(\dfrac{\partial}{\partial t}\right)$. Explicity, we have

$$D_t^\alpha v(t) = D_t \left\{\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v(s) \, ds\right\}, \qquad \text{for a.e. } t \in [0,T].$$

Example 2.37. *Let $\alpha \in (0,1)$, $\beta \in (-1,\infty)$ and consider the function $v(t) = ct^\beta$. Then*

$$D_t^\alpha v(t) = c\frac{\Gamma(\beta+1)}{\Gamma(1-\alpha+\beta)} t^{\beta-\alpha}.$$

Proof. Computing the Riemann-Liouville fractional derivative

$$D_t^\alpha v(t) = D_t^1 \left\{\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} cs^\beta ds\right\}$$

$$
\begin{aligned}
&= D_t^1\left\{\frac{ct^{1+\beta-\alpha}}{\Gamma(1-\alpha)}\int_0^1 (1-s)^{-\alpha}s^{\beta}ds\right\}\\
&= (1+\beta-\alpha)\frac{ct^{\beta-\alpha}}{\Gamma(1-\alpha)}\mathcal{B}(1-\alpha,\beta+1)\\
&= c\frac{\Gamma(\beta+1)}{\Gamma(1-\alpha+\beta)}t^{\beta-\alpha}, \qquad \text{for } t>0.
\end{aligned}
$$

□

Remark 2.38. From Example 2.37, observe that the Riemann-Liouville fractional derivative of a constant function c is given by

$$D_t^{\alpha}c = c\frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \qquad \text{for } t>0.$$

Thus, when computing the Riemann-Liouville derivative of a function, we cannot generally expect non-singular behavior at $t=0$. To avoid such difficulties, we adopt the Caputo fractional derivative, which can be viewed as a regularization of the Riemann-Liouville derivative at the origin and satisfies the relevant property of vanishing when applied to constant functions.

The previous remark, along with several physical and practical considerations, motivates the definition of the Caputo fractional derivative.

Definition 2.39. Let $\alpha\in(0,1)$, and $v\in C([0,T],X)$ with $v*g_{1-\alpha}\in W^{1,1}(0,T;X)$. The Caputo fractional derivative of order α, denoted as $cD_t^{\alpha}v(t)$, is defined by

$$cD_t^{\alpha}v(t) := D_t^{\alpha}(v(t)-v(0)), \qquad \text{for a.e. } t\in[0,T].$$

Proposition 2.40. *Let $\alpha\in(0,1)$ and suppose that $v\in C^1([0,T];X)$. Then $cD_t^{\alpha}v(t)\in C([0,T],X)$ and*

$$cD_t^{\alpha}v(t) = J_t^{1-\alpha}v'(t), \qquad \textit{for a.e. } t\in[0,T].$$

Proof. First, observe that

$$cD_t^{\alpha}v(t) = D_t^{\alpha}(v(t)-v(0)) = D_t^1\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}(v(s)-v(0))ds\right\}.$$

Since v is differentiable in the usual sense, we integrate by parts

$$cD_t^{\alpha}v(t) \;=\; D_t^1\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t\left[\frac{1}{\alpha-1}D_s^1(t-s)^{1-\alpha}\right](v(s)-v(0))ds\right\}$$

$$= D_t^1 \left\{ \frac{(t-s)^{1-\alpha}(v(s)-v(0))}{\Gamma(1-\alpha)\alpha - 1}\Big|_{s=0}^{s=t} - \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{\alpha-1}(t-s)^{1-\alpha} v'(s)\, ds \right\}$$

$$= D_t^1 \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{1-\alpha}(t-s)^{1-\alpha} v'(s)\, ds \right\}$$

Thus, by applying Leibniz Rule (see [42, Theorem 8.1]), we conclude that

$$cD_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} v'(s)\, ds = J_t^{1-\alpha} v'(t), \qquad \text{for a.e. } t \in [0,T].$$

Finally, Theorem 2.35 ensures that $cD_t^\alpha v(t)$ belongs to $C([0,T];X)$. □

Corollary 2.41. *Let $\alpha \in (0,1)$ and suppose $v \in C^1([0,T];X)$ with $v(0) = 0_X$. Then $D_t^{1-\alpha} v(t) \in C([0,T],X)$, and*

$$D_t^{1-\alpha} v(t) = D_t^1 J_t^\alpha v(t) = J_t^\alpha v'(t), \qquad \textit{for a.e. } t \in [0,T].$$

The next proposition revisits a classical set of computation commonly found in textbooks on fractional calculus, such as [25, 34, 36, 39].

Proposition 2.42. *Let α_1, α_2, with $v \in L^1(0,T;X)$ and $h \in C([0,T],X)$. Then the following properties hold*

(i) $J_t^{\alpha_1} J_t^{\alpha_2} v(t) = J_t^{\alpha_1+\alpha_2} v(t)$, *for almost every $t \in [0,T]$;*

(ii) $D_t^{\alpha_1} J_t^{\alpha_1} v(t) = v(t)$, *for almost every $t \in [0,T]$;*

(iii) If $J_t^{1-\alpha_1} v \in W^{1,1}(0,T;X)$, then

$$J_t^{\alpha_1} D_t^{\alpha_1} v(t) = v(t) - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1 - 1} \{J_s^{1-\alpha_1} v\}|_{s=0}, \ \textit{for a.e. } t \in [0,T].$$

Moreover, if there exists an integral function ϕ such that $v = J_t^{\alpha_1}\phi(t)$, then

$$J_t^{\alpha_1} D_t^{\alpha_1} v(t) = v(t), \ \textit{for a.e. } t \in [0,T].$$

(iv) $cD_t^{\alpha_1} J_t^{\alpha_1} h(t) = h(t)$, *for almost every $t \in [0,T]$;*

(v) If $J_t^{1-\alpha_1} h \in W^{1,1}(0,T;X)$, then $J_t^{\alpha_1} cD_t^{\alpha_1} h(t) = h(t) - h(0)$, for almost every $t \in [0,T]$.

Proof. We prove each item separately.

(i) We just observe that by Proposition 2.32

$$J_t^{\alpha_1} J_t^{\alpha_2} v(t) = g_{\alpha_1} * (g_{\alpha_2} * v)(t) = (g_{\alpha_1} * g_{\alpha_2}) * v(t) = J_t^{\alpha_1+\alpha_2} v(t).$$

(ii) Using item (i), we compute

$$D_t^{\alpha_1} J_t^{\alpha_1} v(t) = D_t J_t^{1-\alpha} J_t^{\alpha_1} v(t) = D_t J_t^1 v(t).$$

By the Fundamental Theorem of Calculus, we have

$$D_t^1 J_t^1 v(t) = \frac{d}{dt}\left(\int_0^t v(s)ds\right) = v(t), \text{ for a.e. } t \in [0,T].$$

(iii) Since $D_t^{\alpha_1} v(t) \in L^1(0,T;X)$, we write

$$\begin{aligned} J_t^{\alpha_1} D_t^{\alpha_1} v(t) &= \frac{1}{\Gamma(\alpha_{1)}} \int_0^t (t-s)^{\alpha_1 - 1} D_s^{\alpha_1} v(s)\, ds \\ &= \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t D_t^1 (t-s)^{\alpha_1} D_s^{\alpha_1} v(s)\, ds. \end{aligned}$$

Now, using Leibniz's rule

$$\begin{aligned} J_t^{\alpha_1} D_t^{\alpha_1} v(t) &= D_t^1 \left\{ \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1} D_s^{\alpha_1} v(s)\, ds \right\} \\ &= D_t^1 \left\{ \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1} D_s^1 J_s^{1-\alpha_1} v(s)\, ds \right\}. \end{aligned}$$

Integrating by parts

$$\begin{aligned} J_t^{\alpha_1} D_t^{\alpha_1} v(t) &= D_t^1 \left\{ \frac{1}{\alpha_1 \Gamma(\alpha_1)} \{(t-s)^{\alpha_1} J_s^{1-\alpha_1} v(s)\}|_0^t \right. \\ &\qquad \left. - \frac{1}{\alpha_1 \Gamma(\alpha_1)} \int_0^t D_s^1 (t-s)^{\alpha_1} J_s^{1-\alpha_1} v(s)\, ds \right\} \\ &= D_t^1 \left\{ -\frac{1}{\alpha_1 \Gamma(\alpha_1)} t^{\alpha_1} \{J_s^{1-\alpha_1} v(s)\}|_{s=0} \right. \\ &\qquad \left. + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} J_s^{1-\alpha_1} v(s)\, ds \right\} \end{aligned}$$

$$= -\frac{1}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}\{J_s^{1-\alpha_1} v(s)\}|_{s=0} + D_t^1 J_t^{\alpha_1} J_t^{1-\alpha_1} v(t)$$

$$= v(t) - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}\{J_s^{1-\alpha_1} v(t)\}|_{s=0}, \quad \text{for a.e. } t \in [0,T].$$

Now, if $v(t) = J_t^{\alpha_1}\phi(t)$, then by item (ii),

$$J_t^{\alpha_1} D_t^{\alpha_1} v(t) = J_t^{\alpha_1} D_t^{\alpha_1} J_t^{\alpha_1} \phi(t) = J_t^{\alpha_1}\phi(t) = v(t).$$

(iv) Using the estimate

$$\|J_s^{\alpha_1} h(s)|_{s=0}\| \leq \|h\|_{C([0,T],X)} \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1 - 1} d\tau \,\Big|_{s=0} = 0,$$

we conclude that

$$cD_t^{\alpha_1} J_t^{\alpha_1} h(t) = D_t^{\alpha_1}(J_t^{\alpha_1} h(t) - J_s^{\alpha_1} h(s)|_{s=0}) = D_t^{\alpha_1} J_t^{\alpha_1} h(t) = h(t).$$

(v) Finally, observe that if $H(t) = h(t) - h(0)$, by item (iii) and by the fact observed in item (iv), we obtain

$$J_t^{\alpha_1} cD_t^{\alpha_1} h(t) = J_t^{\alpha_1} D_t^{\alpha_1} H(t) = H(t) - \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}\{J_s^{1-\alpha_1} H(s)\}|_{s=0} = h(t) - h(0).$$

□

By finishing this section, we establish a conection between fractional calculus and the Radon-Nikodym property discussed in Section 2.1.

Proposition 2.43. *Let* $\alpha \in (0,1)$ *and let* X *a Banach Space with the Radon-Nikodym property. If* $v : [0,T] \to X$ *is absolutely continuous, then* $J_t^{1-\alpha} v$ *is absolutely continuous on* $[0,T]$ *and* $J_t^{1-\alpha} v \in W^{1,1}(0,T;X)$*. Moreover,*

$$D_t^\alpha v(t) = \frac{v(0) t^{-\alpha}}{\Gamma(1-\alpha)} + J_t^\alpha v'(t).$$

for almost every $t \in [0,T]$.

Proof. Since v is absolutely continuous, item (ii) in Remark 2.7 ensure that v' belongs to $L^1(0,T;X)$ and

$$v(t) = v(0) + \int_0^t v'(s)\, ds$$

for every $t \in [a,b]$. Thus,

$$J_t^{1-\alpha}v(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t v(0)(t-s)^{1-\alpha}\,ds + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{1-\alpha}\left[\int_0^s v'(w)\,dw\right]ds.$$

By the Fubini-Tonelli theorem for Banach-valued functions, we deduce

$$J_t^{1-\alpha}v(t) = \frac{v(0)t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_0^t v'(s)(t-s)^{1-\alpha}ds. \tag{2.11}$$

Since functions defined as integrals of $L^1(0,T;X)$ functions are absolutely continuous, it follows that $J_t^{1-\alpha}v$ is absolutely continuous on $[0,T]$. Again, by item *(ii)* in Remark 2.7, we conclude that $J_t^{1-\alpha}v$ belongs to $W^{1,1}(0,T;X)$. Differentiating equation (2.11) using the Libniz Rule (see in [8, Theorem 53]), we obtain

$$D_t^{\alpha}v(t) = \frac{v(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t v'(s)(t-s)^{-\alpha}ds, \tag{2.12}$$

for almost every $t \in [0,T]$. ☐

Remark 2.44. Let X be a Banach Space with the Radon-Nikodym property. Suppose $\alpha \in (0,1)$ and $v : [0,T] \to X$ is a absolutely continuous function, then $D_t^{\alpha}v(t) \in L^r(0,T;X)$ for all $1 \le r < 1/\alpha$. Indeed, by Proposition 2.43, representation (2.12) holds. Therefore,

$$\|D_t^{\alpha}v(t)\|_{L^r(0,T;X)} \le \left\|\frac{v(0)t^{-\alpha}}{\Gamma(1-\alpha)}\right\|_{L^r(0,T;X)} + \left\|\frac{1}{\Gamma(1-\alpha)}\int_0^t v'(s)(t-s)^{-\alpha}ds\right\|_{L^r(0,T;X)}.$$

For the second term, by the Minkowski's integral ineqaulity for Banach-valued function, we obtain

$$\left\|\int_0^t v'(s)(t-s)^{-\alpha}ds\right\|_{L^r(0,T;X)} \le \int_0^T \|v'(s)\|_X \left(\int_s^T (w-s)^{-\alpha r}dw\right)^{1/r} ds.$$

Since $1 \le r < 1/\alpha$, we have

$$\int_0^T \|v'(s)\|_X \left(\int_s^T (w-s)^{-\alpha r}dw\right)^{1/r} ds \le \frac{T^{(1/r)-\alpha}}{(1-\alpha r)^{1/r}}\int_0^T \|v'(s)\|\,ds.$$

Concluding that for every $1 \le r < 1/\alpha$,

$$\|D_t^{\alpha}v(t)\|_{L^r(0,T;X)} \le \frac{T^{(1/r)-\alpha}}{\Gamma(1-\alpha)(1-\alpha r)^{1/r}}\left[\|v(0)\|_X + \|v'\|_{L^1(0,T;X)}\right].$$

Chapter 3

Carathéodory's Problem and the Spaces $L^p_\alpha(0,T;X)$

With the aim of formulating and solving the fractional version of the unsteady Stokes equations, the third chapter is devoted to the study of two important tools. The first concerns the conditions for the existence of solutions to the fractional Cauchy problem. The second introduces the space $L^p_\alpha(0,T;X)$, along with a study of its most important properties.

3.1 The Fractional Carathéodory's Problem

Our aim in this section is to establish the existence and uniqueness of global solution to the Cauchy problem

$$\begin{cases} cD^\alpha_t u(t) &= G(t,u(t)), \text{ for a.e. } t\in[0,T], \\ u(0) &= u_0\in\mathbb{R}^m, \end{cases} \tag{3.1}$$

where $\alpha\in(0,1)$ and cD^α_t is the Caputo fractional derivative, $T>0$ is a fixed value and $G:[0,T]\times\mathbb{R}^m\to\mathbb{R}^m$ is a Carathéodory map. It is worth noting that Lan and Webb discussed a similar problem in [26, Theorems 5 and 6] for the case $m=1$, where the Carathéodory map grows at most linearly on the second variable. However, our focus here lies on addressing a slightly different problem using a different method.

To this end, below we introduce some conventional concepts and a classical theorem. For the proofs and further details, we refer to [37, Theorem 1.27].

Definition 3.1. (i) We say that $a:[0,T]\times\mathbb{R}^m\to\mathbb{R}^m$ is a Carathéodory map if the function $a(\cdot,x) : [0,T] \to \mathbb{R}^m$ is measurable for each $x \in \mathbb{R}^m$, and the function $a(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous for almost every $t \in [0,T]$.

(ii) The Nemytskii Mapping $\mathcal{N}_a$, is a function that maps a function $\phi : [0,T] \to \mathbb{R}^m$ into a function $\mathcal{N}_a(\phi) : [0,T] \to \mathbb{R}^m$ which is defined by

$$\big[\mathcal{N}_a(\phi)\big](t) = a(t,\phi(t)).$$

From the above definition, we may deduce the following result.

Proposition 3.2. *If $a : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory map and $\phi : [0,T] \to \mathbb{R}^m$ is measurable, then $t \mapsto \big[\mathcal{N}_a(\phi)\big](t)$ is measurable. Moreover, if $a(t,x)$ satisfies*

$$\|a(t,x)\|_m \leq \gamma(t) + C\|x\|_m^{q/p},$$

for some $\gamma \in L^p(0,T)$, with $q \in [1,\infty)$, then $\mathcal{N}_a$ is a bounded continuous mapping from $L^q(0,T;\mathbb{R}^m)$ into $L^p(0,T;\mathbb{R}^m)$.

In what follows, we adopt a structured approach. First, we establish a criterion that allows us to transform the fractional Cauchy problem (3.1) into an equivalent integral problem. Next, we prove the existence and uniqueness of a local solution, followed by a discussion on the extension of this local solution. Finally, we present a theorem concerning the existence and uniqueness of the global solution.

To begin, we introduce two definitions for the solution of the Cauchy problem (3.1).

Definition 3.3. Let $\tau \in (0,T)$ and $\phi : [0,\tau] \to \mathbb{R}^m$ a continuous function. Then, if ϕ satisfies both equations of (3.1) in $[0,\tau]$, we say that it is a local solution of (3.1).

Definition 3.4. Let $\phi : [0,T] \to \mathbb{R}^m$ a continuous function. If ϕ satisfies both equations of (3.1) in $[0,T]$, we say that it is a global solution of (3.1).

First, we focus on establishing the existence of a local solution to Problem (3.1). To achieve this, we begin by proving its equivalent integral formulation.

Lemma 3.5. *Let $\tau \in (0,T)$, $p \in (1,\infty)$, $\alpha \in (1/p,1)$, $\phi : [0,\tau] \to \mathbb{R}^m$ a continuous function and assume that $G : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory map that satisfies*

$$\|G(t,x)\|_m \leq \gamma(t) + C\|x\|_m^{q/p}, \tag{3.2}$$

for some $\gamma \in L^p(0,T)$*, with* $q \in [1,\infty)$*. Then* ϕ *is a local solution of* (3.1) *in* $[0,\tau]$*, if and only if,* ϕ *satisfies the integral equation*

$$\phi(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\phi(s))\, ds, \quad \forall t \in \ [0,\tau]. \tag{3.3}$$

Proof. Since ϕ is continuous in $[0,\tau]$, Proposition 3.2 ensures that $G(\cdot,\phi(\cdot)) \in L^p(0,T;\mathbb{R}^m)$. Now, if we assume that ϕ is a local solution of (3.1), since

$$cD_t^\alpha \phi(t) = G(t,\phi(t)), \text{ for a.e. } t \in [0,\tau],$$

item (ii) of Remark 3.6 allows us to apply J_t^α to both sides of the above equality, which, together with the Proposition 2.42 [item (v)], ensures that

$$\phi(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\phi(s))\, ds, \quad \forall t \in \ [0,\tau].$$

Conversely, since $\phi \in C([0,\tau],\mathbb{R}^m)$ and satisfies the integral equation,

$$\phi(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\phi(s))\, ds, \quad \forall t \in \ [0,\tau].$$

the continuity of the Riemann-Liouville fractional integral is sufficient for us to deduce that $\phi(0) = u_0$. Moreover, item (ii) of Proposition 2.42 allows us to apply D_t^α in both sides of integral equation, obtaining

$$cD_t^\alpha \phi(t) = G(t,\phi(t)), \quad \textit{a.e. in } [0,\tau].$$

□

Remark 3.6. The hypothesis imposed on p and α in Lemma 3.5 can be justified by the following observations:

(i) To establish the fixed-point argument used in the proof of Carathéodory's classical theorem, it is necessary that $G(\cdot,\phi(\cdot)) \in L^p(0,T;\mathbb{R}^m)$ for some $p \in [1,\infty)$, for all continuous, $\phi : [0,T] \to \mathbb{R}^m$. This condition ensures that the integral formulation of the Cauchy problem is well-defined as a mapping between subspaces of continuous functions. This property is closely tied to the fact that, for every $g \in L^1(0,T)$ we have that function

$$[0,T] \ni t \to \int_0^t g(s)\, ds,$$

defines a continuous function. However, this assertion does not hold in general when dealing with the Riemann-Liouville fractional integral of order $\alpha \in (0,1)$. For example, if we consider $\alpha < \beta < 1$ and $g(t) = t^{-\beta}$, we see that $g \in L^1(0,T)$ but

$$J_t^\alpha g(t) = \left[\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\right] t^{\alpha-\beta},$$

which does not define a continuous function on $[0,T]$.

(ii) In 1928, Hardy and Littlewood established that for all functions in $L^p(0,T)$ with $p \in (1,\infty]$, the fractional integral of order $\alpha \in (1/p, 1+1/p)$ is Hölder continuous of order $\alpha - (1/p)$, which proves to be sufficient for the continuity of the fractional integral (see [21, Theorem 12] for details). Recently, Carvalho-Neto and Fehlberg Júnior improved this result for Bochner-Lebesgue spaces (see [9, Theorem 7]). Consequently, to ensure that the integral formulation of the Cauchy problem related to the Caputo fractional derivative of order $\alpha \in (0,1)$ is well-defined as a mapping between subspace of continuous functions, it is essential that $G(\cdot,\phi(\cdot)) \in L^p(0,T;\mathbb{R}^m)$, for some $p \in (1/\alpha,\infty)$, for all continuous functions $\phi : [0,T] \to \mathbb{R}^m$.

Now we present our first main theorem, which discusses the existence and uniqueness of local solutions to problem (3.1).

Theorem 3.7. *Let* $p \in (1,\infty)$ *and* $\alpha \in (1/p,1]$. *Assume* $G : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ *is a Carathéodory map that satisfies* (3.2) *for some* $\gamma \in L^p(0,T)$, *with* $q \in [1,\infty)$, *and is a locally Lipschitz contimuous in the second variable for almost every* $t \in [0,T]$; *that is, for each fixed* $u_0 \in \mathbb{R}^m$, *there exist* $r = r(u_0) > 0$ *and constant* $L = L\big(B_r(u_0)\big) \geq 0$ *such that*

$$\|G(t,x) - G(t,y)\|_m \leq L\|x-y\|_m,$$

for all $x,\ y \in B_r(u_0)$ *and almost every* $t \in [0,T]$, *where* $B_r(u_0)$ *denotes the open ball centered at* u_0 *with radius* r. *Then, there exists* $\tau \in [0,T]$ *such that* (3.1) *has a unique local solution on* $[0,\tau]$.

Proof. Given $u_0 \in \mathbb{R}^m$, let $r > 0$ and $L > 0$ denote the constants associated with the Lipschitz continuity of G at u_0. Choose $\beta \in (0,r)$ and $\tau \in (0,T)$ such that

$$\frac{\tau^\alpha L}{\Gamma(\alpha+1)} \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-(1/p)} \tau^{\alpha-(1/p)}\|G(\cdot,u_0)\|_{L^p(0,T;\mathbb{R}^m)} \leq \frac{\beta}{2}.$$

Consider

$$K := \{\phi \in C([0,\tau],\mathbb{R}^m) : \phi(0) = u_0 \ \text{ and } \ \|\phi(t) - u_0\|_m \leq \beta, \quad \forall t \in [0,\tau]\}$$

a nonempty and closed subset of $C([0,\tau],\mathbb{R}^m)$. Due to Proposition 3.2 and item *(ii)* of Remark 3.6, we can define the operator $T : K \to C([0,\tau],\mathbb{R}^m)$ by

$$T\phi(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\phi(s))\, ds.$$

To apply Banach fixed-point theorem, we shall prove that $T(K) \subset K$ and that T is a contraction. First, observe that for $\psi \in K$, as argued before, $J_t^\alpha G(t,\psi(t))$ is continuous, which ensures that $T\psi \in C([0,\tau];\mathbb{R}^m)$ and $T\psi(0) = u_0$. Moreover, notice that

$$\|T\psi(t) - u_0\|_m \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\|G(s,\psi(s)) - G(s,u_0)\|_m + \|G(s,u_0)\|_m\right] \ ds,$$

for all $t \in [0,T]$. Therefore, since Proposition 3.2 ensures that $G(\cdot,u_0) \in L^p(0,t;\mathbb{R}^n)$, we have

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|G(s,u_0)\|_m \ ds \ &\leq \ \frac{1}{\Gamma(\alpha)} \|(t-\cdot)^{\alpha-1}\|_{L^{p/(p-1)}(0,t)} \ \|G(\cdot,u_0)\|_{L^p(0,t;\mathbb{R}^m)} \\ &\leq \ \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-(1/p)} \tau^{\alpha-(1/p)} \|G(\cdot,u_0)\|_{L^p(0,T;\mathbb{R}^m)}, \end{aligned}$$

for all $t \in [0,\tau]$, what implies that

$$\begin{aligned} \|T\psi(t) - u_0\|_m \ &\leq \ \frac{\tau^\alpha L \beta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-(1/p)} \tau^{\alpha-(1/p)} \|G(\cdot,u_0)\|_{L^p(0,T;\mathbb{R}^m)} \\ &\leq \ \frac{\beta}{2} + \frac{\beta}{2} = \beta. \end{aligned}$$

Now, if $\psi_1, \psi_2 \in K$, we have that

$$\begin{aligned} \|T\psi_1(t) - T\psi_2(t)\|_m \ &\leq \ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|G(s,\psi_1(s)) - G(s,\psi_2(s))\|_m \ ds \\ &\leq \ \frac{L\tau^\alpha}{\Gamma(\alpha+1)} \sup_{s\in[0,\tau]} \|\psi_1(s) - \psi_2(s)\|_m \\ &\leq \ \frac{1}{2} \sup_{s\in[0,\tau]} \|\psi_1(s) - \psi_2(s)\|_m, \end{aligned}$$

for all $t \in [0, \tau]$, or in other words, T is a contraction. Summarizing the conclusions above, and applying the Banach fixed-point theorem, we conclude that there exists $\tau \in (0, T)$ and a unique local solution $u : [0, \tau] \to \mathbb{R}^m$ to problem (3.1) in K.

To complete the proof, we note that any continuous function from $[0, \tau]$ into $\mathbb{R}^n$ that also satisfies the integral formulation (3.3) must coincide with the unique local solution $\phi \in K$ whose existence we have just established. However, we omit the details, as the result follows from a standard argument based on classical fractional Gronwall inequality, see for instance Theorem 2.28. ☐

Next, our attention turns to extending the local solution, which exist due to the conclusion of Theorem 3.7. To accomplish this, we initially introduce the concept of continuation of local solutions, followed by the results that prove the existence and uniqueness of such extensions.

Definition 3.8. (i) Let $\phi : [0, \tau] \to \mathbb{R}^m$ be a local solution on $[0, \tau]$ of (3.1). If $\tau^* > \tau$ and $\phi^* : [0, \tau^*] \to \mathbb{R}^m$ is a local solution of (3.1) on $[0, \tau^*]$, then we say that ϕ^* is a continuation of ϕ on $[0, \tau^*]$.

(ii) We say that a function $\phi : [0, \tau^*) \to \mathbb{R}^m$ is a local solution of (3.1) on $[0, \tau^*)$, if it is a local solution of (3.1) on $[0, \tau]$ for all $\tau \in (0, \tau^*)$.

(iii) If $\phi : [0, \tau^*) \to \mathbb{R}^m$ is a local solution of (3.1) on $[0, \tau^*)$ but is not a local solution of (3.1) in $[0, \tau^*]$, then we call it a maximal local solution.

Before addressing our next Theorem, we establish a simple technical lemma.

Lemma 3.9. *Let $t_2 > t_1 > 0$, $p \in (1, \infty)$, $\alpha \in (1/p, 1)$ and $v \in L^p(0, t_1; \mathbb{R}^m)$. Then we have that*

$$\left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\right] v(s) ds \right\|_m \leq \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1} \right)^{1-(1/p)} (t_2 - t_1)^{\alpha-(1/p)} \|v\|_{L^p(0,t_1;\mathbb{R}^m)}.$$

Proof. Since we know that for $0 \leq a \leq b$ and $m \in [1, \infty)$ it holds that

$$(b - a)^m \leq b^m - a^m,$$

we may deduce that

$$\left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] v(s)ds \right\|_m$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \left[(t_1-s)^{(\alpha-1)p/(p-1)} - (t_2-s)^{(\alpha-1)p/(p-1)}\right] ds \right\}^{(p-1)/p} \|v\|_{L^p(0,t_1;\mathbb{R}^m)}$$

$$= \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \left[- \int_{t_1}^{t_2} \frac{d}{dw}(w-s)^{(\alpha-1)p/(p-1)}\, dw \right] ds \right\}^{(p-1)/p} \|v\|_{L^p(0,t_1;\mathbb{R}^m)}. \quad (3.4)$$

As a result, by applying Fubini-Tonelli's theorem to (3.4), we arrive at

$$\left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] v(s)ds \right\|_m$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \int_{t_1}^{t_2} \left[(w-t_1)^{(\alpha-1)p/(p-1)} - w^{(\alpha-1)p/(p-1)}\right] dw \right\}^{(p-1)/p} \|v\|_{L^p(0,t_1;\mathbb{R}^m)}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1} \right)^{1-(1/p)} (t_2-t_1)^{\alpha-(1/p)} \|v\|_{L^p(0,t_1;\mathbb{R}^m)}.$$

□

Theorem 3.10. *Let $p \in (1,\infty)$, $\alpha \in (1/p,1)$, and $G : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory map that satisfies (3.2) for some $\gamma \in L^p(0,T)$, with $q \in [1,\infty)$, and is a locally Lipschitz in the second variable for almost everywhere on $[0,T]$. If $\phi : [0,\tau] \to \mathbb{R}^m$ is the unique local solution of (3.1) on $[0,\tau]$, for some $\tau \in (0,T)$, then there exists a unique continuation ϕ^* of ϕ on some interval $[0,\tau+\tau_1] \subset [0,T)$, for some $\tau_1 > 0$.*

Proof. Let $\phi : [0,\tau] \to \mathbb{R}^m$ be the unique local solution to (3.1) on $[0,\tau]$. Let $B_{r_1}(\phi(\tau))$ be the open ball centered at $\phi(\tau)$ with radius $r_1 > 0$, and let $L = L(B_{r_1}(\phi(\tau)))$ be the Lipschitz constant of G associated with its local Lipschitz continuity on $B_{r_1}(\phi(\tau))$. Fix $\beta_1 \in (0,r_1)$ and choose $\tau_1 > 0$ with $\tau+\tau_1 \in (0,T)$, such that the following conditions are satisfied

$$\frac{\tau_1^\alpha L}{\Gamma(\alpha+1)} \leq \frac{1}{3} \quad \text{and} \quad \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1} \right)^{1-(1/p)} \tau_1^{\alpha-(1/p)} D \leq \frac{\beta_1}{3},$$

where

$$D = \max\left\{ \|G(\cdot,\phi(\tau))\|_{L^p(0,T;\mathbb{R}^m)}, \|G(\cdot,\phi(\cdot))\|_{L^p(0,\tau;\mathbb{R}^m)} \right\}.$$

Consider

$$K := \left\{ \varphi \in C([0,\tau+\tau_1],\mathbb{R}^m) : \begin{array}{c} \varphi(t) = \phi(t) \ \ \forall t \in [0,\tau] \ \text{ and } \\ \|\varphi(t) - \phi(\tau)\|_m \leq \beta_1, \quad \forall t \in [\tau, \tau+\tau_1] \end{array} \right\}$$

a nonempty and closed subset of $C([0,\tau+\tau_1];\mathbb{R}^m)$ and $T : K \to C([0,\tau+\tau_1];\mathbb{R}^m)$ given by

$$T\varphi(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\varphi(s))ds.$$

Let us prove that $T(K) \subset K$.

(i) If $\varphi \in K$, then $\varphi(t) = \phi(t)$ in $[0,\tau]$ with ϕ the local solution of (3.1) in $[0,\tau]$. So, if $t \in [0,\tau]$

$$\begin{aligned} T\varphi(t) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\varphi(s))ds \\ &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\phi(s))ds = T\phi(t) = \phi(t). \end{aligned}$$

(ii) If $t \in [\tau,\tau+\tau_1]$, then

$$\begin{aligned} \|T\varphi(t) - \phi(\tau)\|_m &\leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s,\varphi(s))ds - \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} G(s,\phi(s))ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} G(s,\phi(\tau))ds - \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} G(s,\phi(\tau))ds \right\|_m \\ &\leq \left\| \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} [G(s,\varphi(s)) - G(s,\phi(\tau))]ds \right\|_m \\ &\quad + \left\| \frac{1}{\Gamma(\alpha)} \int_0^\tau [(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1}] G(s,\phi(s))ds \right\|_m \\ &\quad + \left\| \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} G(s,\phi(\tau))ds \right\|_m := \mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t). \end{aligned}$$

Now, we proceed to estimate each of the functions $\mathcal{J}_1(t)$, $\mathcal{J}_2(t)$, and $\mathcal{J}_3(t)$ defined above.

(a) A direct estimate shows that

$$\mathcal{J}_1(t) \leq \frac{L\beta_1\, \tau_1^\alpha}{\Gamma(\alpha+1)};$$

(b) Proposition 3.2 and Lemma 3.9 ensure

$$\mathcal{J}_2(t) \leq \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-(1/p)} \tau_1^{\alpha-(1/p)} \|G(\cdot, \phi(\cdot))\|_{L^p(0,\tau;\mathbb{R}^m)};$$

(c) Again, a direct computation guarantees

$$\mathcal{J}_3(t) \leq \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-(1/p)} \tau_1^{\alpha-(1/p)} \|G(\cdot, \phi(\tau))\|_{L^p(0,T;\mathbb{R}^m)}.$$

Therefore

$$\|T(\varphi(t)) - \phi(\tau)\|_m \leq \beta_1, \quad \text{for all } t \in [\tau, \tau+\tau_1].$$

Now, let us prove that T is a contraction. For $\varphi_1, \varphi_2 \in K$ and $t \in [0, \tau+\tau_1]$, it holds that

$$\begin{aligned} \|T(\varphi_1(t)) - T(\varphi_2(t))\|_m &\leq \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} \|G(s,\varphi_1(s)) - G(s,\varphi_2(s))\|_m \; ds \\ &\leq \frac{L\tau_1^\alpha}{\Gamma(\alpha+1)} \sup_{s\in[0,\tau+\tau_1]} \|\varphi_1(s) - \varphi_2(s)\|_m \\ &\leq \frac{1}{3} \sup_{s\in[0,\tau+\tau_1]} \|\varphi_1(s) - \varphi_2(s)\|_m. \end{aligned}$$

Therefore, Banach's fixed-point theorem ensures the existence of a unique fixed point $\phi^* \in K$ for the integral equation, which is, in fact, the only continuation of ϕ over the interval $[0, \tau+\tau_1]$. The uniqueness of this continuation can be established by applying the same argument used at the end of the proof of Theorem 3.7

□

Since the radius r and the parameter β are given in Theorem 3.7, and the radius r_1 and the parameter β_1 in Theorem 3.10, Figure 3.1 below, illustrates a possible local solution ϕ to the fractional Cauchy problem and its extension ϕ^*, assuming that r_1 is smaller than r, that is, $B_{r_1}(\phi(\tau)) \subset B_r(u_0)$.

Next, we present an important Lemma, which provides support for the theorem concerning the existence and uniqueness of the global solution to problem (3.1).

Lemma 3.11. *Let* $p \in (1,\infty)$, $\alpha \in (1/p, 1)$, $\tau \in (0,T]$, *and* $G : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ *a Carathéodory map that satisfies* (3.2) *for some* $\gamma \in L^p(0,T)$, *with* $q \in [1,\infty)$. *If* $\phi : [0,\tau) \to \mathbb{R}^m$ *is a local solution of* (3.1) *and belongs to* $L^q(0,\tau;\mathbb{R}^m)$, *then we can extend* ϕ *continuously on* $[0,\tau]$ *in a unique manner. Furthermore, this extension is a solution of* (3.1) *on the interval* $[0,\tau]$.

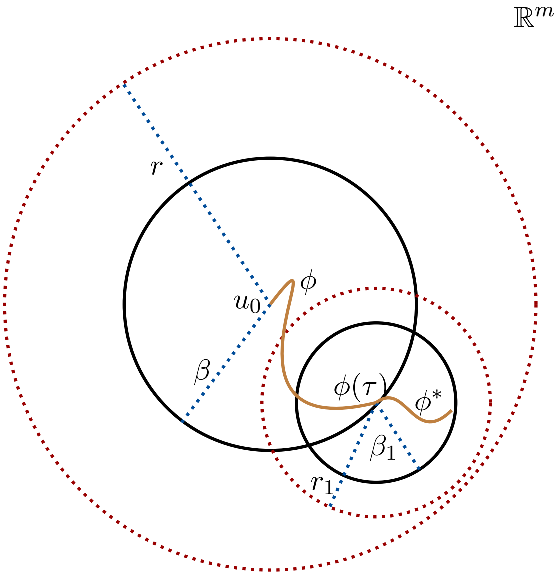

Figure 3.1: Representation of the balls $B_r(u_0)$ and $B_{r_1}(\phi(\tau))$, and a possible local solution ϕ.

Proof. Assume that ϕ is a solution of (3.1) on the interval $[0,\tau)$, for $t, w \in [0,\tau)$, where $t > w > 0$, we have

$$\|\phi(t)-\phi(w)\|_m \leq \frac{1}{\Gamma(\alpha)} \int_w^t (t-s)^{\alpha-1} \|G(s,\phi(s))\|_m \, ds + \int_0^w \left[(w-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \|G(s,\phi(s))\|_m \, ds.$$

By Proposition 3.2 and Lemma 3.9, it follows that

$$\|\phi(t)-\phi(w)\|_m \leq \frac{2}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-(1/p)} (t-w)^{\alpha-(1/p)} \|G(\cdot,\phi(\cdot))\|_{L^p(0,\tau;\mathbb{R}^m)}.$$

Now, let $\big(t_k\big) \subset [0,\tau)$ be a sequence such that $t_k \to \tau$ as $k \to \infty$. The above inequality shows that $(\phi(t_k))$ is a Cauchy sequence. Since $\mathbb{R}^m$ is a Banach space, there exists $u_\tau \in \mathbb{R}^m$ such that $\lim_{k \longrightarrow \infty} \phi(t_k) = u_\tau$.

Next, consider another sequence $\big(s_k\big) \subset [0,\tau)$ such that $s_k \to \tau$ as $k \to \infty$. We show that $\phi(s_k) \to u_\tau$ by observing that:

$$\|\phi(t_k)-\phi(s_k)\|_m \leq \frac{2}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p - 1}\right)^{1-(1/p)} \left[(t_k-\tau)+(\tau-s_k)\right]^{\alpha-(1/p)} \|G(\cdot,\phi(\cdot))\|_{L^p(0,\tau;\mathbb{R}^m)}.$$

Thus,

$$\begin{aligned}\|\phi(s_k)-u_\tau\|_m &\leq \|\phi(s_k)-\phi(t_k)\|_m+\|\phi(t_k)-u_\tau\|_m\\ &\leq C[(t_k-\tau)+(\tau-s_k)]^{\alpha-(1/p)}+\|\phi(t_k)-u_\tau\|_m,\end{aligned}$$

which implies that $\phi(s_k)\to u_\tau$ as $k\to\infty$.

Define now the function $\tilde{\phi}:[0,\tau]\longrightarrow\mathbb{R}^m$ as follows:

$$\tilde{\phi}(t):=\begin{cases}\phi(t) & \text{if } t\in[0,\tau),\\ u_\tau & \text{if } t=\tau.\end{cases}$$

It follows that $\tilde{\phi}$ is continuous on $[0,\tau]$ and is the only possible continuous extension of $\phi(t)$.

Finally, observe that

$$\begin{aligned}\tilde{\phi}(\tau)=\lim_{t\longrightarrow\tau^-}\phi(t)=\lim_{t\longrightarrow\tau^-}\left[u_0+\int\limits_0^t(t-s)^{\alpha-1}G(s,\phi(s))ds\right]\\ =u_0+\int\limits_0^\tau(\tau-s)^{\alpha-1}G(s,\tilde{\phi}(s))ds.\end{aligned}$$

Therefore, $\tilde{\phi}$ is a solution of (3.1) on $[0,\tau]$, as claimed. □

Theorem 3.12. *Let $p\in(1,\infty)$, $\alpha\in(1/p,1]$, $G:[0,T]\times\mathbb{R}^m\to\mathbb{R}^m$ a Carathéodory map that satisfies (3.2) for some $\gamma\in L^p(0,T)$, with $p=q$ and is a locally Lipschitz continuous in the second variable for almost every $t\in[0,T]$. Then, the problem (3.1) has a unique global solution in $[0,T]$.*

Proof. Consider

$$\begin{aligned}H:=\big\{\tau\in(0,T]:\text{exists a unique }\phi_\tau:[0,\tau]\to\mathbb{R}^m\\ \text{that is a local solution of (3.1) on }[0,\tau]\big\}.\end{aligned}$$

Note that Theorem 3.7 guarantees that H is nonempty. Therefore, if we denote $\tau^*=\sup H$, we can define a continuous function $\phi:[0,\tau^*)\to\mathbb{R}^m$, which is the unique solution of problem (3.1) in $[0,\tau]$, for all $\tau\in(0,\tau^*)$. Now notice that due to (3.2) we have

$$\|\phi(t)\|_m \leq \underbrace{\|u_0\|_m + \frac{1}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{1-(1/p)} T^{\alpha-(1/p)}\|\gamma\|_{L^p(0,T)}}_{:=M} + \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\phi(s)\|_m\,ds,$$

for all $t \in [0,\tau^*)$. But then, since $\phi \in L^1_{loc}(0,\tau^*;\mathbb{R}^m)$, the fractional version of Gronwall's inequality (cf. Theorem 2.28) ensures that

$$\|\phi(t)\|_m \leq M E_\alpha(C^{1/\alpha}t),$$

for all $t \in [0,\tau^*)$, where $E_\alpha(z)$ is the Mittag-Leffler function. This implies that $\phi \in L^\infty(0,\tau^*;\mathbb{R}^m)$.

Consequently, we can deduce that $\tau^* = T$. Indeed, if this were not the case, that is, if $\tau^* < T$, Theorem 3.10 and Lemma 3.11 would guarantee the existence and uniqueness of a solution defined on an interval properly containing $[0,\tau^*]$, contradicting the definition of τ^*. Therefore, since $\tau^* = T$, we may apply Lemma 3.11 once more to ensure the existence of a unique extension of ϕ on $[0,T]$, as desired. ☐

Remark 3.13. It is worth noting that Theorem 3.12 becomes significantly more challenging to prove when $q \neq p$, as extending Gronwall's inequality to this scenario results in a far more complex task. Since such an improvement does not contribute to our objectives, we have opted not to address this generalization here.

3.2 The Spaces $L^p_\alpha(0,T;X)$

As mentioned in the introduction, it is essential to discuss a new space that is isometrically isomorphic to a Bochner-Lebesgue spaces and plays a crucial role in the main results of this work. Therefore, in this section, assuming that X is a Banach space, we first introduce the space $L^p_\alpha(0,T;X)$, along with the definition of a suitable norm. We then present a series of results aimed at establishing relationships between the spaces $L^p_\alpha(0,T;X)$ for different values of $\alpha \in [0,1]$ and $p \in [1,\infty)$, as well as their connection with the classical space $L^p(0,T;X)$. Finally, we show that $L^p_\alpha(0,T;X)$ is a Banach space and address its reflexivity properties.

Definition 3.14. For $\alpha \in (0,1]$ and $p \in [1,\infty)$, the (α,p) Bochner integrable functions is denoted by $L^p_\alpha(0,T;X)$ and consists of all Bochner measurable functions $v : [0,T] \to X$

such that

$$\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|v(s)\|_X^p ds < \infty.$$

For completeness, we also define $L_0^p(0,T;X) = L^\infty(0,T;X)$ for any $p \in [1,\infty)$.

Remark 3.15. (i) It is interesting to note that the choice of the interval $[0,T]$, in the definition of the sets $L_\alpha^p(0,T;X)$, was made solely for the purposes of the studies developed in this manuscript. Nevertheless, replacing $[0,T]$ with any other interval $[t_0,t_1]$ does not affect the results obtained in this section, except for slight changes in the constants involved in the estimates.

(ii) Observe that this definition could be extended to $\alpha > 1$. However, since it falls outside the scope of the results we are proving in this manuscript, we prefer not to discuss this case at this moment, leaving such an analysis for a future related work.

Let us begin our study of $L_\alpha^p(0,T;X)$ by proving that each of these sets forms a normed vector space, which is also a subspace of $L^p(0,T;X)$.

Proposition 3.16. *Let $\alpha \in [0,1]$ and $p \in [1,\infty)$. Then $L_1^p(0,T;X) = L^p(0,T;X)$ and $L_0^p(0,T;X) = L^\infty(0,T;X)$, both of which are trivially subspaces of $L^p(0,T;X)$. On the other hand, for $\alpha \in (0,1)$, $L_\alpha^p(0,T;X)$ is a vector subspace of $L^p(0,T;X)$.*

Proof. Assume that $\alpha \in (0,1)$. First, observe that for any $v \in L_\alpha^p(0,T;X)$, we have

$$\int_0^T \|v(s)\|^p \, ds = \int_0^T (T-s)^{1-\alpha}(T-s)^{\alpha-1}\|v(s)\|^p \, ds \leq \Gamma(\alpha)T^{1-\alpha}\|v\|_{L_\alpha^p(0,T;X)} < \infty,$$

where $\|\cdot\|_{L_\alpha^p(0,T;X)}$ is defined below. Thus, $L_\alpha^p(0,T;X)$ is contained in $L^p(0,T;X)$. Now, let $\lambda \in \mathbb{R}$ and $v_1, v_2 \in L_\alpha^p(0,T;X)$. Then,

$$\begin{aligned}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|\lambda v_1(s) + v_2(s)\|_X^p \, ds \quad &\leq \quad 2^{p-1}\max\{1,|\lambda|^p\} \left[\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|v_1(s)\|_X^p \, ds \right. \\ &\quad + \left. \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|v_2(s)\|_X^p \, ds\right] < \infty.\end{aligned}$$

Hence, $\lambda v_1 + v_2 \in L_\alpha^p(0,T;X)$, which proves that $L_\alpha^p(0,T;X)$ is a vector subspace of $L^p(0,T;X)$. ☐

Now, we define an appropriate norm for the vector space $L_\alpha^p(0,T;X)$.

Proposition 3.17. *For* $\alpha \in [0,1]$ *and* $p \in [1,\infty)$*, consider* $\|\cdot\|_{L^p_\alpha(0,T;X)} : L^p_\alpha(0,T;X) \to \mathbb{R}$ *given by*

$$\|v\|_{L^p_\alpha(0,T;X)} := \begin{cases} \|v\|_{L^\infty(0,T;X)}, & \text{if } \alpha = 0, \\ \left(\displaystyle\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|v(s)\|_X^p ds\right)^{1/p}, & \text{if } \alpha \in (0,1]. \end{cases}$$

With this, $\Big(L^p_\alpha(0,T;X), \|\cdot\|_{L^p_\alpha(0,T;X)}\Big)$ *defines a normed vector space.*

Proof. Assume that $\alpha \in (0,1)$.

(i) If $v \in L^p_\alpha(0,T;X)$ and $\|v\|_{L^p_\alpha(0,T;X)} = 0$, since $(T-s)^{\alpha-1}\|v(s)\|_X^p \geq 0$ for almost every $s \in [0,T]$, it follows that $\|v(s)\|_X^p = 0$ for almost every $s \in [0,T]$. Therefore $v = 0$.

(ii) It is straightforward to verify that for $v \in L^p_\alpha(0,T;X)$ and $\lambda \in \mathbb{R}$, we have $\|\lambda v\|_{L^p_\alpha(0,T;X)} = |\lambda|\|v\|_{L^p_\alpha(0,T;X)}$.

(iii) For $v_1, v_2 \in L^p_\alpha(0,T;X)$ we have

$$\begin{aligned} & J_t^\alpha \|v_1(t) + v_2(t)\|_X^p \Big|_{t=T} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{(\alpha-1)/p} \|v_1(s) + v_2(s)\|_X (T-s)^{(\alpha-1)(p-1)/p} \|v_1(s) + v_2(s)\|_X^{p-1} ds, \end{aligned}$$

what ensures that

$$\begin{aligned} & J_t^\alpha \|v_1(t) + v_2(t)\|_X^p \Big|_{t=T} \\ &\leq \int_0^T \left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\right]^{1/p} \|v_1(s)\|_X \left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{p-1}{p}} \|v_1(s) + v_2(s)\|_X^{p-1} ds \\ &\quad + \int_0^T \left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\right]^{1/p} \|v_2(s)\|_X \left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\right]^{\frac{p-1}{p}} \|v_1(s) + v_2(s)\|_X^{p-1} ds. \end{aligned}$$

Thus, Hölder's inequality allows us to achieve that

$$\begin{aligned} & J_t^\alpha \|v_1(t) + v_2(t)\|_X^p \Big|_{t=T} \\ &\leq \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|v_1(s)\|_X^p ds\right)^{\frac{1}{p}} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|v_1(s) + v_2(s)\|_X^p ds\right)^{\frac{p-1}{p}} \end{aligned}$$

$$+\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|v_2(s)\|_X^p ds\right)^{\frac{1}{p}}\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|v_1(s)+v_2(s)\|_X^p ds\right)^{\frac{p-1}{p}}.$$

Therefore

$$\|v_1+v_2\|_{L^p_\alpha(0,T;X)}^p \leq \left[\|v_1\|_{L^p_\alpha(0,T;X)}+\|v_2\|_{L^p_\alpha(0,T;X)}\right]\|v_1+v_2\|_{L^p_\alpha(0,T;X)}^{p-1}. \tag{3.5}$$

It follows now directly from (3.5) that

$$\|v_1+v_2\|_{L^p_\alpha(0,T;X)} \leq \|v_1\|_{L^p_\alpha(0,T;X)}+\|v_2\|_{L^p_\alpha(0,T;X)}.$$

completing the proof that $\|\cdot\|_{L^p_\alpha(0,T;X)}$ defines a norm in $L^p_\alpha(0,T;X)$.

□

Now, in order to establish some relationships between $L^p_\alpha(0,T;X)$ for different values of $\alpha\in[0,1]$ and $p\in[1,\infty)$, as well as its connection with the classical $L^p(0,T;X)$, we present the following series of results.

Proposition 3.18. *Let $\alpha,\beta\in[0,1]$, with $\alpha<\beta$, and $p\in[1,\infty)$. Then, $L^p_\alpha(0,T;X)\subsetneq L^p_\beta(0,T;X)$, and for all $f\in L^p_\alpha(0,T;X)$ it holds that*

$$\|v\|_{L^p_\beta(0,T;X)} \leq \begin{cases}\left(\dfrac{T^\beta}{\Gamma(\beta+1)}\right)^{1/p}\|v\|_{L^p_\alpha(0,T;X)}, & \text{for } \alpha=0,\\ \left(\dfrac{T^{\beta-\alpha}\Gamma(\alpha)}{\Gamma(\beta)}\right)^{1/p}\|v\|_{L^p_\alpha(0,T;X)}, & \text{for } \alpha\in(0,1].\end{cases} \tag{3.6}$$

Proof. First, let $\alpha=0$. If $v\in L^p_0(0,T;X)$, it follows from Theorem 2.35 that $J_t^\beta\|v(t)\|_X^p$ belongs to $C([0,T];\mathbb{R})$. Consequently, we obtain the inclusion $L^p_0(0,T;X)\subset L^p_\beta(0,T;X)$ for all $\beta\in(0,1]$. To show that this inclusion is strict, consider the function $v(t)=t^{-\beta/p}$, which belongs to $L^p_\beta(0,T;\mathbb{R})$ but not to $L^p_0(0,T;\mathbb{R})$. Indeed, observe that

$$\begin{aligned}\|t^{-\beta/p}\|_{L^p_\beta(0,T;\mathbb{R})} &= \left(\frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}s^{-\beta}ds\right)^{1/p}\\ &= \left(\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}s^{(1-\beta)-1}ds\right)^{1/p}\\ &= \left(\frac{\mathcal{B}(1-\beta,\beta)}{\Gamma(\beta)}\right)^{1/p} = (\Gamma(1-\beta))^{1/p}.\end{aligned}$$

Now, assume that $0 < \alpha < \beta \leq 1$. For any $v \in L^p_\alpha(0,T;X)$, we have

$$\int_0^T (T-s)^{\beta-1}\|v(s)\|^p_X\,ds = \int_0^T (T-s)^{\beta-\alpha}(T-s)^{\alpha-1}\|v(s)\|^p_X\,ds \leq T^{\beta-\alpha}\int_0^T (T-s)^{\alpha-1}\|v(s)\|^p_X\,ds.$$

This implies the norm bound (3.6), which establishes the inclusion

$$L^p_\alpha(0,T;X) \subset L^p_\beta(0,T;X),$$

for $0 < \alpha < \beta \leq 1$.

Finally, to verify that this inclusion is strict, consider $v(t) = (T-t)^{-\gamma/p}$ for any $\gamma \in (\alpha,\beta)$. In this case, we have that $v \in L^p_\beta(0,T;\mathbb{R})$ but $v \notin L^p_\alpha(0,T;\mathbb{R})$. Indeed, since $\alpha - \gamma < 0$, the integral

$$\left\|(T-t)^{-\gamma/p}\right\|^p_{L^p_\alpha(0,T;\mathbb{R})} = \int_0^T (T-s)^{\alpha-1-\gamma}ds = \left.\frac{s^{\alpha-\gamma}}{\alpha-\gamma}\right|_0^T$$

diverges.

□

Corollary 3.19. *Let $\alpha \in (0,1)$, and $p \in (1,\infty)$ such that $\alpha > 1/p$. If $v \in L^p_\alpha(0,T;X) \cup L^p(0,T;X)$, then $J^\alpha_t v \in C([0,T];X)$ and $J^\alpha_t v(t)\big|_{t=0} = 0_X$.*

Proof. First suppose that $v \in L^p(0,T;X)$. According to [9, Theorem 7], it follows that $J^\alpha_t v \in C([0,T];X)$. Moreover, observe that

$$\|J^\alpha_t v(t)\|_X \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|v(s)\|_X\,ds \leq \left(\frac{\|v\|_{L^p(0,T;X)}}{\Gamma(\alpha)^{(p-1)/p}}\right)\left(\int_0^t (t-s)^{p(\alpha-1)/(p-1)}\,ds\right)^{(p-1)/p} \leq C_{(p,\alpha,v)}\,t^{\frac{p\alpha-1}{p}},$$

Thus, $\lim_{t\to 0^+}\|J^\alpha_t v(t)\|_X = 0$, and consequently $J^\alpha_t v(t)\big|_{t=0} = 0_X$.

Now suppose that $v \in L^p_\alpha(0,T;X)$. By Proposition 3.18, we have $L^p_\alpha(0,T;X) \subset L^p(0,T;X)$, and hence the continuity of $J^\alpha_t v$ follows from the previous case. It remains to prove that $J^\alpha_t v(t)|_{t=0} = 0_X$. Consider

$$\|J^\alpha_t v(t)\|_X \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(T-s)^{(1-\alpha)/p}(T-s)^{(\alpha-1)/p}\|v(s)\|_X\,ds$$

$$\leq \left(\frac{\|v\|_{L^p_\alpha(0,T;X)}}{\Gamma(\alpha)^{(p-1)/p}}\right)\left(\int_0^t (t-s)^{p(\alpha-1)/(p-1)}(T-s)^{(1-\alpha)/(p-1)}\,ds\right)^{(p-1)/p}, \quad (3.7)$$

for all $t \in (0,T]$.

Now notice that by applying the change of variable $s = T-(T-t)h^{-1}$, we obtain

$$\int_0^t (t-s)^{p(\alpha-1)/(p-1)}(T-s)^{(1-\alpha)/(p-1)}\,ds$$
$$= \underbrace{(T-t)^\alpha \int_{(T-t)/T}^1 (1-h)^{p(\alpha-1)/(p-1)}h^{-\alpha-1}\,dh}_{=:\psi_\alpha(t)}. \quad (3.8)$$

It follows from (3.7) and (3.8) that

$$\|J_t^\alpha v(t)\|_X\Big|_{t=0} = \lim_{t\to 0}\|J_t^\alpha v(t)\|_X \leq \left(\frac{\|v\|_{L^p_\alpha(0,T;X)}}{\Gamma(\alpha)^{(p-1)/p}}\right)\lim_{t\to 0}\psi_\alpha(t) = 0.$$

Thus, the result holds, completing the proof. □

Theorem 3.20. *Let $\alpha \in (0,1]$ and $p,q \in [1,\infty)$, with $p<q$. Then $L^q_\alpha(0,T;X) \subsetneq L^p_\alpha(0,T;X)$, and for all $v \in L^p_\alpha(0,T;X)$ it holds that*

$$\|v\|_{L^q_\alpha(0,T;X)} \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{\frac{q-p}{pq}}\|v\|_{L^p_\alpha(0,T;X)}.$$

Proof. Assume that $\alpha \in (0,1)$. For $v \in L^p_\alpha(0,T;X)$, we have

$$\begin{aligned}\|v\|^p_{L^p_\alpha(0,T;X)} &= \int_0^T \left(\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{p}{q}}\|v(s)\|_X^p\left(\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{q-p}{q}}ds\\ &\leq \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|v(s)\|_X^q ds\right)^{\frac{p}{q}}\left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right)^{\frac{q-p}{q}}\\ &= \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{\frac{q-p}{q}}\|v\|^p_{L^q_\alpha(0,T;X)}.\end{aligned}$$

Now, to prove the strictness of the inclusion, consider $\gamma \in (p,q)$ and let $v(t) = (T-t)^{-\alpha/\gamma}$. Then $v \in L^p_\alpha(0,T;\mathbb{R})$ but $v \notin L^q_\alpha(0,T;\mathbb{R})$.

□

Theorem 3.21. *For $\alpha \in (0,1)$ and $p,q \in [1,\infty)$, with $q > p/\alpha$, we have the strict* *inclusion* $L^q(0,T;X) \subsetneq L^p_\alpha(0,T;X)$*. Furthermore, for all* $v \in L^q(0,T;X)$*, it holds that*

$$\|v\|_{L^p_\alpha(0,T;X)} \leq \left(\frac{q-p}{\alpha q - p}\right)^{(q-p)/pq} \left(\frac{T^{(\alpha q - p)/pq}}{\Gamma(\alpha)^{1/p}}\right) \|v\|_{L^q(0,T;X)}. \tag{3.9}$$

Additionally, we have

$$L^p_\alpha(0,T;X) \not\subset L^{p/\alpha}(0,T;X) \quad \textit{and} \quad L^{p/\alpha}(0,T;X) \not\subset L^p_\alpha(0,T;X).$$

Proof. For $q > p/\alpha$ and $v \in L^q(0,T;X)$, observe that

$$\int_0^T (T-s)^{\alpha-1}\|v(s)\|^p\,ds \leq \left(\int_0^T (T-s)^{(\alpha-1)q/(q-p)}\,ds\right)^{(q-p)/q} \left(\int_0^T \|v(s)\|^q\,ds\right)^{p/q},$$

which implies (3.9).

To verify that $L^q(0,T;X) \subsetneq L^p_\alpha(0,T;X)$ for all $q > p/\alpha$, choose $\gamma \in (1/q, \alpha/p)$ and define $v(t) = (T-t)^{-\gamma}$. Then, $v \in L^p_\alpha(0,T;\mathbb{R})$ but $v \notin L^q(0,T;\mathbb{R})$. Indeed, since $1 - \gamma q < 0$, the integral

$$\left\|(T-t)^{-\gamma}\right\|^q_{L^q(0,T;\mathbb{R})} = \int_0^T (T-s)^{-\gamma q} ds = \left.\frac{s^{1-\gamma q}}{1-\gamma q}\right|_0^T$$

diverges.

Finally, to establish the last part of this theorem, we need to provide two counterexamples showing that the sets $L^p_\alpha(0,T;X)$ and $L^{p/\alpha}(0,T;X)$ are not contained in each other. It suffices to consider the scalar case $X = \mathbb{R}$. First, consider $v(t) = t^{-\alpha/p}$ and observe that $v \in L^p_\alpha(0,T;\mathbb{R})$ but $v \notin L^{p/\alpha}(0,T;\mathbb{R})$. To prove the remaining non-inclusion, assume, just to simplify the computations, that $T = 1$, and consider $v(t) : (0,1) \to \mathbb{R}$ defined by

$$v(t) = \frac{1}{(1-t)^{\alpha/p}\big[\log(e/(1-t))\big]^{1/p}}.$$

Observe that, by applying the change of variable $u = \log(e/(1-s))$, we obtain

$$\|v\|^{p/\alpha}_{L^{p/\alpha}(0,1;\mathbb{R})} = \int_0^1 \left\{\frac{1}{(1-s)^{\alpha/p}\big[\log(e/(1-s))\big]^{1/p}}\right\}^{p/\alpha} ds = \int_1^\infty u^{-1/\alpha} du = \frac{\alpha}{1-\alpha}.$$

which shows that $v \in L^{p/\alpha}(0,1;\mathbb{R})$. On the other hand, for $t \in (0,1)$ we have

$$J^\alpha_t |v(t)|^p = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{(1-s)^\alpha \log(e/(1-s))} ds \geq \frac{1}{\Gamma(\alpha)} \int_1^t \frac{1}{(1-s)\log(e/(1-s))} ds.$$

By making the substitution $u = \log(e/(1-s))$, it follows that

$$J_t^\alpha |v(t)|^p \geq \frac{1}{\Gamma(\alpha)} \int_0^{\log(r/1-t)} \frac{du}{u} = \frac{1}{\Gamma(\alpha)} \log(\log(e/(1-t))),$$

which diverges as $t \to 1^-$. Consequently, $v \notin L^p_\alpha(0,1;\mathbb{R})$. □

Remark 3.22. We have $L^p_1(0,T;X) = L^p(0,T;X)$ and $L^p_0(0,T;X) = L^\infty(0,T;X)$. Therefore, at this point, by Proposition 3.18, we can assert that the sets $L^p_\alpha(0,T;X)$ interpolate between $L^p(0,T;X)$ and $L^\infty(0,T;X)$ as α decreases from 1 to 0.

We conclude our analysis of the relationship between the spaces $L^p_\alpha(0,T;X)$ and $L^p(0,T;X)$ with the following result.

Theorem 3.23. *For $\alpha \in (0,1)$ and $p \in [1,\infty)$, let $q \in \big(p, p/\alpha\big)$. Then neither of the following inclusions holds:*

$$L^q(0,T;X) \subset L^p_\alpha(0,T;X) \quad \textit{nor} \quad L^p_\alpha(0,T;X) \subset L^q(0,T;X).$$

Proof. The first inclusion does not hold. Indeed, if there existed $q \in (p, p/\alpha)$ such that $L^q(0,T;X) \subset L^p_\alpha(0,T;X)$, this would imply that $L^{p/\alpha}(0,T;X) \subset L^p_\alpha(0,T;X)$, which contradicts Theorem 3.21.

To prove the other case, we restrict ourselves to the scalar case $X = \mathbb{R}$, since if $\varphi : (0,T) \to \mathbb{R}$ satisfies the desired properties, then for any $x_0 \in X$ with $\|x_0\|_X = 1$, the function $v(t) = \varphi(t)x_0$ satisfies $\|v(t)\|_X = |\varphi(t)|$. Consequently, the same inclusion relations follow.

Let us prove that $L^p_\alpha(0,T;\mathbb{R}) \not\subset L^q(0,T;\mathbb{R})$. Since $q > p$, there exists δ such that

$$\frac{1}{q} \leq \delta < \frac{1}{p}. \tag{3.10}$$

Choose $t_0 \in (0,T)$ and $\varepsilon \in \big(0, \min\{t_0, T-t_0\}\big)$. Consider function $\varphi_2 : (0,T) \to \mathbb{R}$, given by $\varphi_2(s) = |s-t_0|^{-\delta}\, \chi_{(t_0-\varepsilon,\, t_0+\varepsilon)}(s)$, where $\chi_{(t_0-\varepsilon,\, t_0+\varepsilon)}$ denotes the characteristic function of the interval $(t_0-\varepsilon,\, t_0+\varepsilon)$.

On $(t_0-\varepsilon, t_0+\varepsilon)$ the weight $(T-s)^{\alpha-1}$ is bounded above and below by positive constants. Hence

$$\|\varphi_2\|^p_{L^p_\alpha} \leq M \int_{t_0-\varepsilon}^{t_0+\varepsilon} |s-t_0|^{-\delta p}\, ds = 2\int_0^\varepsilon u^{-\delta p}\, du < \infty \iff \delta p < 1,$$

which holds by (3.10). Thus $\varphi_2 \in L^p_\alpha(0,T;\mathbb{R})$.

On the other hand,

$$\int_0^T |\varphi_2(s)|^q\, ds = \int_{t_0-\varepsilon}^{t_0+\varepsilon} |\varphi_2(s)|^q\, ds = 2\int_0^\varepsilon r^{-\delta q}\, dr = +\infty, \quad \text{whenever } \delta q \geq 1,$$

and since $\delta \geq 1/q$, we have $\delta q \geq 1$. Therefore $\varphi_2 \notin L^q(0,T;\mathbb{R})$. $\square$

The next result is dedicated to prove that $L^p_\alpha(0,T;X)$ is a Banach space.

Theorem 3.24. *Assume that $\alpha \in [0,1]$ and $p \in [1,\infty)$. Then, $L^p_\alpha(0,T;X)$, equipped with the norm $\|\cdot\|_{L^p_\alpha(0,T;X)}$, is a Banach space.*

Proof. The case $\alpha = 0$ and $\alpha = 1$ are classical. Hence, assume that $\alpha \in (0,1)$. Let $(v_k)_{k\in\mathbb{N}} \subset L^p_\alpha(0,T;X)$ be a Cauchy sequence. Observe that

$$\int_0^T (T-s)^{\alpha-1}\|v_m(s)-v_n(s)\|_X^p ds = \int_0^T \left\|(T-s)^{\frac{\alpha-1}{p}}[v_m(s)-v_n(s)]\right\|_X^p ds.$$

Then, $\left((T-\cdot)^{\frac{\alpha-1}{p}} v_k\right)$ is a Cauchy sequence in $L^p(0,T;X)$. Therefore, there exists $v \in L^p(0,T;X)$ such that $\left((T-\cdot)^{\frac{\alpha-1}{p}} v_k\right) \to v$ in $L^p(0,T;X)$.

It is not difficult to verify that $(T-\cdot)^{\frac{1-\alpha}{p}} v \in L^p_\alpha(0,T;X)$, and that

$$\int_0^T (T-s)^{\alpha-1}\left\|v_k(s)-(T-s)^{\frac{1-\alpha}{p}}v(s)\right\|_X^p ds = \left\|(T-\cdot)^{\frac{\alpha-1}{p}}v_k - v\right\|_{L^p(0,T;X)}^p,$$

what implies that $v_k \to (T-\cdot)^{\frac{1-\alpha}{p}} v$ in $L^p_\alpha(0,T;X)$, as we wanted. $\square$

Finally, we address the reflexivity properties of the space $L^P_\alpha(0,T;X)$.

Theorem 3.25. *For $\alpha \in (0,1]$ and $p \in (1,\infty)$, we have that $L^p_\alpha(0,T;X)$ is isometrically isomorphic to $L^p(0,T;X)$. Therefore, if X is reflexive, then $L^p_\alpha(0,T;X)$ is also reflexive.*

Proof. To begin the first part of the proof, let us consider the set

$$\big(L^p(0,T;X)\big)_\alpha := \left\{\left(\frac{(T-\cdot)^{\alpha-1}}{\Gamma(\alpha)}\right)^{1/p} v(\cdot) : \text{ such that } v \in L^p_\alpha(0,T;X)\right\}.$$

Observe that $(L^p(0,T;X))_\alpha$ is a subset of $L^p(0,T;X)$ since, for any $v \in L^p_\alpha(0,T;X)$, we have

$$\left\|\left(\frac{(T-\cdot)^{\alpha-1}}{\Gamma(\alpha)}\right)^{1/p} v(\cdot)\right\|_{L^p(0,T;X)} = \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|v(s)\|_X^p ds\right)^{1/p} = \|v\|_{L^p_\alpha(0,T;X)}. \tag{3.11}$$

Thus, if we define

$$\begin{array}{rcl} H_p : L^p_\alpha(0,T;X) & \to & L^p(0,T;X) \\ v(\cdot) & \to & \left(\dfrac{(T-\cdot)^{\alpha-1}}{\Gamma(\alpha)}\right)^{1/p} v(\cdot), \end{array} \tag{3.12}$$

due to (3.11), we deduce that H_p is an isometry from $L^p_\alpha(0,T;X)$ into $(L^p(0,T;X))_\alpha$. Since $L^p_\alpha(0,T;X)$ is a Banach space, we deduce that $H_p(L^p_\alpha(0,T;X)) = (L^p(0,T;X))_\alpha$ is a closed subset of $L^p(0,T;X)$.

Now observe that if $\phi \in C_c((0,T);X)$, which means that ϕ is a continuous function with compact support contained in $(0,T)$, then by considering

$$\psi(t) := \left[(T-t)^{(1-\alpha)}\Gamma(\alpha)\right]^{1/p}\phi(t),$$

we deduce that $\psi \in L^p_\alpha(0,T;X)$. This implies that $\phi = H_p(\psi) \in (L^p(0,T;X))_\alpha$, which means that $C_c((0,T);X) \subset (L^p(0,T;X))_\alpha$. Since $C_c((0,T);X)$ is dense in $L^p(0,T;X)$, it follows that $(L^p(0,T;X))_\alpha$ is dense in $L^p(0,T;X)$. However, since we already know that it is closed, we have proved that $L^p_\alpha(0,T;X)$ is isometrically isomorphic to $L^p(0,T;X)$.

Finally, if X is reflexive, then $L^p(0,T;X)$ is reflexive (see Corollary 2.16), which implies, thanks to the isometric isomorphism, that $L^p_\alpha(0,T;X)$ is also reflexive. □

We have decided to conclude this section after a last result, which provides a complete characterization of the dual space of $L^p_\alpha(0,T;X)$. For the remainder of this thesis, we recall that for a Banach space X, its dual will be denoted by X'.

Lemma 3.26. *Let X and Y be normed vector spaces, and let $T : X \to Y$ be an isometric isomorphism. Then there exists an isometric isomorphism $T' : Y' \to X'$ given by $T'(\varphi) = \varphi \circ T$. We recall that, for any normed space X, the simbol $\langle \cdot,\cdot\rangle_{X',X}$ denotes the duality paring between X' and X.*

Proof. It is straightforward to prove that T' is linear. To prove surjectivity, note that since $T : X \to Y$ is an isometric isomorphism, it admits an inverse $T^{-1} : Y \to X$, which is also an isometric isomorphism. Then, for any $h \in X'$, we have $T'(h \circ T^{-1}) = h$. To prove that it is an isometry, observe that

$$\begin{aligned} \|T'\varphi\|_{X'} &= \sup\{|\langle T'\varphi, x\rangle_{X',X}| : \|x\|_X = 1\} = \sup\{|\langle \varphi\circ T, x\rangle_{X',X}| : \|x\|_X = 1\} \\ &= \sup\{|\langle \varphi, T(x)\rangle_{Y',Y}| : \|x\|_X = 1\} = \sup\{|\langle \varphi, y\rangle_{Y',Y}| : \|y\|_Y = 1\} = \|\varphi\|_{Y'}. \end{aligned}$$

□

Theorem 3.27. *If X is reflexive, $\alpha \in (0,1]$ and $p,q \in (1,\infty)$ satisfy $(1/p)+(1/q)=1$, then for each $\varphi \in (L^q_\alpha(0,T;X'))'$, there exists a unique $v_\varphi \in L^p_\alpha(0,T;X)$, such that*

$$\langle \varphi, v\rangle_{(L^q_\alpha(0,T;X'))',L^q_\alpha(0,T;X')} = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\langle v(s), v_\varphi(s)\rangle_{X',X}\, ds, \tag{3.13}$$

for all $v \in L^q_\alpha(0,T;X')$ and $\|\varphi\|_{(L^q_\alpha(0,T;X'))'} = \|v_\varphi\|_{L^p_\alpha(0,T;X)}$.

Proof. Let $\alpha \in (0,1)$. Recall the classical isometric isomorphism

$$\begin{array}{rcl} T_{p,q} : (L^p(0,T;X)) & \rightarrow & (L^q(0,T;X'))' \\ f & \rightarrow & K_f, \end{array}$$

where $K_f : L^q(0,T;X') \rightarrow \mathbb{R}$ is given by

$$K_f(g) = \int_0^T \langle g(s), f(s)\rangle_{X',X}\, ds,$$

for all $g \in L^q(0,T;X')$.

Next, recall that the proof of Theorem 3.25 ensures that for any $r>1$, the mapping $H_r : L^r_\alpha(0,T;X) \rightarrow L^r(0,T;X)$, given in (3.12), is an isometric isomorphism. Moreover, due to Lemma 3.26, we already know that H_r induces an isometric isomorphism between their dual spaces, which is given by

$$\begin{array}{rcl} H'_r : (L^r(0,T;X'))' & \rightarrow & (L^r_\alpha(0,T;X'))' \\ \psi & \rightarrow & \psi \circ H_r. \end{array}$$

Finally consider the isometric isomorphism $S_{p,q} = H'_q \circ T_{p,q} \circ H_p$ and observe that

$$\begin{array}{rcl} S_{p,q} : L^p_\alpha(0,T;X) & \rightarrow & (L^q_\alpha(0,T;X'))' \\ f & \rightarrow & W_f. \end{array}$$

where $W_f : L^q_\alpha(0,T;X') \rightarrow \mathbb{R}$ is given by

$$W_f(h) = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\langle h(s), f(s)\rangle_{X',X}\, ds,$$

for all $h \in L^q_\alpha(0,T;X')$. In other words, the isometric isomorphism $S_{p,q}$ allows us to identify any $\varphi \in (L^q_\alpha(0,T;X'))'$ with a unique element $v_\varphi \in L^p_\alpha(0,T;X)$ such that

$\|\varphi\|_{(L^q_\alpha(0,T;X'))'} = \|u_\varphi\|_{L^p_\alpha(0,T;X)}$ and (3.13) holds. □

Corollary 3.28. *If H is a Hilbert space, $\alpha \in (0,1]$ and $p, q \in (1,\infty)$ satisfy $(1/p)+(1/q) = 1$, then for each $\varphi \in (L^q_\alpha(0,T;H))'$, there exists a unique $u_\varphi \in L^p_\alpha(0,T;H)$, such that*

$$\langle \varphi, v\rangle_{(L^q_\alpha(0,T;H))', L^q_\alpha(0,T;H)} = \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} (v(s), u_\varphi(s))_H \, ds,$$

for all $v \in L^q_\alpha(0,T;H)$ and $\|\varphi\|_{(L^q_\alpha(0,T;H))'} = \|u_\varphi\|_{L^p_\alpha(0,T;H)}$.

Chapter 4

Fractional Weak Formulation for Unsteady Stokes Equations

Building on the discussion from the previous sections, we now address the main questions introduced in the introduction of this work. This involves formulating a new notion of weak solution for problem (2.8) and proving its existence and uniqueness.

Following the standard approach for deriving a variational formulation of the classical unsteady Stokes equations (see [46, Chapter III] for details), we obtain the following fractional formulation of (2.3).

Definition 4.1. Consider $\alpha \in (1/2, 1)$, $f \in L^2(0, T; V')$ and $u_0 \in H$. A fractional solution to (2.3) is a function $u \in L^2_{1-\alpha}(0, T; H) \cap L^2(0, T; V)$ such that $cD^\alpha_t u \in L^1(0, T; H)$ and

$$cD^\alpha_t (u(t), v)_H + \nu((u(t), v)) = \langle f(t), v\rangle_{V',V} \tag{4.1}$$

for almost every $t \in [0, T]$, for all $v \in V$ and $u(0) = u_0$.

Since we seek a solution $J^\alpha_t (u(t) - u(0)) \in W^{1,1}(0, T; H)$, the above definition can be reformulated. For this purpose, we first introduce the following remarks.

Remark 4.2. Let $\alpha \in (0, 1)$. Assume that $J^\alpha_t (u(t) - u(0)) \in W^{1,1}(0, T; H)$, and let $v \in H$ be fixed. Then, we can deduce that

$$(cD^\alpha_t u(t), v)_H = \frac{\partial}{\partial t}(J^{1-\alpha}_t (u(t) - u(0)), v)_H.$$

Indeed, let $\varphi \in C^\infty_0(0, T; \mathbb{R})$. By the Fubini-Tonelli Theorem, we have, for $n = 2, 3, 4$,

$$\begin{aligned}
&\int_0^T \left(J_t^{1-\alpha}(u(t)-u(0)),v\right)_H \varphi'(t)dt \\
&\qquad = \int_0^T \left[\sum_{i=1}^n \int_\Omega J_t^{1-\alpha}(u_i(t,x)-u_i(0,x))\cdot v_i(x)\,dx\right]\varphi'(t)dt \\
&\qquad\qquad = \sum_{i=1}^n \int_\Omega \left[\int_0^T J_t^{1-\alpha}(u_i(t,x)-u_i(0,x))\,v_i(x)\,\varphi'(t)\,dt\right]dx.
\end{aligned}$$

Now, integrating by parts in the time variable and applying Fubini-Tonelli again, we obtain

$$\begin{aligned}
&\int_0^T \left(J_t^{1-\alpha}(u(t)-u(0)),v\right)_H \varphi'(t)dt = -\sum_{i=1}^n \int_\Omega \left[\int_0^T cD_t^\alpha u_i(t,x)\,v_i(x)\,\varphi(t)\,dt\right]dx \\
&\quad = -\int_0^T \left[\sum_{i=1}^n \int_\Omega cD_t^\alpha u_i(t,x)\,v_i(x)\,dx\right]\varphi(t)dt = -\int_0^T \left(cD_t^\alpha u(t),v(t)\right)_H \varphi(t)dt.
\end{aligned}$$

Since this identity holds for all test functions $\varphi \in C_0^\infty(0,T;\mathbb{R})$, we conclude that the previous relation is valid in the weak sense.

Moreover, by similar argument, we obtain

$$\left(J_t^{1-\alpha}(u(t)-u(0)),v\right)_H = J_t^{1-\alpha}\left(u(t)-u(0),v\right)_H.$$

Therefore, we conclude that

$$\left(cD_t^\alpha u(t),v\right)_H = cD_t^\alpha\left(u(t),v\right)_H, \quad \forall v \in H.$$

Remark 4.3. Let $\eta \in V$. Consider the linear and continuous functional on V defined by $\langle A\eta, v\rangle_{V',V} = ((\eta,v))$. Then,

$$\begin{aligned}
\|A\eta\|_{V'} &= \sup_{\|v\|_V=1} \left|\langle A\eta,v\rangle_{V',V}\right| \\
&= \sup_{\|v\|_V=1} |((\eta,v))| \le \sup_{\|v\|_V=1} \|v\|_V\|\eta\|_V = \|\eta\|_V.
\end{aligned}$$

Now, assume u is a solution in the sense of the Definition 4.1. According to the above remarks, one can rewrite (4.1) as

$$(cD_t^\alpha u(t),v)_H = \langle f(t),v\rangle_{V',V} - \nu\langle Au(t),v\rangle_{V',V}$$

for all $v \in V$.

Since $u \in L^2(0,T;V)$, the functional $f - \nu Au$ belongs to $L^2(0,T;V')$. Therefore, we can write that $cD_t^\alpha u(t) = f(t) - \nu Au(t)$ for almost every $t \in [0,T]$, and hence $cD_t^\alpha u \in L^2(0,T;V')$. Thus, an alternative and equivalent formulation of Definition 4.1 is the following.

Definition 4.4. Let $\alpha \in (1/2,1)$, $f \in L^2(0,T;V')$ and $u_0 \in H$. A fractional solution to (2.3) is a function $u \in L^2_{1-\alpha}(0,T;H) \cap L^2(0,T;V)$ such that $cD_t^\alpha u \in L^2(0,T;V')$ and

$$cD_t^\alpha u(t) = f(t) - \nu Au(t)$$

for almost every $t \in [0,T]$, and with $u(0) = u_0$.

Remark 4.5. Notice that Definition 4.1 is sufficient to guarantee Definition 4.4, but not necessary. This is because the condition $cD_t^\alpha u \in L^2(0,T;V')$ does not imply that $cD_t^\alpha u \in L^1(0,T;H)$.

In the definitions above, since we seek a solution $u \in L^2_{1-\alpha}(0,T;H) \cap L^2(0,T;V)$, the initial condition $u(0) = u_0$ may not be well defined, as u is only defined almost everywhere. To overcome this issue, when developing a solution corresponding to Definition 4.4, we prove, analogously to the classical case, that $u \in C([0,T],V')$. This result gives meaning to the initial condition, particularly when $u_0 \in H$, since $H \subset V'$ by Remark 2.19.

4.1 The Approximate Solution

In this section, we present an approximate weak formulation for the unsteady Stokes equations involving the Caputo fractional derivative with respect to time.

To achieve this goal, let (w_j) be a sequence of orthonormal basis functions in H, which are orthogonal in V and complete in V (see [6, Theorem IV 5.5]). For each $m \in \mathbb{N}^*$, we define the subspace $V_m = span\{w_1, \dots, w_m\}$, and, in the context of Definition 4.1, we consider the following approximate variational problem.

FRACTIONAL WEAK FORMULATION APPROXIMATION: Given $m \in \mathbb{N}^*$, $\alpha \in (1/2,1)$, $f \in L^2(0,T;V')$ and $u_0 \in H$, we seek a sequence functions (g_{im}), where $g_{im} : [0,T] \to \mathbb{R}$, such that by defining

$$u_m(t) := \sum_{i=1}^{m} g_{im}(t) w_i, \quad w_i \in V_m, \ \ \forall i \in \{1,...,m\}, \tag{4.2}$$

we have

$$\begin{cases} cD_t^\alpha (u_m(t), w_j)_H + \nu((u_m(t), w_j)) = \langle f(t), w_j \rangle_{V',V}, \quad \forall\, j \in \{1, ..., m\}, \\ u_m(0) = u_{0m}, \end{cases} \tag{4.3}$$

for all every $t \in [0, T]$, where u_{0m} denotes the orthogonal projection of u_0 onto V_m in H, that is,

$$u_{0m} = \sum_{j=1}^{m} (u_0, w_j)_H w_j \to u_0 \text{ in } H, \quad m \to \infty.$$

Using the definition of u_m in the approximate problem (4.3), we obtain the following system of equations:

$$\sum_{i=1}^{m} (w_i, w_j)_H cD_t^\alpha g_{im}(t) + \nu \sum_{i=1}^{m} ((w_i, w_j)) g_{im}(t) = \langle f(t), w_j \rangle_{V',V},$$
$$g_{im}(0) = (u_0, w_i) \;\; for\; i \in \{1, 2, ...\}.$$

This system can be written equivalently as a system of fractional ordinary differential equations:

$$A_m \Big[cD_t^\alpha U_m(t) \Big] + \nu B_m U_m(t) = F_m(t),$$
$$U_m(0) = U_{0m},$$

with

$$A_m = \begin{bmatrix} (w_1, w_1) & \dots & (w_m, w_1) \\ \vdots & \ddots & \vdots \\ (w_1, w_m) & \dots & (w_m, w_m) \end{bmatrix}, \quad B_m = \begin{bmatrix} ((w_1, w_1)) & \dots & ((w_m, w_1)) \\ \vdots & \ddots & \vdots \\ ((w_1, w_m)) & \dots & ((w_m, w_m)) \end{bmatrix},$$

$$F_m(t) = \begin{bmatrix} \langle f(t), w_1 \rangle \\ \vdots \\ \langle f(t), w_m \rangle \end{bmatrix}, \quad U_m(t) = \begin{bmatrix} g_{1m}(t) \\ \vdots \\ g_{mm}(t) \end{bmatrix}, \quad U_m(0) = \begin{bmatrix} (u_0, w_1) \\ \vdots \\ (u_0, w_1) \end{bmatrix}.$$

Since $\{w_i\}_{i=1}^m$ is an orthonormal family in H, we have $A = Id$. Therefore, the above system can be reformulated as the following Cauchy problem:

$$\begin{cases} cD_t^\alpha U_m(t) & = & G_m(t, U_m(t)), \quad \text{a.e. in } [0,T] \\ U_m(0) & = & U_{0m} \in \mathbb{R}^m, \end{cases} \tag{4.4}$$

where $G_m : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ given by

$$G_m(t,x) = F_m(t) - \nu B_m x. \tag{4.5}$$

From the theory developed in Section 3.1, we can directly establish the following theorem.

Theorem 4.6. *Let $\alpha \in (1/2, 1)$. Then (4.4) has a unique global solution U_m in $[0,T]$ for all $m \in \mathbb{N}^*$ such that $cD_t^\alpha U_m(t) \in L^2(0,T;\mathbb{R}^m)$.*

Proof. Observe that $G_m : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$, defined in (4.5), satisfies the following conditions:

(i) For each $x \in \mathbb{R}^m$, the function $G_m(\cdot, x) : [0,T] \to \mathbb{R}^m$ is measurable.

(ii) For almost every $t \in [0,T]$, the function $G_m(t, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous. Indeed, observe that for $M = \max\big\{((w_i, w_i)) : i \in \{1, \ldots, m\}\big\}$, we have

$$\|G_m(t,x) - G_m(t,y)\|_m \le \nu M \|x - y\|_m,$$

for all $x, y \in \mathbb{R}^m$ and almost every $t \in [0,T]$.

Thus, $G_m(t,x)$ is a Carathéodory map.

Moreover, we obtain

$$\|G_m(t,x)\|_m \quad \le \quad \underbrace{\Big(\|w_1\|_V^2 + \ldots + \|w_m\|_V^2\Big)^{1/2} \|f(t)\|_{V'}}_{=:\gamma(t)} + \nu M \|x\|_m.$$

Applying Theorem 3.12 for $p = q = 2$, we ensure the existence and uniqueness of a continuous function $U_m : [0,T] \to \mathbb{R}^m$ that satisfies both equations in (4.4). $\square$

Now, since for $m \in \mathbb{N}^*$, the Fractional Weak Formulation Approximation u_m is defined in (4.2), and in analogy to the classical treatment of the unsteady Stokes equation, we need to examine the energy estimates. In other words, use the identity

$$D_t^1 \|u_m(t)\|_H^2 = 2(D_t^1 u_m(t), u_m(t))_H, \qquad \text{for a.e. } t \in [0,T],$$

which is equivalent to the identity

$$\sum_{i=1}^{m} \frac{d}{dt}\left[g_{im}^2(t)\right] = 2\sum_{i=1}^{m} g_{im}'(t) g_{im}(t), \qquad \text{for a.e. } t \in [0,T],$$

see [46, page 284] for more details. However, adapting this approach becomes intricate when fractional derivatives are involved. Following a similar methodology, Carvalho Neto and Fehlberg Junior demonstrated in [10, Theorem 4.16, item (b)] that under the conditions $u \in L^2(0,T;V)$, $cD_t^\alpha u \in L^2(0,T;V')$ and $J_t^{1-\alpha}\|u\|_H^2 \in W^{1,1}(0,T;\mathbb{R})$, the inequality

$$cD_t^\alpha \|u(t)\|_H^2 \leq 2\big(cD_t^\alpha u(t), u(t)\big)_H, \qquad \text{for a.e. } t \in [0,T],$$

is established. With this inequality, we can follow the same steps described in the classical case. However, a significant technical challenge arises: since, by construction, for $m \in \mathbb{N}^*$, we only know that $u_m \in C([0,T];V)$ and $cD_t^\alpha u_m \in L^2(0,T;V')$, how can we ensure that $J_t^{1-\alpha}\|u_m\|_H^2 \in W^{1,1}(0,T;\mathbb{R})$?

Remark 4.7. For $m \in \mathbb{N}^*$ we have the following:

(i) From equation (4.2), we obtain u_m belongs to $C([0,T];H)$.

(ii) Observe that by Theorem 4.6, we have

$$\int_0^T \|cD_t^\alpha u_m\|_H^2 \, dt = \int_0^T \left[\sum_{i=1}^{m} (cD_t^\alpha g_{im}(t))^2\right] dt < \infty.$$

Therefore, $cD_t^\alpha u_m \in L^2(0,T;H)$.

Theorem 4.8. *Let $\alpha \in (1/2,1)$. Assume $\vartheta \in C([0,T];H)$ and $cD_t^\alpha \vartheta \in L^2(0,T;H)$. Then, $J_t^{1-\alpha}\|\vartheta\|_H^2 \in W^{1,1}(0,T;\mathbb{R})$.*

To begin the proof of the previous theorem, we first introduce several auxiliary lemmas, propositions, and definitions that are needed. We start by presenting a short proof of the following classical result.

Proposition 4.9. *If $v \in W^{1,2}(0,T;H)$, then*

$$\frac{d}{dt}\|v(t)\|_H^2 = 2\big(v'(t), v(t)\big)_H,$$

for almost every $t \in [0,T]$. Note that the derivatives above are understood in the weak sense.

Proof. Let $\varphi \in C_0^\infty(0,T;\mathbb{R})$. By the Fubini-Tonelli Theorem, we have

$$\begin{aligned}\int_0^T \|v(t)\|_H^2 \varphi'(t)dt &= \int_0^T \left[\sum_{i=1}^n \int_\Omega |v_i(t,x)|^2\, dx\right] \varphi'(t)dt \\ &= \sum_{i=1}^n \int_\Omega \left[\int_0^T \left[v_i(t,x)\right]^2 \varphi'(t)\, dt\right] dx.\end{aligned}$$

Now, integrating by parts in the time variable and applying Fubini-Tonelli again, we obtain

$$\begin{aligned}\int_0^T \|v(t)\|_H^2 \varphi'(t)dt &= -\sum_{i=1}^n \int_\Omega \left[\int_0^T 2v_i(t,x)v_i'(t,x)\varphi(t)\, dt\right] dx \\ &= -\int_0^T \left[\sum_{i=1}^n \int_\Omega 2v_i(t,x)v_i'(t,x)\, dx\right] \varphi(t)dt = -\int_0^T 2\big(v'(t),v(t)\big)_H \varphi(t)dt.\end{aligned}$$

Since this holds for all test functions $\varphi \in C_0^\infty(0,T)$, we conclude that the weak derivative of $\|v(\cdot)\|_H^2$ exists and satisfies

$$\frac{d}{dt}\|v(t)\|_{L^2(\Omega)}^2 = 2\big(v'(t),v(t)\big)_H,$$

for almost every $t \in [0,T]$. □

Lemma 4.10. *Let $\alpha \in (0,1)$ and $\varphi \in C^\infty([0,T];X)$ such that $\varphi^{(k-1)}(0) = 0_X$ for all $k \in \mathbb{N}^*$. Then $\psi := J_t^\alpha \varphi$ belongs to $C^\infty([0,T];X)$ and satisfies $\psi^{(k-1)}(0) = 0_X$ for all $k \in \mathbb{N}^*$.*

Proof. From the regularity of φ, the condition $\varphi^{(k-1)}(0) = 0_X$ for all $k \in \mathbb{N}^*$, and Corollary 2.41, we deduce that

$$\begin{aligned}\psi^{(k-1)}(t) = \frac{d^{(k-1)}}{dt^{(k-1)}}\Big\{J_t^\alpha \varphi(t)\Big\} &= \frac{d^{(k-2)}}{dt^{(k-2)}}\Big\{J_t^\alpha \varphi'(t)\Big\} \\ &= \frac{d^{(k-3)}}{dt^{(k-3)}}\Big\{J_t^\alpha \varphi''(t)\Big\} = \ldots = J_t^\alpha \varphi^{(k-1)}(t),\end{aligned}$$

for all $k \in \mathbb{N}^*$ and almost every $t \in [0,T]$. Then, Corollary 3.19 implies that $\psi \in C^\infty([0,T];X)$ and $\psi^{(k-1)}(0) = 0_X$ for all $k \in \mathbb{N}^*$. □

Proposition 4.11. *Let $\alpha \in (0,1)$ and $v \in L^\infty(0,T;X)$ such that $D_t^\alpha v \in L^2(0,T;X)$. Then there exists a sequence $(\psi_j) \subset C^\infty([0,T];X)$ satisfying $\psi_j^{(k-1)}(0) = 0_X$ for all $j,k \in$*

$\mathbb{N}^*$, *such that*

$$\psi_j \to v \quad \text{and} \quad D_t^\alpha \psi_j \to D_t^\alpha v, \ \text{in } L^2(0,T;X).$$

Proof. Since $D_t^\alpha v \in L^2(0,T;X)$, there exists a sequence $(\phi_j) \subset C_c^\infty((0,T);X)$ such that

$$\phi_j \to D_t^\alpha v, \quad \text{in } L^2(0,T;X).$$

Define $\psi_j(t) := J_t^\alpha \phi_j(t)$. Then, thanks to Lemma 4.10, each ψ_j belongs to $C^\infty([0,T];X)$ and satisfies $\psi_j^{(k-1)}(0) = 0_X$ for all $j,k \geq 1$. Moreover, Theorem 2.34 allows us to deduce that

$$\psi_j = J_t^\alpha \phi_j \to J_t^\alpha\big(D_t^\alpha v\big), \ \text{in } L^2(0,T;X). \tag{4.6}$$

Now, since $v \in L^\infty(0,T;X)$, items (iii) from Proposition 2.42 together with Corollary 3.19 imply that

$$J_t^\alpha\big(D_t^\alpha v(t)\big) = v(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[J_s^{1-\alpha} v(s)\right]\Big|_{s=0} = v(t), \tag{4.7}$$

for almost every $t \in [0,T]$. Therefore, from (4.6) and (4.7), we deduce the convergence

$$\psi_j(t) \to v(t), \quad \text{in } L^2(0,T;X).$$

Finally, observe that, from item (ii) of Proposition 2.42, we also obtain

$$D_t^\alpha \psi_j(t) = D_t^\alpha J_t^\alpha \phi_j(t) = \phi_j(t) \to D_t^\alpha v(t), \ \text{in } L^2(0,T;X).$$

This completes the proof. □

An analogous version of the previous proposition, but involving the Caputo fractional derivative, is the following:

Corollary 4.12. *If* $\alpha \in (0,1)$ *and* $v \in C([0,T];X)$ *is such that* $cD_t^\alpha v \in L^2(0,T;X)$*, then there exists* $(\psi_j) \subset C^\infty([0,T];X)$ *such that* $\psi_j(0) = v(0)$*,* $\psi_j^{(k)}(0) = 0_X$ *for all* $j,k \in \mathbb{N}^*$*, and*

$$\psi_j \to v \quad \text{and} \quad cD_t^\alpha \psi_j \to cD_t^\alpha v, \ \text{in } L^2(0,T;X).$$

Proof. Define $g(t) = v(t) - v(0)$, then $g \in L^\infty(0,T;X)$ and $D_t^\alpha g = cD_t^\alpha v \in L^2(0,T;X)$. Hence, Proposition 4.11 ensures the existence of a sequence $\big(\tilde{\psi}_j\big)$ that belongs to $C^\infty([0,T];X)$ such that $\tilde{\psi}_j^{(k-1)}(0) = 0_X$ for all $j,k \geq 1$, and

$$\tilde{\psi}_j \to g \quad \text{and} \quad D_t^\alpha \tilde{\psi}_j \to D_t^\alpha g \text{ in } L^2(0,T;X).$$

Finally, define $\psi_j(t) = \tilde{\psi}_j(t) + v(0)$. The desired result then follows. $\square$

At this point, we need to extend the result of Alsaedi et al. in [1, Lemma 1] so that it can be applied to vector-valued functions. To this end, we first recall the notion of Hölder continuity, as well as its relationship with the fractional integral operator.

Definition 4.13. Let Ω be an open subset of $\mathbb{R}^n$ and let X be a Banach space. For any real $\alpha \in (0,1)$, the Hölder spaces $C^\alpha(\Omega; X)$ is defined by

$$C^\alpha(\Omega; X) := \left\{ v \in C(\Omega, X) : \begin{array}{c} v \text{ is bounded in } \Omega, \text{ and there exists } k > 0 \text{ such that} \\ \|v(x) - v(y)\|_X \leq k \|x - y\|_n^\alpha, \quad \forall x, y \in \Omega \end{array} \right\}.$$

This space, equipped with the norm

$$\|v\|_{C^\alpha(\Omega;X)} := \|v\|_\infty + [v]_{C^\alpha(\Omega;X)},$$

where

$$[v]_{C^\alpha(\Omega;X)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{\|v(x) - v(y)\|_X}{\|x - y\|_n^\alpha} \right\},$$

is a Banach space.

Lemma 4.14. *Let $\alpha \in (0,1)$ and let X be a Hilbert space with inner product $(\cdot,\cdot)$. Suppose that $u \in C^\beta([0,T]; X)$ and $v \in C^\delta([0,T]; X)$, where $\alpha < \beta + \delta$ and $0 < \beta \leq \delta < 1$. Moreover, assume that $J_t^{1-\alpha} u \in W^{1,1}(0,T; X)$ and $J_t^{1-\alpha} v \in W^{1,1}(0,T; X)$. Then, we can deduce that for almost every $t \in [0,T]$*

$$\begin{aligned} D_t^\alpha \left(u(t), v(t)\right) &= \left(u(t), D_t^\alpha v(t)\right) + \left(v(t), D_t^\alpha u(t)\right) \\ &\qquad - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{\left(u(s) - u(t), v(s) - v(t)\right)}{(t-s)^{\alpha+1}} ds - \frac{\left(u(t), v(t)\right)}{t^\alpha \Gamma(1-\alpha)}. \end{aligned}$$

Proof. First, assume without loss of generality that k is the Hölder constant for u and v. Now, observe that for s and t in $[0,T]$, we have

$$\left(u(s), v(s)\right) = \left(u(s) - u(t), v(s) - v(t)\right) + \left(u(t), v(s)\right) + \left(u(s), v(t)\right) - \left(u(t) v(t)\right). \quad (4.8)$$

We know that by definition

$$D_t^\alpha \left(u(t), v(t)\right) =$$

$$\frac{1}{\Gamma(1-\alpha)}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^{t+\epsilon}(t+\epsilon-s)^{-\alpha}\left(u(s),v(s)\right)\,ds-\int_0^{t}(t-s)^{-\alpha}\left(u(s),v(s)\right)\,ds\right]. \qquad (4.9)$$

Inserting (4.8) into identity (4.9), and assuming that the limit is taken from the right of 0 (i.e., $\lim_{\varepsilon\to 0^+}$), we obtain an expression that can be rearranged as the sum of four pairs of terms. For clarity, we denote these terms by

$$\frac{1}{\Gamma(1-\alpha)}\left\{\lim_{t\to 0^+}\frac{1}{\varepsilon}\left[\int_0^{t+\varepsilon}(t+\varepsilon-s)^{-\alpha}\big(u(s)-u(t),v(s)-v(t)\big)ds\right.\right.$$
$$\left.\left.-\int_0^{t}(t-s)^{-\alpha}\big(u(s)-u(t),v(s)-v(t)\big)\,ds\right]\right\}=:\mathcal{I}_1(t),$$

$$\frac{1}{\Gamma(1-\alpha)}\left\{\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\int_0^{t+\varepsilon}(t+\varepsilon-s)^{-\alpha}\big(u(s),v(t)\big)\,ds\right.\right.$$
$$\left.\left.-\int_0^{t}(t-s)^{-\alpha}\big(u(s),v(t)\big)ds\right]\right\}=:\mathcal{I}_2(t),$$

$$\frac{1}{\Gamma(1-\alpha)}\left\{\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\int_0^{t+\varepsilon}(t+\varepsilon-s)^{-\alpha}\big(u(t),v(s)\big)\,ds\right.\right.$$
$$\left.\left.-\int_0^{t}(t-s)^{-\alpha}\big(u(t),v(s)\big)ds\right]\right\}=:\mathcal{I}_3(t),$$

and finally

$$\frac{1}{\Gamma(1-\alpha)}\left\{\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\int_0^{t+\varepsilon}(t+\varepsilon-s)^{-\alpha}\big(u(t),v(t)\big)\,ds\right.\right.$$
$$\left.\left.-\int_0^{t}(t-s)^{-\alpha}\big(u(t),v(t)\big)\,ds\right]\right\}=:\mathcal{I}_4(t),$$

for almost every $t\in[0,T]$ such that the four limits above exist. In fact, we shall prove that each of these limits does indeed exist.

We begin with $\mathcal{I}_2(t)$ and $\mathcal{I}_3(t)$, for which it is straightforward to verify that

$$\mathcal{I}_2(t)=\big(D_t^\alpha u(t),v(t)\big) \qquad \text{and} \qquad \mathcal{I}_3(t)=\big(u(t),D_t^\alpha v(t)\big).$$

For $\mathcal{I}_4(t)$, since we have

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\int_0^{t+\varepsilon}(t+\varepsilon-s)^{-\alpha}ds-\int_0^{t}(t-s)^{-\alpha}ds\right]=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left[\frac{(t+\varepsilon)^{1-\alpha}}{1-\alpha}-\frac{t^{1-\alpha}}{1-\alpha}\right]=t^{-\alpha},$$

we obtain

$$\mathcal{J}_4(t) = \frac{\big(u(t), v(t)\big)}{\Gamma(1-\alpha)t^{\alpha}}.$$

Finally, let us address the most complex term, which requires a more careful analysis. First, we rewrite $\Gamma(1-\alpha)\mathcal{J}_1(t) = \mathcal{H}_1(t) + \mathcal{H}_2(t)$, where

$$\mathcal{H}_1(t) = \lim_{\varepsilon\to 0^+}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(t+\varepsilon-s)^{-\alpha}\big(u(s)-u(t), v(s)-v(t)\big)\,ds\right]$$

and

$$\mathcal{H}_2(t) = \lim_{\varepsilon\to 0^+}\int_0^t\left[\frac{(t+\varepsilon-s)^{-\alpha}-(t-s)^{-\alpha}}{\varepsilon}\right]\big(u(s)-u(t), v(s)-v(t)\big)\,ds$$

To deal with $\mathcal{H}_1(t)$, we apply the change of variables $w = t+\varepsilon-s$, which gives

$$\mathcal{H}_1(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon} w^{-\alpha}\big(u(t+\varepsilon-w)-u(t), v(t+\varepsilon-w)-v(t)\big)\,dw.$$

Then, the Cauchy–Schwarz inequality, along with the Hölder regularity, ensures that

$$\begin{aligned}\mathcal{H}_1(t) \leq \lim_{\varepsilon\to 0}\int_0^{\varepsilon}\frac{w^{-\alpha}}{\varepsilon}k^2(\varepsilon-w)^{\beta+\delta}\,dw &= \lim_{\varepsilon\to 0}\int_0^1\frac{(\varepsilon h)^{-\alpha}}{\varepsilon}k^2(\varepsilon-\varepsilon h)^{\beta+\delta}\varepsilon\,dh\\ &= \lim_{\varepsilon\to 0}\varepsilon^{\beta+\delta-\alpha}k^2B(1-\alpha,\beta+\delta+1) = 0,\end{aligned}$$

where, above, $B(\cdot,\cdot)$ stands for the Beta function.

On the other hand, to handle $\mathcal{H}_2(t)$, let us first denote by $g_\varepsilon : [0,t]\to\mathbb{R}$ the integrand function that defines it, given by

$$g_\varepsilon(s) := \left[\frac{(t+\varepsilon-s)^{-\alpha}-(t-s)^{-\alpha}}{\varepsilon}\right]\big(u(s)-u(t), v(s)-v(t)\big).$$

Note that

$$\lim_{\varepsilon\to 0^+} g_\varepsilon(s) = -\alpha(t-s)^{-\alpha-1}\big(u(s)-u(t), v(s)-v(t)\big), \quad \text{for a.e. } s\in[0,t].$$

Moreover, we have

$$|g_\varepsilon(s)| \leq \left|\frac{(t+\varepsilon-s)^{-\alpha}-(t-s)^{-\alpha}}{\varepsilon}\right| k^2(t-s)^{\beta+\delta}, \quad \text{for a.e. } s\in[0,t],$$

thanks to the Hölder continuity of u and v.

Consider now the function $h : [0,\varepsilon]\to\mathbb{R}$ given by $h(w) = (t+w-s)^{-\alpha}$. By the Mean

Value Theorem, there exists $\theta \in (0, \varepsilon)$ such that

$$\left| \frac{(t+\varepsilon-s)^{-\alpha} - (t-s)^{-\alpha}}{\varepsilon} \right| = \left| \frac{h(\varepsilon) - h(0)}{\varepsilon} \right| = |h'(\theta)| = \alpha(t+\theta-s)^{-\alpha-1} \leq \alpha(t-s)^{-\alpha-1}.$$

Therefore,

$$|g_\varepsilon(s)| \leq C(t-s)^{\beta+\delta-\alpha-1},$$

for almost every $s \in [0, t]$.

The aforementioned properties are enough for us to apply the Dominated Convergence Theorem in order to obtain

$$\mathcal{H}_2(t) = \lim_{\varepsilon \to 0^+} \int_0^t g_\varepsilon(s) ds = \int_0^t -\alpha(t-s)^{-\alpha-1} \big(u(s) - u(t), v(s) - v(t)\big)\, ds,$$

for almost every $t \in (0, T)$.

By combining the results obtained for $\mathcal{I}_1(t), \mathcal{I}_2(t), \mathcal{I}_3(t)$, and $\mathcal{I}_4(t)$ in (4.9), and noting that the limit from the left of 0 can be treated analogously, we conclude the proof.

□

Definition 4.15. Fractional Sobolev-type spaces are also referred to as Aronszajn, Gagliardo, or Slobodeckij spaces, named after the mathematicians who introduced them; see [3, 18, 45].

Let X a Banach space. For any $\alpha \in (0, 1)$ and $p \geq 1$, we define:

(i) The fractional Sobolev-Bochner spaces

$$W^{\alpha,p}(0, T; X) := \left\{ v \in L^p(0, T; X) : \frac{\|v(x) - v(y)\|_X}{|x-y|^{(1/p)+\alpha}} \in L^p((0, T) \times (0, T)) \right\},$$

which is a Banach space with the norm

$$\|v\|_{W^{\alpha,p}(0,T;X)} = \left(\int_0^T \|v\|_X^p dx + \int_0^T \int_0^T \frac{\|v(x) - v(y)\|^p}{|x-y|^{1+p\alpha}} dx\, dy \right)^{1/p}.$$

(ii) For $p = 2$, we denote $W^{\alpha,2}(0, T; X)$ by $H^\alpha(0, T; X)$. It is equipped with the fractional Sobolev norm

$$\|v\|_{H^\alpha(0,T;X)} = \left(\|v\|_{L^2(0,T;X)}^2 + \int_0^T \int_0^T \frac{\|v(x) - v(y)\|_X^2}{|x-y|^{1+2\alpha}} dx\, dy \right)^{1/2}. \tag{4.10}$$

Proposition 4.16. *Let $\alpha \in (0,1)$. Then, there exist constants $m, M > 0$, such that*

$$m\|J_t^\alpha v\|_{H^\alpha(0,T)} \leq \|v\|_{L^2(0,T;\mathbb{R})} \leq M\|J_t^\alpha v\|_{H^\alpha(0,T)}$$

for all $v \in L^2(0,T;\mathbb{R})$.

Proof. See [20, Theorem 2.1]. □

Before presenting the proof of Theorem 4.8, we generalize Proposition 4.16 to vector-valued functions.

Corollary 4.17. *There exist constants $m, M > 0$ such that*

$$m\|J_t^\alpha v\|_{H^\alpha(0,T;H)} \leq \|v\|_{L^2(0,T;H)} \leq M\|J_t^\alpha v\|_{H^\alpha(0,T;H)},$$

for every $v \in L^2(0,T;H)$.

Proof. Let $v \in L^2(0,T;H)$. Observe that, for almost every $x \in \Omega$, we have $v_i(\cdot,x) \in L^2(0,T;\mathbb{R})$ for $1 \leq i \leq n$. Therefore, from Proposition 4.16, it follows that for $1 \leq i \leq n$,

$$m\|J_t^\alpha v_i(x,\cdot)\|_{H^\alpha(0,T)} \leq \|v_i(\cdot,x)\|_{L^2(0,T;\mathbb{R})} \leq M\|J_t^\alpha v_i(\cdot,x)\|_{H^\alpha(0,T)},$$

for almost every $x \in \Omega$, with constants m and M independent of x. Squaring both sides of the inequality, integrating it over Ω, applying the Fubini–Tonelli Theorem, and summing over $i = 1, ..., n$, we obtain the desired result. □

We have now established all the necessary results to prove Theorem 4.8. It is worth emphasizing that the next theorem, is built from a sequence of ideas that are carefully detailed in the proof, in order to make the argument as clear and organized as possible.

Proof. To clarify the structure of the proof, we divide it into three steps. In the first step, we construct an auxiliary sequence $(\xi_j) \subset C^\infty([0,T];H)$ such that

$$J_t^{1-\alpha}\|J_t^\alpha \xi_j'\|_H^2 + 2J_t^{1-\alpha}(J_t^\alpha \xi_j', \vartheta(0))_H + J_t^{1-\alpha}\|\vartheta(0)\|_H^2 \to J_t^{1-\alpha}\|\vartheta\|_H^2 \quad \text{in} \quad L^1(0,T;\mathbb{R}).$$

In the second step, we explain why it is necessary to prove that the sequence

$$\left(D_t^\alpha \left\|J_t^\alpha \xi_j'\right\|_H^2\right)_{j\in\mathbb{N}}$$

is Cauchy in $L^1(0,T;\mathbb{R})$. Finally, in the third step, we show that the existence of such a limit allows us to conclude that $D_t^\alpha\|\vartheta\|_H^2$ exists and belongs to $\in L^1(0,T;\mathbb{R})$.

Step 1: Construction of an approximating sequence. We begin by recalling that Corollary 4.12 ensures the existence of a sequence $(\phi_j) \subset C^\infty([0,T];H)$ such that $\phi_j(0) = \vartheta(0)$, $\phi_j^{'}(0) = 0_H$ for each $j \in \mathbb{N}^*$, and

$$\phi_j \to \vartheta \quad \text{and} \quad cD_t^\alpha\phi_j \to cD_t^\alpha\vartheta, \quad \text{in } L^2(0,T;H). \tag{4.11}$$

Define the auxiliary sequence $\xi_j(t) := J_t^{1-\alpha}\big[\phi_j(t) - \phi_j(0)\big]$ for all $j \in \mathbb{N}$. Lemma 4.10 ensures that $(\xi_j) \subset C^\infty([0,T];H)$ and satisfies $\xi_j^{(k)}(0) = 0_H$ for all $k \in \mathbb{N}$. Moreover, by the continuity of the fractional integral operator $J_t^{1-\alpha}$ in $L^2(0,T;H)$, we obtain

$$\xi_j \to J_t^{1-\alpha}\big[\vartheta(\cdot) - \vartheta(0)\big], \quad \text{in } L^2(0,T;H).$$

Consequently, using the second convergence in (4.11), we deduce

$$\xi_j' = \frac{d}{dt}J_t^{1-\alpha}\big[\phi_j(\cdot) - \phi_j(0)\big] = cD_t^\alpha\phi_j \to cD_t^\alpha\vartheta, \quad \text{in } L^2(0,T;H). \tag{4.12}$$

Since $\vartheta \in C([0,T];H)$, items (v) of Proposition 2.42 implies that $J_t^\alpha cD_t^\alpha\vartheta(t) = \vartheta(t) - \vartheta(0)$ for almost every $t \in [0,T]$. Hence,

$$J_t^\alpha\xi_j' \to J_t^\alpha cD_t^\alpha\vartheta = \vartheta(\cdot) - \vartheta(0), \quad \text{in } L^2(0,T;H). \tag{4.13}$$

Thus, we have

$$\big\|J_t^\alpha\xi_j' + \vartheta(0)\big\|_H^2 \to \|\vartheta\|_H^2 \quad \text{in } L^1(0,T;\mathbb{R}).$$

Furthermore, by the continuity of the operator $J_t^{1-\alpha}$ in $L^1(0,T;\mathbb{R})$, together with the definition of the norm in H (see (2.1)), it follows that

$$J_t^{1-\alpha}\|J_t^\alpha\xi_j'\|_H^2 + 2J_t^{1-\alpha}(J_t^\alpha\xi_j',\vartheta(0))_H + J_t^{1-\alpha}\|\vartheta(0)\|_H^2 \to J_t^{1-\alpha}\|\vartheta\|_H^2 \tag{4.14}$$

in $L^1(0,T;\mathbb{R})$. Therefore, we conclude that $J_t^{1-\alpha}\,\|\vartheta\|_H^2 \in L^1(0,T;\mathbb{R})$.

Step 2: The Cauchy sequence in $L^1(0,T)$.

In this step, we begin by observing that, by a recursive argument already used in the previous step, it holds that

$$J_t^\alpha\xi_j'(t) = J_t^\alpha cD_t^\alpha\phi_j(t) = \phi_j(t) - \phi_j(0),$$

for almost every $t \in [0,T]$. Thus, Proposition 4.9 guarantees that

$$\frac{d}{dt}\left\|J_t^\alpha \xi_j'(t)\right\|_H^2 = \frac{d}{dt}\left\|\phi_j(t)-\phi_j(0)\right\|_H^2 = 2\left(\phi_j'(t), \phi_j(t)-\phi(0)\right)_H.$$

Since $\phi_j \in C^\infty([0,T];H)$, it follows that $\left(\phi_j'(t), \phi_j(t)-\phi(0)\right)_H$ belongs to $C([0,T];\mathbb{R})$. Hence, by Proposition 2.40, we obtain

$$D_t^\alpha \left\|J_t^\alpha \xi_j'(t)\right\|_H^2 = \frac{d}{dt}\left[J_t^{1-\alpha}\left\|\phi_j(t)-\phi_j(0)\right\|_H^2\right] = 2J_t^{1-\alpha}\left(\phi_j'(t), \phi_j(t)-\phi_j(0)\right)_H,$$

for almost every $t \in [0,T]$. Consequently

$$D_t^\alpha \left\|J_t^\alpha \xi_j'(t)\right\|_H^2 \in L^1(0,T;\mathbb{R}).$$

Moreover, Remark 4.2 guarantees that $D_t^\alpha\big(J_t^\alpha \xi_j', f(0)\big)_H = \big(cD_t^\alpha \phi_j, f(0)\big)_H$.

Now, following the classical definition of weak derivative and searching for a weak derivative of $J_t^{1-\alpha}\|\vartheta\|_H^2$, let $\varphi \in C_c^\infty([0,T];\mathbb{R})$. Then, we have

$$\begin{aligned}\int_0^T &\left[J_t^{1-\alpha}\|J_t^\alpha \xi_j'\|_H^2 + 2J_t^{1-\alpha}\big(J_t^\alpha \xi_j', \vartheta(0)\big)_H + J_t^{1-\alpha}\|\vartheta(0)\|_H^2\right]\varphi'(t)dt \\ &= -\int_0^T \left[D_t^\alpha\|J_t^\alpha \xi_j'\|_H^2 + 2\big(cD_t^\alpha \phi_j, \vartheta(0)\big)_H + D_t^\alpha\|\vartheta(0)\|_H^2\right]\varphi(t)\, dt.\end{aligned}$$

If the sequence $\big(D_t^\alpha\|J_t^\alpha \xi_j'\|_H^2\big)$ converges in $L^1(0,T;\mathbb{R})$ to the some function v, then by (4.14)

$$\int_0^T J_t^{1-\alpha}\|\vartheta\|_H^2\varphi'(t)dt = -\int_0^T \left[v(t) + 2(cD_t^\alpha\vartheta, \vartheta(0))_H + D_t^\alpha\|\vartheta(0)\|_H^2\right]\varphi(t)\, dt,$$

which implies that $J_t^{1-\alpha}\|\vartheta\|_H^2$ has a weak derivative in $L^1(0,T;\mathbb{R})$.

Step 3: Existence of the fractional derivative.

From the previous step shows, it remains to verify that $\big(D_t^\alpha\|J_t^\alpha \xi_j'\|_H^2\big)$ forms a Cauchy sequence in $L^1(0,T;\mathbb{R})$. We now turn to this task. To begin, observe that for every $j,k \in \mathbb{N}^*$,

$$\left\|J_t^\alpha \xi_j'(t)\right\|_H^2 - \left\|J_t^\alpha \xi_k'(t)\right\|_H^2 = \big(J_t^\alpha \xi_j'(t) - J_t^\alpha \xi_k'(t), J_t^\alpha \xi_j'(t) + J_t^\alpha \xi_k'(t)\big)_H, \tag{4.15}$$

for almost every $t \in [0,T]$. To simplify notation, we temporarily set

$$u_{j,k}(t) := J_t^\alpha \xi_j'(t) - J_t^\alpha \xi_k'(t) \quad \text{and} \quad v_{j,k}(t) := J_t^\alpha \xi_j'(t) + J_t^\alpha \xi_k'(t). \tag{4.16}$$

If $\big(\xi_j\big) \subset C^\infty([0,T];H)$ and $\xi_j' = 0_H$ for all $j \in \mathbb{N}^*$, then both $J_t^\alpha(\xi_j' + \xi_i')$ and $J_t^\alpha(\xi_j' - \xi_i')$ belong to $C^1([0,T];H)$, by Corollary 2.41. Consequently $u_{j,k}$ and $v_{j,k}$ belong to $C^\alpha([0,T];H)$ for every $0 < \alpha < 1$.

Therefore, applying Lemma 4.14 to equation (4.15), we obtain

$$\begin{aligned} D_t^\alpha \big\| J_t^\alpha \xi_j'(t) \big\|_H^2 - D_t^\alpha \big\| J_t^\alpha \xi_k'(t) \big\|^2 &= D_t^\alpha \big(u_{j,k}(t), v_{j,k}(t)\big)_H \\ &= \big(D_t^\alpha u_{j,k}(t), v_{j,k}(t)\big)_H + \big(u_{j,k}(t), D_t^\alpha v_{j,k}(t)\big)_H - \frac{\big(u_{j,k}(t), v_{j,k}(t)\big)_H}{\Gamma(1-\alpha)t^\alpha} \\ &\qquad - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{\big(u_{j,k}(s) - u_{j,k}(t), v_{j,k}(s) - v_{j,k}(t)\big)_H}{(t-s)^{\alpha+1}} ds. \end{aligned}$$

for almost every $t \in [0,T]$. In other words,

$$D_t^\alpha \big\| J_t^\alpha \xi_j'(t) \big\|_H^2 - D_t^\alpha \big\| J_t^\alpha \xi_k'(t) \big\|_H^2 = \mathcal{I}_{jk}(t) + \mathcal{J}_{jk}(t) + \mathcal{K}_{jk}(t) + \mathcal{L}_{jk}(t),$$

for almost every $t \in [0,T]$, where

$$\mathcal{I}_{jk}(t) := \sum_{i=1}^n \int_\Omega \left[\xi_{ji}'(x,t) - \xi_{ki}'(x,t)\right] \cdot J_t^\alpha \left[\xi_{ji}'(x,t) + \xi_{ki}'(x,t)\right] \, dx,$$

$$\mathcal{J}_{jk}(t) := \sum_{i=1}^n \int_\Omega J_t^\alpha \left[\xi_{ji}'(x,t) - \xi_{ki}'(x,t)\right] \cdot \left[\xi_{ji}'(x,t) + \xi_{ki}'(x,t)\right] \, dx,$$

$$\mathcal{K}_{jk}(t) := -\frac{1}{\Gamma(1-\alpha)t^\alpha} \sum_{i=1}^n \int_\Omega J_t^\alpha \left[\xi_{ji}'(x,t) - \xi_{ki}'(x,t)\right] \cdot J_t^\alpha \left[\xi_{ji}'(x,t) + \xi_{ki}'(x,t)\right] \, dx,$$

and

$$\begin{aligned} &-\frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \sum_{i=1}^n \int_\Omega \left\{ \frac{J_s^\alpha \left[\xi_{ji}'(x,s) - \xi_{ki}'(x,s)\right] - J_t^\alpha \left[\xi_{ji}'(x,t) - \xi_{ki}'(x,t)\right]}{(t-s)^{\frac{\alpha+1}{2}}} \right\} \\ &\qquad \left\{ \frac{J_s^\alpha \left[\xi_{ji}'(x,s) + \xi_{ki}'(x,s)\right] - J_t^\alpha \left[\xi_{ji}'(x,t) + \xi_{ki}'(x,t)\right]}{(t-s)^{\frac{\alpha+1}{2}}} \right\} dx\, ds =: \mathcal{L}_{jk}(t). \end{aligned}$$

Let us now analyze each of these terms. First, observe that Hölder's inequality yields

$$|\mathcal{I}_{jk}(t)| \le \big\|\xi_j'(t) - \xi_k'(t)\big\|_H \, \big\|J_t^\alpha[\xi_j'(t) + \xi_k'(t)]\big\|_H,$$

for almost every $t \in [0,T]$. Then, we integrate with respect to the variable t over $[0,T]$

and apply Hölder's inequality once again, thereby obtaining

$$\|\mathcal{I}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq \|\xi_j' - \xi_k'\|_{L^2(0,T;H)} \cdot \|J_t^\alpha[\xi_j' + \xi_k']\|_{L^2(0,T;H)}.$$

Hence, it follows from (4.12) and (4.13) that the values $\left(\|\mathcal{I}_{jk}\|_{L^1(0,T;\mathbb{R})}\right)$ tends to zero as j and k increase. A similar argument shows that the same holds for $\left(\|\mathcal{J}_{jk}\|_{L^1(0,T;\mathbb{R})}\right)$.

Regarding the third term, we apply the Cauchy–Schwarz inequality, to obtain

$$\begin{aligned} |\mathcal{K}_{jk}(t)| &\leq \sum_{i=1}^{n} \int_\Omega \left\{ \frac{J_t^\alpha\left[\xi_{ji}'(x,t) - \xi_{ki}'(x,t)\right]}{[\Gamma(1-\alpha)t^\alpha]^{\frac{1}{2}}} \right\} \left\{ \frac{J_t^\alpha\left[\xi_{ji}'(x,t) + \xi_{ki}'(x,t)\right]}{[\Gamma(1-\alpha)t^\alpha]^{\frac{1}{2}}} \right\} dx \\ &\leq \left\{ \frac{\left\|J_t^\alpha\left[\xi_j'(t) - \xi_k'(t)\right]\right\|_H}{[\Gamma(1-\alpha)t^\alpha]^{\frac{1}{2}}} \right\} \left\{ \frac{\left\|J_t^\alpha\left[\xi_j'(t) + \xi_k'(t)\right]\right\|_H}{[\Gamma(1-\alpha)t^\alpha]^{\frac{1}{2}}} \right\}, \end{aligned} \tag{4.17}$$

for almost every $t \in [0,T]$. Thus, integrating with respect to the variable t over $[0,T]$ and applying Hölder's inequality, we conclude from (4.17) that

$$\begin{aligned} \|\mathcal{K}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq &\left(\int_0^T \frac{t^{-\alpha}\left\|J_t^\alpha\left[\xi_j'(t) - \xi_k'(t)\right]\right\|_H^2}{\Gamma(1-\alpha)}\, dt \right)^{\frac{1}{2}} \\ &\left(\int_0^T \frac{t^{-\alpha}\left\|J_t^\alpha\left[\xi_j'(t) + \xi_k'(t)\right]\right\|_H^2}{\Gamma(1-\alpha)}\, dt \right)^{\frac{1}{2}}. \end{aligned}$$

Since $J_t^\alpha\left[\xi_j' - \xi_k'\right]$ and $J_t^\alpha\left[\xi_j' + \xi_k'\right]$ belong to $C([0,T];H)$, it follows that

$$\|\mathcal{K}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq \frac{1}{\Gamma(1-\alpha)} \left(\int_0^T t^{-\alpha}\, dt \right) \left\|J_t^\alpha\left[\xi_j' - \xi_k'\right]\right\|_{C([0,T];H)} \left\|J_t^\alpha\left[\xi_j' + \xi_k'\right]\right\|_{C([0,T];H)}.$$

Moreover, for $1/2 < \alpha < 1$, Corollary 3.19 ensures that the operator

$$J_t^\alpha : L^2(0,T;H) \to C([0,T];H),$$

is a linear and bounded. Therefore, we finally obtain

$$\|\mathcal{K}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq C_\alpha \left\|\xi_j' - \xi_k'\right\|_H \left\|\xi_j' + \xi_k'\right\|_H. \tag{4.18}$$

Consequently, it follows from (4.12) and (4.18) that the sequence of real values $\left(\|\mathcal{K}_{jk}\|_{L^1(0,T;\mathbb{R})}\right)$ tends to zero as $j,k \to \infty$.

To conclude that $\left(D_t^\alpha\|J_t^\alpha \xi_j'\|_H^2\right)$ defines a Cauchy sequence, it remains to estimate the

$L^1(0,T;\mathbb{R})$ norm of the fourth term, $\mathcal{L}_{jk}$. For every $t, s \in (0,T)$ with $0 < s < t$, Hölder's inequality yields

$$\left| \sum_{i=1}^{n} \int_\Omega \left\{ \frac{J_s^\alpha \left[\xi_{ji}'(x,s) - \xi_{ki}'(x,s)\right] - J_t^\alpha \left[\xi_{ji}'(x,t) - \xi_{ki}'(x,t)\right]}{(t-s)^{\frac{\alpha+1}{2}}} \right\} \right.$$
$$\left. \left\{ \frac{J_s^\alpha \left[\xi_{ji}'(x,s) + \xi_{ki}'(x,s)\right] - J_t^\alpha \left[\xi_{ji}'(x,t) + \xi_{ki}'(x,t)\right]}{(t-s)^{\frac{\alpha+1}{2}}} \right\} dx \right|$$

which is bounded above by

$$\frac{\left\| J_s^\alpha \left(\xi_j'(s) - \xi_k'(s)\right) - J_t^\alpha \left(\xi_j'(t) - \xi_k'(t)\right) \right\|_H}{(t-s)^{\frac{\alpha+1}{2}}}$$
$$\frac{\left\| J_s^\alpha \left(\xi_j'(s) + \xi_k'(s)\right) - J_t^\alpha \left(\xi_j'(t) + \xi_k'(t)\right) \right\|_H}{(t-s)^{\frac{\alpha+1}{2}}}.$$

Integrating first with respect to the variable s over $[0,t]$, then with respect to t over $[0,T]$, and applying Hölder's inequality twice in succession, we obtain the inequality (the full expression follows from identity (4.16))

$$\|\mathcal{L}_{jk}\|_{L^1(0,T;\mathbb{R})}$$
$$\leq \frac{\alpha}{\Gamma(1-\alpha)} \left\{ \int_0^T \int_0^t (t-s)^{-\alpha-1} \|u_{j,k}(s) - u_{j,k}(t)\|_H^2 \, ds\, dt \right\}^{1/2}$$
$$\left\{ \int_0^T \int_0^t (t-s)^{-\alpha-1} \|v_{j,k}(s) - v_{j,k}(t)\|_H^2 \, ds\, dt \right\}^{1/2}.$$

Using the definition of the norm in $H^{\frac{\alpha}{2}}(0,T;H)$ (see equation (4.10)), this yields

$$\|\mathcal{L}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq \frac{\alpha}{\Gamma(1-\alpha)} \left\| J_t^\alpha \left[\xi_j' - \xi_k'\right] \right\|_{H^{\frac{\alpha}{2}}(0,T;H)} \left\| J_t^\alpha \left[\xi_j' + \xi_k'\right] \right\|_{H^{\frac{\alpha}{2}}(0,T;H)}.$$

By recalling the semigroup property of the Riemann-Liouville fractional integral (Proposition 2.42 (i)) and applying Corollary 4.17, we further obtain

$$\|\mathcal{L}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq \frac{\alpha}{m^2\Gamma(1-\alpha)} \left\| J_t^{\frac{\alpha}{2}} \left[\xi_j' - \xi_k'\right] \right\|_{L^2(0,T;H)} \left\| J_t^{\frac{\alpha}{2}} \left[\xi_j' + \xi_k'\right] \right\|_{L^2(0,T;H)}.$$

Finally, Theorem 2.34 ensures that

$$\|\mathcal{L}_{jk}\|_{L^1(0,T;\mathbb{R})} \leq N_\alpha \left\| \xi_j' - \xi_k' \right\|_{L^2(0,T;H)} \left\| \xi_j' + \xi_k' \right\|_{L^2(0,T;H)}.$$

As before, we conclude that the sequence $\big(\|\mathcal{L}_{jk}\|_{L^1(0,T;\mathbb{R})}\big)$ tends to zero as j and k increase.

Taking into account the entire discussion above, we conclude that $\big(D_t^\alpha\|J_t^\alpha\xi_j'\|_H^2\big)$ defines a Cauchy sequence in $L^1(0,T;\mathbb{R})$. Consequently, by Step 2, we obtain

$$D_t^\alpha \|\vartheta\|_H^2 = \frac{d}{dt}\left(J^{1-\alpha}\|\vartheta\|_H^2\right) \in L^1(0,T;\mathbb{R}).$$

□

Corollary 4.18. *Let $\alpha \in (1/2, 1)$ and $m \in \mathbb{N}^*$. The unique solution $u_m : [0,T] \to V$ of the Fractional Weak Formulation Approximation satisfies the inequality*

$$cD_t^\alpha\|u_m(t)\|_H^2 \leq 2\big(cD_t^\alpha u_m(t), u_m(t)\big)_H, \qquad \textit{for a.e. } t \in [0,T].$$

Proof. For any $m \in \mathbb{N}^*$, Remark 4.7 ensures that the solution u_m satisfies the hipotheses of Theorem 4.8. Hence, we have $u_m \in L^2(0,T;V)$, $cD_t^\alpha u_m \in L^2(0,T;V')$ and $J_t^{1-\alpha}\|u_m\|_H^2 \in W^{1,1}(0,T;\mathbb{R})$. Therefore, by [10, Theorem 4.16, item (b)], the desired inequality follows. □

The previous result allows us to continue the steps of the Fadeo-Galerkin method. Multiplying equation (4.3) by g_{jm} for each $1 \leq j \leq m$, and then summing the m resuluting equations, we obtain

$$\big(cD_t^\alpha u_m(t), u_m(t)\big)_H + \nu((u_m(t), u_m(t))) = \langle f(t), u_m(t)\rangle_{V',V}, \tag{4.19}$$

for almost every $t \in [0,T]$. From Corollary 4.18, it follows that

$$\frac{1}{2}cD_t^\alpha\|u_m(t)\|_H^2 + \nu\|u_m(t)\|_V^2 \leq \langle f(t), u_m(t)\rangle_{V',V},$$

for almost every $t \in [0,T]$. Thus, for $0 \leq t < T$, we have

$$\begin{aligned} 2\,|\langle f(t), u_m(t)\rangle_{V',V}| &\leq \frac{2\sqrt{\nu}}{\sqrt{\nu}}\|f(t)\|_{V'}\|u_m(t)\|_V \\ \text{(Young's Inequality)} &\leq \frac{\|f(t)\|_{V'}^2}{\nu} + \nu\|u_m(t)\|_V^2. \end{aligned}$$

Consequently,

$$cD_t^\alpha\|u_m(t)\|_H^2 + \nu\|u_m(t)\|_V^2 \leq \frac{\|f(t)\|_{V'}^2}{\nu}, \text{ for a.e. } t \in [0,T]. \tag{4.20}$$

Now, observe that when $\alpha = 1$, as is classically justified in the literature, we can deduce from (4.20) that u_m belongs to $L^2(0,T;V) \cap L^\infty(0,T;H)$. However, when $\alpha \in (1/2, 1)$, integrating both sides of the equality (4.20) yields

$$\begin{aligned} &J_t^{1-\alpha}\|u_m(t)\|_H^2 + \nu \int_0^t \|u_m(s)\|_V^2 \, ds \\ &\qquad\qquad \le \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\|u_0\|_H^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_{V'}^2 \, ds, \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

Since u_m belongs to $C([0,T];V)$, we then obtain, for all $t \in [0,T]$,

$$J_t^{1-\alpha}\|u_m(t)\|_H^2 \Big|_{t=T} + \nu \|u_m\|_{L^2(0,T;V)}^2 \le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|u_0\|_H^2 + \frac{1}{\nu}\|f\|_{L^2(0,T;V')}^2. \quad (4.21)$$

It is not difficult to notice that the aproximation solution u_m belongs to $L^2(0,T;V) \cap L^2_{1-\alpha}(0,T;H)$.

On the other hand, note that from equation (4.19) we have

$$(cD_t^\alpha u_m(t), v) = \langle f(t), v\rangle_{V',V} - \nu \langle A u_m(t), v\rangle_{V',V}, \quad \forall\, v \in V,$$

for almost every $t \in [0,T]$. Thus, Remark 4.3 ensures that

$$\|cD_t^\alpha u_m\|_{L^2(0,T;V')} \le \|f\|_{L^2(0,T;V')} + \|u_m\|_{L^2(0,T;V)}.$$

Therefore, by inequality (4.21), there exists a constant $C > 0$ such that

$$\|cD_t^\alpha u_m\|_{L^2(0,T;V')} \le C \ \text{ for all } \ m \in \mathbb{N}^*. \quad (4.22)$$

4.2 Boundedness and Convergence of the Sequence (u_m)

It follows from (4.21) that the sequence of aproximation solutions (u_m) belongs to a bounded subset of $L^2(0,T;V)$ and to a bounded subset of $L^2_{1-\alpha}(0,T;H)$. Therefore, by the reflexivity of the spaces $L^2(0,T;V)$ and $L^2_{1-\alpha}(0,T;H)$, there exists $u^* \in L^2(0,T;V)$ and $v \in L^2_{1-\alpha}(0,T;H)$ such that, up to a subsequence,

$$u_m \rightharpoonup v \quad \text{in } L^2_{1-\alpha}(0,T;H), \quad \text{and} \quad u_m \rightharpoonup u^* \quad \text{in } L^2(0,T;V).$$

Next, in order to ensure that the sequence $\big(u_m\big)$ converges to a solution of the problem defined in 4.4, we establish in this section an additional convergence criterion: namely, that $\big(u_m\big)$ converges in $L^2(0,T;H)$. To this end, we recall the following results concerning compact sets in $L^p(0,T;H)$, as proved by J. Simon in [44, Theorem 5].

Proposition 4.19. *Assume V, H, and V' are as in (2.2). Suppose $p \in [1,\infty)$ and let $W \subset L^1_{loc}(0,T;V)$ satisfy the following conditions:*

(i) W is bounded in $L^p(0,T;V)$.

(ii) $\|\tau_h u - u\|_{L^p(0,T-h;V')} \to 0$ uniformly for $u \in W$ as $h \to 0$.

Then W is relatively compact in $L^p(0,T;H)$. Here, τ_h denotes the shift operator, defined by $\tau_h u := u(t+h)$.

On the other hand, L. Li and J. Liu in [31, Proposition 3.4] established a criterion for the uniform convergence $\|\tau_h u - u\|_{L^p(0,T-h;V')} \to 0$, based on the boundedness of the Caputo fractional derivative on W. We present below the case in which the Caputo fractional derivative belongs to a bounded subset of $L^1(0,T;V')$.

Proposition 4.20. *Let $\alpha \in (1/2,1)$. Suppose (u_m) is the sequence of global solutions of the Fractional Weak Formulation Approximation, as stated in Theorem 4.6. Then there exists $C>0$, independent of $h>0$ and u_m, such that for $r_0 = 1/(1-\alpha)$,*

$$\|\tau_h u_m - u_m\|_{L^r(0,T-h,V')} \leq C h^{\alpha+\frac{1}{r}-1} \tag{4.23}$$

for all $m \in \mathbb{N}^*$ and $1 \leq r < r_0$.

Proof. First, note that from equation (4.22), there exists a constant $K>0$ such that

$$\|cD_t^\alpha u_m\|_{L^1(0,T;V')} \leq K \text{ for all } m \in \mathbb{N}^*.$$

Now, observe that by Proposition 2.42, item (iv), we have

$$u_m(t) = u_{0m} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, cD_s^\alpha u_m \, ds, \quad \text{for a.e. } t \in [0,T].$$

To prove of the inequality (4.23), first define the following kernels

$$\begin{aligned} K_1(s,t;h) &:= (t+h-s)^{\alpha-1}, \\ K_2(s,t;h) &:= (t-s)^{\alpha-1} - (t+h-s)^{\alpha-1}, \end{aligned}$$

and observe that for $h > 0$

$$\tau_h u_m(t) - u_m(t) = \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+h} K_1(s,t;h)\ cD_s^\alpha u_m\ ds - \int_0^t K_2(s,t;h)\ cD_s^\alpha u_m\ ds \right).$$

Thus, we can to deduce the estimate

$$\int_0^{T-h} \|\tau_h u_m(t) - u_m(t)\|_{V'}^r\ dt \leq \frac{2^{r-1}}{\Gamma(\alpha)^r} \left[\int_0^{T-h} \left(\int_t^{t+h} K_1(s,t;h)\ \|cD_s^\alpha u_m\|_{V'}\ ds \right)^r dt \right.$$

$$\left. + \int_0^{T-h} \left(\int_0^t K_2(s,t;h)\ \|cD_s^\alpha u_m\|_{V'}\ ds \right)^r dt \right].$$

Furthemore, applying the Hölder's inequality with $1/r + (r-1)/r = 1$, we obtain

$$\int_t^{t+h} K_1(s,t;h) \|cD_s^\alpha u_m\|_{V'} \leq \left(\int_t^{t+h} K_1^r \|cD_s^\alpha u_m\|_{V'} ds \right)^{1/r} \left(\int_t^{t+h} \|cD_s^\alpha u_m\|_{V'} ds \right)^{(r-1)/r}$$

$$\int_0^t K_2(s,t;h) \|cD_s^\alpha u_m\|_{V'} \leq \left(\int_0^t K_2^r \|cD_s^\alpha u_m\|_{V'} ds \right)^{1/r} \left(\int_0^t \|cD_s^\alpha u_m\|_{V'} ds \right)^{(r-1)/r}.$$

Then, by Fubini–Tonelli's theorem, we have

$$\int_0^{T-h} \left(\int_t^{t+h} K_1^r \|cD_s^\alpha u_m\|_{V'} ds \right) dt \leq \int_0^T \|cD_s^\alpha u_m\|_{V'} \left[\int_{\max\{s-h,0\}}^s K_1^r dt \right] ds,$$

$$\int_0^{T-h} \left(\int_0^t K_2^r \|cD_s^\alpha u_m\|_{V'} ds \right) dt \leq \int_0^{T-h} \|cD_s^\alpha u_m\|_{V'} \left[\int_s^{T-h} K_2^r dt \right] ds.$$

Now, note that

$$\int_{\max\{s-h,0\}}^s (t+h-s)^{r(\alpha-1)} dt \leq \frac{h^{r(\alpha-1)+1}}{r(\alpha-1)+1}.$$

Moreover, since we know that for $0 \leq a \leq b$ and $m \in [1,\infty)$ it holds that $(b-a)^m \leq b^m - a^m$, we may deduce the inequality

$$\int_s^{T-h} \left[(t-s)^{\alpha-1} - (t+h-s)^{\alpha-1} \right]^r dt \leq \int_s^{T-h} (t-s)^{r(\alpha-1)} - (t+h-s)^{r(\alpha-1)} dt$$

$$= \frac{(T-h-s)^{r(\alpha-1)+1} + h^{r(\alpha-1)+1} - (T-s)^{r(\alpha-1)+1}}{r(\alpha-1)+1} \leq \frac{h^{r(\alpha-1)+1}}{r(\alpha-1)+1},$$

for $1 \leq r < 1/(1-\alpha)$. Therefore, we conclude that

$$\int_0^{T-h} \|\tau_h u_m(t) - u_m(t)\|_{V'}^r\ dt \ \ \leq \ \ \frac{2^r \|cD_s^\alpha u_m\|_{L^1(0,T;V')}^r}{[\Gamma(\alpha)]^r [r(\alpha-1)+1]}\ h^{r(\alpha-1)+1}.$$

In other words,

$$\|\tau_h u_m - u_m\|_{L^r(0,T-h;V')} \leq C\ h^{\alpha+\frac{1}{r}-1}$$

for all $m \in \mathbb{N}^*$, as desired. $\square$

The conclusion of Proposition 4.20 allows us to assert that if $\alpha \in (1/2, 1)$, then

$$\|\tau_h u_m - u_m\|_{L^2(0,T-h;V')} \to 0$$

uniformly as $h \to 0$, for all $m \in \mathbb{N}^*$. Therefore, for $p = 2$, Proposition 4.19 guarantees that the sequence (u_m) is relatively compact in $L^2(0,T;H)$. That is, there exists a function $u \in L^2(0,T;H)$ such that, up to a subsequence,

$$u_m \to u \ \text{ in } \ L^2(0,T;H).$$

Remark 4.21. For $\alpha \in (1, 1/2)$, the limits of the sequence (u_m) obtained via different types of convergence are actually the same function.

(i) $u = u^*$. Let $f \in (L^2(0,T;H))'$. From the stronger convergence of $u_m \to u$ in $L^2(0,T;H)$, it follows that $f(u_m) \to f(u)$. On the other hand, since $(L^2(0,T;H))' \subset (L^2(0,T;V))'$, and we also have weak convergence $u_m \rightharpoonup u^*$ in $L^2(0,T;V)$, we obtain $f(u_m) \to f(u^*)$. Therefore, by the uniqueness of weak limits, we conclude that $u = u^*$.

(ii) $u = v$. Let $f \in (L^2(0,T;H))'$. By Proposition 3.18, we have

$$|f(h)| \leq \|f\|_{(L^2(0,T;H))'} \|h\|_{L^2(0,T;H)} \leq T^\alpha \|f\|_{(L^2(0,T;H))'} \|h\|_{L^2_{1-\alpha}(0,T;H)},$$

for all $h \in L^2_{1-\alpha}(0,T;H)$. Thus, f belongs to $(L^2_{1-\alpha}(0,T;H))'$, and we conclude the result by applying an argument analogous to item (i).

Summarizing this section, we obtain

$$\begin{aligned} u_m &\to u \quad in \quad L^2(0,T;H), \\ u_m &\rightharpoonup u \quad in \quad L^2(0,T;V), \ \text{ and} \\ u_m &\rightharpoonup u \quad in \quad L^2_{1-\alpha}(0,T;H). \end{aligned}$$

Remark 4.22. Some observations about the limit function u:

(i) By the same argument used in this section, we obtain that for every $t \in (0,T]$, there exits a finction u_t such that

$$\begin{array}{lllll} u_m & \to & u_t & in & L^2(0,t;H), \\ u_m & \rightharpoonup & u_t & in & L^2(0,t;V), \text{ and} \\ u_m & \rightharpoonup & u_t & in & L^2_{1-\alpha}(0,t;H). \end{array}$$

Moreover, the strong convergence implies that $u_t(s) = u(s)$ for almost every $s \in [0,t]$.

(ii) From the previous item and the inequality (4.21), we deduce that for all $t \in (0,T]$

$$\|u\|_{L^2_{1-\alpha}(0,t;H)} \leq \liminf_{m\to\infty} \|u_m\|_{L^2_{1-\alpha}(0,t;H)} \leq \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|u_0\|^2_H + \frac{1}{\nu}\|f\|^2_{L^2(0,T;V')}.$$

4.2.1 Passage to the Limit

The purpose of this section is to show that the limit function u approximates a solution of Definition 4.4 in a suitable sense.

Let ϕ be a test function in $C_c^\infty([0,T];\mathbb{R})$. If we multiply the equation (4.19) by ϕ and integrate over $[0,T]$, for a fixed $j \leq m$, we obtain

$$\int_0^T \left(cD_t^\alpha u_m(t), w_j\right)_H \phi(t)\, dt + \nu\int_0^T ((u_m(t), w_j))\phi(t)\, dt = \int_0^T \langle f(t), w_j\rangle_{V',V}\ \phi(t)\, dt.$$

This can be rewritten as

$$\int_0^T \left(cD_t^\alpha u_m(t), \phi(t)w_j\right)_H dt + \nu\int_0^T ((u_m(t), \phi(t)w_j))\, dt = \int_0^T \langle f(t), \phi(t)w_j\rangle_{V',V}\ dt.$$

Using Remark 4.2 and the definition of weak derivative, we have

$$\int_0^T \frac{d}{dt}\left(J_t^{1-\alpha}(u_m(t) - u_{0m}), \phi(t)w_j\right)_H dt = -\int_0^T \left(J_t^{1-\alpha}(u_m(t) - u_{0m}), \phi'(t)w_j\right)_H dt.$$

Therefore,

$$-\int_0^T \left(J_t^{1-\alpha}(u_m - u_{0m}), \phi'(t)w_j\right)_H dt + \nu\int_0^T ((u_m(t), \phi(t)w_j))\, dt = \int_0^T \langle f(t), \phi(t)w_j\rangle_{V,V'}\ dt. \quad (4.24)$$

Our next step is to use the convergence of sequence (u_m) in the respective spaces to establish the convergence of each terms.

Since ϕw_j belongs to $L^2(0,T;H)$ for all $j \in \mathbb{N}^*$. We consider the first term

$$\left|\int_0^T \left(J_t^{1-\alpha}(u_m(t)-u_{0m}), \phi'(t)w_j\right)_H dt - \int_0^T \left(J_t^{1-\alpha}(u(t)-u_0), \phi'(t)w_j\right)_H dt\right|$$
$$\leq \left[\left\|J_t^{1-\alpha}u_m(t) - J_t^{1-\alpha}u(t)\right\|_{L^2(0,T;H)} + \left\|J_t^{1-\alpha}u_{0m} - J_t^{1-\alpha}u_0\right\|_{L^2(0,T;H)}\right] \|\phi w_j\|_{L^2(0,T;H)}.$$

Thus, the convergences $J_t^{1-\alpha}u_m \to J_t^{1-\alpha}u$ in $L^2(0,T;H)$, and $u_{0m} \to u_0$ in H, implies the convergence of the first term. Moreover, since $u_m \rightharpoonup u$ in $L^2(0,T;V)$, we also have

$$\int_0^T ((u_m(t), \phi(t)w_j))\, dt \to \int_0^T ((u(t), \phi(t)w_j))\, dt.$$

This allows us conclude that, after passing to the limit in equation (4.24), we obtain

$$-\int_0^T \left(J_t^{1-\alpha}(u(t)-u_0), \phi'(t)w_j\right)_H dt + \nu \int_0^T ((u(t), \phi(t)w_j))dt = \int_0^T \langle f(t), \phi(t)w_j\rangle_{V',V}\, dt.$$

Now, observe that the linearity of the summation guarantees that the previous equality holds for all elements in $span(w_j)$, that is, for all $\varphi \in\ span(w_j)$

$$-\int_0^T \left(J_t^{1-\alpha}(u(t)-u_0), \phi'(t)\varphi\right)_H dt + \nu \int_0^T ((u(t), \phi(t)\varphi))dt = \int_0^T \langle f(t), \phi(t)\varphi\rangle_{V',V}\, dt.$$

To extend this further to arbitrary $v \in V$, take a sequence $(\varphi_n) \subset span(w_j)$ such that $\varphi_n \to v$ in V. Focusing on the first term (the others follow analogously). we have

$$\left|\int_0^T \left(J_t^{1-\alpha}(u(t)-u_0), \phi'(t)\varphi_n\right)_H - \left(J_t^{1-\alpha}(u(t)-u_0), \phi'(t)v\right)_H dt\right|$$
$$\leq \|\varphi_n - v\|_H \int_0^T \left\|J_t^{1-\alpha}(u(t)-u_0)\right\|_H |\phi'(t)|\, dt$$
$$\leq C\|\varphi_n - v\|_V \int_0^T \left\|J_t^{1-\alpha}(u(t)-u_0)\right\|_H |\phi'(t)|\, dt.$$

This tends to zero as $n \to \infty$. Therefore,

$$-\int_0^T \left(J_t^{1-\alpha}(u(t)-u_0), v\right)_H \phi'(t)\, dt + \nu \int_0^T ((u(t), v))\phi(t)\, dt = \int_0^T \langle f(t), v\rangle_{V',V}\, \phi(t)\, dt. \tag{4.25}$$

Concluding, in the sense of distributions, that for all $v \in V$, we have

$$\frac{d}{dt}\left(J_t^{1-\alpha}(u(t)-u_0), v\right)_H + \nu((u(t), v)) = \langle f(t), v\rangle_{V',V}$$

for almost every $t \in [0, T]$.

On the other hand, observe that Remark 2.19 ensures that, for almost every $t \in [0, T]$

$$\langle J_t^{1-\alpha}(u(t)-u_0), \cdot\rangle := (J_t^{1-\alpha}(u(t)-u_0), \cdot)_H$$

defines a duality pairing on V. Then, we can rewrite equality (4.25) as

$$-\int_0^T \langle \phi'(t) J_t^{1-\alpha}(u(t)-u_0), v\rangle_{V',V}\, dt + \nu \int_0^T \langle \phi(t) Au(t), v\rangle_{V',V}\, dt = \int_0^T \langle \phi(t) f(t), v\rangle_{V',V}\, dt,$$

where Au is defined in Remark 4.3. By Proposition 2.5, we conclude that

$$\left\langle -\int_0^T \phi'(t) J_t^{1-\alpha}(u(t)-u_0) dt, v\right\rangle_{V',V} = \left\langle \int_0^T \phi(t)[f(t) - \nu Au(t)] dt, v\right\rangle_{V',V}.$$

Therefore, at V', we obtain

$$-\int_0^T \phi'(t) J_t^{1-\alpha}(u(t)-u_0) dt = \int_0^T \phi(t)[f(t) - \nu Au(t)] dt.$$

That is, $J_t^{1-\alpha}(u(t)-u_0) \in W^{1,1}(0, T, V')$, and as a duality in V

$$D_t^{\alpha}(u-u_0) = f(t) - \nu Au(t), \tag{4.26}$$

almost every $t \in [0, T]$.

From the above, it only remains to show that $u(0) = u_0$ in order to guarantee that u is a solution in the sense of Definition 4.4.

Remark 4.23. In the convergence arguments used above, we relied on strong convergence in $L^2(0, T; H)$ together with the boundedness of the Riemann–Liouville fractional integral $J_t^{1-\alpha} : L^2(0, T; H) \to L^2(0, T; H)$, rather than on weak convergence in $L^2_{1-\alpha}(0, T; H)$. Both notions of convergence could, in principle, have been employed at this stage, and neither would lead to a weaker convergence result.

However, the introduction of the space $L^2_{1-\alpha}(0, T; H)$ is not motivated by convergence issues. Its role is to encode additional time regularity of the solution. Indeed, since $L^2_{1-\alpha}(0, T; H)$ is a proper subset of $L^2(0, T; H)$, the corresponding integrability condition reflects a finer behavior near the initial time, which is intrinsic to fractional evolution problems. Therefore, although the convergence argument is carried out in the larger space $L^2(0,T;H)$, the weak solution exhibits the improved regularity inherited from this new framework.

4.3 Initial Condition and Regularity of the Solution

In this section, we show that the solution u belongs to $C([0,T];V')$, thereby ensuring that $u_0 = u(0)$ in the sense of the duality in V.

Since f and Au belong to $L^2(0,T;V')$, Corollary 3.19 guarantees that, for $\alpha > 1/2$, both $J_t^\alpha Au$ and $J_t^\alpha f$ are in $C([0,T];V')$. Moreover, the same corollary ensures that

$$J_t^\alpha Au(t)|_{t=0} = J_t^\alpha f(t)|_{t=0} = 0_{V'}.$$

In summary, for $\alpha \in \big(1/2, 1\big)$, equation (4.26) guarantees that

$$J_t^\alpha[D_t^\alpha(u-u_0)] \in C([0,T],V'),$$

and $J_t^\alpha[D_t^\alpha(u-u_0)]\,|_{t=0} = 0_{V'}$.

Now, Proposition 2.42, item (iii), ensures that

$$\begin{aligned} u(t)-u_0 &= J_t^\alpha[D_t^\alpha(u-u_0)] + \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \left\{J_s^{1-\alpha}(u(s)-u_0)\right\}\Big|_{s=0} \\ &= J_t^\alpha[D_t^\alpha(u-u_0)] + \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \left\{J_s^{1-\alpha}u(s)\right\}\Big|_{s=0}, \end{aligned}$$

for almost every $t \in [0,T]$. Moreover, since $u \in L^2_{1-\alpha}(0,t,H) \cap L^2(0,t,V)$ for all $t \in (0,T]$, see Remark 4.22, and using the embeddings $V \hookrightarrow H \equiv H' \hookrightarrow V'$, we obtain

$$\begin{aligned} \left\| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u(s)\, ds \right\|_{V'} &\le C \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \|u(s)\|_{H'}\, ds \\ &\le C \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \left(\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \|u(s)\|_H^2\, ds\right)^{1/2} \\ &\le C \left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|u_0\|_H^2 + \frac{1}{\nu} \|f\|^2_{L^2(0,T;V')}\right)^{1/2}. \end{aligned}$$

The last inequality follows again from Remark 4.22, item (ii). Thus, without loss of generality, we can assume that $J_t^{1-\alpha}u\big|_{t=0} = 0_{V'}$. Therefore, we conclude that

$$u(t) - u_0 = J_t^\alpha[D_t^\alpha(u - u_0)],$$

for almost every t in $[0,T]$. That is, u admits a representation in $C([0,T];V')$ such that $u(0) = u_0$ in V', as desired.

4.4 Uniqueness of the solution

Under the condition that the solution u is a continuous function over V' when $\alpha \in (1/2, 1)$, we establish in this section that it is unique, except possibly on a set of measure zero.

Assume that u_1 and u_2 are solutions in the sense of Definition 4.4, and define $u = u_1 - u_2$. Thus, $u(0) = 0_H$ and as duality on V, we obtain

$$D_t^\alpha u + \nu A u(t) = 0_{V'}, \tag{4.27}$$

for almost every $t \in [0,T]$. Applying J_t^α to both sides of (4.27) and using item (iii) of Proposition 2.42, we have

$$u(t) + \nu J_t^\alpha A u(t) = 0_{V'},$$

for almost every $t \in [0,T]$. Since $u \in L^2(0,T;V)$, taking the duality paring with $u(t)$ yields

$$\langle u(t), u(t)\rangle_{V',V} + \nu\langle J_t^\alpha A u(t), u(t)\rangle_{V',V} = 0, \tag{4.28}$$

for almost every $t \in [0,T]$.

Remark 4.24. Concerning the duality term $J_t^\alpha A u(t)$, note that Proposition 2.5 ensures that, for all $v \in V$,

$$\langle J_t^\alpha A u(t), v\rangle_{V',V} = J_t^\alpha \langle A u(t), v\rangle_{V',V},$$

for almost every $t \in [0,T]$. Then, seeing $u(t) = (u_1(t), ..., u_n(t))$, by Remark 4.3, recalling the definition of the inner product $((\cdot,\cdot))$ in V, and applying Fubini-Tonelli theorem, we deduce that for all $v = (v_1, ..., v_n) \in V$

$$\langle J_t^\alpha A u(t), v\rangle_{V',V} = \sum_{i=1}^{n} \big(J_t^\alpha \nabla u_i(t), \nabla v_i\big)_{\mathbb{L}^2(\Omega)},$$

for almost every $t \in [0,T]$.

Thus, the above remark and Remak 2.19 imply that equation (4.28) is equivalent to

$$\|u(t)\|_H^2 + \nu \sum_{i=1}^{n} \left(J_t^\alpha \nabla u_i(t), \nabla u_i(t)\right)_{\mathbb{L}^2(\Omega)} = 0, \tag{4.29}$$

for almost every $t \in [0, T]$.

Now, since $\nabla u_i \in L^2(0, T; \mathbb{L}^2(\Omega))$ for $i = 1, ..., n$, item (ii) of Proposition 2.42 gives

$$D_t^\alpha\left[J_t^\alpha \nabla u_i(t)\right] = \nabla u_i(t)$$

for almost every $t \in [0, T]$. Hence, (4.29) can rewritten as

$$\|u(t)\|_H^2 + \nu \sum_{i=1}^{n} \left(D_t^\alpha\left[J_t^\alpha \nabla u_i(t)\right], J_t^\alpha \nabla u_i(t)\right)_{\mathbb{L}^2(\Omega)} = 0, \tag{4.30}$$

for almost every $t \in [0, T]$.

Because $\alpha > 1/2$, Corollary 3.19 ensures that $J_t^\alpha \nabla u_i \in C([0, T]; \mathbb{L}^2(\Omega))$ for all $i = 1, ..., n$. This allows us to apply Theorem 4.8 for case the $H = \mathbb{L}^2(\Omega)$ (see [11, Corollary 19]). We then conclude that $D_t^\alpha \|J_t^\alpha \nabla u_i(t)\|_{\mathbb{L}^2(\Omega)}^2 \in L^1(0, T; \mathbb{R})$ for all $i = 1, ..., n$, and

$$\|u(t)\|_H^2 + (\nu/2) \sum_{i=1}^{n} D_t^\alpha \|J_t^\alpha \nabla u_i(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq 0, \tag{4.31}$$

for almost every $t \in [0, T]$. Finally, by item (iii) of Proposition 2.42, (4.31) is equivalent to

$$J_t^\alpha \|u(t)\|_H^2 + (\nu/2) \sum_{i=1}^{n} \|J_t^\alpha \nabla u_i(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq 0,$$

for almost every $t \in [0, T]$. Hence $J_t^\alpha \|u(t)\|_{\mathbb{L}^2(\Omega)}^2 = 0$ a.e. in $[0, T]$. Since $J_t^\alpha : L^2(0, T; \mathbb{R}) \to L^2(0, T; \mathbb{R})$ is injective (if follows from item (ii) of Proposition 2.42), we conclude that $u = 0$, or equivalently, $u_1 = u_2$. This proves uniqueness.

Chapter 5

Pressure Recovery in our Variational Problem

Let us first recall the fractional order approach to the Stokes equations. Let Ω be a bounded Lipschitz subset of $\mathbb{R}^n$, and let $\alpha \in (0,1)$, $T > 0$, and $\nu > 0$ be fixed constants. Consider $f : [0,T] \times \Omega \to \mathbb{R}^n$ as the forcing term and $u_0 : \Omega \to \mathbb{R}^n$ as the initial condition. The Stokes equations with a Caputo fractional derivative in the time variable are given by

$$\begin{cases} cD_t^\alpha u(t,x) - \nu \Delta u(t,x) + \nabla p(t,x) = f(t,x), & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u(t,x) = 0, & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0, & \text{on } [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & \text{in } \Omega. \end{cases} \tag{5.1}$$

In this setting, following the standard procedure for deriving a variational formulation of the classical time-dependent Stokes equations (see Section 2.2 of this paper), we arrive at the following weak formulation, which is developed in the beginning of the Section 4.

Definition 5.1. Let $\alpha \in (1/2, 1)$, $f \in L^2(0,T;V')$ and $u_0 \in H$. A fractional weak solution to (5.1) is a function $u \in L^2_{1-\alpha}(0,T;H) \cap L^2(0,T;V)$ such that $cD_t^\alpha u \in L^2(0,T;V')$ and

$$(cD_t^\alpha u(t), v)_H = \langle f(t) - \nu A u(t), v \rangle_{V',V} \tag{5.2}$$

for almost every $t \in [0,T]$, for all $v \in V$ and $u(0) = u_0$. Here, the operator Au is defined in Remark 4.3.

Note that the pressure term $\nabla p(t)$ is absent from Definition 5.1. The goal of this section is to recover the pressure, in some sense, given a solution $u \in L^2_{1-\alpha}(0,T;H) \cap$

$L^2(0,T;V)$ of problem (5.2), thereby establishing a connection between problems (5.1) and (5.2). For this purpose, we rely on the following propositons.

Proposition 5.2. *Let* Ω *be an open subset of* $\mathbb{R}^n$*, and let* $g = (g_1, ..., g_n)$ *with* $g_i \in \mathcal{D}'(\Omega)$*,* $i = 1, ..., n$*. A necessary and sufficient condition for the existence of a distribution* $p \in \mathcal{D}'(\Omega)$ *such that* $g = \nabla p$ *is that*

$$\langle g, v \rangle = 0 \quad \forall v \in \mathcal{V}.$$

Proof. See [46, Chapter I, Proposition 1.1] or [46, Chapter I, Remark 1.9]. □

Proposition 5.3. *Let* Ω *be a bounded Lipschitz open subset of* $\mathbb{R}^n$*.*

(i) *If a distribution* p *has all its first-order derivatives* $\frac{\partial p}{\partial x_i} \in L^2(\Omega)$*, for* $1 \leq i \leq n$*, then* $p \in L^2(\Omega)$*, and there exists* $k > 0$*, and* $c \in \mathbb{R}$ *such that*

$$\|p + c\|_{L^2(\Omega)} \leq k\|\nabla p\|_{\mathbb{L}^2(\Omega)}.$$

(ii) *If a distribution* p *has all its first-order derivatives* $\frac{\partial p}{\partial x_i} \in (H_0^1(\Omega))'$*, for* $1 \leq i \leq n$*, then* $p \in L^2(\Omega)$*, and there exists* $k > 0$*, and* $c \in \mathbb{R}$ *such that*

$$\|p + c\|_{L^2(\Omega)} \leq k\|\nabla p\|_{(\mathbb{H}_0^1(\Omega))'}.$$

Proof. See [46, Chapter I, Proposition 1.2]. □

To introduce the pressure into our variational problem, let us define

$$\alpha(t) = \int_0^t Au(s)\, ds \quad \text{and} \quad F(t) = \int_0^t f(s)\, ds.$$

If u is a solution in the sense of Definition 5.1, then Section 4.3 ensures that $u \in C([0,T];V')$, and:

(i) By Theorem 2.35, $J_s^{1-\alpha}u \in C([0,T];V')$;

(ii) By Remark 4.3, we know $Au \in L^1(0,T;V')$, and hence $\alpha \in C([0,T];V')$;

(iv) Since $f \in L^2(0,T;V')$, we obtain $F \in C([0,T];V')$.

Therefore, the functions α and F also belong to $C([0,T];V')$.

Now, Proposition 2.5 ensures that, upon integrating the equation (5.2), we obtain

$$\left(\int_0^t cD_s^\alpha u(s)\, ds, v\right)_H = \left\langle \int_0^t [f(s) - \nu Au(s)]\, ds, v \right\rangle_{V',V}$$

for all $v \in V$. Thus, for almost every $t \in [0, T]$

$$\left\langle -J_t^{1-\alpha}[u(t) - u(0)] + \int_0^t [f(s) - \nu Au(s)]\, ds, v \right\rangle_{V',V} = 0.$$

Observe that the function $-J_t^{1-\alpha}[u(t)-u(0)]+\int_0^t[f(s)-\nu Au(s)]ds$ belongs to $C([0,T]; V')$. Therefore, since $V \subset \mathbb{H}_0^1$, this function also belongs to $C([0,T]; (\mathbb{H}_0^1(\Omega))')$.

By applying Proposition 5.2 and Proposition 5.3 , item (ii), we obtain the existence of a distribution p such that $p(t) \in L^2(\Omega)$ and

$$-J_t^{1-\alpha}[u(t) - u(0)] + \int_0^t [f(s) - \nu Au(s)]\, ds = \nabla p(t) \quad \text{for a.e. } t \in [0, T].$$

Hence, we conclude that $\nabla p \in C([0,T]; (\mathbb{H}_0^1)')$, and thus, by Proposition 5.3, there exists $c \in \mathbb{R}$ such that $p + c \in C([0,T]; L^2(\Omega))$.

Now, since $u \in L^2(\Omega \times (0,T))$, and in the distributional sense on $\Omega \times (0,T)$ we have

$$\left\langle \frac{d}{dt}\partial_{x_i} p, \phi \right\rangle = \left\langle p, \partial_{x_i} \frac{d}{dt}\phi \right\rangle = \left\langle p, \frac{d}{dt}\partial_{x_i} \phi \right\rangle = \left\langle \partial_{x_i} \frac{d}{dt} p, \phi \right\rangle$$

for all $\phi \in \mathcal{D}(\Omega \times (0,T))$; that is

$$\frac{d}{dt}(\nabla p) = \nabla\left(\frac{d}{dt}p\right).$$

We obtain in $\mathcal{D}'(\Omega \times (0,T))$ that

$$-cD_t^\alpha u(t) + f(t) - \nu Au(t) = \nabla P(t) \quad \text{for a.e. } t \in [0, T],$$

where $P := \frac{d}{dt}p$. Recalling the definitions of the operator A, we conclude that, in the distributional sense on $\Omega \times (0,T)$,

$$cD_t^\alpha u(t) - \nu \Delta u(t) + \nabla P(t) = f(t) \quad \text{for a.e. } t \in [0, T],$$

as desired.

Chapter 6

Final Considerations and Future Work

The research developed in this thesis has already led to concrete scientific contributions. In particular, part of the results obtained throughout this work has been consolidated in the following manuscripts:

- P. M. Carvalho-Neto, C. L. Frota, J. C. Oyola Ballesteros, P. G. P. Torelli, *Energy Estimates for Fractional Evolution Equations*, arXiv:2508.05780.
- P. M. Carvalho-Neto, J. C. Oyola Ballesteros, *A New Approach for the Unsteady Stokes Equations with Time Fractional Derivative in Bounded Domains*, arXiv:2511-.13896.

These works reflect the main theoretical developments achieved during this PhD project, particularly concerning fractional Cauchy problems, weighted Bochner spaces, and fractional formulations of fluid dynamics models.

In conclusion, the preparation of this thesis involved several technical aspects that posed substantial theoretical and mathematical challenges, many of which were successfully addressed. We now highlight the three contributions that we consider most relevant to the structure and completion of this work.

(i) Section 3.1 introduces the fractional Cauchy problem (equation (3.1)) under the assumption that the map $G : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory function. Assuming further that G is locally Lipschitz continuous, Theorem 3.12 establishes the existence and uniquenesse of a global solution when the associated Nemytskii operator satisfies the condition $p = q$.

(ii) Section 3.2 presents the space $L^p_\alpha(0,T;X)$ and a series of results aimed at establishing relationships between the spaces $L^p_\alpha(0,T;X)$ for different values of $\alpha \in [0,1]$ and $p \in [1,\infty)$, as well as their connection with the classical space $L^p(0,T;X)$. We show that $L^p_\alpha(0,T;X)$ is a Banach space and that it is reflexive whenever X is reflexive Banach space, as stated in Theorem 3.25. Finally, Theorem 3.27 and Corollary 3.28 provide a caracterization of dual space of $L^p_\alpha(0,T;X)$.

(iii) Section 4 presents a new formulation for the weak solution of the unsteady Stokes equations when considering fractional derivatives on the time variable; as described in Definition 4.4. By applying the Faedo–Galerkin method in a framework specifically adapted to the fractional derivatives, we prove the existence and uniqueness of weak solution.

It is important to highlight that, by imposing the condition $\alpha \in (1/2,1)$, the fractional Cauchy problem introduced in Section 3 guarantees the existence of approximate solutions, and therefore a sequence of approximate solutions corresponding to equation 4.3. Moreover, the energy estimates derived for this sequence via the Faedo–Galerkin method demonstrate the relevance of the space $L^p_\alpha(0,T;X)$.

On the other hand, regarding future work, we note that throughout most of this PhD project we have focused on studying a weak formulation of the Navier-Stokes equations, starting from the strong form

$$\begin{cases} D_t u(t,x) - \nu\Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) = f(t,x), & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u(t,x) = 0, & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0, & \text{on } [0,T]\times\partial\Omega, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{6.1}$$

The nonlinear term in this formulation is the expression $(u(t,x)\cdot\nabla)u(t,x)$, which is precisely the term that distinguishes the Navier-Stokes-equations from the Stokes Equations.

Following the standard approach for deriving a variational formulation of the classical time-dependent Navier–Stokes equations (see [46, Chapter III] for details), we obtain the following fractional weak formulation of the system above.

WEAK FRACTIONAL FORMULATION: Let $\Omega \subset \mathbb{R}^2$, $\alpha \in (2/3,1)$, $f \in L^2(0,T;V')$, and $u_0 \in H$. A fractional weak solution to (6.1) is a function $u \in L^2_{1-\alpha}(0,T;H) \cap L^2(0,T;V)$ such that $cDt^\alpha u \in L^1(0,T;V')$ and

$$cD_t^\alpha u(t) = f(t) - \nu Au(t) - Bu(t) \tag{6.2}$$

for almost every $t \in [0,T]$, and $u(0) = u_0$.

This weak fractional formulation naturally raises two important questions within the context of this work:

(i) Following the steps of the Faedo-Galerkin method, the weak fractional formulation of Navier-stokes equations can be reinterpreted as the following Cauchy problem:

$$\begin{cases} cD_t^\alpha U_m(t) & = & G_m(t, U_m(t)), \quad \text{a.e. in } [0,T], \\ U_m(0) & = & U_{0m} \in \mathbb{R}^m, \end{cases} \tag{6.3}$$

where $G_m : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\|G_m(t,x)\|_m \leq \underbrace{\left(\|w_1\|_V^2 + \ldots + \|w_m\|_V^2\right)^{1/2} \|f(t)\|_{V'} + m^{3/2}\nu M}_{=:\gamma(t)} + m^{3/2}\left(\nu M + N\right)\|x\|_m^2.$$

In this setting, the Nemytskii parameters satisfy $q = 4$ and $p = 2$. Therefore, Theorem 3.12 (in section 3.1) cannot ensure the existence and uniqueness of a continuous function $U_m : [0,T] \to \mathbb{R}^m$ satisfying both equations in (6.3).

(ii) Under the assumptions of the weak fractional formulation of the Navier-Stokes equations ($2/3 < \alpha < 1$) and assuming a solution to item (i) exists, we can obtain a solution u such that $cD_t^\alpha u$ belongs to $L^1(0,T;V')$. However, this condition is too weak to apply Theorem 4.8 or [10, Theorem 4.16, item (b)], and thus insufficient to guarantee uniqueness of the solution u.

Consequently, the open problems that naturally emerge from this work, and which we intend to address in future research, are the following. First, it is necessary to establish a result analogous to Theorem 3.12 for the case $q \neq p$, in particular for $q = 4$ and $p = 2$. Second, additional conditions on the solution u must be identified in order to allow the application of Theorem 4.8 or [10, Theorem 4.16(b)]. For instance, assumptions ensuring that $cD_t^\alpha u \in L^2(0,T;H)$ or that $J_t^{1-\alpha}\|u\|_H^2 \in W^{1,1}(0,T;\mathbb{R})$ would be sufficient to guarantee uniqueness of the solution.

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