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Robustness results for nonuniform exponential dichotomies

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To my parents and Sofia.

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''As invenções são, sobretudo, o resultado de um trabalho teimoso.''
(DUMONT, 1918)

Abstract

In this dissertation, we investigated nonuniform exponential dichotomies in discrete and continuous evolution processes in Banach spaces. In the discrete setting, we demonstrated that the existence and uniqueness of bounded solutions to nonhomogeneous equations in weighted sequence spaces, combined with the exponential decay of forward and backward homogeneous solutions, imply the admissibility of a nonuniform exponential dichotomy. We further showed that, under the condition that the dichotomy exponent is strictly greater than the nonuniformity exponent, the family of unstable projections is unique, depends continuously on the process, and the dichotomy is robust under small linear perturbations. We extended these results to the continuous setting via the discretization technique, allowing for the direct transposition of the discrete properties. Finally, we analyze another type of non-uniform exponential dichotomy, showing that the two types are complementary and apply to different examples. We also establish a robustness result for this new dichotomy for invertible evolution processes in reflexive spaces.

Keywords: Nonuniform exponential dichotomies. Evolution processes. Robustness.

Resumo

Nesta dissertação, investigamos dicotomias exponenciais não uniformes em processos de evolução discretos e contínuos em espaços de Banach. No caso discreto, demonstramos que a existência e unicidade de soluções limitadas para equações não homogêneas em espaços de sequências ponderadas, juntamente com o decaimento exponencial das soluções homogêneas para frente e para trás, implicam a admissibilidade de uma dicotomia exponencial não uniforme. Demonstramos também que, sob a condição de que o expoente da dicotomia seja estritamente maior que o expoente de não uniformidade, a família de projeções instáveis é única, depende continuamente do processo e a dicotomia é robusta sob pequenas perturbações lineares. Estendemos esses resultados ao cenário contínuo através da técnica de discretização, possibilitando a transposição direta das propriedades discretas. Por fim, analisamos um outro tipo de dicotomia exponencial não uniforme, mostrando que os dois tipos são complementares e se aplicam em diferentes exemplos. Também estabelecendo um resultado de robustez desta nova dicotomia para processos de evolução invertíveis em espaços reflexivos.

Palavras-chave: Dicotomias exponenciais não uniformes. Processos de evolução. Robustez.

Resumo expandido

Introdução

Na Matemática, a área de Sistemas Dinâmicos dedica-se ao estudo de modelos que descrevem a evolução de sistemas ao longo do tempo. Fenômenos como a dinâmica populacional, a valorização de ativos financeiros, a propagação de epidemias e a previsão meteorológica são exemplos clássicos que utilizam as ferramentas desenvolvidas nesta teoria. Duas abordagens fundamentais para a modelagem desses problemas são as *equações de diferenças* e as *equações diferenciais*.

Um sistema é modelado por uma equação diferença quando a transição de um estado para o outro se dá por um passo discreto. De maneira abstrata, considere $(\xi_n)_{n\in\mathbb{Z}}$ uma sequência em um espaço de Banach X tal que ξ_n representa o estado do sistema no instante de tempo n e suponha que exista uma sequência de operadores lineares limitados $(A_n)_{n\in\mathbb{Z}}$ tal que

$$\xi_{n+1} = A_n\xi_n \quad \text{para todo } n \in \mathbb{Z}. \tag{0.1}$$

A partir desta equação, para todo $n \geq m$ definimos $S_{n,m}$ por

$$S_{n,m} := \begin{cases} \mathrm{I}, & n = m; \\ A_{n-1}A_{n-2}\cdots A_m, & n > m. \end{cases}.$$

O conjunto $\mathcal{S} := \{S_{n,m} \mid n,m \in \mathbb{Z},\, n \geq m\}$ é o *processo de evolução discreto* induzido pela Equação (0.1) e para todo $n \geq m$ temos $\xi_n = S_{n,m}\xi_m$, ou seja, $S_{n,m}$ descreve a evolução do sistema do instante m até o instante n. De maneira análoga, podemos induzir um processo de evolução *contínuo* a partir das soluções de uma equação diferencial ordinária linear não autônoma, dado por $\mathcal{T} := \{T(t,s) \mid t,s \in \mathbb{R},\, t \geq s\}$.

Uma propriedade importante dos processos de evolução é a admissibilidade de uma *dicotomia exponencial não-uniforme* (ou NED). Dizemos que um processo de evolução discreto admite NED se existe uma família de projeções limitadas $\Pi^u := \{\Pi^u_n \mid n \in \mathbb{Z}\}$ tal que $\Pi^u_n S_{n,m} = S_{n,m}\Pi^u_m$ para todo $n \geq m$, $S_{n,m}|_{\Pi^u_m}$ é um isomorfismo para todo $n \leq m$ e existem $M \geq 1$, $\beta \geq 0$ e $\gamma > 0$ tais que

$$\left\|S_{n,m}\Pi^u_m\right\| \leq Me^{\beta|m|-\gamma|n-m|} \quad \text{para todo } n \leq m$$

e

$$\left\|S_{n,m}\Pi^s_m\right\| \leq Me^{\beta|m|-\gamma|n-m|} \quad \text{para todo } n \geq m,$$

onde $\Pi^s_n := \mathrm{I} - \Pi^u_n$. Esta propriedade nos diz que nosso espaço pode ser decomposto em duas famílias de subespaços para cada instante de tempo de tal maneira que em um dos subespaços a norma do processo é limitada exponencialmente *para trás* e no outro limitada *para frente*. A definição de dicotomia exponencial não-uniforme para o caso contínuo é semelhante e a chamamos de NEDI.

Fundamentando-se nos trabalhos (Caraballo *et al.*, 2022), (Zhou; Lu; Zhang, 2013), (Barreira; Silva; Valls, 2009) e (Henry, 1981), esta dissertação investiga as condições suficientes para a existência de dicotomias exponenciais não uniformes (NED) em processos de evolução discretos, bem como a robustez dessa propriedade sob perturbações lineares. Estendemos a análise ao caso contínuo utilizando a técnica de *discretização*, estabelecendo uma ponte que permite a transposição direta dos resultados obtidos no cenário discreto. Por fim, examinamos a robustez de um novo tipo de dicotomias não uniformes para processos contínuos, introduzida em (Langa; Obaya; Sousa, 2024).

Objetivos

- Formalizar os conceitos de processos de evolução (discretos e contínuos) e de dicotomias exponenciais não uniformes, motivando as definições por meio de exemplos;
- Estabelecer condições suficientes para que um processo de evolução discreto admita uma dicotomia exponencial não uniforme;
- Enunciar resultados de existência e unicidade para equações não homogêneas associadas a processos de evolução;
- Provar a robustez da admissibilidade de dicotomias exponenciais não uniformes no caso discreto;
- Introduzir a técnica de discretização e utilizá-la para transpor resultados do caso discreto para o caso contínuo;
- Definir um novo tipo de dicotomia exponencial não uniforme (Tipo II) e apresentar as suas principais diferenças em relação à definição clássica de dicotomia exponencial não uniforme (Tipo I);
- Mostrar que os dois conceitos de dicotomia são complementares, mostrando exemplos onde a definição clássica não se aplica;
- Provar um resultado de robustez para esta nova dicotomia em espaços reflexivos para processos invertíveis.

Metodologia

No cenário discreto, esta pesquisa fundamentou-se em (Sousa, 2022) e (Caraballo *et al.*, 2022), sendo o resultado de robustez adaptado de (Zhou; Lu; Zhang, 2013) e (Barreira; Silva; Valls, 2009), utilizando técnicas de (Henry, 1981). Para o contexto contínuo, as referências principais foram (Sousa, 2022) e (Caraballo *et al.*, 2022), enquanto o estudo do novo tipo de dicotomia exponencial não uniforme baseou-se em (Langa; Obaya; Sousa, 2024).

Resultados e Discussão

Inicialmente estudamos o caso discreto, formalizando o conceito de processos de evolução, caracterizando as dicotomias exponenciais não uniformes (NED) e apresentando exemplos.

Provamos que se soluções limitadas da equação não homogênea associada ao processo, com o termo de forçamento pertencente a um espaço de sequências com peso, existirem e forem únicas, e, além disso, se houver decaimento exponencial de soluções limitadas para frente e para trás, então o processo admite uma NED. Adicionalmente, utilizando a função de Green, demonstramos que quando a condição $\gamma > \beta$ é imposta aos expoentes da dicotomia, garantimos a unicidade das projeções, sua dependência contínua e a robustez da dicotomia sob pequenas perturbações lineares.

Depois estendemos a teoria onde agora o tempo não são mais os Inteiros e sim os Reais, ou seja, saimos do tempo discreto para o tempo contínuo. Definimos o que é uma NEDI, o análogo de NED mas agora para o caso contínuo e, utilizando a técnica de discretização, estabelecemos uma ponte com os resultados obtidos no caso discreto. Demonstramos que, se um processo contínuo admite uma NEDI, suas discretizações admitem uma NED. Reciprocamente, provamos que se as discretizações de um processo admitem uma NED, sob uma certa condição de regularidade, então o processo original também admite uma NEDI. Essa correspondência permite transpor os resultados do caso discreto, garantindo a unicidade das projeções, a dependência contínua e a robustez da dicotomia sob perturbações lineares.

Por fim, definimos as dicotomias exponenciais não uniformes de Tipo II (NEDII) onde, ao contrário do Tipo I (NEDI), o limitante da norma da dinâmica projetada depende do tempo final, e não do tempo inicial. Além disso, por meio de contraexemplos, demonstramos que as duas noções de dicotomia são independentes. Por fim, definimos o dual de um processo de evolução invertível e provamos um resultado de robustez para dicotomias de Tipo II em espaços reflexivos.

Considerações Finais

Nesta dissertação, investigamos dicotomias exponenciais não uniformes em espaços de Banach. No cenário discreto, caracterizamos a existência da dicotomia via admissibilidade de espaços de sequências e demonstramos sua robustez sob pequenas perturbações lineares, desde que o expoente da dicotomia supere o da não uniformidade $\gamma > \beta$. Estendemos esses resultados para o caso contínuo ao estabelecer uma relação entre um processo contínuo e suas discretizações, permitindo provar a robustez de NEDIs a partir da robustez de NEDs. Por fim, abordamos as dicotomias de Tipo II, provando que a robustez desta classe em sistemas invertíveis pode ser garantida em espaços de Banach reflexivos.

Palavras-chave: Dicotomias exponenciais não uniformes. Processos de evolução. Robustez.

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Contents

1 Introduction

In Mathematics, the field of Dynamical Systems is dedicated to the study of models that describe the evolution of systems over time. Phenomena such as population dynamics, the fluctuation of financial assets, the spread of epidemics, and weather forecasting are classic examples that make use of the tools developed in this theory. Two fundamental approaches for modeling these problems are *difference equations* and *differential equations*.

A system is modeled by a difference equation when the transition from one state to another occurs in a discrete step. Abstractly, consider $(\xi_n)_{n\in\mathbb{Z}}$ a sequence in a Banach space X such that ξ_n represents the state of the system at time instant n and assume that there exists a sequence of bounded linear operators $(A_n)_{n\in\mathbb{Z}}$ such that

$$\xi_{n+1} = A_n\xi_n \quad \text{for all } n \in \mathbb{Z}. \tag{1.2}$$

From this equation, for all $n \geq m$ we define $S_{n,m}$ by

$$S_{n,m} := \begin{cases} \mathrm{I}, & n = m; \\ A_{n-1}A_{n-2}\cdots A_m, & n > m. \end{cases}.$$

The set $\mathcal{S} := \{S_{n,m} \mid n,m \in \mathbb{Z},\, n \geq m\}$ is the *discrete evolution process* induced by Equation (1.2) and for all $n \geq m$ we have $\xi_n = S_{n,m}\xi_m$, that is, $S_{n,m}$ describes the evolution of the system from instant m to instant n. Analogously, we can induce a *continuous* evolution process from the solutions of a non-autonomous linear ordinary differential equation, given by $\mathcal{T} := \{T(t,s) \mid t,s \in \mathbb{R},\, t \geq s\}$.

An important property of evolution processes is the admissibility of a *nonuniform exponential dichotomy* (or NED). We say that a discrete evolution process admits a NED if there exists a family of bounded projections $\Pi^u := \{\Pi^u_n \mid n \in \mathbb{Z}\}$ such that $\Pi^u_n S_{n,m} = S_{n,m}\Pi^u_m$ for all $n \geq m$, $S_{n,m}|_{\Pi^u_m}$ is an isomorphism for all $n \geq m$ and there exist $M \geq 1$, $\beta \geq 0$ and $\gamma > 0$ such that

$$\left\|S_{n,m}\Pi^u_m\right\| \leq Me^{\beta|m|-\gamma|n-m|} \quad \text{for all } n \leq m$$

and

$$\left\|S_{n,m}\Pi^s_m\right\| \leq Me^{\beta|m|-\gamma|n-m|} \quad \text{for all } n \geq m,$$

where $\Pi^s_n := \mathrm{I} - \Pi^u_n$. This property tells us that our space can be decomposed into two families of subspaces for each time instant in such a way that in one of the subspaces the process norm is exponentially bounded *backward* and in the other bounded *forward*. The definition of nonuniform exponential dichotomy for the continuous case is similar, and we refer to it as NEDI.

Based on the works (Caraballo *et al.*, 2022), (Zhou; Lu; Zhang, 2013), (Barreira; Silva; Valls, 2009), and (Henry, 1981), this dissertation investigates sufficient conditions for the existence of nonuniform exponential dichotomies (NED) in discrete evolution processes, as well as the robustness of this property under linear perturbations. We extend the analysis to the continuous case using the *discretization* technique, establishing a bridge that allows the direct transposition of the results obtained in the discrete setting. Finally, we examine the robustness of a new type of nonuniform dichotomy for continuous processes (NEDII), introduced in (Langa; Obaya; Sousa, 2024).

In Chapter 2, we study the discrete case, formalize the concept of evolution processes, characterize nonuniform exponential dichotomies (NED), and present examples. We prove that if bounded solutions to the nonhomogeneous equation associated with the process, with the forcing term belonging to a weighted sequence space, exist and are unique, and furthermore, if there is exponential decay of forward and backward bounded solutions, then the process admits a NED. Additionally, using the Green's function, we demonstrate that when the condition $\gamma > \beta$ is imposed on the dichotomy exponents, we guarantee the uniqueness of the projections, their continuous dependence, and the robustness of the dichotomy under small linear perturbations.

In Chapter 3, we extend the theory to the continuous case and, with the discretization technique, establish a bridge with the results obtained in the discrete case. We demonstrate that if a continuous process admits a NEDI, its discretizations admit a NED. Conversely, we prove that if the discretizations of a process admit a NED, under a certain regularity condition, then the original process also admits a NEDI. This correspondence allows transposing the results from the discrete case, ensuring the uniqueness of the projections, continuous dependence, and robustness of the dichotomy under linear perturbations.

In Chapter 4, we define Type II nonuniform exponential dichotomies (NEDII), in which, unlike Type I (NEDI), the bound on the norm of the projected dynamics depends on the final time rather than the initial time. Furthermore, through counterexamples, we demonstrate that the two notions of dichotomy are independent. Lastly, we define the dual of an invertible evolution process and prove a robustness result for Type II dichotomies in reflexive spaces.

2 Discrete nonuniform exponential dichotomies

In this chapter, motivated by nonautonomous difference equations, we define discrete evolution processes as a generalization to semigroups. Through examples we show that the classical notion of uniform exponential dichotomy is often insufficient to describe the decay of some systems and require a more general decay rate. This limitation motivates the introduction of nonuniform exponential dichotomies, where the decay rates depend explicitly on the initial time.

A central result of this chapter is the establishment of sufficient conditions for the existence of such dichotomies. We prove that the existence of a unique bounded solution to the nonhomogeneous equation in appropriate sequence spaces together with decayment of backward/forward solutions implies the existence of a nonuniform exponential dichotomy. Furthermore, we construct the Green's function and analyze the forward, backward, and global solutions, showing that the condition where the dichotomy exponent dominates the nonuniformity growth guarantees the uniqueness of solutions and projections. Finally, we prove the continuous dependence of projections and the robustness of nonuniform exponential dichotomies under small perturbations.

This chapter is inspired by (Caraballo *et al.*, 2022, Section 2), (Sousa, 2022, Chapter 2) and (Zhou; Lu; Zhang, 2013).

2.1 Discrete evolution process

2.1.1 Motivation

Modeling time-dependent phenomena is a central motivation across various scientific disciplines. Whether in meteorology, population dynamics, or epidemiology, the fundamental goal is often one of prediction: given the current state of a system and a rule governing its transition, we seek to determine its future states. When time is modeled as a discrete variable, the appropriate mathematical formalism to describe this evolution is that of *difference equations*.

Let $(\xi_n)_{n\in\mathbb{Z}}$ be a sequence in a Banach space X, where ξ_k represents the state of the system at time k. Assume that the dynamics are governed by a linear bounded operator $A : X \to X$ satisfying

$$\xi_{n+1} = A\xi_n, \qquad \text{for all } n \in \mathbb{Z}. \tag{2.3}$$

This recurrence relation determines the *evolution* of the system. Indeed, given the operator A and an initial state ξ_0, any future state can be obtained through direct iteration. For instance, for $n = 3$, we have $\xi_3 = A^3\xi_0$. More generally, for any integers $n \geq m$:

$$\xi_n = A^{n-m}\xi_m.$$

Motivated by the preceding equation, we define the family of operators $\mathcal{S} \coloneqq \{S_{n,m} \mid n,m \in \mathbb{Z},\ n \geq m\}$ by

$$S_{n,m} \coloneqq A^{n-m}, \qquad \text{for all } n \geq m.$$

The operator $S_{n,m}$ describes the evolution of the system from time m to time n. Moreover, this family satisfies the following properties:

$$S_{n,n} = \mathrm{I} \qquad \text{for all } n \in \mathbb{Z} \qquad \text{and} \qquad S_{n,k}S_{k,m} = S_{n,m} \qquad \text{for all } n \geq k \geq m.$$

The first property implies that if there is no time lapse, the state remains unchanged. The second property ensures the consistency of the evolution: evolving from m to k and subsequently from k to n is equivalent to evolving directly from m to n.

We now consider a more general difference equation than the previous one. Let $(\zeta_n)_{n\in\mathbb{Z}}$ be a sequence in a Banach space X. Assume the existence of a sequence of bounded linear operators $(A_n)_{n\in\mathbb{Z}}$ on X satisfying

$$\zeta_{n+1} = A_n\zeta_n, \qquad \text{for all } n \in \mathbb{Z}. \tag{2.4}$$

Analogously to the previous case, for any integers $n \geq m$, we have

$$\zeta_n = A_{n-1}\zeta_{n-1} = A_{n-1}A_{n-2}\zeta_{n-2} = \cdots = A_{n-1}A_{n-2}\cdots A_{m+1}A_m\zeta_m$$

and we define the family of operators $\mathcal{T} \coloneqq \{T_{n,m} \mid n,m \in \mathbb{Z},\ n \geq m\}$ by

$$T_{n,m} \coloneqq \begin{cases} A_{n-1}A_{n-2}\cdots A_m & \text{if } n > m, \\ \mathrm{I} & \text{if } n = m. \end{cases}$$

that also satisfies the properties

$$T_{n,n} = \mathrm{I} \qquad \text{for all } n \in \mathbb{Z} \qquad \text{and} \qquad T_{n,k}T_{k,m} = T_{n,m} \qquad \text{for all } n \geq k \geq m.$$

A crucial distinction between $\mathcal{S}$ and $\mathcal{T}$ lies in their dependence on time parameters. The evolution described by the former depends solely on the *elapsed time* $n-m$, whereas the latter depends also on the initial time m. We say that $\mathcal{S}$ is *autonomous* when:

$$S_{n,m} = S_{p,q} \qquad \text{whenever } n - m = p - q.$$

This distinction comes from the fact that in Equation (2.3), the evolution law is time-invariant, whereas in Equation (2.4), it depends on the specific time instance. In other words, the transition from ξ_0 to ξ_1 is identical to the transition from ξ_{100} to ξ_{101}. Conversely, the transition from ζ_0 to ζ_1 generally differs from that of ζ_{100} to ζ_{101}, as they are governed by the operators A_0 and A_{100}, respectively. This motivates the following definition.

2.1.2 Definition and examples

Definition 2.1 (Discrete evolution process). Let X be a Banach space. A DISCRETE EVOLUTION PROCESS on X is a family of bounded linear operators $\mathcal{S} := \{S_{n,m} \mid n,m \in \mathbb{Z},\ n \geq m\}$ that satisfies the following conditions:

(C1) $S_{n,n} = \mathrm{I}$ for all $n \in \mathbb{Z}$;

(C2) $S_{n,k}S_{k,m} = S_{n,m}$ for all $n,k,m \in \mathbb{Z}$ such that $n \geq k \geq m$.

Moreover, if

$$S_{n,m} = S_{p,q} \qquad \text{whenever } n - m = p - q,$$

we call $\mathcal{S}$ AUTONOMOUS.

In this work, we consider general evolution processes and, therefore, do not assume the autonomous property. For a comprehensive study of autonomous processes (often referred to as *semigroups*), we refer the reader to (Costa, 2012).

Following the motivation from difference equations, we reiterate the possibility of generating evolution processes from a sequence of operators and vice-versa.

Remark 2.2. Let $S := (S_n)_{n\in\mathbb{Z}}$ be a sequence of bounded operators. We can generate a discrete evolution process from S defined by

$$S_{n,n} := \mathrm{I} \quad \text{for all } n \in \mathbb{Z} \quad \text{and} \quad S_{n,m} := S_{n-1}S_{n-2}\cdots S_m = \prod_{i=m}^{n-1} S_i \quad \text{for all } n > m.$$

Conversely, if $\mathcal{S} := \{S_{n,m} \mid n,m \in \mathbb{Z},\ n \geq m\}$ is a discrete evolution process, we can generate a sequence of bounded operators from $\mathcal{S}$ defined by

$$S_n := S_{n+1,n} \qquad \text{for all } n \in \mathbb{Z}.$$

We now present two examples of evolution processes. The first arises from an autonomous difference equation in $\mathbb{R}^2$, whereas the second is generated by a non-autonomous difference equation in $\mathbb{R}$.

Example 2.3. Let A be a matrix in $\mathbb{R}^2$ defined by

$$A := \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \text{for all } n \in \mathbb{Z}.$$

By Remark 2.2 we can generate an evolution process $\mathcal{S}$ defined by

$$S_{n,m} = A^{n-m} = \begin{bmatrix} 3^{n-m} & 0 \\ 0 & 2^{-(n-m)} \end{bmatrix} \qquad \text{for all } n \geq m.$$

Let $(x,y) \in \mathbb{R}^2$ and notice that

$$S_{n,m}(x,y) = (3^{n-m}x, 2^{-(n-m)}y).$$

Example 2.4. Let $a, b > 0$ with $b > a$ and consider the difference equation in $\mathbb{R}$ where

$$A_n := e^{-b-a(-1)^n(2n+1)}$$

Applying Remark 2.2, we can generate an evolution process

$$S_{n,m} := e^{an(-1)^n - am(-1)^m - b(n-m)}.$$

Indeed,

$$\begin{aligned}
S_{n,m} &= \prod_{k=m}^{n-1} A_n \\
&= \prod_{k=m}^{n-1} e^{-b-a(-1)^k(2k+1)} \\
&= \exp\left(\sum_{k=m}^{n-1} -b - a(-1)^k(2k+1)\right) \\
&= \exp\left(-b(n-m) - a\sum_{k=m}^{n-1}(-1)^k(2k+1)\right) \\
&= \exp\left(-b(n-m) - a\sum_{k=m}^{n-1} k(-1)^k - (k+1)(-1)^{k+1}\right)
\end{aligned}$$

and notice that the sum is telescopic. Therefore,

$$S_{n,m} = e^{-b(n-m) - am(-1)^m - an(-1)^n}.$$

2.2 Uniform and nonuniform exponential dichotomies

Consider Example 2.3 again. Let us analyze its asymptotic behavior.

Example (Example 2.3 revisited)**.** Let $(x,y) \in \mathbb{R}^2$ and notice that

$$S_{n,m}(x,y) = (3^{n-m}x, 2^{-(n-m)}y) \qquad \text{for all } n, m \in \mathbb{Z}.$$

If we fix $m \in \mathbb{Z}$ and let $n \to +\infty$, then the first variable of the resulting vector would diverge but the second would converge to zero. Similarly, if $n \to -\infty$, then the first variable of the resulting vector would converge to zero but the second would diverge.

What we have here is a situation where the asymptotic behavior of the dynamic can be analyzed in two separate subspaces: the y-axis, where the process converges *forward* in time, and the x-axis, where the process converges *backward* in time. Let us define two projections, one in each axis.

$$\Pi_n^u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Pi_n^s = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for all } n \in \mathbb{Z}.$$

When estimating the process norm on the y axis we have

$$\begin{aligned} \left\| S_{n,m} \Pi_m^s \right\| &= 2^{-(n-m)} \\ &= e^{\ln\left(2^{-(n-m)}\right)} \\ &= e^{-\ln 2 \cdot |n-m|}, \end{aligned}$$

and similarly for the x-axis we have $\left\| S_{n,m} \Pi_m^u \right\| = e^{-\ln 3 \cdot |n-m|}$. Therefore, our evolution processes have a *exponential decay*.

This example leads us to reflect on when it is possible to ''decompose'' the entire space into two subspaces, where in one the evolution process decays exponentially forward while in the other it decays exponentially backward, such that the decay rate depends on $n - m$. This ''decomposition'' is also called *hyperbolicity* property. The next definition establishes abstract criteria in any Banach space for this to happen.

Definition 2.5 (Discrete uniform exponential dichotomy). A discrete evolution process $\mathcal{S} := \{S_{n,m} \mid n,m \in \mathbb{Z},\ n \geq m\}$ admits a UNIFORM EXPONENTIAL DICHOTOMY (UED) if there exist a family of bounded projections $\Pi^u := \{\Pi_n^u \mid n \in \mathbb{Z}\}$, constants $M \geq 1$ and $\gamma > 0$ such that, defining $\Pi^s := \{\Pi_n^s := \mathrm{I} - \Pi_n^u \mid n \in \mathbb{Z}\}$, the following conditions hold:

(C1) $\Pi_n^u S_{n,m} = S_{n,m} \Pi_m^u$ for all $n \geq m$;

(C2) The restriction $S_{n,m}|_{\mathrm{Im}(\Pi_m^u)} \colon \mathrm{Im}(\Pi_m^u) \to \mathrm{Im}(\Pi_n^u)$ is an isomorphism for all $n \geq m$ and we define $S_{m,n}$ as its inverse;

(C3) For all $n \geq m$,
$$\left\| S_{n,m} \Pi_m^s \right\| \leq M e^{-\gamma |n-m|};$$

(C4) For all $n \leq m$,
$$\left\| S_{n,m} \Pi_m^u \right\| \leq M e^{-\gamma |n-m|}.$$

We call Π^u the family of UNSTABLE PROJECTIONS, Π^s the family of STABLE PROJECTIONS and γ the DICHOTOMY EXPONENT.

Intuitively, the subspaces generated by the stable projections are where the process decays in forward time, whereas the unstable projections correspond to decay in backward time.

Condition (C1) ensures that the process within the unstable space remains in the unstable space, while Condition (C2) enables ‘‘backward evolution’’ via its inverse.

Remark 2.6. Condition (C1) in Definition 2.5 also holds for the stable projections. Indeed, for all $n \geq m$, we have

$$\Pi_n^s S_{n,m} = (\mathrm{I} - \Pi_n^u) S_{n,m} = S_{n,m} - \Pi_n^u S_{n,m} = S_{n,m} - S_{n,m}\Pi_m^u = S_{n,m}(\mathrm{I} - \Pi_m^u) = S_{n,m}\Pi_m^s.$$

Example 2.3 is an example of a discrete evolution process that admits uniform exponential dichotomy with $M := 1$ and $\gamma := \min\{\ln 2, \ln 3\} = \ln 2$. For any finite-dimensional autonomous processes, we can establish the following result:

Remark 2.7. For evolution processes defined on $\mathbb{C}^n$, as in Example 2.3, where a fixed matrix generates the system, it follows that if the spectrum of this matrix contains no eigenvalues with modulus equal to 1, then the process admits a uniform exponential dichotomy. In this case, the stable space is generated by the eigenspaces associated to eigenvalues with modulus less than 1, and the unstable space by those with modulus greater than 1.

Besides that, this property is an open property in the space of square matrices. Let us show that the complement of the space of matrices that have this property is closed. Indeed, let $Y \subset \mathcal{M}_n(\mathbb{C})$ be the space of square matrices that at least one of its eigenvalues have modulus equal to 1. Consider a sequence $(M_i)_{i\in\mathbb{N}}$ of matrices in Y such that $M_i \to M \in \mathcal{M}_n(\mathbb{C})$. By the continuity of the determinant and characteristic polynomial functions in any norm (in finite-dimensional spaces all norms are equivalent), the sequence limit M must also admit a root with norm 1 in its characteristic polynomial. Therefore Y is closed and its complement is open.

Now consider Example 2.4 again. Let us analyze its asymptotic behavior and show that it *does not* admit a uniform exponential dichotomy.

Example (Example 2.4 revisited)**.** Let $x \in X$ and notice that

$$S_{n,m}x := xe^{an(-1)^n - am(-1)^m - b(n-m)} \qquad \text{for all } n \geq m.$$

Let us show that $\mathcal{S}$ does not admit any uniform exponential dichotomy. In $\mathbb{R}$ the only possible projections are 0 or I. For the uniform exponential dichotomy Condition (C1) to hold, the unstable projections family must be constant equal to I or 0. Assume that $\Pi^s = \mathrm{I}$ and that there exists $M \geq 1$ and $\gamma > 0$ and such that

$$\|S_{n,m}\| \leq Me^{-\gamma|n-m|} = e^{\ln(M) - \gamma|n-m|} \qquad \text{for all } n \geq m.$$

If the above inequality holds true, then, by the fact that logarithms preserve inequalities, we have

$$an(-1)^n - am(-1)^m - b(n-m) \leq \ln(M) - \gamma|n-m|.$$

Since the inequality holds for all $n \geq m$, consider $m = 2k+1$ and $n = m+1$ for some $k \in \mathbb{N}$. Substituting the values and rearranging the inequality, we have

$$\begin{aligned} n(-1)^n - m(-1)^m &\leq \frac{\ln(M) + b - \gamma}{a} \\ n + m &\leq \frac{\ln(M) + b - \gamma}{a} \\ 2(k+1) &\leq \frac{\ln(M) + b - \gamma}{a}. \end{aligned}$$

Choosing k big enough, the inequality is false, therefore we achieve a contradiction, implying that $\mathcal{S}$ can not admit a uniform exponential dichotomy. An analogous argument holds for $\Pi^s = 0$.

This example demonstrates that it is not always possible to estimate the process norm using a bound that depends solely on the elapsed time $|n-m|$. Let us observe the following details:

$$\begin{aligned} \|S_{n,m}\| &= e^{an(-1)^n - am(-1)^m - bn + bm} \\ &\leq e^{a|n| + a|m| - b|n-m|} \\ &\leq e^{a(|n-m| + |m|) + a|m| - b|n-m|} \\ &= e^{2a|m| - (b-a)|n-m|}. \end{aligned}$$

In other words, we are unable to estimate the process norm relying solely on the elapsed time $|n-m|$ but we can establish a bound if we allow it to depend on the initial time $|m|$. Drawing a parallel to the transition from autonomous to nonautonomous systems where the dependence shifts from the time interval length to the initial time, we introduce the second fundamental concept of this theory: *nonuniform* exponential dichotomies. In this framework, the decay estimates are not uniform, rather, they depend explicitly on the value of $|m|$.

Definition 2.8 (Discrete nonuniform exponential dichotomy). A discrete evolution process $\mathcal{S} := \{S_{n,m} \mid n,m \in \mathbb{Z}, n \geq m\}$ admits a NONUNIFORM EXPONENTIAL DICHOTOMY (NED) if there exist a family of bounded projections $\Pi^u := \{\Pi^u_n \mid n \in \mathbb{Z}\}$, constants $M \geq 1$, $\beta \geq 0$ and $\gamma > 0$, and a sequence of positive numbers $K := (K_n)_{n\in\mathbb{Z}}$ satisfying

$$K_n \leq Me^{\beta|n|} \qquad \text{for all } n \in \mathbb{Z},$$

such that, defining $\Pi^s := \{\Pi^s_n := \mathrm{I} - \Pi^u_n \mid n \in \mathbb{Z}\}$, the following conditions hold:

(C1) $\Pi^u_n S_{n,m} = S_{n,m}\Pi^u_m$ for all $n \geq m$;

(C2) The restriction $S_{n,m}|_{\mathrm{Im}(\Pi^u_m)}\colon \mathrm{Im}(\Pi^u_m) \to \mathrm{Im}(\Pi^u_n)$ is an isomorphism for all $n \geq m$ and we define $S_{m,n}$ as its inverse;

(C3) For all $n \geq m$,

$$\left\|S_{n,m}\Pi^s_m\right\| \leq K_m e^{-\gamma|n-m|};$$

(C4) For all $n \leq m$,

$$\left\|S_{n,m}\Pi^u_m\right\| \leq K_m e^{-\gamma|n-m|}.$$

We call Π^u the family of UNSTABLE PROJECTIONS, Π^s the family of STABLE PROJECTIONS, K the BOUND SEQUENCE, β the NONUNIFORMITY EXPONENT and γ the DICHOTOMY EXPONENT.

As the title of this work suggests, our focus will be on *nonuniform* exponential dichotomies. For an in-depth study of the uniform case, see (Henry, 1981).

Another fundamental concept in the theory is that of nonhomogeneous equations associated with evolution processes. A solution to the nonhomogeneous equation can be understood as a sequence that satisfies the process dynamics subject to a perturbation, similar to difference equations.

Definition 2.9 (Nonhomogeneous equation solutions). Let $\mathcal{S}$ be a discrete evolution process, $f := (f_n)_{n\in\mathbb{Z}}$ and $\xi := (\xi_n)_{n\in\mathbb{Z}}$ sequences in X and $m \in \mathbb{Z}$. For all $n \in \mathbb{Z}$, consider the NONHOMOGENEOUS EQUATION:

$$\xi_{n+1} = S_{n+1,n}\xi_n + f_n. \tag{2.5}$$

We define:

(D1) ξ is called a NONHOMOGENEOUS FORWARD BOUNDED SOLUTION at m if the equation is satisfied for all $n \geq m$ and $\sup_{n\geq m} \|\xi_n\| < +\infty$.

(D2) ξ is called a NONHOMOGENEOUS BACKWARD BOUNDED SOLUTION at m if the equation is satisfied for all $n < m$ and $\sup_{n\leq m} \|\xi_n\| < +\infty$.

(D3) ξ is called a NONHOMOGENEOUS GLOBALLY BOUNDED SOLUTION if the equation is satisfied for all $n \in \mathbb{Z}$ and $\sup_{n\in\mathbb{Z}} \|\xi_n\| < +\infty$.

If $f = 0$, the equation is referred to as the HOMOGENEOUS EQUATION.

Remark 2.10 (Variation of parameters). For all $m,n \in \mathbb{Z}$, if $(\xi_n)_{n\in\mathbb{Z}}$ satifies the nonhomogeneous equation for all $n \geq k \geq m$, we have

$$\begin{aligned}
\xi_n &= S_{n,n-1}\xi_{n-1} + f_{n-1} \\
&= S_{n,n-1}(S_{n-1,n-2}\xi_{n-2} + f_{n-2}) + f_{n-1} \\
&= S_{n,n-2}\xi_{n-2} + S_{n,n-1}f_{n-2} + S_{n,n}f_{n-1} \\
&= S_{n,n-3}\xi_{n-3} + S_{n,n-2}f_{n-3} + S_{n,n-1}f_{n-2} + S_{n,n}f_{n-1} \\
&= S_{n,m}\xi_m + \sum_{k=m}^{n-1} S_{n,k+1}f_k.
\end{aligned}$$

This formulation is often referred to as the VARIATION OF PARAMETERS formula.

Next, we denote some sequence spaces to simplify the notation throughout the propositions and proofs.

Notation 2.11 (Sequences spaces). Let $(K_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers. We define

(D1) The space of BOUNDED SEQUENCES

$$f := (f_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, X) := \ell^\infty \qquad \text{with} \qquad \|f\| := \sup_{k\in\mathbb{Z}} \{\|f_k\|\} < +\infty.$$

(D2) The space of WEIGHTED BOUNDED SEQUENCES

$$f := (f_n)_{n\in\mathbb{Z}} \in \ell_K^\infty(\mathbb{Z}, X) := \ell_K^\infty \qquad \text{with} \qquad \|f\|_K := \sup_{k\in\mathbb{Z}} \{K_{k+1}\|f_k\|\} < +\infty.$$

We now state a theorem establishing sufficient conditions for an evolution process to admit a nonuniform exponential dichotomy.

2.3 Sufficient conditions to discrete nonuniform exponential dichotomies

Before proving the first main theorem of this work, we establish some key properties of projections in Banach spaces.

To show that an evolution process admits NED, we need to find families of suitable projections. We will prove below an important result that, under certain conditions, we can define projections from the subspaces formed by their image. This will allow us to define a family of subspaces where the NED inequalities hold and then construct a family of projections that satisfies our exponential decayment.

Lemma 2.12 (Properties of projections). Let X be a Banach space. Let U and V be subspaces such that $X = U \oplus V$. Let P be the projection onto U along V, and let Q be the projection onto V along U. The following properties hold true:

(P1) $\mathrm{Ker}(P) = V$;

(P2) $Q = \mathrm{I} - P$;

(P3) P is a bounded linear operator if and only if U and V are closed.

Proof.

(P1) First, we show that P is indeed a projection (linear and idempotent). For any $x \in X$, the assumption $X = U \oplus V$ implies a unique decomposition $x = u_x + v_x$ with $u_x \in U$ and $v_x \in V$. Define $P\colon X \to X$ by $Px = u_x$. To see that P is linear, let $x, y \in X$ and $\lambda \in \mathbb{R}$. Decomposing $x = u_x + v_x$ and $y = u_y + v_y$, we have

$$x + \lambda y = u_x + v_x + \lambda(u_y + v_y) = (u_x + \lambda u_y) + (v_x + \lambda v_y).$$

Since U and V are subspaces, $u_x + \lambda u_y \in U$ and $v_x + \lambda v_y \in V$. By the uniqueness of the decomposition, we must have

$$P(x + \lambda y) = u_x + \lambda u_y = Px + \lambda Py,$$

confirming linearity. To see that $P^2 = P$, observe that for any $x \in X$, since $Px = u_x \in U$, the explicit decomposition of Px is $u_x + 0$. Thus, $P(Px) = u_x = Px$. Now we prove that $\mathrm{Ker}(P) = V$. Clearly, if $x \in V$, we have $x = 0 + x$ (where $0 \in U$), implying $Px = 0$. Conversely, if $x \in \mathrm{Ker}(P)$, then $Px = u_x = 0$. Since $x = u_x + v_x$, this implies $x = 0 + v_x = v_x \in V$.

(P2) Notice that for any $x \in X$, we have

$$(\mathrm{I} - P)x = x - u_x = (u_x + v_x) - u_x = v_x.$$

Therefore, $\mathrm{I} - P = Q$.

(P3) We first prove the direct implication ($\Rightarrow$). Assume P is bounded. Since P is continuous, its kernel $V = \mathrm{Ker}(P)$ is closed. To show U is closed, let $(x_n)_{n\in\mathbb{N}} \subset U$ be a sequence such that $x_n \to x$ for some $x \in X$. We must show that $x \in U$. Since $x_n \in U$, we have $Px_n = x_n$. By the continuity of P,

$$Px = P(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} x_n = x.$$

Thus $x = Px \in \mathrm{Im}(P) = U$, so U is closed. Finally, we prove the converse ($\Leftarrow$). Assume U and V are closed. Since X is a Banach space, it suffices to show that P has a closed graph (by the Closed Graph Theorem). Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence such that $x_n \to x$ and $Px_n \to y$ for some $x, y \in X$. We must show that $y = Px$. For each $n \in \mathbb{N}$, let $x_n = u_n + v_n$ be the unique decomposition with $u_n \in U$ and $v_n \in V$. By definition, $Px_n = u_n$. Since $Px_n \to y$ and U is closed, we have $y \in U$. Furthermore, $v_n = x_n - u_n \to x - y$. Since V is closed, we conclude that $x - y \in V$. Finally, since $x - y \in V = \mathrm{Ker}(P)$, we have $0 = P(x - y) = Px - Py$. Since $y \in U$, we have $Py = y$, which implies $0 = Px - y$, proving that $Px = y$.

□

The first theorem of this work seeks to establish three conditions that an evolution process must satisfy to admit a nonuniform exponential dichotomy. The first condition concerns the existence of solutions to the nonhomogeneous equation for a specific class of perturbations. The second and third conditions guarantee the exponential decay of the process norm on two specific subspaces. This theorem was inspired by (Zhou; Lu; Zhang, 2013) arguments and by Henry's technique in (Henry, 1981).

Theorem 2.13 (Sufficient conditions for NED). Let $\mathcal{S}$ be a discrete evolution process. Let $M \geq 1$, $\beta \geq 0$ and $\gamma > 0$ be constants, and let $K := (K_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers satisfying $K_n \leq Me^{\beta|n|}$ for all $n \in \mathbb{Z}$. Suppose the following conditions hold:

(H1) For all $f \in \ell^\infty_K$ there exists a unique nonhomogeneous globally bounded solution;

(H2) For all $m \in \mathbb{Z}$, if ξ is a homogeneous backward bounded solution at m, then

$$\|\xi_n\| \leq K_m e^{-\gamma|n-m|} \|\xi_m\| \qquad \text{for all } n \leq m; \tag{2.6}$$

(H3) For all $m \in \mathbb{Z}$, if ξ is a homogeneous forward bounded solution at m, then

$$\|\xi_n\| \leq K_m e^{-\gamma|n-m|} \|\xi_m\| \qquad \text{for all } n \geq m. \tag{2.7}$$

Then $\mathcal{S}$ admits a nonuniform exponential dichotomy with family of unstable projections defined in the proof, bound sequence $(\widehat{K}_n)_{n\in\mathbb{Z}}$ defined by $\widehat{K}_n := CK_n = CMe^{\beta|n|}$ for all $n \in \mathbb{Z}$ and some $C \geq 1$ and dichotomy exponent γ.

Proof. First, we must find the candidates for the projection families. Fix $m \in \mathbb{Z}$ and consider the following subspaces:

$$V_m^u := \{x \mid \text{there exists a homogeneous backward bounded solution } \xi \text{ at } m \text{ s.t. } \xi_m = x\}, \tag{2.8}$$

$$V_m^s := \{x \mid \text{there exists a homogeneous forward bounded solution } \xi \text{ at } m \text{ s.t. } \xi_m = x\}. \tag{2.9}$$

We shall prove that both are closed and that $X = V_m^u \oplus V_m^s$.

Claim 1 *For all $m \in \mathbb{Z}$ we have $X = V_m^s + V_m^u$.*

Fix $x \in X$ and define

$$f_k := \begin{cases} K_m^{-1}x, & k = m-1 \\ 0, & k \neq m-1 \end{cases}.$$

By Definition 2.11, since

$$\|f\|_K = \sup_{k\in\mathbb{Z}} \{K_{k+1}\|f_k\|\} = K_m \left\|K_m^{-1}x\right\| = \|x\| < +\infty,$$

clearly $f \in \ell_K^\infty$. By Hypothesis (H1) and applying Remark 2.10, we know that the unique nonhomogeneous globally bounded solution ξ satisfies

$$\xi_n = S_{n,m}\xi_m + \sum_{k=m}^{n-1} S_{n,k+1}f_k = S_{n,m}\xi_m \qquad \text{for all } n \geq m.$$

Then, by the definition of f, notice that

$$\xi_m = S_{m,m-1}\xi_{m-1} + K_m^{-1}x \qquad \text{and} \qquad \xi_n = S_{n,n-1}\xi_{n-1} \qquad \text{for all } n \neq m,$$

and since ξ is bounded, then ξ is a homogeneous forward bounded solution at m and $\xi_m \in V_m^s$. Define

$$\zeta_k := \begin{cases} S_{k,k-1}\zeta_{k-1}, & k > m \\ \xi_k, & k \leq m \end{cases}.$$

By the boundedness of ξ we have that ζ is a homogeneous backward bounded solution at m, therefore $\zeta_m \in V_m^u$. We conclude that

$$x = K_m^{-1}\xi_m - K_m^{-1}\zeta_m,$$

and since V_m^s and V_m^u are subspaces, it is true that $K_m^{-1}\xi_m \in V_m^s$ and $K_m^{-1}\zeta_m \in V_m^u$, proving $X = V_m^s + V_m^u$ as desired.

Claim 2 *For all $m \in \mathbb{Z}$ we have $V_m^s \cap V_m^u = \{0\}$.*

Consider $x \in V_m^s \cap V_m^u$. Let ξ be a homogeneous forward bounded solution at m such that $\xi_m = x$ and ζ a homogeneous backward bounded solution at m such that $\zeta_m = x$. Define

$$\omega_n = \begin{cases} \xi_n, & n > m, \\ x, & n = m, \\ \zeta_n, & n < m. \end{cases}$$

Notice that ω is a homogeneous globally bounded solution and, by Hypothesis (H1), the zero solution is the unique homogeneous globally bounded solution, therefore $\omega = 0$ and $x = 0$.

Claim 3 *For all $m \in \mathbb{Z}$ we have that V_m^s is closed.*

Let $(x_i)_{i \in \mathbb{N}} \subset V_m^s$ be a sequence such that $x_i \to x$ for some $x \in X$. For each $i \in \mathbb{N}$, there exists a homogeneous forward bounded solution $\xi^{(i)}$ at m such that $\xi_m^{(i)} = x_i$. By linearity, for all $i,j \in \mathbb{N}$, the difference $\omega_n^{(i,j)} := \xi_n^{(i)} - \xi_n^{(j)}$ is a homogeneous forward bounded solution with $\omega_m^{(i,j)} = x_i - x_j$. By Hypothesis (H3), we have

$$\left\| \xi_n^{(i)} - \xi_n^{(j)} \right\| \leq K_m e^{-\gamma|n-m|} \left\| x_i - x_j \right\| \qquad \text{for all } n \geq m,$$

implying that $(\xi_n^{(i)})_{i \in \mathbb{N}}$ is a Cauchy sequence for all $n \geq m$ and therefore convergent. Let $\xi_n^{(i)} \to \xi_n$ as $i \to \infty$. Notice that

$$S_{n+1,n}\xi_n = S_{n+1,n} \lim_{i \to \infty} \xi_n^{(i)} = \lim_{i \to \infty} S_{n+1,n}\xi_n^{(i)} = \lim_{i \to \infty} \xi_{n+1}^{(i)} = \xi_{n+1}.$$

Thus, ξ satisfies the homogeneous equation for $n \geq m$. Furthermore, $\xi_m = \lim_{i \to \infty} \xi_m^{(i)} = \lim_{i \to \infty} x_i = x$. Taking the limit as $i \to \infty$ in the inequality

$$\left\| \xi_n^{(i)} \right\| \leq K_m e^{-\gamma|n-m|} \left\| x_i \right\|,$$

we obtain

$$\|\xi_n\| \leq K_m e^{-\gamma|n-m|} \|x\| \leq K_m \|x\| \qquad \text{for all } n \geq m.$$

Since m is fixed, we conclude that $\sup_{n \geq m} \|\xi_n\| < +\infty$ and ξ is indeed a homogeneous forward bounded solution, proving that $x = \xi_m \in V_m^s$.

Claim 4 *For all $m \in \mathbb{Z}$ we have that V_m^u is closed.*

Analogous to **Claim 3**, but instead of forward bounded solutions we consider backward bounded solutions.

By Lemma 2.12, we can define Π_m^u as the projection onto V_m^u along V_m^s, and $\Pi_m^s = \mathrm{I} - \Pi_m^u$.

Claim 5 *For all $n \geq m$ we have $S_{n,m}(V_m^s) \subset V_n^s$*

Consider $x \in V_m^s$. Then there exists a homogeneous forward bounded solution ξ at m such that $\xi_m = x$. Define $y := S_{n,m}x$. We want to prove that there exists a homogeneous forward bounded solution ζ at n such that $\zeta_n = y$. Notice that $\xi_n = S_{n,m}x$ and therefore ξ itself is the desired solution, showing that $S_{n,m}(V_m^s) \subset V_n^s$.

Claim 6 *For all $n \geq m$ we have $S_{n,m}(V_m^u) = V_n^u$*

Let us show that $V_n^u \subset S_{n,m}(V_m^u)$. Consider $x \in V_n^u$. Then there exists a homogeneous backward bounded solution ξ at n such that $\xi_n = x$. Since $m \leq n$, ξ is also a homogeneous backward bounded solution at m, therefore $\xi_m \in V_m^u$ and $S_{n,m}\xi_m = x$, showing that $V_n^s \subset S_{n,m}(V_m^s)$.

Now, we show that $S_{n,m}(V_m^u) \subset V_n^u$. Consider $x \in V_m^u$. Then there exists a homogeneous backward bounded solution ξ at m such that $\xi_m = x$. Define $y := S_{n,m}x$. We want to prove that there exists a homogeneous backward bounded solution ζ at n such that $\zeta_n = y$. Define

$$\zeta_k = \begin{cases} S_{k,m}\xi_m, & n > k \geq m \\ \xi_k, & k < m \end{cases}$$

and notice that ζ is a homogeneous backward bounded solution at n and $\zeta_n = S_{n,m}\xi_m = y$, showing that $S_{n,m}(V_m^u) \subset V_n^u$.

Claim 7 *For all $n \geq m$, $S_{n,m}|_{\mathrm{Im}(\Pi_m^u)} : \mathrm{Im}(\Pi_m^u) \to \mathrm{Im}(\Pi_n^u)$ is an isomorphism*

From **Claim 6**, we know that $S_{n,m}|_{\mathrm{Im}(\Pi_m^u)}$ is surjective onto $\mathrm{Im}(\Pi_n^u)$. Let us prove injectivity by characterizing the kernel. Fix $x \in \mathrm{Im}(\Pi_m^u)$ such that $S_{n,m}x = 0$ and let ξ be a homogeneous backward bounded solution satisfying $\xi_m = x$. Since ξ is a solution, we conclude that $\xi_k = S_{k,m}\xi_m = 0$ for all $k \geq m$. Therefore, ξ is a homogeneous globally bounded solution. By uniqueness of solutions, we conclude that $\xi = 0$, and thus $x = 0$.

Claim 8 *For all $n \geq m$ we have $S_{n,m}\Pi_m^u = \Pi_n^u S_{n,m}$*

Choose $x \in X$ and notice that

$$S_{n,m}x = S_{n,m}\Pi_m^s x + S_{n,m}\Pi_m^u x,$$
$$\Pi_n^u S_{n,m}x = \Pi_n^u S_{n,m}\Pi_m^s x + \Pi_n^u S_{n,m}\Pi_m^u x.$$

By **Claim 5**, we have $\Pi_n^u(S_{n,m}\Pi_m^s x) = 0$. Returning to the equation:

$$\begin{aligned}\Pi_n^u S_{n,m}x &= 0 + \Pi_n^u(S_{n,m}\Pi_m^u x) \\ &= S_{n,m}\Pi_m^u x\end{aligned}$$

where the last equality holds by **Claim 6**.

For the last set of claims we define $\mathcal{F} : \ell_K^\infty \to \ell^\infty$ as $\mathcal{F}(f)$ being the unique nonhomogeneous global solution provided in Hypothesis (H1).

Claim 9 *$\mathcal{F}$ is a bounded linear operator.*

First let us show that $\mathcal{F}$ is linear. Fix $f,g \in \ell_K^\infty$ and $\lambda \in \mathbb{R}$. Let $\xi := \mathcal{F}(f)$ and $\zeta := \mathcal{F}(g)$. Notice that

$$\begin{aligned} \xi_n + \lambda\zeta_n &= S_{n,n-1}\xi_{n-1} + f_{n-1} + \lambda S_{n,n-1}\zeta_{n-1} + \lambda g_{n-1} \\ &= S_{n,n-1}(\xi_{n-1} + \lambda\zeta_{n-1}) + f_{n-1} + \lambda g_{n-1}, \end{aligned}$$

therefore, by uniqueness of solutions, $\mathcal{F}(f) + \lambda\mathcal{F}(g) = \mathcal{F}(f + \lambda g)$. To prove the boundedness we apply the Closed Graph Theorem. Let $(f^{(i)})_{i\in\mathbb{N}} \subset \ell_K^\infty$ be a sequence of sequences such that $f^{(i)} \to f$ for some $f \in \ell_K^\infty$ and consider that $\mathcal{F}(f^{(i)}) = \xi^{(i)} \to \xi$. We want to prove that $\mathcal{F}(f) = \xi$. Note that for all $i \in \mathbb{N}$ we have

$$\xi_{n+1}^{(i)} = S_{n+1,n}\xi_n^{(i)} + f_n^{(i)} \qquad \forall n \in \mathbb{Z}.$$

Passing the limits in both sides of the equation concludes that

$$\xi_{n+1} = S_{n+1,n}\xi_n + f_n \qquad \forall n \in \mathbb{Z}.$$

as desired.

Claim 10 *For all $m \in \mathbb{Z}$ we have $\|\Pi_m^s\| \leq \|\mathcal{F}\|$.*

Fix $x \in X$ and define $f \in \ell_K^\infty$ as defined in **Claim 1**. Let $\xi := \mathcal{F}(f)$. As proved in **Claim 1** we have that $\Pi_m^s x = \xi_m$. Therefore,

$$\|\Pi_m^s x\| = \|\xi_m\| \leq \|\xi\| = \|\mathcal{F}(f)\| \leq \|\mathcal{F}\| \cdot \|f\|_K = \|\mathcal{F}\| \cdot \|x\|$$

as desired.

Claim 11 *For all $m \in \mathbb{Z}$ we have $\|\Pi_m^u\| \leq (1 + \|\mathcal{F}\|)$.*

Using **Claim 10** we have

$$\|\Pi_m^u\| = \|\mathrm{I} - \Pi_m^s\| \leq 1 + \|\mathcal{F}\|.$$

Claim 12 *For all $n \leq m$ we have $\left\|S_{n,m}\Pi_m^u\right\| \leq (1 + \|\mathcal{F}\|)K_m e^{-\gamma|n-m|}$.*

For $x \in X$ consider ζ from **Claim 1**. Applying Hypothesis (H2)

$$\begin{aligned}\left\|S_{n,m}\Pi^u_m x\right\| &= \left\|S_{n,m}\zeta_m\right\| \\ &= \|\zeta_n\| \\ &\leq K_m e^{-\gamma|n-m|}\|\zeta_m\| \\ &\leq K_m e^{-\gamma|n-m|}\|\Pi^u_m x\| \\ &\leq \|\Pi^u_m\| K_m e^{-\gamma|n-m|}\|x\| \\ &\leq (1+\|\mathcal{F}\|)K_m e^{-\gamma|n-m|}\|x\|\end{aligned}$$

Claim 13 *For all $n \geq m$ we have* $\left\|S_{n,m}\Pi^s_m\right\| \leq (1+\|\mathcal{F}\|)K_m e^{-\gamma|n-m|}$

Similarly to **Claim 12**, for $x \in X$ consider ξ from **Claim 1**, we apply Hypothesis (H3) and conclude

$$\begin{aligned}\left\|S_{n,m}\Pi^s_m x\right\| &= \left\|S_{n,m}\xi_m\right\| \\ &= \|\xi_n\| \\ &\leq K_m e^{-\gamma|n-m|}\|\Pi^s_m x\| \\ &\leq \|\Pi^s_m\| K_m e^{-\gamma|n-m|}\|x\| \\ &\leq \|\mathcal{F}\| K_m e^{-\gamma|n-m|}\|x\| \\ &\leq (1+\|\mathcal{F}\|)K_m e^{-\gamma|n-m|}\|x\|\end{aligned}$$

Finally, the constant $C \geq 1$ from the statement is defined as $C := 1 + \|\mathcal{F}\|$.

□

The hypotheses of Theorem 2.13 highlight the need for a deeper understanding of the existence, uniqueness, and asymptotic behavior of forward, backward, and globally bounded solutions. These topics will be addressed in the next section.

2.4 Nonhomogeneous solutions

In this section, we discuss explicit solutions to the nonhomogeneous equation. However, we must first define the Green's function, which unifies the forward and backward decay properties of a dichotomy into a single operator.

Definition 2.14 (Green's function). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u. The function $G \colon \mathbb{Z}^2 \to$

$\mathcal{B}(X)$ defined by

$$G_{n,m} := \begin{cases} S_{n,m}\Pi^s_m, & n \geq m, \\ -S_{n,m}\Pi^u_m, & n < m \end{cases}$$

is called the GREEN'S FUNCTION associated with $\mathcal{S}$ and Π^u.

The following three subsections share a similar structure and are of crucial importance to our study. We establish criteria for the existence and uniqueness of forward, backward, and globally bounded solutions, which relate directly to the hypotheses assumed in Theorem 2.13. Throughout this analysis, we impose the condition $\gamma > \beta$ on the exponents of the nonuniform dichotomy to ensure that, for sufficiently large time scales, the uniform decay rate dominates the nonuniform growth bound, thereby guaranteeing the convergence of the solutions.

2.4.1 Forward bounded solutions

We shall prove that forward bounded solutions are directly related to stable projections for evolution processes admitting a nonuniform exponential dichotomy with $\gamma > \beta$. Furthermore, we show that, although these solutions are not unique, they admit a general form. Although the homogeneous case is a special case of the nonhomogeneous one, proving it separately allows for a cleaner proof of the nonhomogeneous case.

Lemma 2.15 (Homogeneous forward bounded solution). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and dichotomy exponent γ. For all $m \in \mathbb{Z}$ and $v \in X$ define

$$\xi_n = S_{n,m}\Pi^s_m v \qquad \text{for all } n \geq m. \tag{2.10}$$

Then ξ is a homogeneous forward bounded solution at m. Moreover, if $\gamma > \beta$, then any homogeneous forward bounded solution at m can be written as (2.10) for some $v \in X$.

Proof.

Claim 1 $\sup_{n\geq m} \|\xi_n\| < +\infty.$

By NED stable decay property, for all $n \geq m$ we have

$$\|\xi_n\| = \left\|S_{n,m}\Pi^s_m v\right\| \leq K_m e^{-|n-m|} \|v\| \leq K_m \|v\|.$$

Claim 2 *ξ is a homogeneous forward solution at m.*

For all $n \geq m$, notice that

$$S_{n+1,n}\xi_n = S_{n+1,n}S_{n,m}\Pi^s_m v = S_{n+1,m}\Pi^s_m = \xi_{n+1},$$

therefore ξ is an homogeneous forward bounded solution.

Claim 3 *If $\gamma > \beta$, then any homogeneous forward bounded solution at m can be written as (2.10) for some $v \in X$.*

Let ζ be a homogeneous forward bounded solution at m. Consider ξ with $v = \zeta_m$. Notice that $\omega_n := \zeta_n - \xi_n$ is a homogeneous forward bounded solution at m, therefore

$$\zeta_n - \xi_n = S_{n,m}(\zeta_m - \xi_m) = S_{n,m}(\zeta_m - \Pi^s_m \zeta_m) = S_{n,m}\Pi^u_m \zeta_m.$$

Let $k \geq n$ and notice that

$$\Pi^u_k \zeta_k = S_{k,m}\Pi^u_m \zeta_m = S_{k,n}S_{n,m}\Pi^u_m \zeta_m.$$

Since $S_{k,n}$ is restricted to Π^u_n, we can apply its inverse $S_{n,k}$ to obtain

$$S_{n,k}\Pi^u_k \zeta_k = S_{n,m}\Pi^u_m \zeta_m.$$

Estimating the left-hand side, we have

$$\left\| S_{n,k}\Pi^u_k \zeta_k \right\| \leq \left\| S_{n,k}\Pi^u_k \right\| \cdot \|\zeta\| \leq Me^{\beta|k| - \gamma|n-k|} \|\zeta\|,$$

and since $\gamma > \beta$, as $k \to +\infty$ we have that $\beta|k| - \gamma|n-k| \to -\infty$, proving that $\left\| S_{n,k}\Pi^u_k \zeta_k \right\| \to 0$ and

$$0 = \left\| S_{n,m}\Pi^u_m \zeta_m \right\| = \|\zeta_n - \xi_n\|.$$

This shows that $\zeta = \xi$.

□

We now prove that, analogous to the homogeneous case, forward bounded solutions admit a general form under the condition on the exponents.

Proposition 2.16 (Nonhomogeneous forward bounded solution)**.** Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and dichotomy exponent γ. For all $m \in \mathbb{Z}$ and $v \in X$ define

$$\xi_n = S_{n,m}\Pi^s_m v + \sum_{k=m}^{+\infty} G_{n,k+1} f_k \qquad \text{for all } n \geq m. \tag{2.11}$$

Then ξ is a nonhomogeneous forward bounded solution at m. Moreover, if $\gamma > \beta$, then any nonhomogeneous forward bounded solution at m can be written as (2.11) for some $v \in X$.

Proof.

Claim 1 $\sup_{n\geq m} \|\xi_n\| < +\infty$.

For any $n \geq m$, we estimate the norm of ξ_n. By the definition of the Green's function and the dichotomy hypothesis we have:

$$\begin{aligned}\left\|G_{n,k+1}f_k\right\| &\leq \left\|G_{n,k+1}\right\| \|f_k\| \\ &\leq K_{k+1}e^{-\gamma|n-(k+1)|} \|f_k\| \\ &\leq \|f\|_K \, e^{-\gamma|n-k-1|}.\end{aligned}$$

Summing over $k \in \mathbb{Z}$, and using the substitution $i = n - k - 1$, we obtain:

$$\begin{aligned}\sum_{k=m}^{+\infty} \left\|G_{n,k+1}f_k\right\| &\leq \|f\|_K \sum_{k=m}^{+\infty} e^{-\gamma|n-k-1|} \\ &= \|f\|_K \sum_{i=-\infty}^{n-m-1} e^{-\gamma|i|} \\ &= \|f\|_K \cdot \frac{1 + e^{-\gamma} - e^{-\gamma|n-m|}}{1 - e^{-\gamma}}.\end{aligned}$$

By the Triangle Inequality and the comparison test, we have

$$\begin{aligned}\|\xi_n\| &\leq \left\|S_{n,m}\Pi^s_m v\right\| + \sum_{k=m}^{+\infty} \left\|G_{n,k+1}f_k\right\| \\ &\leq K_m e^{-\gamma|n-m|} \|v\| + \|f\|_K \cdot \frac{1 + e^{-\gamma} - e^{-\gamma|n-m|}}{1 - e^{-\gamma}}.\end{aligned}$$

Since m is fixed and $n \geq m$, at $n = m$ the right-hand side reaches its maximum value, therefore

$$\|\xi_n\| \leq K_m \|v\| + \|f\|_K \cdot \frac{e^{-\gamma}}{1 - e^{-\gamma}}$$

Claim 2 *ξ is a nonhomogeneous forward solution at m.*

Let $n \geq m$. Notice that

$$\begin{aligned}\xi_{n+1} - S_{n+1,n}\xi_n &= S_{n+1,m}\Pi^s_m v + \sum_{k=m}^{+\infty} G_{n+1,k+1}f_k - S_{n+1,n}\xi_n \\ &= \sum_{k=m}^{+\infty} G_{n+1,k+1}f_k - S_{n+1,n} \sum_{k=m}^{+\infty} G_{n+1,k+1}f_k \\ &= \sum_{k=m}^{+\infty} G_{n+1,k+1}f_k - S_{n+1,n}G_{n+1,k+1}f_k \\ &= \sum_{k=m}^{+\infty} (G_{n+1,k+1} - S_{n+1,n}G_{n+1,k+1})f_k\end{aligned}$$

We analyze the term inside the summation for each $k \geq m$:

- If $k > n$, then

$$\begin{aligned} G_{n+1,k+1} - S_{n+1,n} G_{n,k+1} &= -S_{n+1,k+1} \Pi^u_{k+1} - S_{n+1,n}(-S_{n,k+1} \Pi^u_{k+1}) \\ &= 0. \end{aligned}$$

- If $k < n$, then

$$\begin{aligned} G_{n+1,k+1} - S_{n+1,n} G_{n,k+1} &= S_{n+1,k+1} \Pi^s_{k+1} - S_{n+1,n} S_{n,k+1} \Pi^s_{k+1} \\ &= 0. \end{aligned}$$

- If $k = n$, then

$$\begin{aligned} G_{n+1,n+1} - S_{n+1,n} G_{n,n+1} &= \Pi^s_{n+1} - S_{n+1,n}(-S_{n,n+1} \Pi^u_{n+1}) \\ &= \Pi^s_{n+1} + S_{n+1,n+1} \Pi^u_{n+1} \\ &= \Pi^s_{n+1} + \Pi^u_{n+1} \\ &= \mathrm{I}\,. \end{aligned}$$

Thus, the only non-zero term in the series corresponds to $k = n$. The equation simplifies to:

$$\xi_{n+1} - S_{n+1,n} \xi_n = f_n,$$

as desired.

Claim 3 *If $\gamma > \beta$, then any nonhomogeneous forward bounded solution at m can be written as (2.11) for some $v \in X$.*

Let ζ be a nonhomogeneous forward bounded solution at m. Consider ξ with $v = \zeta_m$. Define $\omega_n := \zeta_n - \xi_n$ and for all $n \geq m$ notice that

$$\omega_{n+1} = \zeta_{n+1} - \xi_{n+1} = S_{n+1,n} \zeta_n + f_n - S_{n+1,n} \xi_n - f_n = S_{n+1,n} \omega_n,$$

therefore ω is a homogeneous forward bounded solution at m. Notice that

$$\begin{aligned} \Pi^s_m \xi_m &= \Pi^s_m \left[S_{m,m} \Pi^s_m \zeta_m + \sum_{k=m}^{+\infty} G_{m,k+1} f_k \right] \\ &= \Pi^s_m \zeta_m + \sum_{k=m}^{+\infty} \Pi^s_m S_{m,k+1} \Pi^u_{k+1} f_k \\ &= \Pi^s_m \zeta_m, \end{aligned}$$

since $k+1>m$ inside the series and $\Pi^s_{k+1}\Pi^u_{k+1}=0$. By Lemma 2.15, we have that

$$\omega_n = S_{n,m}\Pi^s_m\omega_m = S_{n,m}\Pi^s_m(\zeta_m - \Pi^s_m\zeta_m) = 0.$$

This shows that $\zeta = \xi$.

□

2.4.2 Backward bounded solutions

Analogously to the forward case, we establish the relationship between backward bounded solutions and unstable projections, in addition to presenting the general form of these solutions.

Lemma 2.17 (Homogeneous backward bounded solution). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and dichotomy exponent γ. For all $m \in \mathbb{Z}$ and $v \in X$ define

$$\xi_n = S_{n,m}\Pi^u_m v \qquad \text{for all } n \leq m. \tag{2.12}$$

Then ξ is a homogeneous backward bounded solution at m. Moreover, if $\gamma > \beta$, then any homogeneous backward bounded solution at m can be written as (2.12) for some $v \in X$.

Proof.

Claim 1 $\sup\limits_{n\leq m} \|\xi_n\| < +\infty$.

Analogous to Lemma's 2.15 **Claim 1**

Claim 2 *ξ is a homogeneous backward solution at m.*

Analogous to Lemma's 2.15 **Claim 2**

Claim 3 *If $\gamma > \beta$, then any homogeneous backward bounded solution at m can be written as (2.12) for some $v \in X$.*

Analogous to Lemma's 2.15 **Claim 3**

□

Now the nonhomogeneous backward case, with proof similar to the forward case.

Proposition 2.18 (Nonhomogeneous backward bounded solution). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family

Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and dichotomy exponent γ. For all $m \in \mathbb{Z}$ and $v \in X$ define

$$\xi_n = S_{n,m}\Pi^u_m v + \sum_{k=-\infty}^{m-1} G_{n,k+1} f_k \qquad \text{for all } n \leq m. \tag{2.13}$$

Then ξ is a nonhomogeneous backward bounded solution at m. Moreover, if $\gamma > \beta$, then any nonhomogeneous backward bounded solution at m can be written as (2.13) for some $v \in X$.

Proof.

Claim 1 $\sup_{n\leq m} \|\xi_n\| < +\infty$.

Analogous to Proposition's 2.16 **Claim 1**.

$$\begin{aligned} \|\xi_n\| &\leq \left\|S_{n,m}\Pi^u_m v\right\| + \|f\|_K \cdot \sum_{k=-\infty}^{m-1} e^{-\gamma|n-k-1|} \\ &\leq K_m e^{-\gamma|n-m|} \|v\| + \|f\|_K \cdot \frac{1 + e^{-\gamma} - e^{-\gamma|m-n+1|}}{1 - e^{-\gamma}} \\ &\leq K_m \|v\| + \|f\|_K \cdot \frac{1}{1 - e^{-\gamma}} \end{aligned}$$

Claim 2 *ξ is a nonhomogeneous backward solution at m.*

Analogous to Proposition's 2.16 **Claim 2**.

Claim 3 *If $\gamma > \beta$, then any nonhomogeneous backward bounded solution at m can be written as (2.13) for some $v \in X$.*

Analogous to Proposition's 2.16 **Claim 3**.

□

2.4.3 Globally bounded solutions

Finally, for the global case, we show that if $\gamma > \beta$, then the homogeneous and nonhomogeneous solutions are indeed unique, in contrast to the forward and backward cases. By virtue of linearity, proving that the trivial solution is the unique globally bounded homogeneous solution suffices to guarantee uniqueness for the nonhomogeneous case.

Proposition 2.19 (Homogeneous globally bounded solution). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and dichotomy exponent $\gamma > \beta$. The trivial solution is the unique homogeneous globally bounded solution.

Proof. The trivial solution is clearly a homogeneous globally bounded solution even without the $\gamma > \beta$ hypothesis. Let ξ be a homogeneous globally bounded solution and $m \in \mathbb{Z}$. Clearly ξ is both a homogeneous forward/backward bounded solution at m. By Lemmas 2.15 and 2.17 we have that

$$\xi_m = \Pi_m^s \xi_m = \Pi_m^u \xi_m,$$

and this can only be true if $\xi_m = 0$, therefore $\xi = 0$. □

Proving existence and uniqueness for the global case is directly related to the first Hypothesis of Theorem 2.13.

Proposition 2.20 (Nonhomogeneous globally bounded solution)**.** Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and dichotomy exponent γ. Define

$$\xi_n = \sum_{k=-\infty}^{+\infty} G_{n,k+1} f_k. \tag{2.14}$$

Then ξ is a nonhomogeneous globally bounded solution. Moreover, if $\gamma > \beta$, then ξ is unique.

Proof.

Claim 1 $\sup_{n \in \mathbb{Z}} \|\xi_n\| < +\infty$.

Analogous to Proposition's 2.16 **Claim 1** and Proposition's 2.18 **Claim 1**.

$$\begin{aligned}
\|\xi_n\| &\leq \sum_{k=-\infty}^{+\infty} \left\|G_{n,k+1} f_k\right\| \\
&\leq \|f\|_K \cdot \sum_{k=-\infty}^{+\infty} e^{-\gamma|n-k-1|} \\
&\leq \|f\|_K \cdot \frac{1+e^{-\gamma}}{1-e^{-\gamma}}.
\end{aligned}$$

Claim 2 *ξ is a nonhomogeneous global solution at m.*

Analogous to Proposition's 2.16 **Claim 2** and Proposition's 2.18 **Claim 2**.

Claim 3 *If $\gamma > \beta$, then ξ is the unique nonhomogeneous globally bounded solution.*

Let ζ and ω be nonhomogeneous globally bounded solutions. Define $\chi_n := \zeta_n - \omega_n$ and notice that χ is a homogeneous globally bounded solution. By Proposition 2.19 and 2.17 we have that $\chi = 0$, therefore $\zeta = \omega$.

□

These three sections establish a partial converse to Theorem 2.13, as the existence of a NEDI with $\gamma > \beta$ implies the uniqueness of the bounded solution to the homogeneous equation. In the next section, we will examine which additional properties can be derived when $\gamma > \beta$ and how they relate to the spaces constructed in Theorem 2.13.

2.5 Uniqueness and continuous dependence of projections

We will now show that if the condition on the exponents holds, the family of unstable projections constructed in the proof of Theorem 2.13 is *unique*. The main sources for these propositions were (Caraballo *et al.*, 2022) and (Sousa, 2022).

Proposition 2.21 (Uniqueness of projections). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound sequence K satisfying $K_n \leq Me^{\beta|n|}$, and exponent $\gamma > \beta$. Suppose Γ^u is another family of unstable projections associated with a NED for $\mathcal{S}$, with bound $K'_n \leq M'e^{\beta'|n|}$ and exponent $\gamma' > \beta'$. Then $\Pi^u = \Gamma^u$.

Proof. Let G be the Green's function associated with Π^u and let Q be the Green's function associated with Γ^u. Fix $x \in X$, $m \in \mathbb{Z}$ and define

$$f_k := \begin{cases} K_m^{-1}x, & k = m-1 \\ 0, & k \neq m-1 \end{cases}.$$

Clearly $f \in \ell^\infty_K$. Applying Proposition 2.20, we know that there exists a unique nonhomogeneous globally bounded solution ξ. Notice that

$$\xi_m = \sum_{k\in\mathbb{Z}} G_{m,k+1}f_k = G_{m,m}f_{m-1} = \Pi^s_m x = (\mathrm{I} - \Pi^u_m)x.$$

Using Q (associated with Γ^u):

$$\xi_m = \sum_{k\in\mathbb{Z}} Q_{m,k+1}f_k = Q_{m,m}f_{m-1} = \Gamma^s_m x = (\mathrm{I} - \Gamma^u_m)x.$$

By the uniqueness of the solution ξ, we must have:

$$(\mathrm{I} - \Pi^u_m)x = (\mathrm{I} - \Gamma^u_m)x,$$

which implies $\Pi^s_m x = \Gamma^s_m x$. Since $x \in X$ and $m \in \mathbb{Z}$ are arbitrary, we conclude that $\Pi^s = \Gamma^s$, and consequently $\Pi^u = \Gamma^u$. □

We will show that, provided the conditions on the exponents are satisfied, if two evolution processes admitting a NED with the same bound are "close" in some sense, then their respective projections are also "close".

Proposition 2.22 (Continuous dependence of projections). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound $K_n \leq Me^{\beta|n|}$, and exponent $\gamma_{\mathcal{S}} > \beta$. Let $\mathcal{T}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Γ^u, bound $K_n \leq Me^{\beta|n|}$, and exponent $\gamma_{\mathcal{T}} > \beta$. If there exists $\varepsilon > 0$ such that

$$\sup_{k\in\mathbb{Z}}\{K_{k+1}\left\|S_{k+1,k} - T_{k+1,k}\right\|\} \leq \varepsilon,$$

then

$$\sup_{k\in\mathbb{Z}}\{K_k^{-1}\left\|\Pi_k^u - \Gamma_k^u\right\|\} \leq \varepsilon \cdot \frac{e^{-\gamma_{\mathcal{S}}} + e^{-\gamma_{\mathcal{T}}}}{1 - e^{-(\gamma_{\mathcal{S}}+\gamma_{\mathcal{T}})}}.$$

Proof. Fix $x \in X$ and $m \in \mathbb{Z}$, and define

$$f_n := \begin{cases} K_m^{-1}x, & n = m-1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $f \in \ell_K^\infty$. Applying Proposition 2.20, we know that there exist unique nonhomogeneous globally bounded solutions $\xi \in \ell^\infty$ and $\zeta \in \ell^\infty$ such that

$$\xi_{n+1} = S_{n+1,n}\xi_n + f_n,$$
$$\zeta_{n+1} = T_{n+1,n}\zeta_n + f_n,$$

for all $n \in \mathbb{Z}$. Subtracting the second equation from the first and defining $\omega_n := \xi_n - \zeta_n$, we obtain:

$$\begin{aligned}
\omega_{n+1} &= S_{n+1,n}\xi_n - T_{n+1,n}\zeta_n \\
&= S_{n+1,n}\xi_n - S_{n+1,n}\zeta_n + S_{n+1,n}\zeta_n - T_{n+1,n}\zeta_n \\
&= S_{n+1,n}(\xi_n - \zeta_n) + (S_{n+1,n} - T_{n+1,n})\zeta_n \\
&= S_{n+1,n}\omega_n + g_n,
\end{aligned}$$

where $g_n := (S_{n+1,n} - T_{n+1,n})\zeta_n$.

By the hypothesis on $\left\|S_{k+1,k} - T_{k+1,k}\right\|$ and since $\zeta \in \ell^\infty$, we conclude that $g \in \ell_K^\infty$. Applying Proposition 2.20 again, let G be the Green's function associated with $\mathcal{S}$ and Q the one associated with $\mathcal{T}$. We can express ω as:

$$\begin{aligned}
\omega_n &= \sum_{k\in\mathbb{Z}} G_{n,k+1}g_k \\
&= \sum_{k\in\mathbb{Z}} G_{n,k+1}(S_{k+1,k} - T_{k+1,k})\zeta_k \\
&= \sum_{k\in\mathbb{Z}} G_{n,k+1}(S_{k+1,k} - T_{k+1,k})\left(\sum_{l\in\mathbb{Z}} Q_{k,l+1}f_l\right).
\end{aligned}$$

By the definition of f, only the term $l = m-1$ is non-zero, so the inner sum becomes $Q_{k,m}K_m^{-1}x$. Thus:

$$\omega_n = \sum_{k\in\mathbb{Z}} G_{n,k+1}(S_{k+1,k} - T_{k+1,k})Q_{k,m}K_m^{-1}x.$$

Evaluating at $n = m$ and estimating the norm:

$$\begin{aligned}
\|\omega_m\| &\le \sum_{k\in\mathbb{Z}} \left\|G_{m,k+1}\right\| \left\|S_{k+1,k} - T_{k+1,k}\right\| \left\|Q_{k,m}\right\| \|w\| K_m^{-1} \\
&\le \sum_{k\in\mathbb{Z}} K_{k+1}e^{-\gamma_S|m-k-1|} \left\|S_{k+1,k} - T_{k+1,k}\right\| K_m e^{-\gamma_T|k-m|} \|w\| K_m^{-1} \\
&\le \|w\| \sum_{k\in\mathbb{Z}} K_{k+1} \left\|S_{k+1,k} - T_{k+1,k}\right\| e^{-\gamma_S|m-k-1|} e^{-\gamma_T|k-m|} \\
&\le \|w\| \,\varepsilon \sum_{k\in\mathbb{Z}} e^{-\gamma_S|m-k-1|} e^{-\gamma_T|m-k|}.
\end{aligned}$$

Let $i = m-k$. Then $m-k-1 = i-1$ and $k-m = -i$. The sum becomes:

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} e^{-\gamma_S|i-1|-\gamma_T|i|} &= \sum_{i=-\infty}^{0} e^{-\gamma_S|i-1|-\gamma_T|i|} + \sum_{i=1}^{+\infty} e^{-\gamma_S|i-1|-\gamma_T|i|} \\
&= \sum_{i=-\infty}^{0} e^{\gamma_S(i-1)+\gamma_T i} + \sum_{i=1}^{+\infty} e^{-\gamma_S(i-1)-\gamma_T i} \\
&= e^{-\gamma_S} \sum_{i=-\infty}^{0} e^{(\gamma_S+\gamma_T)i} + e^{\gamma_S} \sum_{i=1}^{+\infty} e^{-(\gamma_S+\gamma_T)i} \\
&= e^{-\gamma_S} \cdot \frac{1}{1-e^{-(\gamma_S+\gamma_T)}} + e^{\gamma_S} \cdot \frac{e^{-(\gamma_S+\gamma_T)}}{1-e^{-(\gamma_S+\gamma_T)}} \\
&= \frac{e^{-\gamma_S} + e^{-\gamma_T}}{1-e^{-(\gamma_S+\gamma_T)}}.
\end{aligned}$$

Substituting this back into the inequality for $\|z_m\|$:

$$\|\omega_m\| \le \varepsilon \cdot \frac{e^{-\gamma_S} + e^{-\gamma_T}}{1-e^{-(\gamma_S+\gamma_T)}} \|x\|.$$

Finally, using the definitions of the solutions at m:

$$\begin{aligned}
\omega_m &= \xi_m - \zeta_m \\
&= \sum_{i=-\infty}^{+\infty} G_{m,k+1}f_{k+1} - \sum_{i=-\infty}^{+\infty} Q_{m,k+1}f_{k+1} \\
&= (\mathrm{I} - \Pi_m^u)K_m^{-1}x - (\mathrm{I} - \Gamma_m^u)K_m^{-1}x \\
&= K_m^{-1}(\Gamma_m^u - \Pi_m^u)x.
\end{aligned}$$

Therefore,
$$K_m^{-1}\left\|(\Pi_m^u - \Gamma_m^u)x\right\| = \|\omega_m\| \leq \varepsilon \cdot \frac{e^{-\gamma_S} + e^{-\gamma_T}}{1 - e^{-(\gamma_S+\gamma_T)}}\|x\|.$$
Since $x \in X$ and $m \in \mathbb{Z}$ are arbitrary, the proof is complete. □

As we shall see, continuous dependence and robustness are closely linked. Under appropriate conditions, if an evolution process with $\gamma > \beta$ is subjected to a small perturbation and the perturbed process also admits a NED, then the respective families of projections will remain close, ensuring stability.

2.6 Robustness of NEDs

From the perspective of applying mathematical models to real-world problems, it is important to consider the possibility of errors and noise. These may arise as measurement errors due to equipment imprecision or as noise caused by external interference during an experiment. We will determine a maximum admissible perturbation for an evolution process, ensuring that it still admits a unique solution to the nonhomogeneous equation. Unfortunately (as is to be expected), we will not be able to explicitly find the solution for a perturbed process, instead we will only prove its existence and uniqueness using the Banach Fixed Point Theorem.

Proposition 2.23 (Perturbed nonhomogeneous equation solution). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound $K_n \leq Me^{\beta|n|}$, and exponent $\gamma > \beta$. Let $f \in \ell_K^\infty$ and $B := (B_n)_{n\in\mathbb{Z}}$ be a sequence of bounded operators such that

$$\|B_k\| \leq \delta K_{k+1}^{-1} \qquad \text{for all } k \in \mathbb{Z},$$

where $0 < \delta < \dfrac{1-e^{-\gamma}}{1+e^{-\gamma}}$. Consider $\mathcal{T} := \{T_{n,m} \mid n,m \in \mathbb{Z}\, n \geq m\}$ the evolution process generated by $(S_n + B_n)_{n\in\mathbb{Z}}$. There exists a unique $\xi \in \ell^\infty$ such that

$$\xi_{n+1} = T_{n+1,n}\xi_n + f_n \qquad \text{for all } n \in \mathbb{Z}. \tag{2.15}$$

Proof. Applying Proposition's 2.20 **Claim 2**, if there exists $\xi \in \ell^\infty$ such that

$$\xi_n = \sum_{k\in\mathbb{Z}} G_{n,k+1}(B_k\xi_k + f_k),$$

then ξ is a nonhomogeneous globally bounded solution to (2.15). Define the operators $\mathcal{G}$ and $\mathcal{B}$ by

$$(\mathcal{G}x)_n := \sum_{k\in\mathbb{Z}} G_{n,k+1}x_k, \qquad \text{and} \qquad (\mathcal{B}x)_n := B_n x_n.$$

Our desired solution is a fixed point of the map $\mathcal{F}$ defined by

$$\mathcal{F}x = \mathcal{G}\mathcal{B}x + \mathcal{G}f.$$

To prove the existence of this fixed point, we invoke the Banach Fixed Point Theorem. Note that

$$\begin{aligned}\|\mathcal{F}x - \mathcal{F}y\| &= \|\mathcal{G}\mathcal{B}x - \mathcal{G}\mathcal{B}y\| \\ &= \|\mathcal{G}\mathcal{B}(x-y)\| \\ &\leq \|\mathcal{G}\mathcal{B}\| \, \|x-y\| .\end{aligned}$$

Thus, we must prove that the operator norm satisfies $\|\mathcal{G}\mathcal{B}\| < 1$. Fix $m \in \mathbb{Z}$ and $x \in \ell^\infty$. Notice that

$$\begin{aligned}\|(\mathcal{G}\mathcal{B}x)_m\| &= \left\| \sum_{k\in\mathbb{Z}} G_{m,k+1} B_k x_k \right\| \\ &\leq \sum_{k\in\mathbb{Z}} \|G_{m,k+1}\| \, \|B_k\| \, \|x_k\| \\ &\leq \sum_{k\in\mathbb{Z}} K_{k+1} e^{-\gamma|m-k-1|} \delta K_{k+1}^{-1} \|x\| \\ &= \|x\| \, \delta \sum_{k\in\mathbb{Z}} e^{-\gamma|m-k-1|} \\ &= \|x\| \, \delta \cdot \frac{1+e^{-\gamma}}{1-e^{-\gamma}}.\end{aligned}$$

Using the hypothesis $\delta < \frac{1-e^{-\gamma}}{1+e^{-\gamma}}$, we have:

$$\begin{aligned}\|(\mathcal{G}\mathcal{B}x)_m\| &< \|x\| \left(\frac{1-e^{-\gamma}}{1+e^{-\gamma}}\right) \cdot \left(\frac{1+e^{-\gamma}}{1-e^{-\gamma}}\right) \\ &= \|x\| .\end{aligned}$$

Since this holds for all m, taking the supremum yields $\|\mathcal{G}\mathcal{B}x\| \leq \lambda \|x\|$ for some $\lambda < 1$. Therefore, $\mathcal{F}$ is a contraction and admits a unique fixed point by the Banach Fixed Point Theorem. □

The following proposition will not be proven as the proof is beyond the scope of this study. The proof can be found in (Barreira; Silva; Valls, 2009, Lemma 8, Lemma 17).

Lemma 2.24 (Bound for perturbed solutions). Let $M \geq 1$, $\beta,\nu \geq 0$, $\gamma > 0$ and $\delta > 0$ be constants. Define

$$\widetilde{\gamma} := -\log\left(\cosh\gamma - \sqrt{\cosh^2\gamma - 1 - 2\delta M \sinh\gamma}\right)$$

and

$$\widehat{\gamma} := \log\left(\cosh\gamma + \sqrt{\cosh^2\gamma - 1 - 2\delta M \sinh\gamma}\right).$$

These are well defined for a sufficiently small δ. Also define

$$\widetilde{M}_1 := \frac{M}{1 - \delta D \dfrac{e^{-\gamma}}{1 - e^{-(\gamma+\widetilde{\gamma})}}}$$

and

$$\widetilde{M}_2 := \frac{M}{1 - \delta D \dfrac{e^{\gamma}}{e^{\gamma+\widehat{\gamma}} - 1}}.$$

Let $m \in \mathbb{Z}$ be a fixed index. Suppose that a bounded nonnegative sequence $(x_n)_{n\in\mathbb{Z}}$ satisfies

$$x_n \le M e^{\beta|n| - \gamma|n-m|}\nu + \delta M \sum_{k=m}^{+\infty} e^{-\gamma|n-k-1|} x_k \qquad \text{for all } n \ge m,$$

then

$$x_n \le \widetilde{M}_1 e^{\beta|m| - \widetilde{\gamma}|n-m|}\nu \qquad \text{for all } n \ge m.$$

Similarly, suppose that a bounded nonnegative sequence $(x_n)_{n\in\mathbb{Z}}$ satisfies

$$x_n \le M e^{\beta|n| - \gamma|n-m|}\nu + \delta M \sum_{k=-\infty}^{m-1} e^{-\gamma|n-k-1|} x_k \qquad \text{for all } n \le m,$$

then

$$x_n \le \widetilde{M}_2 e^{\beta|m| - \widehat{\gamma}|n-m|}\nu \qquad \text{for all } n \ge m.$$

Finally, the robustness of nonuniform exponential dichotomies will be demonstrated. In mathematics, a property is considered "robust" if it persists even under small perturbations. In our case, using perturbed nonhomogeneous equations, we will demonstrate the conditions a dichotomy must satisfy to be preserved even when the process admitting it is perturbed. This theorem is a direct follow up from Theorem 2.13 and is based on the work of (Zhou; Lu; Zhang, 2013).

Theorem 2.25 (Robustness of NEDs). Let $\mathcal{S}$ be a discrete evolution process that admits a nonuniform exponential dichotomy with unstable projection family Π^u, bound K satisfying $K_n \le M e^{\beta|n|}$, and exponent $\gamma > \beta$. Let $B = (B_n)_{n\in\mathbb{Z}}$ be a sequence of bounded operators. Assume that

$$\|B_k\| \le \delta K_{k+1}^{-1} \qquad \text{for all } k \in \mathbb{Z}, \qquad \text{where } 0 < \delta < \frac{1 - e^{-\gamma}}{1 + e^{-\gamma}}.$$

Then the evolution process $\mathcal{T} := \{T_{n,m} \mid n \ge m\}$ generated by the sequence of operators $(S_n + B_n)_{n\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy with bound $\widetilde{K}_m \le C \max\{\widetilde{M}_1, \widetilde{M}_2\} e^{\beta|m|}$

as defined in Lemma 2.24 and $C \geq 1$ defined in Theorem 2.13, and dichotomy exponent $\min\{\widetilde{\gamma}, \widehat{\gamma}\}$ also defined in Lemma 2.24.

Proof. We will prove that $\mathcal{T}$ satisfies the three hypothesis of Theorem 2.13.

(H1) The first hypothesis follows directly from Proposition 2.23, which guarantees the existence and uniqueness of globally bounded solutions for the perturbed nonhomogeneous equation.

(H2) Fix $m \in \mathbb{Z}$ and consider a homogeneous backward bounded solution ξ at m for the process $\mathcal{T}$. Applying Proposition 2.10, we have

$$\xi_n = S_{n,m}\Pi_m^u \xi_m + \sum_{k=-\infty}^{m-1} G_{n,k+1} B_k \xi_k \qquad \text{for all } n \leq m.$$

Estimating the norm using the triangle inequality and the hypothesis on B_k:

$$\begin{aligned}
\|\xi_n\| &\leq \left\|S_{n,m}\Pi_m^u\right\| \|\xi_m\| + \sum_{k=-\infty}^{m-1} \left\|G_{n,k+1}\right\| \|B_k\| \|\xi_k\| \\
&\leq M e^{\beta|m|-\gamma|n-m|} \|\xi_m\| + \sum_{k=-\infty}^{m-1} \left(K_{k+1} e^{-\gamma|n-k-1|}\right)\left(\delta K_{k+1}^{-1}\right) \|\xi_k\| \\
&= M e^{\beta|m|-\gamma|n-m|} \|\xi_m\| + \delta \sum_{k=-\infty}^{m-1} e^{-\gamma|n-k-1|} \|\xi_k\| \\
&\leq M e^{\beta|m|-\gamma|n-m|} \|\xi_m\| + \delta M \sum_{k=-\infty}^{m-1} e^{-\gamma|n-k-1|} \|\xi_k\| .
\end{aligned}$$

Using Lemma 2.24 with $\nu := \|\xi_m\|$, we conclude that ξ decays exponentially in backward time, satisfying the second hypothesis.

(H3) Analogous to the claim above.

Thus, $\mathcal{T}$ satisfies all hypothesis of Theorem 2.13 and admits a nonuniform exponential dichotomy.

□

This ends our study in discrete evolution processes. In the next chapter, we study evolution processes that depend on *continuous* time, rather than *discrete* time instants.

3 Continuous nonuniform exponential dichotomies

In this chapter, we extend the framework of evolution processes to the continuous-time setting. We begin by motivating continuous evolution processes through linear differential equations and define nonuniform exponential dichotomies (NEDI), illustrating their relevance with examples where uniform dichotomies fail to exist.

Using the *discretization* technique, we establish a correspondence between continuous and discrete dynamics. We prove that a continuous process admits a dichotomy if and only if its discretizations do, provided a regularity condition is met. This connection allows us to transfer the results obtained in the previous chapter to the continuous context. Specifically, we utilize this bridge to establish the uniqueness of the projection families, their continuous dependence, and the robustness of nonuniform exponential dichotomies under small linear perturbations.

The main sources for this chapter are (Caraballo *et al.*, 2022, Section 3) and (Sousa, 2022, Chapter 2).

3.1 Continuous evolution processes

3.1.1 Motivation

Certain real-world problems are modeled within a *continuous* time framework, such as planetary orbits around the Sun or the heat distribution on a metal plate. These phenomena are typically described by *differential equations*.

Analogous to difference equations, we can generate the continuous counterpart of discrete evolution processes through the solutions of differential equations. Let $x : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function and A a linear operator on $\mathbb{R}^n$. Assume that

$$x'(t) = Ax(t), \qquad \text{for all } t \in \mathbb{R}. \tag{3.16}$$

We can solve this autonomous linear ODE using the matrix exponential (see (Hale, 1980, Chapter 3)) of A:

$$x(t) = e^{At}x_0 = \sum_{n=0}^{+\infty} \frac{A^n t^n}{n!} x_0, \qquad \text{for all } t \in \mathbb{R},$$

for some $x_0 \in \mathbb{R}^n$. For each $s \in \mathbb{R}$ and $v \in \mathbb{R}^n$, define $S(\cdot,s)v$ as the unique solution of Equation (3.16) satisfying the initial condition $x(s) = v$. We then have

$$S(t,s)v := e^{A(t-s)}v, \qquad \text{for all } t \geq s.$$

In this case, the family of operators exhibits properties similar to discrete evolution processes:

$$S(t,t) = \mathrm{I} \quad \text{for all } t \in \mathbb{R} \quad \text{and} \quad S(t,r)S(r,s) = S(t,s) \quad \text{for all } t \geq r \geq s.$$

More generally, let $y : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function and $A : \mathbb{R} \to \mathcal{B}(\mathbb{R}^n)$ be a continuous function. Assume that

$$y'(t) = A(t)y(t), \qquad \text{for all } t \in \mathbb{R}. \tag{3.17}$$

For each $s \in \mathbb{R}$ and $v \in \mathbb{R}^n$, define $T(\cdot,s)v$ as the unique solution of Equation (3.17) satisfying the initial condition $y(s) = v$. By existence and uniqueness of solutions to linear autonomous ODEs, this new family of operators also satisfies the following properties:

$$T(t,t) = \mathrm{I} \quad \text{for all } t \in \mathbb{R} \quad \text{and} \quad T(t,r)T(r,s) = T(t,s) \quad \text{for all } t \geq r \geq s.$$

We can now define *continuous evolution processes*.

3.1.2 Definition

Definition 3.1 (Continuous evolution process). Let X be a Banach space. A CONTINUOUS EVOLUTION PROCESS on X is a family of bounded linear operators $\mathcal{S} := \{S(t,s) \mid t,s \in \mathbb{R},\, t \geq s\}$ that satisfies the following conditions:

(C1) $S(t,t) = \mathrm{I}$ for all $t \in \mathbb{R}$;

(C2) $S(t,r)S(r,s) = S(t,s)$ for all $t,r,s \in \mathbb{R}$ such that $t \geq r \geq s$;

(C3) The map $H\colon \{(t,s) \in \mathbb{R}^2 \mid t \geq s\} \times X \to X$ defined by $H(t,s,x) := S(t,s)x$ is continuous.

Moreover, if

$$S(t,s) = S(u,v) \qquad \text{whenever } t - s = u - v,$$

we call $\mathcal{S}$ AUTONOMOUS.

Following the motivation from differential equations, we reiterate the possibility of inducing an evolution process via the solutions of linear ODEs.

Remark 3.2. Let $A\colon \mathbb{R} \to \mathcal{B}(\mathbb{R}^n)$ be a continuous function and consider the linear differential equation

$$x' = A(t)x.$$

For each $s \in \mathbb{R}$ and $v \in \mathbb{R}^n$, define $S(\cdot,s)v$ as the unique solution satisfying the initial condition $x(s) = v$. Then the family $\mathcal{S}$ constituted by these operators is a continuous evolution process.

In particular, if there exists a differentiable family of invertible operators $T\colon \mathbb{R} \to \mathcal{B}(\mathbb{R}^n)$ satisfying $T'(t) = A(t)T(t)$ for all $t \in \mathbb{R}$ (often referred to as a FUNDAMENTAL SOLUTION, see (Hale, 1980, Chapter 4)), then the evolution process is explicitly given by

$$S(t,s) = T(t)T(s)^{-1} \qquad \text{for all } t \geq s.$$

3.2 Continuous nonuniform exponential dichotomy

We extend the theory of exponential dichotomies to the continuous-time setting. Following similar motivations presented in the last chapter, we define nonuniform exponential dichotomies (NEDI), mirroring the structure established for the discrete case.

Definition 3.3 (Continuous nonuniform exponential dichotomy of type I)**.** A continuous evolution process $\mathcal{S} := \{S(t,s) \mid t,s \in \mathbb{R},\ t \geq s\}$ admits a CONTINUOUS NONUNIFORM EXPONENTIAL DICHOTOMY OF TYPE I (NEDI) if there exist a family of bounded projections $\Pi^u := \{\Pi^u(t) \mid t \in \mathbb{R}\}$, constants $M \geq 1$, $\beta \geq 0$ and $\gamma > 0$, and a function $K\colon \mathbb{R} \to [1,+\infty)$ satisfying

$$K(t) \leq Me^{\beta|t|} \qquad \text{for all } t \in \mathbb{R}$$

such that, defining $\Pi^s := \{\Pi^s(t) := \mathrm{I} - \Pi^u(t) \mid t \in \mathbb{R}\}$, the following conditions hold:

(C1) $\Pi^u(t)S(t,s) = S(t,s)\Pi^u(s)$ for all $t \geq s$;

(C2) The restriction $S(t,s)|_{\mathrm{Im}(\Pi^u(s))}\colon \mathrm{Im}(\Pi^u(s)) \to \mathrm{Im}(\Pi^u(t))$ is an isomorphism for all $t \geq s$ and we define $S(s,t)$ as its inverse;

(C3) For all $t \geq s$,

$$\|S(t,s)\Pi^s(s)\| \leq K(s)e^{-\gamma|t-s|};$$

(C4) For all $t \leq s$,

$$\|S(t,s)\Pi^u(s)\| \leq K(s)e^{-\gamma|t-s|}.$$

We call Π^u the family of UNSTABLE PROJECTIONS, Π^s the family of STABLE PROJECTIONS, K the BOUND FUNCTION, β the NONUNIFORMITY EXPONENT, and γ the DICHOTOMY EXPONENT. If $\beta = 0$, then the dichotomy is UNIFORM.

In this text, we refer to NED whenever we are talking about the discrete case and to NEDI for the continuous case. The terminology "*type I*" is based on (Langa; Obaya; Sousa, 2024) and serves to distinguish these from the "*type II*" dichotomies that will be introduced in the next chapter. Since we do not deal with type II dichotomies in this chapter, we will omit the type designation when referring to them.

Analogous to the discrete setting, one can find examples of continuous evolution processes that admit a nonuniform exponential dichotomy but fail to admit a uniform one. The following example is inspired from (Barreira; Valls, 1998).

Example 3.4. Let $a, b > 0$ with $b > a$ and consider the linear differential equation $x' = (-b - at\sin(t))x$. The general solution is given by:

$$T(t) = c_1 e^{-a\sin(t)+at\cos(t)-bt}.$$

Applying Remark 3.2, for all $t, s \in \mathbb{R}$ we can induce an evolution process $\mathcal{S}$ defined by:

$$\begin{aligned} S(t,s)x := T(t)T^{-1}(s)x &= x \cdot \frac{e^{-a\sin(t)+at\cos(t)-bt}}{e^{-a\sin(s)+as\cos(s)-bs}} \\ &= xe^{-a\sin(t)+a\sin(s)+at\cos(t)-as\cos(s)-bt+bs}. \end{aligned}$$

Then $\mathcal{S}$ admits NEDI with $\Pi^s(t) = \mathrm{I}$ for all $t \in \mathbb{R}$, bound $K(s) := e^{2a}e^{2a|s|}$ and dichotomy exponent $b - a$. Moreover, $\mathcal{S}$ does not admit NEDI.

Indeed, restricting our analysis to $t \geq s$, we estimate the trigonometric functions and obtain

$$\begin{aligned} \|S(t,s)\| &= e^{a(t\cos t-\sin t-s\cos s+\sin s)-b|t-s|} \\ &\leq e^{a(|t|\cos t-\sin t+|s|\cos s+\sin s)-b|t-s|} \\ &= e^{a+a+a|t|+a|s|-b|t-s|} \\ &= e^{2a}e^{a|t|+a|s|-b|t-s|} \\ &\leq e^{2a}e^{a|s|+a|t-s|+a|s|-b|t-s|} \\ &= e^{2a}e^{2a|s|-(b-a)|t-s|}. \end{aligned}$$

Therefore, the process admits a nonuniform exponential dichotomy with family of projections, bound and exponent stated above.

Now let us show that $\mathcal{S}$ does not admit any uniform exponential dichotomy. In $\mathbb{R}$ the only possible projections are 0 or I. For the nonuniform exponential dichotomy Condition (C4) to hold, the unstable projections family must be constant equal to I or 0. Assume that $\Pi^s = \mathrm{I}$ and that there exists $M \geq 1$ and $\gamma > 0$ and such that

$$\|S(t,s)\| \leq Me^{-\gamma|t-s|} = e^{\ln(M)-\gamma|t-s|} \qquad \text{for all } t \geq s.$$

If the above inequality holds true, then, by the fact that logarithms preserve inequalities, we have

$$a(t\cos t - \sin t - s\cos s + \sin s) - b|t-s| \leq \ln(M) - \gamma|t-s|.$$

Since the inequality holds for all $t \geq s$, consider $s = (2k+1)\pi$ and $t = s + \pi$ for some $k \in \mathbb{N}$. Substituting the values and rearranging the inequality, we have

$$\begin{aligned} t\cos t - \sin t - s\cos s + \sin s &\leq \frac{\ln(M) + \pi(b-\gamma)}{a} \\ t\cos t - s\cos s &\leq \frac{\ln(M) + \pi(b-\gamma)}{a} \\ t + s &\leq \frac{\ln(M) + \pi(b-\gamma)}{a} \\ 2\pi(k+1) &\leq \frac{\ln(M) + \pi(b-\gamma)}{a}. \end{aligned}$$

Choosing k big enough, the inequality is false, therefore we achieve a contradiction, implying that $\mathcal{S}$ cannot admit a uniform exponential dichotomy. An analogous argument holds for $\Pi^s = 0$.

Note that the preceding example resembles Example 2.4. Indeed, Example 2.4 can be derived from this continuous process via the *discretization* technique, which will be discussed in the next section. The next example is taken from (Barreira; Valls, 1998, Chapter 2) and provides a bidimensional extension of the scalar case presented above.

Example 3.5. Let $a, b, c > 0$ with $b > a$ and $c > a$. Consider the family of operators $A : \mathbb{R} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(t) := \begin{bmatrix} -b - at\sin t & 0 \\ 0 & c + at\sin t \end{bmatrix}.$$

Let $\mathcal{T}$ be the evolution process in $\mathbb{R}$ induced by $x' = (-b - at\sin t)x$ and $\mathcal{U}$ by $x' = (c + at\sin t)x$. Then the evolution process $\mathcal{S}$ in $\mathbb{R}^2$ induced by $x' = A(t)x$ is

$$\begin{aligned} S(t,s) &:= \begin{bmatrix} T(t,s) & 0 \\ 0 & U(t,s) \end{bmatrix} \\ &= \begin{bmatrix} e^{-a\sin t + at\cos t + a\sin s - as\cos s - bt + bs} & 0 \\ 0 & e^{a\sin t - at\cos t - a\sin s + as\cos s + ct - cs} \end{bmatrix}. \end{aligned}$$

Consider the projections

$$\Pi^s(s) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi^u(s) := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

then $\mathcal{S}$ admits NEDI with unstable projection family Π^u, bound $K(s) := e^{2a}e^{2a|s|}$ and exponent $\gamma := \min\{c-a, b-a\}$.

Indeed, the stable direction estimates follow Example 3.4. For the unstable direction, consider $t \leq s$. We have:

$$\begin{aligned}\|S(t,s)\Pi^u(s)\| &= \|U(t,s)\| \\ &= e^{a\sin t - at\cos t - a\sin s + as\cos s + ct - cs} \\ &\leq e^{a + a|t| + a + a|s| - c|t-s|} \\ &= e^{2a} e^{a|t| + a|s| - c|t-s|} \\ &\leq e^{2a} e^{2a|s| - (c-a)|t-s|}\end{aligned}$$

Therefore, the process admits a nonuniform exponential dichotomy with family of projections, bound and exponent stated above.

We now prove that $\mathcal{S}$ does not admit a uniform exponential dichotomy by showing that any family of unstable projections must coincide with Π^u. Suppose there exists a family Γ^u of unstable projections such that $\mathcal{S}$ admits a uniform dichotomy with exponent $\gamma > 0$. To show that $\Pi^u = \Gamma^u$ we will prove that $\mathrm{Im}(\Gamma^u(s)) = \ker(\Pi^s(s))$ for all $s \in \mathbb{R}$.

Fix $s \in \mathbb{R}$ and assume, by contradiction, that there exists $u \in \mathrm{Im}(\Gamma^u(s))$ such that $u \notin \ker(\Pi^s(s))$. Observe that

$$\lim_{t \to -\infty} \|S(t,s)u\| = \lim_{t \to -\infty} \|S(t,s)\Gamma^u(s)u\| \leq \lim_{t \to -\infty} Me^{-\gamma|t-s|} = 0,$$

and by the continuity of projections we have

$$\lim_{t \to -\infty} \|\Pi^s S(t,s)u\| = 0.$$

By assumption, we have

$$\begin{aligned}\lim_{t \to -\infty} \|\Pi^s S(t,s)u\| &= \lim_{t \to -\infty} \|S(t,s)\Pi^s u\| \\ &= \lim_{t \to -\infty} \|S(t,s)(u_x, 0)\| \\ &= \lim_{t \to -\infty} \|T(t,s)u_x\|,\end{aligned}$$

and as $u \notin \ker(\Pi^s(s))$, then $u_x \neq 0$. Analyzing the exponent of $T(t,s)$ with $t < 0$, we have

$$-a\sin t + at\cos t - bt + a\sin s - as\cos s + bs \geq -a + at - bt + a\sin s - as\cos s + bs.$$

Knowing that s is fixed and $b > a$, we conclude that $\lim_{t \to -\infty}(a-b)t = +\infty$. Therefore,

$$\lim_{t \to -\infty} \|\Pi^s S(t,s)u\| = \lim_{t \to -\infty} \|T(t,s)u_x\| \geq +\infty$$

and this is a contradiction. Therefore, there cannot exist $u \in \mathrm{Im}(\Gamma^u(s))$ such that $u \notin \ker(\Pi^s(s))$. This implies that $\mathrm{Im}(\Gamma^u(s)) \subset \ker(\Pi^u(s))$, but as $\ker(\Pi^u(s))$ is unidimensional we have that $\mathrm{Im}(\Gamma^u(s)) = \ker(\Pi^u(s))$. Finally, this proves that $\Gamma^u(s) = \Pi^u$ and $\Gamma^s(s) = \Pi^s$ for all $s \in \mathbb{R}$. However, as previously demonstrated in Example 3.4, this family of projections does not admit a uniform exponential dichotomy. Thus, there exists no family of unstable projections that allows $\mathcal{S}$ to admit a uniform exponential dichotomy.

We now study the *discretization* technique, which allows us to derive discrete-time processes from continuous-time processes.

3.3 Discretization and its properties

This section introduces the technique of discretization, a central concept that establishes a bridge between continuous and discrete dynamics. We formally define the discretization of a continuous evolution process by sampling the system at fixed time intervals with a specific time offset.

Definition 3.6 (Discretization). Let $\mathcal{S}$ be a continuous evolution process, let $r \in \mathbb{R}$, and let $l > 0$. The family of bounded operators $\widetilde{\mathcal{S}}(r) := \{\widetilde{S}(r)_{n,m} := S(r+nl, r+ml) \mid n,m \in \mathbb{Z},\ n \geq m\}$ is called the DISCRETIZATION of $\mathcal{S}$ with OFFSET r and STEP SIZE l. When l is not explicitly specified, we assume $l = 1$.

We prove that the discretization of a continuous process generates a discrete evolution process and that it inherits the property of nonuniform exponential dichotomy. This correspondence serves as the mathematical foundation for transferring the robust stability results established in the discrete setting to continuous-time systems.

Proposition 3.7 (Discretization of an evolution process). If $\mathcal{S}$ is a continuous evolution process and $\widetilde{\mathcal{S}}(r)$ is its discretization with offset $r \in \mathbb{R}$ and step size $l > 0$, then $\widetilde{\mathcal{S}}(r)$ is a discrete evolution process.

Proof. We verify the conditions of Definition 2.1. Note that:

(C1) For all $n \in \mathbb{Z}$, we have $\widetilde{S}(r)_{n,n} = S(r+nl, r+nl) = \mathrm{I}$;

(C2) For all $n,m,k \in \mathbb{Z}$ such that $n \geq m \geq k$, we have

$$\widetilde{S}(r)_{n,m}\widetilde{S}(r)_{m,k} = S(r+nl, r+ml)S(r+ml, r+kl) = S(r+nl, r+kl) = \widetilde{S}(r)_{n,k}.$$

Thus, $\widetilde{\mathcal{S}}(r)$ satisfies the required properties. □

We now show that every discretization of an evolution process admitting a NEDI admits a NED.

Proposition 3.8 (Discretization and NEDI). Let $\mathcal{S}$ be a continuous evolution process and let $\widetilde{\mathcal{S}}(r)$ be its discretization with offset $r \in \mathbb{R}$ and step size $l = 1$. If $\mathcal{S}$ admits a NEDI, then $\widetilde{\mathcal{S}}(r)$ admits a discrete NED with bound sequence $\widetilde{K}(r) := (\widetilde{K}(r)_n)_{n\in\mathbb{Z}}$ defined by $\widetilde{K}(r)_n := K(r+n)$, satisfying

$$\widetilde{K}(r)_n \leq Me^{\beta|r|}e^{\beta|n|} \qquad \text{for all } n \in \mathbb{Z},$$

and with the same dichotomy exponent.

Proof. By Proposition 3.7, we know that $\widetilde{\mathcal{S}}(r)$ is a discrete evolution process. Define the family of projections $\widetilde{\Pi}^u(r) := \{\widetilde{\Pi}^u(r)_n := \Pi^u(r+n) \mid n \in \mathbb{Z}\}$. We verify Definition 2.8 conditions:

(C1) For $n \geq m$, we have

$$\begin{aligned}\widetilde{\Pi}^u(r)_n\widetilde{S}(r)_{n,m} &= \Pi^u(r+n)S(r+n,r+m)\\ &= S(r+n,r+m)\Pi^u(r+m)\\ &= \widetilde{S}(r)_{n,m}\widetilde{\Pi}^u(r)_m.\end{aligned}$$

(C2) For $n \geq m$, the restriction

$$\widetilde{S}(r)_{n,m}|_{\mathrm{Im}(\widetilde{\Pi}^u(r)_m)} = S(r+n,r+m)|_{\mathrm{Im}(\Pi^u(r+m))}$$

is clearly an isomorphism onto $\mathrm{Im}(\Pi^u(r+n)) = \mathrm{Im}(\widetilde{\Pi}^u(r)_n)$, with inverse given by $S(r+m,r+n) = \widetilde{S}(r)_{m,n}$.

(C3) Finally, define $\widetilde{\Pi}^s(r) := \{\widetilde{\Pi}^s(r)_n := \mathrm{I} - \widetilde{\Pi}^u(r)_n \mid n \in \mathbb{Z}\}$. Note that $\widetilde{\Pi}^s(r)_n = \Pi^s(r+n)$. If $n \geq m$, then

$$\begin{aligned}\left\|\widetilde{S}(r)_{n,m}\widetilde{\Pi}^s(r)_m\right\| &= \|S(r+n,r+m)\Pi^s(r+m)\|\\ &\leq K(r+m)e^{-\gamma|(r+n)-(r+m)|}\\ &= K(r+m)e^{-\gamma|n-m|}\\ &\leq Me^{\beta|r+m|}e^{-\gamma|n-m|}\\ &\leq (Me^{\beta|r|})e^{\beta|m|}e^{-\gamma|n-m|}.\end{aligned}$$

Similarly, if $n \leq m$, then

$$\begin{aligned}\left\|\widetilde{S}(r)_{n,m}\widetilde{\Pi}^u(r)_m\right\| &= \|S(r+n,r+m)\Pi^u(r+m)\|\\ &\leq K(r+m)e^{-\gamma|n-m|}\\ &\leq (Me^{\beta|r|})e^{\beta|m|}e^{-\gamma|n-m|}.\end{aligned}$$

Thus, $\widetilde{\mathcal{S}}(r)$ admits a NED with bound sequence $\widetilde{K}(r)$ and dichotomy exponent γ. □

Similar to the last chapter, we prove the uniqueness of the unstable projections and their continuous dependence on the underlying evolution process.

Proposition 3.9 (Uniqueness of projections, continuous case)**.** Let $\mathcal{S}$ be a continuous evolution process. If $\mathcal{S}$ admits a NEDI with $\gamma > \beta$, then Π^u is the unique family of unstable projections that satisfies the NEDI conditions (assuming any other candidate family also satisfies the condition $\gamma' > \beta'$).

Proof. Let Γ^u be another family of unstable projections that satisfies the NEDI conditions for $\mathcal{S}$, with bound function K' characterized by β' and exponent γ', such that $\gamma' > \beta'$.

Fix any $t \in \mathbb{R}$. Consider the discretization $\widetilde{\mathcal{S}}(t)$ with offset t. By Proposition 3.8, both Π^u and Γ^u induce discrete families of unstable projections, $\widetilde{\Pi}^u(t)$ and $\widetilde{\Gamma}^u(t)$ respectively, for the discrete process $\widetilde{\mathcal{S}}(t)$. At index $n = 0$, we have by definition:

$$\widetilde{\Pi}^u(t)_0 = \Pi^u(t) \quad \text{and} \quad \widetilde{\Gamma}^u(t)_0 = \Gamma^u(t).$$

Applying Proposition 2.21, we conclude that $\widetilde{\Pi}^u(t) = \widetilde{\Gamma}^u(t)$. In particular, their components at $n = 0$ must coincide:

$$\Pi^u(t) = \Gamma^u(t).$$

Since $t \in \mathbb{R}$ was arbitrary, we conclude that $\Pi^u = \Gamma^u$. □

Proposition 3.10 (Continuous dependence of projections, continuous case)**.** Let $\mathcal{S}$ and $\mathcal{T}$ be continuous evolution processes that admit a NEDI with families of unstable projections Π^u and Γ^u, and exponents $\gamma_{\mathcal{S}}$ and $\gamma_{\mathcal{T}}$, respectively, sharing the same bound function K satisfying $K(t) \leq M e^{\beta|t|}$ such that $\gamma_{\mathcal{S}}, \gamma_{\mathcal{T}} > \beta$. If there exists $\varepsilon > 0$ such that

$$\sup_{0 \leq t-s \leq 1} \{K(t)\, \|S(t,s) - T(t,s)\|\} \leq \varepsilon,$$

then

$$\sup_{t \in \mathbb{R}} \{K(t)^{-1}\, \|\Pi^u(t) - \Gamma^u(t)\|\} \leq \varepsilon \frac{e^{-\gamma_{\mathcal{S}}} + e^{-\gamma_{\mathcal{T}}}}{1 - e^{-(\gamma_{\mathcal{S}} + \gamma_{\mathcal{T}})}}.$$

Proof. Fix $r \in \mathbb{R}$. Let $\widetilde{\mathcal{S}}(r)$ and $\widetilde{\mathcal{T}}(r)$ be the discretizations of $\mathcal{S}$ and $\mathcal{T}$, respectively, with offset r. As proved in Proposition 3.8, both admit a discrete NED with the same exponents as their continuous counterparts and with bound sequence $\widetilde{K}(r)_n = K(r+n)$. From the hypothesis on the continuous perturbation, we have:

$$\begin{aligned}\varepsilon &\geq \sup_{0\leq t-s\leq 1}\{K(t)\,\|S(t,s)-T(t,s)\|\}\\ &\geq \sup_{k\in\mathbb{Z}}\{K(r+k+1)\,\|S(r+k+1,r+k)-T(r+k+1,r+k)\|\}\\ &= \sup_{k\in\mathbb{Z}}\left\{\widetilde{K}(r)_{k+1}\left\|\widetilde{S}(r)_{k+1,k}-\widetilde{T}(r)_{k+1,k}\right\|\right\}.\end{aligned}$$

Thus, the discrete processes satisfy the condition of Proposition 2.22, and applying it we obtain:

$$\sup_{k\in\mathbb{Z}}\left\{\widetilde{K}(r)_k^{-1}\left\|\widetilde{\Pi}^u(r)_k-\widetilde{\Gamma}^u(r)_k\right\|\right\}\leq \varepsilon\frac{e^{-\gamma_S}+e^{-\gamma_T}}{1-e^{-(\gamma_S+\gamma_T)}}.$$

Substituting the definitions of the discrete projections, this becomes:

$$\sup_{k\in\mathbb{Z}}\left\{K(r+k)^{-1}\,\|\Pi^u(r+k)-\Gamma^u(r+k)\|\right\}\leq \varepsilon\frac{e^{-\gamma_S}+e^{-\gamma_T}}{1-e^{-(\gamma_S+\gamma_T)}}.$$

Since this inequality holds for any arbitrary $r\in\mathbb{R}$ (and any $t\in\mathbb{R}$ can be represented as $r+k$ for some choice of r), we conclude that:

$$\sup_{t\in\mathbb{R}}\left\{K(t)^{-1}\,\|\Pi^u(t)-\Gamma^u(t)\|\right\}\leq \varepsilon\frac{e^{-\gamma_S}+e^{-\gamma_T}}{1-e^{-(\gamma_S+\gamma_T)}}.$$

□

3.4 Robustness of NEDIs

This section is dedicated to establishing the robustness of nonuniform exponential dichotomies in the continuous setting. First, we prove a converse result demonstrating that a continuous evolution process admits a NEDI provided all its discretizations admits NED and satisfy a specific regularity condition. Building on this framework, we derive the main robustness theorem by discretizing the perturbed continuous process, applying the discrete robustness result established in the previous chapter, and subsequently recovering the continuous dichotomy for the perturbed system.

Similar to the discrete case, we will define global solutions for the continuous case. Previously, it was sufficient for the equation to be satisfied for each step, but now it is necessary for it to be satisfied for all instants of time.

Definition 3.11 (Global solution). Let $\mathcal{S}$ be a continuous evolution process on X. A function $\xi\colon\mathbb{R}\to X$ is called a GLOBAL SOLUTION for $\mathcal{S}$ if

$$\xi(t)=S(t,s)\xi(s)\quad\text{for all } t\geq s.$$

The following proposition shows that if all discretizations of a continuous process admit a NEDI, then, under a certain regularity condition, the process itself also admits a NEDI.

Proposition 3.12 (Obtaining NEDI from NED). Let $\mathcal{S}$ be a continuous evolution process. Suppose that:

(H1) There exist constants $M \geq 1$ and $\beta \geq 0$ such that

$$L := \sup_{0 \leq t-s \leq 1} \{K(t)^{-1} \|S(t,s)\|\} < +\infty,$$

where $K(t)$ is a function satisfying $K(t) \leq Me^{\beta|t|}$ for all $t \in \mathbb{R}$.

(H2) For all $r \in \mathbb{R}$, the discretization $\widetilde{\mathcal{S}}(r)$ admits a NED with family of unstable projections $\widetilde{\Pi}^u(r)$, bound sequence $\widetilde{K}(r)_n := K(r+n)$ for all $n \in \mathbb{Z}$ and dichotomy exponent γ.

If $\gamma > \beta$, then $\mathcal{S}$ admits a NEDI with bound function $\widehat{K}(t) = LM^2 e^{\gamma} e^{2\beta|t|}$ and nonuniformity exponent $\gamma - \beta$.

Proof. First, we define a candidate family of projections for $\mathcal{S}$. For all $t \in \mathbb{R}$, define

$$\Pi^u(t) := \widetilde{\Pi}^u(t)_0.$$

We now verify the NEDI conditions through several claims.

Claim 1 *For all $t \geq s$ we have $\|S(t,s)\Pi^s(s)\| \leq LM^2 e^{\gamma} e^{2\beta|s|-(\gamma-\beta)|t-s|}$*

Let $t \geq s$ and choose $n \geq 0$ such that $s+n \leq t \leq s+(n+1)$. We have

$$\begin{aligned}
\|S(t,s)\Pi^s(s)\| &= \|S(t,s+n)S(s+n,s)\Pi^s(s)\| \\
&= \left\|S(t,s+n)\widetilde{S}(s)_{n,0}\widetilde{\Pi}^s(s)_0\right\| \\
&\leq \|S(t,s+n)\| \left\|\widetilde{S}(s)_{n,0}\widetilde{\Pi}^s(s)_0\right\| \\
&\leq \|S(t,s+n)\| \widetilde{K}(s)_0 e^{-\gamma|n|} \\
&= \|S(t,s+n)\| K(s) e^{-\gamma|n|}.
\end{aligned}$$

Multiplying by $K(t)K(t)^{-1}$ and using Hypothesis (H1) and the fact that $0 \leq t-(s+n) \leq 1$ we conclude

$$\begin{aligned}
\|S(t,s+n)\| K(s)e^{-\gamma|n|} &= K(t)K(t)^{-1} \|S(t,s+n)\| K(s)e^{-\gamma|n|} \\
&\leq LK(t)K(s)e^{-\gamma|n|} \\
&= LM^2 e^{\beta|t|+\beta|s|-\gamma|n|}.
\end{aligned}$$

Since $n \geq 0$ and $n \geq t-s-1$, we have

$$-\gamma|n| = -\gamma n \leq -\gamma(t-s-1) = \gamma - \gamma(t-s),$$

thus

$$e^{\beta|t|+\beta|s|-\gamma|n|} \leq e^{\gamma} e^{\beta|t|+\beta|s|-\gamma|t-s|}.$$

Applying Proposition 4.5 we conclude:

$$\|S(t,s)\Pi^s(s)\| \leq LM^2 e^{\gamma} e^{2\beta|s|-(\gamma-\beta)|t-s|}.$$

Claim 2 *For all* $t \leq s$ *we have* $\|S(t,s)\Pi^u(s)\| \leq LM^2 e^{\gamma} e^{2\beta|s|-(\gamma-\beta)|t-s|}$.

Let $t \leq s$ and choose $n \leq 0$ such that $s+n \leq t \leq s+(n+1)$. For $x \in \mathrm{Im}(\Pi^u(s))$, we define the inverse map by

$$S(t,s)x := S(t, s+n)\widetilde{S}(s)_{n,0}x.$$

By a similar argument as in **Claim 1**, we have

$$\begin{aligned}
\|S(t,s)\Pi^u(s)\| &= \left\| S(t,s+n)\widetilde{S}(s)_{n,0}\widetilde{\Pi}^u(s)_0 \right\| \\
&\leq \|S(t,s+n)\| \left\| \widetilde{S}(s)_{n,0}\widetilde{\Pi}^u(s)_0 \right\| \\
&\leq LK(t)K(s)e^{-\gamma|n|} \\
&\leq LM^2 e^{\beta|t|+\beta|s|} e^{-\gamma|n|}
\end{aligned}$$

Since $n \leq 0$ and $n \leq t-s$, we have

$$-\gamma|n| = \gamma n \leq \gamma(t-s),$$

and by the fact that $e^{\gamma} > 1$ follows

$$e^{\beta|t|+\beta|s|-\gamma|n|} \leq e^{\beta|t|+\beta|s|-\gamma|t-s|} \leq e^{\gamma} e^{\beta|t|+\beta|s|-\gamma|t-s|}.$$

Therefore

$$\|S(t,s)\Pi^u(s)\| \leq LM^2 e^{\gamma} e^{2\beta|s|-(\gamma-\beta)|t-s|}.$$

Claim 3 *For all* $t_0 \in \mathbb{R}$, $\mathrm{Ker}(\Pi^u(t_0)) = \{x \in X \mid \sup_{t \geq t_0} \|S(t,t_0)x\| < +\infty\}$.

Let $z \in \mathrm{Ker}(\Pi^u(t_0)) = \mathrm{Im}(\Pi^s(t_0))$. Then

$$\|S(t,t_0)z\| = \|S(t,t_0)\Pi^s(t_0)z\| \leq Ce^{2\beta|t_0|-(\gamma-\beta)|t-t_0|} \|z\|.$$

Since $\gamma > \beta$, this function decays and is bounded on $[t_0, +\infty)$. Now consider the converse. Let $z \notin \mathrm{Ker}(\Pi^u(t_0))$ and $n \in \mathbb{N}$. Using the fact that $S(t_0, t_0 + n)$ is the inverse of $S(t_0 + n, t_0)$ on the unstable subspace, we can write:

$$\begin{aligned}
\|\Pi^u(t_0)z\| &= \left\|\widetilde{\Pi}^u(t_0)_0 z\right\| \\
&= \left\|\widetilde{S}(t_0)_{0,n}\widetilde{S}(t_0)_{n,0}\widetilde{\Pi}^u(t_0)_0 z\right\| \\
&= \left\|\widetilde{S}(t_0)_{0,n}\widetilde{\Pi}^u(t_0)_n\widetilde{S}(t_0)_{n,0} z\right\| \\
&\leq \left\|\widetilde{S}(t_0)_{0,n}\widetilde{\Pi}^u(t_0)_n\right\| \left\|\widetilde{S}(t_0)_{n,0} z\right\| \\
&\leq \widetilde{K}(t_0)_n e^{-\gamma|n|} \|S(t_0 + n, t_0)z\| \\
&= K(t_0 + n)e^{-\gamma|n|} \|S(t_0 + n, t_0)z\| \\
&\leq M e^{\beta|t_0+n|} e^{-\gamma|n|} \|S(t_0 + n, t_0)z\| \\
&\leq M e^{\beta|t_0|} e^{(\beta-\gamma)|n|} \|S(t_0 + n, t_0)z\| .
\end{aligned}$$

Rearranging the terms, we obtain

$$\|S(t_0 + n, t_0)z\| \geq M^{-1} e^{-\beta|t_0|} e^{(\gamma-\beta)n} \|\Pi^u(t_0)z\| .$$

Since $\gamma > \beta$, the term $e^{(\gamma-\beta)n} \to +\infty$ as $n \to +\infty$. Thus, the sequence $\|S(t_0 + n, t_0)z\|$ is unbounded, which implies the supremum is unbounded on $[t_0, +\infty)$. By contraposition, the claim is proved.

Claim 4 $S(t,t_0)|_{\mathrm{Im}(\Pi^u(t_0))} \colon \mathrm{Im}(\Pi^u(t_0)) \to X$ *is injective for all* $t \geq t_0$.

Let $z \in \mathrm{Im}(\Pi^u(t_0))$ such that $S(t,t_0)z = 0$. Choose $n \in \mathbb{N}$ such that $t_0 + n \geq t$. Then

$$\widetilde{S}(t_0)_{n,0} z = S(t_0 + n, t)S(t,t_0)z = 0.$$

Since $\widetilde{S}(t_0)_{n,0}$ restricted to $\mathrm{Im}(\widetilde{\Pi}^u(t_0)_0)$ is an isomorphism (from the discrete NED), we must have $z = 0$.

Claim 5 *For all* $t_0 \in \mathbb{R}$ *we have that*

$$\mathrm{Im}(\Pi^u(t_0)) = \{x \mid \text{there exists a backwards bounded solution } \xi \text{ s.t. } \xi(t_0) = z\}.$$

Let $z \in \mathrm{Im}(\Pi^u(t_0))$ and $t < t_0$. Take $n \in \mathbb{Z}_-$ such that $t_0 + n \leq t \leq t_0 + (n + 1)$ and define

$$\xi(t) := S(t, t_0 + n)T_{n,0}(t_0)z = S(t,t_0)z.$$

Choose $x \in X$ such that $z = \Pi^u(t_0)x$ and notice that

$$\begin{aligned} \|\xi(t)\| &= \|S(t,t_0)z\| \\ &= \|S(t,t_0)\Pi^u(t_0)x\| \\ &\leq De^{2\beta|t_0|}e^{-(\gamma-\beta)|t-t_0|}\|x\|, \end{aligned}$$

and as $t \to -\infty$ we have that $\|\xi(t)\|$ vanishes because $\gamma > \beta$, proving that ξ is backwards bounded.

Now suppose that $z \notin \mathrm{Im}(\Pi^u(t_0))$ and exists ξ global solution such that $\xi(t_0) = z$. For all $n \in \mathbb{Z}_-$ we have that $z = S(t_0,t_0+n)\xi(t_0+n)$, therefore

$$\begin{aligned} \|\Pi^s(t_0)z\| &= \|\Pi^s(t_0)S(t_0,t_0+n)\xi(t_0+n)\| \\ &\leq \|S(t_0,t_0+n)\Pi^s(t_0+n)\| \, \|\xi(t_0+n)\| \\ &= \left\|T_{0,n}(t_0)\Pi_0^s(t_0)\right\| \|\xi(t_0+n)\| \\ &\leq Me^{\beta|t_0|}e^{\beta|n|-\gamma|n|}\|\xi(t_0+n)\| \\ &= Me^{\beta|t_0|}e^{(\beta-\gamma)|n|}\|\xi(t_0+n)\|, \end{aligned}$$

therefore

$$e^{-(\beta-\gamma)l|n|}\|\Pi^s(t_0)z\| \leq Me^{\beta|t_0|}\|\xi(t_0+n)\|.$$

Since $\gamma > \beta$ and $n < 0$, then ξ cannot be backwards bounded. By contraposition we proved the desired.

Claim 6 $S(t,t_0)\,\mathrm{Im}(\Pi^u(t_0)) = \mathrm{Im}(\Pi^u(t))$.

Let $z \in \mathrm{Im}(\Pi^u(t_0))$. There exists global backwards bounded solution ξ such that $\xi(t_0) = z$. Notice that

$$\xi(t) = S(t,t_0)\xi(t_0) = S(t,t_0)z,$$

therefore $S(t,t_0)z \in \mathrm{Im}(\Pi^u(t))$ because $\xi(t)$ is a global backwards bounded solution that pass through it at time t. Now let $z \in \mathrm{Im}(\Pi^u(t))$ and notice that there is a global backwards bounded solution ξ such that $\xi(t) = z$. Choosing $n \in \mathbb{Z}$ such that $n + t \leq t_0 \leq t$. Define

$$x \coloneqq S(t_0, n+t)S(n+t,t)z,$$

therefore $S(t,t_0)x = z$ and this implies that $z \in S(t,t_0)\,\mathrm{Im}(\Pi^u(t_0))$. Claim 4 and Claim 6 guarantee that $S(t,t_0)|_{\mathrm{Im}(\Pi^u(t_0))}$ is an isomorphism.

Claim 7 *For all $t \geq s$, we have $\Pi^u(t)S(t,s) = S(t,s)\Pi^u(s)$.*

Choose $z \in X$ and notice that

$$S(t,s)z = S(t,s)(\mathrm{I} - \Pi^u(s))z + S(t,s)\Pi^u(s)z,$$

and since $(\mathrm{I} - \Pi^u(s))z \in \ker(\Pi^u(s))$ (Claim 3) and $S(t,s)\Pi^u(s)z \in \mathrm{Im}(\Pi^u(t))$ (Claim 6), we apply $\Pi^u(t)$ in the equation above and conclude

$$\begin{aligned}\Pi^u(t)S(t,s)z &= \Pi^u(t)S(t,s)(I - \Pi^u(s))z + \Pi^u(t)S(t,s)\Pi^u(s)z \\ &= 0 + S(t,s)\Pi^u(s)z\end{aligned}$$

All those claims prove that $\mathcal{S}$ admits NEDI.

□

The following remark ensures that the regularity hypothesis of the previous proposition is equivalent even when the multiplicative factor of the norm of $S(t,s)$ depends on s.

Remark 3.13. Proposition's 3.12 Hypothesis (H1) is equivalent to

(H1') There exist constants $M \geq 1$ and $\beta \geq 0$ such that

$$L := \sup_{0 \leq t-s \leq 1} \{K(s)^{-1} \|S(t,s)\|\} < +\infty,$$

where $K(t)$ is a function satisfying $K(t) \leq Me^{\beta|t|}$ for all $t \in \mathbb{R}$.

To prove this note that if $0 \leq t - s \leq 1$, then

$$K(t) \leq Me^{\beta|t|} \leq Me^{\beta|t-s|+\beta|s|} \leq Me^{\beta+\beta|s|} = Me^{\beta}e^{\beta|s|} = e^{\beta}K(s)$$

and $K(s)^{-1} \geq e^{-\beta}K(t)^{-1}$. Similarly, we have

$$K(s) \leq Me^{\beta|s|} \leq Me^{\beta|t-s|+\beta|t|} \leq e^{\beta}K(t).$$

Hence, if one supremum is finite then the other is.

Finally, we present the robustness theorem for the continuous case. The proof begins by using Proposition 3.8 to construct discretizations of the perturbed process. Then, applying the robustness theorem for the discrete case, we show that these perturbed discretizations must admit a NEDI. Subsequently, we use Proposition 3.12 to demonstrate that the continuous perturbed process itself admits a NEDI.

Theorem 3.14 (Robustness of NEDIs)**.** Let $\mathcal{S}$ be a continuous evolution process that admits a NEDI with bound function K satisfying $K(t) \leq Me^{\beta|t|}$ and dichotomy exponent $\gamma > \beta$. Suppose that

$$L := \sup_{0 \leq t-s \leq 1} \{K(t)^{-1} \|S(t,s)\|\} < +\infty.$$

Then there exists $0 < \delta < \dfrac{1-e^{-\gamma}}{1+e^{-\gamma}}$ such that if $\mathcal{T}$ is a continuous evolution process satisfying

$$\sup_{0 \leq t-s \leq 1} \{K(t) \|S(t,s) - T(t,s)\|\} < \delta,$$

then $\mathcal{T}$ admits a NEDI with bound $\widehat{K}(s) := L \max\{\widetilde{M}_1, \widetilde{M}_2\}^2 e^{\min\{\widetilde{\gamma},\widehat{\gamma}\}} e^{2\beta|s|}$ and dichotomy exponent $\min\{\widetilde{\gamma}, \widehat{\gamma}\} - \beta$ as defined in Theorem 2.25.

Proof. Choose $r \in \mathbb{R}$. By Proposition 3.8, the discretization $\widetilde{\mathcal{S}}(r)$ of $\mathcal{S}$ with offset r admits a discrete NED with bound $\widetilde{K}(r)_n = K(r+n)$ and exponent γ.

Choose $\delta > 0$ such that $0 < \delta < \dfrac{1-e^{-\gamma}}{1+e^{-\gamma}}$. Let $\mathcal{T}$ be a continuous evolution process satisfying

$$\sup_{0 \leq t-s \leq 1} \{K(t) \|S(t,s) - T(t,s)\|\} < \delta$$

and consider its discretization $\widetilde{\mathcal{T}}(r)$. For all $n \in \mathbb{Z}$, define the perturbation sequence

$$B(r)_n := \widetilde{T}(r)_{n+1,n} - \widetilde{S}(r)_{n+1,n}.$$

Notice that

$$\begin{aligned} \|B(r)_n\| &= \|T(r+n+1, r+n) - S(r+n+1, r+n)\| \\ &< \delta K(r+n+1)^{-1} \\ &= \delta \widetilde{K}(r)_{n+1}^{-1}. \end{aligned}$$

Since this bound holds for all n and noting that $\widetilde{\mathcal{S}}(r)$ admits a NED, we apply Theorem 2.25 to conclude that $\widetilde{\mathcal{T}}(r)$ admits a NED. Since δ and the exponents of $\widetilde{\mathcal{S}}(r)$ do not depend on r, the resulting dichotomy exponents for $\widetilde{\mathcal{T}}(r)$ are independent of r. Now we verify the Proposition's 3.12 Hypothesis (H1) for $\mathcal{T}$. Consider $0 \leq t-s \leq 1$. We have

$$\begin{aligned} K(t)^{-1} \|T(t,s)\| &= K(t)^{-1} \|T(t,s) - S(t,s) + S(t,s)\| \\ &\leq K(t)^{-1} \|T(t,s) - S(t,s)\| + K(t)^{-1} \|S(t,s)\| \\ &\leq \delta K(t)^{-2} + K(t)^{-1} \|S(t,s)\|. \end{aligned}$$

Since $K(t) \geq 1$, we have $K(t)^{-2} \leq 1$. Thus

$$K(t)^{-1} \|T(t,s)\| \leq \delta + L.$$

This implies that $\sup_{0\le t-s\le 1}\{K(t)^{-1}\|T(t,s)\|\}<+\infty$. Finally, applying Proposition 3.12, we conclude that $\mathcal{T}$ admits a NEDI. □

Remark 3.15. In Theorem 2.25, we can choose δ sufficiently small such that the exponents of the perturbed process satisfy the robustness condition. In contrast, this does not necessarily hold in the continuous setting, as we cannot always choose δ such that $\min\{\widetilde{\gamma},\widehat{\gamma}\}-\beta>\beta$. This implies that, in the discrete setting, if δ is sufficiently small, the perturbed process is also robust, however this is not guaranteed in the continuous setting.

We now establish a criterion for the existence of a NEDI in systems governed by integral equations.

Proposition 3.16 (Integral equations and NEDI)**.** Let $\mathcal{S}$ be a continuous evolution process that admits a NEDI with bound function $K(t)\le Me^{\beta|t|}$ and dichotomy exponent $\gamma>\beta$. Assume that $\mathcal{S}$ satisfies the growth condition

$$L:=\sup_{0\le t-s\le 1}\{K(t)^{-1}\|S(t,s)\|\}<+\infty.$$

Let $\{B(t)\mid t\in\mathbb{R}\}\subset\mathcal{B}(X)$ be a family of bounded operators such that the map $t\mapsto B(t)x$ is continuous for all $x\in X$. Then, there exists $\delta>0$ such that if

$$\|B(t)\|<\delta e^{-3\beta|t|}\qquad\text{for all } t\in\mathbb{R},$$

any continuous evolution process $\mathcal{T}$ satisfying the integral equation

$$T(t,s)=S(t,s)+\int_s^t S(t,r)B(r)T(r,s)dr\qquad\text{for all } t\ge s,$$

admits a NEDI.

Proof. Consider that $\mathcal{T}$ solves the integral equation, then

$$\|T(t,s)\|\le\|S(t,s)\|+\int_s^t\|S(t,r)\|\,\|B(r)\|\,\|T(r,s)\|\,dr.$$

Fix $s\in\mathbb{R}$ and define $\phi(t):=K(t)^{-1}\|T(t,s)\|$. For $0\le t-s\le 1$ we have

$$\phi(t)\le L+L\int_s^t\|B(r)\|\,K(r)\phi(r).$$

Applying Grönwall's inequality we conclude

$$\phi(t)\le Le^{L\int_s^t\|B(r)\|K(r)dr}<+\infty,$$

and unwrapping ϕ we conclude

$$Q:=\sup_{0\le t-s\le 1}\{K(t)^{-1}\|T(t,s)\|\}<+\infty.$$

Now notice that for $0 \le t - s \le 1$ we have

$$\begin{aligned}\|S(t,s) - T(t,s)\| &\le \left\| \int_s^t S(t,r)B(r)T(r,s)dr \right\| \\ &\le \int_s^t LK(t)\,\|B(r)\|\,QK(r)dr \\ &\le LQK(t) \int_s^t K(r)\,\|B(r)\|\,dr \\ &< LQM^2 e^{\beta|t|} \int_s^t e^{-2\beta|r|}\delta dr,\end{aligned}$$

and multiplying by $K(t)$ leads to

$$K(t)\,\|S(t,s) - T(t,s)\| < \delta LQM^3 e^{2\beta|t|} \int_s^t e^{-2\beta|r|}dr = \delta D \int_s^t e^{2\beta(|t|-|r|)}dr$$

where $D := LQM^3$. Knowing that $t \ge r \ge s$ and $0 \le t - s \le 1$ we can conclude that $|t| - |r| \le 1$, therefore

$$K(t)\,\|S(t,s) - T(t,s)\| < \delta D \int_s^t e^{2\beta}dr = \delta D(t-s)e^{2\beta} \le \delta D e^{2\beta}.$$

Choosing δ small enough we can apply Theorem 3.14 and conclude that $\mathcal{T}$ admits NEDI. $\square$

A significant application of the robustness theorem is the construction of processes admitting a NEDI from known ones. We illustrate this in the following example.

Proposition 3.17 (Robustness of Example 3.5). Consider the evolution process $\mathcal{S}$ defined in Example 3.5 with parameters $a, b, c > 0$. Let $\{B(t) \mid t \in \mathbb{R}\} \subset \mathcal{M}_{2\times 2}(\mathbb{R})$ be a continuous family of matrices such that

$$\|B(t)\| < \delta e^{-6a|t|} \qquad \text{for all } t \in \mathbb{R}.$$

If $b > 3a$ and $c > 3a$, then for $\delta > 0$ sufficiently small, the evolution process $\mathcal{T}$ generated by the perturbed system

$$x' = (A(t) + B(t))x$$

admits a nonuniform exponential dichotomy.

Proof. In Example 3.5, we established that $\mathcal{S}$ admits a NEDI with nonuniformity exponent $\beta = 2a$ and dichotomy exponent $\gamma = \min\{b - a, c - a\}$.

First, we observe that the condition $b > 3a$ implies $b - a > 2a$, and $c > 3a$ implies $c - a > 2a$. Therefore:

$$\gamma = \min\{b - a, c - a\} > 2a = \beta.$$

This satisfies the condition $\gamma > \beta$ required for robustness.

Next, we must verify the growth condition of Proposition 3.16. We need to show that

$$\sup_{0\leq t-s\leq 1} \{e^{-2a|t|}\,\|S(t,s)\|\} < +\infty.$$

Recall from the calculations in Example 3.5 that the stable component $T(t,s)$ satisfies:

$$\|T(t,s)\| \leq e^{2a}e^{a|t|+a|s|-(b-a)|t-s|} \qquad \text{for } t \geq s.$$

Consequently, for all $0 \leq t-s \leq 1$ we have

$$\begin{aligned} e^{-2a|t|}\,\|T(t,s)\| &\leq e^{2a}e^{a|s|-a|t|-(b-a)|t-s|} \\ &\leq e^{2a}e^{a(|s|-|t|)} \end{aligned}$$

By the reverse triangle inequality, we know that $\big||s|-|t|\big| \leq |s-t|$. Since $t \geq s$, we have $|s-t| = t-s$. Therefore:

$$|s|-|t| \leq t-s \leq 1.$$

Consequently,

$$e^{-2a|t|}\,\|T(t,s)\| \leq e^{2a}e^{a(|s|-|t|)} \leq e^{2a}e^{a} = e^{3a}.$$

Thus, the stable component satisfies the growth condition. An analogous analysis applies to the unstable component.

Finally, since $\beta = 2a$, the hypothesis on the perturbation becomes:

$$\|B(t)\| < \delta e^{-3(2a)|t|} = \delta e^{-6a|t|}.$$

Also, $\mathcal{T}$ satisfies the integral equation by variation of constants in ODE theory. Since all assumptions of Proposition 3.16 are met, we conclude that the perturbed evolution process $\mathcal{T}$ admits a nonuniform exponential dichotomy. □

In (Barreira; Valls, 2008), Barreira and Valls presented a result similar to Proposition 3.16 for invertible evolution processes. They assumed that $\mathcal{S}$ is invertible (i.e., $S(t,s)$ is invertible for all $t \geq s$) and that the perturbation B satisfies $\|B(t)\| \leq \delta e^{-2\gamma|t|}$ for all $t \in \mathbb{R}$. In their approach, the invertibility assumption allows deriving explicit expressions for the projections of the perturbed process. However, this hypothesis is restrictive since in many scenarios, particularly when $A(t) \notin \mathcal{B}(X)$, we cannot expect the evolution process to be invertible.

Barreira and Valls (Barreira; Valls, 2015) also establish a version of Proposition 3.16 under different assumptions, considering a general growth rate $\rho(t)$ for the nonuniform exponential dichotomy. They proved that if $\gamma > 2\beta$ and the continuous function $B\colon \mathbb{R} \to \mathcal{B}(X)$ satisfies

$$\|B(t)\| \leq \delta e^{-3\beta|\rho(t)|}\rho'(t), \quad \text{for all } t \in \mathbb{R},$$

then the perturbed problem admits a ρ-nonuniform exponential dichotomy. Although our method does not apply to general growth rates, for the particular case $\rho(t) = t$ we obtain an improvement of their robustness result, as our condition on the exponents requires only $\gamma > \beta$.

4 Nonuniform exponential dichotomies of type II

In this chapter, we address the limitations of the nonuniform exponential dichotomy definition presented in the previous chapters. Motivated by examples where the nonuniform growth depends on the final time t rather than the initial time s, we study the concept of *nonuniform exponential dichotomy of type II* (NEDII), introduced by (Langa; Obaya; Sousa, 2024).

We investigate the relationship between these two types of dichotomies, distinguishing their asymptotic behaviors and providing conditions under which a process admits one, the other, or both. A central aspect of our analysis is the construction of the *dual evolution process*. We establish that the dual of a process admitting a type I dichotomy admits a type II dichotomy, and vice-versa. This duality allows us to leverage the results from Chapter 3 to prove the robustness for invertible processes that admit type II dichotomies in reflexive Banach spaces.

4.1 Motivation

Although non-uniform exponential dichotomies encompass a broad family of examples, one may ask which evolution processes do not admit a NEDI. We now investigate an example that addresses precisely this question.

Example 4.1. Consider the ODE on $\mathbb{R}$ given by

$$x'(t) = \tanh(t)x(t),$$

and the evolution process induced by it:

$$S(t,s)u = u \cdot \frac{\cosh(t)}{\cosh(s)} = u \cdot \frac{e^t + e^{-t}}{e^s + e^{-s}}.$$

We have that $\mathcal{S}$ does not admit NEDI.

Indeed, for all $t, s \in \mathbb{R}$ we can estimate the operator norm as follows:

$$\begin{aligned} \|S(t,s)\| &= \frac{e^t + e^{-t}}{e^s + e^{-s}} \\ &\geq \frac{\frac{1}{2}e^{|t|}}{e^s + e^{-s}} \\ &\geq \frac{\frac{1}{2}e^{|t|}}{2e^{|s|}} \\ &= \frac{e^{|t|-|s|}}{4}. \end{aligned}$$

Suppose that $\mathcal{S}$ admits a NEDI with $\Pi^s = \mathrm{I}$. Thus, for $t \geq s$, we must have

$$\frac{e^{|t|-|s|}}{4} \leq Me^{\beta|s|-\gamma|t-s|}.$$

For any fixed $s \in \mathbb{R}$, letting $t \to +\infty$ leads to a contradiction, as the left-hand side grows exponentially while the right-hand side decays to zero. Therefore, $\mathcal{S}$ does not admit a NEDI with $\Pi^s = \mathrm{I}$. A similar argument shows us that $\mathcal{S}$ does not admit NEDI with $\Pi^s = 0$.

On the other hand, for all $t,s \in \mathbb{R}$, we can establish the following estimate:

$$\|S(t,s)\| \leq \frac{2e^{|t|}}{\frac{1}{2}e^{|s|}} = 4e^{|t|-|s|} \leq 4e^{2|t|-|t-s|}.$$

This implies that the norm cannot be nonuniformly estimated by the initial time s, but rather by the final time t. To encompass such examples, (Langa; Obaya; Sousa, 2024) introduced the concept of *nonuniform exponential dichotomy of type II*, which we define below.

4.2 Nonuniform exponential dichotomy of type II

In the previous chapters, we studied the concept of nonuniform exponential dichotomy where the bound K depends on the initial time s. From now on, this will be referred to as *type I dichotomy*, following the terminology used by (Langa; Obaya; Sousa, 2024). However, in certain dynamical systems, particularly those arising from specific differential equations, the nonuniformity may naturally manifest with dependence on the final time t. To address this, we introduce the concept of "nonuniform exponential dichotomy of type II" (NEDII).

Definition 4.2 (Nonuniform exponential dichotomy of type II)**.** A continuous evolution process $\mathcal{S} := \{S(t,s) \mid t,s \in \mathbb{R},\, t \geq s\}$ admits a NONUNIFORM EXPONENTIAL DICHOTOMY OF TYPE II (NEDII) if there exist a family of bounded projections $\Pi^u := \{\Pi^u(t) \mid t \in \mathbb{R}\}$, constants $M \geq 1$, $\alpha \geq 0$ and $\gamma > 0$, and a function $K\colon \mathbb{R} \to [1,+\infty)$ satisfying

$$K(t) \leq Me^{\alpha|t|} \qquad \text{for all } t \in \mathbb{R}$$

such that, defining $\Pi^s := \{\Pi^s(t) := \mathrm{I} - \Pi^u(t) \mid t \in \mathbb{R}\}$, the following conditions hold:

(C1) $\Pi^u(t)S(t,s) = S(t,s)\Pi^u(s)$ for all $t \geq s$;

(C2) The restriction $S(t,s)|_{\mathrm{Im}(\Pi^u(s))}\colon \mathrm{Im}(\Pi^u(s)) \to \mathrm{Im}(\Pi^u(t))$ is an isomorphism for all $t \geq s$ and we define $S(s,t)$ as its inverse;

(C3) For all $t \geq s$,

$$\|S(t,s)\Pi^s(s)\| \leq K(t)e^{-\gamma|t-s|};$$

(C4) For all $t \leq s$,

$$\|S(t,s)\Pi^u(s)\| \leq K(t)e^{-\gamma|t-s|}.$$

We call Π^u the family of UNSTABLE PROJECTIONS, Π^s the family of STABLE PROJECTIONS, K the BOUND FUNCTION, α the NONUNIFORMITY EXPONENT, and γ the DICHOTOMY EXPONENT.

It is crucial to understand that type I and type II are distinct properties. As demonstrated in Example 4.1, there exist processes that admit a NEDII but fail to admit a NEDI. We now present an example similar to Example 4.1 where the converse holds, that is, an evolution process that admits a NEDI but does not admit a NEDII.

Example 4.3. Consider the ODE on $\mathbb{R}$ given by

$$x'(t) = -\tanh(t)x(t),$$

and the evolution process induced by it:

$$S(t,s)u := u \cdot \frac{\operatorname{sech} t}{\operatorname{sech} s} = u \cdot \frac{e^s + e^{-s}}{e^t + e^{-t}}$$

then $\mathcal{S}$ admits NEDI with unstable projection family $\Pi^u = \mathrm{I}$, bound $K(s) := e^{2|s|}$ and exponent $\gamma := 1$. Moreover, $\mathcal{S}$ does not admit NEDII.

Indeed, for all $t, s \in \mathbb{R}$ we can estimate the operator norm as follows:

$$\frac{e^{|s|-|t|}}{4} \leq \|S(t,s)\| \leq 4e^{|s|-|t|} \leq 4e^{2|s|-|t-s|}.$$

Therefore, the process admits a NEDI with family of projections, bound and exponent stated above.

Now let us prove that $\mathcal{S}$ does not admit NEDII. Suppose, to obtain contraction, that $\mathcal{S}$ does indeed admit a NEDII with $\Pi^s = \mathrm{I}$. Thus, for $t \geq s$, we must have

$$\frac{e^{|s|-|t|}}{4} \leq Me^{\beta|t|-\gamma|t-s|}.$$

For any fixed $t \in \mathbb{R}$, letting $s \to -\infty$ leads to a contradiction, as the left-hand side grows exponentially while the right-hand side decays to zero. Therefore, $\mathcal{S}$ does not admit a NEDII with $\Pi^s = \mathrm{I}$. A similar argument shows us that $\mathcal{S}$ does not admit NEDII with $\Pi^s = 0$.

Note that Examples 4.1 and 4.3 are defined similarly yet exhibit a fundamental distinction. Whereas the first shows exponential growth in both directions, the second features exponential decay. This highlights that the specific growth dynamics of the system determine whether a type I or type II dichotomy is more appropriate.

Inspired by (Langa; Obaya; Sousa, 2024, Proposition 2.10), we now present a generalization of the two previous examples.

Proposition 4.4. Let $a,b \in \mathbb{R}$ and consider a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that

- $\lim_{t\to+\infty} f(t) = a$;
- $\lim_{t\to-\infty} f(t) = b$.

Let $\mathcal{S}$ be the evolution process induced by $x' = f(t)x$. Then,

(P1) If $\max\{a,-b\} < 0$, then $\mathcal{S}$ admits a NEDI but does not admit any NEDII (i.e., $a < 0$ and $b > 0$);

(P2) If $\max\{-a,b\} < 0$, then $\mathcal{S}$ admits a NEDII but does not admit any NEDI (i.e., $a > 0$ and $b < 0$).

Proof. Consider the function $\delta\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\delta(x) := \begin{cases} a, & x \geq 0 \\ b, & x < 0 \end{cases}.$$

For all $t,s \in \mathbb{R}$, we can estimate its integral by

$$\int_s^t \delta(r)dr \leq \max\{a,-b\}|t| + \max\{-a,b\}|s|.$$

Note that for all $t,s \in \mathbb{R}$, we have

$$\lim_{|t-s|\to+\infty} \frac{1}{t-s}\int_s^t |f(r)-\delta(r)|dr = 0.$$

Thus, for any $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$\int_s^t |f(r)-\delta(r)|dr \leq K_\varepsilon + \varepsilon|t-s|.$$

Therefore,

$$\begin{aligned}\int_s^t f(r)dr &\leq \int_s^t \delta(r)dr + K_\varepsilon + \varepsilon|t-s| \\ &\leq \max\{a,-b\}|t| + \max\{-a,b\}|s| + K_\varepsilon + \varepsilon|t-s|.\end{aligned}$$

Knowing that $S(t,s)u = u\cdot e^{\int_s^t f(r)dr}$, for all $t,s \in \mathbb{R}$ we obtain the estimate:

$$\|S(t,s)\| \leq e^{K_\varepsilon} e^{\max\{a,-b\}|t|+\max\{-a,b\}|s|+\varepsilon|t-s|}. \tag{4.18}$$

We now prove the items of the proposition.

(P1) Suppose $\max\{a,-b\}<0$. This implies $\min\{-a,b\}>0$. Using the triangle inequality $|t-s|\leq|t|+|s|$, we have $|t|\geq|t-s|-|s|$. Since $\max\{a,-b\}<0$, multiplying by this negative factor reverses the inequality, yielding:

$$\max\{a,-b\}|t|\leq\max\{a,-b\}(|t-s|-|s|).$$

Substituting this into (4.18), we conclude:

$$\|S(t,s)\|\leq e^{K_\varepsilon}e^{(\max\{-a,b\}-\max\{a,-b\})|s|+(\varepsilon+\max\{a,-b\})|t-s|}.$$

Choosing $\varepsilon>0$ sufficiently small such that $\gamma:=-(\varepsilon+\max\{a,-b\})>0$, and setting $\beta:=\max\{-a,b\}-\max\{a,-b\}$, we conclude that $\mathcal{S}$ admits a NEDI with $\Pi^u=\mathrm{I}$ (or $\Pi^s=0$).

Now we show that $\mathcal{S}$ does not admit a NEDII. Suppose, for the sake of contradiction, that $\mathcal{S}$ admits a NEDII with $\Pi^s=\mathrm{I}$ and exponents $\alpha\geq0$ and $\gamma>0$. Then, for any $\varepsilon>0$, there exists $K_\varepsilon>0$ such that

$$\int_s^t\delta(r)dr\leq\int_s^t f(r)dr+K_\varepsilon+\varepsilon|t-s|.$$

Observe that we also have the lower bound:

$$\int_s^t\delta(r)dr\geq\min\{a,-b\}|t|+\min\{-a,b\}|s|.$$

If $\mathcal{S}$ admits a NEDII, then for all $t\geq s$ we must have

$$\min\{a,-b\}|t|+\min\{-a,b\}|s|\leq\ln M+\alpha|t|-(\gamma-\varepsilon)|t-s|.$$

Since $\max\{a,-b\}<0$, it follows that $a<0$ and $b>0$. Consequently, $\min\{-a,b\}>0$. Fixing $t\in\mathbb{R}$ and letting $s\to-\infty$, the left-hand side grows linearly with $|s|$ (with positive coefficient $\min\{-a,b\}$), while the right-hand side goes to $-\infty$ (due to the term $-\gamma|s|$). This is a contradiction. Thus, $\mathcal{S}$ does not admit a NEDII with $\Pi^s=\mathrm{I}$.

(P2) The proof is analogous to the previous item. We conclude that $\mathcal{S}$ admits a NEDII with exponents $\alpha:=\max\{a,-b\}-\max\{-a,b\}$ and $\gamma:=-(\varepsilon+\max\{-a,b\})$ for any $\varepsilon\in(0,-\max\{-a,b\})$.

□

While the previous examples illustrated cases admitting one dichotomy but not the other, the following proposition presents some inequalities that allow us to manipulate the dichotomy exponents. These inequalities enable us to determine a sufficient condition for an evolution process to admit both types of dichotomies.

Proposition 4.5 (Useful equalities and inequalities). Let $\alpha, \beta \geq 0$ and $\gamma > 0$ be constants and let $t, s \in \mathbb{R}$. The following statements hold:

(1) If $t \geq s \geq 0$ or $0 \geq s \geq t$, then $\beta|s| - \gamma|t-s| = \beta|t| - (\gamma+\beta)|t-s|$.

(2) If $0 \geq t \geq s$ or $s \geq t \geq 0$, then $\beta|s| - \gamma|t-s| = \beta|t| - (\gamma-\beta)|t-s|$.

(3) For all $t, s \in \mathbb{R}$, we have $\alpha|t| + \beta|s| - \gamma|t-s| \leq (\alpha+\beta)|t| - (\gamma-\beta)|t-s|$.

(4) For all $t, s \in \mathbb{R}$, we have $\alpha|t| + \beta|s| - \gamma|t-s| \leq (\alpha+\beta)|s| - (\gamma-\alpha)|t-s|$.

(5) For all $t, s \in \mathbb{R}$, we have $\alpha|t| + \beta|s| - \gamma|t-s| \leq \beta|t| + \alpha|s| - (\gamma-\alpha-\beta)|t-s|$.

Proof. Follows from the equalities $|t| = |s| + |t-s|$ when $t \geq s \geq 0$, $|t| = |s| - |t-s|$ when $0 \geq t \geq s$, $|t| \leq |t-s| + |s|$ and $|s| \leq |t-s| + |t|$ for all $t,s \in \mathbb{R}$. □

Corollary 4.6. Let $\mathcal{S}$ be an evolution process. By Proposition 4.5 we have

(P1) If $\mathcal{S}$ admits NEDI with nonuniformity exponent β and dichotomy exponent $\gamma > \beta$, then it admits NEDII with same nonuniformity exponent β but dichotomy exponent $\gamma - \beta$;

(P2) If $\mathcal{S}$ admits NEDII with nonuniformity exponent α and dichotomy exponent $\gamma > \alpha$, then it admits NEDI with same nonuniformity exponent α but dichotomy exponent $\gamma - \alpha$.

As an application of Proposition 4.5 and Corollary 4.6, consider Example 3.4.

Example (Example 3.4 revisited). Let $\mathcal{S}$ be the evolution process defined in Example 3.4. Manipulating the exponents we have that

- $\mathcal{S}$ admits NEDII on $\mathbb{R}^+$ with $\Pi^s(t) = \mathrm{I}$, nonuniformity exponent $2a$ and dichotomy exponent $b + a$.
- $\mathcal{S}$ admits NEDI on $\mathbb{R}^-$ with $\Pi^s(t) = \mathrm{I}$, nonuniformity exponent $2a$ and dichotomy exponent $b + a$.
- $\mathcal{S}$ admits NEDI on $\mathbb{R}^+$ with $\Pi^s(t) = \mathrm{I}$, nonuniformity exponent $2a$ and dichotomy exponent $b - a$.
- $\mathcal{S}$ admits NEDII on $\mathbb{R}^-$ with $\Pi^s(t) = \mathrm{I}$, nonuniformity exponent $2a$ and dichotomy exponent $b - a$.
- $\mathcal{S}$ admits NEDI and NEDII on $\mathbb{R}$ with $\Pi^s(t) = \mathrm{I}$, nonuniformity exponent $2a$ and dichotomy exponent $b - a$.

Notice that on $\mathbb{R}^+$ if $b < 3a$, $\mathcal{S}$ admits NEDII with dichotomy exponent $b+a > 2a$ and NEDI with dichotomy exponent $b-a < 2a$. Hence, in some situations, it is possible to choose NEDII with "better" relation in the exponents than NEDI, that is, we can apply NEDII robustness result but not NEDI robustness. Similarly, on $\mathbb{R}^-$ NEDI has a "better" relationship of exponents than NEDII.

Next, we define the dual of an evolution process, which allows us to construct evolution processes on the dual of the space on which the original process is defined.

4.3 Dual of an evolution process and its properties

In this section, we explore the "dual" of an evolution process and verify that it retains the fundamental properties of the original system, provided that the latter is invertible. A central result is the relationship between the two types of dichotomies: we demonstrate that the dual of a process admitting a dichotomy of type I admits a dichotomy of type II, and vice-versa.

First, we define the dual operator. In Functional Analysis, the dual space of X (denoted by X^*) consists of all bounded linear functionals on X. The dual of an operator $T\colon X \to X$ is the operator $T^*\colon X^* \to X^*$ defined by composition: for any functional $\varphi \in X^*$, the functional $T^*\varphi$ is given by $\varphi \circ T$.

Definition 4.7 (Dual operator). Let $A \in \mathcal{B}(X)$. The DUAL OPERATOR of A, denoted by $A^* : X^* \to X^*$, is defined as the unique bounded linear operator satisfying

$$\langle Ax, x^* \rangle = \langle x, A^* x^* \rangle \qquad \text{for all } x \in X \text{ and } x^* \in X^*.$$

Due to the properties of dual operators, taking the dual of an evolution process on X yields a new process defined on X^*.

Proposition 4.8 (Dual of an evolution process). Let S be an invertible evolution process. For all $t,s \in \mathbb{R}$ define $T(t,s)$ by

$$T(t,s) := [S(s,t)]^* .$$

Then $\mathcal{T} := \{T(t,s) \mid t,s \in \mathbb{R}\}$ is an invertible evolution process in X^*.

Proof. We shall prove Definition's 3.1 conditions.

(C1) For all $t \in \mathbb{R}$ we have

$$T(t,t) = [S(t,t)]^* = [\mathrm{I}_X]^* = \mathrm{I}_{X^*} .$$

(C2) For all $t,s,r \in \mathbb{R}$ such that $t \geq s \geq r$ we have

$$T(t,s)T(s,r) = [S(s,t)]^* \, [S(r,s)]^* = [S(r,s)S(s,t)]^* = [S(r,t)]^* = T(t,r)$$

(C3) Let $(t_n, s_n, x_n^*)_{n\in\mathbb{N}}$ be a sequence such that $(t_n, s_n, x_n^*) \to (t, s, x^*)$. We have

$$\begin{aligned}\|T(t_n, s_n)x_n^* - T(t,s)x^*\| &= \sup_{\|x\|=1} |\langle x, T(t_n, s_n)x_n^*\rangle - \langle x, T(t,s)x^*\rangle| \\ &= \sup_{\|x\|=1} |\langle S(s_n, t_n)x, x_n^*\rangle - \langle S(s,t)x, x^*\rangle| \,.\end{aligned}$$

For all $x \in X$ we have

$$\begin{aligned}|\langle S(s_n, t_n)x, x_n^*\rangle - \langle S(s,t)x, x^*\rangle| &= |\langle S(s_n, t_n)x - S(s,t)x, x_n^*\rangle - \langle S(s,t)x, x_n^* - x^*\rangle| \\ &\leq |\langle S(s_n, t_n)x - S(s,t)x, x_n^*\rangle| + |\langle S(s,t)x, x_n^* - x^*\rangle| \\ &\leq \|x_n^*\| \, \|S(s_n, t_n)x - S(s,t)x\| + \|S(s,t)x\| \, \|x_n^* - x^*\| \,.\end{aligned}$$

Since $(x_n^*)_{n\in\mathbb{N}}$ is convergent, it is bounded. Hence, by the continuity of $H(s,t,x) := S(s,t)x$, we have

$$\lim_{n\to+\infty} \|x_n^*\| \, \|S(s_n, t_n)x - S(s,t)x\| + \|S(s,t)x\| \, \|x_n^* - x^*\| = 0,$$

implying that

$$\lim_{n\to+\infty} \|T(t_n, s_n)x_n^* - T(t,s)x^*\| = 0.$$

Therefore $G : \mathbb{R}^2 \times X \to X$ defined by $G(t,s,x) := T(t,s)x$ is continuous.

□

We now show that if an invertible evolution process admits a dichotomy of one type, its dual admits a dichotomy of the other type.

Proposition 4.9 (Dual of an exponential dichotomy). Let S be an invertible evolution process that admits nonuniform exponential dichotomy of type I (or type II). If $\mathcal{T}$ is the dual of $\mathcal{S}$, then $\mathcal{T}$ admits a nonuniform exponential dichotomy of type II (or type I) with same bound and dichotomy exponent.

Proof. We shall prove Definition's 4.2 conditions. Define $\widetilde{\Pi}^u := \{\widetilde{\Pi}^u(t) := [\Pi^u(t)]^* \mid t \in \mathbb{R}\}$.

(C1) For all $t,s \in \mathbb{R}$ such that $t \geq s$ we have

$$T(t,s)\widetilde{\Pi}^u(s) = [S(s,t)]^* \, [\Pi^u(s)]^* = [\Pi^u(s)S(s,t)]^* = [S(s,t)\Pi^u(t)]^* = \widetilde{\Pi}^u(t)T(t,s)$$

(C2) Due to the properties of dual operators, for all $t,s \in \mathbb{R}$ the restriction $T(t,s)|_{\widetilde{\Pi}^u(s)}$ is an isomorphism.

(C3) Define $\widetilde{\Pi}^s := \{\widetilde{\Pi}^s(t) := I - \widetilde{\Pi}^u(t) \mid t \in \mathbb{R}\}$. If $t \geq s$, we have

$$\left\|T(t,s)\widetilde{\Pi}^s(s)\right\| = \left\|[\Pi^s(s)S(s,t)]^*\right\| = \|S(s,t)\Pi^s(t)\| \leq Me^{\beta|t|-\gamma|t-s|},$$

(C4) If $t \leq s$, we have

$$\left\|T(t,s)\widetilde{\Pi}^u(s)\right\| = \|S(s,t)\Pi^u(t)\| \leq Me^{\beta|t|-\gamma|t-s|}.$$

Proving that if $\mathcal{S}$ admits NEDII, then $\mathcal{T}$ admits NEDI is similar. □

Finally, we can state a robustness result for NEDII.

4.4 A robustness result for NEDII

We now prove that, subject to specific conditions on the exponents, process invertibility, and space reflexivity, an evolution process admitting a NEDII is robust. The strategy is straightforward: if a process $\mathcal{S}$ admits a NEDII, its dual $\mathcal{S}^*$ admits a NEDI. We can therefore apply the robustness theorem for NEDI to the dual process. Finally, if the space is reflexive, we transfer the result back to the original operator.

Theorem 4.10. (Robustness of NEDII). Let $\mathcal{S}_1$ be a continuous invertible evolution process that admits NEDII with bound $K(t) \leq Me^{\alpha|t|}$ and exponent $\gamma > \alpha$ such that

$$L := \sup_{0\leq|t-s|\leq 1} \left\{K(s)^{-1}\|S_1(t,s)\|\right\} < +\infty.$$

Then there exists $\varepsilon > 0$ such that if $\mathcal{S}_2$ is another invertible evolution process satisfying

$$\sup_{0\leq|t-s|\leq 1} \{K(s)\|S_1(t,s) - S_2(t,s)\|\} < \varepsilon,$$

then $\mathcal{T}_2 := \{T_2(t,s) := [S_2(s,t)]^* \mid t,s \in \mathbb{R}\}$ admits NEDI with bound

$$\widehat{K}(t) := L\max\{\widetilde{M}_1, \widetilde{M}_2\}^2 e^{\min\{\widetilde{\gamma},\widehat{\gamma}\}} e^{2\alpha|t|}$$

and dichotomy exponent $\min\{\widetilde{\gamma},\widehat{\gamma}\} - \alpha$ as defined in Theorem 2.25. Additonally, if X is reflexive, then $\mathcal{S}_2$ admits NEDII with same bound and exponent defined before.

Proof. Let $\mathcal{T}_1$ be the dual evolution process of $\mathcal{S}_1$. Then applying Proposition 4.9 we guarantee that $\mathcal{T}_1$ admits NEDI with same bound and expontent as $\mathcal{S}_1$. By hypothesis, $\mathcal{T}_1$ satisfies

$$\sup_{0\leq|t-s|\leq 1} \left\{K(t)^{-1}\|T_1(t,s)\|\right\} = \sup_{0\leq|t-s|\leq 1} \left\{K(t)^{-1}\|S_1(s,t)\|\right\} < +\infty,$$

then, by Theorem 3.14, there exists $\varepsilon > 0$ such that if $\mathcal{T}$ is a continuous evolution process on X^* satisfying

$$\sup_{0\leq|t-s|\leq 1} \{K(t)\, \|T_1(t,s) - T(t,s)\|\} < \varepsilon,$$

we have that $\mathcal{T}$ admits NEDI. Let $\mathcal{S}_2$ be an evolution process and $\mathcal{T}_2$ its dual. Suppose that

$$\sup_{0\leq|t-s|\leq 1} \{K(s)\, \|S_1(t,s) - S_2(t,s)\|\} < \varepsilon.$$

Then, by properties of dual operators, we have

$$\begin{aligned}\sup_{0\leq|t-s|\leq 1} \{K(t)\, \|T_1(t,s) - T_2(t,s)\|\} &= \sup_{0\leq|t-s|\leq 1} \{K(t)\, \|[S_1(s,t)]^* - [S_2(s,t)]^*\|\} \\ &= \sup_{0\leq|t-s|\leq 1} \{K(t)\, \|S_1(s,t) - S_2(s,t)\|\} \\ &< \varepsilon,\end{aligned}$$

implying that $\mathcal{T}_2$ admits NEDI.

Now assume that X is reflexive. We shall prove that $\mathcal{S}_2$ admits NEDII. Let $\mathcal{T}_2^* = \{[T_2(s,t)]^* = [S_2(t,s)]^{**} \mid t,s \in \mathbb{R}\}$ be the dual of $\mathcal{T}_2$. As proved before, $\mathcal{T}_2$ admits NEDI with family of unstable projections $\{\Gamma^u(t) \mid t \in \mathbb{R}\}$, therefore, by Proposition 4.9, $\mathcal{T}_2^*$ admits NEDII with family of unstable projections $\{[\Gamma^u(t)]^* \mid t \in \mathbb{R}\}$. By reflexivity, the evaluation map $J : X \to X^{**}$ defined by

$$\langle x^*, Jx\rangle = \langle x, x^*\rangle \qquad \text{for all } x \in X \text{ and } x^* \in X^*$$

is an isometric isomorphism. For all $t,s \in \mathbb{R}$, $x \in X$ and $x^* \in X^*$, we have

$$\begin{aligned}\langle x^*, [S_2(t,s)]^{**}(Jx)\rangle &= \langle [S_2(t,s)]^* x^*, (Jx)\rangle \\ &= \langle x, [S_2(t,s)]^* x^*\rangle \\ &= \langle S_2(t,s)x, x^*\rangle \\ &= \langle x^*, J(S_2(t,s)x)\rangle.\end{aligned}$$

Therefore

$$\langle x^*, [S_2(t,s)]^{**}(Jx)\rangle = \langle x^*, J(S_2(t,s)x)\rangle \qquad \text{for all } x \in X \text{ and } x^* \in X^*,$$

implying that $JS_2(t,s) = [S_2(t,s)]^{**}J$ and by consequence

$$S_2(t,s) = J^{-1}[S_2(t,s)]^{**}J$$

Define $\Pi^u := \{\Pi^u(t) := J^{-1}[\Gamma^u(t)]^* J \mid t \in \mathbb{R}\}$ and notice that by the isometric properties of the evaluation map, we have

$$\|S_2(t,s)\Pi^u(s)\| = \left\|J^{-1}[S_2(t,s)]^{**}[\Gamma^u(t)]^* J\right\| = \|[S_2(t,s)]^{**}[\Gamma^u(t)]^*\|\,.$$

Therefore $\mathcal{S}_2$ admits NEDII with family of unstable projections Π^u. □

Remark 4.11. The reasoning in Remark 3.13 applies analogously to Theorem 4.10. That is, instead of requiring

$$\sup_{0\le|t-s|\le1}\left\{K(s)^{-1}\|S_1(t,s)\|\right\}<+\infty,$$

we could equivalently assume

$$\sup_{0\le|t-s|\le1}\left\{K(t)^{-1}\|S_1(t,s)\|\right\}<+\infty.$$

We conclude this chapter by presenting illustrative example that elucidate the distinctions between type I and type II robustness. The next example comes from (Langa; Obaya; Sousa, 2024) and shows us an ODE that induces a process that admits NEDI and NEDII, but only one of the robustness results can be applied, showing that the two types of robustness do not necessarily coincide.

Example 4.12. Let $a,b,c,d>0$ with $b>a$ and $d>c$. Consider the function

$$f(t):=\begin{cases}-b-at\sin t, & t\ge0\\ -d-ct\sin t, & t<0\end{cases}$$

and let $\mathcal{S}$ be the evolution process induced by $x'=f(t)x$. Then $\mathcal{S}$ admits NEDI with $\Pi^s(t)=\mathrm{I}$, bound $K_n:=e^{2a}e^{\beta|n|}$ with $\beta:=\max\{2a,2c\}$ and dichotomy exponent $\gamma_1:=\min\{b-a,d+c\}$. Also, $\mathcal{S}$ admits NEDII with same family of projections and nonuniformity exponent, but with dichotomy exponent $\gamma_2:=\min\{b+a,d-c\}$.

Indeed, from Example 3.4 we know that if $t\ge s\ge0$, then

$$\|S(t,s)\|\le e^{2a}e^{2a|s|-(b-a)|t-s|},$$

if $0>t\ge s$, then

$$\|S(t,s)\|\le e^{2c}e^{2c|s|-(d+c)|t-s|},$$

and if $t\ge0>s$,

$$\begin{aligned}\|S(t,s)\|&=e^{-a\sin t+at\cos t-bt+c\sin s-cs\cos s+ds}\\&\le e^{a+at-bt+c-cs+ds}\\&=e^{a+c}e^{-(b-a)t-cs+ds}\\&\le e^{\beta}e^{-\gamma_1t-cs+ds}\\&=e^{\beta}e^{-\gamma_1t-cs+ds+cs-cs}\\&=e^{\beta}e^{-\gamma_1t-cs-(d+c)|s|-cs}\\&\le e^{\beta}e^{-\gamma_1t-cs-\gamma_1|s|-cs}\\&=e^{\beta}e^{-\gamma_1t-cs+\gamma_1s-cs}\\&=e^{\beta}e^{2c|s|-\gamma_1|t-s|}\end{aligned}$$

Hence, for all $t \geq s$ we conclude

$$\|S(t,s)\| \leq e^{\beta} e^{\beta|s| - \gamma_1|t-s|}.$$

Therefore, the process admits a NEDI with family of projections, bound and exponent stated above. A similar reasoning can be used to prove that $\mathcal{S}$ also admits NEDII with same family of projections and nonuniformity exponent, but with dichotomy exponent γ_2.

Is possible to choose values of a,b,c and d such that $\gamma_1 > \beta$ and $\gamma_2 \leq \beta$, i.e., Theorem 3.14 can be applied while Theorem 4.10 does not. Similarly, is possible to choose values of a,b,c and d such that $\gamma_2 > \beta$ and $\gamma_1 \leq \beta$, i.e., Theorem 4.10 can be applied while Theorem 3.14 does not.

Some sufficient conditions for $\gamma_1 > \beta$ and $\gamma_2 \leq \beta$ to both be true are

$$c < d \leq 3c \quad \text{and} \quad b > 3a \quad \text{and} \quad b > 2c + a \quad \text{and} \quad |d - 2a| < c.$$

A concrete example of those conditions might be $a = 1$ and $c = 2$, implying $d \in (2,4)$ and $b > 3$. Indeed,

$$\min\{b - a, d + c\} = \min\{b - 1, d + 2\} > \min\{3 - 1, 2 + 2\} = 4 = \max\{2a,2c\}$$
$$\min\{b + a, d - c\} = \min\{b + 1, d - 2\} \leq \min\{b + 1, 4 - 2\} = \min\{b + 1, 2\} = 2 < \max\{2a,2c\}.$$

While for $\gamma_2 > \beta$ and $\gamma_1 \leq \beta$ to both be true, the conditions are

$$a < b \leq 3a \quad \text{and} \quad d > 3c \quad \text{and} \quad d > 2a + c \quad \text{and} \quad |b - 2c| < a.$$

Based on Example 4.1, we demonstrated that not every evolution process admits a nonuniform exponential dichotomy with nonuniformity dependent on the initial time as some depend on the final time. This motivated the definition of a type II nonuniform exponential dichotomy (NEDII). We showed that certain processes fail to admit the first type but admit the second, and vice versa. Proposition 4.4 revealed that this relationship of admissibility in $\mathbb{R}$ (and consequently in $\mathbb{R}^n$, as we can extend the result to any dimension as in Example 3.5) is linked to the asymptotic behavior of the process. Theorem 4.10 showed that, under sufficient conditions, type II dichotomies are robust, while Example 4.12 illustrated that the robustness properties of type I and type II do not necessarily coincide, even for processes admitting both. Ultimately, we conclude that both concepts are indeed distinct and serve different and complementary purposes, since in some examples type II robustness can be applied while type I robustness is uncertain, and vice versa.

5 Conclusion

In this dissertation, we studied nonuniform exponential dichotomies in Banach spaces. In the discrete setting, we characterized the existence of a nonuniform exponential dichotomy through the admissibility of the pair of sequence spaces $(\ell^\infty_K, \ell^\infty)$ and decay of forward/backward solutions. By constructing the Green's function, we obtained explicit representations for the projections and bounded solutions of the nonhomogeneous equation. We proved that if the dichotomy exponent γ is strictly greater than the nonuniformity growth rate β, the dichotomy is robust under small linear perturbations.

For the continuous case of type I, we utilized the discretization technique to establish a bridge between continuous evolution processes and discrete dynamics. We proved that a continuous process admits a nonuniform exponential dichotomy if and only if its discretizations satisfy the corresponding discrete conditions, provided that a regularity hypothesis is met. This correspondence enabled us to transfer the discrete robustness theorem directly to the continuous setting.

Regarding type II dichotomies, we introduced the concept of the dual evolution process and proved that the dual of an invertible process admitting a type I dichotomy admits a type II dichotomy, and vice versa. This duality provided a mechanism to prove the robustness of type II dichotomies by transferring the problem to the dual space. However, the result relies on the properties of the evaluation map, limiting the validity of our proof for type II robustness in this framework to reflexive Banach spaces.

Future work

The results established in this work suggest several directions for further investigation. One of them concerns the reflexivity requirement for the robustness of type II dichotomies. A natural open problem is to determine whether this robustness holds in non-reflexive spaces. Addressing this would likely require the development of a direct discretization technique for type II processes or a variation of parameters formula adapted to the dependence on the final time, avoiding the transition to the dual space. In this regard, it is worth noting that the direct proof with type I relies on a Gronwall-like bound, which would need to be adapted to the specific growth behavior of type II dichotomies. Furthermore, such an approach requires establishing the existence and uniqueness of global nonhomogeneous solutions, which may necessitate altering the underlying solution/function space to be more compatible with the structure of type II dichotomies.

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