



UNIVERSIDADE FEDERAL DE SANTA CATARINA
CENTRO DE CIÊNCIAS FÍSICAS E MATEMÁTICAS
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA PURA E APLICADA

Laura Ribeiro dos Santos

Blown-up singular Riemannian foliations

Florianópolis
2026

Laura Ribeiro dos Santos

Blown-up singular Riemannian foliations

Dissertação submetida ao Programa de Pós-Graduação em Matemática Pura e Aplicada da Universidade Federal de Santa Catarina como requisito parcial para a obtenção do título de Mestre em Geometria e Topologia.

Orientador: Prof. Francisco Carlos Caramello Junior, Dr.

Florianópolis

2026

Ficha catalográfica gerada por meio de sistema automatizado gerenciado pela BU/UFSC.
Dados inseridos pelo próprio autor.

dos Santos, Laura Ribeiro

Blown-up singular Riemannian foliations / Laura Ribeiro
dos Santos ; orientador, Francisco Carlos Caramello
Junior, 2026.

67 p.

Dissertação (mestrado) - Universidade Federal de Santa
Catarina, Centro de Ciências Físicas e Matemáticas,
Programa de Pós-Graduação em Matemática, Florianópolis, 2026.

Inclui referências.

1. Matemática. 2. Desingularizações . 3. Folheações
Singulares Riemannianas. 4. Cohomologia Básica . 5. Feixe
de Molino. I. Junior, Francisco Carlos Caramello. II.
Universidade Federal de Santa Catarina. Programa de Pós
Graduação em Matemática. III. Título.

Laura Ribeiro dos Santos
Blown-up singular Riemannian foliations

O presente trabalho em nível de mestrado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

Prof. Francisco Carlos Caramello Junior, Dr.
Universidade Federal de Santa Catarina

Prof. Dirk Töben, Dr.
Universidade Federal de São Carlos

Prof. Ivan Struchiner, Dr.
Instituto de Matemática e Estatística da Universidade de São Paulo

Luca Accornero, Dr. [suplente]
Instituto de Matemática e Estatística da Universidade de São Paulo

Profa. Marianna Ravara Vago, Dra. [suplente]
Universidade Federal de Santa Catarina

Certificamos que esta é a **versão original** do trabalho de conclusão que foi julgado adequado para obtenção do título de mestre em geometria e topologia.



Documento assinado digitalmente
Dirceu Bagio
Data: 27/03/2026 14:58:46-0300
CPF: ***.915.569-**
Verifique as assinaturas em <https://v.ufsc.br>

Coordenação do Programa de Pós-Graduação



Documento assinado digitalmente
FRANCISCO CARLOS CARMELLO JUNIOR
Data: 27/03/2026 16:49:56-0300
CPF: ***.801.148-**
Verifique as assinaturas em <https://v.ufsc.br>

Prof. Francisco Carlos Caramello Junior, Dr.
Orientador

Florianópolis, 2026.

Agradecimentos

“Se eu vi mais longe, foi por estar sobre os ombros de gigantes.”

-Isaac Newton

Inicio estes agradecimentos com o já conhecido verso de Newton, pois acredito que a gratidão não deve ser apenas um sentimento, mas algo institucionalizado. Sem ela, este trabalho não teria sido realizado (talvez nem mesmo iniciado). Ele se ergue a partir da gentileza, da companhia e do apoio daqueles que me cercam, contando com contribuições das mais diversas naturezas, algumas delas inclusive póstumas.

Gostaria de agradecer, em primeiro lugar, ao meu orientador de longa data, o professor Francisco Carlos Caramello Junior, que se tornou uma influência central em minha trajetória acadêmica. Além de me ensinar praticamente tudo o que sei (merecendo o título de mestre, apesar do mais nobre de doutor), ele me ensinou também muitas lições sobre a vida, mesmo sem que tenhamos trocado tantas ideias para além da matemática. Para além das aulas de geometria e folheações, aprendi com ele a ser uma (futura) professora melhor, uma aluna mais consciente e, sobretudo, uma pessoa mais preparada para a vida (a parte boa dela, diga-se de passagem). A ele se soma meu agradecimento à Thais, por ter me permitido “roubá-lo” por tantas horas ao longo desses anos e por ter me apresentado Daisy Jones & The Six (por isso, devo-lhe muito).

Agradeço ao professor Ivan Pontual, por ter me ouvido e aconselhado ao longo de todos esses anos. Sendo o físico mais geômetra e o geômetra mais físico que conheço, ele me ensinou que essa dualidade existe apenas burocraticamente. Seus conselhos foram fundamentais para que eu concluísse a graduação em Física e, posteriormente, migrasse de área sem medo de errar, caso fosse necessário. Por isso, e por tantos outros ensinamentos, deixo aqui minha profunda gratidão.

Ao professor Eliezer Batista, o físico mais algebrista e o algebrista mais físico que conheço. Considero-o um de meus padrinhos acadêmicos, ao lado do professor Ivan, guiando-me em escolhas difíceis e fazendo-me acreditar que nada era tão complicado quanto parecia. Sua sala abarrotada de livros sempre me acolheu calorosamente, permeada por histórias de sua trajetória que, mesmo quando anedóticas, serviram como fonte de inspiração e conselhos valiosos.

Este trabalho também contou com a valiosa colaboração de professores de diferentes instituições. Em especial, agradeço ao professor Dirk Töben, que gentilmente aceitou se reunir comigo em pleno mês de janeiro para discutir matemática. Sou grata por seu tempo, paciência e hospitalidade. Agradeço igualmente ao professor Ivan Struchiner, que nos últimos meses se mostrou extremamente receptivo e disposto a fazer parte da minha história, bem como a construir um futuro de muita matemática e grupóides de Lie. Pela disponibilidade, paciência e ensinamentos, meu sincero agradecimento. À professora Mariana, agradeço pela constante disponibilidade em oferecer conselhos e auxílio nas

decisões sobre o futuro, bem como por sua contribuição direta a este trabalho, trazendo novas perspectivas sobre desingularizações e sendo, por vezes, o nosso “sanity check”.

Ao professor Marcos Alexandrino, por ser um grande apoiador, conselheiro e pesquisador em todos esses anos. Sua preocupação e apoio constante me permitiram abrir meus olhos para as possibilidades e construir caminhos que me guiaram até onde me encontro agora. Minha profunda gratidão.

Aos meus colegas acadêmicos, em especial Daniel Bergossa, cuja participação no último ano foi profundamente inspiradora. Tive o privilégio de acompanhar como um grande matemático pensa em seu estado mais puro: enquanto aprende. Sou também muito grata à Ana, cuja convivência, junto com Daniel, tornou este último ano muito mais leve. Pelos momentos compartilhados, pelas risadas e pelo aprendizado conjunto, agradeço imensamente. Vocês fizeram (e fazem) tudo parecer menos pesado do que realmente é.

Entre a vida acadêmica e a vida para além da UFSC, agradeço à Júlia, que entrou recentemente em minha vida e acompanhou o estopim deste trabalho, inclusive com participações tão inusitadas que acabaram registradas neste texto. Obrigada pela paciência durante a verdadeira gangorra emocional que foi construir este trabalho em meio a tantas mudanças, e por todo o amor e carinho.

Chego agora àqueles que, sem necessariamente fazer matemática, foram essenciais para que este trabalho existisse, tornando minha vida um espaço mais seguro e fértil para que eu pudesse realizar o que almejava ao ingressar no mestrado. Agradeço, antes de tudo, aos meus pais, Katiane e Paulo André. À minha mãe, devo toda a coragem e inspiração: nunca me faltou apoio, nem alguém genuinamente interessado no que faço, graças a ela. Ao meu pai, que não esteve presente em vida para acompanhar toda a minha trajetória acadêmica, mas foi fundamental para que eu tivesse anos mais tranquilos do que poderiam ter sido. Obrigada também por me ensinar a rir quando tudo parecia difícil demais, esse aprendizado foi essencial para que este trabalho fosse feito com leveza.

Ao meu irmão, Igor, que não sabe, mas me ensinou geometria quando ainda era muito pequeno, por definir tão simplesmente o que era um triângulo. Seu jeito de ser criança me fez lembrar todos esses anos do amor que tenho pelo o que faço, florescendo em mim a origem de toda essa trajetória: os sonhos de quando eu era criança.

À minha família adotiva, a família Mastrocola, deixo meu agradecimento mais carinhoso. Sempre me receberam com amor, afeto e portas abertas. Grande parte deste trabalho é fruto do apoio de vocês, em especial da mãe Ana, que constantemente me lembrou de quem eu sou e das coisas que eu não precisava ter medo de querer para mim.

Aos meus amigos, que trouxeram leveza nos momentos difíceis, agradeço por todo o amor, carinho, paciência e sabedoria. Em especial, agradeço a Adrian, Jenni, Tori, Vini, Lucas, Aghata, André, Vic, Jadna, Vivi e a tantos outros que talvez não estejam citados nominalmente, mas que estiveram presentes durante este período.

Por fim, dedico este último parágrafo à pessoa que é responsável por grande parte

da confiança que tenho em mim mesma e por tornar meu mundo um lugar mais divertido e seguro. Àquela que esteve presente desde o primeiro dia em que tudo isso começou, em um elevador do EFI. Obrigada, Bia, por todo o amor, carinho e confiança; por vibrar com cada conquista; e por me permitir aprender tanto sobre tudo. Da engenharia à culinária, aprendi mais sobre o mundo porque você me ensinou uma maneira diferente de enxergar a vida.

Por fim, à Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), cujo financiamento foi fundamental para minha subsistência ao longo destes anos.

“Definitions are vital starting points for the imagination. What we cannot imagine cannot come into being. A good definition marks our starting point and lets us know where we want to end up. As we move toward our desired destination we chart the journey, creating a map.”

— Bell Hooks

Resumo

Nesta dissertação de mestrado investigamos novas propriedades do método de desingularização por blow-up no contexto de folheações Riemannianas singulares. Primeiramente, relacionamos a dinâmica de tais folheações (governada pelo chamado feixe de Molino) com a dinâmica de seu blow-up. No caso particular das folheações de Killing singulares, isso conduz a uma forte restrição sob hipóteses topológicas: a saber, as folhas de tais folheações são todas fechadas, desde que a característica de Euler da variedade ambiente seja não nula e que todos os seus estratos singulares tenham codimensão ímpar. Em seguida, relacionamos a cohomologia básica de uma folheação Riemanniana singular com a de seu blow-up, generalizando resultados clássicos bem conhecidos da geometria algébrica e da geometria complexa. Por fim, mostramos que o espaço dos fechos das folhas de uma folheação de Killing singular é o limite de Gromov–Hausdorff de uma sequência de orbifolds, cujas dimensões são iguais à codimensão da folheação.

Palavras chave: Blow-up, Folheações Riemannianas Singulares, Dessingularização, Co-homologia Básica, Feixe de Molino.

Abstract

In this master dissertation we investigate new properties of the blow-up desingularization method in the context of singular Riemannian foliations. First, we relate the dynamics of such a foliation, which is governed by the so called Molino sheaf, with that of its blow-up. In the particular case of singular Killing foliations, this leads to a strong constraint under mild topological assumptions: namely, the leaves of such foliations are all closed, provided the Euler characteristic of its ambient manifold is non-vanishing and its singular strata are all odd-codimensional. Next, we relate the basic cohomology of a singular Riemannian foliation with that of its blow-up, generalizing well-known, classical analogous results in algebraic and complex geometry. Finally, we show that the space of leaf closures of a singular Killing foliation is the Gromov–Hausdorff limit of a sequence of orbifolds, whose dimensions are the codimension of the foliation.

Keywords: Blow-up, Singular Riemannian foliations, Desingularization, Basic Cohomology, Molino Sheaf.

List of Figures

Figure 1.1 -	Haefliger cocycle . Source: [CT19] .	19
Figure 1.2 -	Kronecker foliation . Source: [CT19] .	20
Figure 1.3 -	Foliate fields . Source: [CT19] .	22
Figure 1.4 -	Generalized Hopf Fibration via stereographic projection . Source: [CT19] .	23
Figure 1.5 -	Sliding along the leaves . Source: [CT19] .	24
Figure 1.6 -	Homothetic transformation . Source: [CT19] .	29
Figure 1.7 -	Molino bundle . Source: [CT19] .	35
Figure 1.8 -	Molino sheaf . Source: [CT19] .	37
Figure 2.1 -	Projectivization of the normal bundle .	41
Figure 2.2 -	Blow-up of the origin in $\mathbb{R}^2$.	42
Figure 3.1 -	Blow-up of circle foliation in $\mathbb{R}^2$.	61

List of symbols

$C^\infty(M)$	Smooth functions in M
$T_x M$	Tangent space in x
df_x	Differential of f in x
TM	Tangent bundle of M
$\nu(N)$	Normal bundle of a submanifold N
$\Omega^k(M)$	k -differential forms in M
$H_{\text{dR}}^*(M)$	Differential graded algebra of differential forms given by the de Rham cohomology
$H^i(M)$	De Rham cohomology group of i -forms
$\cong$	Isomorphism
d	Exterior differential on differential forms
ι	Interior differential of vector fields

Resumo Expandido

A construção de blow-up é uma técnica geométrica central quando se trata de resolver singularidades. Historicamente, blow-ups foram introduzidos pela primeira vez na geometria algébrica: dada uma subvariedade (complexa) $Z \subset X$, o blow-up $\text{Bl}_Z X$ substitui Z pela projetivização de seu fibrado normal, produzindo uma nova variedade na qual o conjunto singular é “espalhado” ao longo de um divisor excepcional. Esse refinamento atingiu um ponto de virada com o célebre teorema de Hironaka sobre a resolução de singularidades para variedades analíticas e algébricas complexas (ver, por exemplo, [Hir62], [GH14, Pg. 621]). No contexto suave (real), o blow-up é análogo: dada uma variedade suave M e uma subvariedade mergulhada fechada $N \subset M$, o blow-up real substitui N pelo fibrado de suas direções normais, frequentemente realizado como o fibrado normal esférico ou projetivo.

Mais recentemente, a técnica de blow-up tem sido aplicada no contexto de folheações riemannianas singulares em [Mol84] e [Ale10], seguindo resultados semelhantes provenientes da teoria de ações de grupos de Lie [DK12, Section 2.9]. Para essa classe de folheações existe uma teoria estrutural robusta, devida principalmente a P. Molino [MC88]. Um de seus principais resultados, conhecido como conjectura de Molino, afirma que os fechos das folhas de tal folheação $\mathcal{F}$ formam outra folheação riemanniana singular $\overline{\mathcal{F}}$; recentemente foi demonstrado que isso é verdadeiro em [AR17]. A teoria de Molino também implica que $\overline{\mathcal{F}}$ é descrita por um feixe localmente constante $\mathcal{C}_{\mathcal{F}}$ de álgebras de Lie de campos vetoriais transversais (veja a Seção 1.7 para mais detalhes). Quando $\mathcal{C}_{\mathcal{F}}$ é suave e globalmente constante, $\mathcal{F}$ é chamada de folheação singular de *Killing*, seguindo a terminologia estabelecida em [Moz85] para o caso regular e em [Car24] para o caso singular.

A primeira aplicação do método de blow-up nesse contexto foi feita pelo próprio Molino, dessingularizando uma folheação riemanniana regular $\mathcal{F}$ ao longo do conjunto Σ constituído pelos estratos mínimos (mais profundos, mais singulares) de $\overline{\mathcal{F}}$, levando a uma folheação blow-up com dinâmica mais simples. Em [Ale10] essa construção foi generalizada para uma folheação riemanniana já singular, resultando em uma folheação regular $\text{Bl}\mathcal{F}$ após uma sequência finita de blow-ups. Ambos os trabalhos focaram na construção de uma métrica em $\text{Bl}M$ que é adaptada à folheação blow-up $\text{Bl}\mathcal{F}$, tornando-a uma folheação riemanniana singular.

Na presente dissertação encontramos novas aplicações do método de blow-up, além de estabelecer propriedades adicionais, com o objetivo de generalizar resultados do caso regular. Por exemplo, generalizamos [Mol84, page 1102] para folheações riemannianas singulares arbitrárias, mostrando que a dinâmica de $\text{Bl}\mathcal{F}$ é descrita pelo blow-up do feixe de Molino (central) $\mathcal{C}_{\mathcal{F}}$ de $\mathcal{F}$; ou, mais precisamente, que tomar blow-ups comuta com

tomar o feixe de Molino:

Theorem A (Teorema 3.1.4). *Seja $\mathcal{F}$ uma folheação riemanniana singular em uma variedade compacta M . Então a aplicação de blow-down induz um isomorfismo*

$$\mathcal{C}_{\text{Bl}\mathcal{F}} \cong \text{Bl}\mathcal{C}_{\mathcal{F}}.$$

Em particular, se $\mathcal{C}_{\mathcal{F}}$ é suave, então $\mathcal{C}_{\text{Bl}\mathcal{F}}$ também é suave.

Segue do Teorema G que, se $\mathcal{F}$ é uma folheação singular de Killing, então $\text{Bl}\mathcal{F}$ é uma folheação de Killing (regular). Isso, por sua vez, nos leva a uma forte restrição dinâmica para folheações singulares de Killing em uma variedade compacta M com $\chi(M) \neq 0$:

Theorem B (Teorema 3.1.9). *Seja $\mathcal{F}$ uma folheação singular de Killing em uma variedade compacta M com $\chi(M) \neq 0$, e sejam $\Sigma^1, \dots, \Sigma^\ell$ os loci mínimos de codimensão par que aparecem no processo de blow-up. Se*

$$\chi(M) > \sum_{j=1}^{\ell} \chi(\Sigma^j),$$

então $\mathcal{F}$ é fechada (todas as folhas de $\mathcal{F}$ são fechadas). Em particular, se todos os estratos singulares de $\mathcal{F}$ têm codimensão ímpar, então $\mathcal{F}$ é fechada.

Também investigamos o comportamento da *cohomologia básica* sob blow-ups, estabelecendo uma relação entre os grupos de cohomologia básica da folheação blow-up $\text{Bl}\mathcal{F}$ e os de $\mathcal{F}$.

Theorem C (Teorema 3.2.4). *Seja $\mathcal{F}$ uma folheação riemanniana singular em uma variedade compacta M , com locus mínimo Σ , e seja $\pi: \text{Bl}_{\Sigma}M \rightarrow M$ a aplicação de blow-up. Então existe um complexo de cochains colapsado*

$$\dots \longrightarrow 0 \longrightarrow \frac{H^i(\text{Bl}_{\Sigma}\mathcal{F})}{\pi^*(H^i(\mathcal{F}))} \longrightarrow \frac{H^i(\text{Bl}_{\Sigma}\mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_{\Sigma}))} \longrightarrow 0 \longrightarrow \dots,$$

que é exato quando $\pi^*: H^i(\mathcal{F}) \rightarrow H^i(\text{Bl}_{\Sigma}\mathcal{F})$ é injetora. Em particular, nesse caso

$$H^i(\text{Bl}_{\Sigma}\mathcal{F}) \cong \pi^*(H^i(\mathcal{F})) \oplus \frac{H^i(\text{Bl}_{\Sigma}\mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_{\Sigma}))}, \quad (1)$$

para todo i , como espaços vetoriais.

Isso fornece uma generalização da situação clássica em que a cohomologia básica coincide com a cohomologia de de Rham, isto é, o caso em que a folheação é trivial, consistindo de pontos:

Corollary D (Corolário 3.2.7). *Para uma variedade compacta orientável M , seja Σ uma subvariedade fechada de M tal que $\text{Bl}_\Sigma M$ é orientável. Então*

$$H_{\text{dR}}^i(\text{Bl}_\Sigma M) \cong H_{\text{dR}}^i(M) \oplus \frac{H_{\text{dR}}^i(E)}{\pi^*(H_{\text{dR}}^i(\Sigma))}$$

para todo i , onde $E = \pi^{-1}(\Sigma)$.

A prova do Teorema I segue essencialmente o caso clássico mencionado [GH14; Git82], com a diferença de que este último depende da construção de uma aplicação de Gysin [Git82, Section 2] via dualidade de Poincaré. Não é trivial nem computacionalmente simples determinar quando a aplicação π^* é injetora. Em geral, a dualidade de Poincaré falha para folheações riemannianas singulares (ver, por exemplo, Seção 1.4.1). Ainda assim, analisamos um caso particular em que a cohomologia do espaço blow-up é, de fato, isomorfa à do espaço original.

Theorem E (Teorema 3.2.9). *Seja $\mathcal{F}$ uma folheação riemanniana singular em uma variedade compacta M , com locus mínimo Σ , e seja $\pi: \text{Bl}_\Sigma M \rightarrow M$ a aplicação de blow-up. Se $(\text{Bl}_\Sigma \mathcal{F})|_E = \pi^*(\mathcal{F}|_\Sigma)$ então,*

$$H^i(\text{Bl}_\Sigma \mathcal{F}) \cong H^i(\mathcal{F}),$$

para todo i , como espaços vetoriais.

Finalmente, combinando [Ale10, Corollary 1.6], que exhibe o espaço das folhas de uma folheação riemanniana singular fechada como o limite de Gromov–Hausdorff de uma sequência de orbifolds (colapsados), com um resultado recente relacionado de [CN25] sobre deformações de folheações de Killing, obtemos o seguinte resultado. Esse resultado generaliza [CN25, Theorem 4.1] para o caso singular e pode ser visto como uma versão “descolapsada” de [Ale10, Theorem 1.5 and Corollary 1.6].

Theorem F (Teorema 3.3.2). *Seja $\mathcal{F}$ uma folheação singular de Killing de codimensão q em uma variedade compacta conexa M . Então existe uma sequência de folheações riemannianas fechadas $\mathcal{G}_i$ de codimensão q em espaços blow-up $(\text{Bl}M)_i$ de M tais que*

$$(\text{Bl}M)_i/\mathcal{G}_i \xrightarrow{\text{GH}} M/\overline{\mathcal{F}}.$$

Em particular, $M/\overline{\mathcal{F}}$ é o limite de Gromov–Hausdorff de uma sequência de orbifolds de dimensão q .

Introduction

The blow-up construction is a central geometric technique when it comes to resolving singularities. Historically, blow-ups were first introduced in algebraic geometry: given a (complex) subvariety $Z \subset X$, the blow-up $\text{Bl}_Z X$ replaces Z by the projectivization of its normal bundle, producing a new variety in which the singular locus is “spread out” along an exceptional divisor. This refinement reached a turning point with Hironaka’s celebrated theorem on the resolution of singularities for complex analytic and algebraic varieties (see, for instance, [Hir62], [GH14, Pg. 621]). In the smooth (real) setting, the blow-up is analogous: given a smooth manifold M and an closed embedded submanifold $N \subset M$, the real blow-up replaces N by the bundle of its normal directions, often realized as the spherical or projective normal bundle.

More recently, the blow-up technique has been applied in the context of singular Riemannian foliations in [Mol84] and [Ale10], following similar results coming from the theory of Lie group actions [DK12, Section 2.9]. For this class of foliations there is a robust structural theory, due mainly to P. Molino [MC88]. One of its main results, known as Molino’s conjecture, states that the leaf closures of such a foliation $\mathcal{F}$ form another singular Riemannian foliation $\overline{\mathcal{F}}$; it was established to hold true recently in [AR17]. Molino’s theory also implies that $\overline{\mathcal{F}}$ is described by a locally constant sheaf $\mathcal{C}_{\mathcal{F}}$ of Lie algebras of transverse vector fields (see Section 1.7 for more details). When $\mathcal{C}_{\mathcal{F}}$ is smooth and globally constant, $\mathcal{F}$ is called a singular *Killing* foliation, following the terminology established in [Moz85] for the regular case and in [Car24] for the singular case.

The first application of the blow-up method in this context was by Molino himself, desingularizing a regular Riemannian foliation $\mathcal{F}$ along the locus Σ consisting of the minimal (deepest, most singular) strata of $\overline{\mathcal{F}}$, leading to a blown-up foliation with simpler dynamics. In [Ale10] this construction was generalized to an already singular Riemannian foliation, resulting in a regular foliation $\text{Bl}\mathcal{F}$ after a finite sequence of blow-ups. Both works focused on the construction of a metric on $\text{Bl}M$ that is adapted to the blow-up foliation $\text{Bl}\mathcal{F}$, turning it into singular Riemannian foliation.

In the present dissertation we find new applications of the blow-up method, also establishing further properties of it, aiming at generalizing results from the regular setting. For instance, we generalize [Mol84, page 1102] to arbitrary singular Riemannian foliations, showing that the dynamics of $\text{Bl}\mathcal{F}$ is described by the blow-up of the Molino (central) sheaf $\mathcal{C}_{\mathcal{F}}$ of $\mathcal{F}$; or, more precisely, that taking blow-ups commute with taking the Molino sheaf:

Theorem G (Theorem 3.1.4). *Let $\mathcal{F}$ be a singular Riemannian foliation on a compact*

manifold M . Then the blow-down map induces an isomorphism

$$\mathcal{C}_{\text{Bl}\mathcal{F}} \cong \text{Bl}\mathcal{C}_{\mathcal{F}}.$$

In particular, if $\mathcal{C}_{\mathcal{F}}$ is smooth, then $\mathcal{C}_{\text{Bl}\mathcal{F}}$ is smooth.

It follows from Theorem G that, if $\mathcal{F}$ is a singular Killing foliation, then $\text{Bl}\mathcal{F}$ is a (regular) Killing foliation. This in turn leads us to a strong dynamics constraint for singular Killing foliations of a compact manifold M with $\chi(M) \neq 0$:

Theorem H (Theorem 3.1.9). *Let $\mathcal{F}$ be a singular Killing foliation on a compact manifold M with $\chi(M) \neq 0$, and let $\Sigma^1, \dots, \Sigma^\ell$ be the even-codimensional minimal loci appearing in the blow-up process. If*

$$\chi(M) > \sum_{j=1}^{\ell} \chi(\Sigma^j),$$

then $\mathcal{F}$ is closed (all the leaves of $\mathcal{F}$ are closed). In particular, if all the singular strata of $\mathcal{F}$ are odd-codimensional, then $\mathcal{F}$ is closed.

We also investigate the behavior of *basic cohomology* under blow-ups, establishing a relationship between the basic cohomology groups of the blow-up foliation $\text{Bl}\mathcal{F}$ and those of $\mathcal{F}$.

Theorem I (Theorem 3.2.4). *Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M , with minimal locus Σ , and let $\pi: \text{Bl}_{\Sigma}M \rightarrow M$ be the blow-up map. Then there exists a collapsed cochain complex*

$$\cdots \longrightarrow 0 \longrightarrow \frac{H^i(\text{Bl}_{\Sigma}\mathcal{F})}{\pi^*(H^i(\mathcal{F}))} \longrightarrow \frac{H^i(\text{Bl}_{\Sigma}\mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_{\Sigma}))} \longrightarrow 0 \longrightarrow \cdots,$$

which is exact when $\pi^*: H^i(\mathcal{F}) \rightarrow H^i(\text{Bl}_{\Sigma}\mathcal{F})$ is injective. In particular, in this case

$$H^i(\text{Bl}_{\Sigma}\mathcal{F}) \cong \pi^*(H^i(\mathcal{F})) \oplus \frac{H^i(\text{Bl}_{\Sigma}\mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_{\Sigma}))}, \quad (2)$$

for every i , as vector spaces.

This provides a generalization of the classical situation in which the basic cohomology coincides with the de Rham cohomology, namely the case where the foliation is trivial, consisting of points:

Corollary J (Corollary 3.2.7). *For a compact orientable manifold M , let Σ be a closed submanifold of M such that $\text{Bl}_{\Sigma}M$ is orientable. Then*

$$H_{\text{dR}}^i(\text{Bl}_{\Sigma}M) \cong H_{\text{dR}}^i(M) \oplus \frac{H_{\text{dR}}^i(E)}{\pi^*(H_{\text{dR}}^i(\Sigma))}$$

for every i , where $E = \pi^{-1}(\Sigma)$.

The proof of Theorem I essentially follows the aforementioned classical case [GH14; Git82], with the difference that the later relies on the construction of a Gysin map [Git82, Section 2] via Poincaré duality. It is neither trivial nor computationally straightforward to determine when the map π^* is injective. In general, Poincaré duality fails for singular Riemannian foliations (see, for instance, Section 1.4.1). Nevertheless, we analyze a particular in which the cohomology of the blown-up space is, in fact, isomorphic to that of the original space.

Theorem K (Theorem 3.2.9). *Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M , with minimal locus Σ , and let $\pi: \text{Bl}_\Sigma M \rightarrow M$ be the blow-up map. If $(\text{Bl}_\Sigma \mathcal{F})|_E = \pi^*(\mathcal{F}|_\Sigma)$ then,*

$$H^i(\text{Bl}_\Sigma \mathcal{F}) \cong H^i(\mathcal{F}),$$

for every i , as vector spaces.

Finally, by combining [Ale10, Corollary 1.6], which exhibits the leaf space of a closed singular Riemannian foliation as the Gromov–Hausdorff limit of a sequence of (collapsed) orbifolds, with a related recent theorem from [CN25] concerning deformations of Killing foliations, we obtain the following result. This result generalizes [CN25, Theorem 4.1] to the singular setting and may be viewed as a “de-collapsed” version of [Ale10, Theorem 1.5 and Corollary 1.6].

Theorem L (Theorem 3.3.2). *Let $\mathcal{F}$ be a singular Killing foliation of codimension q of a connected, compact manifold M . Then there exists a sequence of closed q -codimensional Riemannian foliations $\mathcal{G}_i$ on blow-up spaces $(\text{Bl}M)_i$ of M such that*

$$(\text{Bl}M)_i/\mathcal{G}_i \xrightarrow{\text{GH}} M/\overline{\mathcal{F}}.$$

In particular, $M/\overline{\mathcal{F}}$ is the Gromov–Hausdorff limit of a sequence of q -dimensional orbifolds.

1 Preliminaries

In this chapter, we review the foundational theory of singular Riemannian foliations. We begin by recalling the notions of basic cohomology, the homothetic transformation lemma, and the stratification naturally associated with a foliation. We then introduce Killing foliations and present an overview of Molino's theory in both the regular and singular settings. Throughout this chapter, we assume that the reader is familiar with the basic theory of smooth manifolds and Lie groups, as covered, for instance, in Lee's textbook [Lee03].

1.1 Regular Foliations

Regular Riemannian foliations are relatively well-understood geometric structures that received extensive attention in the 1980s, mainly due to the work of P. Molino [MC88]. His contributions provided a deep understanding of the dynamical behavior of the leaves of a regular Riemannian foliation, culminating in what is now known as Molino theory (see [MM03, Chapter 4] and [MC88, Chapter 5]). In this section, we briefly review the definitions for the regular case (not necessarily Riemannian), which will subsequently be generalized to the singular setting in Section 1.2.

Definition 1.1.1 (Foliated Atlas). Let M be a smooth n -dimensional manifold. A *foliated atlas* of dimension $p \in \mathbb{N}$ for M is a smooth atlas $\{\varphi_i: U_i \rightarrow \mathbb{R}^p \times \mathbb{R}^{n-p}\}_{i \in I}$ for M such that the transition charts are locally of the form $\varphi_{ij}(x, y) = (\varphi_{ij}^1(x, y), \varphi_{ij}^2(y))$, with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$.

The connected components of the sets $\varphi_i^{-1}(\mathbb{R}^p \times \{y\})$ are called *plaques*, which glue together to form immersed connected submanifolds of M that we call *leaves*. This gives a partition $\mathcal{F}$ of M by the leaves of such foliated atlas.

Definition 1.1.2 (Regular Foliation). The partition $\mathcal{F}$ of M into leaves of a foliated atlas is a *regular foliation* of M of dimension p .

We call $\text{codim}(\mathcal{F}) = n - p$ the *codimension* of $\mathcal{F}$. For every $x \in M$, denote L_x as the leaf that passes through x . There are several other equivalent definitions of regular foliations (see, for instance, [MM03, Section 1.2]), for example:

- A smooth involutive distribution Δ of rank p of TM , such that $\Delta_x = T_x L_x$.
- An integrable subbundle $T\mathcal{F} \subset TM$ of rank p , called the *tangent bundle* of $\mathcal{F}$.

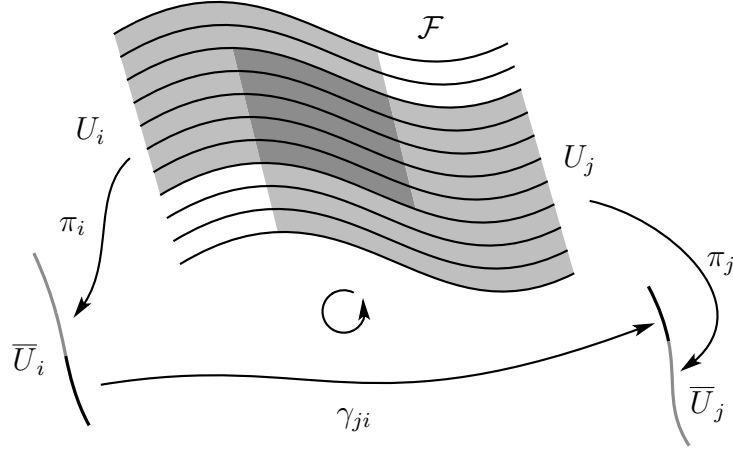


Figure 1.1 – Haefliger cocycle. Source: [CT19].

- A locally trivial differential graded ideal $\mathcal{I} \subset \Omega^*(M)$ of rank q . That is, if $U \subset M$ is an open set on which $\mathcal{I}$ is trivial, then $\mathcal{I}|_U$ is generated by q linearly independent 1-forms $\alpha^1, \dots, \alpha^q$ such that

$$T_x L_x = \bigcap_{i=1}^q \ker(\alpha_x^i)$$

- An open cover $\{U_i\}_{i \in I}$ of M , submersions $\pi_i: U_i \rightarrow \tilde{U}_i$, with $\tilde{U}_i \subset \mathbb{R}^q$ open, and diffeomorphisms $\gamma_{ij}: \pi_j(U_i \cap U_j) \rightarrow \pi_i(U_i \cap U_j)$ satisfying $\gamma_{ij} \circ \pi_j|_{U_i \cap U_j} = \pi_i|_{U_i \cap U_j}$ for all $i, j \in I$. The collection $\{\gamma_{ij}\}$ is a *Haefliger cocycle* representing $\mathcal{F}$ and each U_i is a *simple open set* for $\mathcal{F}$ (see figure 1.1).

Example 1.1.3 (Pullback Foliation). Given a foliation $(M, \mathcal{F})$ and a smooth map $f: N \rightarrow M$ transverse to every leaf of $\mathcal{F}$, we may define the *pullback foliation* $f^*(\mathcal{F})$ on N via the following construction. Take a cocycle $(U_i, \pi_i, \gamma_{ij})$ representing $\mathcal{F}$. Then $f^*(\mathcal{F})$ is represented by the cocycle $(V_i, \pi'_i, \gamma_{ij})$, where $V_i = f^{-1}(U_i)$ and $\pi'_i = \pi_i \circ f|_{V_i}$. In this case the leaves $f^*(\mathcal{F})$ are the preimages of leaves of $\mathcal{F}$. We note that $Tf^*(\mathcal{F}) = df^{-1}(T\mathcal{F})$ and that the codimension is preserved: $\text{codim}(f^*(\mathcal{F})) = \text{codim}(\mathcal{F})$.

Example 1.1.4 (Group Actions). Recall that for a Lie group action $\mu: G \times M \rightarrow M$, the connected components of its orbits $G(x)$ provide a partition of M into immersed submanifolds given by the connected components of the image of the injective immersion $G/G_x \rightarrow M$, where G_x is the *stabilizer* of x (check, for instance, [AB+15, Proposition 3.14]). If $x \mapsto \dim(G_x)$ is constant, it provides a regular foliation. Indeed, note that as $T_x G(x) = d(\mu_x)_e(\mathfrak{g})$, the action fields $V^* \in \mathfrak{X}$ generates $T\mathcal{F}$, showing that this is an involutive distribution.

A standard example is the *Kronecker foliation*, given as follows. Consider the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For each $\lambda \in (0, +\infty)$, we have a smooth $\mathbb{R}$ -action

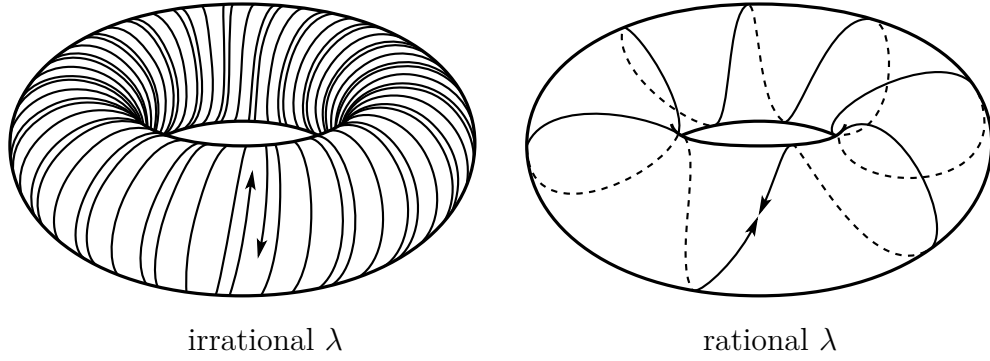


Figure 1.2 – Kronecker foliation. Source: [CT19].

$$\begin{aligned}
 \mathbb{R} \times \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\
 (t, [x, y]) &\longmapsto [x + t, y + \lambda t]
 \end{aligned}$$

with $\dim(\mathbb{R}_{[x,y]}) \equiv 0$. Observe that when λ is irrational each leaf is dense in $\mathbb{T}^2$, while a rational λ yields closed leaves (see figure 1.2).

1.2 Singular Riemannian Foliations

In the past two decades, the study of singular Riemannian foliations (a generalization of Molino’s framework to the singular setting) has developed substantially. Building upon Molino’s foundational ideas, recent research has yielded new insights into fundamental questions in the theory of isometric group actions, as well as significant progress on long-standing conjectures and extensions of classical results in the area (see, for instance, [AT08; AR17; Töb07; MR19]). In this section we present the basic and fundamental theory of singular Riemannian foliations, presenting the results of regular *Riemannian* foliations as a particular case.

Definition 1.2.1 (Singular Riemannian Foliation). Given (M, g) a Riemannian manifold and $\mathcal{F}$ a partition of M into immersed, connected submanifolds, called *leaves*. Let $\mathfrak{X}(\mathcal{F})$ denote the submodule of $\mathfrak{X}(M)$ consisting of vector fields which are tangent to the leaves of $\mathcal{F}$. We say that $\mathcal{F}$ is a *singular Riemannian foliation* if

1. It is a *singular foliation*; that is, for every $v \in T_x \mathcal{F}$, there exists a tangent field $X \in \mathfrak{X}(\mathcal{F})$ such that $X(x) = v$. This means that the module $\mathfrak{X}(\mathcal{F})$ is transitive.
2. It is a *transnormal system* (in the terminology of Bolton [Bol73]); that is, every geodesic that is perpendicular to a leaf in some point remains perpendicular to all the leaves it meets. In this case g is said to be *adapted* to $\mathcal{F}$.

Similar to the regular case, given $x \in M$, we will denote the leaf containing it by

L_x . Moreover, we define the *dimension* of $\mathcal{F}$ as

$$\dim(\mathcal{F}) = \max_{L \in \mathcal{F}} \dim(L).$$

and its *codimension* by $\text{codim}(\mathcal{F}) = \dim(M) - \dim(\mathcal{F})$. Let r be the number of dimensions occurring among the leaves of $\mathcal{F}$. The *depth* of $\mathcal{F}$ is $\text{depth}(\mathcal{F}) := r - 1$.

We say that $\mathcal{F}$ is a *regular* Riemannian foliation when $\dim(L)$ is constant for every $L \in \mathcal{F}$, that is, when $\text{depth}(\mathcal{F}) = 0$. In the regular case one also says that an adapted metric g is *bundle-like* for $\mathcal{F}$ ¹. In that case, it coincides with the classical notion of a regular foliation (Definition 1.1.2), since the underlying module $\mathfrak{X}(\mathcal{F})$ integrates to a smooth, involutive distribution $T\mathcal{F} \subset TM$, due to the transitive condition of the module $\mathfrak{X}(\mathcal{F})$.

For the regular case, the bundle-like metric g has the following property: for any open set U and any $Y, Z \in \mathfrak{L}(\mathcal{F}|_U)$ perpendicular to the leaves, we have $g(Y, Z) \in \Omega^0(\mathcal{F}|_U)$ [MC88, Proposition 6.1].

Definition 1.2.2 (Transverse Metric). For $\mathcal{F}$ a Riemannian foliation, we say the metric g is *transverse* for the foliation $\mathcal{F}$ if it induces a symmetric, basic $(2, 0)$ -tensor field g^T on M satisfying $g^T(v, v) > 0$ for every v not tangent to $\mathcal{F}$.

Proposition 1.2.3. For a Riemannian foliation $\mathcal{F}$, the metric g induces a transverse metric for $\mathcal{F}$.

Proof. Define

$$g^T(X, Y) := g(X^\perp, Y^\top)$$

where $X, Y \in \mathfrak{X}(M)$ and $X = X^\perp + X^\top$ with respect to the decomposition $TM = T\mathcal{F} \oplus T\mathcal{F}^\perp$ given by the metric g . By the previous property of the bundle-like metric g , it defines a transverse metric for $\mathcal{F}$. $\square$

The converse is also true: any transverse metric can be completed to a bundle-like metric [MC88, Proposition 3.3].

Remark 1.2.4. Let $\mathcal{F}$ denote a singular Riemannian foliation on a Riemannian manifold (M, g) . When needed, we explicitly write $(M, g, \mathcal{F})$. If $\mathcal{F}$ is only a singular foliation, we sometimes write $(M, \mathcal{F})$. When $\mathcal{F}$ is a *regular* Riemannian foliation, we simply call it a Riemannian foliation.

Definition 1.2.5 (Complete Singular Riemmanian Foliation). We say $(M, \mathcal{F}, g)$ is a *complete* singular Riemannian foliation when (M, g) is a complete Riemannian manifold.

¹In the regular setting there exists a definition of *bundle-like* metric which is equivalent to the transverse system of Bolton (check, for instance, [MC88, Proposition 3.5 and Proposition 6.1]).

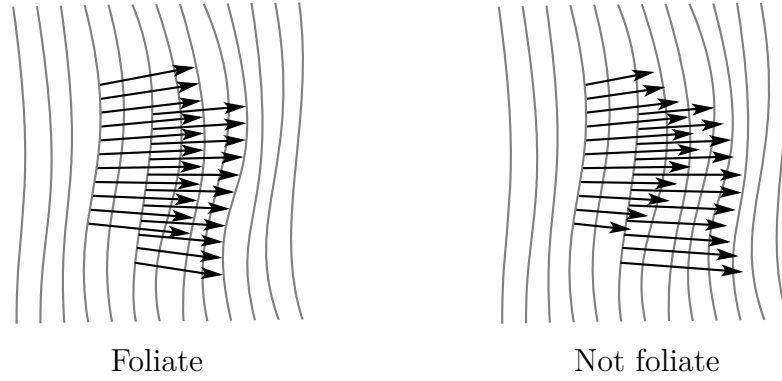


Figure 1.3 – Foliate fields. Source: [CT19].

Definition 1.2.6 (Foliate Map). Suppose $(N, \mathcal{G})$ is a singular foliation. A smooth map $f: (M, \mathcal{F}) \rightarrow (N, \mathcal{G})$ is called a *foliate map* if it maps the leaves of $\mathcal{F}$ into leaves of $\mathcal{G}$.

We denote by $C^\infty(\mathcal{F}) = \Omega^0(\mathcal{F})$ the ring of foliate functions $f: (M, \mathcal{F}) \rightarrow \mathbb{R}$ (in which $\mathbb{R}$ is endowed with the trivial foliation by points), whose elements are called *basic functions*.

Definition 1.2.7 (Foliate Fields). A vector field $X \in \mathfrak{X}(M)$ is called *foliate* when $[X, Y] \in \mathfrak{X}(\mathcal{F})$ for every $Y \in \mathfrak{X}(\mathcal{F})$.

These are the vector fields whose flows are foliate maps [MC88, Proposition 2.2] (see figure 1.3). We denote the Lie algebra of foliate fields by $\mathfrak{L}(\mathcal{F})$.

Definition 1.2.8 (Transverse Fields). The *transverse vector fields* are the elements in the quotient Lie algebra $\mathfrak{l}(\mathcal{F}) := \mathfrak{L}(\mathcal{F})/\mathfrak{X}(\mathcal{F})$.

These hence combine into the exact sequence of $\Omega^0(\mathcal{F})$ -modules

$$0 \longrightarrow \mathfrak{X}(\mathcal{F}) \longrightarrow \mathfrak{L}(\mathcal{F}) \longrightarrow \mathfrak{l}(\mathcal{F}) \longrightarrow 0.$$

Example 1.2.9 (Product Foliation). Let $(M, g, \mathcal{F})$ and $(N, \eta, \mathcal{G})$ be singular Riemannian foliations. The partition $\mathcal{F} \times \mathcal{G}$ on $(M \times N, g + \eta)$ defined by $L_{(x,y)} := L_x \times L'_y$, where $L_x \in \mathcal{F}$ and $L'_y \in \mathcal{G}$ is a singular Riemannian foliation.

As we saw in example 1.1.4, group actions are naturally generalized by the foliation theory. In particular, the same goes for isometric group actions:

Example 1.2.10 (Homogeneous Singular Riemannian Foliations). Let G be a Lie group acting isometrically on a Riemannian manifold (M, g) . Moreover, it is a singular Riemannian foliation, because g is such that $\mathcal{F}_G$ is a transnormal system [MC88, Section 6.1].

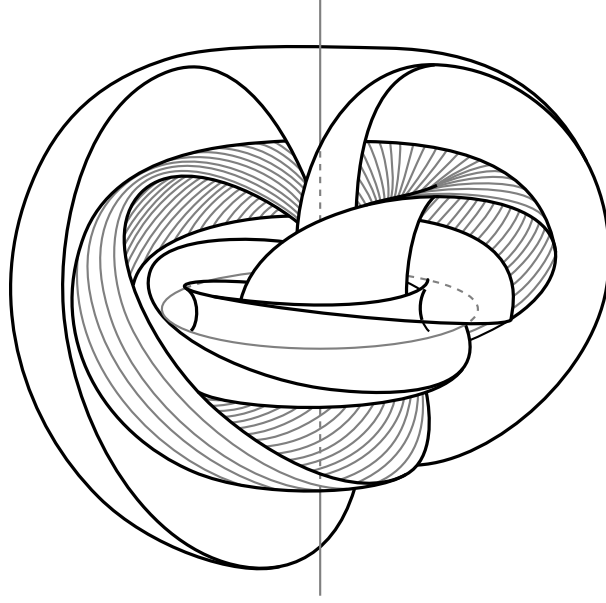


Figure 1.4 – Generalized Hopf Fibration via stereographic projection. Source: [CT19].

Example 1.2.11 (Generalized Hopf Fibration). Consider the action of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ on $\mathbb{S}^3$ by

$$(t_0, t_1) \cdot (z_0, z_1) = (t_0 z_0, t_1 z_1)$$

which has two singular orbits, namely $\mathbb{T}^2(1, 0)$ and $\mathbb{T}^2(0, 1)$, both diffeomorphic to $\mathbb{S}^1$. The other are tori and coincide with the distance tubes of the two singular orbits. Note that the 1-dimensional Lie subalgebras of $\mathbb{R}^2 \cong \text{lie}(\mathbb{T}^2)$ also acts on $\mathbb{S}^3$ and integrates to 1-dimensional Riemannian foliations of $\mathbb{S}^3$. In this example, such foliations are Kronecker foliations of the regular $\mathbb{T}^2$ -orbits (see figure 1.4).

1.3 Holonomy

This section reviews the basic definitions and notation related to the holonomy pseudogroup of a *regular* foliation. Although results concerning holonomy in the singular setting are available (see, for instance, [Ale04]), the present work is restricted to the regular case, which is sufficient for the purposes considered here and therefore the only one addressed. A thorough presentation may be found in [CC00, Chapter 2].

Definition 1.3.1 (Pseudogroups). Let S be a smooth manifold. A *pseudogroup* $\mathcal{H}$ of local diffeomorphisms of S consists of a set of diffeomorphisms $h: U \rightarrow V$, where U and V are open subsets of S , such that:

1. $\text{Id}_U \in \mathcal{H}$ for any open set $U \subset S$,
2. if $h \in \mathcal{H}$ then $h^{-1} \in \mathcal{H}$,

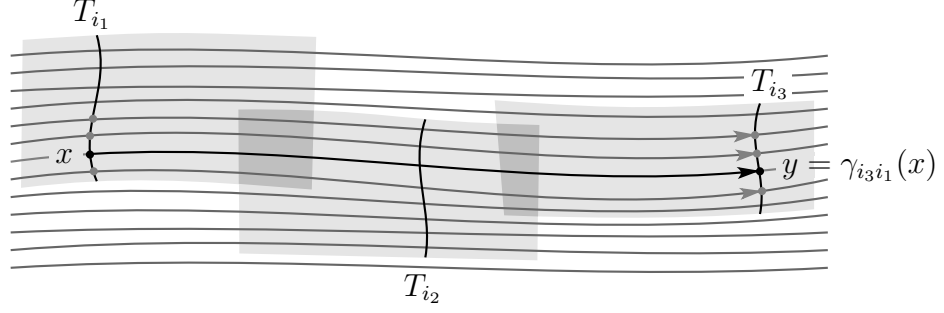


Figure 1.5 – Sliding along the leaves. Source: [CT19].

3. if $h_1: U_1 \rightarrow V_1$ and $h_2: U_2 \rightarrow V_2$ are in $\mathcal{H}$, then their composition

$$h_2 \circ h_1: h_1^{-1}(V_1 \cap U_2) \longrightarrow h_2(V_1 \cap U_2)$$

also belongs to $\mathcal{H}$, and

4. if $U \subset S$ is open and $k: U \rightarrow V$ is a diffeomorphism such that U admits an open cover $\{U_i\}$ with $k|_{U_i} \in \mathcal{H}$ for all i , then $k \in \mathcal{H}$.

The $\mathcal{H}$ -orbit of $x \in S$ consists of the points $y \in S$ for which there exists some $h \in \mathcal{H}$ satisfying $h(x) = y$. The quotient by the corresponding equivalence relation, endowed with the quotient topology, is the *space of orbits* of $\mathcal{H}$, which we denote by $S/\mathcal{H}$.

Let $\mathcal{F}$ be a regular foliation represented by the cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$. The pseudogroup of local diffeomorphisms generated by $\gamma = \{\gamma_{ij}\}$ acting on

$$T_\gamma := \bigsqcup_i U_i$$

is the *holonomy pseudogroup* of $\mathcal{F}$ associated to γ , which we denote by $\mathcal{H}_\gamma$. If δ is another Haefliger cocycle defining $\mathcal{F}$, then $\mathcal{H}_\delta$ is equivalent to $\mathcal{H}_\gamma$, so we can define, up to equivalence, the holonomy pseudogroup of $\mathcal{F}$, which can be denoted by $\mathcal{H}(\mathcal{F})$. We write $(T(\mathcal{F}), \mathcal{H}(\mathcal{F}))$ to denote both this equivalence class and a specific representative, since this rarely leads to confusion. It is clear that $T(\mathcal{F})/\mathcal{H}(\mathcal{F})$ is precisely the leaf space $M/\mathcal{F}$ of $\mathcal{F}$ endowed with the quotient topology.

If $L_x = L_y$, choose a path $c: [0, 1] \rightarrow L$ joining x to y . Fix a cocycle $\{(U_i, \pi_i, \gamma_{ij})\}$ representing $\mathcal{F}$ and a subdivision $0 = t_1 < \dots < t_{m+1} = 1$ such that $c([t_k, t_{k+1}]) \subset U_{i_k}$ for some U_{i_k} . Then there exists a diffeomorphism

$$\gamma_{i_m i_{m-1}} \circ \gamma_{i_{m-1} i_{m-2}} \circ \dots \circ \gamma_{i_2 i_1} = \gamma_{i_m i_1}$$

between sufficiently small neighborhoods of $\bar{x} = \pi_1(x)$ and $\bar{y} = \pi_m(y)$. If we identify $T_\mathcal{F}$ with a total transversal $\bigsqcup_i T_i$ for $\mathcal{F}$ containing x and y , this corresponds to the classical notion of *sliding along the leaves* [MC88, Section 1.7] (see Figure 1.5). Let us denote the

germ of $\gamma_{i_m i_1}$ at $\bar{x}$ by h_c . This germ depends only on the $[0, 1]$ -relative homotopy class of c (see [CC00, Proposition 2.3.2]). In particular, if we consider the holonomy group of L at x , that is,

$$\text{Hol}_x(L) := \{h_c \mid c: [0, 1] \rightarrow L \text{ is a loop}\},$$

we obtain a surjective homomorphism

$$h: \pi_1(L, x) \twoheadrightarrow \text{Hol}_x(L).$$

Since the isomorphism class of $\text{Hol}_x(L)$ does not depend on x , we often omit x from the notation. In particular, we say that L is a *leaf without holonomy* (or a *generic leaf*) when $\text{Hol}(L) = 0$. It follows immediately from the surjectivity of h that simply connected leaves are without holonomy. Moreover, it can be shown that leaves without holonomy are generic, in the sense that

$$\{x \in M \mid \text{Hol}(L_x) = 0\}$$

is residual in M (see [CC00, Theorem 2.3.12]).

1.4 Basic Cohomology

Definition 1.4.1 (Foliate Homotopy). Let $(M, \mathcal{F})$ be a singular foliation. A smooth map $f: M \times [0, 1] \rightarrow N$ is a *foliate homotopy* when, for every $t \in [0, 1]$, the maps

$$\begin{aligned} f_t: M &\longrightarrow M \\ x &\longmapsto f(x, t) \end{aligned}$$

are foliate.

If $V \subset M$ is a saturated submanifold (that is, $V = \pi^{-1}(\pi(V))$ for the canonical projection $\pi: M \rightarrow M/\mathcal{F}$), then a *foliate deformation retraction* $f: M \times [0, 1] \rightarrow M$ onto V is a foliate homotopy which is also a deformation retraction of M onto V .

Definition 1.4.2 (Basic Forms). A differential form $\omega \in \Omega^k(M)$ is called *basic* if, for every $X \in \mathfrak{X}(\mathcal{F})$, it satisfies $\iota_X \omega = 0$ and $\mathcal{L}_X \omega = 0$.

The space of basic k -forms is denoted by $\Omega^k(\mathcal{F})$. By Cartan's magic formula

$$\mathcal{L}_X = \iota_X d + d\iota_X,$$

a form $\omega \in \Omega^k(M)$ is basic if and only if $\iota_X \omega = 0$ and $\iota_X(d\omega) = 0$, for all $X \in \mathfrak{X}(\mathcal{F})$. Let

$$\Omega(\mathcal{F}) := \bigoplus_k \Omega^k(\mathcal{F})$$

denote the *algebra of basic forms of $\mathcal{F}$* . This is a $\mathbb{Z}$ -graded differential algebra (defining $\Omega^k(\mathcal{F}) = \{0\}$ for $k < 0$), since the exterior differential d preserves basic forms. Therefore we can compute the cohomology groups of the complex of basic forms

$$\dots \xrightarrow{d} \Omega^{k-1}(\mathcal{F}) \xrightarrow{d} \Omega^k(\mathcal{F}) \xrightarrow{d} \Omega^{k+1}(\mathcal{F}) \xrightarrow{d} \dots$$

which we call the *basic cohomology groups of $\mathcal{F}$* , denoted by $H^k(\mathcal{F})$. We have $H(\mathcal{F}) = \bigoplus H^k(\mathcal{F})$, which is a graded algebra with the usual exterior product. Notice that when $\mathcal{F}$ is the trivial foliation by points, $H(\mathcal{F})$ reduces to the de Rham cohomology of M . In general, we can intuitively think of $H(\mathcal{F})$ as the de Rham cohomology of $M/\mathcal{F}$, as one usually does in basic cohomology for group actions. If $f: (M, \mathcal{F}) \rightarrow (N, \mathcal{G})$ is foliated, then the pullback of a $\mathcal{G}$ -basic form is an $\mathcal{F}$ -basic form, and thus there is an induced linear map

$$f^*: H(\mathcal{G}) \longrightarrow H(\mathcal{F}).$$

For $\mathcal{F}$ -saturated open sets $U, V \subset M$, the short exact sequence

$$0 \longrightarrow \Omega(\mathcal{F}|_{U \cup V}) \xrightarrow{(i_U^*, i_V^*)} \Omega(\mathcal{F}|_U) \oplus \Omega(\mathcal{F}|_V) \xrightarrow{i_U^* - i_V^*} \Omega(\mathcal{F}|_{U \cap V}) \longrightarrow 0$$

induces a Mayer–Vietoris sequence in basic cohomology (see [Car24, Section 2.1]), where $i_U: U \rightarrow U \cup V$, $i_V: V \rightarrow U \cup V$, $j_U: U \cap V \rightarrow U$ and $j_V: U \cap V \rightarrow V$ are the natural inclusions.

The classical homotopy invariance of de Rham cohomology (see, for instance, [Lee03, Theorem 17.11]) admits a natural extension to the foliated setting. This extension was first established for regular foliations [Töb14, Corollary 2.2] and has been obtained more recently in the singular case:

Proposition 1.4.3 ([Car24, Proposition 2.4]). If there exists a foliated homotopy $f: M \times [0, 1] \rightarrow N$ between two singularly foliated manifolds $(M, \mathcal{F})$ and $(N, \mathcal{G})$, then $f_0^* = f_1^*: H^i(\mathcal{G}) \rightarrow H^i(\mathcal{F})$.

Definition 1.4.4 (Basic Homotopic Equivalence). We say that $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are *homotopy equivalent* when there exist foliated maps $f: M \rightarrow N$ and $g: N \rightarrow M$ such that $g \circ f \cong \text{Id}_M$ and $f \circ g \cong \text{Id}_N$ by foliated homotopies.

Corollary 1.4.5 ([Car24, Corollary 2.5]). If $\mathcal{F}$ and $\mathcal{G}$ are homotopy equivalent singular foliations, then $H(\mathcal{F}) \cong H(\mathcal{G})$. In particular, if $h: (M, \mathcal{F}) \rightarrow (M, \mathcal{F})$ is a foliate deformation retraction onto a saturated submanifold $V \subset M$, then $H(\mathcal{F}) \cong H(\mathcal{F}|_V)$.

1.4.1 Poincaré Duality

As basic cohomology models the cohomology of the leaf space $M/\mathcal{F}$ whenever this quotient is a smooth manifold, it is natural to ask under which conditions *Poincaré duality*

holds for a regular foliation $\mathcal{F}$ of codimension q , namely, when one has an isomorphism

$$H^i(\mathcal{F}) \cong H^{q-i}(\mathcal{F}).$$

This question has been settled in the regular case for the class of *taut foliations*. Recall that a Riemannian foliation $\mathcal{F}$ is said to be *taut* if M admits a Riemannian metric g for which all leaves of $\mathcal{F}$ are minimal submanifolds². In fact, for a transversely orientable Riemannian foliation $\mathcal{F}$ on a closed manifold M (that is, such that the normal bundle $\nu\mathcal{F} := TM/T\mathcal{F}$ is orientable) the following conditions are equivalent (see [KH86; KT06; Mas92; Dom98]):

1. $\mathcal{F}$ is taut;
2. $H^q(\mathcal{F}) \neq 0$;
3. $H^i(\mathcal{F}) \cong H^{q-i}(\mathcal{F})$ for all i .

However, these equivalences do not extend naturally to the singular setting. Indeed, as shown by Miquel and Wolak [MW06, Theorem 1.1], tautness fails in general for singular Riemannian foliations.

Theorem 1.4.6 (Miquel–Wolak). Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M . Then there exists no Riemannian metric adapted to $\mathcal{F}$ for which all leaves of $\mathcal{F}$ are minimal.

Moreover, it was presented in [PSW05, Example 2.4] that even in the case of flows, replacing the tautness condition by that of being isometric (which is equivalent to tautness in the regular Riemannian flow setting) does not ensure the validity of Poincaré duality:

Example 1.4.7. Consider the isometric action $\varphi: \mathbb{R} \times \mathbb{S}^{2d+2} \rightarrow \mathbb{S}^{2d+2}$ given by

$$\varphi(t, (z_0, \dots, z_d, x)) = (e^{a_0\pi it} \cdot z_0, \dots, e^{a_d\pi it} \cdot z_d, x),$$

with $(a_0, \dots, a_d) \neq (0, \dots, 0)$. We are considering $\mathbb{S}^{2d+2} := \{(z_0, \dots, z_d) \mid |z_0|^2 + \dots + |z_d|^2 + x^2 = 1\}$. Notice that there exist two singular strata, namely, the north pole $N = (0, \dots, 0, 1)$ and the south pole $S = (0, \dots, 0, -1)$. The regular stratum Σ_{reg} is homeomorphic to $\mathbb{S}^{2d+1} \times (-1, 1)$. The authors compute $H(\mathbb{S}^{2d+2}/\mathcal{F}) \cong H(\mathcal{F})$, using intersection cohomology, developed in [GM80], theory used to compute cohomology of stratified pseudomanifolds (check [GM80] for a precise definition). Using the techniques developed in [HS93] for $\mathbb{S}^1$ -actions, they obtained $H(\mathbb{S}^{2d+2}/\mathcal{F})$:

$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = \dots$	$i = 2d$	$i = 2d + 1$
1	0	0	$[dr \wedge e]$	0	$[dr \wedge e^2]$	$\dots$	0	$[dr \wedge e^d]$

²In fact, this property can be achieved with a bundle-like metric; see [KT06, Corollary 2.31].

Here $e \in \Omega_{\mathbb{Z}}^2(\mathbb{S}^{2d+2}/\mathcal{F})$ denotes an Euler form (see [HS93])³ and $r: \Sigma_{\text{reg}} \rightarrow \mathbb{R}$ is a smooth map such that $r \equiv 0$ on $(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ and $\int_0^1 r = 1$.

Notice that, for the particular case where $d = 2$, Poincaré duality does not hold.

To address this issue, the authors introduced a cohomological model that incorporates the stratification induced by the singular foliation $\mathcal{F}$ (see Section 1.4.2), known as *basic intersection cohomology*. This theory provides a more suitable framework in the singular context, as it restores Poincaré duality under appropriate conditions [PSW05, Example 2.4].

1.4.2 Homothetic Transformation Lemma and Stratification

Let $L \in \mathcal{F}$ be a leaf of a complete singular Riemannian foliation on M . Consider a connected, relatively compact, open subset $P \subset L$, and let $U := \text{Tub}_\varepsilon(P)$ be its tubular neighborhood of radius $\varepsilon > 0$. Explicitly, U is the image of $B_\varepsilon^\nu P := \{v \in \nu P \mid \|v\| < \varepsilon\}$ under the normal exponential map $\exp^\perp: \nu L \rightarrow M$, where ε is chosen small enough so that the restriction $\exp^\perp|_{B_\varepsilon^\nu P}: B_\varepsilon^\nu P \rightarrow U$ is a diffeomorphism. The orthogonal projection onto P is denoted by $\pi_P: U \rightarrow P$. By possibly shrinking P and decreasing ε , we may assume that U is a *distinguished tubular neighborhood*, defined as follows:

Definition 1.4.8 (Distinguished Tubular Neighborhood). In the above setting, U is called a *distinguished tubular neighborhood* if it additionally satisfies: For each $y \in U$ with $x = \pi_P(y)$, the leaf L_y is transverse to the slice

$$S_x := \pi_P^{-1}(x) = \exp_x^\perp(B_\varepsilon(0)).$$

The connected component of $L_y \cap U$ that contains y is called the *plaque* through y and is denoted by P_y .

Definition 1.4.9 (Slice Foliation). Let U be a distinguished tubular neighborhood. For each $x \in P$, the *slice foliation* at x is the restriction of $\mathcal{F}$ to the slice S_x , defined by the intersections of the plaques P_y (for $y \in U$) with S_x .

Lemma 1.4.10 ([MC88, Lemma 6.2]). Let $\mathcal{F}$ be a singular Riemannian foliation and $U = \text{Tub}_\varepsilon(P)$ be a distinguished tubular neighborhood. If $\varepsilon_1, \varepsilon_2 < \varepsilon$ and for $\lambda > 0$ such that $\varepsilon_2 = \lambda \varepsilon_1$, then the homothetic transformation

$$\begin{aligned} h_\lambda: \text{Tub}_{\varepsilon_1}(P) &\longrightarrow \text{Tub}_{\varepsilon_2}(P) \\ \exp(x) &\longmapsto \exp(\lambda x) \end{aligned}$$

is foliated with respect to $\mathcal{F}|_U$.

³The symbol $\mathbb{Z}$ indicates the *perversity* of the form, a fundamental notion in intersection homology theory; we refer to [GM80] for a detailed discussion.

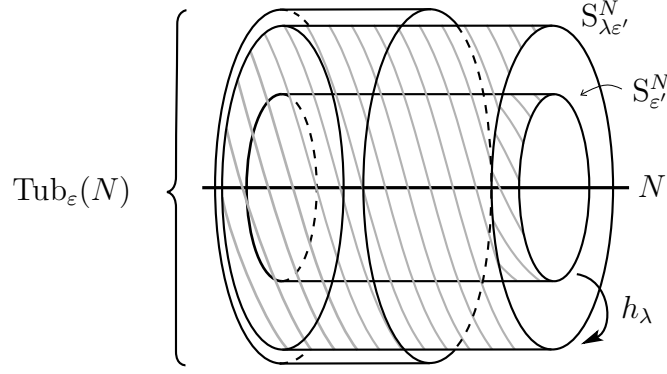


Figure 1.6 – Homothetic transformation. Source: [CT19].

We say that the subset Σ^r consisting of all leaves of $\mathcal{F}$ of dimension r is the *locus* of r -dimensional leaves. A fundamental consequence of Lemma 1.4.10 is that every connected component Σ_α^r of it is an embedded submanifold of M [MC88, Proposition 6.3]. Moreover, these components are the *strata* of a stratification

$$M = \bigsqcup_{r, \alpha} \Sigma_\alpha^r,$$

of M . We will often omit the super-index r when it is not relevant. By construction, for each α , the restriction $\mathcal{F}_\alpha := \mathcal{F}|_{\Sigma_\alpha}$ forms a regular foliation.

Definition 1.4.11 (Regular and Singular Locus). The locus of leaves of maximal dimension $\Sigma_{\dim \mathcal{F}}$ is the *regular locus* of $\mathcal{F}$, that we will denote by Σ_{reg} . Its complement is the *singular locus* Σ_{sing} .

Proposition 1.4.12 ([MC88, Section 6.2]). With the notation above:

1. Every geodesic which is perpendicular to a leaf $L \in \Sigma_\alpha$ in a stratum Σ_α and tangent to Σ_α remains within Σ_α for some positive time, which implies that it is a geodesic of Σ_α with respect to the restriction of g in Σ_α .
2. $g|_{\Sigma_\alpha}$ turns $\mathcal{F}_\alpha$ into a regular Riemannian foliation.
3. If $L \subset \Sigma_\alpha$, then $\bar{L} \subset \Sigma_\alpha$.
4. All the singular strata have codimension at least 2, so Σ_{reg} is an open, dense submanifold of M .
5. Lemma 1.4.10 extends, *mutatis mutandis*, to the situation in which P is replaced by a saturated closed submanifold $n \subset M$ that is contained in a stratum (see Figure 1.6).

We refer to the singular locus of leaves with minimal dimension as the *minimal locus*. It is usually denoted by Σ_{min} , but we will favor the cleaner notation Σ whenever

this doesn't lead to confusion, since this object will be recurrently used throughout the work. We notice that it is a compact submanifold of M , due to the inferior semi-continuity of the function that associates a point to the dimension of the leaf containing it.

1.5 Foliation on Spheres

In this section we follow [AT08], which provides some technical results that are important for the construction of the subsequent sections (see Section 2.3).

Definition 1.5.1 (Transverse Vectors and Curves). Let $\mathcal{F}$ be a singular Riemannian foliation. A vector $v \in T_x M$ is said to be *transverse* if $v \in \nu_x(L_x)$. A smooth curve $\gamma: I \rightarrow M$ is said to be *transverse* if $\gamma'(t)$ is transverse for every $t \in I$.

Theorem 1.5.2 (Singular Riemannian Foliations on Spheres). Let $\mathcal{F}$ be a singular Riemannian foliation on $\mathbb{R}^n$ containing the origin as a leaf. Then the restriction of $\mathcal{F}$ to the unit sphere $\mathbb{S}^{n-1}$ is a singular Riemannian foliation.

Proof. Notice first that every leaf is contained in a distance sphere, because the leaves are equidistant (transnormal system condition) and, in particular, that the unit sphere is given by a union of leaves. Also, it is clear that $(\mathbb{S}^{n-1}, \mathcal{F}|_{\mathbb{S}^{n-1}})$ is a singular foliation. If $\mathbb{S}^{n-1}$ consists of one leaf, the result follows trivially. Let us assume that there is a $p \in \mathbb{S}^{n-1}$ such that L_p is a proper submanifold of $\mathbb{S}^{n-1}$. Let now $\gamma(t)$ be a geodesic in $\mathbb{S}^{n-1}$ with $\gamma(0) = p$ and $\gamma'(0) = v$. We want to show that $\gamma(t)$ is perpendicular to all the leaves it meets, and it is enough to check that for a point close enough to p , say $\gamma(t) = q$. Consider in $\mathbb{R}^n$ the orthogonal projection $\pi_p(q)$ of q on $\text{span}\{p\}$. Then $\pi_p(q) = \lambda p$ for some positive λ , and the line from $\pi_p(q)$ to q is given by $\pi_p(q) + tv$. This is a geodesic and $h_\lambda(p + tv) = \lambda p + tv = \pi_p(q) + tv$, where h_λ is a homothetic transformation with respect to the origin. It follows that the curve $\pi_p(q) + tv$ is transverse to the foliation $\mathcal{F}|_{\mathbb{S}^{n-1}}$. In particular, the vector $v \in T_q \mathbb{R}^n$ is also transverse to $\mathcal{F}$ and so is the radial vector q . These two vectors thus span a 2-plane at q , and the intersection of this 2-plane with $\mathbb{S}^{n-1}$ is precisely γ . It follows that $\gamma'(t) \in T_q \mathbb{R}^n$ lies in this 2-plane, and therefore it is a transverse vector in $\mathbb{R}^n$ (and thus in $\mathbb{S}^{n-1}$ as well). $\square$

Proposition 1.5.3 (Antipodal Map is Foliate). Let $(\mathbb{S}^n, \mathcal{F})$ be a singular Riemannian foliation. Then the antipodal map $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ given by $z \mapsto -z$, is foliate.

Proof. The foliation is uniquely determined by the (singular) distribution $p \mapsto \nu_p(L_p)$, so it is enough to check that $df(\nu_p(L_p)) = \nu_{f(p)}(L_{f(p)})$. For any $x \in \nu_p(L_p)$, let γ be the unit speed transverse geodesic such that $\gamma(0) = p$, $\gamma'(0) = \frac{v}{|v|}$. Then $\gamma(\pi) = f(p) = -p$, and $\gamma'(\pi) = df(x)$. Since γ is transverse, $\gamma'(\pi) \in \nu_{f(p)}(L_{f(p)})$, and therefore $df(\nu_p(L_p)) \subset \nu_{f(p)}(L_{f(p)})$. The same argument holds for its inverse f^{-1} , which is f (f is involutive), and the other inclusion follows. $\square$

Corollary 1.5.4. Suppose $(\mathbb{R}^n, \mathcal{F})$ is a singular Riemannian foliation with $0 \in \mathcal{F}$. Then for any $\lambda \in \mathbb{R}$, the map $h_\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $h_\lambda(p) = \lambda p$ is foliate.

Proof. The case of $\lambda \geq 0$ is a special case of the Homothetic Transformation Lemma 3.2.1. Since for $\lambda < 0$ we can write $h_\lambda = h_{-1} \circ h_{|\lambda|}$, it is enough to check the proposition for $\lambda = -1$. However, for any $L \in \mathbb{S}^{n-1}$, the map $(h_\lambda)|_L$ decomposes as

$$L \xrightarrow{\cong} S^{n-1} \xrightarrow{a} S^{n-1} \xrightarrow{h_r} \mathbb{R}^n,$$

which is a composition of foliate maps, by Lemma 1.4.10 and 1.5.3. Therefore, h_λ is foliate. $\square$

1.6 Sheaf Theory

In this section, we briefly review the basic notions of sheaf theory that will be needed in the sequel. The purpose of this interlude is not to develop the theory in full generality, but rather to introduce the language and tools required to formulate and state the results of Section 1.7 in a more precise and transparent manner. The presentation follows [AC22, Section 3.2]; we refer to [Bre97] for a more deep and detailed presentation.

Definition 1.6.1 (Presheaf). Let (X, τ) be a topological space. A *presheaf* $\mathcal{P}$ on X consists of:

1. For each open set $U \in \tau$, a set $\mathcal{P}(U)$ (whose elements are called *sections over U*),
 2. For each inclusion $V \subset U$ of open sets, a *restriction map* $\text{res}_V^U: \mathcal{P}(U) \rightarrow \mathcal{P}(V)$,
- satisfying $\text{res}_U^U = \text{id}$ and $\text{res}_W^U = \text{res}_W^V \circ \text{res}_V^U$ whenever $W \subset V \subset U$.

If each $\mathcal{P}(U)$ carries additional algebraic structure (group, ring, Lie algebra, etc.) and the restriction maps preserve this structure, we speak of a *presheaf of groups, rings, Lie algebras, etc.*

Example 1.6.2 (Smooth functions). Let M be a smooth manifold. The assignment $U \mapsto C^\infty(U) = \{f: U \rightarrow \mathbb{R} \text{ smooth}\}$, with the usual restriction of functions, is a presheaf of rings, denoted by $\mathcal{C}_M^\infty$.

For a presheaf $\mathcal{P}$ and a point $x \in X$, consider the family $\mathcal{U}_x$ of open neighborhoods of x . For $s_1 \in \mathcal{P}(U_1)$ and $s_2 \in \mathcal{P}(U_2)$ with $U_1, U_2 \in \mathcal{U}_x$, we say $s_1 \sim s_2$ if there exists $V \in \mathcal{U}_x$ with $V \subset U_1 \cap U_2$ such that

$$\text{res}_V^{U_1}(s_1) = \text{res}_V^{U_2}(s_2).$$

The equivalence class of $s \in \mathcal{P}(U)$ is the *germ* of s at x , denoted $[s]_x$. The set $\mathcal{P}_x$ of all germs at x is the *stalk* of $\mathcal{P}$ at x .

Definition 1.6.3 (Sheaf). A presheaf $\mathcal{S}$ is a *sheaf* if for any open set $U \subset X$ and any open cover $\{U_i\}_{i \in I}$ of U , the following hold:

1. (*Locality*) If $s, t \in \mathcal{S}(U)$ satisfy $\text{res}_{U_i}^U(s) = \text{res}_{U_i}^U(t)$ for all i , then $s = t$.
2. (*Gluing*) Given sections $s_i \in \mathcal{S}(U_i)$ such that

$$\text{res}_{U_i \cap U_j}^{U_i}(s_i) = \text{res}_{U_i \cap U_j}^{U_j}(s_j)$$

for all i, j , there exists $s \in \mathcal{S}(U)$ with $\text{res}_{U_i}^U(s) = s_i$ for all i .

If $\mathcal{S}$ is a presheaf of algebraic structures and satisfies these axioms, it is a *sheaf of groups, rings, etc.*

Example 1.6.4. As one can easily prove, the presheaf of smooth functions $\mathcal{C}_M^\infty$ is a sheaf of rings.

Example 1.6.5 (Sections of a Vector Bundle). If $\pi : E \rightarrow M$ is a smooth vector bundle, then the presheaf $U \mapsto \Gamma(U, E) = \{\text{smooth local sections of } E \text{ over } U\}$ is a sheaf of $\mathcal{C}_M^\infty$ -modules.

Given a presheaf $\mathcal{P}$, there is a canonical sheaf associated to it, which is the sections of its *étale space*. It is defined as the following set

$$\text{Et}(\mathcal{P}) = \bigsqcup_{x \in X} \mathcal{P}_x,$$

which is a topological space with basis topology given by

$$V_{U,s} = \{[s]_x \mid x \in U\}, \quad U \in \tau, \quad s \in \mathcal{P}(U).$$

This basis provides that the canonical projection $\pi_{\mathcal{P}} : \text{Et}(\mathcal{P}) \rightarrow X$ is a local homeomorphism. The sheaf $\Gamma(\text{Et}(\mathcal{P}))$ of continuous sections of $\pi_{\mathcal{P}}$ is called the *sheafification* of $\mathcal{P}$. If $\mathcal{P}$ is already a sheaf, then $\mathcal{P} \cong \Gamma(\text{Et}(\mathcal{P}))$.

Example 1.6.6 (Constant and Locally Constant Sheaves). A sheaf $\mathcal{S}$ on X is *constant* if all stalks $\mathcal{S}_x$ are equal to the same set Z . In that case, its étale space is $\text{Et}(\mathcal{S}) \cong X \times Z$ (with Z discrete) and sections correspond to locally constant maps $U \rightarrow Z$. A sheaf is *locally constant* if every point has an open neighborhood on which the sheaf is constant.

Finally, there is a canonical way to obtain sheaves, induced by continuous maps. The following examples are important to the next section, when we define the Molino sheaf of a Riemannian foliation (see Definition 1.7.4) and when we blow-up the Molino sheaf of a singular Riemannian foliation (see 3.1.4).

Definition 1.6.7 (Direct Image Sheaf). Let $f : X \rightarrow Y$ be continuous and $\mathcal{R}$ a sheaf on X . The *direct image sheaf* $f_*\mathcal{R}$ on Y is defined by

$$f_*\mathcal{R}(U) = \mathcal{R}(f^{-1}(U)), \quad U \subset Y \text{ open.}$$

The restriction maps $f_*\mathcal{R}$ are such that $f_*\text{res}_V^U = \text{res}_{f^{-1}(V)}^{f^{-1}(U)}$. It defines a sheaf on Y .

Although a continuous map sends open sets to open sets via inverse images, it does not, in general, map open sets to open sets. For this reason, the notion of “pulling” back sheaves is subtler than the direct image.

Definition 1.6.8 (Inverse Image). Let $f : X \rightarrow Y$ be continuous and $\mathcal{S}$ be a sheaf on Y . Define the *inverse image of $\mathcal{S}$ by f* as the sheaf of germs of sections of the étale space

$$f^{-1}\text{Et } \mathcal{S} := \{(x, [s]_y) \in X \times \text{Et}(\mathcal{S}) \mid f(x) = y\}.$$

It is instructive to describe $f^{-1}\mathcal{S}$ in terms of sections of $\mathcal{S}$ over open sets of Y . For this, we need to generalize the notion of a germ, since $f(U)$ is not necessarily open. Let $A \subset Y$ be an arbitrary subset, and let U_1 and U_2 be open neighborhoods of A . We say that two sections $s_1 \in \mathcal{S}(U_1)$ and $s_2 \in \mathcal{S}(U_2)$ are *equivalent at A* if there exists an open neighborhood $W \subset U_1 \cap U_2$ of A such that $\text{res}_W^{U_1}(s_1) = \text{res}_W^{U_2}(s_2)$. We denote by $\mathcal{S}_A$ the set of equivalence classes and by $[s]_A$ the equivalence class of a section s . If $B \subset A$, then the restriction map $\text{res}_B^A : \mathcal{S}_A \rightarrow \mathcal{S}_B$ is well defined, since any open neighborhood of A is also an open neighborhood of B . Using this notion, one defines a presheaf $f_{\text{pre}}^{-1}\mathcal{S}$ on X by

$$f_{\text{pre}}^{-1}\mathcal{S}(U) := \mathcal{S}_{f(U)}.$$

This presheaf need not be a sheaf, even when $\mathcal{S}$ itself is a sheaf. Nevertheless, the inverse image sheaf $f^{-1}\mathcal{S}$ is canonically isomorphic to the sheafification of $f_{\text{pre}}^{-1}\mathcal{S}$, namely

$$f^{-1}\mathcal{S} \cong \Gamma(\text{Et}(f_{\text{pre}}^{-1}\mathcal{S})).$$

Finally, the restriction maps of $f^{-1}\mathcal{S}$ satisfy $f^{-1}\text{res}_V^U = \text{res}_{f(V)}^{f(U)}$.

Remark 1.6.9. Note that when $f : X \rightarrow Y$ is an open map, the inverse image sheaf coincides with the sheaf with sections given by $f^{-1}(\mathcal{S})(U) = \mathcal{S}(f(U))$, for every open set $U \subset X$.

1.7 Molino's Theory

In this section, we briefly recall the structure theorems for (singular) Riemannian foliations, known as Molino theory, following [GT18, Section 4.1] and [Töb14, Section 3.2]; we refer to [MC88] for a detailed treatment. For the singular case, we refer to [MC88, Chapter 6]. This section is not strictly necessary for understanding the remainder of the work, but is included for the sake of completeness.

1.7.1 Regular Case

Let $(M, \mathcal{F}, g)$ be a Riemannian foliation. The transverse metric g^T induced by a bundle-like metric determines, for each foliated chart U , a Riemannian metric $\pi_*(g^T)$ on the local quotient manifold $U/\mathcal{F}$ [MC88, Section 3.2]. Equivalently, the transverse metric on U is the pullback via the projection $\pi: U \rightarrow U/\mathcal{F}$ of the Riemannian metric on the local quotient.

Each local quotient manifold $U/\mathcal{F}$ carries a canonical Levi-Civita connection, and by uniqueness these local connections glue (via pullback) to define the *basic Levi-Civita connection* ∇^B . The term *basic* reflects the fact that local foliated vector fields in $\mathfrak{L}(\mathcal{F}|_U)$ are parallel along the leaves with respect to ∇^B . Moreover, passing to the quotient, ∇^B induces a covariant derivative on the space of transverse vector fields $\mathfrak{l}(\mathcal{F})$, which we denote by the same symbol.

Definition 1.7.1 (Molino Bundle). For $(M, g, \mathcal{F})$ a Riemannian foliation, consider $\pi^\perp: M^\perp \rightarrow M$ the principal $O(q)$ -bundle of $\mathcal{F}$ -transverse orthonormal frames, where q is the codimension of $\mathcal{F}$. We call $M^\perp$ the *Molino bundle* of the foliated manifold $(M, \mathcal{F}, g)$.

If $\mathcal{F}$ is transversely orientable, $M^\perp$ consist of two $SO(q)$ -invariant connected components. In this case we assumed that one component was chosen and we also denote it by $M^\perp$, by abuse of notation. There is a unique torsion free linear connection $\omega_{\mathcal{F}}$ associated with the basic Levi-Civita connection ∇^B [MC88, Lemma 3.3], which allows us to horizontally lift the leaves of $\mathcal{F}$ by lifting the flows of the tangent vector fields, which therefore induces an involutive distribution $T\mathcal{F}^\perp$, whose integral foliation $\mathcal{F}^\perp$ is called the *lifted foliation*.

There is a *tautological form* $\theta_{\mathcal{F}}: \nu\mathcal{F}^\perp \rightarrow \mathbb{R}^q$ associated to the Molino bundle, defined by $\theta_{\mathcal{F}}(X_{x^\perp}) := (x^\perp)^{-1}(d\pi^\perp(X_{x^\perp}))$, where, by definition of the orthonormal bundle, $x^\perp$ is an orthonormal basis of $\nu_x\mathcal{F}$, viewed as an isomorphism $x^\perp: \mathbb{R}^q \rightarrow \nu_x\mathcal{F}$. The tautological form $\theta_{\mathcal{F}}$ is $\mathcal{F}^\perp$ -basic [MC88, Lemma 2.1], therefore there is an $\mathcal{F}^\perp$ -basic, $O(q)$ -equivariant map $\omega_{\mathcal{F}} \oplus \theta_{\mathcal{F}}: \nu\mathcal{F}^\perp \rightarrow \mathfrak{so}(q) \oplus \mathbb{R}^q$ which restricts to an isomorphism at each fiber $\nu_{x^\perp}\mathcal{F}^\perp$ ⁴.

⁴Beside Molino himself, we also refer here to [MM03, Chapter 4] for a more detailed presentation of these results and its connection with transverse parallelizable foliations.

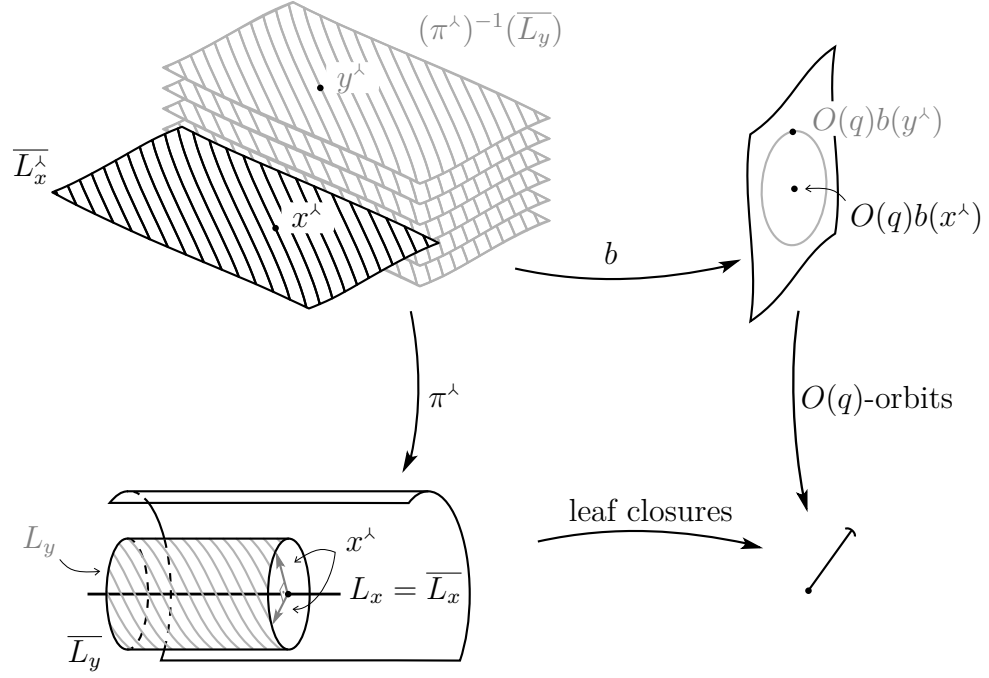


Figure 1.7 – Molino bundle. Source: [CT19].

This allows us to define a natural transverse metric for the lifted foliation as follows. The pullback via $\omega_{\mathcal{F}} \oplus \theta_{\mathcal{F}}$ of the sum of an arbitrary bi-invariant scalar product on $\mathfrak{so}(q)$ (unique up to a scalar λ), and the standard scalar product on $\mathbb{R}^q$ yields an $O(q)$ -invariant transverse metric $(g^T)^\lambda$ for $\mathcal{F}^\lambda$. Consequently, $\mathcal{F}^\lambda$ is a Riemannian foliation. The parameter λ can be fixed by requiring that the fibers of π^λ satisfy $\text{vol}((\pi^\lambda)^{-1}(x)) = 1$, for all x in the base. For $x \in M$, we now fix λ by requiring that the fibers of π^λ satisfy $\text{vol}((\pi^\lambda)^{-1}(x)) = 1$.

The key feature of the lifted foliation $\mathcal{F}^\lambda$ is that it admits a global transverse parallelism; that is, $\nu\mathcal{F}^\lambda$ is parallelizable by fields in $\mathfrak{l}(\mathcal{F}^\lambda)$ [MC88, p. 82, p. 148]. If we assume that $\mathcal{F}$ is complete, then those fields admit complete representatives in $\mathfrak{L}(\mathcal{F}^\lambda)$ [GT18, Section 4.1]. Transversely parallelizable foliations are dynamically well behaved: the partition $\overline{\mathcal{F}^\lambda}$ of M^λ is a *simple foliation*; that is, $B := M^\lambda / \overline{\mathcal{F}^\lambda}$ is a manifold and $\overline{\mathcal{F}^\lambda}$ is given by the fibers of a locally trivial fibration $b : M^\lambda \rightarrow B$ (check [MC88, Proposition 4.1] and [MM03, Theorem 4.3]), called the *basic fibration*. Since $\mathcal{F}^\lambda$ is $O(q)$ -invariant, by continuity so is $\overline{\mathcal{F}^\lambda}$; hence the action of $O(q)$ on M^λ descends to an action on B such that b is $O(q)$ -equivariant. A leaf closure $\overline{L} \in \overline{\mathcal{F}}$ is the image by π^λ of a leaf closure of $\mathcal{F}^\lambda$, which implies that each leaf closure is an embedded submanifold of M [MC88, Lemma 5.1] that also corresponds to an $O(q)$ -orbit in B . This induces an identification $M/\overline{\mathcal{F}} \cong B/O(q)$ and gives the commutative diagram (see figure 1.7):

$$\begin{array}{ccc}
 (M^\lambda, \mathcal{F}^\lambda, O(q)) & \xrightarrow{b} & (B, O(q)) \\
 \downarrow \pi^\lambda & & \downarrow \\
 (M, \mathcal{F}) & \longrightarrow & M/\overline{\mathcal{F}} \cong B/O(q).
 \end{array}$$

Fix a leaf $L^\lambda \in \mathcal{F}^\lambda$ and set $J := \overline{L^\lambda}$. Consider the restricted foliation $(J, \mathcal{F}^\lambda|_J)$, and define

$$\mathfrak{g}_{\mathcal{F}} := \mathfrak{l}(\mathcal{F}^\lambda|_J).$$

Then $\mathcal{F}^\lambda|_J$ is a complete Lie $\mathfrak{g}_{\mathcal{F}}$ -foliation in the sense of E. Fedida [Fed06]. Fedida's theory implies that such foliations are *developable*, meaning that they lift to simple foliations on the universal covering space [MC88, see Theorem 4.1]. Moreover, the restriction of $\mathcal{F}^\lambda$ to the closure of any other leaf is isomorphic to $(J, \mathcal{F}^\lambda|_J)$. Consequently, $\mathfrak{g}_{\mathcal{F}}$ depends only on $\mathcal{F}$ and defines an algebraic invariant.

Definition 1.7.2 (Structural Algebra). The lie algebra $\mathfrak{g}_{\mathcal{F}}$ is called the *structural algebra* of $\mathcal{F}$.

Definition 1.7.3 (Transverse Killing Field). We say $\overline{X} \in \mathfrak{l}(\mathcal{F})$ is a *transverse Killing vector field* when $\mathcal{L}_X g^\top = 0$, for a foliated representative $X \in \mathfrak{L}(\mathcal{F})$.

These fields form a Lie subalgebra of $\mathfrak{L}(\mathcal{F})$ [MC88, Lemma 3.5], called the Lie algebra of *transverse Killing vector fields*, denoted by $\mathfrak{iso}(\mathcal{F}, g^\top)$, which we denote as $\mathfrak{iso}(\mathcal{F})$, when the context is clear. Notice that these are the vector fields that project to Killing vector fields on the local quotients of $\mathcal{F}$.

Definition 1.7.4 (Molino Sheaf). Let $\mathcal{F}$ be a complete Riemannian foliation. Consider on M^λ the presheaf $\mathcal{C}_{\mathcal{F}^\lambda}$ that associated each open set U^λ with the lie algebra of transverse vector fields in U^λ that commute with all the global fields in $\mathfrak{l}(\mathcal{F}^\lambda)$, when they are restricted to U^λ . Consider its sheafification and also denote it by $\mathcal{C}_{\mathcal{F}^\lambda}$. The direct image sheaf $\pi_*(\mathcal{C}_{\mathcal{F}^\lambda}) =: \mathcal{C}_{\mathcal{F}}$ is called the *Molino sheaf* of $\mathcal{F}$.

Each field in $\mathcal{C}_{\mathcal{F}^\lambda}(U^\lambda)$ is the natural lift of an $\mathcal{F}$ -transverse Killing vector field on $\pi(U^\lambda)$ [MC88, Proposition 3.4]. Therefore, $\mathcal{C}_{\mathcal{F}}$ consists of a sheaf of Lie algebras of local transverse Killing vector fields that lift to local sections of $\mathcal{C}_{\mathcal{F}^\lambda}$.

Definition 1.7.5 (Orbit of Transverse Killing Fields). Let $\mathcal{F}$ be a Riemannian foliation. For every $\overline{X} \in \mathfrak{iso}(\mathcal{F})$, define the orbit $\overline{X} \cdot x$ of $x \in M$ as the saturation of the orbit of x under the flow of a representative $X \in \mathfrak{L}(\mathcal{F})$.

Remark 1.7.6. Notice the definition above is well-defined (is independent of the representative X), because foliated fields differ by a vector field in $\mathfrak{X}(\mathcal{F})$ and thus provides the same saturation under its flows.

The Molino sheaf is Hausdorff [MC88, Lemma 4.6] and its stalk $(\mathcal{C}_{\mathcal{F}})_x$ on each point is isomorphic to the Lie algebra $\mathfrak{g}_{\mathcal{F}}^{\text{op}}$ opposed to $\mathfrak{g}_{\mathcal{F}}$ [MC88, Proposition 4.4]. Now we can enunciate Molino structural theorem [MC88, Theorem 5.2]⁵:

⁵The structural theorem also states about Lie foliations, which we are going to omit, because it is not needed in the rest of the work, and we refer to [MC88; MM03] for a more detailed presentation of the topic.

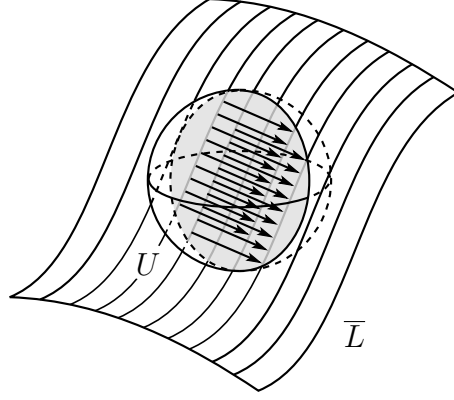


Figure 1.8 – Molino sheaf. Source: [CT19].

Theorem 1.7.7 (Structural Theorem for Riemannian Foliations). For a complete Riemannian foliation $\mathcal{F}$, the orbits of the vector fields in the stalk of $\mathcal{C}_{\mathcal{F}}$ give the closure of the leaves of $\mathcal{F}$. This means that

$$\{X_x \mid X \in (\mathcal{C}_{\mathcal{F}})_x\} \oplus T_x L_x = T_x \overline{L}_x.$$

Since $\mathcal{C}_{\mathcal{F}}$ is locally constant, with stalk isomorphic to $\mathfrak{g}_{\mathcal{F}}$, the structural theorem means that for a small open set U , taking a basis $\overline{X}_1, \dots, \overline{X}_d$ for a stalk $(\mathcal{C}_{\mathcal{F}})_x$, with $d = \dim(\mathfrak{g}_{\mathcal{F}})$ (consequently for every point in U), we obtain

$$T\overline{L}|_U = TL|_U \oplus \text{span}(X_1, \dots, X_d)$$

for any $L \in \mathcal{F}$, where $X_1, \dots, X_d \in \mathfrak{L}(\mathcal{F})$ are representatives of that basis (see 1.8).

When M is simply connected, the sheaf $\mathcal{C}_{\mathcal{F}}$ has no holonomy, in the sense that every section $\mathcal{C}_{\mathcal{F}}(U)$ has a smooth extension to $\mathcal{C}_{\mathcal{F}}(M)$, because $\mathcal{C}_{\mathcal{F}}$ is locally constant and therefore a natural gluing process leads to a smooth extension of the vector field in $\mathcal{C}_{\mathcal{F}}(U)$. That case motivates the following definition:

Definition 1.7.8 (Killing Foliation). A complete Riemannian foliation is a *Killing foliation* when its Molino sheaf is globally constant.

Example 1.7.9 ([Mol84, Lemme III]). If $\mathcal{F}$ is the regular Riemannian foliation of a compact manifold M given by the orbits of an isometric action of a Lie group H on M , then $\mathcal{F}$ is a Killing foliation, because the Molino sheaf $\mathcal{C}_{\mathcal{F}}$ consists of the transverse Killing vector fields induced by the action of $\overline{H}$.

1.7.2 Singular Case

Let us now cover the singular setting. Let $\mathcal{F}$ be a complete *singular* Riemannian foliation. Using its stratification, one can generalize the concept of transverse Killing vector field:

Definition 1.7.10. We say that $X \in \mathfrak{X}(\mathcal{F})$ is a *transverse Killing vector field* if its restriction to each stratum Σ_α is a transverse Killing vector field of the regular Riemannian foliation $(\Sigma_\alpha, \mathcal{F}_\alpha, g|_{\Sigma_\alpha})$.

Extending the Molino sheaf to the singular case is possible, but not straightforward. First, even though a restricted foliation $\mathcal{F}_\alpha$ may fail to be complete, Molino's theory can still be applied to it. This is possible thanks to the pseudogroup formulation of Molino's theory due to Haefliger and Salem [MC88, Appendix D], since the holonomy pseudogroup associated with $\mathcal{F}_\alpha$ is complete. In particular, for the regular part $\mathcal{F}_{\text{reg}}$, one can show that the corresponding Molino sheaf $\mathcal{C}_{\text{reg}}$ admits a continuous extension to a locally constant sheaf $\mathcal{C}_{\mathcal{F}}$ on M , referred to as the *Molino sheaf of $\mathcal{F}$* [MC88, Lemma 6.5]. Here lies the subtlety: although every section of $\mathcal{C}_{\text{reg}}$ extends continuously through the singular stratum, it is not known whether such an extension is necessarily smooth. Equivalently, a section of $\mathcal{C}_{\mathcal{F}}$ defines a *continuous* transverse Killing vector field $\bar{X}$, which restricts to a transverse Killing vector field on each stratum, but is represented on M by a merely continuous local vector field.

Molino proved that the closure of a complete regular Riemannian foliation is a singular Riemannian foliation [MC88], and he conjectured that the closure of a complete singular Riemannian foliation is a singular Riemannian foliation. This conjecture was finally proved to hold true recently in [AR17]. Nevertheless, the *strong Molino conjecture* is still open:

Conjecture 1.7.11 (Strong Molino Conjecture). The sheaf $\mathcal{C}_{\mathcal{F}}$ is a sheaf of Lie algebras of germs of transverse Killing vector fields.

Basically, it states that the extension $\mathcal{C}_{\mathcal{F}}$ described above is always smooth. It is true, for instance, in the particular case of the so-called *orbit-like* foliations (see [Car24, Theorem 3.4] for details).

Definition 1.7.12. The Molino sheaf $\mathcal{C}_{\mathcal{F}}$ is said to be *smooth* when Conjecture 1.7.11 is satisfied for $\mathcal{F}$.

For each stratum Σ_α , the corresponding Molino sheaf $\mathcal{C}_\alpha$ associated to $\mathcal{F}_\alpha$ is obtained as the quotient of the restriction of $\mathcal{C}_{\mathcal{F}}$ to Σ_α by the kernel of the restriction map on sections. Equivalently, this kernel is the subsheaf consisting of sections whose restrictions to Σ_α vanish [MC88, Proposition 6.8]. Consequently, the orbits of $\mathcal{C}_{\mathcal{F}}$ describe precisely the closures of the leaves of $\mathcal{F}$. In analogy with the regular setting, when M is connected we call the typical stalk $\mathfrak{g}_{\mathcal{F}}$ of $\mathcal{C}_{\mathcal{F}}$ the *structural Lie algebra* of $\mathcal{F}$. It agrees with the structural algebra of the regular foliation $\mathcal{F}_{\text{reg}}$. Moreover, from the relationship between $\mathcal{C}_\alpha$ and $\mathcal{C}_{\mathcal{F}}$, it follows that the structural algebra $\mathfrak{g}_\alpha$ of $\mathcal{F}_\alpha$ is a quotient of $\mathfrak{g}_{\mathcal{F}}$ [MC88, Proposition 6.8].

In summary, we obtain the structural theorem for singular Riemannian foliations (see [MC88, Theorem 6.2] and [AR17, Main Theorem]):

Theorem 1.7.13 (Structural Theorem for Singular Riemannian Foliations). Let $\mathcal{F}$ be a complete singular Riemannian foliation of M and let g be an adapted metric. Then

1. The closure $\overline{\mathcal{F}}$ is a singular Riemannian foliation with adapted metric g .
2. There exists a locally constant sheaf of Lie algebras $\mathcal{C}_{\mathcal{F}}$ on M which induces $\mathcal{C}_{\text{reg}}$ on Σ_{reg} and whose restriction to a singular stratum Σ_{α} admits $\mathcal{C}_{\alpha}$ as a quotient sheaf.

Definition 1.7.14 (Killing Singular Foliation). A complete singular Riemannian foliation $\mathcal{F}$ is called *Killing* when the sheaf $\mathcal{C}_{\mathcal{F}}$ is a globally constant sheaf of Lie algebras of germs of (smooth) transverse Killing vector fields.

Again, singular Riemannian foliations of simply connected manifolds are automatically Killing, whenever their sheaves are smooth. Homogeneous foliations provide another class of examples.

Example 1.7.15. The exact same setting as in example 1.7.9 applies without the regularity assumption. Moreover, $\overline{\mathcal{F}}$ is also homogeneous, given by the connected components of the orbits of $\overline{H}$.

2 Blowing-up

In this chapter we want to introduce the basic properties of the blow-up of singular Riemannian foliations, as prescribed in [Ale10] and [Mol84], where the latter described the process of desingularization for the singular Riemannian foliation given by the closure of a regular Riemannian foliation. First, we present a section where we construct the classical blow-up process, for closed submanifolds.

2.1 Blowing-up Submanifolds

We are going to present in this section the more general construction of blowing-up a closed submanifold. We choose the approach of Arone and Kankaanrinta [AK10], which has the nice description of the blow-up as the orbit space of a free $\mathbb{Z}_2$ -action. There are other equivalent constructions, most commonly used in the literature, which we refer to [GH14, Pg. 602] (using charts) or [Obs21, Section 5.2] (using deformation to the normal cone).

Before giving the precise definitions, we briefly sketch the expected properties of the construction and outline its main features. Let M be a smooth manifold and let $\Sigma \subset M$ be a closed submanifold¹. Intuitively, the blow-up of M along Σ is a smooth manifold $\text{Bl}_\Sigma M$ together with a map

$$\pi: \text{Bl}_\Sigma M \rightarrow M,$$

called the *blow-up projection*. It is obtained by removing a tubular neighborhood of Σ and replacing it with the projectivization of the normal bundle of Σ in M (see Figure 2.1). The map π satisfies two fundamental properties: it restricts to a diffeomorphism from $\pi^{-1}(M \setminus \Sigma)$ onto $M \setminus \Sigma$, and its inverse image $\pi^{-1}(\Sigma)$ is the total space of the projective bundle $\mathbb{P}(\nu(\Sigma))$, whose fibers are the projectivizations of the fibers of the normal bundle $\nu(\Sigma)$.

Local Construction

Throughout this section, we identify $\mathbb{R}^m$ with the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$. With this identification, we have a canonical decomposition

$$\mathbb{R}^{m+n} \setminus \mathbb{R}^m \cong \mathbb{R}^m \times (\mathbb{R}^n \setminus \{0\}).$$

¹The choice of the symbol Σ is intentional: in the subsequent sections we will apply this construction to the blow-up of the minimal stratum of a singular Riemannian foliation.

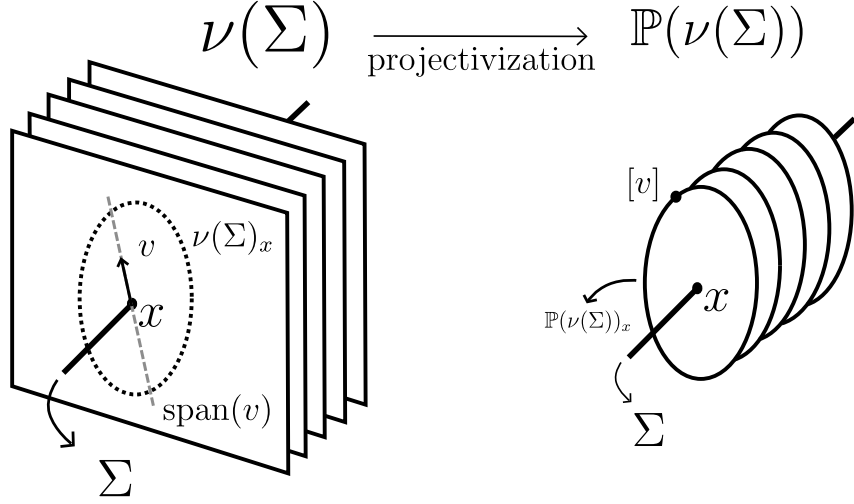


Figure 2.1 – Projectivization of the normal bundle.

We denote by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ the sets of positive and non-negative real numbers, respectively. Then there is a smooth diffeomorphism

$$\begin{aligned} \mu' : \mathbb{R}^m \times \mathbb{R}_{>0} \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R}^m \times (\mathbb{R}^n \setminus \{0\}) \cong \mathbb{R}^{m+n} \setminus \mathbb{R}^m, \\ (x, r, z) &\longmapsto (x, rz). \end{aligned}$$

This diffeomorphism extends to a smooth map

$$\mu : \mathbb{R}^m \times \mathbb{R} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^{m+n},$$

defined by the same formula. We denote by $\mu_{\geq 0}$ the restriction of μ to $\mathbb{R}^m \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1}$. Moreover, we define a $\mathbb{Z}_2$ -action on $\mathbb{R}^m \times \mathbb{R} \times \mathbb{S}^{n-1}$ by

$$g \cdot (x, r, z) := (x, -r, -z),$$

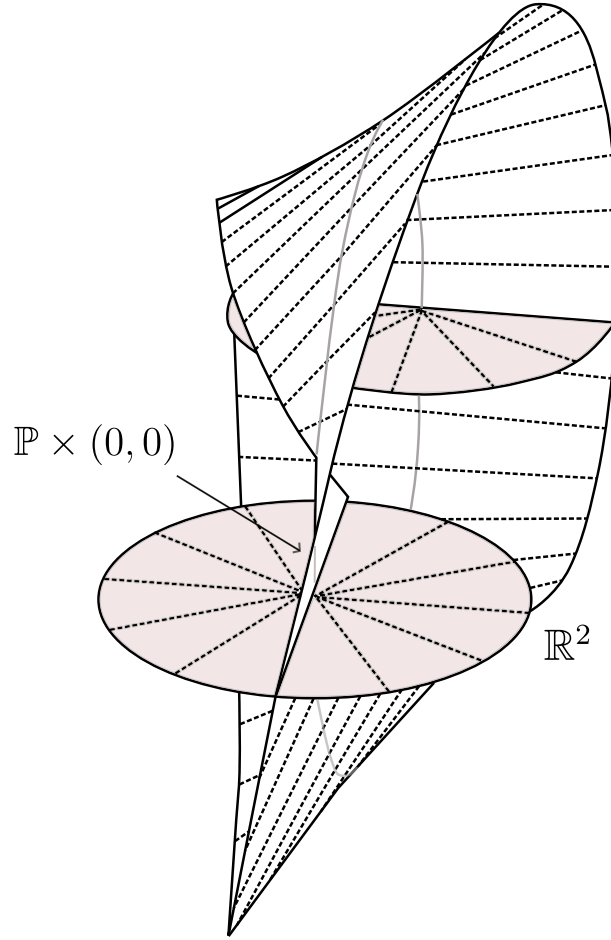
that is, by the antipodal map on the last two factors.

We recall the quotient manifold theorem [Lee03, Theorem 21.10].

Theorem 2.1.1 (Quotient Manifold Theorem). Suppose G is a Lie group acting smoothly, freely, and properly on a smooth manifold M . Then the orbit space M/G is a topological manifold of dimension $\dim M - \dim G$, and it admits a unique smooth structure for which the quotient map $\pi : M \rightarrow M/G$ is a smooth submersion.

Remark 2.1.2. From the characterizations of proper actions [Lee03, Proposition 21.5], it follows that compact Lie groups act properly [Lee03, Corollary 21.6].

Considering the $\mathbb{Z}_2$ -action on $\mathbb{R}^m \times \mathbb{R} \times \mathbb{S}^{n-1}$, note that $\mathbb{Z}_2$ is a compact Lie group, since it is finite. Hence, by Remark 2.1.2, this action is proper. Moreover, it is easily seen

Figure 2.2 – Blow-up of the origin in $\mathbb{R}^2$.

to be free. Therefore, we can define the *blow-up* of $\mathbb{R}^{m+n}$ along $\mathbb{R}^m$ as the quotient space

$$\text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n} := \mathbb{R}^m \times \mathbb{R} \times \mathbb{S}^{n-1} / \mathbb{Z}_2,$$

which is a smooth manifold by Theorem 2.1.1.

Observe that $\mu(x, r, z) = \mu(g \cdot (x, r, z))$, that is, μ is $\mathbb{Z}_2$ -invariant. This implies the existence of the *blow-up map*

$$\pi: \text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n},$$

defined by $\pi([(x, r, z)]) = \mu(x, r, z)$. The map π is well defined by $\mathbb{Z}_2$ -invariance. Moreover, π is the unique map making the following diagram commute:

$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R} \times \mathbb{S}^{n-1} & \xrightarrow{\mu} & \mathbb{R}^{m+n} \\ \downarrow & \nearrow \pi & \\ \text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n} & & \end{array}$$

Definition 2.1.3 (Projective Blow-up). The *projective blow-up* of $\mathbb{R}^{m+n}$ along $\mathbb{R}^m$ is the pair $(\text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n}, \pi)$.

Example 2.1.4. Identifying the origin $(0, 0)$ with $\mathbb{R}^0$, a classical example of a blow-up in the literature is $\text{Bl}_{(0,0)} \mathbb{R}^2$. In this construction, the associated projectivized normal bundle can be visualized as a Möbius band obtained by gluing a disk along its boundary (see Figure 2.2). Under this blow-up, the origin $(0, 0)$ is replaced by the central circle of the Möbius band, which is precisely the locus carrying nontrivial holonomy.

Lemma 2.1.5 ([AK10, Lemma 2.2]). Both π and $\mu_{\geq 0}$ are smooth and proper maps. Each of these maps restricts to a smooth diffeomorphism between $\mathbb{R}^{m+n} \setminus \mathbb{R}^m$ and its inverse image. In addition,

$$\pi^{-1}(\mathbb{R}^m) \cong \mathbb{R}^m \times \mathbb{RP}^{n-1} \quad \text{and} \quad \mu_{\geq 0}^{-1}(\mathbb{R}^m) \cong \mathbb{R}^m \times \mathbb{S}^{n-1}.$$

For a smooth map $G: \mathbb{R}^n \rightarrow \mathbb{R}^l$, we denote by $D_0 G$ the derivative of G at 0. Let

$$F: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^k \times \mathbb{R}^l$$

be a smooth map, which we write as $F = (F_1, F_2)$, where $(F_1, F_2)(x, y) = (F_1(x, y), F_2(x, y))$. Assume that

$$F^{-1}(\mathbb{R}^k \times \{0\}) = \mathbb{R}^m \times \{0\}.$$

For each $x \in \mathbb{R}^m$, let $F_{2,x}: \mathbb{R}^n \rightarrow \mathbb{R}^l$ be the map $y \mapsto F_2(x, y)$, and assume that the derivative $D_0 F_{2,x}$ is injective for all $x \in \mathbb{R}^m$. Equivalently, F induces a fiberwise injective map from the normal bundle of $\mathbb{R}^m \times \{0\}$ to the normal bundle of $\mathbb{R}^k \times \{0\}$.

Define the map

$$\tilde{F}: \mathbb{R}^m \times \mathbb{R} \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}^k \times \mathbb{R} \times \mathbb{S}^{l-1}$$

by

$$\tilde{F}(x, r, z) = \begin{cases} \left(F_1(x, rz), \frac{r}{\|F_2(x, rz)\|}, \frac{F_2(x, rz)}{\|F_2(x, rz)\|} \right), & \text{if } r \neq 0, \\ \left(F_1(x, rz), 0, \frac{D_0 F_{2,x}(z)}{\|D_0 F_{2,x}(z)\|} \right), & \text{if } r = 0. \end{cases} \quad (2.1)$$

Lemma 2.1.6 ([AK10, Lemma 2.5]). Under the above assumptions on F , the map $\tilde{F}$ is smooth.

Note that $\tilde{F}$ is equivariant with respect to the $\mathbb{Z}_2$ -action. Therefore, it induces a map

$$\overline{F}: \text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n} \longrightarrow \text{Bl}_{\mathbb{R}^k} \mathbb{R}^{k+l}.$$

Corollary 2.1.7 ([AK10, Corollary 2.6]). The induced map $\overline{F}: \text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n} \rightarrow \text{Bl}_{\mathbb{R}^k} \mathbb{R}^{k+l}$ is smooth.

Global Constructions

First, assume that all manifolds under consideration are without boundary. We define the blow-up along $\mathbb{R}^m$ for arbitrary codimension-zero submanifolds of $\mathbb{R}^{m+n}$. Let $S \subset \mathbb{R}^{m+n}$ be a codimension-zero submanifold. We define the blow-up of S along $S \cap \mathbb{R}^m$ as

$$\text{Bl}_{S \cap \mathbb{R}^m} S := \pi^{-1}(S),$$

where $\pi: \text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ denotes the blow-up map. In this situation, we also denote by π the induced map

$$\pi: \text{Bl}_{S \cap \mathbb{R}^m} S \rightarrow S.$$

Note that π restricts to a diffeomorphism

$$\pi^{-1}(S \setminus (S \cap \mathbb{R}^m)) \xrightarrow{\cong} S \setminus (S \cap \mathbb{R}^m),$$

and that

$$\pi^{-1}(S \cap \mathbb{R}^m) \cong (S \cap \mathbb{R}^m) \times \mathbb{R}P^{n-1}.$$

Moreover, for open subsets $U, V \subset \mathbb{R}^{m+n}$, we have

$$\begin{cases} \text{Bl}_{U \cap \mathbb{R}^m} U \cap \text{Bl}_{V \cap \mathbb{R}^m} V &= \text{Bl}_{U \cap V \cap \mathbb{R}^m} U \cap V, \\ \text{Bl}_{U \cap \mathbb{R}^m} U \cup \text{Bl}_{V \cap \mathbb{R}^m} V &= \text{Bl}_{(U \cup V) \cap \mathbb{R}^m} U \cup V, \end{cases}$$

where the intersection and union on the left-hand side are taken by viewing all spaces as subspaces of $\text{Bl}_{\mathbb{R}^m} \mathbb{R}^{m+n}$.

Let M be a smooth $(m+n)$ -dimensional manifold and let $\Sigma \subset M$ be a closed embedded m -dimensional submanifold. Let $\{(U, \varphi)\}$ be a collection of charts of M such that

$$\varphi(U \cap \Sigma) = \varphi(U) \cap \mathbb{R}^m,$$

and such that $\varphi(U \cap \Sigma)$ is embedded in $\varphi(U)$. Let (U, φ) and (V, ψ) be two charts in this collection with $U \cap V \neq \emptyset$. The transition map $\psi \circ \varphi^{-1}$ induces a smooth map between the corresponding blow-ups, denoted by $\overline{\psi \circ \varphi^{-1}}$, such that the following diagram commutes:

$$\begin{array}{ccc} \text{Bl}_{\varphi(U \cap V \cap \Sigma)} \varphi(U \cap V) & \xrightarrow{\overline{\psi \circ \varphi^{-1}}} & \text{Bl}_{\psi(U \cap V \cap \Sigma)} \psi(U \cap V) \\ \pi \downarrow & & \downarrow \pi \\ \varphi(U \cap V) & \xrightarrow{\psi \circ \varphi^{-1}} & \psi(U \cap V). \end{array}$$

Let T be the disjoint union of all spaces $\text{Bl}_{\varphi(U \cap \Sigma)} \varphi(U)$. Define an equivalence

relation $\sim$ on T by declaring $x \sim y$ if

$$x \in \text{Bl}_{\varphi(U \cap V \cap \Sigma)} \varphi(U \cap V), \quad y \in \text{Bl}_{\psi(U \cap V \cap \Sigma)} \psi(U \cap V), \quad \text{and} \quad y = \overline{\psi \circ \varphi^{-1}}(x).$$

Then $\sim$ is an equivalence relation.

Proposition 2.1.8 ([AK10, Proposition 3.2]). Let $\text{Bl}_{\Sigma} M := T / \sim$. Then $\text{Bl}_{\Sigma} M$, endowed with the quotient topology, is a smooth manifold.

The local blow-up maps

$$\pi: \text{Bl}_{\varphi(U \cap \Sigma)} \varphi(U) \rightarrow \varphi(U)$$

glue together to define a global smooth map, also denoted by

$$\pi: \text{Bl}_{\Sigma} M \rightarrow M.$$

Definition 2.1.9 (Blow-up Manifold). The pair $(\text{Bl}_{\Sigma} M, \pi)$ is called the *blow-up of M along Σ* .

The blow-up map π satisfies the following properties [AK10]:

1. π is a proper map.
2. $\pi^{-1}(\Sigma)$ is isomorphic to the projectivized normal bundle $\mathbb{P}(\nu(\Sigma))$.
3. π restricts to a smooth diffeomorphism

$$\pi^{-1}(M \setminus \Sigma) \xrightarrow{\cong} M \setminus \Sigma.$$

Definition 2.1.10 (Exceptional Divisor). Let $(\text{Bl}_{\Sigma} M, \pi)$ be the blow-up of M along Σ . The pre-image $\pi^{-1}(\Sigma) \cong \mathbb{P}(\nu(\Sigma)) =: E$ is called the *exceptional divisor*.

Lifting Group Actions

A fundamental result concerning the lifting of vector fields (see Section 3.1.4) is the following.

Theorem 2.1.11 (Theorem 5.1, [AK10]). Let G be a Lie group acting smoothly on a manifold M , and let $\Sigma \subset M$ be a closed G -invariant submanifold. Then G acts smoothly on $\text{Bl}_{\Sigma} M$, and the blow-up map

$$\pi: \text{Bl}_{\Sigma} M \rightarrow M$$

is G -equivariant, that is, $g \circ \pi = \pi \circ g$ for all $g \in G$.

In this situation, the action of G on $\text{Bl}_\Sigma M$ is given by the lifted diffeomorphisms

$$\bar{g}: \text{Bl}_\Sigma M \rightarrow \text{Bl}_\Sigma M.$$

The proof in [AK10] relies on the smoothness of the lifted action map $\bar{\phi}: G \times \text{Bl}_\Sigma M \rightarrow \text{Bl}_\Sigma M$.

2.2 Molino's Blow-up

The first construction of desingularizations of SRF is from Molino [Mol84], which desingularized leaves coming from the closure of a regular foliation. His construction differs slightly from that of Alexandrino, which we going to present in the next section, as he explicitly constructs foliated charts for the blow-up foliation, using the foliated charts for the regular foliation whose closure is desingularized.

For a regular Riemannian foliation $\mathcal{F}$ whose closure foliation is a singular Riemannian foliation $\bar{\mathcal{F}}$ in a compact manifold M , consider the minimal stratum Σ of the closure foliation. Molino constructs the blow-up of M as the union of $M \setminus \Sigma$ with $\mathbb{P}(\nu(\Sigma))$ equipped with the map $\pi: M \setminus \Sigma \cup \mathbb{P}(\nu(\Sigma)) \rightarrow M$ such that

$$\pi(x) = \begin{cases} x & \text{if } x \in M \setminus \Sigma, \\ p(x) & \text{if } x \in \mathbb{P}(\nu(\Sigma)), \text{ where } p: \mathbb{P}(\nu(\Sigma)) \rightarrow \Sigma \text{ is the fiber bundle projection.} \end{cases}$$

Using foliated charts for $\mathcal{F}$, he showed that $M \setminus \Sigma \cup \mathbb{P}(\nu(\Sigma))$ has a smooth structure such that $\pi: M \setminus \Sigma \cup \mathbb{P}(\nu(\Sigma)) \rightarrow M$ is smooth and a diffeomorphism when restricted to $M \setminus \Sigma$. Moreover, for a foliated chart $\varphi: U \rightarrow \tilde{U}$, denote $\text{Bl}_\Sigma U$ the blow-up of U along $\Sigma \cap U$ and $\pi_U: \text{Bl}_\Sigma U \rightarrow U$ its blow-up map. Notice that Σ projects by φ to $\bar{\Sigma}$ and we can also blow-up $\tilde{U}$ along $\bar{\Sigma}$, which we denote by $\text{Bl}_{\bar{\Sigma}} \tilde{U}$ and $\pi_{\tilde{U}}: \text{Bl}_{\bar{\Sigma}} \tilde{U} \rightarrow \tilde{U}$ the respective blow-up map. Hence, we obtain the following commutative diagram

$$\begin{array}{ccc} \text{Bl}_\Sigma U & \xrightarrow{\text{Bl}_\Sigma \varphi} & \text{Bl}_{\bar{\Sigma}} \tilde{U} \\ \pi_U \downarrow & & \downarrow \pi_{\tilde{U}} \\ U & \xrightarrow{\varphi} & \tilde{U}, \end{array}$$

where $\text{Bl}_\Sigma \varphi$ is the induced map such that the diagram commutes. Observe that $\text{Bl}_\Sigma \varphi$ is a submersion and therefore defines in $\text{Bl}_\Sigma U$ a Riemannian foliation $\text{Bl}_{\Sigma \cap U} \mathcal{F}$ of codimension q , where $\text{codim } \mathcal{F} = q$. This foliation is defined globally in $M \setminus \Sigma \cup \mathbb{P}(\nu(\Sigma))$ by gluing foliated charts and the foliation outside $\mathbb{P}(\nu(\Sigma))$ is a copy of $\mathcal{F}$ in $M \setminus \Sigma$. We are now able to state Molino's main theorem:

Theorem 2.2.1 ([Mol84], Lemme II and Theoreme II). For a regular Riemannian

foliation $\mathcal{F}$ whose closure foliation is a singular Riemannian foliation $\overline{\mathcal{F}}$ in a compact manifold M and Σ the minimal stratum of the foliation $\overline{\mathcal{F}}$, there exists a foliated manifold $(\text{Bl}_\Sigma M, \text{Bl}_\Sigma \mathcal{F}, \text{Bl}_\Sigma g)$ with a Riemannian foliation $\text{Bl}_\Sigma \mathcal{F}$ such that $\text{depth}(\overline{\text{Bl}_\Sigma \mathcal{F}}) < \text{depth}(\overline{\mathcal{F}})$.

2.3 Alexandrino's Blow-up

In this section we present the results of desingularizations of singular Riemannian foliation of [Ale10]. First, we need to briefly recall the theory of smooth (singular) distributions, in order to present Alexandrino's blow-up foliation.

Definition 2.3.1 (Smooth Distribution). Let M be a smooth manifold M . A distribution $\mathcal{D}$ is an assignment $M \rightarrow TM$ given by $x \in M \mapsto \mathcal{D}_x \in T_x M$, where $\mathcal{D}_x$ is a subspace of $T_x M$, for every x . A distribution is *smooth* at a point x if any tangent vector $X(x) \in \mathcal{D}_x$ can be locally extended to a smooth vector field $X \in \mathfrak{X}(U)$, where U is an open set, such that $X(y) \in \mathcal{D}_y$ for every $y \in U$.

Note that, for a singular foliation, the module $\mathfrak{X}(\mathcal{F})$ being transitive implies that $T\mathcal{F}$ is a smooth distributions, in the previous notion.

Definition 2.3.2 (Distribution Generated by Vector Fields). Let F be a family of smooth vector fields (possibly locally defined). A distribution $\mathcal{D}$ is said to be *generated by* F if

$$\mathcal{D}_x = \text{span}(X(x) \mid X \in F)$$

for every $x \in M$.

In this case it is easy to see that the distribution is automatically smooth.

Definition 2.3.3 (Invariant Distribution). Let F be a family of smooth vector fields. A distribution $\mathcal{D}$ is said to be *F-invariant* if it is invariant under the action of the elements of the flows of F . This means that, for every $X \in F$, with φ_t^X denoting its flow, then

$$(\varphi_t^X)_*(\mathcal{D}_y) \subset \mathcal{D}_{\varphi_t^X(y)}$$

for every y in the domain of X and $t \in \mathbb{R}$.

Definition 2.3.4 (Integrable Distribution). Let $\mathcal{D}$ be a smooth distribution. The distribution is said to be *integrable* if, for every $x \in M$, there exists a connected immersed submanifold L of M such that $T_x L = \mathcal{D}_x$.

Note that a smooth integrable distribution determines a singular foliation whose leaves are precisely its integral submanifolds. Consequently, the study of integrability

conditions for smooth distributions provides natural examples of singular foliations. In this context, a fundamental characterization of integrable smooth distributions is given by the Stefan–Sussmann theorem [Lav18, Theorem 4]:

Theorem 2.3.5 (Stefan-Sussmann). Let M be a smooth manifold and $\mathcal{D}$ be a smooth distribution. Then $\mathcal{D}$ is integrable if and only if it is generated by a family F of smooth vector fields, and is invariant with respect to F .

Now, turning back to the blow-up setup: let $(M, \mathcal{F}, g)$ be a compact foliated manifold endowed with a singular Riemannian foliation $\mathcal{F}$, and let Σ denote its minimum locus. Consider the blow-up $(\text{Bl}_\Sigma M, \pi)$ of M along Σ , which is well defined since Σ is compact (see Section 1.4.2). Alexandrino constructed a Riemannian metric $\text{Bl}_\Sigma g$ on $\text{Bl}_\Sigma M$ by extending a metric $\text{Bl}_\Sigma^U g$ defined on $\pi^{-1}(U) =: \text{Bl}_\Sigma U$, where U is a tubular neighborhood of the minimum locus. This construction yields a singular Riemannian foliation $\text{Bl}_\Sigma \mathcal{F}$ on $\text{Bl}_\Sigma M$ such that $(\text{Bl}_\Sigma M, \text{Bl}_\Sigma \mathcal{F}, \text{Bl}_\Sigma g)$ is a foliated manifold. In what follows, this construction is described in detail.

Since M is compact, every $X \in \mathfrak{L}(\mathcal{F})$ generates a global $\mathbb{R}$ -action on M via its flow. By Theorem 2.1.11, this action lifts to a smooth $\mathbb{R}$ -action on the blow-up $\text{Bl}_\Sigma M$, because the orbits of $X \in \mathfrak{L}(\mathcal{F})$ preserve the foliation (in particular, the singular locus Σ).

Definition 2.3.6 (Blow-up Foliate Field). For every $X \in \mathfrak{L}(\mathcal{F})$, the fundamental vector field of the lifted flow is called the *blow-up vector field of X* , denoted by $\text{Bl}_\Sigma X$.

Let $\text{Bl}_\Sigma \mathfrak{X}(\mathcal{F})$ be the smooth distribution given by the family of vector fields $F = \{\text{Bl}_\Sigma X \mid X \in \mathfrak{X}(\mathcal{F})\}$.

Proposition 2.3.7 (Blow-up Foliation). The smooth distribution $\text{Bl}_\Sigma \mathfrak{X}(\mathcal{F})$ is integrable and therefore provides a singular foliation on $\text{Bl}_\Sigma M$, denoted by $\text{Bl}_\Sigma \mathcal{F}$.

Proof. By the Stefan–Sussmann theorem, it suffices to verify that $\text{Bl}_\Sigma \mathfrak{X}(\mathcal{F})$ is invariant under the flows of $\text{Bl}_\Sigma X$, for each $X \in \mathfrak{X}(\mathcal{F})$. Moreover, by the construction of the lifted flows (given by the lifted $\mathbb{R}$ -action in Definition 2.1) there are two invariant subsets for this action, namely $\text{Bl}_\Sigma M \setminus E$ and E . Therefore, it is enough to check invariance on each of these subsets separately, since $\text{Bl}_\Sigma M$ is the union of the two.

Outside the exceptional divisor, the lifted action induces a diffeomorphic copy of the orbits of the original $\mathbb{R}$ -action on $M \setminus \Sigma$. Consequently, the distribution in $\text{Bl}_\Sigma M \setminus E$ is naturally invariant by the flows of $\text{Bl}_\Sigma X$.

The $\mathbb{R}$ -action on E is the projectivization of the lifted action defined in 2.1 for $r = 0$, and it corresponds precisely to the directions determined by vector fields $X \in \mathfrak{X}(\mathcal{F})$ along the slices of Σ . Indeed, since $\mathcal{F}$ is Riemannian, the flows of vector fields in $\mathfrak{X}(\mathcal{F})$ are contained in the normal tubular neighborhoods $S(r)$ around Σ . Moreover, the antipodal map on the normal spheres is a foliated diffeomorphism (see Proposition 1.5.3), ensuring

that the lifted action preserves leaves. This proves that the distribution on E is invariant and therefore concludes that $\text{Bl}_\Sigma \mathfrak{X}(\mathcal{F})$ is an integrable distribution. $\square$

Note that, from the previous proposition, the restriction of $\text{Bl}_\Sigma \mathcal{F}$ to the exceptional divisor E is given by the quotient

$$\text{Bl}_\Sigma \mathcal{F}|_E \cong (\mathcal{F}|_{S(r)})/\mathbb{Z}_2,$$

while its restriction to $\text{Bl}_\Sigma M \setminus E$ is diffeomorphic to $\mathcal{F}|_{M \setminus \Sigma}$.

Remark 2.3.8. The previous arguments implies that, in particular, $\pi: \text{Bl}_\Sigma M \rightarrow M$ is a foliated map.

We now sketch the construction of the metric appearing in item (4) of Theorem 2.3.9, following [AC22, Section 9.2]. First, recall that a minimal stratum $\Sigma^\alpha \subset \Sigma$ is a compact, connected subset of M , and therefore admits a tubular neighborhood $\text{Tub}_\varepsilon(\Sigma^\alpha)$, for $\varepsilon > 0$ sufficiently small, given by the image of its normal exponential map. Choose ε small enough so that the neighborhoods $\text{Tub}_\varepsilon(\Sigma^\alpha)$ are pairwise disjoint when α varies, and consider the *tubular neighborhood*

$$U = \bigcup_{\alpha} \text{Tub}_\varepsilon(\Sigma^\alpha)$$

of the locus Σ .

Let $\mathcal{S}$ be the smooth distribution on U defined by $\mathcal{S}_y = T_y S_x$, where S_x denotes a slice of the leaf L_x at x with respect to the original metric, and where $y = \exp_x(X)$ for some $X \in \nu_x \Sigma$. Recall that there exists a Riemannian metric $\tilde{g}$ on U such that the $\tilde{g}$ -normal distribution P to $\mathcal{S}$ is tangent to the leaves, and such that $(U, \mathcal{F}, \tilde{g})$ has the same transverse geometry as $(U, \mathcal{F}, g)$ [AT08, Proposition 2.2]. In particular, the distance between plaques is the same for both metrics. With respect to $\tilde{g}$, the tangent space at y admits the orthogonal decomposition

$$T_y M = P_y \oplus \mathcal{S}_y^s \oplus \mathcal{S}_y^r \oplus \mathcal{S}_y^c,$$

where:

1. P_y is the $\tilde{g}$ -orthogonal complement of $\mathcal{S}_y$,
2. $\mathcal{S}_y^s \subset \mathcal{S}_y$ is tangent to $\exp_x(\nu_x \Sigma \cap B_{\|X\|}(0))$,
3. $\mathcal{S}_y^r$ is the one-dimensional subspace generated by $\frac{d}{dt} \exp_x(tX) \Big|_{t=1}$, and
4. $\mathcal{S}_y^c$ is the $\tilde{g}$ -orthogonal complement of $\mathcal{S}_y^s \oplus \mathcal{S}_y^r$ inside $\mathcal{S}_y$.

We now define a metric $\tilde{g}$ on $U \setminus \Sigma$ adapted to the foliation $\mathcal{F}$. Let $f: (0, r) \rightarrow \mathbb{R}$ be a smooth function satisfying

$$f(t) = \frac{r^2}{t^2} \quad \text{for } 0 < t < \frac{r}{4}, \quad f(t) = 1 \quad \text{for } \frac{r}{4} < t < \frac{r}{2}.$$

and set

$$\hat{g}_y(Z, W) = \tilde{g}(Z^\perp, W^\perp) + f(\|X\|)\tilde{g}(Z^s, W^s) + \tilde{g}(Z^r, W^r) + \tilde{g}(Z^c, W^c).$$

It follows that $\hat{g}_{\exp(X)} = \tilde{g}_{\exp(X)}$ if $\frac{1}{4} \leq \|X\| \leq \frac{1}{2}$. Moreover, $\hat{g}$ is adapted to $\mathcal{F}$, because $f(\|\exp^{-1}(x)\|)$ is constant along L_x and therefore $\hat{g}$ is basic on each stratum [Ale10, Lemma 3.5 and Lemma 3.7]. Since the distribution $\mathcal{S}$ (equivalently, the $\tilde{g}$ -normal distribution to P) can be smoothly deformed to the g -normal distribution to P without altering the transverse metric, the metric $\hat{g}$ defined on $\text{Tub}_{r/4}(\Sigma)$ extends to a Riemannian metric $\hat{g}$ on U satisfying

$$\hat{g}_{\exp(rX/4)} = \tilde{g}_{\exp(rX/4)} \quad \text{and} \quad \hat{g}_{\exp(tX)} = g_{\exp(tX)} \quad \text{for } t > r/2 \text{ and } \|X\| = 1.$$

Finally, the pullback metric $\pi_\Sigma^* \tilde{g}$ defines a smooth Riemannian metric on $\text{Bl}_\Sigma U$, denoted by $\text{Bl}_\Sigma^U g$ with respect to which the foliation $\text{Bl}_\Sigma \mathcal{F}|_U$ is a singular Riemannian foliation, where π_Σ is the orthogonal projection $\pi_\Sigma: U \rightarrow \Sigma$.

Using a suitable partition of the unity, one can extend the metric $\text{Bl}_\Sigma^U g$ to an $\text{Bl}_\Sigma \mathcal{F}$ -adapted Riemannian metric $\text{Bl}_\Sigma g$ on the whole $\text{Bl}_\Sigma M$, so that furthermore $\pi: \text{Bl}_\Sigma M \setminus \text{Bl}_\Sigma U \rightarrow M \setminus U$ is an isometry [Ale10, Theorem 1.2].

Theorem 2.3.9 (Theorem 1.2, [Ale10]). With the notation above:

1. The foliation $\mathcal{F}$ blows-up to a singular foliation $\text{Bl}_\Sigma \mathcal{F}$ on $\text{Bl}_\Sigma M$ such that

$$\pi: \text{Bl}_\Sigma M \setminus E \rightarrow M \setminus \Sigma$$

is a foliated diffeomorphism.

2. $\text{depth}(\text{Bl}_\Sigma^U \mathcal{F}) = \text{depth}(\mathcal{F}|_U) - 1$.
3. There exists a Riemannian metric $\text{Bl}_\Sigma g$ on $\text{Bl}_\Sigma M$ adapted to $\text{Bl}_\Sigma \mathcal{F}$ and such that

$$\pi: (E, \text{Bl}_\Sigma g) \rightarrow (\Sigma, g)$$

is a foliated Riemannian submersion.

In particular, since $\text{depth}(\text{Bl}_\Sigma \mathcal{F}) = \text{depth}(\mathcal{F}) - 1$, one can repeatedly apply this blow-up process until the depth drops to 0, thus obtaining a regular foliation, i.e., desingularizing $\mathcal{F}$. In more detail, let us denote the minimal locus of $\mathcal{F}$ by Σ_0 and $\text{Bl}_1 M := \text{Bl}_{\Sigma_0} M$.

For $k \geq 1$ we define by induction $\text{Bl}_{k+1}M := \text{Bl}_{\Sigma_k} \text{Bl}_k M$, so that $\pi_{k+1}: \text{Bl}_{k+1}M \rightarrow \text{Bl}_k M$ is the blow-up map along the minimal locus Σ_k of $\text{Bl}_k \mathcal{F}$. Similarly, we apply this induction to the foliation and the metric, obtaining a sequence $(\text{Bl}_k M, \text{Bl}_k \mathcal{F}, \text{Bl}_k g)$. This process ends (becomes trivial) since $\text{depth}(\mathcal{F})$ is finite and the blow up construction is trivial for regular foliations. This final stage, called the *blow-up of* $(M, \mathcal{F}, g)$, will be denoted simply by $(\text{Bl}M, \text{Bl}\mathcal{F}, \text{Bl}g)$, with projection $\pi := \pi_{\text{depth}(\mathcal{F})} \circ \dots \circ \pi_1: \text{Bl}M \rightarrow M$.

When $\mathcal{F}$ is closed, by controlling the radius ε Alexandrino proves that this process leads to a sequence of orbifolds converging, in the Gromov–Hausdorff sense, to $M/\mathcal{F}$:

Theorem 2.3.10 (Theorem 1.5 and Corollary 1.6, [Ale10]). Let $(M, \mathcal{F}, g)$ be a closed singular Riemannian foliation on a compact Riemannian manifold. Then for every $\varepsilon > 0$, one can find a closed (regular) Riemannian foliation $\text{Bl}\mathcal{F}$ of a blow-up $\text{Bl}M$ such that, for every $x, y \in \text{Bl}M$,

$$|d_g(L_{\pi(x)}, L_{\pi(y)}) - d_{\text{Bl}g}(L_x, L_y)| < \varepsilon.$$

In particular, this implies that $M/\mathcal{F}$ is a Gromov–Hausdorff limit of Riemannian orbifolds $\{(\text{Bl}M)_i/(\text{Bl}\mathcal{F})_i\}$.

3 Results

In this chapter, we establish new results concerning the desingularization of singular Riemannian foliations. As outlined in the introduction, the chapter is divided into three sections. The first is devoted to the dynamics of the blown-up foliation: we show that the blow-up sheaf coincides with the sheaf of the blow-up and derive conditions under which conditions a singular Killing foliation is closed. The second section studies the behavior of basic cohomology under the blow-down map, identifying conditions under which the corresponding cohomology groups are isomorphic. The third section generalizes recent results on the Gromov–Hausdorff convergence of orbifolds to the leaf space of a closed Killing foliation.

3.1 Dynamics of the Blown-up Foliation

In [Mol84], Molino showed that, for a regular Riemannian foliation, the Molino sheaf of the blow-up foliation $\text{Bl}_\Sigma \mathcal{F}$ is the blow-up sheaf of $\mathcal{F}$. In this section we will generalize this result for the general blow-up construction of Alexandrino for singular Riemannian foliations seen in Section 2.3.

First, recall that for $X \in \mathfrak{X}(\mathcal{F})$, its blow-up vector field $\text{Bl}_\Sigma X$ is tangent to the leaves of $\text{Bl}_\Sigma \mathcal{F}$. In fact, it follows by the construction of $\text{Bl}_\Sigma \mathcal{F}$ (check Section 2.3).

Proposition 3.1.1. If $X \in \mathfrak{L}(\mathcal{F})$, then $\text{Bl}_\Sigma X \in \mathfrak{L}(\text{Bl}_\Sigma \mathcal{F})$. Moreover, the map $\text{Bl}_\Sigma: \mathfrak{L}(\mathcal{F}) \rightarrow \mathfrak{L}(\text{Bl}_\Sigma \mathcal{F})$ is linear and injective, and $\text{Bl}_\Sigma(\mathfrak{X}(\mathcal{F})) \subset \mathfrak{X}(\text{Bl}_\Sigma \mathcal{F})$.

Proof. Since X respects $\mathcal{F}$, its flow sends leaves to leaves, hence the flow of $\text{Bl}_\Sigma X$ also sends leaves to leaves: outside the exceptional divisor $\text{Bl}_\Sigma \mathcal{F}$ is a diffeomorphic copy of $\mathcal{F}|_{M \setminus \Sigma}$, and in the exceptional divisor it is a projectivization of the normal cylinders along the locus Σ . In the same fashion, one argues that, if $X \in \mathfrak{X}(\mathcal{F})$, then $\text{Bl}_\Sigma X \in \mathfrak{X}(\text{Bl}_\Sigma \mathcal{F})$. The linearity and injectivity of Bl_Σ also follows from this observation (and continuity): on the dense open set $M \setminus \Sigma$ it is just the pullback by π . $\square$

It follows that, for a transverse field $\overline{X} \in \mathfrak{l}(\mathcal{F})$, we can define the blow-up $\text{Bl}_\Sigma \overline{X}$ by $\overline{\text{Bl}_\Sigma X}$. In fact, if $X' \in \mathfrak{L}(\mathcal{F})$ is another representative, then $X' = X + T$, for $T \in \mathfrak{X}(\mathcal{F})$, thus

$$\text{Bl}_\Sigma X' = \text{Bl}_\Sigma X + T = \text{Bl}_\Sigma X + \text{Bl}_\Sigma T,$$

hence $\overline{\text{Bl}_\Sigma X'} = \overline{\text{Bl}_\Sigma X}$.

Covariant objects lift straightforwardly via $\pi^*: \Omega^k(\mathcal{F}) \rightarrow \Omega^k(\text{Bl}_\Sigma \mathcal{F})$ (or, more generally, $\mathcal{F}$ -basic covariant tensor fields pull via π back to $\text{Bl}_\Sigma \mathcal{F}$ -basic covariant tensor

fields). Each π^* is also injective, again by continuity, since π restricts to a diffeomorphism $\text{Bl}_\Sigma M \setminus E \rightarrow M \setminus \Sigma$. In particular, there is a natural inclusion $C^\infty(\mathcal{F}) \rightarrow C^\infty(\text{Bl}_\Sigma \mathcal{F})$. From this discussion it is easy to conclude the following:

Proposition 3.1.2. The blow-up of foliate vector fields yields a $C^\infty(\mathcal{F})$ -linear map $\text{Bl}_\Sigma: \mathfrak{l}(\mathcal{F}) \rightarrow \mathfrak{l}(\text{Bl}_\Sigma \mathcal{F})$, with respect to the $C^\infty(\mathcal{F})$ -module structure on $\mathfrak{l}(\text{Bl}_\Sigma \mathcal{F})$ given by $\pi^*: C^\infty(\mathcal{F}) \rightarrow C^\infty(\text{Bl}_\Sigma \mathcal{F})$.

Remark 3.1.3. We stated Propositions 3.1.1 and 3.1.2 for global vector fields only in order to have a simpler notation. The construction is local, therefore they are, *mutatis mutandis*, equally valid for local foliate/transverse vector fields.

We define the blow-up of $\mathcal{C}_\mathcal{F}$ as the inverse image

$$\text{Bl}_\Sigma(\mathcal{C}_\mathcal{F}) := \pi^{-1}(\mathcal{C}_\mathcal{F}).$$

Recall that, via sections, this can be seen, at the level of presheaves, as “generalized germs”: given an open set $\tilde{U} \subset \text{Bl}_\Sigma M$, the value of $\text{Bl}_\Sigma(\mathcal{C}_\mathcal{F})(\tilde{U})$ is the equivalence class of sections $\bar{Y}$ of $\mathcal{C}_\mathcal{F}$ on open sets $U \subset M$ containing $\pi(\tilde{U})$, under the relation that identifies $\bar{Y} \in \mathcal{C}_\mathcal{F}(U)$ with $\bar{Y}' \in \mathcal{C}_\mathcal{F}(U')$ whenever they agree on a smaller $U'' \subset U \cap U'$.

Theorem 3.1.4. Let $\mathcal{F}$ be a singular Riemannian foliation of a compact manifold M . Then π induces an isomorphism

$$\mathcal{C}_{\text{Bl}_\Sigma \mathcal{F}} \cong \text{Bl}_\Sigma \mathcal{C}_\mathcal{F}.$$

In particular, if $\mathcal{C}_\mathcal{F}$ is smooth, then $\mathcal{C}_{\text{Bl}_\Sigma \mathcal{F}}$ is smooth.

Proof. Since π restricts to a foliated diffeomorphism $\pi^{-1}(M_{\text{reg}}) \rightarrow M_{\text{reg}}$, it is clear that $\mathcal{C}_{\text{Bl}_\Sigma \mathcal{F}}$ and $\text{Bl}_\Sigma(\mathcal{C}_\mathcal{F})$ are isomorphic when restricted to $\pi^{-1}(M_{\text{reg}})$: the pullback by π identifies their sections. But this determines both sheaves completely, since $\mathcal{C}_{\text{Bl}_\Sigma \mathcal{F}}$ is, by definition, the continuous extension of the Molino sheaf of $(\text{Bl}_\Sigma \mathcal{F})_{\text{reg}}$, and $\pi^{-1}(M_{\text{reg}}) \subset (\text{Bl}_\Sigma M) \setminus E$ is dense in $(\text{Bl}_\Sigma M)_{\text{reg}}$. If $\mathcal{C}_\mathcal{F}$ is smooth, then, by continuity, the isomorphism is realized by the lift Bl_Σ of local transverse fields, hence sections of $\mathcal{C}_{\text{Bl}_\Sigma \mathcal{F}}$ are also smooth. $\square$

Since inverse images of constant sheaves are constant, we immediately obtain:

Corollary 3.1.5. If $\mathcal{F}$ is a singular Killing foliation of a compact manifold M , then $\text{Bl}_\Sigma \mathcal{F}$ is a singular Killing foliation.

Applying this repeatedly, we get that singular Killing foliations desingularize to Killing foliations:

Corollary 3.1.6. If $\mathcal{F}$ is a singular Killing foliation of a compact manifold M , then $\text{Bl}_\Sigma \mathcal{F}$ is a (regular) Killing foliation.

Still another immediate consequence of Theorem 3.1.4 is:

Corollary 3.1.7. If $\mathcal{F}$ is a singular Riemannian foliation on a connected, compact manifold M , then the structural algebra $\mathfrak{g}$ of the foliation $\mathcal{F}$ is isomorphic to the structural algebra $\mathfrak{g}_{\text{Bl}\mathcal{F}}$ of the blow-up foliation $\text{Bl}\mathcal{F}$.

Proof. By Theorem 3.1.4,

$$\mathfrak{g}_{\mathcal{F}} \cong \text{stalk}(\mathcal{C}_{\mathcal{F}})^{\text{op}} \cong \text{stalk}(\text{Bl}\mathcal{C}_{\mathcal{F}})^{\text{op}} \cong \text{stalk}(\mathcal{C}_{\text{Bl}\mathcal{F}})^{\text{op}} \cong \mathfrak{g}_{\text{Bl}\mathcal{F}}. \quad \square$$

We now recall a result of Gitler [Git82], who computed the singular cohomology (with $\mathbb{Z}_2$ coefficients) of the blow-up $\text{Bl}_{\Sigma}M$. This result serves as the main motivation for the next section, where we adapt Gitler's argument to the foliated setting. Gitler proved that

$$H^i(\text{Bl}_{\Sigma}M, \mathbb{Z}_2) \cong \pi^*(H^i(M, \mathbb{Z}_2)) \oplus \frac{H^*(E, \mathbb{Z}_2)}{\pi^*(H^i(\Sigma, \mathbb{Z}_2))} \quad (3.1)$$

We also recall that the Euler characteristic of M can be computed as

$$\chi(M) = \sum_n (-1)^n \dim H_n(M; F)$$

where $H_n(M; F)$ are the homology group with coefficients in an arbitrary field F [Hat02, Theorem 2.44].

We may now easily compute the Euler characteristic of $\text{Bl}_{\Sigma}M$. The injectivity of the induced pullback map π^* follows from Poincaré duality for homology with $\mathbb{Z}_2$ coefficients, as established by via the Gysin map in [Git82]. Therefore, from equation (3.1), it follows that

$$\dim H^i(\text{Bl}_{\Sigma}M, \mathbb{Z}_2) + \dim H^i(\Sigma, \mathbb{Z}_2) = \dim H^i(E, \mathbb{Z}_2) + \dim H^i(M, \mathbb{Z}_2).$$

The alternate sum on i then leads to

$$\chi(\text{Bl}_{\Sigma}M) + \chi(\Sigma) = \chi(E) + \chi(M).$$

Furthermore, from the fiber bundle property for Euler characteristic we know that $\chi(E) = \chi(\Sigma)\chi(\mathbb{RP}^{k-1})$, where $k = \text{codim}(\Sigma)$. Thus

$$\chi(\text{Bl}_{\Sigma}M) = \chi(\Sigma)(\chi(\mathbb{RP}^{k-1}) - 1) + \chi(M),$$

and so

$$\chi(\text{Bl}_{\Sigma}M) = \begin{cases} \chi(M) & \text{if } k \text{ is odd,} \\ \chi(M) - \chi(\Sigma) & \text{if } k \text{ is even.} \end{cases} \quad (3.2)$$

We will now apply this to prove Theorem H, by reducing it to the regular case,

which was proven in [CT19, Theorem 9.1] (see also [AC22, Theorem 6.4]):

Theorem 3.1.8 (Caramello-Töben). Let $\mathcal{F}$ be a regular Killing foliation of a compact manifold M . If $\chi(M) \neq 0$, then $\mathcal{F}$ is closed.

Let us understand what happens with the Euler characteristic in the blow-up process. Using (3.2), we observe that, for every blow-up along a minimal locus Σ , the possible cases in which $\chi(M) \neq 0$ are:

1. If Σ has odd codimension, then $\chi(\text{Bl}_\Sigma M) \neq 0$.
2. If Σ has even codimension and $\chi(M) > \chi(\Sigma)$, then $\chi(\text{Bl}_\Sigma \mathcal{F}) > 0$.

In order to desingularize $\mathcal{F}$ we must do a finite number of blow-ups, and in each step both cases may occur. So, proceeding inductively, we are led to consider the collection $\Sigma^1, \dots, \Sigma^\ell$ of all the even-codimensional minimal loci appearing in the blow-up process, and the number

$$L := \sum_{j=1}^{\ell} \chi(\Sigma^j).$$

Theorem 3.1.9. Let $\mathcal{F}$ be a singular Killing foliation on a compact manifold M with $\chi(M) \neq 0$. If $\chi(M) > L$, then $\mathcal{F}$ is closed.

Proof. Applying (3.2) inductively one easily concludes that $\chi(\text{Bl} M) = \chi(M) - L$. Hence $\chi(\text{Bl} M) \neq 0$ if and only if $\chi(M) \geq L$. By Theorem 3.1.8, if that is the case then the foliation $\text{Bl} \mathcal{F}$ is closed. This implies that $\mathfrak{g}_{\text{Bl} \mathcal{F}}$ is trivial, and, by Corollary 3.1.7 it follows that $\mathfrak{g}$ is trivial as well. Therefore $\mathcal{F}$ is closed, by Molino's structural theorem 1.7.13. $\square$

It is worth highlighting the following special case:

Corollary 3.1.10. Let $\mathcal{F}$ be a singular Killing foliation on a compact manifold M with $\chi(M) \neq 0$ and suppose all singular loci of $\mathcal{F}$ are odd-codimensional. Then $\mathcal{F}$ is closed.

3.2 Basic Cohomology of the Blown-up Foliation

Consider a singular Riemannian foliation $\mathcal{F}$ on a compact manifold M , and let U be a distinguished tubular neighborhood of the minimal stratum Σ (see Definition 1.4.8), as constructed in Section 2.3. Define

$$H: [0, 1] \times U \longrightarrow U, \quad H(t, x) = \exp^\perp\left((1-t)(\exp^\perp)^{-1}(x)\right).$$

This map is a deformation retraction: indeed, $H(0, x) = x$, while $H(1, x) = \exp^\perp(0) = \pi_\Sigma(x) \in \Sigma$, where $\pi_\Sigma: U \rightarrow \Sigma$ denotes the orthogonal projection, and $H(1, q) = q$ for every $q \in \Sigma$. Moreover, H is a foliated map, since the normal exponential map preserves

the slice foliation induced by $\mathcal{F}$ on U by the Homothetic Transformation Lemma 1.4.10. By the Basic Homotopy Lemma 1.4.5, it follows that

$$H^i(\mathcal{F}|_U) \cong H^i(\mathcal{F}|_\Sigma).$$

Likewise, let $\pi : \text{Bl}_\Sigma M \rightarrow M$ be the blow-up of M along Σ . Set $\text{Bl}_\Sigma U = \pi^{-1}(U)$, and define

$$\widehat{H} : [0, 1] \times \text{Bl}_\Sigma U \longrightarrow \text{Bl}_\Sigma U, \quad \widehat{H}(t, (x, [\xi])) = (H(t, x), [\xi]).$$

This is a foliated deformation retraction of $\text{Bl}_\Sigma U$ onto the exceptional divisor $E = \pi^{-1}(\Sigma)$, which follows directly from the defining properties of H and from the fact that π restricts to a foliated diffeomorphism outside E . Consequently,

$$H^i(\text{Bl}_\Sigma \mathcal{F}|_{\text{Bl}_\Sigma U}) \cong H^i(\text{Bl}_\Sigma \mathcal{F}|_E).$$

The following lemma is a technical tool to be used in the main theorem of this section:

Lemma 3.2.1. Let

$$\begin{array}{ccccccc} \dots & \xrightarrow{f^{k-2}} & A^{k-1} & \xrightarrow{f^{k-1}} & A^k & \xrightarrow{f^k} & A^{k+1} \xrightarrow{f^{k+1}} \dots \\ & & \uparrow \pi^{k-1} & & \uparrow \pi^k & & \uparrow \pi^{k+1} \\ \dots & \xrightarrow{g^{k-2}} & B^{k-1} & \xrightarrow{g^{k-1}} & B^k & \xrightarrow{g^k} & B^{k+1} \xrightarrow{g^{k+1}} \dots \end{array}$$

be a commutative diagram of vector spaces, where both rows are exact sequences. Then the following sequence is a cochain complex

$$\dots \xrightarrow{\overline{f^{k-2}}} \frac{A^{k-1}}{\pi^{k-1}(B^{k-1})} \xrightarrow{\overline{f^{k-1}}} \frac{A^k}{\pi^k(B^k)} \xrightarrow{\overline{f^k}} \frac{A^{k+1}}{\pi^{k+1}(B^{k+1})} \xrightarrow{\overline{f^{k+1}}} \dots$$

where $\overline{f^k}([a]) = [f^k(a)]$. Moreover, if all maps $(\pi_k)_{k \in \mathbb{N}}$ are injective, this sequence is exact.

Proof. First we need to show that the induced maps are well-defined: let $a = \tilde{a} + \pi^k(b)$, where $b \in B^k$, then $\overline{f^k}[a] = [f^k(a)] = [f^k(\tilde{a}) + f^k(\pi^k(b))] = [f^k(\tilde{a})] + [f^k(\pi^k(b))]$. By the commutativity of the diagram, we have that $f^k \circ \pi^k = \pi^{k+1} \circ g^k$, which results in $[f^k(\pi^k(b))] = [\pi^{k+1}(g^k(b))]$, thus $\overline{f^k}[a] = [f^k(\tilde{a})] + [\pi^{k+1}(g^k(b))] = [f^k(\tilde{a}) + \pi^{k+1}(g^k(b))] = \overline{f^k}[\tilde{a}]$. We obtain therefore that $\overline{f^k}$ is well-defined, for every $k \in \mathbb{N}$. It follows that the induced sequence is a cochain complex.

Now suppose all maps π_k are injective. Consider $[f^k(a)] \in \text{im } \overline{f^k}$. Since the rows

are exact sequences, we know that $\text{im } f^k \subset \ker f^{k+1}$, therefore

$$\overline{f^{k+1}}([f^k(a)]) = [f^{k+1}(f^k(a))] = 0.$$

as $f^k(a) \in \text{im } f^k$. To show that it is exact, it remains to prove that $\ker \overline{f^{k+1}} \subset \text{im } \overline{f^k}$. Let $[a] \in \ker \overline{f^{k+1}}$, then

$$\overline{f^{k+1}}([a]) = [f^{k+1}(a)] = 0,$$

thus $f^{k+1}(a) = \pi^{k+2}(b)$, for some $b \in B^{k+1}$. As $\ker f^{k+2} = \text{im } f^{k+1}$, we have $f^{k+2}f^{k+1}(a) = 0$. Moreover, due to commutativity, we know that $f^{k+2} \circ \pi^{k+2} = \pi^{k+3} \circ g^{k+2}$, which implies

$$0 = f^{k+2}(f^{k+1}(a)) = f^{k+2}(\pi^{k+2}(b)) = \pi^{k+3}(g^{k+2}(b)).$$

By hypothesis, π^{k+3} is injective, therefore $g^{k+2}(b) = 0$, that is, $b \in \ker g^{k+2} = \text{im } g^{k+1}$, as it follows from the exactness of the second row. Therefore, there exists a $\tilde{b} \in B^k$ such that $b = g^{k+1}(\tilde{b})$. We obtain

$$f^{k+1}(a) = \pi^{k+2}(g^{k+1}(\tilde{b})) = f^{k+1}(\pi^{k+1}(\tilde{b})) \implies f^{k+1}(a - \pi^{k+1}(\tilde{b})) = 0,$$

therefore $a - \pi^{k+1}(\tilde{b}) \in \ker f^{k+1} = \text{im } f^k$, which implies that there exists a $\tilde{a}$ such that $f^k(\tilde{a}) = f^{k+1}(a) - \pi^{k+1}(\tilde{b})$. Finally,

$$\overline{f^k}[\tilde{a}] = [f^k(\tilde{a})] = [f^{k+1}(a) - \pi^{k+1}(\tilde{b})] = [f^{k+1}(a)],$$

that shows $\ker \overline{f^{k+1}} \subset \text{im } \overline{f^k}$. Since k was arbitrary, we conclude that the induced sequence is exact: $\ker \overline{f^{k+1}} = \text{im } \overline{f^k}$ for every $k \in \mathbb{N}$. $\square$

Moreover, we need the four and five lemmas, technical standard tools in homological theory (check [Hat02]):

Lemma 3.2.2 (Five Lemma for Abelian Groups). Consider the following commutative diagram of abelian groups:

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ l \downarrow & & m \downarrow & & n \downarrow & & p \downarrow & & q \downarrow \\ A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E' \end{array}$$

if the rows are exact and if the homomorphisms l, m, p, q are all isomorphisms, then n is an isomorphism.

Lemma 3.2.3 (Four Lemma for Abelian Groups). Consider the following commu-

tative diagram of abelian groups:

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
 l \downarrow & & m \downarrow & & n \downarrow & & p \downarrow \\
 A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D'
 \end{array}$$

if the rows are exact and if the homomorphisms m and p are injective and l is surjective, then n is injective.

Theorem 3.2.4. Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M , with minimal locus Σ , and let $\pi: \text{Bl}_\Sigma M \rightarrow M$ be the blow-up map. Then there exists a collapsed cochain complex

$$\dots \longrightarrow 0 \longrightarrow \frac{H^i(\text{Bl}_\Sigma \mathcal{F})}{\pi^*(H^i(\mathcal{F}))} \longrightarrow \frac{H^i(\text{Bl}_\Sigma \mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_\Sigma))} \longrightarrow 0 \longrightarrow \dots,$$

which is exact when $\pi^*: H^i(\mathcal{F}) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F})$ is injective. In particular, in this case

$$H^i(\text{Bl}_\Sigma \mathcal{F}) \cong \pi^*(H^i(\mathcal{F})) \oplus \frac{H^i(\text{Bl}_\Sigma \mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_\Sigma))}, \quad (3.3)$$

for every i , as vector spaces.

Proof. In order to clean up the notation, we will write π^* even when this map is restricted/co-restricted to specific subsets; we will explicitly indicate the domain and co-domain of the map when necessary.

Fix a distinguished tubular neighborhood U of Σ and consider $\text{Bl}_\Sigma U = \pi^{-1}(U)$. From now on, we denote

$$U^* := M \setminus U, \quad M^* := M \setminus \Sigma, \quad (\text{Bl}_\Sigma M)^* = \text{Bl}_\Sigma M \setminus E, \quad (\text{Bl}_\Sigma U)^* = \text{Bl}_\Sigma U \setminus E.$$

Observe that $\text{Bl}_\Sigma M = (\text{Bl}_\Sigma U)^* \cup \text{Bl}_\Sigma U$ and $M = U \cup U^*$, hence we can use the Mayer Vietoris sequence for basic cohomology to obtain the commutative diagram

$$\begin{array}{ccccc}
 H^i(\mathcal{F}) & \xrightarrow{i} & H^i(\mathcal{F}|_{M^*}) \oplus H^i(\mathcal{F}|_\Sigma) & \xrightarrow{j} & H^i(\mathcal{F}|_{U^*}) \\
 \downarrow \pi^* & & \downarrow \sigma & & \downarrow \cong \\
 H^i(\text{Bl}_\Sigma \mathcal{F}) & \xrightarrow{\tilde{i}} & H^i(\text{Bl}_\Sigma \mathcal{F}|_{(\text{Bl}_\Sigma M)^*}) \oplus H^i(\text{Bl}_\Sigma \mathcal{F}|_E) & \xrightarrow{\tilde{j}} & H^i(\text{Bl}_\Sigma \mathcal{F}|_{(\text{Bl}_\Sigma U)^*})
 \end{array} \quad (3.4)$$

where we used $H^i(\mathcal{F}|_U) \cong H^i(\mathcal{F}|_\Sigma)$ and $H^i(\text{Bl}_\Sigma \mathcal{F}|_{\text{Bl}_\Sigma U}) \cong H^i(\text{Bl}_\Sigma \mathcal{F}|_E)$ and the fact that

$\pi^*: H^i(\mathcal{F}|_{U^*}) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F}|_{(\text{Bl}_\Sigma U)^*})$ is an isomorphism. The map σ is given by

$$([\eta], [\omega]) \mapsto ([\pi^*(\eta)]|_{\text{Bl}_\Sigma M^*}, [\pi^*(\omega)]|_E),$$

which is an isomorphism on the first factor. Since the horizontal rows of diagram (3.4) are exact, it follows directly from Lemma 3.2.1 that this diagram induces a collapsed cochain complex

$$\cdots \rightarrow 0 \xrightarrow{\bar{\delta}} \frac{H^i(\text{Bl}_\Sigma \mathcal{F})}{\pi^*(H^i(\mathcal{F}))} \xrightarrow{\bar{i}} \frac{H^i(\text{Bl}_\Sigma \mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_\Sigma))} \xrightarrow{\bar{j}} 0 \rightarrow \cdots$$

because $H^i(\text{Bl}_\Sigma \mathcal{F}|_{(\text{Bl}_\Sigma U)^*})/\pi^*(H^i(\mathcal{F}|_{U^*}))$ trivializes for every i , since π^* is an isomorphism when restricted to such subsets.

Claim 3.2.5. If $\pi^*: H^i(\mathcal{F}) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F})$ is injective, then

$$\pi^*: H^i(\mathcal{F}|_\Sigma) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F}|_E)$$

is injective.

Extending diagram (3.4) with:

$$\begin{array}{ccccc} H^{i-1}(\mathcal{F}|_{U^*}) & \longrightarrow & H^i(\mathcal{F}) & \longrightarrow & \cdots \\ \cong \downarrow & & \pi^* \downarrow & & \\ H^{i-1}(\text{Bl}_\Sigma \mathcal{F}|_{\tilde{U}^*}) & \longrightarrow & H^i(\text{Bl}_\Sigma \mathcal{F}) & \longrightarrow & \cdots \end{array}$$

and applying the Four Lemma 3.2.3 gives that $\pi^*: H^i(\mathcal{F}|_\Sigma) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F}|_E)$ is injective, since σ is. This is trivial for any $i > 0$, then let us check the case $i = 0$. For this case, just observe that $\Sigma/\mathcal{F}$ and $E/\text{Bl}_\Sigma \mathcal{F}$ have the same number of connected components (for instance, it is the number of minimal strata).

Therefore all the vertical arrows of (3.4) are injective, and Lemma 3.2.1 guarantees that the collapsed cochain is exact. In particular,

$$\frac{H^i(\text{Bl}_\Sigma \mathcal{F})}{\pi^*(H^i(\mathcal{F}))} \cong \frac{H^i(\text{Bl}_\Sigma \mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_\Sigma))}$$

as vector spaces. Which gives our desired result:

$$H^i(\text{Bl}_\Sigma \mathcal{F}) \cong \pi^*(H^i(\mathcal{F})) \oplus \frac{H^i(\text{Bl}_\Sigma \mathcal{F}|_E)}{\pi^*(H^i(\mathcal{F}|_\Sigma))},$$

□

Remark 3.2.6. It is worth noticing that the same proof holds, more generally, for the blow-up along a closed, saturated submanifold V contained within the minimal locus Σ , since V admits a distinguished tubular neighborhood with the same properties. This further generalization can be used, for instance, when blowing-up an already regular Riemannian foliation along a closed saturated submanifold. For instance, a stratum of $\overline{\mathcal{F}}$ or a holonomy stratum.

In the classical case (see [Git82; GH14]) the hypothesis on the injectivity of π^* is not needed, since it is guaranteed by Poincaré duality. We had to keep this technicality here, since there is no analog property for basic cohomology, in the general singular setting.

Corollary 3.2.7 (De Rham cohomology). For a compact orientable manifold M , let Σ be a closed submanifold of M such that $\text{Bl}_\Sigma M$ is orientable. Then

$$H_{\text{dR}}^i(\text{Bl}_\Sigma M) \cong H_{\text{dR}}^i(M) \oplus \frac{H_{\text{dR}}^i(E)}{\pi^*(H_{\text{dR}}^i(\Sigma))}$$

where $E = \pi^{-1}(\Sigma)$.

Proof. This is the particular case in which we take $\mathcal{F}$ to be the trivial foliation of M by points, and therefore $H(\mathcal{F}) = H_{\text{dR}}(M)$. Notice that $\text{Bl}_\Sigma \mathcal{F}$ is also trivial. Furthermore, when M is orientable, one can construct a left inverse (a Gysin map) for $\pi^*: H_{\text{dR}}^i(\text{Bl}_\Sigma M) \rightarrow H_{\text{dR}}^i(M)$ by means of Poincaré duality [Git82, Section 2]. In particular, this shows that π^* is injective. $\square$

There is a special situation in which the basic cohomology of the original foliation and the basic cohomology of its blow-up are naturally isomorphic. This phenomenon highlights the inherently transverse nature of basic cohomology. In contrast, for the de Rham cohomology of manifolds (3.2.7), such an isomorphism only holds under more restrictive hypotheses, typically depending on how differential forms pull back along the projectivized normal bundle of the blow-up.

Recall that a smooth map $\pi: E \rightarrow B$ is *transverse* to a foliation $\mathcal{F}$ of B when

$$T_{\pi(x)}B = T_{\pi(x)}L_{\pi(x)} + d\pi_x(T_xE),$$

for every $x \in E$. In particular, for submersions this is always true.

Proposition 3.2.8. If $\pi: E \rightarrow B$ is a fiber bundle with connected fiber F and B has a regular foliation $\mathcal{F}$, then π induces an isomorphism between the holonomy pseudogroups of $\pi^*(\mathcal{F})$ and $\mathcal{F}$. Consequently, $\pi^*: H^i(\mathcal{F}) \rightarrow H^i(\pi^*(\mathcal{F}))$ is an isomorphism.

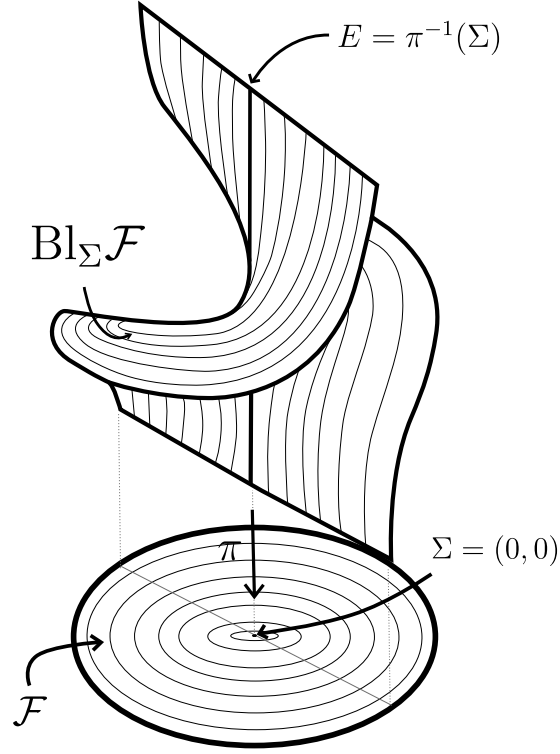


Figure 3.1 – Blow-up of circle foliation in $\mathbb{R}^2$.

Proof. The result follows since in this situation one can choose a total transversals S and $\tilde{S}$ for $\mathcal{F}$ and $\pi^*(\mathcal{F})$, respectively, such that the restriction

$$\tilde{\pi} = \pi|_{\tilde{S}}: \tilde{S} \rightarrow S$$

is a diffeomorphism (for instance, $\tilde{S}$ can be constructed from a transversal S whose components are contained in E -trivializing open sets, as intersections of $\pi^{-1}(S)$ with local sections on the trivializations). The equivalence $\mathcal{H}(\pi^*(\mathcal{F})) \cong \mathcal{H}(\mathcal{F})$ is then induced by $\tilde{\pi}$: it is the family of all restrictions $\pi|_W$, where W is an open subset of $\tilde{S}$. Indeed, $\mathcal{H}(\mathcal{F})$ is generated by elements of the form $\tilde{\pi}|_{W'} \circ \tilde{h} \circ (\tilde{\pi}|_W)^{-1}$, for sufficiently small W and W' (see [Wal12, Chapter 1] for more details).

The space of holonomy-invariant k -forms on S , denoted $\Omega^k(S)^{\mathcal{H}}$, is naturally identified with the space of basic k -forms $\Omega^k(\mathcal{F})$ (see, e.g. [Hae80]). Hence

$$\Omega^k(\pi^*(\mathcal{F})) \cong \Omega^k(\tilde{S})^{\tilde{\mathcal{H}}} \cong \Omega^k(S)^{\mathcal{H}} \cong \Omega^k(\mathcal{F}).$$

where the middle isomorphism is induced by π , as we saw, so π^* is an isomorphism already on the level of basic forms. $\square$

Theorem 3.2.9 (Pullback Foliation on the Exceptional Divisor). Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M , with minimal locus Σ , and let

$\pi: \text{Bl}_\Sigma M \rightarrow M$ be the blow-up map. If $(\text{Bl}_\Sigma \mathcal{F})|_E = \pi^*(\mathcal{F}|_\Sigma)$ then,

$$H^i(\text{Bl}_\Sigma \mathcal{F}) \cong H^i(\mathcal{F}),$$

for every i , as vector spaces.

Proof. As in Theorem 3.2.4, denote

$$U^* := M \setminus U, \quad M^* := M \setminus \Sigma, \quad (\text{Bl}_\Sigma M)^* = \text{Bl}_\Sigma M \setminus E, \quad (\text{Bl}_\Sigma U)^* = \text{Bl}_\Sigma U \setminus E$$

for a distinguished tubular neighborhood U of Σ and consider $\text{Bl}_\Sigma U = \pi^{-1}(U)$. This gives the Mayer-Vietoris diagram (3.4). By Proposition 3.2.8, $\pi^*: H^i(\mathcal{F}|_\Sigma) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F}|_E)$ is an isomorphism, which implies that σ is an isomorphism. Now, by the Five Lemma 3.2.2, we obtain that $\pi^*: H^i(\mathcal{F}) \rightarrow H^i(\text{Bl}_\Sigma \mathcal{F})$ is an isomorphism for every i . Indeed, note that the same observation from Theorem 3.2.4 applies to this case. $\square$

Example 3.2.10. A classical example in which the restriction of a foliation to the normal axial cylinders of a singular locus coincides with the pullback foliation along the locus occurs when the singular stratum has codimension 2 (see the discussion in [MC88, Section 6.2]). This encompasses the paradigmatic example underlying the geometric intuition of the blow-up: the desingularization of the smooth isometric action of $O(2)$ on $\mathbb{R}^2$, in which the singular locus reduces to a single point and the leaves are concentric circles centered at the origin. See figure 3.1.

3.3 Blown-up Leaf Spaces

In this final section we will couple Alexandrino's result on the Gromov–Hausdorff approximation of the leaf spaces of blow-ups (already seen in Theorem 2.3.10) with a similar one, by Caramello and Neubauer [CN25], in order to realize the leaf closure space $M/\overline{\mathcal{F}}$ of a singular Killing foliation as a limit of $\text{codim}(\mathcal{F})$ -dimensional orbifolds. As we discussed in the introduction, the novelty here is the control in the dimension, which we hope can lead to further consequences via convergence of Alexandrov spaces. Let us start by stating the mentioned result in [CN25, Theorem 4.1]:

Theorem 3.3.1 (Caramello-Neubauer). Let $\mathcal{F}$ be a Killing foliation of a connected, compact manifold M . Then there exists a sequence of closed approximations $\mathcal{G}_k$ of $\mathcal{F}$ such that $M/\mathcal{G}_i \xrightarrow{\text{GH}} M/\overline{\mathcal{F}}$.

Let us now prove Theorem L of the introduction.

Theorem 3.3.2. Let $\mathcal{F}$ be a d -codimensional singular Killing foliation of a connected, compact manifold M . Then there exists a sequence of d -codimensional closed Riemannian foliations $\mathcal{G}_i$ of blow-up spaces $(\text{Bl}M)_k$ of M such that $(\text{Bl}M)_k/\mathcal{G}_k \xrightarrow{\text{GH}} M/\overline{\mathcal{F}}$.

Notice that, in the above setting, the quotients $(\text{Bl}M)_k/\mathcal{G}_k$ are d -dimensional orbifolds [MM03, Theorem 2.15].

Proof. By Theorem 2.3.10, there exists a sequence $\{((\text{Bl}M)_i, (\text{Bl}\mathcal{F})_i)\}_{i \in \mathbb{N}}$ such that

$$(\text{Bl}M)_i/\overline{(\text{Bl}\mathcal{F})_i} \xrightarrow{\text{GH}} M/\overline{\mathcal{F}}.$$

By Theorem 3.1.5 we know that each $(\text{Bl}\mathcal{F})_i$ is a (regular) Killing foliation, so Theorem 3.3.2 provides, for each $i \in \mathbb{N}$, a sequence $\{\mathcal{G}_{j_i}\}_{j_i \in \mathbb{N}}$ of closed foliations verifying

$$(\text{Bl}M)_i/\mathcal{G}_{j_i} \xrightarrow{\text{GH}} (\text{Bl}M)_i/\overline{(\text{Bl}\mathcal{F})_i}.$$

Now for each $k \in \mathbb{N}$, there is $N(k) \in \mathbb{N}$ such that $i > N(k)$ implies

$$d_{\text{GH}}\left((\text{Bl}M)_i/\overline{(\text{Bl}\mathcal{F})_i}, M/\overline{\mathcal{F}}\right) < \frac{1}{2k}.$$

Likewise, for each i , we can choose $M_i(k) \geq N(k)$ such that $j_i > M_i(k)$ implies

$$d_{\text{GH}}\left((\text{Bl}M)_i/\mathcal{G}_{j_i}, (\text{Bl}M)_i/\overline{(\text{Bl}\mathcal{F})_i}\right) < \frac{1}{2k}.$$

So, defining $\mathcal{G}_k := \mathcal{G}_{M_i(k)}$, the triangle inequality gives us

$$d_{\text{GH}}\left((\text{Bl}M)_k/\mathcal{G}_k, M/\overline{\mathcal{F}}\right) < \frac{1}{k} \longrightarrow 0$$

□

Bibliography

- [AB+15] Marcos M Alexandrino, Renato G Bettiol, et al. *Lie groups and geometric aspects of isometric actions*. Vol. 82. Springer, 2015.
- [AC22] Marcos M Alexandrino and Francisco C Caramello Jr. “Leaf closures of Riemannian foliations: a survey on topological and geometric aspects of Killing foliations”. In: *Expositiones Mathematicae* 40.2 (2022), pp. 177–230.
- [AK10] Gregory Arone and Marja Kankaanrinta. “On the functoriality of the blow-up construction”. In: *Bulletin of the Belgian Mathematical Society-Simon Stevin* 17.5 (2010), pp. 821–832.
- [Ale04] Marcos M Alexandrino. “Singular Riemannian foliations with sections”. In: *Illinois Journal of Mathematics* 48.4 (2004), pp. 1163–1182.
- [Ale10] Marcos M Alexandrino. “Desingularization of singular Riemannian foliation”. In: *Geometriae Dedicata* 149.1 (2010), pp. 397–416.
- [AR17] Marcos M Alexandrino and Marco Radeschi. “Closure of singular foliations: the proof of Molino’s conjecture”. In: *Compositio Mathematica* 153.12 (2017), pp. 2577–2590.
- [AT08] Marcos Alexandrino and Dirk Töben. “Equifocality of a singular Riemannian foliation”. In: *Proceedings of the American Mathematical Society* 136.9 (2008), pp. 3271–3280.
- [Bol73] John Bolton. “Transnormal systems”. In: *The Quarterly Journal of Mathematics* 24.1 (1973), pp. 385–395.
- [Bre97] Glen E Bredon. *Sheaf theory*. Vol. 170. Springer Science & Business Media, 1997.
- [Car24] Francisco C Caramello Jr. “Equivariant basic cohomology of singular Riemannian foliations”. In: *Monatshefte für Mathematik* 203.4 (2024), pp. 779–808.
- [CC00] Alberto Candel and Lawrence Conlon. *Foliations I*. Vol. 23. American Mathematical Soc., 2000.
- [CN25] Francisco C Caramello Jr and Francisco A Neubauer. “Transverse sphere theorems for Riemannian foliations”. In: *arXiv preprint arXiv:2505.01378* (2025).
- [CT19] Francisco Caramello Jr and Dirk Töben. “Positively curved Killing foliations via deformations”. In: *Transactions of the American Mathematical Society* 372.11 (2019), pp. 8131–8158.

- [DK12] Johannes Jisse Duistermaat and Johan AC Kolk. *Lie groups*. Springer Science & Business Media, 2012.
- [Dom98] Demetrio Domínguez. “Finiteness and tenseness theorems for Riemannian foliations”. In: *American journal of Mathematics* 120.6 (1998), pp. 1237–1276.
- [Fed06] Edmond Fedida. “Sur la théorie des feuilletages associée au repère mobile: cas des feuilletages de Lie”. In: *Differential Topology, Foliations and Gelfand-Fuks Cohomology: Proceedings of the Symposium held at the Pontifica Universidade Católica do Rio de Janeiro, 5–24 January, 1976*. Springer. 2006, pp. 183–195.
- [GH14] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons, 2014.
- [Git82] SAMUEL Gitler. “The cohomology of blow-ups”. In: *Bol. Soc. Mat. Mexicana* 37 (1982), pp. 167–175.
- [GM80] Mark Goresky and Robert MacPherson. “Intersection homology theory”. In: *Topology* 19.2 (1980), pp. 135–162.
- [GT18] Oliver Goertsches and Dirk Töben. “Equivariant basic cohomology of Riemannian foliations”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2018.745 (2018), pp. 1–40.
- [Hae80] André Haefliger. “Some remarks on foliations with minimal leaves”. In: *Journal of Differential Geometry* 15.2 (1980), pp. 269–284.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.
- [Hir62] Heisuke Hironaka. “Resolution of singularities”. In: *Proceedings of the International Congress of Mathematicians*. Stockholm, 1962, pp. 507–525.
- [HS93] Gilbert Hector and Martin Saralegi. “Intersection cohomology of ^{1} -actions”. In: *Transactions of the American Mathematical Society* 338.1 (1993), pp. 263–288.
- [KH86] Aziz El Kacimi-Alaoui and Gilbert Hector. “Décomposition de Hodge basique pour un feuilletage riemannien”. In: *Annales de l’institut Fourier*. Vol. 36. 3. 1986, pp. 207–227.
- [KT06] Franz W Kamber and Philippe Tondeur. “Harmonic foliations”. In: *Harmonic Maps: Proceedings of the NSF-CBMS Regional Conference, Held at Tulane University, New Orleans December 15–19, 1980*. Springer. 2006, pp. 87–121.
- [Lav18] Sylvain Lavau. “A short guide through integration theorems of generalized distributions”. In: *Differential Geometry and its Applications* 61 (2018), pp. 42–58.
- [Lee03] John M Lee. “Smooth manifolds”. In: *Introduction to smooth manifolds*. Springer, 2003, pp. 1–29.

- [Mas92] Xosé Masa. “Duality and minimality in Riemannian foliations”. In: *Commentarii Mathematici Helvetici* 67.1 (1992), pp. 17–27.
- [MC88] Pierre Molino and Grant Cairns. *Riemannian foliations*. Vol. 73. Springer, 1988.
- [MM03] Ieke Moerdijk and Janez Mrcun. *Introduction to foliations and Lie groupoids*. Vol. 91. Cambridge university press, 2003.
- [Mol84] Pierre Molino. “Désingularisation des feuilletages Riemanniens”. In: *American Journal of Mathematics* 106.5 (1984), pp. 1091–1106.
- [Moz85] Witold Mozgawa. “Feuilletages de Killing”. In: *Collectanea Mathematica* (1985), pp. 285–290.
- [MR19] Ricardo Mendes and Marco Radeschi. “A slice theorem for singular Riemannian foliations, with applications”. In: *Transactions of the American Mathematical Society* 371.7 (2019), pp. 4931–4949.
- [MW06] Vicente Miquel and Robert A Wolak. “Minimal singular Riemannian foliations”. In: *Comptes Rendus Mathématique* 342.1 (2006), pp. 33–36.
- [Obs21] Lennart Obster. “Blow-ups of Lie groupoids and Lie algebroids”. In: *arXiv preprint arXiv:2110.12247* (2021).
- [PSW05] JI Prieto, Martintxo Saralegi-Aranguren, and Robert Wolak. “Top dimensional group of the basic intersection cohomology for singular riemannian foliations”. In: *arXiv preprint math/0505676* (2005).
- [Töb07] Dirk Töben. “Singular Riemannian foliations on nonpositively curved manifolds”. In: *Mathematische Zeitschrift* 255.2 (2007), pp. 427–436.
- [Töb14] Dirk Töben. “Localization of basic characteristic classes”. In: *Annales de l’Institut Fourier*. Vol. 64. 2. 2014, pp. 537–570.
- [Wal12] Pawel Walczak. *Dynamics of foliations, groups and pseudogroups*. Vol. 64. Birkhäuser, 2012.